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ADDENDUM TO "ON SMOOTH MAPS WITH FINITELY MANY CRITICAL POINTS"

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1. *Statements*

1.1.

The proof of the Proposition 4.1 from [**2**] is incomplete and contains several inaccuracies which might lead to confusions. The aim of the present note is to give alternative proofs and to add a few comments and corrections for clarifying the picture.

Recall that we considered smooth maps $f : M \to N$ with $\partial M = f^{-1}(\partial N)$ such that f has no critical points on ∂M, where M, N are compact manifolds, possibly with boundary. We denoted by $\varphi(M, N)$ the minimal number of critical points of such maps. The Proposition alluded above reads:

Proposition 1.1. *If $\varphi(M^m, N^n)$ is finite, $n \geq 2$ and either $m = n + 1 \neq 4$, $m = n+2 \neq 4$, or $m = n+3 \notin \{5, 6, 8\}$ (when one assumes the Poincaré conjecture to be true) then M is homeomorphic to a fibration of base N. In particular, if $m = 3, n = 2$ then $\varphi(M^3, N^2) \in \{0, \infty\}$, except possibly for M^3 a non-trivial homotopy sphere and $N^2 = S^2$.*

Remark 1.1. The condition $n \geq 2$ was not explicit in [**2**]. This is essential since any compact manifold (not necessarily a fibration) admits smooth functions with finitely many critical points. Notice also that estimations of the minimal number of critical points (not necessarily non-degenerate) of functions (i.e. for $n = 1$) have been obtained first by Takens ([**19**]) in terms of the Lusternik-Schnirelman category.

Remark 1.2. The present and former papers are concerned only with compact manifolds, though as the results are valid more generally for proper maps between open manifolds. Notice that the properness is essential. In fact, an old result of Morris Hirsch states ([**8**]) that any non-closed smooth manifold admits a smooth function without critical points.

1.2. *A direct proof of Proposition 1.1*

In local coordinates centered around the singular point (and value) we consider the link of the singularity $L = S^{m-1}_{\varepsilon} \cap f^{-1}(0)$, where S^{m-1}_{ε} is a sphere of radius ε, chosen small enough. This might well depend on ε and not stabilize as ε goes to zero. We can perturb f by an arbitrary small smooth isotopy so that $f^{-1}(0)$ becomes

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transverse to the sphere S_ε^{m-1}. Then the link is a smooth $(m-n)$-submanifold of S_ε^{m-1} with trivial normal bundle. Notice that the perturbed function will still have an isolated critical point. Thus $f^{-1}(0)$ is a manifold everywhere but at the critical point.

The result of [**1**] implies that L^{m-n} is the link of an isolated real polynomial singularity $p:\mathbb{R}^m\to\mathbb{R}^n$, i.e. (S_ε^{m-1},L) is diffeomorphic to $(S_\varepsilon^{m-1},S_\varepsilon^{m-1}\cap p^{-1}(0))$, and p has no other critical points inside the ball of radius ε.

One further knows from [**4**] that p should be a topological fibration under the dimensional assumptions stated in Proposition 1.1. Let us remark that p is not necessarily equivalent to f, at least unless the codimension is small. One can handle this difficulty by considering the conical extension $\hat{f}$ of f to the ball B_ε^m, as follows: $\hat{f}(x)=|x|f\left(\frac{x}{|x|}\right)$. Then $\hat{f}$ is a continuous function which is smooth and has no critical points outside the origin, and its (topological) critical point is cone-like. Moreover, $\hat{f}$ coincides with f outside the ball of radius ε.

We claim now that $\hat{f}$ and p are locally topologically equivalent, and hence $\hat{f}$ is a topological submersion. By replacing f with the corresponding $\hat{f}$ around each critical point we obtain therefore a topological submersion.

The claim is a consequence of the classification given in [**10**] of cone-like isolated singularities. Roughly speaking two cone-like isolated singularities are equivalent if their links are invertible cobordant and their Milnor-type fibrations are equivalent.

Moreover, the link L^{m-n} of the polynomial p is diffeomorphic to a $(m-n)$-sphere trivially embedded in S_ε^{m-1}. Thus any manifold invertible cobordant (thus h-cobordant) to L^{m-n} is also homeomorphic (and thus diffeomorphic) to a sphere because the dimension $m-n\leq 3$, with the use of the Poincaré Conjecture for $m-n=3$. Further, let W^{m-1} be the complement in S_ε^{m-1} of a regular neighborhood of the link. The Milnor-type fibration defined in [**10**] is a fibration $W^{m-1}\to S^{n-1}$. In particular the fiber F^{m-n} should be a homotopy ball and thus diffeomorphic to a $(m-n)$-ball, by assuming again the Poincaré Conjecture. Then any two fibrations as above are fiberwise homeomorphic, and thus the result of [**10**] implies the claim.

1.3. *Local equivalence to a polynomial singularity*

The proof outlined in [**2**] aimed at showing first:

LEMMA 1.1. *Assume that $f:M^m\to N^n$ is a smooth map with finitely many critical points between the compact manifolds M^m and N^n, and $0\leq m-n\leq 3$. Then, for each $x\in M^m$ there exists a polynomial map $p:(\mathbb{R}^m,0)\to(\mathbb{R}^n,0)$ having an isolated singularity at the origin so that f is locally topologically equivalent at x with the polynomial p.*

REMARK 1.3. The condition $m-n\leq 3$ was omitted in Lemma 4.1 from [**2**], while the requirements on the polynomial function were somewhat implicit in the proof.

1.4. *End of the proof of Proposition 1.1 when assuming Lemma 1.1*

Further, a polynomial map with an isolated critical point is a locally topological submersion if the respective dimensions verify the requirements of Proposition 0.1, according to Church and Lamotke (see [**4**]). Thus the map f is everywhere locally topologically equivalent to the projection $\mathbb{R}^m \to \mathbb{R}^n$. Eventually, the map f is proper and hence one can apply the topological analogue of a classical theorem by Ehresmann - proved by Cheeger and Kister in [**3**]) - to obtain that f is the projection map of a topological fiber bundle.

2. *Comments*

2.1. *The condition $m - n \leq 3$ is necessary in Lemma 1.1: smooth functions with isolated singularities not $\mathcal{C}^0$-equivalent to real analytic functions*

The requirements on the codimension are essential. In fact, for each pair (m, n) with $m - n \geq 4$ there exist smooth maps $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ having an isolated critical point at the origin, which are not topologically equivalent to a real analytic function. Actually for such counterexamples the singular fibers are not tame, in the following sense: a topological space is tame if each point has a closed neighborhood which is homeomorphic to the cone on its boundary. A well-known result of Lojasewicz (see [**12**, **13**]) says that fibers of analytic maps are triangulable and hence tame. In particular, functions with a non-tame fiber cannot be topologically equivalent to analytic functions. Explicit examples of such smooth functions for $m = 1$ are given in [**18**] and by suspending them one can construct examples for all (m, n) as above.

Remark 2.1. The local homeomorphism transforming f to a polynomial map cannot be made differentiable, in general (see below).

2.2. *An alternative proof for Proposition 1.1 when $m - n \leq 2$*

This result is already present in the literature. Let us call non-exceptional the dimensions verifying the conditions from Proposition 1.1. Then, in the non-exceptional dimensions the map f should be locally topologically equivalent to the projection map $\mathbb{R}^m \to \mathbb{R}^n$, in the following situations:

(1) if $m - n = 1$, under the more general assumption that the critical set is zero dimensional. This was proved by Church and Timourian in [**5**, **7**].

(2) if $m - n = 2$ and both the critical set and its image by f are zero dimensional (see [**6**]).

However, it should be stressed that their method of proof works also in the case when $m - n = 3$, when assuming that the (three dimensional) Poincaré Conjecture holds true.

2.3. *A sketch of proof for Proposition 1.1 when $m - n = 3$, following Church and Timourian*

All arguments needed in this case can be found in [**5**, **6**, **7**, **20**]. Let x_0 be a critical point for f. If the connected component $\Gamma(x_0)$ of $f^{-1}(f(x_0))$ containing x_0 is $\{x_0\}$ then the dimensions are from the exceptional range, by the main theorem from [**20**]. Otherwise, one shows that f is open (see [**5**]) and thus the regular fiber $f^{-1}(y)$ has finitely many components. Further, the singular fibers have less components than the regular ones. One shows then that the lemmas (2.1) and (2.2) from [**6**] hold for $n - m = 3$. Specifically, there exists a manifold with boundary $L^m \subset M^m$ so that the restriction of f yields a smooth map $f : L^m \to B^n$ onto a small n-ball $B^n \subset N^n$ centered at a singular value, with no critical points on the boundary. Moreover, the singular fiber is the one point union of k closed 3-balls and the regular fiber F^3 is a manifold with boundary. By the exact sequences associated to the fibrations over $B^m - \{0\}$ (where $m \geq 4$) one finds then that the regular fiber is a homotopy disk. Thus, by assuming the Poincaré Conjecture, it follows that F^3 is diffeomorphic to the 3-ball, and so $k = 1$. Eventually, this implies that the branch locus should be empty.

2.4. *A weaker lemma for arbitrary (m, n) which is sufficient for Proposition 1.1*

Assume that f is locally analytic around its critical points. Then a certain claim slightly weaker than Lemma 1.1 holds true in any dimensions (m, n), and this will be sufficient to derive Proposition 1.1.

In fact, according to Kuo ([**11**]) the jet (of sufficiently high order) of an analytic function having an isolated singularity at the origin is V-sufficient. This means that for any two functions with the same given jet the singular fibers at the origin are (ambiently) homeomorphic. It follows that there exists a polynomial with an isolated critical point at the origin which is V-equivalent to f.

Remark 2.2. If $m-n \leq 3$ then the V-equivalence implies the topological right-left equivalence of real analytic germs. This is well-known for $m \leq 3$, $n = 1$ (see e.g. [**9**], p.195). The general case can be obtained by means of the classification of isolated singularities from [**10**], but the proof is somewhat involved. The main ideas are at follows: two isolated singularities are topologically equivalent iff their Milnor-type fibrations are equivalent up to invertible cobordisms between their links (see [**10**]). Further the links are of dimension $m-n \leq 3$ and hence invertible cobordisms (which are h-cobordisms) are trivial if $m - n \leq 2$. In the case when $m - n = 3$ one has to prove first that the link is simply connected, to assume that the Poincaré Conjecture holds true and then to use Freedman's classification of topological 4-manifolds. Eventually, remark that there is only one way to fiber $M^{m-1} \to S^{n-1}$ if the fiber has dimension at most 2. In fact, fibrations like that are in one-to-one correspondence with elements of $\pi_{n-2}(\mathrm{Homeo}(F^{m-n}))$, where F is the fiber, and Homeo denotes the group of homeomorphisms. If F is a disk or a sphere then the connected components of Homeo(F) are contractible (by using Alexander's trick). The same holds true if F is a surface by a deep result of Eells and Sampson. If $m - n = 3$ we need again that F be identified with a 3-disk.

Let us choose a small ball B^n around the origin so that none of the functions $tj^k(f) + (1-t)f$ has critical values within that ball, for $t \in [0, 1]$. Notice that all functions of this family have an isolated critical point at the origin. Consider then the function:

$$\psi(x) = \begin{cases} j^k(f)(x), & \text{if } x \in \frac{1}{2}B^m \\ f(x), & \text{if } x \notin B^m \\ \lambda(t)j^k(f)(x) + (1-\lambda(t))f(x), & \text{if } x \in \frac{1+t}{2}S^{m-1} \subset B^m - \frac{1}{2}B^m \end{cases}$$

where the real function $\lambda(t)$ is chosen so that ψ is smooth.

We obtained therefore a smooth map around each critical point which is polynomial around the singularity, it has only an isolated singular point and it coincides with the former map f outside a ball.

In particular one can construct a smooth map F by using f on the complementary of the finitely many balls corresponding to the critical points and the respective maps ψ around the critical points.

If the dimensions (m, n) are non-exceptional then the real polynomials with isolated singularities are topological submersions at the critical points by [**4**]. In particular the map F must be a topological submersion since all its singularities are associated to polynomials.

2.5. *Another proof of Lemma 1.1 for locally real analytic maps*

An alternative solution in this case is to use the theorem of Tougeron ([**21**] Cor.1, p.215) which states that: a locally really analytic (and more generally a smooth function whose Jacobian ideal has vanishing co-height) $f : \mathbb{R}^m \to \mathbb{R}^n$ having only isolated critical points is $\mathcal{C}^r$-equivalent to a polynomial mapping.

However the assumption for smooth f is nontrivial. In fact, when $m = n = 1$ this extra assumption amounts to saying that the derived function is nowhere flat. Recall that a function is flat at a point if all its derivatives (of arbitrary high order) vanish at that point.

The result of Tougeron has been geometrically expressed in the following form by Shiota ([**17**]): A $\mathcal{C}^\infty$ function on $\mathbb{R}^n (n \neq 4, 5)$ is equivalent to a polynomial if it is proper, the number of critical points is finite, and the Milnor number of the germ at each critical point is finite. Recall that the Milnor number of a germ f of a $\mathcal{C}^\infty$ function at 0 in $\mathbb{R}^n$ is the dimension of the real vector space $\mathcal{E}_n/(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$. Here $\mathcal{E}_n$ is the ring of C^∞ function germs at 0 in $\mathbb{R}^n$.

Remark 2.3. For instance there is a well-known example due to Lojasewicz of a smooth function $f(x, y, z) = (x^2 + y^2)\left(x + \exp\left(-\frac{1}{z^2}\right)\right)$ which is not locally smoothly equivalent to a real analytic function around the origin.

2.6. *Real analytic functions not $\mathcal{C}^\infty$-equivalent to polynomial ones*

A classical example of Whitney (see [**22**]) is the analytic function of 3 variables $F(x, y, z) = xy(y - x)\,(y - (3 + z)x)\,(y - \gamma(z)x)$, where γ is a transcendental analytic function and $\gamma(0) = 4$. Whitney has shown that F is not equivalent to a polynomial by using real analytic coordinate transformations. Then Shiota ([**14**]) strenghted this result by proving that F is not equivalent to a polynomial in a neighborhood of zero, by using a C^∞ coordinate transformation.

It is worth noticing that 3 is the smallest number of variables needed for such a counterexample, because real analytic functions $\mathbb{R}^2 \to \mathbb{R}$ are C^∞-equivalent to a polynomial (see [**15**]).

2.7. *Continuous mappings which are $\mathcal{C}^0$-equivalent to smooth maps*

According to Shiota ([**16**]) if a continuous function $F : M^m \to \mathbb{R}$ has only isolated topological singularities and $m \neq 4, 5$ then f is topologically equivalent to a smooth function. The arguments in [**16**] extend to show that a continuous map $F : \mathbb{R}^m \to \mathbb{R}^n$, with finitely many topological singularities, and $m - n \neq 4, 5$ is locally equivalent to a smooth function around each point. In particular the local triviality from Proposition 1.1 can be obtained under the weaker assumption that f is continuous and has only finitely many topological critical points.

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