L. Funar (2013) “Zariski Density and Finite Quotients,”
International Mathematics Research Notices, Vol. 2013, No. 9, pp. 2078–2096
Advance Access Publication March 29, 2012
doi:10.1093/imrn/rns097

Zariski Density and Finite Quotients of Mapping Class Groups

Louis Funar

Institut Fourier BP 74, UMR 5582, University of Grenoble I, 38402
Saint-Martin-d’Hères Cedex, France

Correspondence to be sent to: funar@fourier.ujf-grenoble.fr

Our main result is that the image of the quantum representation of a central extension of the mapping class group of the genus $g \geq 3$ closed orientable surface at a prime $p \geq 5$ is a Zariski dense discrete subgroup of some higher rank algebraic semi-simple Lie group $\mathbb{G}_p$ defined over $\mathbb{Q}$. As an application, we find that, for any prime $p \geq 5$, a central extension of the genus g mapping class group surjects onto the finite groups $\mathbb{G}_p(\mathbb{Z}/q\mathbb{Z})$, for all but finitely many primes q. This method provides infinitely many finite quotients of a given mapping class group outside the realm of symplectic groups.

1 Introduction and Statements

The aim of the this paper was to obtain a largeness result for the images of quantum representations of mapping class groups in genus at least 3. The main motivation is the construction of large families of finite quotients of (central extensions of the) mapping class groups by using the strong approximation theorem. This method furnishes a large supply of finite quotients of mapping class groups outside the realm of symplectic groups. Similar results were obtained independently by Masbaum and Reid [26]. Earlier, Looijenga [22] has proved that the images of Prym representations (associated with finite abelian groups) of suitable finite index subgroups of mapping class groups and of subgroups from the Johnson filtration are arithmetic groups.

Received October 10, 2011; Accepted January 24, 2012

Some largeness results in this direction are already known. In [11], we showed that the images are infinite and nonabelian (for all but finitely many explicit cases) using earlier results of Jones [19] who proved that the same holds true for the braid group representations factorizing through the Temperley–Lieb algebra at roots of unity. Masbaum then found in [24] explicit elements of infinite order in the image. General arguments concerning Lie groups actually show that the image should contain a free nonabelian group. Furthermore, Larsen and Wang [21] proved that the image of the quantum representations of the mapping class groups at roots of unity of the form $\pm\exp(\frac{2(p+1)\pi i}{4p})$, for prime $p\geq 5$, is dense in the projective unitary group.

In a previous paper [13], the authors proved that the images are large in the sense that they contain explicit free nonabelian groups. Now, each image is contained into some unitary group of matrices with cyclotomic integers entries, by Gilmer and Masbaum [16]. The latter group can be embedded as an irreducible higher-rank lattice in a semi-simple Lie group $\mathbb{G}_p(\mathbb{R})$ (depending on the genus and the order of the roots of unity) obtained by restriction of scalars. The main result of this paper strengthens the largeness property above by showing that, in general, the image of a quantum representation is Zariski dense in the noncompact group $\mathbb{G}_p(\mathbb{R})$.

Let us introduce now some terminology. Recall that in [2] the authors defined the topological quantum field theory (TQFT) functor $\mathcal{V}_p$, for every $p\geq 3$, and a primitive root of unity A of order $2p$. These TQFTs should correspond to the so-called SU(2)-TQFT, for even p, and to the SO(3)-TQFT, for odd p (see also [21] for another SO(3)-TQFT).

Definition 1.1. Let $p\in\mathbb{Z}_+$, $p\geq 3$ and A be a primitive $2p$th root of unity. The quantum representation $\rho_{p,A}$ is the projective representation of the mapping class group associated with the TQFT $\mathcal{V}_p$ at the root of unity A. We denote therefore by $\tilde{\rho}_{p,A}$ the linear representation of the central extension $\widetilde{M_g}$ of the mapping class groups M_g (of the genus g closed orientable surface) which resolves the projective ambiguity of $\rho_{p,A}$ (see [15, 27]). Furthermore, $N(p,g)$ will denote the dimension of the space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the closed orientable surface of genus g. □

Remark 1.1. The unitary TQFTs arising usually correspond to the following choices of the root of unity:

$$A_p=\begin{cases}-\exp\left(\dfrac{2\pi i}{2p}\right) & \text{if } p\equiv 0 \pmod 2;\\ -\exp\left(\dfrac{(p+1)\pi i}{p}\right) & \text{if } p\equiv 1 \pmod 2.\end{cases}$$

□

For prime $p \geq 5$, we denote by $\mathcal{O}_p$ the ring of cyclotomic integers $\mathcal{O}_p = \mathbb{Z}[\zeta_p]$, if $p \equiv -1 (\bmod 4)$ and $\mathcal{O}_p = \mathbb{Z}[\zeta_{4p}]$, if $p \equiv 1 (\bmod 4)$, respectively, where ζ_p is a primitive pth root of unity. The main result of Gilmer and Masbaum [16] states that, for every prime $p \geq 5$, there exists a free $\mathcal{O}_p$-lattice $S_{g,p}$ in the $\mathbb{C}$-vector space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the genus g closed orientable surface and a nondegenerate Hermitian $\mathcal{O}_p$-valued form on $S_{g,p}$ such that (a central extension of) the mapping class group preserves $S_{g,p}$ and keeps invariant the Hermitian form. Therefore, the image of the mapping class group consists of unitary matrices (with respect to the Hermitian form) with entries in $\mathcal{O}_p$. Let $P\mathbb{U}(\mathcal{O}_p)$ be the group of all such matrices, up to scalar multiplication. For the sake of simplicity of the exposition we will consider only primes p satisfying $p \equiv -1 (\bmod 4)$, from now on. Similar results hold for the remaining primes with only minor modifications.

It is known that $P\mathbb{U}(\mathcal{O}_p)$ is an irreducible lattice in a semi-simple Lie group $\mathbb{PG}_p(\mathbb{R})$ obtained by the so-called restriction of scalars construction from the totally real cyclotomic field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ to $\mathbb{Q}$. Specifically, let us denote by $\mathbb{G}_p(\mathbb{R})$ the product $\prod_{\sigma \in S(p)} S\mathbb{U}^{\sigma}$. Here $S(p)$ stands for a set of representatives for the classes of complex valuations σ of $\mathcal{O}_p$ modulo complex conjugacy. The factor $S\mathbb{U}^{\sigma}$ is the special unitary group associated with the Hermitian form conjugated by σ, thus corresponding to some Galois conjugate root of unity. Denote also by $\tilde{\rho}_p$ and ρ_p the representations $\prod_{\sigma \in S(p)} \tilde{\rho}_{p,\sigma(A_p)}$ and $\prod_{\sigma \in S(p)} \rho_{p,\sigma(A_p)}$, respectively. In [9], the authors proved that the restriction of $\tilde{\rho}_p$ to the universal central extension $\widetilde{M_g}^{\text{univ}}$ of M_g—which is a subgroup of $\widetilde{M_g}$ of index 12—takes values in $S\mathbb{U}$. This implies that $\tilde{\rho}_p(\widetilde{M_g}) \subset S\mathbb{U}$ for $g \geq 3$ and prime $p \geq 5$.

Note that the $\mathbb{G}_p(\mathbb{R})$ is the set of real points of a semi-simple algebraic group $\mathbb{G}_p$ defined over $\mathbb{Q}$.

Our main result can be stated now as follows.

Theorem 1.2. Suppose that $g \geq 3$ and $p \geq 5$ is a prime such that $p \equiv -1 (\bmod 4)$. Then $\tilde{\rho}_p(\widetilde{M_g})$ is a discrete Zariski dense subgroup of $\mathbb{G}_p(\mathbb{R})$ whose projections onto the simple factors of $\mathbb{G}_p(\mathbb{R})$ are topologically dense. □

Remark 1.2. A similar result holds for the SU(2)-TQFT. Specifically, let $p = 2r$, where $r \geq 5$, is prime. According to [2, Section 1.5] there is an isomorphism of TQFTs between $\mathcal{V}_{2r}$ and $\mathcal{V}_2' \otimes \mathcal{V}_r$, and hence the projection on the second factor gives us a homomorphism $\pi : \tilde{\rho}_{2r}(\widetilde{M_g}) \to \mathbb{G}_r(\mathbb{R})$. Furthermore, the image of the TQFT representation associated with $\mathcal{V}_2'$ is finite. Therefore, $\pi \circ \tilde{\rho}_{2r}(\widetilde{M_g})$ is a discrete Zariski dense subgroup of $\mathbb{G}_r(\mathbb{R})$. Note that the result holds also for $g=2$ and prime $p\geq 7$ using the modifications from [13] in the constructions of free nonabelian subgroups in the image. We skip the details. □

We now consider the Johnson filtration by the subgroups $I_g(k)$ of the mapping class group M_g of the closed orientable surface of genus g, consisting of those elements having a trivial outer action on the kth nilpotent quotient of the fundamental group of the surface, for some $k\in\mathbb{Z}_+$.

The main application of our density result is the following.

Theorem 1.3. For every $g\geq 3$ and prime $p\geq 5$ such that $p\equiv -1 (\mathrm{mod}\ 4)$, there exists some homomorphism $\widetilde{M}_g\to\mathbb{G}_p(\mathbb{Z}/q^k\mathbb{Z})$, whose restriction to $I_g(3)$ is surjective for all large enough primes q and all $k\geq 1$. In particular, the surjectivity holds also for $\widetilde{M}_g$, the Torelli group $I_g(1)$, the Johnson kernel $I_g(2)$, or any finite index subgroup of $\widetilde{M}_g$. □

Proof. The pull-back of the central extension $\widetilde{M}_g$ on the Torelli subgroup $I_g(1)$ is trivial, because the generator of $H^2(M_g)$ is the pull-back of the generator of $H^2(\mathrm{Sp}(2g,\mathbb{Z}))$, when $g\geq 3$. Thus $I_g(3)$ embeds into $\widetilde{M}_g$. The image of the central factor is finite and $\rho_p(I_g(3))$ is of finite index in $\rho_p(M_g)$, according to Proposition 4.7. Therefore, $\tilde{\rho}_p(I_g(3))$ is of finite index in $\tilde{\rho}_p(\widetilde{M}_g)$ and hence Zariski dense in $\mathbb{G}_p(\mathbb{R})$. Furthermore, we will use the following version of the strong approximation theorem due to Nori [30, Theorem 5.4] (see also [38]): let G be a connected linear algebraic group G defined over $\mathbb{Q}$ and $\Lambda\subset G(\mathbb{Z})$ be a Zariski dense subgroup. Assume that $G(\mathbb{C})$ is simply connected. Then the completion of Λ with respect to the congruence topology induced from $G(\mathbb{Z})$ is an open subgroup in the group $G(\hat{\mathbb{Z}})$ of points of G over the pro-finite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$. We now consider the group $G=\mathbb{G}_p$ which satisfies the assumptions of Nori's theorem. If we take Λ to be a finite index subgroup of $\tilde{\rho}_p(\widetilde{M}_g)$, then the strong approximation theorem implies our claim for $k=1$. Then a classical result due to Serre [34] for GL(2) and extended by Vasiu [36] to all reductive linear algebraic groups defined over $\mathbb{Q}$ improves the surjectivity statement to all $k\geq 1$. ■

Remark 1.3. The arithmetic group $\mathbb{G}_p(\mathbb{Z})$, for $g\geq 3$ and prime $p\geq 5$, has the congruence property. This follows from results of Tomanov [35, Main Theorem (a)] and Prasad and Rapinchuk [31, Theorems 2.(1) and 3] on the congruence kernel for $\mathbb{Q}$-anisotropic algebraic groups of type ${}^2A_{n-1}$, with $n\geq 4$. In this respect it would be interesting to construct finite quotients of the residually finite (see [12]) group $\widetilde{M}_g$, other than the quotients of groups of the form $\mathbb{G}_p(\mathbb{Z}/q\mathbb{Z})$. Moreover, $\mathbb{G}_p(\mathbb{Z})$ is cocompact in $\mathbb{G}_p(\mathbb{R})$, since it is $\mathbb{Q}$-anisotropic, by a classical result of Borel and Harish-Chandra [3]. □

Corollary 1.4. For any prime $p \geq 5$ such that $p \equiv -1 (\mathrm{mod}\ 4)$ and $g \geq 3$ there exists a homomorphism $M_g \to \mathbb{PG}_p(\mathbb{Z}/q^k\mathbb{Z})$ whose restriction to a given finite index subgroup of M_g is surjective for all large enough primes $q \geq 5$ and all $k \geq 1$. Here $\mathbb{PG}_p(\mathbb{R})$ is the product of the projective unitary groups whose associated special unitary groups occur as factors of $\mathbb{G}_p(\mathbb{R})$. □

Proof. The image of the center of $\widetilde{M_g}$ by the homomorphism $\tilde{\rho}_p$ is contained in the centralizer of $\tilde{\rho}_p(\widetilde{M_g})$. The Zariski density result above implies that the centralizer is contained in the product of the centers of each simple factor of $\mathbb{G}_p(\mathbb{R})$. This proves the claim. ■

Remark 1.4.

1. The first construction of finite quotients of mapping class group by this method was given by Masbaum [25].
2. The set of finite quotients of a particular M_g (with $g \geq 3$) provided by Theorem 1.3 is rather large. Indeed, $\mathbb{PG}_p(\mathbb{Z}/q\mathbb{Z})$ are finite groups of Lie type of arbitrarily large rank. In particular, the alternate group on m elements is contained into some $\mathbb{PG}_p(\mathbb{Z}/q\mathbb{Z})$, for large enough p and q, and hence into some finite quotient of M_g. Therefore, every finite group embeds in some finite quotient of the genus $g \geq 3$ mapping class group. This answers a question of Hamenstaedt, first settled by Masbaum and Reid [26].
3. In [14], we already obtained results showing that a given mapping class group has many more finite quotients than the family of all symplectic groups, as it can be measured by the torsion of their (essential) 2-homology groups.
4. The number $n(p)$ of the noncompact factors in $\mathbb{G}_p(\mathbb{R})$ goes to infinity with p. □

Let $QH(G)$ denote the vector space of quasi-homomorphisms of the group G, namely of maps $\varphi : G \to \mathbb{R}$ for which $\sup_{a,b \in G} |\partial\varphi(a,b)| < \infty$, where $\partial\varphi(a,b) = \varphi(ab) - \varphi(a) - \varphi(b)$ is the boundary 2-cocyle. The quasi-homomorphism φ is homogeneous if $\varphi(a^n) = n\varphi(a)$, for every $a \in G$ and $n \in \mathbb{Z}$. We consider the quotient $\widetilde{\mathrm{QH}}(G)$of $\mathrm{QH}(G)$ by the submodule generated by the bounded quasi-homomorphisms of G and the group homomorphisms. It is well known (see [7]) that $\widetilde{\mathrm{QH}}(G)$ is isomorphic to the kernel of the comparison homomorphism $H_b^2(G,\mathbb{R}) \to H^2(G,\mathbb{R})$, where $H_b^2(G,\mathbb{R})$ denotes the second bounded cohomology group of G.

If $\mathcal{X}$ is an irreducible Hermitian symmetric space of noncompact type and $I(\mathcal{X})$ its isometry group, then denote by $\mathcal{K}_{I(\mathcal{X})}$ the generator of the continuous bounded second cohomology group $H_{cb}^2(I(\mathcal{X}),\mathbb{R})$ of $I(\mathcal{X})$. This class is defined, for instance, by the Dupont cocycle $c_{I(\mathcal{X})} : I(\mathcal{X}) \times I(\mathcal{X}) \to \mathbb{Z}$, given by

$$c_{I(\mathcal{X})}(g_1, g_2) = \frac{1}{4\pi} \int_{\Delta(g_1(x_0), g_2(x_0), g_1 g_2(x_0))} \omega, \quad g_1, g_2 \in I(\mathcal{X}),$$

where ω is the Kähler form on $\mathcal{X}$, $x_0 \in \mathcal{X}$ and $\Delta(x, y, z)$ denotes an oriented smooth triangle on $\mathcal{X}$ with geodesic sides. Although the interior of the triangle with geodesic sides is not uniquely defined, the value of the cocycle is well defined because ω is closed (see [7] for details). It is also known that the class of $c_{I(\mathcal{X})}$ in the continuous second cohomology group $H_c^2(I(\mathcal{X}),\mathbb{R}) \cong \mathbb{R}$ is a generator. Bestvina and Fujiwara [1] proved that $\widetilde{\mathrm{QH}}(M_g)$ is infinitely generated. Quantum representations yield an explicit family of quasi-homomorphisms on the mapping class groups generalizing the Rademacher function on $\mathrm{SL}(2,\mathbb{Z})$ (which can be obtained for the one-holed torus), as follows.

Corollary 1.5. Let $g \geq 3$, $p \geq 5$ be a prime number such that $p \equiv -1 (\mathrm{mod}\ 4)$, $\mathrm{SU}(m, n)$ be a noncompact simple factor of $\mathbb{G}_p$ corresponding to the primitive root of unity A. Let $\tilde{M}_g^{\mathrm{univ}} \subset \widetilde{M}_g$ be the universal central extension of M_g. Then there exists a unique quasi-homomorphism $L_A : \tilde{M}_g^{\mathrm{univ}} \to \mathbb{R}$ verifying

$$\partial L_A = \frac{1}{4\pi(m+n)} \tilde{\rho}_{p,A}^{*} c_{\mathrm{SU}(m,n)}.$$

Furthermore, the classes of those L_A, for which $1 \leq m < n$ are linearly independent over $\mathbb{Q}$ in $\widetilde{\mathrm{QH}}(\widetilde{M}_g{}^{\mathrm{univ}})$. □

Proof. Burger and Iozzi [7, Theorem 1.3] proved that for each set of pairwise nonequivalent (i.e., not conjugate by an isometry of the corresponding symmetric spaces), Zariski dense representations $\rho_j : \Gamma \to \mathrm{SU}(m_j, n_j)$, of a finitely generated group Γ, for which $1 \leq m_j < n_j$, the elements $\rho_j^*(\mathcal{K}_{\mathrm{SU}(m_j,n_j)}) \in H_b^2(\Gamma,\mathbb{R})$ are linearly independent over $\mathbb{Z}$. On the other hand, $H^2(\tilde{M}_g^{\mathrm{univ}}) = 0$ because $\tilde{M}_g^{\mathrm{univ}}$ is a universal central extension and hence it does not have any nontrivial central extension. Therefore, the cocycle $\tilde{\rho}_{p,A}^{*} c_{\mathrm{SU}(m,n)}$ corresponds to an element of $\widetilde{\mathrm{QH}}(M_g)$ and the corollary follows from the above cited result in [7]. ■

Remark 1.5. If $\rho' : \widetilde{M_g}^{\text{univ}} \to \mathrm{SU}(m,n)$ is some Zariski dense representation and L' is the corresponding quasi-homomorphism, then $L_A = L'$ in $\widetilde{\mathrm{QH}}(M_g)$ only if ρ' is conjugate to $\tilde{\rho}_{p,A}$ (see [7]). Moreover, let $\bar{L}_A$ denote the unique homogeneous quasi-homomorphism in the class of L_A. Then $\bar{L}_A$ is a class function (i.e., invariant on conjugacy classes) on $\widetilde{M_g}^{\text{univ}}$ which encodes all information about the representation $\tilde{\rho}_{p,A}$. □

Remark 1.6. If the groups $\rho_p(M_g)$ were finitely presented and the number of relations in some group presentation were uniformly bounded (for fixed g), then the rank of $H^2(\rho_p(M_g))$ would also be uniformly bounded and our density theorem would imply that $\widetilde{\mathrm{QH}}(\rho_p(M_g))$ cannot be trivial for large enough p. For instance, the uniform bound holds if we replace $\ker \rho_p$ above by the subgroup generated by pth powers of all Dehn twists and one might conjecture that the two subgroups coincide in genus $g \geq 3$. Eventually, if $\widetilde{\mathrm{QH}}(\rho_p(M_g))$ were infinite dimensional, then $\rho_p(M_g)$ would not be boundedly generated. □

Recall now the following recent result due to Salehi Golsefidy and Varjú [33, Theorem 1 and Corollary 5]: let $\Gamma \subset \mathrm{GL}(N,\mathbb{Z})$ be a group generated by a symmetric set S and denote by $\pi_q : \mathrm{GL}(N,\mathbb{Z}) \to \mathrm{GL}(N,\mathbb{Z}/q\mathbb{Z})$ the reduction mod q. Then the family of Cayley graphs of the groups $\pi_q(\Gamma)$ with generator systems $\pi_q(S)$ forms a family of expanders (see [23] for details about expanders), where q runs through square-free integers, if and only if the connected component of the Zariski closure of Γ is perfect. This family of Cayley graphs forms expanders iff Γ has property τ with respect to the family of finite quotients $\pi_q(\Gamma)$, where q runs through square-free integers, namely if there exists some $\varepsilon > 0$ (depending on S) such that for every unitary representation ρ of Γ into some Hilbert space H factorizing through some $\pi_q(\Gamma)$ without invariant nontrivial vector and every unit vector $v \in H$ we have $|\rho(s)v - v| \geq \varepsilon$, for some $s \in S$. Here $|*|$ denotes the norm in H. Observe now that the (complex) Zariski closure of $\mathrm{SU}(m,n)$ in $\mathrm{GL}(m+n,\mathbb{C})$ is $\mathrm{SL}(m+n,\mathbb{C})$. In particular, our density result implies that the Zariski closure of $\tilde{\rho}_p(\widetilde{M_g})$ (within the appropriate product of copies of $\mathrm{GL}(N(p,g),\mathbb{C})$) is the complex Zariski closure of $\mathbb{G}_p$, namely the product of several copies of $\mathrm{SL}(N(p,g),\mathbb{C})$. Since the Zariski closure is perfect, then by Salehi Golsefidy and Varjú ([33], Theorem 1, but Corollary 5 would also suffice), we obtain the following.

Corollary 1.6. For every prime $p \geq 5$ such that $p \equiv -1 (\mathrm{mod}\ 4)$, the groups $\tilde{\rho}_p(\widetilde{M_g})$ (and $\tilde{\rho}_p(I_g(k))$, $k \leq 3$) have property τ with respect to the family of finite quotients induced by π_q, where q runs through square-free integers. □

Remark 1.7. Bourgain and Gamburd [4, 5] proved that a dense subgroup of SU(N) generated by matrices with algebraic entries, and, in particular, $\tilde{\rho}_p(\widetilde{M}_g)$ has a spectral gap with respect to the natural action on $L^2(\mathrm{SU}(N))$. □

Remark 1.8. On the other hand, these results cannot be considered as evidence in favor of the claim that $\tilde{\rho}_p(\widetilde{M}_g)$ has property τ with respect to the family of *all* its finite quotients. In fact, by Breuillard and Gelander [6], there exists a free subgroup on two generators $L \subset \tilde{\rho}_p(\widetilde{M}_g)$ which is Zariski dense in $\mathbb{G}_p(\mathbb{R})$. Although a free group does not have property τ with respect to all its finite quotients, the group L has property τ with respect to the family of finite quotients induced by π_q, where q runs through square-free integers, by the result from [33]. It would be very interesting to find whether $\widetilde{M}_g$ has property τ with respect to the family of finite quotients $\pi_q(\tilde{\rho}_p(\widetilde{M}_g))$, where $q \geq q(p)$ are sufficiently large, possibly square-free and p runs over the primes. □

2 Proof of Theorem 1.2

The starting point is the following result from [21], subsequently improved in [9].

Proposition 2.1 ([21]). If $g \geq 2$, $p \geq 5$ is prime, and $(g, p) \neq (2, 5)$, then $\rho_{p,A_p}(M_g)$ is (topologically) dense in SU($N(p, g)$), where $N(p, g)$ is the dimension of the space of conformal blocks associated to the closed orientable surface Σ_g of genus g. □

A cautionary remark is in order. There are two TQFTs which might reasonably be called the SO(3)-TQFT and although their differences are rather minimal, they are distinct (see the discussion in [21]). We work here with the version from [2], in order to have a common approach for both SU(2)-TQFT and SO(3)-TQFT. However, the proof from [21], although stated for one version, works also for the other version of the TQFT.

The Hermitian form H associated with the TQFT $\mathcal{V}_p$ and the root of unity A has signature $N_+(p, g, A)$, $N_-(p, g, A)$, where $N_+(p, g, A) + N_-(p, g, A) = N(p, g)$. We prove first the following.

Proposition 2.2. Let $g \geq 3$ and A be any primitive 2pth root of unity. Then $\rho_{p,A}(M_g)$ is (topologically) dense in SU($N_+(p, g, A), N_-(p, g, A)$). □

Two representations into the same group G are called equivalent if there exists an isomorphism of G which intertwines them. It is easy to see that

$$\tilde{\rho}_{p,\bar{A}} = \overline{\tilde{\rho}_{p,A}}$$

and therefore $\tilde{\rho}_{p,A}$ and $\tilde{\rho}_{p,\bar{A}}$ are equivalent. The converse is provided by the following.

Proposition 2.3. Let $g \geq 3$, A and B be distinct primitive $2p$th roots of unity. The representations $\tilde{\rho}_{p,A}$ and $\tilde{\rho}_{p,B}$ are equivalent if and only if $A = \bar{B}$. □

Remark 2.1. The same proof actually shows that the projective representations $\rho_{p,A}$ and $\rho_{p,B}$ are equivalent if and only if $A = B$ or $A = \bar{B}$. □

Let now $\Gamma \to H_i$, $i = 1, \ldots, m$, be a collection of representations of the group Γ. The subgroup $H \subset \prod_{i=1}^m H_i$ is called Γ-diagonal, if there exists a partition $A_1, \ldots, A_s$ of $\{1, 2, \ldots, m\}$ such that:

1. All factors H_i, with $i \in A_t$, $1 \leq t \leq s$ are equivalent as representations of Γ. Pick up some $i_t \in A_t$. Given some intertwining isomorphisms $L_{j,i_t} : H_j \to H_{i_t}$, $j \in A_t \setminus \{i_t\}$, we set:

$$H_{A_t} = \{(x, (L_{j,i_t}(x))_{j \in A_t \setminus \{i_t\}}), \quad x \in H_{i_t}\},$$

   which is the graph of the homomorphism $\oplus_{j \in A_t \setminus \{i_t\}} L_{j,i_t}$.
2. There exist intertwining isomorphisms as above with the property that the group H contains $\prod_{1 \leq t \leq s} H_{A_t}$. In particular, if all representations H_i of Γ are pairwise inequivalent, then $H = \prod_{i=1}^m H_i$.

We then have the following Hall lemma from [20]:

Lemma 2.4 (Hall lemma)**.** Let Γ be a subgroup of the product $\prod_{i=1}^m H_i$ of the adjoint simple (i.e., connected, without center and whose Lie algebra is simple) Lie groups H_i. Assume that the projection of Γ on each factor H_i is Zariski dense. Then the Zariski closure of Γ in $\prod_{i=1}^m H_i$ is a Γ-diagonal subgroup. □

Therefore, the Hall lemma above shows that the Zariski closure of $\rho_p(M_g)$ is all of $\mathbb{PG}_p(\mathbb{R})$. Now using [20, Lemma 3.6] we obtain that $\tilde{\rho}_p(\widetilde{M_g})$ is Zariski dense in $\mathbb{G}_p(\mathbb{R})$. This proves the theorem.

3 Proof of Proposition 2.2

Recall that the (reduced) Burau representation $\beta : B_n \to \mathrm{GL}(n-1, \mathbb{Z}[q, q^{-1}])$ is defined on the standard generators $g_1, g_2, \ldots, g_{n-1}$ of the braid group B_n on $n \geq 3$ strands by the matrices:

$$\beta_q(g_1) = \begin{pmatrix} -q & 1 \\ 0 & 1 \end{pmatrix} \oplus \mathbf{1}_{n-3},$$

$$\beta_q(g_j) = \mathbf{1}_{j-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ q & -q & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus \mathbf{1}_{n-j-2}, \quad \text{for } 2 \le j \le n-2,$$

$$\beta_q(g_{n-1}) = \mathbf{1}_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ q & -q \end{pmatrix}.$$

Denote by $q_p(A)$, where A is a primitive $2p$th root of unity, the following root of unity:

$$q_p(A) = \begin{cases} A^{-4} & \text{if } p = 5 \text{ or } p \equiv 0 \pmod 2; \\ A^{-8} & \text{if } p \equiv 1 \pmod 2,\ p \ge 7. \end{cases}$$

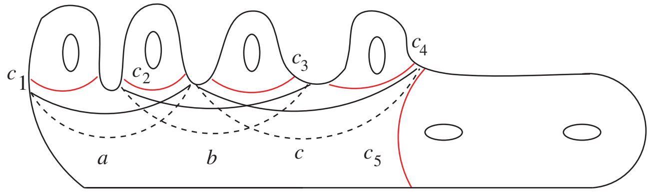
Squier proved that, after a suitable rescaling, $\beta_{q_p(A)}$ preserves a nondegenerate Hermitian form and hence it takes values in either $U(3)$ or $U(2, 1)$, depending on the signature of the invariant form.

Now, whenever $p \ge 5$ is prime, the group $\beta_{q_p(A)}(PB_3)$ is neither finite nor abelian, so that it is dense in $\mathrm{SU}(2) \subset \mathrm{SU}(3)$ and hence $\beta_{q_p(A)}(PB_4)$ is dense in $\mathrm{SU}(3)$.

Lemma 3.1. If the signature of the Hermitian invariant form associated with β_q is $(2, 1)$ or $(1, 2)$, then the image of β_q in $\mathrm{SU}(2, 1)$ is dense. □

Proof. There are several proofs of this statement. It is a consequence of the results of Deligne and Mostow concerning hyper-geometric integrals (see [29]) and the identification of the Burau representation as such one. Alternatively, we might use a result of Freedman et al. [10] subsequently reproved and extended by Kuperberg [20, Theorem 1] saying that the Burau representation of B_4 at a $2p$th root of unity is Zariski dense in the group $\mathrm{SU}(2, 1)$. ■

We will relate now the quantum representation $\rho_{p,A}$ and the Burau representation $\beta_{q_p(A)}$. Specifically we embed $\Sigma_{0,5}$ into Σ_g by means of curves c_1, c_2, c_3, c_4, and c_5 as in the figure below.



Then the curves a, b, and c (each one surrounding two of the holes of $\Sigma_{0,5}$) are separating. The pure braid group PB_4 embeds into $M_{0,5}$ using a non-canonical splitting of the surjection $M_{0,5} \to \mathrm{PB}_4$ by means of the Dehn twists along the curves a, b, and c. Furthermore, $M_{0,5}$ embeds into M_g when $g \geq 5$, by using the homomorphism induced by the inclusion of $\Sigma_{0,5}$ into Σ_g as in the figure. When $g \in \{3, 4\}$, we shall use other embeddings, similar to those used in [13] for the proof that the image of the quantum representation contains free nonabelian subgroups.

Lemma 3.2. Let $p \geq 5$. The restriction of the quantum representation ρ_p at $\mathrm{PB}_4 \subset M_{0,5}$ has an invariant three-dimensional subspace such that the corresponding sub-representation is equivalent to the Burau representation $\beta_{q_p(A)}$. □

Proof. For even p and PB_3 instead of PB_4 this is the content of Funar [11, Proposition 3.2]. The odd case is similar. The invariant subspace is the space of conformal blocks associated with the surface $\Sigma_{0,5}$ with boundary labels $(2, 2, 2, 2, 2)$, when $p = 5$, and $(4, 2, 2, 2, 2)$, when $p \geq 7$, respectively. The eigenvalues of the half-twist can be computed as in [11]. ■

Thus the image $\rho_p(\mathrm{PB}_4)$ of the quantum representation projects onto the image of the Burau representation $\beta_{q_p(A)}(\mathrm{PB}_4)$.

End of the proof of Proposition 2.2. If p is as above, then Proposition 2.1 shows that $\rho_{p,A_p}(M_g)$ is topologically dense in $\mathrm{SU}(N(p, g))$ which is a compact Lie group. In particular, the image is Zariski dense in $\mathrm{SU}(N(p, g))$. The Zariski density is preserved by a Galois conjugacy and thus, $\rho_{p,A}(M_g)$ is Zariski dense in $\mathrm{SU}(N_+(p, g, A), N_-(p, g, A))$ for every A. Furthermore, a Zariski dense subgroup of a reductive almost simple Lie group (in particular, of $\mathrm{SU}(N_+(p, g, A), N_-(p, g, A))$ not contained in the center is either dense or discrete. Now, recall that $\rho_{p,A}(M_g)$ contains $\beta_{q_p}(\mathrm{PB}_4)$. The latter group is topologically dense in $\mathrm{SU}(3)$ and respectively $\mathrm{SU}(2, 1)$. Therefore $\rho_{p,A}(M_g)$ is indiscrete. Since the image group

$\rho_{p,A}(M_g)$ is not contained within the center of $SU(N_+(p, g, A), N_-(p, g, A))$, it follows that it should be topologically dense in $SU(N_+(p, g, A), N_-(p, g, A))$. This proves the claim.

4 Proof of Proposition 2.3

Assume that the representations $\tilde{\rho}_{p,A}$ and $\tilde{\rho}_{p,B}$ were equivalent as representations into the special pseudo-unitary group $SU(N_+(p, g, A), N_-(p, g, A))$. Then $\rho_{p,A}$ and $\rho_{p,B}$ are equivalent as projective representations into $PU(N_+(p, g, A), N_-(p, g, A))$ and hence by results of Walter [37] they are equivalent as representations into $U(N_+(p, g, A), N_-(p, g, A))$. This means that there exists an isomorphism $L: U(N_+(p, g, A), N_-(p, g, A)) \to U(N_+(p, g, B), N_-(p, g, B))$ with the property that

$$L \circ \tilde{\rho}_{p,A} = \tilde{\rho}_{p,B}.$$

According to classical results of Rickart (improving previous results by Dieudonné) [32], any automorphism L of a unitary group $\mathbb{U}$ over an infinite field could be expressed in the form:

$$L(x) = \chi(x) V x V^{-1},$$

where $V: \mathbb{C}^{N(p,g)} \to \mathbb{C}^{N(p,g)}$ is a semi-linear isomorphism, which will be called the intertwiner. This means that there exists some field automorphism $\phi: \mathbb{C} \to \mathbb{C}$ with the following properties:

$$\overline{\phi(a)} = \phi(\bar{a}),$$

$$V(ax + by) = \phi(a)V(x) + \phi(b)V(y), \qquad \text{for } a, b \in \mathbb{C},\ x, y \in \mathbb{C}^{N(p,g)},$$

$$H(V(x), V(y)) = \phi(V(x, y)),$$

where $\bar{a}$ denotes the conjugate of a, and H is the Hermitian form associated with A (or with B). Furthermore, $\chi: \mathbb{U} \to U(1)$ is some homomorphism. It is well known that the only automorphisms of $\mathbb{C}$ commuting with the complex conjugacy are the identity and the complex conjugacy. Thus, V is either a linear or an anti-linear isomorphism preserving (and, respectively, conjugating) the Hermitian form H.

Suppose therefore that $\tilde{\rho}_{p,A}$ and $\tilde{\rho}_{p,B}$ are equivalent. Let V be the intertwiner and χ the character.

Lemma 4.1. If $g \geq 3$, then one can assume that the character χ is trivial. □

Proof. Let $\chi_0 : \widetilde{M_g} \to U(1)$ denote the homomorphism defined by $\chi_0(x) = \chi(\tilde{\rho}_{p,A}(x))$. Since $\tilde{\rho}_{p,B}(x) = \chi(\tilde{\rho}_{p,A}(x)) V \circ \tilde{\rho}_{p,A}(x) \circ V^{-1}$, we obtain that χ_0 is a character on $\widetilde{M_g}$. The presentation given by Gervais [15, Theorem B] shows that lantern relations lift to $\widetilde{M_g}$. Furthermore, $\widetilde{M_g}$ is generated by lifts of pairwise conjugate Dehn twists along nonseparating curves, and one central generator c. We can express c^{12} using the chain relation on 2-holed subtori with nonseparating boundary curves. The lantern relation implies then that $H_1(\widetilde{M_g}) = \mathbb{Z}/12\mathbb{Z}$. Alternatively, we can use the fact that the universal central extension $\widetilde{M_g}^{\text{univ}}$ is a normal subgroup of index 12 of $\widetilde{M_g}$ and $H_1(\widetilde{M_g}^{\text{univ}}) = 0$. Therefore $\chi_0(x)^{12} = 1$, for any $x \in \widetilde{M_g}$. On the other hand, it is known that $\tilde{\rho}_{p,A}(c)$ is the scalar matrix A^{-12} (see, e.g., [2, 14, 27]) and hence $\chi_0(c)^p = 1$. Since $p \geq 5$ is prime, it follows that $\chi_0(c) = 1$ and hence χ_0 factors through M_g. Now, M_g is perfect for $g \geq 3$ and thus the character χ_0 is trivial. In particular, we can take χ to be trivial. ■

For each simple loop γ on the surface, there is defined in [27, 4.3] a canonical lift $\tilde{T}_\gamma \in \widetilde{M_g}$ of the (right) Dehn twist T_γ along γ.

Consider the collection of eigenvalues of Dehn twists along with their multiplicities:

$$E(A, \tilde{T}_\gamma) = \{\lambda;\ \text{there exists } x \neq 0 \text{ such that } \tilde{\rho}_{p,A}(\tilde{T}_\gamma)(x) = \lambda x\}.$$

Lemma 4.2. If $\tilde{\rho}_{p,A}$ and $\tilde{\rho}_{p,B}$ are equivalent, then either $E(A, \tilde{T}_\gamma) = E(B, \tilde{T}_\gamma)$, for every γ or else $E(A, \tilde{T}_\gamma) = \overline{E(B, \tilde{T}_\gamma)}$, for every γ. □

Proof. The linear representations $\tilde{\rho}_{p,A}$ and $\tilde{\rho}_{p,B}$ are either linearly or anti-linearly equivalent by means of some intertwiner isomorphism V. Now, if

$$\tilde{\rho}_{p,A}(\tilde{T}_\gamma)(x) = \lambda x$$

then we have

$$\tilde{\rho}_{p,B}(\tilde{T}_\gamma) V(x) = V \tilde{\rho}_{p,A}(\tilde{T}_\gamma) V^{-1}(V(x)) = \phi(\lambda) V(x).$$

Thus, the set of eigenvalues of $\tilde{\rho}_{p,A}(\tilde{T}_\gamma)$ and $\tilde{\rho}_{p,B}(\tilde{T}_\gamma)$ should correspond to each another by means of V. ■

Let $C = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, $n \leq 3g-3$, be some set of disjoint simple closed curves on the closed surface of genus g. Then the n lifts of Dehn twists $\tilde{T}_{\gamma_i}$, $\gamma_i \in C$, pairwise commute and hence one could diagonalize them simultaneously. Let $W_{p,g}$ denote the space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the closed orientable surface of genus g. We set $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and define

$$W(\Lambda, C, A) = \{x \in W_{p,g};\ \tilde{\rho}_{p,A}(\tilde{T}_{\gamma_i})(x) = \lambda_i x, i \in \{1, 2, \ldots, n\}\} \subset W_{p,g}.$$

Let now $\Sigma_{g',n} \hookrightarrow \Sigma_g$ be an essential embedding of the surface $\Sigma_{g',n}$ of genus g' with n boundary components in Σ_g, that is, such that the homomorphism induced at the level of fundamental groups is injective. We also assume that $\Sigma_{g',n}$ has a pants decomposition. Let C be a maximal system of non null-homotopic and pairwise nonhomotopic simple closed curves in $\Sigma_g \setminus \Sigma_{g',n}$. Then C contains the set C_0 of boundary circles of $\Sigma_{g',n}$. We associate to each circle c_i from C_0 some arbitrary λ_i. Further, to each circle c_s in $C - C_0$ we associate $\lambda_s = 1$. We say that Λ comes from a coloring of C_0 if there is some set of colors $(i_1, \ldots, i_n)$ such that, for all

$$\lambda_j = \begin{cases} A^{i_j(i_j+2)} & \text{for odd } p; \\ (-1)^{i_j} A^{i_j(i_j+2)} & \text{for even } p. \end{cases}$$

Lemma 4.3. The vector space $W(\Lambda, C, A)$ is nonzero only if Λ comes from a coloring of C_0. In this case $W(\Lambda, C, A)$ can be identified with the space of conformal blocks associated to the subsurface $\Sigma_{g',n}$ and the coloring $(i_1, \ldots, i_n)$ of the boundary components. □

Proof. It is well known that the eigenvalues λ_i of Dehn twists in the basis given by colored trivalent graphs from [2, 4.11] are given by the formula from above, in terms of colors (see also [2, 5.8]). Now the space of conformal blocks associated to a subsurface splits as a direct sum of one dimensional eigenspaces $W(\Lambda, C \cup C', A)$ associated to all possible colorings of some maximal system C' of non null-homotopic and pairwise nonhomotopic simple closed curves on the subsurface $\Sigma_{g',n}$. ■

Lemma 4.4. Any intertwiner V induces an isomorphism

$$V : W(\Lambda, C, A) \to W(\Lambda, C, B),$$

and hence an isomorphism between the subspaces of conformal blocks associated to any essential subsurface with colored boundary. □

Proof. This is a consequence of the previous two lemmas. ■

Lemma 4.5. Let A and B be primitive $2p$th roots of unity.

1. If $\tilde{\rho}_{p,A}$ and $\tilde{\rho}_{p,B}$ are linearly equivalent, then $A=B$.
2. If $\tilde{\rho}_{p,A}$ and $\tilde{\rho}_{p,B}$ are anti-linearly equivalent, then $A=\bar{B}$. □

Proof. Consider first the case (of the SO(3)-TQFT) when $p=2r+1$ is odd. Then the set of colors is $\mathcal{C}=\{0,2,4,\ldots,2r-2\}$ and the collection of eigenvalues is (see [2, 5.8]) given by $E(A,\tilde{T}_\gamma)=(A^{i(i+2)})_{i\in\mathcal{C}}$. By assumption, there exists a bijection between the set of colors associated with A and B, namely $f:\mathcal{C}\to\mathcal{C}$ with the property that $A^{i(i+2)}=B^{f(i)(f(i)+2)}$. Moreover, the bijection f between the colors should be compatible with the conformal blocks structure.

We will prove by recurrence on i that $f(i)=i$. First, we have $f(0)=0$. If $f(2)\neq 2$, then $f(2)>2$. The space of conformal blocks associated to a 4-holed sphere $\Sigma_{0,4}$ whose boundary components are colored by 2 has dimension 3, corresponding to the colors $\{0,2,4\}$ on a separating curve (if $p\geq 7$). The bijection f should send this space into the space of conformal blocks associated to a 4-holed sphere whose boundary components are colored by $f(2)$. But the separating curve can be given any even color in the range $0,2,\ldots,f(2)$.Thus, $f(2)>2$ would lead to a contradiction. This implies that $f(2)=2$. Therefore, we have $A^8=B^8$ where both A and B are primitive roots of unity of order $2p$, with odd p, which implies that $A=B$. The case $p=5$ is immediate, by direct calculation.

Consider now the case (of the SU(2)-TQFT) when $p=2r+2$ is even. Then the set of colors is $\mathcal{D}=\{0,1,2,\ldots,r-1\}$ and the collection of eigenvalues is (see [2, Section 5.8]) given by $E(A,T_\gamma)=((-1)^iA^{i(i+2)})_{i\in\mathcal{D}}$. By assumption, there exists a bijection between the set of colors associated with A and B, namely $f:\mathcal{D}\to\mathcal{D}$ with the property that $(-1)^iA^{i(i+2)}=(-1)^{f(i)}B^{f(i)(f(i)+2)}$. Moreover, the bijection f between the colors should be compatible with the conformal blocks structures.

We first have $f(0)=0$. If $f(1)\neq 1$, then $f(1)>1$. The space of conformal blocks associated to a 4-holed sphere $\Sigma_{0,4}$ whose boundary components are colored by 1 has dimension 2, corresponding to the colors $\{0,2\}$ on a separating curve (if $p\geq 4$). The space of conformal blocks associated to a 4-holed sphere $\Sigma_{0,4}$ whose boundary components are colored by $f(1)$ has dimension $f(1)+1$, corresponding to the colors $\{0,2,\ldots,\min(2f(1),2r-2-2f(1))\}$ on a separating curve. This leads to a contradiction when $f(1)>1$ and $r\geq 3$. Thus $f(1)=1$. Moreover, the space of conformal blocks associated to the 3-holed sphere with boundary components colored by $1,1$, and 2 is one-dimensional, and so this is the same for the the coloring $1,1$ and $f(2)$. Therefore

$f(2) \leq 2$ by the Clebsch–Gordan admissibility conditions and hence $f(2) = 2$. Therefore $A^3 = B^3$ and $A^8 = B^8$ which gives us $A = B$.

The above proof works without essential modifications when V is anti-linear. ■

4.1 The first Johnson subgroups and their quantum images

For a group G, we denote by $G_{(k)}$ the lower central series defined by:

$$G_{(1)} = G, \quad G_{(k+1)} = [G, G_{(k)}], k \geq 1.$$

An interesting family of subgroups of the mapping class group is the set of higher Johnson subgroups defined as follows.

Definition 4.6. The kth Johnson subgroup $I_g(k)$ is the group of mapping classes of homeomorphisms of the closed orientable surface Σ_g of genus g whose action by outer automorphisms on $\pi/\pi_{(k+1)}$ is trivial, where $\pi = \pi_1(\Sigma_g)$. □

Thus $I_g(0) = M_g$, $I_g(1)$ is the Torelli group commonly denoted T_g, while $I_g(2)$ is the group generated by the Dehn twists along separating simple closed curves and considered by Johnson and Morita (see, e.g., [18, 28]), which is often denoted by K_g.

Proposition 4.7. For $g \geq 3$, we have the following chain of normal groups of finite index:

$$\rho_p([K_g, K_g]) \subset \rho_p(I_g(3)) \subset \rho_p(K_g) \subset \rho_p(T_g) \subset \rho_p(M_g). \qquad \square$$

Proof. There is a surjective homomorphism $f_1 : \mathrm{Sp}(2g, \mathbb{Z}) \to \frac{\rho_p(M_g)}{\rho_p(T_g)}$. The image of the pth power of a Dehn twist in $\rho_p(M_g)$ is trivial. On the other hand, the image of a Dehn twist in $\mathrm{Sp}(2g, \mathbb{Z})$ is a transvection and taking all Dehn twists, one obtains a system of generators for $\mathrm{Sp}(2g, \mathbb{Z})$. Using the congruence subgroup property for $\mathrm{Sp}(2g, \mathbb{Z})$, where $g \geq 2$, the image of p-th powers of Dehn twists in $\mathrm{Sp}(2g, \mathbb{Z})$ generate the congruence subgroup $\mathrm{Sp}(2g, \mathbb{Z})[p] = \ker(\mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z}))$. Since the mapping class group is generated by Dehn twists, the homomorphism f_1 should factor through $\frac{\mathrm{Sp}(2g,\mathbb{Z})}{\mathrm{Sp}(2g,\mathbb{Z})[p]} = \mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$. In particular, the image of f_1 is finite.

By the work of Johnson [18], one knows that when $g \geq 3$ the quotient $\frac{T_g}{K_g}$ is a finitely generated abelian group A isomorphic to $\bigwedge^3 H/H$, where H is the homology of the surface. Thus there is a surjective homomorphism $f_2 : A \to \frac{\rho_p(T_g)}{\rho_p(K_g)}$. The Torelli group T_g is generated by BP-pairs, namely elements of the form $T_\gamma T_\delta^{-1}$, where γ and δ are nonseparating disjoint simple closed curves bounding a subsurface of genus 1 (see [17]). Now p-th powers of the BP-pairs $(T_\gamma T_\delta^{-1})^p = T_\gamma^p T_\delta^{-p}$ have trivial images in $\frac{\rho_p(T_g)}{\rho_p(K_g)}$. But the classes $T_\gamma T_\delta^{-1}$ also generate the quotient A and hence the classes $(T_\gamma T_\delta^{-1})^p$ will generate the abelian subgroup pA of those elements of A which are divisible by p. This shows that f_2 factors through A/pA, which is a finite group because A is finitely generated. In particular, $\frac{\rho_p(T_g)}{\rho_p(K_g)}$ is finite.

Eventually, $\frac{K_g}{I_g(3)}$ is also a finitely generated abelian group A_3, namely the image of the third Johnson homomorphism. Since K_g is generated by the Dehn twists along separating simple closed curves, the previous argument shows that $\frac{\rho_p(K_g)}{\rho_p(I_g(3))}$ is the image of a surjective homomorphism from A_3/pA_3 and hence is finite.

Recently, Dimca and Papadima [8] proved that $H_1(K_g)$ is finitely generated for $g \geq 3$. The above proof implies that $\rho_p([K_g, K_g])$ is of finite index in $\rho_p(K_g)$. We have also the following alternative argument, which makes the proof independent of the result in [8]. The group $\rho_p(K_g)$ is finitely generated since it is of finite index in the finitely generated group $\rho_p(M_g)$. Thus, $\rho_p(K_g)/\rho_p([K_g, K_g])$ is abelian and finitely generated. Moreover, $\rho_p(K_g)/\rho_p([K_g, K_g])$ is generated by torsion elements of order p, since K_g is generated by Dehn twists along bounding simple closed curves. Now, any finitely generated abelian group generated by order p elements must be finite. ∎

Remark 4.1. A natural question is whether $\rho_p(I_g(k+1))$ is of finite index in $\rho_p(I_g(k))$, for every k. The arguments above break down at $k=3$ since there are no products of powers of commuting Dehn twists in any higher Johnson subgroups. More specifically, we have to know the image of the group $M_g[p] \cap I_g(k)$ in $\frac{I_g(k)}{I_g(k+1)}$ by the Johnson homomorphism. Here $M_g[p]$ denotes the normal subgroup generated by the pth powers of Dehn twists. If the image were a lattice in $\frac{I_g(k)}{I_g(k+1)}$, then we could deduce as above that $\frac{\rho_p(I_g(k))}{\rho_p(I_g(k+1))}$ is finite. □

Acknowledgements

I am indebted to Yves Benoist, Philippe Gille, Olivier Guichard, Greg Kuperberg, Toshitake Kohno, Greg McShane, Gregor Masbaum, Mahan Mj, Alan Reid, Adrian Vasiu, and Tyakal N. Venkataramana for useful discussions and advice.

Funding

The author was partially supported by the ANR grant 2011 BS 01 020 01 ModGroup.

References

[1] Bestvina, M. and K. Fujiwara. "Bounded cohomology of subgroups of mapping class groups." *Geometry and Topology* 6 (2002): 69–89.

[2] Blanchet, C., N. Habegger, G. Masbaum, and P. Vogel. "Topological quantum field theories derived from the Kauffman bracket." *Topology* 34, no. 4 (1995): 883–927.

[3] Borel, A. and Harish-Chandra. "Arithmetic subgroups of algebraic groups." *Annals of Mathematics. Second Series* 75, no. 3 (1962): 485–535.

[4] Bourgain, J. and A. Gamburd. "Spectral gaps in SU(d)." *Comptes Rendus Mathématique. Académie des Sciences Paris, Series I*, 348, no. 11–12 (2010): 609–11.

[5] Bourgain, J. and A. Gamburd. "A spectral gap theorem in SU(d)." (2011): preprint arXiv:1108.6264.

[6] Breuillard, E. and T. Gelander. "Uniform independence in linear groups." *Inventiones Mathematicae* 173, no. 2 (2008): 225–63.

[7] Burger, M. and A. Iozzi. "Bounded Kähler class rigidity of actions on Hermitian symmetric spaces." *Annales Scientifiques de l'École Normale Supérieure. Quatriéme Série* 37, no. 1 (2004): 77–103.

[8] Dimca, A. and S. Papadima. "Arithmetic group symmetry and finiteness properties of Torelli groups." (2010): preprint arXiv:1002.0673.

[9] Dunfield, N. and H. Wong. "Quantum invariants of random 3-manifolds." *Algebraic & Geometric Topology* 11, no. 4 (2011): 2191–205.

[10] Freedman, M. H., M. Larsen, and Z. Wang. "The two-eigenvalue problem and density of Jones representation of braid groups." *Communications in Mathematical Physics* 228, no. 1 (2002): 177–99.

[11] Funar, L. "On the TQFT representations of the mapping class groups." *Pacific Journal of Mathematics* 188, no. 2 (1999): 251–74.

[12] Funar, L. "Two questions on mapping class groups." *Proc. Amer. Math. Soc.* 139, no. 1 (2011): 375–82.

[13] Funar, L. and T. Kohno. "Free subgroups within the images of quantum representations." *Forum Mathematicum* (2011): preprint arxiv:1108.4904.

[14] Funar, L. and W. Pitsch. "Finite quotients of symplectic groups vs mapping class groups." (2011): Preprint arxiv:1103.1855.

[15] Gervais, S. "Presentation and central extensions of mapping class groups." *Transactions of the American Mathematical Society* 348, no. 8 (1996): 3097–132.

[16] Gilmer, P. and G. Masbaum. "Integral lattices in TQFT." *Annales Scientifiques de l'École Normale Supérieure. Quatriéme Série* (4) 40, no. 5 (2007): 815–44.

[17] Johnson, D. L. "Homeomorphisms of a surface which act trivially on homology." *Proceedings of the American Mathematical Society* 75, no. 1 (1979): 119–25.

[18] Johnson, D. L. "The structure of the Torelli group. II. A characterization of the group generated by twists on bounding curves." *Topology* 24, no. 2 (1985): 113–26.

[19] Jones, V. F. R. "Braid Groups, Hecke Algebras and Type II_1 Factors." In *Geometric Methods in Operator Algebras*, edited by H. Araki and E. G. Effros, 242–71. Proceeding of US-Japan Seminar, 1986.

[20] Kuperberg, G. Denseness and Zariski denseness of Jones braid representations. *Geometry and Topology* 15, no. 1 (2011), 11–40.

[21] Larsen, M. and Z. Wang. "Density of the SO(3) TQFT representation of mapping class groups." *Communications in Mathematical Physics* 260, no. 3 (2005): 641–58.

[22] Looijenga, E. "Prym representations of mapping class groups." *Geometriae Dedicata* 64, no. 1 (1997): 69–83.

[23] Lubotzky, A. and A. Zuk. "On property tau." to appear.

[24] Masbaum, G. "An Element of Infinite Order in TQFT-Representations of Mapping Class Groups." *Low-Dimensional Topology (Funchal, 1998)*, 137–39. Contemporary Mathematics 233. Providence, RI: American Mathematical Society, 1999.

[25] Masbaum, G. "On representations of mapping class groups in integral TQFT." *Oberwolfach Reports*, 5 (2008): 1202–5.

[26] Masbaum, G. and A. Reid. "All finite groups are involved in the mapping class groups." (2011): preprint arXiv:1106.4261.

[27] Masbaum, G. and J. Roberts. "On central extensions of mapping class groups." *Mathematische Annalen* 302, no. 1 (1995): 131–50.

[28] Morita, S. "On the structure of the Torelli group and the Casson invariant." *Topology* 30, no. 4 (1991): 603–21.

[29] Mostow, G. D. "On discontinuous action of monodromy groups on the complex n-balls." *Journal of the American Mathematical Society* 1, no. 3 (1988): 555–86.

[30] Nori, M. "On subgroups of $GL_n(\mathbb{F}_p)$." *Inventiones Mathematicae* 88, no. 2 (1987): 257–75.

[31] Prasad, G. and A. S. Rapinchuk. "Developements on the congruence subgroup problem after the work of Bass, Milnor and Serre." Collected papers of John Milnor, Vol. V, 307–25. Providence, RI: American Mathematical Society, 2010.

[32] Rickart, C. E. "Isomorphic groups of linear transformations. II." *American Journal of Mathematics* 73, no. 3 (1951): 697–716.

[33] Salehi Golsefidy, A. and P. Varjú. "Expansion in perfect groups." (2011): arXiv:11084900.

[34] Serre, J.-P. *Abelian l-adic Representations and Elliptic Curves*. With the collaboration of Willem Kuyk and John Labute. Advanced Book Classics. Addison Wesley Publ. Co., 1989.

[35] Tomanov, G. "On the congruence-subgroup problem for some anisotropic algebraic groups over number fields." *Journal für die Reine und Angewandte Mathematik* 402 (1989): 138–52.

[36] Vasiu, A. "Surjectivity criteria for p-adic representations I." *Manuscripta Mathematica* 112, no. 3 (2003): 325–55.

[37] Walter, J. H. "Isomorphisms between projective unitary groups." *American Journal of Mathematics* 77, no. 4 (1955): 805–44.

[38] Weisfeiler, B. "Strong approximation for Zariski-dense subgroups of semi-simple algebraic groups." *Annals of Mathematics* 120, no. 2 (1984): 271–315.