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On the Minimal Number of Critical Points of Smooth Maps between Closed Manifolds

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Abstract. New information concerning the minimal number of critical points of smooth proper mappings between closed connected surfaces (possibly with boundary) without critical points on the boundary is presented.

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1. INTRODUCTION AND STATEMENTS

Let M and N be connected manifolds, possibly with boundary. Consider smooth mappings $f\colon M \to N$ with $\partial M = f^{-1}(\partial N)$ (i.e., proper mappings) such that f has no critical points on ∂M. Let $\varphi(M^m, N^k)$ be the minimal number of critical points of smooth proper mappings of this kind between the manifolds M^m and N^k of dimensions m and k, respectively. In this paper, we consider only the case in which $m \geqslant k \geqslant 2$, unless otherwise explicitly stated. The main problem in this area is to characterize the above pairs of manifolds for which φ is finite and nonzero and then to compute this value (see [7, p. 617]).

Let $V = f^{-1}(f(x))$, where x is a critical point. Following King (see [20]), we say that the singular point x is *cone-like* if it admits a cone neighborhood in V, i.e., there is a closed manifold $L \subset V \setminus \{x\}$ and a neighborhood N of x in V homeomorphic to the cone $C(L)$ over L. Recall that the cone is defined as the quotient $C(L) = L \times (0,1]/L \times \{1\}$. Notice that an isolated critical point is considered as cone-like, as the cone over the empty set. In this case, the manifold L is called the *local link* at x. If x is not cone-like, then x (and also V) are said to be *wild*.

A well-known theorem of Łojasiewicz (see, e.g., [22]) states that real-analytic mappings have cone-like singularities. However, smooth functions can have wild singularities. The first examples of smooth mappings with isolated wild singularities were obtained by Takens (see [29]) in codimension three.

The main result obtained in [1] is the following characterization of φ in small codimension.

Theorem 1.1. *Consider two closed connected manifolds with finite $\varphi(M^m, N^k)$ and $k \geqslant 2$.*

(1) *If $0 \leqslant m - k \leqslant 2$, then $\varphi(M^m, N^k) \in \{0, 1\}$, except for the exceptional pairs of dimensions $(m, k) \in \{(2, 2), (4, 3), (4, 2)\}$.*
(2) *Suppose $m - k = 3$ and assume additionally that there is a smooth mapping $f\colon M \to N$ with finitely many critical points, all of which are cone-like. Then $\varphi(M^m, N^k) \in \{0, 1\}$, except for an additional set of exceptional pairs of dimensions $(m, k) \in \{(5, 2), (6, 3), (8, 5)\}$.*

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Moreover, under the finiteness assumption, $\varphi(M,N)=1$ if and only if M is the connected sum of a smooth fibration over N with an exotic sphere and not a fibration by itself.

Remark 1.1. The proof provided in [1], where the result is stated for all singularities of codimension at most three, actually works only for smooth mappings with cone-like singularities. As explained in the addendum [2] (using [7, 9]), isolated singularities of smooth functions in codimension at most two with $k \geqslant 2$ are cone-like. Thus, the proof is complete for codimension not exceeding two. In [16], we shall analyze wild codimension-three singularities.

There are two essential ingredients in this result. First, there are local obstructions to the existence of isolated singularities, namely, the germs of smooth mappings $\mathbb{R}^m \to \mathbb{R}^k$ having an isolated singularity at the origin are actually locally topologically equivalent to a projection. Thus, these mappings are topological fibrations. Second, singular points located in a disk cluster together.

The behavior of φ in the exceptional dimensions is rather different. For instance, there are topological obstructions preventing singular points to cluster together. Specifically, the authors of [15] proved the following assertion.

Theorem 1.2. *If $n \in \{2,4,8\}$, then $\varphi(\#_e S^n \times S^n, S^{n+1}) = 2e+2$. In particular, φ can take arbitrary large (even) values in the exceptional dimensions $(4,3)$ and $(8,5)$.*

Very little is known for the other exceptional and generic (i.e., $m-k \geqslant 4$) cases, and even the case of pairs of spheres is not completely settled yet. We have proved the following partial results for mappings between spheres in [1].

Theorem 1.3.

(1) *The values of $m > k \geqslant 1$ for which $\varphi(S^m, S^k) = 0$ are exactly those arising in the Hopf fibrations, i.e., $k \in \{2,4,8\}$ and $m = 2k-1$.*
(2) *One has $\varphi(S^4,S^3) = \varphi(S^8,S^5) = \varphi(S^{16},S^9) = 2$.*
(3) *If $1 < k < m \leqslant 2k-3$, then $\varphi(S^m,S^k) = \infty$.*
(4) *If $\varphi(S^{2k-2},S^k)$ is finite and $k \geqslant 2$, then $k \in \{2,3,5,9\}$.*

Remark 1.2. It is not known at present what situation is more common in general: (1) φ is bounded in terms of the dimensions only (as in nonexceptional cases of small codimension) or (2) φ is unbounded (as in the exceptional dimensions $(4,3)$ and $(8,5)$).

A particularly interesting case is the pair of dimensions $(4,2)$. The existence of Lefschetz fibrations provides many examples of closed 4-manifolds M^4 with a finite value $\varphi(M^4,S^2)$.

Let M^4 be a closed, connected, oriented, and smooth 4-manifold and let Σ be a closed connected oriented surface. A *Lefschetz fibration* is a smooth mapping $f\colon M \to \Sigma$ such that f is injective on the set of critical points and, near any critical point, f is locally of the form $f(z_1,z_2) = z_1^2 + z_2^2$ in local complex coordinates compatible with the orientations of M and Σ.

The Lefschetz fibrations are complex analogs of Morse functions. Every symplectic manifold admits a Lefschetz pencil (see [10]) which induces a Lefschetz fibration of the manifold obtained by blowing up finitely many points (see also [11] for a survey). Conversely, Gompf showed that any 4-manifold with a Lefschetz pencil admits a symplectic structure, provided that the fibers are nontrivial in homology (see [17]). Thus, φ can be estimated from above by means of its analog for Lefschetz fibrations between symplectic manifolds.

There is a more general notion, namely, that of an achiral Lefschetz fibration, where one drops the assumption that the local complex coordinates define compatible orientations. Harer proved in [18] that a 4-dimensional manifold having a handlebody decomposition with handles of index less than or equal to two admits an achiral Lefschetz fibration over the disk with bounded fibers. Moreover, Etnyre and Fuller [13] showed that, for any smooth, closed, simply-connected, oriented 4-manifold M, the connected sum $M \# S^2 \times S^2$ admits an achiral Lefschetz fibration over S^2. It is still not known whether or not all smooth, closed, simply connected, oriented 4-manifolds M admit achiral Lefschetz fibrations, and this problem is related to the existence of handlebody decompositions without index-one handles on such manifolds. We believe that the simply connected closed oriented 4-manifolds M with finite $\varphi(M,S^2)$ are precisely those which admit achiral Lefschetz fibrations. There are only a few examples of manifolds not admitting achiral Lefschetz fibrations, for instance, $\#_n S^1 \times S^3$ for $n \geqslant 2$ (see [17]).

The number of critical fibers of a Lefschetz fibration of a four-manifold M^4 and the genera of the fiber and the base determine the Euler characteristic $\chi(M^4)$. A conjecture of Gompf claims that a symplectic four-manifold with $b_+ > 1$ has a nonnegative Euler characteristic and, if this is true, then it would follow in particular that the Lefschetz fibrations on such manifolds over S^2 have at least $4g-4$ singular fibers, where g stands for the fiber genus.

In [27, 28], Stipsicz provided lower bounds for the numbers of singular fibers in Lefschetz fibrations.

Theorem 1.4. *A nontrivial genus g Lefschetz fibration over S^2 has at least $\frac{1}{5}(8g-4)$ irreducible critical fibers. Moreover, if the 4-manifold has $b_+ = 1$ and if $g \geqslant 6$ is even, then we have at least $2g+4$ critical fibers, whereas, for odd $g \geqslant 15$, we have at least $2g+10$ critical fibers.*

Korkmaz and Ozbagci considered the minimal number $N(g,h)$ of singular fibers in a Lefschetz fibration with at least one singular fiber whose generic fiber is assumed to be connected and the fibration is relatively minimal, i.e., no fiber contains a (-1) sphere which is an embedded sphere of self-intersection -1, where g stands for the genus of the fiber and h for that of the base. They proved in [21] the following result which is parallel to Theorem 1.1 in [1].

Theorem 1.5. *The relation $N(g,h) = 1$ holds if and only if $g \geqslant 3$ and $h \geqslant 2$. Moreover, $N(g,1) > 1$ for all $g \geqslant 1$.*

Other inequalities for the number of critical points (which can cluster in a fiber) are proved by Braungardt and Kotschick in [6].

Remark 1.3. Using a construction of Matsumoto for singular fibrations by tori (see [23]), one can prove that $\varphi(S^4, S^2) = 1$. It can readily be seen that $\varphi(S^4, S^2) \leqslant 2$ by considering the composition of the mappings $S^4 \to S^3$ and $S^3 \to S^2$, where the second mapping is the Hopf fibration and the former one is the suspension of the Hopf fibration. This construction yields two smooth mappings, each one with two singularities, which lie either in two distinct fibers or in the same fiber, depending on the position of the suspension points with respect to the fibers of the Hopf fibration.

In the present paper, we add new information by computing $\varphi(M,N)$ for all closed connected surfaces and thus completing the results of [1] with the nonorientable case.

Note that a smooth mapping $f\colon Y \to X$ between surfaces has finitely many critical points if and only if it is a ramified covering. Further, $\varphi(Y,X)$ is the minimal number of ramification points of a covering $Y \to X$.

Denote by $\lceil r \rceil$ the least integer greater than or equal to r and by $\lfloor x \rfloor$ the largest integer lower than or equal to x. For $\chi(N) < 0$ and $\chi(M) \leqslant 0$, write $|\chi(M)| = d|\chi(N)| + v$, where $d, v \in \mathbb{Z}_+$ and $0 \leqslant v < |\chi(N)|$, and thus $d = \left\lfloor \frac{\chi(M)}{\chi(N)} \right\rfloor$.

If N is a nonorientable surface, denote by $\widehat{N}$ the orientable double cover of N.

Theorem 1.6. *Let M and N be connected closed surfaces.*

I. *Assume that M and N are orientable.*

(a) *If $\chi(M) > \chi(N)$, then $\varphi(M,N) = \infty$.*
(b) *If $M \neq S^2$, then $\varphi(M,S^2) = 3$ and $\varphi(S^2,S^2) = 0$.*
(c) *If N is the torus $S^1 \times S^1$, then $\varphi(M, S^1 \times S^1)$ is equal to 1 if $\chi(M) < 0$, to 0 if $M = S^1 \times S^1$, and to ∞ if $M = S^2$.*
(d) *If $\chi(N) < 0$, then $\varphi(M,N)$ is equal to $\left\lceil \frac{v}{d-1} \right\rceil$ if $d \geqslant 2$, to 0 if $M = N$, and to ∞ otherwise.*

II. *Suppose that M and N are nonorientable.*

(a) *If $N = \mathbb{RP}^2$, then $\varphi(M, \mathbb{RP}^2)$ is equal to 0 if $M = \mathbb{RP}^2$ and to 2 otherwise.*
(b) *If N is the Klein bottle, then $\varphi(M, \mathbb{RP}^2 \# \mathbb{RP}^2)$ is equal to 1 if $\chi(M) \equiv 0 \pmod 2$, to 0 if $M = \mathbb{RP}^2 \# \mathbb{RP}^2$, and to ∞ if $\chi(M) \not\equiv 0 \pmod 2$.*
(c) *Assume from now on that $\chi(N) < 0$, and thus N is neither $\mathbb{RP}^2$ nor the Klein bottle.*
   (i) *If $\chi(M) \geqslant 2\chi(N)$, then $\varphi(M,N)$ is equal to 0 if either $\chi(M) = 2\chi(N)$ or $M = N$ and to ∞ otherwise.*

(ii) *Let* $\chi(M) < 2\chi(N)$.

(A) *Assume that* $\chi(N) \equiv 0 \pmod 2$. *Then* $\varphi(M,N)$ *is equal to* $\lceil \frac{v}{d-1} \rceil$ *if* $\chi(M) \equiv 0 \pmod 2$ *and to* ∞ *if* $\chi(M) \equiv 1 \pmod 2$.

(B) *Let* $\chi(N) \equiv 1 \pmod 2$. *Then* $\varphi(M,N)$ *is equal to* $\lceil \frac{v}{d-1} \rceil$ *if* $d \equiv \chi(M) \pmod 2$, *to* $\lceil \frac{v+|\chi(N)|}{d-2} \rceil$ *if* $d \not\equiv \chi(M) \pmod 2$ *and* $d \geqslant 3$, *and to* ∞ *if* $d \not\equiv \chi(M) \pmod 2$ *and* $d = 2$.

III. *Suppose that* M *is nonorientable and* N *is orientable. Then* $\varphi(M,N) = \infty$.

IV. *Suppose that* M *is orientable and* N *is not orientable.*

(a) *If* $\chi(N) < 0$, *then* $\varphi(M,N)$ *is equal to* $\lceil 2\frac{v+|\chi(N)|}{d-1} \rceil$ *if* d *is odd and* $d \geqslant 5$, *to* $\lceil \frac{2v}{d-2} \rceil$ *if* d *is even and* $d \geqslant 4$, *to* 0 *if* $M = \widehat{N}$, *and to* ∞ *if* $M \neq \widehat{N}$ *and* $d \leqslant 3$, *where* $\widehat{N}$ *stands for the orientable double cover of* N.

(b) *If* $N = \mathbb{RP}^2$, *then* $\varphi(M, \mathbb{RP}^2)$ *is equal to* 3 *if* $\chi(M) \leqslant 0$ *and to* 0 *if* $M = S^2$.

(c) *If* $N = \mathbb{RP}^2 \# \mathbb{RP}^2$, *then* $\varphi(M, \mathbb{RP}^2 \# \mathbb{RP}^2)$ *is equal to* 1 *if* $\chi(M) < 0$, *to* 0 *if* $M = S^1 \times S^1$, *and to* ∞ *if* $M = S^2$.

Remark 1.4. Computations were previously done for orientable surfaces in [1], and in [24], it was proved that $\varphi(Y,X)$ is infinite for $\chi(Y) > \chi(X)$.

Moreover, recent results of Bogatyi, Gonçalves, Kudryavtseva, and Zieschang ([4, 5]) show that the minimal number of critical points in cases I and II can be achieved by using mappings $f\colon Y \to X$ which are *primitive* branched coverings, i.e., mappings inducing surjective mappings at the level of fundamental groups.

2. PROOF OF THEOREM 1.6

2.1. Existence of branched coverings with prescribed ramification. The proof is based upon results of Edmonds, Kulkarni, and Stong [12] who gave necessary and sufficient conditions for the existence of a covering of a surface with prescribed degree and branching data (i.e., a family of ramification orders at each branch point). By [12, Prop. 2.8], the following assertion holds.

Proposition 2.1. *Let* M *and* N *be connected closed surfaces and* $\delta \geqslant 2$. *Suppose that* M *is orientable if* N *is. Moreover, suppose that, if* N *is nonorientable and* δ *is odd or* $\delta = 2$, *then* M *is nonorientable. In this case, there is a branched covering* $f\colon M \to N$ *of degree* δ *if and only if* $\chi(M) \leqslant \delta\chi(N)$ *and* $\nu = \delta\chi(N) - \chi(M)$ *is even.*

Consider M and N as in the assumptions of the proposition. Assume that we have a branched covering $M \to N$ of degree δ with r ramification points of ramification degrees d_i, $i = 1, \dots, r$. Then $2 \leqslant d_i \leqslant \delta$, and the Hurwitz formula reads

$$\chi(M) = \delta\chi(N) - \sum_{i=1}^{r}(d_i - 1). \tag{1}$$

According to [19, 14, 3, 12], the following statement holds for any family $2 \leqslant d_i \leqslant \delta$, $i = 1, \dots, r$, satisfying (1). If $N \neq S^2$ and there is a branched covering f of degree δ as claimed in Proposition 2.1, where either N is orientable or M is nonorientable, then there is an f with r ramification points of ramification degrees d_i, $i = 1, \dots, r$.

Therefore, $\varphi(M,N)$ is the minimal possible r for which there is a natural solution δ, d_i of the linear equation (1) such that $\chi(M) - \delta\chi(N)$ is even and $2 \leqslant d_i \leqslant \delta$ (possibly with additional assumptions related to orientability, or $N \neq S^2$). For $N = S^2$, we have only showed the necessity of conditions (1) for the existence of the branched covering. As shown below (see 2.5 and 2.7), such a covering exists if $N = S^2$, $r = 3$, and $\delta = 3 - \chi(M)$.

Note that the condition $\delta \geqslant 2$ is necessary, since a degree-one branched covering is a homeomorphism.

We have the bounds $r \leqslant \sum_{i=1}^{r}(d_i - 1) \leqslant r(\delta - 1)$ which imply

$$r \leqslant -\chi(M) + \delta\chi(N) \leqslant r(\delta - 1). \tag{2}$$

Conversely, if r and δ satisfy inequalities (2), then we can find solutions d_i of equations (1). Thus, if $N \neq S^2$ and either N is orientable or M is nonorientable, then we are to find the smallest r for which there is a natural solution $\delta \geqslant 2$ (of the system of inequalities (2)) for which $\chi(M) - \delta\chi(N)$ is even.

2.2. Generic case in which $\chi(M) \leqslant \chi(N) < 0$ and $d \geqslant 2$. Inequalities (2) are equivalent to

$$(\chi(M) - r)/(\chi(N) - r) \leqslant \delta \leqslant (\chi(M) + r)/\chi(N). \tag{3}$$

Lemma 2.1. *Any solution δ of* (3) *satisfies $\delta \leqslant d$.*

Proof. We have $\delta \leqslant \frac{\chi(M)+r}{\chi(N)} = d + \frac{v-r}{|\chi(N)|} < d + 1$ since $v < |\chi(N)|$.

If we drop the parity condition, then we seek for the smallest natural number r such that

$$\lceil (\chi(M) - r)/(\chi(N) - r) \rceil \leqslant (\chi(M) + r)/\chi(N). \tag{4}$$

Lemma 2.2. *The smallest value of r for which* (4) *is verified is given by $a = \left\lceil \frac{v}{d-1} \right\rceil$.*

Proof. In fact, $r = a$ meets the inequalities because $\left\lceil \frac{\chi(M)-a}{\chi(N)-a} \right\rceil = d + \left\lceil \frac{v-(d-1)a}{a-\chi(N)} \right\rceil \leqslant d \leqslant d + \frac{v-a}{|\chi(N)|} = \frac{\chi(M)+a}{\chi(N)}$. On the other hand, if $r < a$, then $\left\lceil \frac{\chi(M)-r}{\chi(N)-r} \right\rceil = d + \left\lceil \frac{v-(d-1)r}{r-\chi(N)} \right\rceil > d$, and thus $\left\lceil \frac{\chi(M)-r}{\chi(N)-r} \right\rceil \geqslant d+1$. Since the left-hand side is an integer, whereas we have noted in the proof of the previous lemma that $\frac{\chi(M)+r}{\chi(N)} < d + 1$, we see that inequality (4) cannot be satisfied for any $r < a$.

Now consider the parity condition $\chi(M) - \delta\chi(N) \equiv 0 \,(\mathrm{mod}\, 2)$.

A. If $\chi(N) \equiv 0 \,(\mathrm{mod}\, 2)$, then the parity condition

- is either satisfied for any δ, namely, if $\chi(M) \equiv 0 \,(\mathrm{mod}\, 2)$, or
- it cannot be satisfied for any choice of δ if $\chi(M) \equiv 1 \,(\mathrm{mod}\, 2)$.

This proves that if N is orientable or M is nonorientable, then $\varphi(M, N)$ is either a or ∞ as claimed.

B. If $\chi(N) \equiv 1 \,(\mathrm{mod}\, 2)$, then the parity condition is equivalent to $\delta \equiv \chi(M) \,(\mathrm{mod}\, 2)$. Moreover, recall that δ was supposed only to satisfy inequalities (3). Alternatively, by Lemma 2.1, $d + \left\lceil \frac{v-(d-1)a}{a-\chi(N)} \right\rceil \leqslant \delta \leqslant d$ if $r = a$.

- If $d \equiv \chi(M) \,(\mathrm{mod}\, 2)$, then $\delta = d$ satisfies the parity condition and the above inequalities.
- If $d \not\equiv \chi(M) \,(\mathrm{mod}\, 2)$, then $-1 < \frac{v-(d-1)a}{a-\chi(N)} \leqslant 0$ since otherwise $(d-2)a \geqslant v - \chi(N)$, and thus

$d \geqslant 3$ and $\left\lceil \frac{v}{d-1} \right\rceil = a \geqslant \left\lceil \frac{v-\chi(N)}{d-2} \right\rceil$, which is impossible, since $v \geqslant 0$, $\chi(N) < 0$, and $d \geqslant 3$. In this case, there is no appropriate δ for $r = a$ because $0 \leqslant \frac{v-a}{|\chi(N)|} < 1$, $\left\lceil \frac{\chi(M)-a}{\chi(N)-a} \right\rceil = d = \left\lfloor \frac{\chi(M)+a}{\chi(N)} \right\rfloor$, and so the only possible value for δ would be $\delta = d$. This implies that we must consider $r > a$ and to find the smallest r for which there is a $\delta \equiv d - 1 \,(\mathrm{mod}\, 2)$ satisfying the inequalities $d + \left\lceil \frac{v-(d-1)r}{r-\chi(N)} \right\rceil \leqslant \delta \leqslant d + \frac{v-r}{|\chi(N)|}$. This implies that the smallest r with the above property must satisfy the condition $\left\lceil \frac{v-(d-1)r}{r-\chi(N)} \right\rceil \leqslant -1$ or, equivalently, the conditions $d \geqslant 3$ and $r \geqslant \left\lceil \frac{v+|\chi(N)|}{d-2} \right\rceil$. For the values $r = \left\lceil \frac{v+|\chi(N)|}{d-2} \right\rceil$ and $\delta = d-1$, which must verify $\delta \geqslant 2$, both the required inequalities hold. Thus, if $d = 2$, then $\varphi(M, N) = \infty$.

Summing up, it follows that, if $d \not\equiv \chi(M) \,(\mathrm{mod}\, 2)$ and N is orientable or M is nonorientable, then $\varphi(M, N)$ is equal to $\left\lceil \frac{v+|\chi(N)|}{d-2} \right\rceil$ if $d \geqslant 3$ and to ∞ if $d = 2$.

2.3. M and N are nonorientable. If $\chi(N) < 0$, then Case 2.2 and Remark 1.4 prove the claim.

If $N = \mathbb{RP}^2$ and $M \neq \mathbb{RP}^2$, then conditions (2) read $r \leqslant \delta - \chi(M) \leqslant r(\delta - 1)$. Then $r = 1$ does not work since $\chi(M) < 1$, whereas, for $r = 2$, the above inequalities and the parity condition have natural solutions. Thus, $\varphi(M, N) = 2$ if $\chi(M) < 1$.

If $N = \mathbb{RP}^2 \# \mathbb{RP}^2$, then conditions (2) read $r \leqslant -\chi(M) \leqslant r(\delta - 1)$. Thus, $\chi(M) \leqslant -1$, and $r = 1$ is convenient. However, $\nu = -\chi(M)$ should be even. This implies that $\varphi(M, \mathbb{RP}^2 \# \mathbb{RP}^2)$ is equal to 1 if $\chi(M) \equiv 0 \,(\mathrm{mod}\, 2)$, to 0 if $M = \mathbb{RP}^2 \# \mathbb{RP}^2$, and to ∞ if $\chi(M) \not\equiv 0 \,(\mathrm{mod}\, 2)$.

2.4. M **nonorientable and** N **orientable.** It is standard that the ramified covering of an orientable manifold is orientable. Thus, $\varphi(M, N) = \infty$ in this case.

2.5. M **and** N **are orientable.** This was solved in [1], and it corresponds to the above arguments if $\chi(M)$ and $\chi(N) < 0$ are even.

If $N = S^2$ and $\chi(M) \leqslant 0$, then conditions (2) are

$$r \leqslant 2\delta - \chi(M) \leqslant r(\delta - 1).$$

If $r \leqslant 2$, then $\chi(M) \geqslant 2$, which is impossible. For $r = 3$, the above inequalities are necessary for the existence of a branched covering $M \to S^2$ of degree δ with $r = 3$ ramification points. As shown below, these inequalities are also sufficient for the existence of such a covering with $\delta = 3 - \chi(M)$ (see 2.7), and thus $\varphi(M, S^2) = 3$ for $M \neq S^2$.

If $N = S^1 \times S^1$, then conditions (2) read

$$r \leqslant -\chi(M) \leqslant r(\delta - 1),$$

and therefore, $r = 1$ yields natural solutions for any M with $\chi(M) < 0$. Thus, $\varphi(M, S^1 \times S^1) = 1$ if $M \neq S^2, S^1 \times S^1$, and $\varphi(S^2, S^1 \times S^1) = \infty$.

2.6. M **orientable and** N **nonorientable.** Assume that $\chi(M) \leqslant \chi(N) < 0$. Then, in addition to constraints (3), the solution δ must be even, since an odd-degree branched covering of a nonorientable surface is also nonorientable (see, e.g., [12, Prop. 2.3]). Further, if $\delta = 2$, then M should be the orientable (nonramified) double cover $\widehat{N}$ of N. This follows from the fact that the mapping $M \to N$ lifts to a ramified covering $M \to \widehat{N}$ (see [12, Prop. 2.7]). Thus, $\varphi(M, N) = \varphi(M, \widehat{N})$, which was computed in 2.5.

If $\chi(M) \leqslant 2\chi(N) < 0$, then we write $|\chi(M)| = d'|\chi(\widehat{N})| + v'$, where $0 \leqslant v' < |\chi(\widehat{N})|$. It follows from 2.5 that $\varphi(M, \widehat{N})$ is equal to $\left\lceil \frac{v'}{d'-1} \right\rceil$ if $d' \geqslant 2$, to 0 if $M = \widehat{N}$, and to ∞ otherwise. Therefore, if $\chi(N) < 0$, then $\varphi(M, N)$ is equal to $\left\lceil 2\frac{v+|\chi(N)|}{d-1} \right\rceil$ if d is odd and $d \geqslant 5$, to $\left\lceil \frac{2v}{d-2} \right\rceil$ if d is even and $d \geqslant 4$, to 0 if $M = \widehat{N}$, and to ∞ if $M \neq \widehat{N}$ and $d \leqslant 3$.

Alternatively, for any collection $2 \leqslant d_i \leqslant \delta/2$, $i = 1, \dots, r$, satisfying (1), the following statement holds by [19, 14, 3, 12]. If there is a branched covering f of degree δ, as in Proposition 2.1, where N is nonorientable and M is orientable, then there is an f of this kind with r ramification points of ramification degrees d_i, $i = 1, \dots, r$. We can then proceed as in 2.2.

2.7. Triangulations and $N = S^2$**.** In [1], we treated the case $N = S^2$ separately, by making use of Belyi mappings (see [26]); however, the proof was rather sketchy. Morris Hirsch asked us for more details and later gave us the following simple proof using triangulations. Note that a similar construction was used by Prasolov and Sossinsky in [25, Th. 20.6].

Algebraic topology considerations in [1] or in 2.5 show that $\varphi(M, S^2) \geqslant 3$, and it suffices to see that there is a Belyi mapping having precisely 3 critical points, namely, with one critical point above each critical value.

There exist triangulations of the surface M with any number of vertices s, $s \geqslant 1$; in particular, with $s = 3$. In fact, we choose vertices and then add (inductively) a number of disjoint arcs joining the vertices in such a way that no two arcs are homotopic by a homotopy keeping the endpoints fixed. Consider the maximal family of pairwise nonhomotopic arcs of this kind. The complementary regions are triangles, since otherwise, we can add more arcs, contradicting the maximality. We have therefore obtained a triangulation of M with s vertices, $2s - 2\chi(M)$ triangles, and $3s - 3\chi(M)$ edges. Although each cell has its vertices among the three vertices of the triangulation, they are not necessarily distinct.

However, we need a *special* triangulation in which all triangles have the same (distinct) three vertices. For a given $n \geqslant 1$, we consider the regular polygon in the hyperbolic plane (in the Euclidean plane for $n = 1$) with $2(2n + 1)$ vertices and the angles $2\pi/(2n + 1)$. Identify the opposite edges by means of isometries reversing the orientation. There are then two orbits of the vertices, and the total angle around each vertex is 2π. We thus obtain a closed hyperbolic surface. Let us subdivide it into equal triangles with a common vertex at the center of the polygon. This triangulation has three vertices, $2(2n+1)$ triangles, and $3(2n+1)$ edges, and thus the surface is of genus n. Label the central vertex by 1 and the two other vertices by 2 and 3. Then each triangle of the triangulation has the vertices 1, 2, and 3. The hyperbolic surface is oriented, and thus each triangle inherits an orientation. We say that a triangle is *positive* if the cyclic order of the labels of its vertices is $1, 2, 3$, and *negative* otherwise. Note that adjacent triangles have opposite signs since the order of 2 and 3 is reversed.

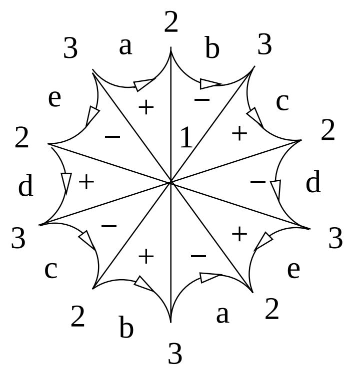

Figure.

Consider the triangulation of S^2 consisting of two triangles whose boundaries are identified. Map the triangulation of M onto that of the sphere S^2 by taking each triangle of M onto one of the triangles on the sphere, according to the sign. This yields a mapping $M \to S^2$, which is ramified at the three vertices only.

There is also a beautiful example, due to John Hubbard, of a Belyi function with three critical points only which is obtained by regarding $\Sigma \subset \mathbb{CP}^2$ as the projective algebraic curve defined by the (inhomogeneous) equation $y^{2g+1} = x^2 - 1$. The projection onto the first coordinate is a holomorphic mapping $M \to \mathbb{CP}^1$, which is ramified over 1, -1, and ∞ and has only three critical points. By the Riemann–Hurwitz formula, $\chi(M) = 2 - 2g$.

One can seek topological classification of smooth functions $f\colon M \to S^2$ with three critical points with ramification orders coinciding with $|\deg(f)|$, i.e., up to the action (by left multiplication) of the diffeomorphisms of M. The pull-back by f of the triangulation of S^2 consisting of two triangles (with vertices at critical values) is a special triangulation of M in which triangles can be equipped with signs according to the triangle covered on the sphere. If we label the vertices by $1, 2, 3$, then the sign of a triangle corresponds to the cyclic order of the boundary labels. Further, choosing a vertex, say, the one labelled by 1, and looking at the edges incident to this vertex, the endpoints of the edges become labelled by 2 and 3 only; moreover, consecutive edges correspond to different labels, and thus the cyclic order of these labels is an alternating sequence $2, 3, 2, \dots, 3$. The union of these triangles is a fundamental polygon P for the surface M. Thus, P has $2(2n+1)$ edges, where n is the genus of M. In particular, P is the polygon drawn above.

Furthermore, one obtains M by gluing the edges of P by means of an involution j on the set of edges. The gluing should satisfy the following conditions:

- the gluing reverses the orientation of the edges inherited from the circle in such a way that the quotient is orientable;
- j preserves the labels;
- the orbit of a vertex under the permutation group generated by the involutions on the set of vertices induced by j and the gluing is the set of all vertices with the same label; this means that there are precisely two vertices in the quotient M;
- adjacent edges are not identified by j.

Thus, up to a homeomorphism of M, the special triangulations of M correspond to polygons P with an involution j as above. By a direct inspection, it follows that there are no other involutions j except for the standard one given above if the genus is at most 3.

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