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On power subgroups of mapping class groups

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Abstract. In the first part of this paper we prove that the mapping class subgroups generated by the D-th powers of Dehn twists (with $D \geq 2$) along a sparse collection of simple closed curves on an orientable surface are right angled Artin groups. The second part is devoted to power quotients, i.e., quotients by the normal subgroup generated by the D-th powers of all elements of the mapping class groups. We show first that for infinitely many values of D, the power quotient groups are non-trivial. On the other hand, if $4g+2$ does not divide D then the associated power quotient of the mapping class group of the genus $g \geq 3$ closed surface is trivial. Eventually, an elementary argument shows that in genus 2 there are infinitely many power quotients which are infinite torsion groups.

1. Introduction and statements

The aim of this paper is to give a sample of results concerning power subgroups of mapping class groups. We denote by $M(S)$ the *mapping class group* of the orientable surface S, namely the group of isotopy classes of orientation-preserving homeomorphisms that fix point-wise the boundary components. Let $\Sigma_{g,k}^r$ denote the orientable surface of genus g with k boundary components and r punctures. We will omit the indices k and r in $\Sigma_{g,k}^r$ when they are zero.

Definition 1.1. Let A be a collection of (isotopy classes of) simple closed curves on the surface S. We denote by $M(S)(A;D)$ the subgroup generated by D-th powers of Dehn twists along curves in A.

When S is a surface, let $SCC(S)$ be the set of representatives for all *simple closed curves up to homotopy* on the surface S. The group $M(S)(SCC(S);D)$ will be denoted by $M(S)[D]$. We will omit the indices k and r in $M(\Sigma_{g,k}^r)[D]$ and $M(\Sigma_{g,k}^r)(A;D)$ when they are zero. For simplicity, when we do not need to specify the surface $\Sigma_{g,k}^r$ we will use the notation $M_{g,k}^r$ for $M(\Sigma_{g,k}^r)$ and respectively $M_{g,k}^r[D]$ for $M(\Sigma_{g,k}^r)[D]$, with the same convention concerning the indices k and r, which we omit when they are zero.

Observe that $M_g[D]$ is a normal subgroup of M_g, whose definition is similar to that of the congruence subgroups of the symplectic groups. In fact, let T_a denote the Dehn twist along the simple closed curve a. Then for every $h \in M_g$ we have $hT_a^D h^{-1} = T_{h(a)}^D$ in

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$M_g[D]$. As $M_g[D]$ is generated by the T_a^D, for a running over the set of all simple closed curves, it follows that $M_g[D]$ is a normal subgroup.

The first results on $M_g[D]$ were obtained by Humphries ([20]) who proved that $M_g/M_g[2]$, for each $g \geq 1$, $M_2/M_2[3]$ and $M_3/M_3[3]$ are finite, while $M_2/M_2[D]$ is infinite when $D \geq 4$.

On the other hand, using quantum topology techniques we proved in [14] that the groups $M_g[D]$ are of infinite index in M_g, if $g \geq 3$, and $D \notin \{1, 2, 3, 4, 6, 8, 12\}$.

Mapping class groups have interesting actions on various moduli spaces, for instance on spaces of $SU(2)$ representations of surface groups. It is known (see [16]) that the whole mapping class group acts ergodically. Actually the same proof extends trivially to show that $M_g[D]$ still acts ergodically. This yields the first examples of infinite index subgroups of the mapping class group acting ergodically.

Methods from quantum topology also show that:

$$\cap_{D \in \mathcal{D}} M_g[D] = 1$$

if $g \geq 2$ and $\mathcal{D}$ is any infinite set of positive integers. In fact, the kernel of the $SO(3)$ quantum representation of level k of M_g contains $M_g[k]$. Then the asymptotic faithfulness theorem from [2, 13] yields the claim.

However, these results seem to exhaust our present knowledge about the groups $M_g[D]$. It is not known, for instance, whether the following holds:

Conjecture 1.2. *The group $H_1(M_g[D])$ is not finitely generated if $D \geq 3$, $g \geq 4$ or $D \geq 4$, $g \in \{2, 3\}$.*

If true, this would imply that $M_g/M_g[D]$ is infinite for the above values of D and g.

Remark 1.3. The groups $M_g[2]$ have finite index in M_g (see [20]) and hence are finitely generated. However the quantum representations at 4-th roots of unity (see [37, 43]) and 6-th roots of unity (see [44]) have finite image. Thus the quantum method used for large D cannot decide whether $M_g[4]$ and $M_g[6]$ have finite index or not. It is likely that $M_g[D]$ is of infinite index for every $D \geq 4$ and $g \geq 3$. Notice also that a similar problem for pure braid groups was considered in [21].

A question of Ivanov (see [25], Question 12) is particularly relevant for the structure of the group $M_g[D]$ by asking about the possible relations between powers of Dehn twists. We formulate it here as a conjecture, under a slight restriction on D:

Conjecture 1.4. *The group $M_g[D]$ (for $D \geq 3$, $g \geq 4$ or $D \geq 4$, $g \in \{2, 3\}$) has the following presentation:*

1. *Generators Z_a (standing for T_a^D), where a belongs to the (infinite) set $SCC(\Sigma_g)$ of simple closed curves on the surface;*
2. *Relations of conjugacy type:*

$$Z_{T_a^D(b)} = Z_a Z_b Z_a^{-1}$$

for each pair $a, b \in SCC(\Sigma_g)$.

We denote by $\mathcal{A}_\Gamma$ the right angled Artin group associated to the graph Γ, which is defined by the following presentation:

(1) Generators Z_a, where a belongs to the set of vertices of Γ;
(2) Relations

$$Z_a Z_b = Z_b Z_a, \text{ if } a \text{ and } b \text{ are connected by an edge in } \Gamma$$

A related (but much weaker) Conjecture is as follows:

Conjecture 1.5. *Let $C_{\Sigma,D} \subset SCC(\Sigma)$ be a set of representatives of the orbits set $SCC(\Sigma)/M(\Sigma)[D]$. Consider the associated intersection graph $\Gamma(C_{\Sigma,D})$, whose vertex set is $C_{\Sigma,D}$ and edges join vertices corresponding to disjoint curves on the surface. Then the homomorphism $A_{\Gamma(C_{\Sigma,D})} \to M(\Sigma)[D]$ which sends the generators Z_a into the elements T_a^D is an isomorphism on its image for $D \geq 3$, $g \geq 4$ or $D \geq 4$, $g \in \{2, 3\}$. Here g denotes the genus of the surface Σ.*

Clay, Leininger and Margalit recently proved in [4] that $M(\Sigma)[D]$ is not abstractly commensurable with any right angled Artin group. In particular, the homomorphism $A_{\Gamma(C_{\Sigma,D})} \to M(\Sigma)[D]$ from above is not surjective.

Remark 1.6. According to Ishida (see [22]) the group generated by two Dehn twists is either free abelian (if the curves are disjoint or coincide) or generate the braid group B_3 in 3 strands (if the curves intersect in one point) or free (if the curves intersect in at least two points). In particular the subgroup generated by two D-th powers of Dehn twists is either free abelian or free, supporting the Conjecture 1.5. See also [8] or ([19], Thm. 3.5) for the braid case. Relations between multi-twists are also given in [36].

Proposition 1.7. *The analogues of Conjecture 1.4 for $D = 2$ and any closed orientable surface Σ of genus $g \geq 3$ are false as stated, namely there are additional relations in a presentation of $M_g[2]$ with the given generators.*

Proof. According to Humphries (see [20]) the subgroup $M_g[2]$ can be identified to the kernel of the homomorphism $M_g \to Sp(2g, \mathbb{Z}/2\mathbb{Z})$. Hain proved in [18] (see also [35]) that any finite index subgroup of M_g (for $g \geq 3$) containing the Torelli subgroup (i.e., the subgroup of mapping classes acting trivially in homology) has trivial first cohomology. This implies that $H^1(M_g[2]) = 0$, which was also proved by McCarthy in [35]. But the abelianization of the group presented by the relations from Conjecture 1.4 is a free abelian group of rank equal to the cardinal of $SCC(\Sigma_g)/M_g[2]$. This contradiction shows that in $M_g[2]$ there are additional relations. □

Remark 1.8. The referee pointed out explicit relations among squares of Dehn twists along nonseparating curves. Choose for instance the nonseparating curves $a_1, a_2, \ldots, a_7$ on Σ_3 such that a_i intersects a_j at one point if $j = i + 1$ and they are disjoint otherwise. Then we have the following relation in M_3:

$$(T_{a_1} T_{a_2} T_{a_3} T_{a_4} T_{a_5} T_{a_6} T_{a_7} T_{a_7} T_{a_6} T_{a_5} T_{a_4} T_{a_3} T_{a_2} T_{a_1})^2 = 1$$

Observe further that

$$T_{a_1}T_{a_2}T_{a_3}T_{a_4}T_{a_5}T_{a_6}T_{a_7}T_{a_7}T_{a_6}T_{a_5}T_{a_4}T_{a_3}T_{a_2}T_{a_1} =$$
$$= (T_{a_7}^2)^{A_6}(T_{a_6}^2)^{A_5}(T_{a_5}^2)^{A_4}(T_{a_4}^2)^{A_3}(T_{a_3}^2)^{A_2}(T_{a_2}^2)^{A_1}(T_{a_1}^2)$$

where we put $A_i = T_{a_1}T_{a_2}\cdots T_{a_i}$, and $x^A = AxA^{-1}$. Now, we can express $(T_{a_i}^2)^{A_{i-1}} = T_{A_{i-1}(a_i)}^2$ as squares of Dehn twists. We obtain therefore the following relation

$$(T_{A_6(a_7)}^2 T_{A_5(a_6)}^2 T_{A_4(a_5)}^2 T_{A_3(a_4)}^2 T_{A_2(a_3)}^2 T_{A_1(a_2)}^2 T_{a_1}^2)^2 = 1$$

which does not follow from those defining a right angled Artin group in these (square Dehn twists) generators.

Remark also that the analogue of Conjecture 1.4 cannot hold when $D = 1$ either. In fact the abelianization of M_g would be a nontrivial free abelian group, contradicting the fact that M_g is perfect when $g \geq 3$ and has torsion abelianization otherwise.

An important step towards a solution to Conjecture 1.5 was taken in the recent paper [30] of Koberda, where the following is proven: For any irredundant (see [30] for the definition) collection $\{f_1, f_2, \ldots, f_k\}$ of mapping classes of homeomorphisms, each one being either a Dehn twist or a pseudo-Anosov homeomorphism supported on a single connected subsurface, there exists N_0 such that $\{f_1^N, f_2^N, \ldots, f_k^N\}$ is a right angled generating system for a right angled Artin subgroup of the mapping class group, for any $N \geq N_0$.

The first result of this paper supports further evidence for the last two conjectures. Let A be a finite collection of simple closed curves on a surface S and denote by $F(A)$ the regular neighborhood of A in S. We assume that curves are isotoped so that for each $a, b \in A$ the number $i(a, b)$ of intersection points between a and b is minimal. We pick up a base point p on the surface S and a set of distinct points $p_a^0 \in a$, for each $a \in A$.

Definition 1.9. The collection A of curves on the surface S is *sparse* if it is finite and for some choice of paths γ_a joining p to p_a^0 the free subgroup $O(A) \subset \pi_1(F(A), p)$ generated by the homotopy classes of based loops $\gamma_a a \gamma_a^{-1}$, $a \in A$, embeds into $\pi_1(S, p)$ under the map induced by the inclusion $F(A) \hookrightarrow S$. The collection A is *nontrivial* if the group $O(A)$ is nontrivial.

Theorem 1.10. *Let $D \geq 2$ and A be a nontrivial sparse collection of curves on $\Sigma_{g,d}$, where $d \geq 1$. Then the subgroup $M(\Sigma_{g,d})(A; D)$ is a right angled Artin group.*

Remark 1.11. One can construct sparse sets A by considering free subgroups generated by primitive elements in $\Sigma_{g,d}$.

Remark 1.12. J. Crisp and L. Paris considered before the question of finding presentations of subgroups generated by non-trivial powers of the standard generators in Artin groups. They established in [8] the Tits conjecture, which claimed that these subgroups are right angled Artin groups. M. Lönne proved in [33] similar results in the braid group setting, by showing that the subgroups generated by the powers of band generators are again right angled Artin groups if the powers are at least 3.

Remark 1.13. Recently, M. Kapovich proved in [28] (making use of our result above) that all right angled Artin groups associated to finite graphs embed into the group of Hamiltonian symplectomorphisms of any symplectic manifold.

The second part of this article is concerned with power subgroups and quotients. Recall the following:

Definition 1.14. Let $X_g[D]$ denote the D-th power subgroup of M_g, namely the subgroup generated by powers h^D of all elements of $h \in M_g$. Then $X_g[D]$ is a normal subgroup of M_g whose quotient is a torsion group.

Remark 1.15. Newman ([40]) proved that the D-th power subgroup of $PSL(2,\mathbb{Z})$ (and hence of $SL(2,\mathbb{Z})$) is of infinite index when $D = 6m \geq 48000$. This was extended by Fine and Spellman (see [12]) to the $2p$-th power subgroups of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/p\mathbb{Z}$ (for odd prime p).

A natural question is whether power quotients of the mapping class group could be non-trivial, or even infinite torsion groups. Our second result gives some answers in particular cases:

Theorem 1.16.
1. *Choose an ordered basis of $H_1(\Sigma_g,\mathbb{Z})$ and denote by P the homomorphism $M_g \to Sp(2g,\mathbb{Z})$ which sends a mapping class into the matrix describing its action in homology. Then, for every $g \geq 2$ there exist infinitely many integers D for which $P(X_g[D])$ is a proper subgroup of $Sp(2g,\mathbb{Z})$. In particular $M_g/X_g[D]$ are non-trivial torsion groups, for these values of D.*
2. *If $4g+2$ does not divide D and $g \geq 3$ then $M_g = X_g[D]$.*

The question concerning the existence of infinite torsion quotients of M_g (see the question of Ivanov in [25]) has an elementary solution for genus $g = 2$. Using arguments similar to those of Korkmaz in [31] we show that:

Theorem 1.17. *The group $M_2/X_2[360D]$ is an infinite torsion group (of exponent $360D$) for $D \geq 8000$.*

2. Subgroups of mapping class groups generated by powers of Dehn twists

2.1. Finitely generated subgroups generated by powers in braid groups

The analogues of the groups $M(\Sigma_g)(A;D)$ in the case of braid groups have been considered long time ago by Coxeter. The braid group B_n in n strands has the usual presentation, due to Artin:

$$B_n = \langle \sigma_1, \sigma_2, ..., \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } \mid i-j \mid > 1, \ \ \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i, 1 \leq i \leq n-2 \rangle$$

It is well-known that the quotient of B_n by the normal subgroup generated by σ_i^2 is the permutation group S_n. Consider, after Coxeter (see [5]):

Definition 2.1. The subgroup $B_n\{D\}$ of B_n is the group generated by the powers σ_i^D of the *standard* generators σ_i. Let also $N(B_n\{D\})$ denote the normal closure of $B_n\{D\}$ in B_n.

Coxeter gave in [5] the list of all those quotients $B_n/N(B_n\{D\})$ which are finite, together with their respective description (see also [6, 7]), as follows:

Proposition 2.2 (Coxeter)**.** *The group $N(B_n\{D\})$ is of finite index in B_n if and only if $(D-2)(n-2) < 4$. Away from the trivial cases $D = 2$ or $n = 2$ we have another five groups:*

(1) *n=3*
   (a) *For $D = 3$ the quotient $B_3/N(B_3\{3\})$ is isomorphic to $SL(2, \mathbb{Z}/3\mathbb{Z})$ and has order 24;*
   (b) *For $D = 4$ the quotient $B_3/N(B_3\{4\})$ is a non-split extension of the symmetric group S_4 on a set of 4 elements by $\mathbb{Z}/4\mathbb{Z}$ and has order 96;*
   (c) *For $D = 5$ the quotient $B_3/N(B_3\{5\})$ is isomorphic to $GL(2, \mathbb{Z}/5\mathbb{Z})$ and has order 600;*

(2) *For $n = 4, D = 3$ the factor group $B_4/N(B_4\{3\})$ has order 648 and is the central extension of the Hessian group $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes PSL(2, \mathbb{Z}/3\mathbb{Z})$ by $\mathbb{Z}/3\mathbb{Z}$.*

(3) *For $n = 5, D = 3$ the factor group $B_5/N(B_5\{3\})$ has order 155 520 and is the central extension of the simple group of order 25 920 by $\mathbb{Z}/6\mathbb{Z}$.*

Remark 2.3. (1) There is an analogue of Conjecture 1.5 for the punctured disk $\Sigma_{0,1}^n$, where we replace powers of Dehn twists by powers of half-twists (i.e., braids). Notice that $N(B_n\{2D\})$ is a subgroup of $M_{0,1}^n[D]$.

(2) J. Tits conjectured that $B_n\{D\}$ and more generally the subgroups generated by powers of the standard generators in Artin groups are right angled Artin groups. The latter conjecture was settled in full generality by Crisp and Paris [8].

(3) It seems unknown whether the analogue of Conjecture 1.5 for $N(B_n\{D\})$ holds for $D \geq 3$. Notice that for $D = 2$ there exist nontrivial relations among squares of band generators (which are Dehn twists) according to [33].

2.2. Proof of Theorem 1.10

Consider the regular neighborhood $F(A)$ of A in $\Sigma_{g,d}$, which is a subsurface of genus $g(A)$ with $k(A)$ boundary components. Then $g(A) \leq g$, but the number $k(A)$ of boundary components of $F(A)$ depends on the geometry of A and can be arbitrarily large. We denote by $i(a, b)$ the minimal number of intersection points between curves in the isotopy classes of a and b, respectively. We assume that curves in A are isotoped so that for each $a, b \in A$ the number of intersection points between a and b equals $i(a, b)$ and there are no triple intersections among curves in A.

We will adapt the proof of the Tits conjecture given in [8]. In the present situation we will be concerned with an Artin group $B(A)$ (to be defined later) associated to the collection A and its representation into the mapping class group of $F(A)$.

We can obtain $F(A)$ as the result of plumbing one annulus neighborhood Ann_a for each curve a in A. In particular the core curves of the annuli are transverse to each other. Pick-up one base point p_a^0 in the boundary of Ann_a, for each $a \in A$. We can suppose that all p_a^0 belong to $\partial F(A)$. Choose one distinguished boundary component a^+ for each annulus Ann_a. There is no loss of generality in assuming that each p_a^0 belongs to a^+ and a small arc of a^+ centered at p_a^0 is contained within $\partial F(A)$.

Give an orientation to every curve $a \in A$ and a total order $<$ on A.

If we travel along a^+ in the direction given by the orientation and starting at p_a^0 we will meet a number of intersection points between a^+ and the other curves b^+, where $b \in A$. We denote them in order $p_a^1, p_a^2, \ldots, p_a^{d(a)}$. Denote then by $S = \cup_{a\in A} \cup_{0\leq j\leq d(a)} \{p_a^j\}$ the set of all these points. It is clear that $S \subset \partial F(A)$.

The groupoid $\pi_1(F(A), S)$ is the fundamental groupoid of $F(A)$ based at the points of S. Since $F(A)$ has boundary it follows that $\pi_1(F(A), S)$ is a free groupoid (see [9], p.7).

Furthermore the mapping class group $M(F(A))$ acts by automorphisms on the fundamental groupoid $\pi_1(F(A), S)$, because $S \subset \partial F(A)$ and elements of $M(F(A))$ are classes of homeomorphisms fixing point-wise the boundary.

Consider the following elements of $\pi_1(F(A), S)$:

(1) For every $s \in A$ the *elementary loop* α_s is s^+ based at p_s^0, with its orientation. Thus α_s is parallel to the central curve s in the annulus Ann_s.
(2) For every $s \in A$ and $i \in \{0, 1, \ldots, d(s)-1\}$ consider the arc $p_s^i p_s^{i+1}$ of s^+ which joins p_s^i to p_s^{i+1}. We call them *admissible arcs*. Observe that the arc $p_s^{d(s)} p_s^0$ is not admissible.

The *dual intersection graph* of A is defined as the graph whose vertices are the elements of A and two vertices are connected by an edge if the corresponding curves intersect. Assume henceforth that the dual intersection graph of A is connected. Then admissible arcs and elementary loops generate the groupoid $\mathbb{F} = \pi_1(F(A), S)$.

Let then Γ_A be the subgroup of $M(F(A))$ generated by the Dehn twists T_a, for all $a \in A$.

Set $\mathbb{B}$ for the sub-groupoid of $\mathbb{F}$ generated by the admissible arcs.

We will need some terminology and facts from [8]. Any element of $\mathbb{F}$ can uniquely be written in the reduced form:

$$w = \mu_0 \alpha_{s_1}^{k_1} \mu_1 \cdots \alpha_{s_m}^{k_m} \mu_m$$

where $\mu_i \in \mathbb{B}$, μ_i is non-trivial if $i \neq 0, m$ and $k_i \neq 0$.

We say that w has a *square* in α_s if for some j we have $s_j = s$ and $|k_j| \geq 2$, and is *without squares in* α_s, otherwise. Moreover w is of type (μ, α_t^p) if its reduced form is

$$w = \mu_0 \alpha_t^{k_1 p} \mu_1 \cdots \alpha_t^{k_m p} \mu_m, \; k_j \in \mathbb{Z} \setminus \{0\}, \text{ and } \mu = \mu_0 \mu_1 \cdots \mu_m$$

By language abuse we will speak about $T_a(w)$, where w is a word in $\mathbb{F}$, using the action of Γ_A by automorphisms on $\mathbb{F}$.

Lemma 2.4. *Let $s \in A$ and $m \in \mathbb{Z} \setminus \{0\}$.*

(1) *If $\mu \in \mathbb{B}$ then $T_s^m(\mu)$ is of type (μ, α_s^m).*
(2) *Let $t \in A$. If $s = t$ or $i(s,t) = 0$ then $T_s^m(\alpha_t) = \alpha_t$.*
(3) *If $i(s,t) \neq 0$ then $T_t^m(\alpha_s)$ is $u\alpha_s$, where u is an element of type $(1, \alpha_t^m)$. Thus, if $|m| \geq 2$ and $i(s,t) \neq 0$ then $T_t^m(\alpha_s)$ has a square in α_t.*

Proof. If s^+, t^+ intersect at p we define $\varepsilon(s,t;p) \in \{-1, 1\}$ as follows. Assume that we travel along s^+ to meet p. At p we use the global orientation of the surface for turning right along t^+ and continue travelling this way. If the direction along t^+ is the orientation of t^+ then we set $\varepsilon(s,t;p) = 1$ and otherwise $\varepsilon(s,t;p) = -1$.

Next, we will identify canonically $\pi_1(F(A), S)$ with $\pi_1(F(A), S')$ where S' is a copy of S, each point p_a^j being slightly moved in the positive direction along the arc a^+ to a point $\tilde{p}_a^j$.

Denote by $\alpha_t(p_t^j)$ the element $\tilde{p}_t^0\tilde{p}_t^j\, \alpha_t\, \tilde{p}_t^j\tilde{p}_t^0$, where $\tilde{p}_t^0\tilde{p}_t^j$ is the unique arc of α_t joining $\tilde{p}_t^0$ to $\tilde{p}_t^j$ and consisting only of admissible sub-arcs. Also α_t is considered as the loop t^+ based at $\tilde{p}_t^0$. Then by direct computation we find:

$$T_{\alpha_t}^m(\tilde{p}_s^i\tilde{p}_s^{i+1}) = \begin{cases} \tilde{p}_s^i\tilde{p}_s^{i+1}, & \text{if } p_s^i p_s^{i+1} \cap \alpha_t = \emptyset, \text{ or } s = t \\ \tilde{p}_s^i\tilde{p}_s^{i+1}\, \alpha_t^{m\varepsilon(s,t;p_s^{i+1})}(p_s^{i+1}) & \text{if } p_s^{i+1} \in t^+ \\ \tilde{p}_s^i\tilde{p}_s^{i+1} & \text{if } p_s^{i+1} \notin t^+ \end{cases}$$

Notice that when the start-point p_s^i belongs to t^+ the action is trivial since the base-point p_s^i is slightly pushed out of t^+ in S'.

Let now $s, t \in A$ be two curves with $i(s,t) \neq 0$. Suppose now that starting from p_s^0 and traveling along s^+ we meet the circle t^+ at the points $p_s^{j_1}, p_s^{j_2}, \ldots, p_s^{j_r}$, $r > 0$. By direct inspection we find that

$$T_{\alpha_t}^m(\alpha_s) = \tilde{p}_s^0\tilde{p}_s^{j_1}\, \alpha_t^{m\varepsilon(s,t;p_s^{j_1})}(p_s^{j_1})\, \tilde{p}_s^{j_1}\tilde{p}_s^{j_2}\, \alpha_t^{m\varepsilon(s,t;p_s^{j_2})}(p_s^{j_2}) \cdots \alpha_t^{m\varepsilon(s,t;p_s^{j_{r-1}})}(p_s^{j_{r-1}})(\tilde{p}_s^0\tilde{p}_s^{j_r})^{-1}\alpha_s$$

It is immediate that $T_{\alpha_t}^m(\alpha_s) = u\alpha_s$, where u is of type $(1, \alpha_t^m)$. □

Lemma 2.5. *Let $x \in \mathbb{F}$, $|m| \geq 2$. If x is without squares in α_t and $T_{\alpha_t}^m(x)$ has a square in α_s then either $s = t$ or else $i(s,t) = 0$ and x has a square in α_s.*

Proof. Let $x = \mu_0\alpha_{s_1}^{k_1}\mu_1 \cdots \alpha_{s_r}^{k_r}\mu_r$ in reduced form. The previous lemma shows that:

(1) If $s_i = t$ then $v_i = T_{\alpha_t}^m(\alpha_{s_i}^{k_i}) = \alpha_{s_i}^{k_i}$, where $k_i \in \{-1, 1\}$, because x is without squares in α_t.
(2) If s_i and t are disjoint then $T_{\alpha_t}^m(\alpha_{s_i}^{k_i}) = \alpha_{s_i}^{k_i}$.

(3) If $i(s_j, t) \neq 0$ then

$$T^m_{\alpha_t}(\alpha^{k_j}_{s_j}) = \begin{cases} u_j(\alpha_{s_j} u_j)^{k_j - 1}\alpha_{s_j} & \text{if } k_j > 0 \\ \alpha^{-1}_{s_j}(u^{-1}_j \alpha^{-1}_{s_j})^{-k_j - 1} u^{-1}_j & \text{if } k_j < 0 \end{cases}$$

where u_j is a non-constant term of type $(1, \alpha^m_t)$.

(4) $T^m_{\alpha_t}(\mu_j)$ has a reduced form y_j of type (μ, α^m_t), for all $j \geq 0$.

Therefore we can write in reduced form $T^m_{\alpha_t}(x) = x_0 v_1 x_1 v_2 \cdots v_r x_r$ as follows:

(1) If either $s_i = t$ or s_i and t are disjoint then $v_i = T^m_{\alpha_t}(\alpha^{k_i}_{s_i}) = \alpha^{k_i}_{s_i}$.
(2) Assume that $i(s_j, t) \neq 0$.
  (a) If $k_j > 0$ then $v_j = (\alpha_{s_j} u_j)^{k_j - 1}\alpha_{s_j}$. Absorb the extra factor u_j into x_{j-1}.
  (b) If $k_j < 0$ then $v_j = \alpha^{-1}_{s_j}(u^{-1}_j \alpha^{-1}_{s_j})^{-k_j - 1}$. Absorb the extra factor u^{-1}_j into x_j.
(3) Eventually x_j are $T^m_{\alpha_t}(\mu_j)$, possibly corrected by the absorption of terms coming from v_j or v_{j+1}. Thus x_j are of reduced form of type (μ_j, α^m_t).

In particular, if $T^m_{\alpha_t}(x)$ has a square in α_s then either $s = t$ or there exists j such that $s_j = s$ and s and t are disjoint. □

To each set of curves $A \subset \Sigma_{g,d}$ we can associate the Artin group $B(A)$, with the following presentation:

$$B(A) = \langle z_a, a \in A \mid z_a z_b = z_b z_a, \text{ if } a \cap b = \emptyset, \; z_a z_b z_a = z_b z_a z_b, \text{ if } i(a, b) = 1 \rangle$$

There is a natural homomorphism $\tau : B(A) \to M(F(A))$ which sends z_a into the Dehn twist T_a.

Consider now the right angled Artin group defined by the presentation:

$$H(A) = \langle w_a, a \in A \mid w_a w_b = w_b w_a, \text{ if } i(a, b) = 0 \rangle$$

There is a map $\iota : H(A) \to B(A)$ given by $\iota(w_a) = z^D_a$. We will suppose that $D \geq 2$ in the sequel. The word $W = w^{n_l}_{s_l} w^{n_{l-1}}_{s_{l-1}} \cdots w^{n_2}_{s_2} w^{n_1}_{s_1}$ is called an *M-reduced expression* of the element $w \in H(A)$ (obtained by interpreting letters as the corresponding generators of $H(A)$) if for any $i < j$ such that $s_i = s_j$ there exists k such that $i < k < j$ and $i(s_i, s_k) \neq 0$. Then the M-reduced expression for w *ends* in s if, up to change the order of commuting generators, we can arrange that $s_l = s$.

Recall now that $\tau(\iota(w))$ is an automorphism of $\mathbb{F}$, for each $w \in H(A)$. We will write simply $w(x)$ or $W(x)$ for $\tau(\iota(w))(x)$, where $w \in H(A)$, $x \in \mathbb{F}$ and W is an M-reduced expression for w.

The following two lemmas are restatements of Propositions 9 and 10 from [8].

Lemma 2.6. *Let W be an M-reduced expression for $w \in H(A)$, $x \in \mathbb{F}$ and $s \in A$. Suppose that x is without squares in α_t for all $t \in A$, and that $w(x)$ has a square in α_s. Then W ends in s.*

Proof. We will proceed by induction on the length l of the M-reduced expression $W = w_{s_l}^{n_l} w_{s_{l-1}}^{n_{l-1}} \cdots w_{s_2}^{n_2} w_{s_1}^{n_1}$ (see also [8], p.30). When $l = 0$, w is identity and thus $w(x) = x$ cannot have squares in α_s, under our assumptions. For the induction step let us now write $W = w_{s_l}^{n_l} W'$, where $l \geq 1$. If $W'(x)$ had a square in α_{s_l} then W' would end in s_l (by the induction hypothesis) and hence W would not be an M-reduced expression. Hence $W'(x)$ is without squares in α_{s_l}.

Now $W(x) = T_{\alpha_{s_l}}^{Dn_l}(W'(x))$ has a square in α_s. By Lemma 2.5 one has:

(1) either $s_l = s$ and so W ends in s;
(2) or else s_l and s are disjoint and $W'(x)$ has a square in α_s. By the induction hypothesis W' ends in s. Since s_l and s commute we switch the position of the last two generators and find that W ends in s.

□

For a fixed $a \in A$ the fundamental group $\pi_1(F(A), p_a^0)$ embeds into the groupoid $\pi_1(F(A), S)$. It is also clear that $\pi_1(F(A)), p_a^0)$ is kept invariant by the action of any element $w \in H(A)$.

Lemma 2.7. *Assume that the dual intersection graph of A (or, equivalently the surface $F(A)$) is connected. If w has a nontrivial M-reduced expression then w acts non-trivially on $O(A)$.*

Proof. It is known (see e.g. [8] and references therein) that an M-reduced expression representing the identity in $H(A)$ is trivial. Take then a non-trivial M-reduced expression W, as above. Since the dual intersection graph of curves is connected there exists some t in A such that $i(s_l, t) \neq 0$. We will show that $W(\alpha_t) \neq \alpha_t$. Since $\alpha_t \in O(A) \subset \pi_1(F(A), p_t^0)$ the action of W is nontrivial on $O(A)$.

Suppose $W(\alpha_t) = \alpha_t$ and write $W = T_{s_l}^{n_l} W'$. Then

$$W'(\alpha_t) = w_{s_l}^{-n_l}(\alpha_t) = T_{s_l}^{-Dn_l}(\alpha_t)$$

Lemma 2.4 shows that $T_{s_l}^{-Dn_l}(\alpha_t)$ has a square in α_{s_l} and further from Lemma 2.5 W' ends in s_l. But then W is not M-reduced, contradiction. This proves the claim. □

Proposition 2.8. *Assume that the dual intersection graph of the finite collection A is connected, A has at least two elements and $D \geq 2$. Then the group $M(F(A))(A; D)$ is a right angled Artin group with presentation:*

$$M(F(A))(A; D) = \langle T_a^D, a \in A \mid T_a^D T_b^D = T_b^D T_a^D, \text{if } a \cap b = \emptyset \rangle$$

Proof. Lemma 2.7 shows that the map $\tau \circ \iota : H(A) \to M(F(A))$ is injective, since $M(F(A))$ is a subgroup of the group of automorphisms of $\mathbb{F}$. Therefore $M(F(A))(A; D)$ is isomorphic to $H(A)$, as claimed. □

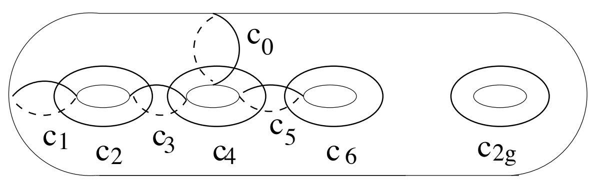
Corollary 2.9. *If A is nontrivial and $\Sigma_{g,d} \setminus F(A)$ has neither disks nor cylinder components joining two distinct boundary components of $F(A)$ and $D \geq 2$ then $M(\Sigma_{g,d})(A, D)$ is a right angled Artin group with presentation:*

$$M(\Sigma_{g,d})(A; D) = \langle T_a^D, a \in A \mid T_a^D T_b^D = T_b^D T_a^D, \text{if } a \cap b = \emptyset \rangle$$

Proof. The embedding $F(A) \subset \Sigma_{g,d}$, with $F(A)$ different from a disk or an inessential annulus induces a group embedding $M(F(A)) \hookrightarrow M(\Sigma_{g,d})$ according to ([41], Corollary 4.2) if and only if $\Sigma_{g,d} \setminus F(A)$ has neither disk nor cylinder components joining two distinct boundary components of $F(A)$. Now, since A is nontrivial $F(A)$ is neither a disk nor an inessential annulus. □

End of proof of Theorem 1.10. It suffices to consider the case when the dual intersection graph of A is connected. The mapping class group $M(\Sigma_{g,d})$ embeds into $\mathrm{Aut}(\pi_1(\Sigma_{g,d}, p))$, where the base point p is chosen on the boundary $\partial\Sigma_{g,d}$. By Lemma 2.7 for every nontrivial element $w \in H(A)$ there is some $z \in O(A)$ such that $\tau(\iota(w))(z) \cdot z^{-1} \neq 1$. Since the homomorphism $j : O(A) \to \pi_1(\Sigma_{g,d})$ was assumed to be injective it follows that $\tau(\iota(w))(j(z)) \cdot j(z)^{-1} = j(\iota(w)(z) \cdot z^{-1}) \neq 1$. Therefore $\tau(\iota(w))$ acts nontrivially on $\pi_1(\Sigma_{g,d}, p)$ and thus $\tau(\iota(w))$ is not identity. This means that $\tau \circ \iota$ is injective and hence the homomorphism of $H(A)$ onto $M(\Sigma_{g,d})(A; D)$ is an isomorphism. This finishes the proof of Theorem 1.10.

Let $B = \{c_0, c_1, \ldots, c_{2g}\}$ and $C = \{c_1, c_2, \ldots, c_{2g}\}$, where c_j are the curves from the figure below.



Let $\Sigma_{g,2}$ and $\Sigma_{g,1}$ be the regular neighborhoods in Σ_g of the union of curves from B and respectively C.

Corollary 2.10. *The groups $M(\Sigma_{g,2})(B; D)$ and $M(\Sigma_{g,1})(C; D)$ are right angled Artin groups with the presentations:*

$$M(\Sigma_{g,2})(B; D) = \langle T_{c_j}^D, j = 0, \ldots, 2g; T_{c_j}^D T_{c_k}^D = T_{c_k}^D T_{c_j}^D, \text{if } j < k, k \neq j+1, (j,k) \neq (0,4) \rangle$$

and respectively:

$$M(\Sigma_{g,1})(C; D) = \langle T_{c_j}^D, j = 1, \ldots, 2g; T_{c_j}^D T_{c_k}^D = T_{c_k}^D T_{c_j}^D, \text{if } j < k, k \neq j+1 \rangle$$

Proof. Here is a direct simpler proof which uses the proof given in [8] for small Artin groups. Let E_{2g} be the Artin group associated to the Dynkin graph of type E_{2g}, which is the tree whose vertices are in one-to-one correspondence with the curves $c_0, c_1, \ldots, c_{2g}$ from the figure above and whose edges join two vertices only if the respective curves have one intersection point. Observe that A_{2g} is the Dynkin subgraph associated to the curves $c_1, c_2, \ldots, c_{2g}$.

Let now $E_{2g}[D]$ denote the subgroup of E_{2g} generated by $T_{c_j}^D$, $j = 0, 1, \ldots, 2g$. Crisp and Paris proved in [8] that the subgroup $E_{2g}[D]$ has the following right angled Artin group presentation:

$$E_{2g}[D] = \langle T_{c_j}^D, j = 0, \ldots, 2g \mid T_{c_j}^D T_{c_k}^D = T_{c_k}^D T_{c_j}^D, \text{if } j < k, k \neq j+1, (j,k) \neq (0,4) \rangle$$

The regular neighborhoods $F(B)$ and $F(C)$ are homeomorphic to $\Sigma_{g,2}$ and $\Sigma_{g,1}$, respectively.

An essential ingredient of the proof in [8] is the natural representation of the Artin group E_{2g} into the mapping class group $M(F(B))$. Consequently E_{2g} acts by automorphisms on the fundamental groupoid $\pi_1(F(B); S)$, where $S = \{s_0, \ldots, s_{2g}\}$ is a set of boundary base points, one base point for each annulus. Set $\tau : E_{2g} \to \mathrm{Aut}(\pi_1(F(B); S))$ for this representation.

Let then $H(B)$ and $H(C)$ be the right angled Artin group

$$H(B) = \langle a_j, j = 0, \ldots, 2g \mid a_j a_k = a_k a_j, \text{if } j < k, k \neq j+1, (j,k) \neq (0,4) \rangle$$

$$H(C) = \langle a_j, j = 1, \ldots, 2g \mid a_j a_k = a_k a_j, \text{if } j < k, k \neq j+1, (j,k) \neq (0,4) \rangle$$

There is a homomorphism $\iota : H(B) \to E_{2g}$ that sends each a_j into $T_{c_j}^D$.

The key point of the proof from [8] is that, given any non-trivial element $w \in H(B)$, the automorphism $\tau(\iota(w))$ acts non-trivially on some element of $\pi_1(F(B), S)$ and hence $\tau(\iota(w)) \neq 1$. This shows that ι injects $H(B)$ into E_{2g}.

However this proof also shows that the right angled Artin group $H(B)$ injects into the mapping class group $M(F(B))$. The corresponding map sends a_j into the Dehn twist $T_{c_j}^D$. As $M(F(B))$ is actually $M(\Sigma_{g,2})(B; D)$ the claim follows.

The same proof works for the sub-family C. □

We can slightly generalize the previous results to subgroups generated by not necessarily equal powers of Dehn twists.

Proposition 2.11. *Let A be a nontrivial sparse collection of curves on $\Sigma_{g,d}$. Then the subgroup of $M(F(A))$ generated by the powers $T_a^{D(a)}$, where $|D(a)| \geq 2$, $a \in A$ is a right angled Artin group.*

Proof. The proof from above applies with only minor modifications. □

Remark 2.12. If $D(a) = D$, for a non-separating curve a and $D(a) = 1$, for all other simple closed curves a, then the subgroup generated by *all* the powers $T_a^{D(a)}$ is the level D subgroup of the mapping class group of Σ_g, namely the kernel of $M(\Sigma_g) \to Sp(2g, \mathbb{Z}/D\mathbb{Z})$. This is proved by McCarthy in ([35], Theorem 2.8). In particular, in this case the subgroup is of finite index.

3. Power subgroups of the mapping class group

3.1. $M_g[D]$ and symplectic groups

We fix once for all a symplectic basis $\{a_i, b_i\}_{i=1,\ldots,g}$ in homology consisting of classes of simple loops and denote by $P : M_g \to Sp(2g, \mathbb{Z})$ the natural homomorphism.

Proposition 3.1. *If $g \geq 2$ then P sends $M_g[D]$ onto the special congruence subgroup*

$$Sp(2g, \mathbb{Z})[D] = \ker(Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/D\mathbb{Z}))$$

Proof. The action of the Dehn twist T_b in homology is given by

$$T_b^D a = a + D\langle a, b\rangle b$$

where $\langle a, b\rangle$ is the algebraic intersection number on Σ_g. Therefore $T_b^D(a) - a$ belongs to the submodule $DH_1(S_g, \mathbb{Z})$ of $H_1(S_g, \mathbb{Z})$, for any $b \in H_1(S_g, \mathbb{Z})$. This implies that $P(T_b^D) \in Sp(2g, \mathbb{Z})[D])$ and hence $P(M_g[D])$ is a normal subgroup of $Sp(2g, \mathbb{Z})[D]$.

Recall that $Sp(2g, \mathbb{Z})$ is the group of matrices A with integer entries which satisfy $AJA^T = J$, where the almost complex structure matrix J is the direct sum of g blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Consider the elementary matrices

$$SE_{i\tau(i)}[D] = I_{2g} + DE_{i\tau(i)}$$

$$SE_{ij}[D] = I_{2g} + DE_{ij} - (-1)^{i+j} DE_{\tau(j)\tau(i)}$$

where τ is the permutation $\tau(2j) = 2j - 1, \tau(2j - 1) = 2j$, for $1 \leq j \leq g$ and E_{ij} denotes the matrix having a single non-zero unit entry at position (ij). By direct computation we find that:

$$SE_{12}[D] = P(T_{a_1}^{-D})$$

$$SE_{13}[D] = P(T_{b_2}^{-D} T_{a_1}^{-D} T_c^D)$$

$$SE_{14}[D] = P(T_{a_2}^D T_{a_1}^D T_f^{-D})$$

where c and f are simple closed curves whose homology classes are $a_1 + b_2$ and $a_1 + a_2$ respectively.

Therefore the elementary congruence subgroup of level D, which is defined as the matrix group generated by the matrices $SE_{ij}[D]$, is contained in $P(M_g[D])$. Now, a deep result of Mennicke (see [38, 39, 3]) says that the elementary congruence subgroup coincides with the congruence subgroup $Sp(2g, \mathbb{Z})[D]$, if $g \geq 2$. Therefore $P(M_g[D])$ equals $Sp(2g, \mathbb{Z})[D]$, as claimed. □

Remark 3.2. If $g = 1$ then $M_g[D]$ might be of infinite index in $SL(2, \mathbb{Z})$ (see [40]).

Corollary 3.3. *The group $M_g[D]$ is torsion-free and consists of pure mapping classes when $D \geq 3$ and $g \geq 2$.*

Proof. Serre's Lemma tells us that torsion elements in the mapping class group act non-trivially on the homology with $\mathbb{Z}/D\mathbb{Z}$ coefficients for any $D \geq 3$.

Recall that a mapping class h is pure if $h^n(\gamma) = \gamma$ implies that $h(\gamma) = \gamma$, for each isotopy class of a simple closed curve γ. Then the second claim is a simple consequence of Ivanov's results (see [23, 24]) concerning pure classes. □

3.2. Power subgroups and symplectic groups

We start by analyzing the images of the power subgroups in the symplectic group. This amounts to finding the power subgroups of the symplectic group. Let $g \geq 2$ and recall that P denotes the homomorphism $M_g \to Sp(2g, \mathbb{Z})$ induced by a homology basis. We already saw in Section 3.1 that $P(M_g[D]) = Sp(2g, \mathbb{Z})[D]$. Moreover since P is surjective $P(X_g[D])$ is a normal subgroup of $Sp(2g, \mathbb{Z})$ containing $Sp(2g, \mathbb{Z})[D]$. We have then an obvious surjective homomorphism:

$$L : Sp(2g, \mathbb{Z}/D\mathbb{Z}) = Sp(2g, \mathbb{Z})/Sp(2g, \mathbb{Z})[D] \to Sp(2g, \mathbb{Z})/P(X_g[D])$$

Our first technical result is the following:

Lemma 3.4. *For any integer $D \not\equiv 0 (\text{mod } 6)$ and any proper ideal $J \subset \mathbb{Z}/D\mathbb{Z}$ there exists an element in the kernel of L which is not central after reduction mod J.*

Proof. It suffices to find a matrix in $C \in Sp(2g, \mathbb{Z}/D\mathbb{Z})$ whose power C^D is neither the identity $\mathbf{1}$ nor $-\mathbf{1}$ modulo the ideal J, since the center of $Sp(2g, \mathbb{Z}/D\mathbb{Z})$ consists of $\{\mathbf{1}, -\mathbf{1}\}$ (see [29], Prop. 2.1). Since C^D belongs to $\ker L$ this will prove the lemma.

We look for C of the form $A \oplus A \oplus \cdots \oplus A$ where A is a 2-by-2 matrix. We take a lift of A with integer entries. Then C^D has the form $A^D \oplus A^D \oplus \cdots \oplus A^D$. Since $A \in SL(2, \mathbb{Z})$ we have

$$A^2 = tA - \mathbf{1}$$

where t is the trace of A. It follows that

$$A^D = Q_{D-1}(t)A - Q_{D-2}(t)\mathbf{1}$$

where $Q_k(t) \in \mathbb{Z}[t]$ are polynomials in the variable t determined by the recurrence relation:

$$Q_n(t) = tQ_{n-1}(t) - Q_{n-2}(t)$$

with initial values $Q_0 = 1, Q_1(t) = t$.

We obtain therefore, by induction on D, the following formulas:

$$Q_{D-1}(0) = \begin{cases} (-1)^{\frac{D-1}{2}}, & \text{if } D \equiv 1 (\text{mod } 2), \\ 0, & \text{if } D \equiv 0 (\text{mod } 2). \end{cases}$$

$$Q_{D-1}(-1) = \begin{cases} 1, & \text{if } D \equiv 1 (\text{mod } 3), \\ -1, & \text{if } D \equiv 2 (\text{mod } 3), \\ 0, & \text{if } D \equiv 0 (\text{mod } 3). \end{cases}$$

If the reduction mod J of C^D is trivial for all C as above then $Q_{D-1}(t) \equiv 0 (\text{mod } J)$ for all t, since there exist matrices A of given trace t having some entry off-diagonal which is congruent to 1 mod D, for instance $A = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}$. Now, either $Q_{D-1}(-1)$ or $Q_{D-1}(0)$ is ± 1 mod D, hence J is trivial. This proves the claim. □

Remark 3.5. The conclusion of Lemma 3.4 does not hold when $D \equiv 0 (\text{mod } 6)$. For instance $Q_5(t) = t(t-1)(t+1)(t^2-3)$ and thus $Q_5(t) \equiv 0 (\text{mod } 6)$ for every integer t. More generally $Q_{6k-1}(t) \equiv 0 (\text{mod } 6)$, for every integer k. It suffices to observe that:

$$Q_{D-1}(1) = \begin{cases} 1, & \text{if } D \equiv 1 (\text{mod } 6), \text{ or } D \equiv 2 (\text{mod } 6), \\ -1, & \text{if } D \equiv 4 (\text{mod } 6), \text{ or } D \equiv 5 (\text{mod } 6), \\ 0, & \text{if } D \equiv 3 (\text{mod } 6), \text{ or } D \equiv 6 (\text{mod } 6). \end{cases}$$

and use the previous computations for $Q_{D-1}(0)$ and $Q_{D-1}(1)$.

Remark 3.6. Observe that Q_n is the n-th Chebyshev polynomial of the second kind

$$Q_n(t) = \frac{\sin (n+1) \arccos(t/2)}{\sin \arccos(t/2)}$$

which can be given by the explicit formula

$$Q_n(t) = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} (-1)^k \frac{(n-k)!}{k!(n-2k)!} t^{n-2k}$$

Notice that the usual definition for the Chebyshev polynomial uses the variable x, where $t = 2x$ (see [42] for more details).

Proposition 3.7. *Suppose that $g \geq 2$, D is of the form p^m for a prime p, $m \in \mathbb{Z}_+$ and additionally $g \geq 3$, $m \geq 2$ when $p \in \{2, 3\}$. Then $P(X_g[D])$ is all of $Sp(2g, \mathbb{Z})$.*

Proof. We want to prove that the image of $L : Sp(2g, \mathbb{Z}/D\mathbb{Z}) \to Sp(2g, \mathbb{Z})/P(X_g[D])$, (introduced at the beginning of Section 3.2) is trivial. Since the homomorphism L is surjective, this will prove our claim. To this purpose we analyze its kernel $\ker L$.

Now, the normal subgroups of symplectic groups over local rings were described by Klingenberg (see [29], Lemma 3.2) and Jehne ([27]), in the case when $D = p^m$, p prime and $p \notin \{2, 3\}$. The most general statement can be found in ([17], Thm. 9.1.7, p.517) where one also considered $p \in \{2, 3\}$ but $g \geq 3$. The above cited result is that under these conditions all normal subgroups of $Sp(2g, \mathbb{Z}/D\mathbb{Z})$ (where $D = p^m$, such that $\mathbb{Z}/D\mathbb{Z}$ is a local ring) are *congruence* subgroups, namely they contain the kernel $Sp(2g, \mathbb{Z}/D\mathbb{Z})[J]$ of the homomorphism $Sp(2g, \mathbb{Z}/D\mathbb{Z}) \to Sp(2g, (\mathbb{Z}/D\mathbb{Z})/J)$, for some ideal J. This implies that there exists an ideal $J \subset \mathbb{Z}/D\mathbb{Z}$ for which $\ker L$ contains $Sp(2g, \mathbb{Z}/D\mathbb{Z})[J]$.

On the other hand, if J were a proper ideal of $\mathbb{Z}/D\mathbb{Z}$, Lemma 3.4 would provide an element of $\ker L$ which does not belong to $Sp(2g, \mathbb{Z}/D\mathbb{Z})[J]$. Therefore $J = \mathbb{Z}/D\mathbb{Z}$, whenever $p \notin \{2, 3\}$ or $m \geq 2$, and hence the map L is trivial. □

Remark 3.8. The projective symplectic group $PSp(2g, \mathbb{Z}/DZ)$ is simple when D is prime, except when $g = 1, D \in \{2, 3\}$ (where it coincides with the permutation group S_3 and respectively the alternating group A_4) and $g = 2$, $D = 2$ (when it coincides with the permutation group S_6).

Remark 3.9. When $g = 2$ and $D = 2$ the image of $P(X_2[2])$ is of index 2 in $Sp(4, \mathbb{Z}/2\mathbb{Z})$. The subgroup generated by squares of elements in S_6 is the index 2 alternating subgroup A_6. In fact any square has even signature and A_6 is also the commutator subgroup of S_6. Observe that $[a, b] = (ab)^2$, if $a^2 = b^2 = 1$ and commutators of transpositions generate A_6. Finally we have the exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to P(X_2[2]) \to A_6 \to 1.$$

In the general case when D is not a power of a prime the image of $X_g[D]$ might be strictly smaller than $Sp(2g, \mathbb{Z}/D\mathbb{Z})$. This is clear when $D \equiv 0 (\text{mod } 6)$, since Remark 3.5 shows that the image of $P(X_g(D)) \subset Sp(2g, \mathbb{Z}/D\mathbb{Z})$ into $Sp(2g, \mathbb{Z}/6\mathbb{Z})$ must be central. A similar result holds more generally. Let us set:

$$o_c(D) = \min\{d; A^d \in Z(Sp(2g, \mathbb{Z}/D\mathbb{Z})), \text{ for any } A \in Sp(2g, \mathbb{Z}/D\mathbb{Z})\}$$

where $Z(G)$ stands for the center of the group G. Write D as $D = q_1 q_2 \cdots q_m D'$, where q_j are powers of distinct primes and $D' \in \mathbb{Z}$. Set $V = \{j; o_c(q_j) \text{ divides } D\} \subset \{1, 2, \ldots, m\}$ and $\nu(D) = \prod_{j \in V} q_j$. Consider also the general congruence subgroup $GSp(2g, \mathbb{Z}/D\mathbb{Z})[F]$ which is the preimage of $Z(Sp(2g, \mathbb{Z}/F\mathbb{Z}))$ under the reduction mod F homomorphism $Sp(2g, \mathbb{Z}/D\mathbb{Z}) \to Sp(2g, \mathbb{Z}/F\mathbb{Z})$.

Proposition 3.10. *The image $P(X_g[D])$ is contained in the general congruence subgroup $GSp(2g, \mathbb{Z})[\nu(D)]$.*

Proof. Consider the homomorphism $p_j : Sp(2g, \mathbb{Z}/D\mathbb{Z}) \to Sp(2g, \mathbb{Z}/q_j\mathbb{Z})$ which reduces entries modulo q_j. If $A \in Sp(2g, \mathbb{Z}/D\mathbb{Z})$ then $p_j(A^D)$ is central for any $A \in Sp(2g, \mathbb{Z}/D\mathbb{Z})$ if $o_c(q_j)$ divides D. Therefore the D-th power subgroup of $Sp(2g, \mathbb{Z}/D\mathbb{Z})$ is contained into $\cap_{j \in V} GSp(2g, \mathbb{Z}/D\mathbb{Z})[q_j]$, which can be identified with $GSp(2g, \mathbb{Z})[\nu(D)]$. □

3.3. Proof of Theorem 1.16

Theorem 1.16 (1) can be restated as follows:

Proposition 3.11. *There exist infinitely many integers D for which $P(X_g[D])$ is a proper subgroup of $Sp(2g, \mathbb{Z})$, for $g \geq 2$. In particular $M_g/X_g[D]$ are non-trivial torsion groups, for these values of D.*

Proof. It is clear that $o_c(q)$ is a divisor of the order of $Sp(2g, \mathbb{Z}/q\mathbb{Z})$, although this upper bound is far from being optimal. Let $D = \text{l.c.m.}(o_c(q), q)$. Thus we can write $D = qD'$ for some integer D', and we know that $o_c(q)$ divides D. Therefore $\nu(D)$ is divisible by q. Henceforth there exist infinitely many integers D for which $P(X_g[D])$ is a proper subgroup of $Sp(2g, \mathbb{Z})$, by Proposition 3.10. In particular $M_g/X_g[D]$ is a non-trivial torsion group. □

Notice however that $P(X_g[D])$ is always of finite index in $Sp(2g,\mathbb{Z})$ since it contains the congruence subgroup $P(M_g[D])$. The second step in the study of $X_g[D]$ is to understand the interactions with the torsion subgroup of M_g. We restate here Theorem 1.16 (2) for the sake of completeness.

Proposition 3.12. *We have $X_g[D] = M_g$, for $g \geq 3$, if $4g+2$ does not divide D.*

Proof. The chain relation (see e.g. [11], 4.4) shows that whenever $c_1, c_2, \ldots, c_k$ are simple closed curves forming a chain i.e., consecutive c_j have a common point and are otherwise disjoint, then:

(1) if k is even we have:
$$(T_{c_1}T_{c_2}\cdots T_{c_k})^{2k+2} = T_d$$
and also:
$$(T_{c_1}^2 T_{c_2}\cdots T_{c_k})^{2k} = T_d$$
where d is the boundary of the regular neighborhood of the union of the c_j.

(2) if k is odd we have:
$$(T_{c_1}T_{c_2}\cdots T_{c_k})^{k+1} = T_{d_1}T_{d_2}$$
and respectively:
$$(T_{c_1}^2 T_{c_2}\cdots T_{c_k})^{k} = T_{d_1}T_{d_2}$$
where d_1, d_2 are the boundary curves of the regular neighborhood of the union of the c_j.

As a consequence the element $a = T_{c_1}T_{c_2}\cdots T_{c_{2g}}$ is of order $4g+2$ and the element $b = T_{c_1}^2 T_{c_2}\cdots T_{c_{2g}}$ is of order $4g$, where $c_1, c_2, \ldots, c_{2g}$ are the curves from the first figure.

Lemma 3.13. *The normal subgroup generated by a^k is M_g when $k \leq 2g$ and $g \geq 3$ and of index 2 when $g = 2$.*

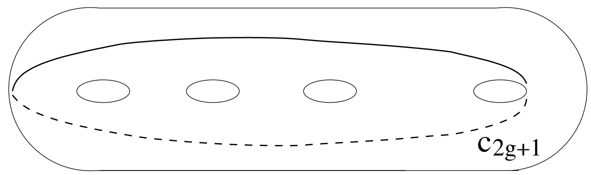
Proof. See ([32], Theorem 4). □

Let $\pi : M_g \to M_g/X_g[D]$ be the projection. We have then $a^{4g+2} = 1$. Let k denote $\gcd(4g+2, D) < 4g+2$. In the quotient $M_g/X_g[D]$ we have also $\pi(a^D) = 1$ and hence $\pi(a^k) = 1$. We have either $k \leq 2g$ or else $k = 2g+1$.

If $k < 2g+1$ Lemma 3.13 shows that the quotient $M_g/X_g[D]$ is trivial.

If $k = 2g+1$ recall that we have also $b^{4g} = 1$ and hence $\pi(b) = 1$. This implies that $\pi(a) = \pi(T_{c_1}T_{c_2}\cdots T_{c_g}) = \pi(T_{c_1}^{-1})$.

By recurrence on k we can show that $a^k(c_1) = c_{k+1}$, if $k \leq 2g$, where c_{2g+1} is the curve from the figure below:



Thus

$$T_{c_1}^{-1} a^k T_{c_1} a^{-k} = T_{c_1}^{-1} T_{a^k(c_1)} = T_{c_1}^{-1} T_{c_{k+1}}$$

Therefore

$$\pi(T_{c_1}^{-1} T_{c_{k+1}}) = \pi(T_{c_1}^{-1} a^k T_{c_1} a^{-k}) = 1$$

so that

$$\pi(T_{c_1}) = \pi(T_{c_2}) = \cdots = \pi(T_{c_{2g}})$$

The braid relations in M_g read

$$T_{c_0} T_{c_4} T_{c_0} = T_{c_4} T_{c_0} T_{c_4}$$

and

$$T_{c_1} T_{c_0} = T_{c_0} T_{c_1}$$

from which one can find

$$\pi(T_{c_0}) = \pi(T_{c_1})$$

Thus the images by π of all standard $2g+1$ generators of M_g coincide and since $H_1(M_g)$ is trivial, for $g \geq 3$, we obtain:

$$\pi(T_{c_i}) = 1, \text{ for all } i = 0, 1, \ldots 2g$$

Thus the quotient group is trivial. □

Remark 3.14. One knows that $M_g/M_g[2]$ is finite (see [20]), when $g \geq 2$, and $M_g/X_g[2]$ is the further quotient obtained by adjoining all squares as relations. Thus the quotient is a finite commutative 2-torsion group. But M_g is perfect (when $g \geq 3$) and hence it does not have surjective morphisms into nontrivial abelian groups. Thus $M_g/X_g[2]$ should be trivial, for $g \geq 3$.

Remark 3.15. For every non-separating curve d we can find a chain $c_1, c_2, \ldots, c_{2g-1}$ whose boundary is made of two curves isotopic to d and hence

$$(T_{c_1}^2 T_{c_2} \cdots T_{c_{2g-1}})^{2g-1} = T_d^2$$

Since T_d and T_{c_i} commute we have

$$((T_{c_1}^2 T_{c_2} \cdots T_{c_{2g-1}})^{1-g} T_d)^{2g-1} = T_d$$

Thus every Dehn twist along a non-separating curve is a $(2g-1)$-power. Since these Dehn twists generate M_g it follows that $X_g[2g-1] = M_g$, for $g \geq 3$.

Corollary 3.16. *The index of a normal subgroup of M_g is a multiple of $4g+2$, when $g \geq 3$.*

Proof. In fact $X_g[N]$ is contained in a normal subgroup of index N. Proposition 3.12 implies the claim. □

3.4. Proof of Theorem 1.17

For a group G denote by $Q(G)[D]$ the quotient of G by its D-th power subgroup $X(G)[D]$. The key ingredient we shall use is the deep result of Adian and Novikov (see [1]), Lysënok ([34]) and Sergei Ivanov (see [26]) that the free Burnside group $Q(\mathbb{F}_2)[D]$ is infinite for large D (e.g., $D \geq 8000$).

Lemma 3.17. *If $G \to H$ is surjective then $Q(G)[D] \to Q(H)[D]$ is also surjective.*

Proof. It suffices to see that it is well-defined and thus surjective. □

Lemma 3.18. *If $G \subset H$ is a subgroup of index n and $Q(G)[D]$ is infinite then $Q(H)[n!D]$ is infinite. When G is a normal subgroup then $Q(H)[nD]$ is infinite.*

Proof. If G is a normal subgroup of H then for every $a \in H$ we have $a^n \in G$. If G is not necessarily normal then we claim that for every $a \in H$ we have $a^{n!} \in G$. In fact, by our assumption there are only n distinct left cosets of G in H. Thus the following $(n+1)$ left cosets G, aG, $a^2G, \ldots, a^nG$ cannot be distinct. This means that there exists some non-zero integer $p \leq n$ such that $a^p \in G$. Since p divides $n!$ it follows that $a^{n!} \in G$, as claimed.

Therefore if G is normal we have $X(H)[nD] \subset X(G)[D] \subset G \subset H$ and otherwise $X(H)[n!D] \subset X(G)[D] \subset G \subset H$. The lemma follows from this. □

Lemma 3.19. *We have $Q(M_0^n)[n(n-1)(n-2)(n-3)D]$ is infinite if $n \geq 4$ and $D \geq 8000$.*

Proof. Observe that M_0^n contains the index $n(n-1)(n-2)(n-3)$ subgroup U which preserves point-wise four punctures. Let PM_0^n denote the subgroup of pure mapping classes in M_0^n which preserve point-wise all punctures. Then U surjects onto PM_0^4, by forgetting all but the four fixed punctures. But PM_0^4 is isomorphic to the free group $\mathbb{F}_2$. Thus Lemmas 3.17 and 3.18 settle the claim. □

The proof of Theorem 1.17 follows now from the following exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to M_2 \to M_0^6 \to 1$$

and Lemmas 3.17 and 3.19.

Remark 3.20. The same proof shows that the group $Q(C_{M_g}(j)((2g+2)!D)$ associated to the centralizer $C_{M_g}(j)$ of the hyper-elliptic involution j is infinite as soon as D is large enough.

Remark 3.21. One might speculate that for large values of D the subgroup $X_g[g!(4g+2)D]$ is of infinite index in M_g and the quotient is a finitely generated torsion group of exponent $g!(4g+2)D$. Moreover, in this case there would exist $N(g)$, which divides $g!(4g+2)$, such that $Q(M_g)[N(g)D]$ is infinite for large enough D, while $Q(M_g)[D]$ is finite for every D not divisible by $N(g)$. This would follow if there exists a finite index subgroup of M_g which surjects onto a free non-abelian group.

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References

[1] S.I. Adian, The Burnside problem and identities in groups, Translated from the Russian by John Lennox and James Wiegold, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 95, Springer-Verlag, Berlin-New York, 1979.

[2] J.E.Andersen, *Asymptotic faithfulness of the quantum* $\mathrm{SU}(n)$ *representations of the mapping class groups*, Ann. of Math. (2) **163** (2006), 347–368.

[3] H. Bass, J. Milnor and J.-P. Serre, *Solution of the congruence subgroup problem for* SL_n $(n \geq 3)$ *and* Sp_{2n} $(n \geq 2)$, Inst. Hautes Études Sci. Publ. Math. No. 33 (1967), 59–137.

[4] M. Clay, C. J. Leininger and D. Margalit, *Abstract commensurators of right-angled Artin groups and mapping class groups*, arXiv:1307.8387.

[5] H.S.M. Coxeter, *On factors of braid groups*, Proc. 4-th Canadian Math. Congress, Banff 1957, Univ. Toronto Press, 1959, 95–122.

[6] H.S.M. Coxeter and W.O.J. Moser, Generators and relations for discrete groups, 3-rd edition, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 14, Springer-Verlag, New York-Heidelberg, 1972.

[7] H.S.M. Coxeter, *Regular complex polytopes*, Cambridge Univ. Press, 1974.

[8] J. Crisp and L. Paris, *The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group*, Invent. Math. **145** (2001), 19–36.

[9] W. Dicks and E. Ventura, *The group fixed by a family of injective endomorphisms of a free group*, Contemporary Math. 195, 1996.

[10] B. Farb, *Some problems on mapping class groups and moduli space*, in Problems on mapping class groups and related topics, 11–55, Proc. Sympos. Pure Math., 74, Amer. Math. Soc., 2006.

[11] B. Farb and D. Margalit, A primer of mapping class groups, Princeton Mathematical Series, 49, Princeton University Press, Princeton, NJ, 2012.

[12] B. Fine and D. Spellman, *Counting subgroups of the Hecke groups*, Internat. J. Algebra Comput. **3** (1993), 43–49.

[13] M.H. Freedman, K. Walker and Z. Wang, *Quantum* $\mathrm{SU}(2)$ *faithfully detects mapping class groups modulo center*, Geom. Topol. **6** (2002), 523–539.

[14] L. Funar, *On the TQFT representations of the mapping class groups*, Pacific J. Math. **188** (1999), 251–274.

[15] S. Gervais, *Presentation and central extensions of mapping class groups*, Trans. Amer. Math. Soc. **348** (1996), 3097–3132.

[16] W. Goldman, *Ergodic theory on moduli spaces*, Ann. of Math. (2) **146** (1997), 475–507.

[17] A.J. Hahn and O.T. O'Meara, *The classical groups and K-theory*, Grund. Math. Wissenschaften, 291, Springer Verlag, 1989.

[18] R.M. Hain, *Torelli groups and geometry of moduli spaces of curves*, Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), Math. Sci. Res. Inst. Publ., 28, Cambridge Univ. Press, Cambridge, 1995, 97–143.

[19] H. Hamidi-Tehrani, *Groups generated by positive multi-twists and the fake lantern problem*, Algebr. Geom. Topol. **2** (2002), 1155–1178.
[20] S.P. Humphries, *Normal closures of powers of Dehn twists in mapping class groups*, Glasgow J. Math. **34** (1992), 313–317.
[21] S.P. Humphries, *Subgroups of pure braid groups generated by powers of Dehn twists*, Rocky Mountain J. Math. **37** (2007), 801–828.
[22] A. Ishida, *The structure of the subgroup of the mapping class group generated by two Dehn twists*, Proc. Japan Acad., Ser. A, Math. Sci. **72** (1996), 240–241.
[23] N.V. Ivanov, *Subgroups of Teichmüller modular groups and their Frattini subgroups*, Funct. Anal. Appl. **21** (1987), 154–155; translation from Funkts. Anal. Prilozh. **21** (1987), 76–77.
[24] N.V. Ivanov, *Subgroups of Teichmüller modular groups*, Translated from the Russian by E. J. F. Primrose, Translations of Mathematical Monographs, 115, Amer. Math. Soc., Providence, RI, 1992.
[25] N.V. Ivanov, *Fifteen problems about the mapping class groups*, in "Problems on mapping class groups and related topics", (B. Farb, Editor) 74–84, Proc. Sympos. Pure Math., 74, Amer. Math. Soc., Providence, RI, 2006.
[26] Sergei V. Ivanov, *The free Burnside groups of sufficiently large exponents*, Internat. J. Algebra Comput. **4** (1994), no. 1-2.
[27] W. Jehne, *Die Struktur der symplektischen Gruppe über lokalen und dedekindschen Ringen*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. 1962/1964, 187–235.
[28] M. Kapovich, *RAAGs in Ham*, Geom. Funct. Analysis **22** (2012), 733–755.
[29] W. Klingenberg, *Symplectic groups over local rings*, Amer. J.Math. **85** (1963), 232–240.
[30] T. Koberda, *Right-angled Artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups*, Geom. Funct. Analysis **22** (2012), 1541–1590.
[31] M. Korkmaz, *On cofinite subgroups of mapping class groups*, Turkish J. Math. **27** (2003), 115–123.
[32] M. Korkmaz and W. Harvey, *Homomorphisms from mapping class groups*, Bull. London Math. Soc. **37** (2005), 275–284.
[33] M. Lönne, *Presentations of subgroups of the braid group generated by powers of band generators*, Topology Appl. **157** (2010), 1127-1135.
[34] I.G. Lysënok, *Infinite Burnside groups of even period*, Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996), no. 3, 3–224; translation in Izv. Math. **60** (1996), no. 3, 453–654.
[35] J. McCarthy, *On the first cohomology group of cofinite subgroups in surface mapping class groups*, Topology **40** (2001), 401–418.
[36] D. Margalit, *A lantern lemma*, Algebr. Geom. Topol. **2** (2002), 1179–1195.
[37] G. Masbaum, *Spin TQFT and the Birman-Craggs homomorphisms*, Turkish J. Math. **19** (1995), 189–199.
[38] Jens L. Mennicke, *Finite factor groups of the unimodular group*, Ann. of Math. (2) **81** (1965), 31–37.
[39] Jens L. Mennicke, *Zur Theorie der Siegelschen Modulgruppe*, Math. Ann. **159** (1965), 115–129.
[40] M. Newman, *The structure of some subgroups of the modular group*, Ill. J. Math. **6** (1962), 480–487.
[41] L. Paris and D. Rolfsen, *Geometric subgroups of mapping class groups*, J. Reine Angew. Math. **521** (2000), 47–83.
[42] Theodore J. Rivlin, *The Chebyshev Polynomials*, John Wiley & Sons, 1974.
[43] G. Wright, *The Reshetikhin-Turaev representation of the mapping class group*, J. Knot Theory Ramifications **3** (1994), 547–574.
[44] G. Wright, *The Reshetikhin-Turaev representation of the mapping class group at the sixth root of unity*, J. Knot Theory Ramifications **5** (1996), 721–739.

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