

An Infinite Genus Mapping Class Group and Stable Cohomology[★]

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Received: 23 June 2005 / Revised: 25 January 2007 / Accepted: 20 September 2008
Published online: 17 February 2009 – © Springer-Verlag 2009

Abstract: We exhibit a finitely generated group \mathcal{M} whose rational homology is isomorphic to the rational stable homology of the mapping class group. It is defined as a mapping class group associated to a surface \mathcal{S}_∞ of infinite genus, and contains all the pure mapping class groups of compact surfaces of genus g with n boundary components, for any $g \geq 0$ and $n > 0$. We construct a representation of \mathcal{M} into the restricted symplectic group $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r)$ of the real Hilbert space generated by the homology classes of non-separating circles on \mathcal{S}_∞ , which generalizes the classical symplectic representation of the mapping class groups. Moreover, we show that the first universal Chern class in $H^2(\mathcal{M}, \mathbb{Z})$ is the pull-back of the Pressley-Segal class on the restricted linear group $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ via the inclusion $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r) \subset \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$.

1. Introduction

1.1. Statements of the main results. The tower of all extended mapping class groups was considered first by Moore and Seiberg ([19]) as part of the conformal field theory data. This object is actually a groupoid, which has been proved to be finitely presented (see [1, 2, 7, 15]). When seeking for a group analog Penner ([24]) investigated a universal mapping class group which arises by means of a completion process and which is closely related to the group of homeomorphisms of the circle, but it seems to be infinitely generated.

In [8], we introduced the universal mapping class group in genus zero \mathcal{B} . The latter is an extension of Thompson's group V (see [5]) by the infinite spherical pure mapping class group. We proved in [8] that the group \mathcal{B} is finitely presented and we exhibited an explicit presentation. Our main difference with the previous attempts is that we consider groups acting on infinite surfaces with a prescribed behaviour at infinity that comes from actions on trees.

[★] L. F. was partially supported by the ANR Reperurf:ANR-06-BLAN-0311.

Following the same kind of approach, we propose a treatment of the arbitrary genus case by introducing a mapping class group \mathcal{M} , called *the asymptotic infinite genus mapping class group*, that contains a large part of the mapping class groups of compact surfaces with boundary. More precisely, the group \mathcal{M} contains all the pure mapping class groups $P\mathcal{M}(\Sigma_{g,n})$ of compact surfaces $\Sigma_{g,n}$ of genus g with n boundary components, for any $g \geq 0$ and $n > 0$. Its construction is roughly as follows. Let \mathcal{S} denote the surface obtained by taking the boundary of the 3-dimensional thickening of the complete trivalent tree, and further let \mathcal{S}_∞ be the result of attaching a handle to each cylinder in \mathcal{S} that corresponds to an edge of the tree (see Fig. 1). Then \mathcal{M} is the group of mapping classes of those homeomorphisms of \mathcal{S}_∞ which preserve a certain *rigid structure* at infinity (see Definition 1.3 for the precise definition). This rigidity condition essentially implies that \mathcal{M} induces a group of transformations on the set of ends of the tree, which is isomorphic to Thompson’s group V . The relation between both groups is enlightened by a short exact sequence $1 \rightarrow P\mathcal{M} \rightarrow \mathcal{M} \rightarrow V \rightarrow 1$, where $P\mathcal{M}$ is the mapping class group of compactly supported homeomorphisms of \mathcal{S}_∞ . The latter is an infinitely generated group. Our first result is:

Theorem 1.1. *The group \mathcal{M} is finitely generated.*

The interest in considering the group \mathcal{M} , outside the framework of the topological quantum field theory where it can replace the duality groupoid, is the following homological property:

Theorem 1.2. *The rational homology of \mathcal{M} is isomorphic to the stable rational homology of the (pure) mapping class groups.*

As a corollary of the argument of the proof (see Proposition 3.1), the group \mathcal{M} is perfect, and $H_2(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}$. For a reason that will become clear in what follows, the generator of $H^2(\mathcal{M}, \mathbb{Z}) \cong \mathbb{Z}$ is called *the first universal Chern class* of \mathcal{M} , and is denoted $c_1(\mathcal{M})$.

Let \mathcal{M}_g be the mapping class group of a closed surface Σ_g of genus g . We show that the standard representation $\rho_g : \mathcal{M}_g \rightarrow \text{Sp}(2g, \mathbb{Z})$ in the symplectic group, deduced from the action of \mathcal{M}_g on $H_1(\Sigma_g, \mathbb{Z})$, extends to the infinite genus case, by replacing the finite dimensional setting by concepts of Hilbertian analysis. In particular, a key role is played by Shale’s *restricted symplectic group* $\text{Sp}_{\text{res}}(\mathcal{H}_r)$ on the real Hilbert space \mathcal{H}_r generated by the homology classes of non-separating closed curves of \mathcal{S}_∞ . We have then:

Theorem 1.3. *The action of \mathcal{M} on $H_1(\mathcal{S}_\infty, \mathbb{Z})$ induces a representation $\rho : \mathcal{M} \rightarrow \text{Sp}_{\text{res}}(\mathcal{H}_r)$.*

The generator c_1 of $H^2(\mathcal{M}_g, \mathbb{Z})$ is called the first Chern class, since it may be obtained as follows (see, e.g., [20]). The group $\text{Sp}(2g, \mathbb{Z})$ is contained in the symplectic group $\text{Sp}(2g, \mathbb{R})$, whose maximal compact subgroup is the unitary group $U(g)$. Thus, the first Chern class may be viewed in $H^2(B\text{Sp}(2g, \mathbb{R}), \mathbb{Z})$. It can be first pulled-back on $H^2(B\text{Sp}(2g, \mathbb{R})^\delta, \mathbb{Z}) = H^2(\text{Sp}(2g, \mathbb{R}), \mathbb{Z})$ and then on $H^2(\mathcal{M}_g, \mathbb{Z})$ via ρ_g . This is the generator of $H^2(\mathcal{M}_g, \mathbb{Z})$. Here $B\text{Sp}(2g, \mathbb{R})^\delta$ denotes the classifying space of the group $\text{Sp}(2g, \mathbb{R})$ endowed with the discrete topology.

The restricted symplectic group $\text{Sp}_{\text{res}}(\mathcal{H}_r)$ has a well-known 2-cocycle, which measures the projectivity of the *Berezin-Segal-Shale-Weil metaplectic representation* in the bosonic Fock space (see [22], Chap. 6 and Notes, p. 171). Contrary to the finite dimension case, this cocycle is not directly related to the topology of $\text{Sp}_{\text{res}}(\mathcal{H}_r)$, since the

latter is a contractible Banach-Lie group. However, $\text{Sp}_{\text{res}}(\mathcal{H}_r)$ embeds into the restricted linear group of Pressley-Segal $\text{GL}_{\text{res}}^0(\mathcal{H})$ (see [25]), where \mathcal{H} is the complexification of \mathcal{H}_r , which possesses a cohomology class of degree 2: the Pressley-Segal class $PS \in H^2(\text{GL}_{\text{res}}^0(\mathcal{H}), \mathbb{C}^*)$. The group $\text{GL}_{\text{res}}^0(\mathcal{H})$ is homotopically equivalent to the classifying space BU , where $U = \varinjlim U(n, \mathbb{C})$, and the class PS does correspond to the universal first Chern class. Its restriction on $\text{Sp}_{\text{res}}(\mathcal{H}_r)$ is closely related to the Berezin-Segal-Shale-Weil cocycle, and reveals the topological origin of the latter. Via the composition of morphisms

$$\mathcal{M} \longrightarrow \text{Sp}_{\text{res}}(\mathcal{H}_r) \hookrightarrow \text{GL}_{\text{res}}^0(\mathcal{H}),$$

we then derive from PS an *integral* cohomology class on \mathcal{M} (see Theorem 5.1 for a more precise statement):

Theorem 1.4. *The Pressley-Segal class $PS \in H^2(\text{GL}_{\text{res}}^0(\mathcal{H}), \mathbb{C}^*)$ induces the first universal Chern class $c_1(\mathcal{M}) \in H^2(\mathcal{M}, \mathbb{Z})$.*

1.2. Definitions.

1.2.1. The infinite genus mapping class group \mathcal{M} . Set $\mathcal{M}(\Sigma_{g,n})$ for the extended mapping class group of the n -holed orientable surface $\Sigma_{g,n}$ of genus g , consisting of the isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ which respect a fixed parametrization of the boundary circles, allowing them to be permuted among themselves.

We wish to construct a mapping class group, containing all mapping class groups $\mathcal{M}(\Sigma_{g,n})$. It seems impossible to construct such a group, but if one relaxes slightly our requirements then we could follow our previous method used for the genus zero case in [8].

The choice of the extra structure involved in the definitions below is important because the final result might depend on it. For instance, using the same planar punctured surface but different decompositions, one obtained in [9] two non-isomorphic braided Ptolemy-Thompson groups.

Definition 1.1 (The infinite genus surface \mathcal{S}_∞). *Let \mathcal{T} be the complete trivalent planar tree and \mathcal{S} be the surface obtained by taking the boundary of the 3-dimensional thickening of \mathcal{T} .*

*By grafting an **edge-loop** (i.e. the graph obtained by attaching a loop to a boundary vertex of an edge) at the midpoint of each edge of \mathcal{T} , one obtains the graph \mathcal{T}_∞ . The surface \mathcal{S}_∞ is the boundary of the 3-dimensional thickening of \mathcal{T}_∞ .*

The graph \mathcal{T} (respectively \mathcal{T}_∞) is embedded in \mathcal{S} (respectively \mathcal{S}_∞) as a cross-section of the fiber projection, as indicated on Fig. 1. Thus, \mathcal{S}_∞ is obtained by removing small disks from \mathcal{S} centered at midpoints of edges of \mathcal{T} and gluing back one holed tori $\Sigma_{1,1}$, called *wrists* which correspond to the thickening of edge-loops.

It is convenient to assume that \mathcal{T} is embedded in a horizontal plane, while the edge-loops are in vertical planes (see Fig. 1).

Definition 1.2 (Pants decomposition of \mathcal{S}_∞). *A **pants decomposition** of the surface \mathcal{S}_∞ is a maximal collection of distinct nontrivial simple closed curves on \mathcal{S}_∞ which are pairwise disjoint and non-isotopic. The complementary regions (which are 3-holed spheres)*

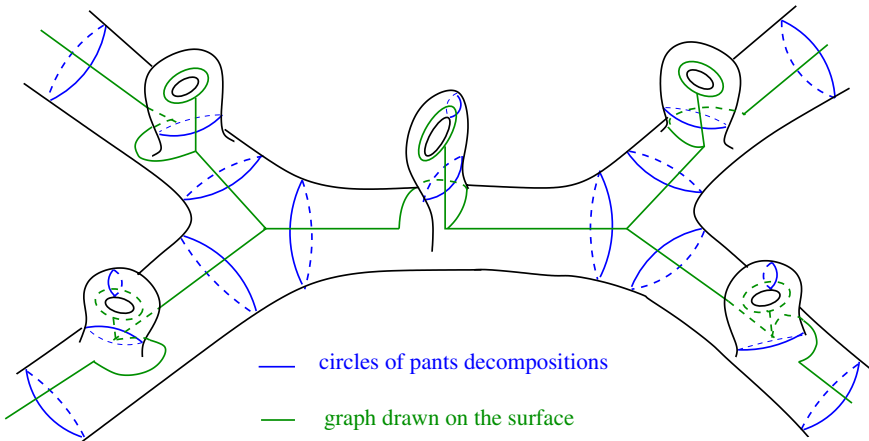


Fig. 1. The infinite genus surface \mathcal{S}_∞ with its canonical rigid structure

are called **pairs of pants**. By construction, \mathcal{S}_∞ is naturally equipped with a pants decomposition, which will be referred to below as the **canonical (pants) decomposition**, as shown in Fig. 1:

- the wrists are decomposed using a meridian circle and the boundary circle of $\Sigma_{1,1}$.
- there is one pair of pants for each edge, which has one boundary circle for attaching the wrists, and two circles to grip to the other type of pants. We call them **edge pants**.
- there is one pair of pants for each vertex of the tree, called **vertex pants**.

A pants decomposition is *asymptotically trivial* if outside a compact subsurface of \mathcal{S}_∞ , it coincides with the canonical pants decomposition.

Definition 1.3. 1. A connected subsurface Σ of \mathcal{S}_∞ is *admissible* if all its boundary circles are from vertex type pair of pants from the canonical decomposition and moreover, if one boundary circle from a vertex type pants is contained in Σ then the entire pants is contained in Σ . In particular, $\mathcal{S}_\infty - \Sigma$ has no compact components.

2. Let φ be a homeomorphism of \mathcal{S}_∞ . One says that φ is **asymptotically rigid** if the following conditions are fulfilled:

- There exists an admissible subsurface $\Sigma_{g,n} \subset \mathcal{S}_\infty$ such that $\varphi(\Sigma_{g,n})$ is also admissible.
- The complement $\mathcal{S}_\infty - \Sigma_{g,n}$ is a union of n infinite surfaces. Then the restriction $\varphi : \mathcal{S}_\infty - \Sigma_{g,n} \rightarrow \mathcal{S}_\infty - \varphi(\Sigma_{g,n})$ is **rigid**, meaning that it maps the pants decomposition into the pants decomposition and maps $\mathcal{T}_\infty \cap (\mathcal{S}_\infty - \Sigma_{g,n})$ onto $\mathcal{T}_\infty \cap (\mathcal{S}_\infty - \varphi(\Sigma_{g,n}))$. Such a surface $\Sigma_{g,n}$ is called a **support** for φ .

One denotes by $\mathcal{M} = \mathcal{M}(\mathcal{S}_\infty)$ the **group of asymptotically rigid homeomorphisms of \mathcal{S}_∞ up to isotopy** and call it the **asymptotic mapping class group of infinite genus**.

In the same way one defines the asymptotic mapping class group $\mathcal{M}(\mathcal{S})$, denoted by \mathcal{B} in [8].

Remark 1.1. In genus zero (i.e. for the surface \mathcal{S}) a homeomorphism between two complements of admissible subsurfaces which maps the restrictions of the tree \mathcal{T} one into

the other is rigid, thus preserves the isotopy class of the pants decomposition. This is not anymore true in higher genus: the Dehn twist along a longitude preserves the edge-loop graph but it is not rigid, as a homeomorphism of the holed torus.

Remark 1.2. Notice that, in general, rigid homeomorphisms φ do not have an invariant support i.e. an admissible $\Sigma_{g,n}$ such that $\varphi(\Sigma_{g,n}) = \Sigma_{g,n}$. Take for instance a homeomorphism which translates the wrists along a geodesic ray in \mathcal{T} .

Remark 1.3. Any admissible subsurface $\Sigma_{g,n} \subset \mathcal{S}_\infty$ has $n = g + 3$. Moreover \mathcal{S}_∞ is the ascending union $\cup_{g=1}^\infty \Sigma_{g,g+3}$. Instead of the wrist $\Sigma_{1,1}$ use a surface of higher genus $\Sigma_{g,1}$ and the same definitions as above. The admissible subsurfaces will be $\Sigma_{kg,k+3}$. The asymptotic mapping class group obtained this way is finitely generated by small changes in the proof below.

Remark 1.4. The surface \mathcal{S}_∞ contains infinitely many compact surfaces of type (g, n) with at least one boundary component. For any such compact subsurface $\Sigma_{g,n} \subset \mathcal{S}_\infty$, there is an obvious injective morphism $i_* : \mathcal{PM}(\Sigma_{g,n}) \hookrightarrow \mathcal{PM} \subset \mathcal{M}$. However, the morphism $i_* : \mathcal{M}(\Sigma_{g,n}) \hookrightarrow \mathcal{M}$ is not always defined. Indeed, it exists if and only if the n connected components of $\mathcal{S}_\infty \setminus \Sigma_{g,n}$ are homeomorphic to each other, by asymptotically rigid homeomorphisms.

In particular, for any admissible subsurface $\Sigma_{g,n}$ (hence $n = g + 3$), i_* extends to an injective morphism $i_* : \mathcal{M}(\Sigma_{g,n}) \hookrightarrow \mathcal{M}$ defined by rigid extension of homeomorphisms of $\Sigma_{g,n}$ to \mathcal{S}_∞ .

1.2.2. The group \mathcal{M} and the Thompson groups

Definition 1.4. 1. Let \mathcal{T} be the planar trivalent tree. A **partial tree automorphism** of \mathcal{T} is an isomorphism of graphs $\varphi : \mathcal{T} \setminus \tau_1 \rightarrow \mathcal{T} \setminus \tau_2$, where τ_1 and τ_2 are two finite trivalent subtrees of \mathcal{T} (each vertex except the leaves are 3-valent). A connected component of $\mathcal{T} \setminus \tau_1$ or $\mathcal{T} \setminus \tau_2$ is a **branch**, that is, a rooted planar binary tree whose vertices are 3-valent, except the root, which is 2-valent. Each vertex of a branch has two descendant edges, and given an orientation to the plane, one may distinguish between the left and the right descendant edges. A partial automorphism $\varphi : \mathcal{T} \setminus \tau_1 \rightarrow \mathcal{T} \setminus \tau_2$ is **planar** if it maps each branch of $\mathcal{T} \setminus \tau_1$ onto the corresponding branch of $\mathcal{T} \setminus \tau_2$ by respecting the left and right ordering of the edges.

2. Two planar partial automorphisms $\varphi : \mathcal{T} \setminus \tau_1 \rightarrow \mathcal{T} \setminus \tau_2$ and $\varphi' : \mathcal{T} \setminus \tau'_1 \rightarrow \mathcal{T} \setminus \tau'_2$ are equivalent, which is denoted $\varphi \sim \varphi'$, if and only if there exists a third $\varphi'' : \mathcal{T} \setminus \tau''_1 \rightarrow \mathcal{T} \setminus \tau''_2$ such that $\tau_1 \cup \tau'_1 \subset \tau''_1$, $\tau_2 \cup \tau'_2 \subset \tau''_2$ and $\varphi|_{\mathcal{T} \setminus \tau''_1} = \varphi'|_{\mathcal{T} \setminus \tau''_1} = \varphi''$. One denotes by $[\varphi]$ the equivalence class of φ .
3. If φ and φ' are planar partial automorphisms, one can find $\varphi_0 \sim \varphi$ and $\varphi'_0 \sim \varphi'$ such that the source of φ_0 and the target of φ'_0 coincide. The product $[\varphi] \cdot [\varphi'] = [\varphi_0 \circ \varphi'_0]$ is well defined, as is easy to check. The set of equivalence classes of such automorphisms endowed with the above internal law, is a group with neutral element the class of $id_{\mathcal{T}}$. This is the **Thompson group** V .

Remark 1.5. We warn the reader that our definition of the group V is different from the standard one (as given in [5]). Nevertheless, the present group V is isomorphic to the group denoted by the same letter in [5].

We introduce Thompson’s group T , the subgroup of V acting on the circle (see [11]), which will play a key role in the proofs.

Definition 1.5 (Ptolemy-Thompson’s group T). Choose a vertex v_0 of \mathcal{T} . Each $g \in V$ may be represented by a planar partial automorphism $\varphi : \mathcal{T} \setminus \tau_1 \rightarrow \mathcal{T} \setminus \tau_2$ such that v_0 belongs to $\tau_1 \cap \tau_2$. Let D_1 (respectively D_2) be a disk containing τ_1 (respectively τ_2), whose boundary circle S_1 (respectively S_2) passes through the leaves of τ_1 (respectively τ_2), giving to them a cycling ordering. If φ preserves this cycling ordering, which amounts to saying that the bijection from the set of leaves of τ_1 onto the set of leaves of τ_2 can be extended to an orientation preserving homeomorphism from S_1 onto S_2 , then any other φ' equivalent to φ also does, and one says that g itself is **circular**. The subset of circular elements of V is a subgroup, called the **Ptolemy-Thompson group T** .

Proposition 1.1. Set $P\mathcal{M}$ for the inductive limit of the pure mapping class groups of admissible subsurfaces of \mathcal{S}_∞ . We have then the following exact sequences:

$$1 \rightarrow P\mathcal{M} \rightarrow \mathcal{M} \rightarrow V \rightarrow 1.$$

Proof. Let φ be an asymptotically rigid homeomorphism of \mathcal{S}_∞ and $\Sigma_{g,n}$ a support for φ . Then it maps $\mathcal{T}_\infty \cap (\mathcal{S}_\infty - \Sigma_{g,n})$ onto $\mathcal{T}_\infty \cap (\mathcal{S}_\infty - \varphi(\Sigma_{g,n}))$, hence $\mathcal{T} \cap (\mathcal{S}_\infty - \Sigma_{g,n})$ onto $\mathcal{T} \cap (\mathcal{S}_\infty - \varphi(\Sigma_{g,n}))$ by forgetting the action on the edge-loops. This may be identified with a planar partial automorphism $\phi : \mathcal{T} \setminus \tau_1 \rightarrow \mathcal{T} \setminus \tau_2$. The map $[\varphi] \in \mathcal{M} \rightarrow [\phi] \in V$ is a group epimorphism. The kernel is the subgroup of isotopy classes of homeomorphisms inducing the identity outside a support, and hence is the direct limit of the pure mapping class groups. \square

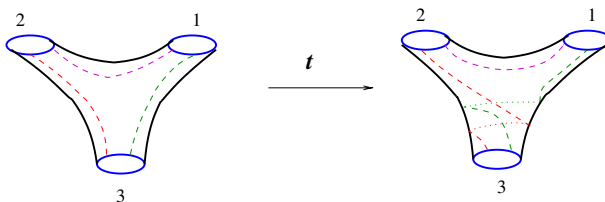
Remark 1.6. In [8] we prove the existence of a similar short exact sequence relating \mathcal{B} to V , which splits over the Ptolemy-Thompson group T . It is worth noticing that the present extension of V is not split over T .

2. The Proof of Theorem 1.1

2.1. Specific elements of \mathcal{M} . Recall that \mathcal{S}_∞ has a canonical pants decomposition, as shown in Fig. 1. We fix an admissible subsurface $A = \Sigma_{1,4}$ which contains a central wrist and an admissible $B = \Sigma_{0,3} \subset \Sigma_{1,4}$ which is not adjacent to the wrist.

Let us consider now the elements of \mathcal{M} described in the pictures below. Specifically:

- Let γ be a circle contained inside B and parallel to the boundary curve labeled 3. Let t be the right Dehn twist around γ . This means that, given an outward orientation to the surface, t maps an arc crossing γ transversely to an arc which turns right as it approaches γ . The dashed arcs (also called *seams*) on the left-hand side picture figure out the boundary of the visible side of B , i.e. the part of B which is located on the side of the surface that contains the graph and which is represented on Fig. 1. Their images by t are represented on the right-hand side picture,



- π is the braiding, acting as a braid in $\mathcal{M}(\Sigma_{0,3})$, with the support B . It rotates the circles 1 and 2 in the horizontal plane (spanned by the circles) counterclockwise.

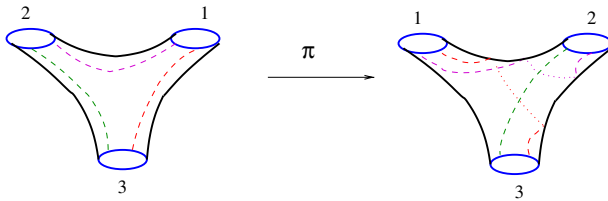
Assume that B is identified with the complex domain $\{|z| \leq 7, |z - 3| \geq 1, |z + 3| \geq 1\} \subset \mathbb{C}$. A specific homeomorphism in the mapping class of π is the composition of the counterclockwise rotation of 180 degrees around the origin—which exchanges the small boundary circles labeled 1 and 2 in the figure—with a map which rotates 180 degrees in the clockwise direction each boundary circle. The latter can be constructed as follows.

Let A be an annulus in the plane, which we suppose for simplicity to be $A = \{1 \leq |z| \leq 2\}$. The homeomorphism $D_{A,C}$ acts as the counterclockwise rotation of 180 degrees on the boundary circle C and keeps the other boundary component pointwise fixed:

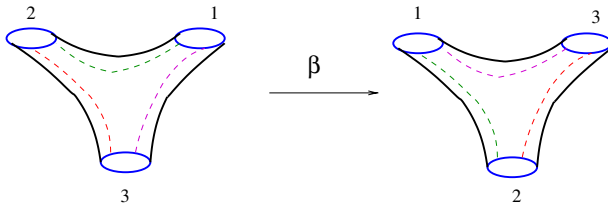
$$D_{A,C}(z) = \begin{cases} z \exp(\pi \sqrt{-1}(2 - |z|)), & \text{if } C = \{|z| = 1\} \\ z \exp(\pi \sqrt{-1}(|z| - 1)), & \text{otherwise} \end{cases}$$

The map we wanted is $D_{A_0, C_0}^{-1} D_{A_1, C_1}^{-1} D_{A_2, C_2}^{-1}$, where $A_0 = \{6 \leq |z| \leq 7\}$, $C_0 = \{|z| = 7\}$, $A_1 = \{1 \leq |z - 3| \leq 2\}$, $C_1 = \{|z - 3| = 1\}$, $A_2 = \{1 \leq |z + 3| \leq 2\}$, and $C_2 = \{|z + 3| = 1\}$.

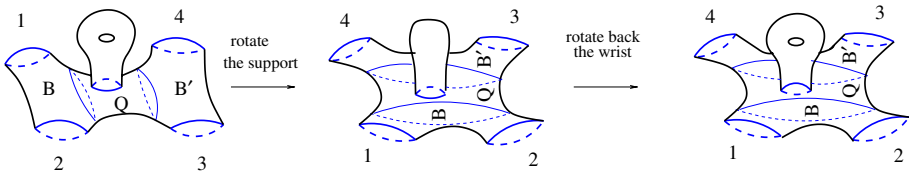
One has pictured also the images of the seams.



- β is the order 3 rotation in the vertical plane of the paper. It is the unique globally rigid mapping class which permutes counterclockwise and cyclically the three boundary circles of B . An invariant support for β is B .



- α is a twisted rotation of order 4 in the vertical plane which moves cyclically the labels of the boundary circles counterclockwise. Its support is a 4-holed torus $A = \Sigma_{1,4}$.

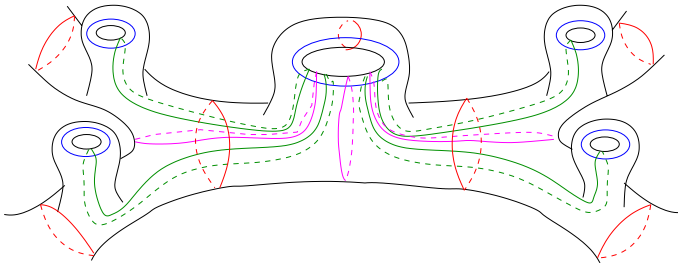


Let $\Sigma_{0,5}$ be the 5-holed sphere consisting of the union of B with the edge pants Q near B and the next vertex pants B' adjacent to Q . There are four boundary circles which are vertex type and one boundary circle which bounds a wrist. We perform first a rotation in \mathbb{R}^3 which preserves globally the pants decomposition and visible side, permutes counterclockwise and cyclically the four vertex type boundary circles of $\Sigma_{0,5}$ and rotates the edge type circle according to one fourth twist. This rotation changes the position of the wrist $\Sigma_{1,1}$ in \mathbb{R}^3 . We consider next the clockwise rotation of this wrist alone, of angle $\frac{\pi}{2}$ around the vertical axis that meets the edge type circle in its center. This rotation restores the initial wrist position. The composition of the two partial rotations above is a homeomorphism of $\Sigma_{1,4}$ that gives a well-defined element of \mathcal{M} .

- Let a_1, b_1 be the meridian and longitude on the basic wrist in A . We denote by t_{a_1} and t_{b_1} the Dehn twists along these curves, and further, t_0 states for the Dehn twist along the boundary circle of the wrist.

Remark 2.1. It is worth noting that we have three types of Dehn twists: those along separating curves (conjugate either to t (the boundary on the vertex pants) or with t_0 (the edge type pants)) and those along non-separating curves which are conjugate to the twist around such a curve on the wrist.

2.2. *Generators for PM .* Consider the following collection of simple curves drawn on S_∞ :



Their description follows:

1. Choose, for each wrist, a longitude b_i , which turns once along the wrist.
2. For each pair of wrists we choose a circle joining them as follows. For each wrist we have an arc going from the base point of its attaching circle to the longitude and back to the opposite point of the circle. Then join these two pairs of points by a pair of parallel arcs in the horizontal surface, asking that the arc which joins the two base points be a geodesic path in the tree \mathcal{T} . We call them *wrist-connecting loops*.
3. Further we associate a loop to each pair consisting of a wrist and a vertex of the tree \mathcal{T} . A vertex gives rise to a pair of pants in S_∞ . Two of the boundary components of these pants correspond to the directions to move away from the wrist. Thus we can define again an arc on the pair of pants which joins a point p of the third circle (closest to the wrist), on the visible side of S_∞ , to its opposite, on the hidden side, and separates the remaining two circles. Consider the loop resulting from gathering the following three kinds of arcs:

- (a) the arc on the wrist;
- (b) the arc on the pair of pants;
- (c) and a pair of parallel arcs which join them, asking that the arc which joins the point p to the base point of the wrist be a geodesic path of the tree \mathcal{T} .

We call them *vertex connecting loops*.

4. Consider the loops that come from the canonical pants decomposition of \mathcal{S} by doubling them. We call them the *horizontal pants decomposition loops*.

Lemma 2.1. *The set \mathcal{H} of Dehn twists along the meridians, the longitudes, the wrist connecting loops associated to edges, the vertex connecting loops and the horizontal pants decomposition loops generates PM .*

Proof. It suffices to consider the finite case of an admissible surface with boundary that contains g wrists and has $g+3$ boundary components. Then the lemma follows from [10], in which it is proved that the pure mapping class group of such a surface is generated by a set of Dehn twists $\mathcal{H}_{g,n}$ (with $n = g + 3$). It suffices to check that all the Dehn twists belonging to $\mathcal{H}_{g,n}$ also belong to the set \mathcal{H} . Referring to the notations of [10], there are four types of Dehn twists in $\mathcal{H}_{g,n}$: the α_i 's, the β_i 's, the γ_{ij} 's and the δ'_i 's. The α_i 's are associated to wrist-connecting or vertex connecting loops, the β_i 's are associated to longitudes b_i 's, the γ_{ij} 's are associated to wrist-connecting loops, except γ_{12} which is associated to a vertex-connecting loop, and finally, the δ_i 's are associated to the circles of the boundary of the surface, hence of the pants decomposition (after doubling them) of \mathcal{S}_∞ . Therefore all of them belong to \mathcal{H} .

Remark also that in ([10], Fig. 1) the 1-handles are cyclically ordered and arranged on one side and then followed by all boundary components of the surface. However, we can arbitrarily permute the position of holes and 1-handles in the picture and keep the same system of generators. \square

2.3. The action of \mathbb{T} on the generators of PM .

2.3.1. The groups \mathbb{T} and T^* . Consider the subgroup \mathbb{T} of \mathcal{M} generated by the elements α and β . We will prove that the set of conjugacy classes for the action of \mathbb{T} on \mathcal{H} is finite by considering the action of \mathbb{T} on some planar subsurface of \mathcal{S}_∞ .

The surface obtained by puncturing (respectively deleting disjoint small open disks from) \mathcal{S} at the midpoints of the edges is denoted by \mathcal{S}^* (and respectively \mathcal{S}^\bullet). The 2-dimensional thickening in \mathcal{S} of the embedded tree \mathcal{T} is an infinite planar surface, which will be called the *visible side* of \mathcal{S} , and will be denoted D . The intersection of D with \mathcal{S}^* and \mathcal{S}^\bullet is denoted D^* and D^\bullet , respectively.

The elements α and β as defined above (i.e. as specific homeomorphisms, not only as mapping classes) keep invariant both \mathcal{S}^\bullet and D^\bullet . If we crush the boundary circles to points then we obtain elements of $\mathcal{M}(D^*)$, and there is a well defined homomorphism $\mathbb{T} \rightarrow \mathcal{M}(D^*)$.

We studied in [9] the asymptotic mapping class group $\mathcal{M}(D^*)$ denoted by T^* there. Recall from [9] that:

Proposition 2.1. *The group T^* is generated by α and β .*

This implies that $\mathbb{T} \rightarrow T^*$ is an epimorphism. The relation between the asymptotic mapping class groups \mathbb{T} and T^* is made precise by the following:

Lemma 2.2. *We have an exact sequence*

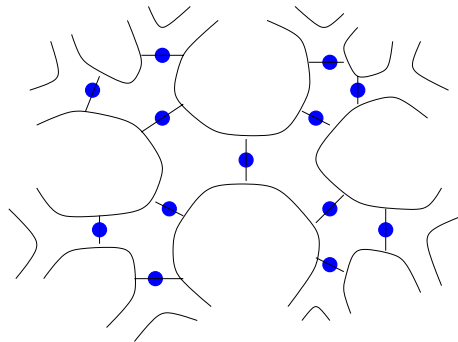
$$0 \rightarrow \mathbb{Z}^\infty \rightarrow \mathbb{T} \rightarrow T^* \rightarrow 1,$$

where the central factor \mathbb{Z}^∞ is the group of Dehn twists along attaching circles, normally generated by t_0 .

Proof. If an asymptotically rigid homeomorphism of S_∞ preserving D^\bullet is isotopically trivial once the circles are crushed to points, then it is isotopic to a finite product of Dehn twists along those circles. Therefore, the kernel of $\mathbb{T} \rightarrow T^*$ is contained in the subgroup denoted \mathbb{Z}^∞ . Observe that $\alpha^4 = t_0$ in \mathbb{T} , so that t_0 belongs to the kernel of $\mathbb{T} \rightarrow T^*$. Consequently, the kernel contains all the \mathbb{T} -conjugates of t_0 , hence \mathbb{Z}^∞ . \square

Thus if we understand the action of T^* on the isotopy classes of arcs embedded in D^* then we can easily recover the action of \mathbb{T} on homotopy classes of loops of S^\bullet , up to some twists along attaching circles.

2.3.2. The action of T^* on the isotopy classes of arcs of D^* . The planar model of D^* is the punctured thick tree obtained from the binary tree by thickening in the plane and puncturing along midpoints of edges. The traces on D^* of the loops coming from the pants decomposition of S are arcs transversal to the edges. Thus D^* has a canonical decomposition into punctured hexagons. Each hexagon has three punctured sides coming from the arcs above, that we call *separating side arcs*. Moreover there are also three sides which are part of the boundary of D^* that we will call *bounding side arcs*. Notice that hexagons correspond to vertices of the binary tree, while separating side arcs. Further β is the rotation of order 3 supported on the hexagon \bar{B} (image of the pants B) and α is the rotation of order 4 that is supported on the union of \bar{B} with an adjacent hexagon.



Lemma 2.3. *Let γ be an arc embedded in D^* that joins two punctures. Then there exists some element of T^* that sends γ in a prescribed arc joining the punctures 0 and 1.*

Proof. Recall from [9] that the infinite braid group associated to the punctures B_∞ is contained in T^* . Further, there exists always a braid mapping class (supported in a compact subsurface of D^*) sending the arc γ in the prescribed one. \square

Lemma 2.4. *The group T^* acts transitively on the set of separating side arcs.*

Proof. The group T^* contains $PSL_2(\mathbb{Z})$, the group of orientation-preserving automorphisms of the tree \mathcal{T} , generated by α^2 and β . It acts transitively on the set of edges of \mathcal{T} , hence on the set of separating sides of the hexagons of D^* . \square

An arc joining a puncture belonging to a hexagon H to a bounding side of H is called *standard* if it is entirely contained in H .

Lemma 2.5. *For any arc joining a puncture to a bounding side arc of a hexagon, there exists some element of T^* sending it into a standard arc joining the puncture 0 to one of the bounding sides of its hexagon.*

Proof. As above, $PSL(2, \mathbb{Z}) \subset T^*$ also acts transitively on the set of all bounding sides. Thus we can use an element of T^* to send one end of our arc on a bounding side of the hexagon \overline{B} . Next, one composes by a braid element in B_∞ that moves the other endpoint of the arc onto the puncture 0 and then makes the arc isotopic to a standard arc. \square

Let t_{a_1} and t_{b_1} denote the Dehn twists along a meridian a_1 and a longitude b_1 on the wrist.

Corollary 2.1. *The elements $\alpha, \beta, t, t_{a_1}, t_{b_1}, \pi$, a Dehn twist along one wrist connecting loop and a Dehn twist along a vertex connecting loop generate \mathcal{M} .*

3. The Rational Homology of \mathcal{M}

Theorem 3.1. *The rational homology of \mathcal{M} is isomorphic to the stable rational homology of the mapping class group: $H_*(\mathcal{M}, \mathbb{Q}) \cong H_*(P\mathcal{M}, \mathbb{Q})$.*

Proof. Recall first the theorem of stability, due to J. Harer (see [14]): Let R be a connected subsurface of genus g_R of a connected compact surface S with at least one boundary component. Then the map $H_n(P\mathcal{M}_R, \mathbb{Z}) \rightarrow H_n(P\mathcal{M}_S, \mathbb{Z})$ induced by the natural morphism $P\mathcal{M}_R \rightarrow P\mathcal{M}_S$ is an isomorphism if $g_R \geq 2n + 1$.

The pure mapping class group $P\mathcal{M}$ is the inductive limit of the pure mapping class groups $P\mathcal{M}_R$, for all the compact subsurfaces $R \subset S_\infty$. It follows that $H_n(P\mathcal{M}, \mathbb{Q}) = \lim H_n(P\mathcal{M}_R, \mathbb{Q}) = H_n(P\mathcal{M}_R, \mathbb{Q})$ for any compact subsurface $R \subset S_\infty$ of genus $\overline{R} \geq 2n + 1$. Therefore, the homology of $P\mathcal{M}$ is what is called the *stable homology of the mapping class group*. By Mumford’s conjecture proved in [18], $H^*(P\mathcal{M}, \mathbb{Q})$ is isomorphic to $\mathbb{Q}[\kappa_1, \dots, \kappa_i, \dots]$, where κ_i , the i^{th} Miller-Morita-Mumford class, has degree $2i$. Since $H^*(P\mathcal{M}, \mathbb{Q}) = \text{Hom}(H_*(P\mathcal{M}, \mathbb{Q}), \mathbb{Q})$, each $H_n(P\mathcal{M}, \mathbb{Q})$ is finite dimensional over \mathbb{Q} .

Write now the Lyndon-Hochschild-Serre spectral sequence in homology associated with

$$1 \rightarrow P\mathcal{M} \rightarrow \mathcal{M} \rightarrow V \rightarrow 1.$$

The second term is $E_{p,q}^2 = H_p(V, H_q(P\mathcal{M}, \mathbb{Q}))$. If we prove that V acts trivially on the finite dimensional \mathbb{Q} -vector space $H_q(P\mathcal{M}, \mathbb{Q})$, and invoke a theorem of K. Brown ([4]) saying that V is rationally acyclic, then the only possibly non-trivial term of the spectral sequence is $E_{0,n}^2 = H_n(P\mathcal{M}, \mathbb{Q})$, and the proof is done.

Thus it remains to justify that V acts trivially on the homology groups $H_q(P\mathcal{M}, \mathbb{Q})$, for any integer $q \geq 0$. This results from the fact that V is not linear, as we explain below.

Indeed, if $\dim_{\mathbb{Q}} H_q(P\mathcal{M}, \mathbb{Q}) = N$, then $\text{Aut}(H_q(P\mathcal{M}, \mathbb{Q})) \cong GL(N, \mathbb{Q})$. So, let $\rho : V \rightarrow GL(N, \mathbb{Q})$ be the representation resulting from the action of V on $H_q(P\mathcal{M}, \mathbb{Q})$. Since V is a simple group, ρ is either trivial or injective. Suppose it is injective, so that V is isomorphic to a finitely generated subgroup of $SL(N, \mathbb{Q})$. Now each finitely generated subgroup of $SL(N, K)$ for any field K is residually finite. But V is not residually finite, since its unique normal subgroup of finite index is V itself. Therefore, ρ is trivial. \square

Proposition 3.1. *The free universal mapping class group \mathcal{M} is perfect, and $H_2(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}$. The generator of $H^2(\mathcal{M}, \mathbb{Z}) \cong \mathbb{Z}$ is called the first universal Chern class of \mathcal{M} , and is denoted $c_1(\mathcal{M})$.*

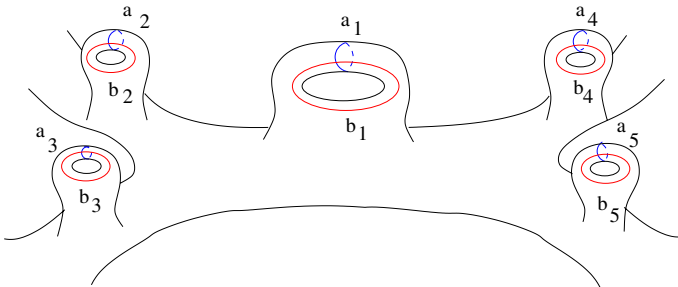
Proof. Recall that the pure mapping class group of a surface of type (g, n) is perfect if $g \geq 3$. Consequently, $P\mathcal{M}$ is perfect. Since V is perfect, \mathcal{M} is perfect as well.

The above spectral sequence may be written with integral coefficients. One obtains $E_{2,0}^2 = H_2(V, \mathbb{Z}) = 0$ (see [4]), $E_{1,1}^2 = H_1(V, H_1(P\mathcal{M}, \mathbb{Z})) = 0$ since $P\mathcal{M}$ is perfect, and $E_{0,2}^2 = H_0(V, H_2(P\mathcal{M}, \mathbb{Z}))$. By Harer’s theorem ([13]) and stability ([14]), $H_2(P\mathcal{M}, \mathbb{Z}) \cong \mathbb{Z}$. The action of V on $\mathbb{Z} = H_2(P\mathcal{M}, \mathbb{Z})$ must be trivial, since V is simple, and it follows that $E_{0,2}^2 = \mathbb{Z}$. Thus, the only non-trivial E^∞ term is $E_{0,2}^\infty = E_{0,2}^2 = \mathbb{Z}$, and this implies $H_2(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}$. \square

4. The Symplectic Representation in Infinite Genus

4.1. Hilbert spaces and symplectic structure associated to \mathcal{S}_∞ . There is a natural intersection form $\omega : H_1(\mathcal{S}_\infty, \mathbb{R}) \times H_1(\mathcal{S}_\infty, \mathbb{R}) \rightarrow \mathbb{R}$ on the homology of the infinite surface, but this is degenerate because it is obtained as a limit of intersection forms on surfaces with boundary. The \mathcal{M} -module $H_1(\mathcal{S}_\infty, \mathbb{R})$ is the direct sum of two submodules: $H_1(\mathcal{S}_\infty, \mathbb{R}) = H_1(\mathcal{S}_\infty, \mathbb{R})_s \oplus H_1(\mathcal{S}_\infty, \mathbb{R})_{ns}$, where $H_1(\mathcal{S}_\infty, \mathbb{R})_s$ is generated by the homology classes of separating circles of \mathcal{S}_∞ , while $H_1(\mathcal{S}_\infty, \mathbb{R})_{ns}$ is generated by the homology classes of non-separating circles of \mathcal{S}_∞ . The kernel $\ker \omega$ of ω is $H_1(\mathcal{S}_\infty, \mathbb{R})_s$, and the restriction of ω to $H_1(\mathcal{S}_\infty, \mathbb{R})_{ns}$ is a symplectic form.

For each wrist torus occurring in the construction of \mathcal{S}_∞ (see Definition 1.1), we consider the meridian a_k and the longitude b_k , with intersection number $\omega(a_k, b_k) = 1$. Note that both collections $\{a_k, k \in \mathbb{N}\}$ and $\{b_k, k \in \mathbb{N}\}$ are invariant by the mapping class group \mathbb{T} , since the generators α and β rigidly map a wrist onto a wrist. Moreover, these collections are almost invariant by \mathcal{M} , meaning that for each $g \in \mathcal{M}$, $g(\{a_k, k \in \mathbb{N}\})$ (respectively $g(\{b_k, k \in \mathbb{N}\})$) coincides up to isotopy with $\{a_k, k \in \mathbb{N}\}$ (respectively $\{b_k, k \in \mathbb{N}\}$) for all but finitely many elements.



The classes $\{a_k, b_k, k \in \mathbb{N}\}$ form a symplectic basis for $H_1(\mathcal{S}_\infty, \mathbb{R})_{ns}$. Each element of \mathcal{M} acts on $H_1(\mathcal{S}_\infty, \mathbb{R})_{ns}$ by preserving the intersection form ω . In particular, there is a representation

$$\rho : P\mathcal{M} \rightarrow \text{Sp}(2\infty, \mathbb{R}),$$

where $\text{Sp}(2\infty, \mathbb{R})$ is the inductive limit of the symplectic groups $\text{Sp}(2k, \mathbb{R})$, with respect to the natural inclusions $\text{Sp}(2k, \mathbb{R}) \subset \text{Sp}(2(k + 1), \mathbb{R})$. Note, though, that if $g \in \mathcal{M}$ is not in $P\mathcal{M}$, it is not represented into $\text{Sp}(2\infty, \mathbb{R})$, but into a larger symplectic group, that we are defining below.

One completes $H_1(\mathcal{S}_\infty, \mathbb{R})_{ns}$ as a real Hilbert space for which this basis orthonormal. Let \mathcal{H}_r be this Hilbert space, and (\cdot, \cdot) denote its scalar product.

Let J be the almost-complex structure induced by ω , i.e. the linear operator defined by $\omega(v, w) = (v, Jw)$ for all v, w in \mathcal{H}_r . We have $J^2 = -\mathbf{1}$. Each linear operator on \mathcal{H}_r decomposes into a J -linear part T_1 and a J -antilinear part T_2 , $T = T_1 + T_2$, where $T_1 = \frac{T - J TJ}{2}$ and $T_2 = \frac{T + J TJ}{2}$.

Recall that ([27]) the restricted symplectic group $\text{Sp}_{\text{res}}(\mathcal{H}_r)$ is defined as the group of symplectic (i.e. ω preserving) bounded invertible operators T whose J -antilinear part T_2 is a Hilbert-Schmidt operator. An operator T is called Hilbert-Schmidt if $\|T\|_{HS}^2 := \sum_i \|T(e_i)\|^2$ is finite, where $(e_i)_{i \in \mathbb{N}}$ is an orthonormal Hilbert basis.

Theorem 4.1. *The symplectic representation of the mapping class group $P\mathcal{M}$ extends to a representation $\hat{\rho} : \mathcal{M} \rightarrow \text{Sp}_{\text{res}}(\mathcal{H}_r)$ of \mathcal{M} into the restricted symplectic group.*

4.2. Proof of Theorem 4.1. Instead of a direct proof we will introduce the complexification of \mathcal{H}_r to be used also in the next section. Let $\mathcal{H} = \mathcal{H}_r \otimes_{\mathbb{R}} \mathbb{C}$. Extend ω and J by \mathbb{C} -linearity, and (\cdot, \cdot) by sesquilinearity, and denote by $\omega_{\mathbb{C}}, J_{\mathbb{C}}$ and $(\cdot, \cdot)_{\mathbb{C}}$ the extensions. Thus, $(\mathcal{H}, (\cdot, \cdot)_{\mathbb{C}})$ is a complex Hilbert space. Let B be the indefinite hermitian form $B(v, w) = \frac{1}{\sqrt{-1}} \omega_{\mathbb{C}}(v, \bar{w})$, for all v, w in \mathcal{H} , where \bar{w} is the complex-conjugate of w .

Let $\text{Aut}(\mathcal{H}, \omega_{\mathbb{C}}, B)$ be the group of bounded invertible operators of \mathcal{H} which preserve $\omega_{\mathbb{C}}$ and B . The morphism $\phi : \text{Sp}(\mathcal{H}_r) \rightarrow \text{Aut}(\mathcal{H}, \omega_{\mathbb{C}}, B)$, given by $\phi(T) = T \otimes \mathbf{1}_{\mathbb{C}}$ is an isomorphism (see [22]), since any $T \in \text{Aut}(\mathcal{H}, \omega_{\mathbb{C}}, B)$ commutes with the complex conjugation and hence stabilizes \mathcal{H}_r .

Since $J_{\mathbb{C}}^2 = -id$, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_{\pm} = \ker(J \pm \sqrt{-1} \cdot \mathbf{1})$. Moreover, the direct sum is orthogonal. The complex conjugation interchanges \mathcal{H}_+ and \mathcal{H}_- . Let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis of \mathcal{H}_+ and $(f_k = \bar{e}_k)_{k \in \mathbb{N}}$ the conjugate basis of \mathcal{H}_- .

According to ([22], 6.2) a symplectic operator T belongs to $\text{Sp}_{\text{res}}(\mathcal{H}_r)$ if and only if the decomposition of $\phi(T)$ relative to the direct sum $\mathcal{H}_+ \oplus \mathcal{H}_-$ in the basis $(e_k)_{k \in \mathbb{N}} \cup (f_k = \bar{e}_k)_{k \in \mathbb{N}}$ reads $\begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix}$, where

1. ${}^t \bar{\Phi} \Phi - {}^t \bar{\Psi} \bar{\Psi} = 1$ and ${}^t \bar{\Phi} \Psi = {}^t \bar{\Psi} \bar{\Phi}$, where ${}^t T$ denotes the adjoint of T with respect to $(\cdot, \cdot)_{\mathbb{C}}$.
2. $\Psi : \mathcal{H}_- \rightarrow \mathcal{H}_+$ is a Hilbert-Schmidt operator.

We will apply this criterion for the action of \mathcal{M} .

Set

$$e_k = \frac{1}{\sqrt{2}}(a_k - \sqrt{-1} \cdot b_k), f_k = \frac{1}{\sqrt{2}}(a_k + \sqrt{-1} \cdot b_k),$$

Then $(e_k)_{k \in \mathbb{N}}$ is an orthonormal basis of \mathcal{H}_+ and $(f_k)_{k \in \mathbb{N}}$ is the conjugate orthonormal basis of \mathcal{H}_- . Moreover, $\omega_{\mathbb{C}}(e_k, e_l) = \omega_{\mathbb{C}}(f_k, f_l) = 0$, $\omega_{\mathbb{C}}(e_k, f_l) = \sqrt{-1}\delta_{kl}$, and $B(e_k, e_l) = -B(f_k, f_l) = \delta_{kl}$, $B(e_k, f_l) = 0$ for all k, l .

Consider now the action $\hat{\rho}(g)$ of $g \in \mathcal{M}$ on the \mathcal{M} -invariant subspace \mathcal{H}_r . We must check that $\Psi(\phi(\hat{\rho}(g)))$ is a Hilbert-Schmidt operator. In fact, it is a finite rank operator. Let $\Sigma_{h,n}$ be an admissible surface for g , that is g is the mapping class of a homeomorphism G so that $G : \mathcal{S}_{\infty} \setminus \Sigma_{h,n} \rightarrow \mathcal{S}_{\infty} \setminus \varphi(\Sigma_{h,n})$ is rigid. Any wrist torus $T_k = \Sigma_{1,1}$ of $\mathcal{S}_{\infty} \setminus \Sigma_{g,n}$ is rigidly mapped by G onto another corresponding wrist torus $T_{\sigma(k)}$, for some infinite permutation σ . Therefore, for any such T_k , the associated matrices are such that

$$\phi(\hat{\rho}(g))(e_k) = e_{\sigma(k)}, \quad \phi(\hat{\rho}(g))(f_k) = f_{\sigma(k)}.$$

In particular, for all but finitely many f_k (i.e. excepting those corresponding to tori $T_k \subset \Sigma_{h,n}$) we have $\phi(\hat{\rho}(g))(f_k) \in \mathcal{H}_-$. Now $\Psi(\phi(\hat{\rho}(g))) : \mathcal{H}_- \rightarrow \mathcal{H}_+$ corresponds to the components of $\phi(\hat{\rho}(g))(f_k)$ in \mathcal{H}_+ . This means that $\Psi(\phi(\hat{\rho}(g)))$ has finite rank, and in particular, it is Hilbert-Schmidt. This proves that $\hat{\rho}(g) \in \text{Sp}_{\text{res}}(\mathcal{H}_r)$, as claimed.

5. The Universal First Chern Class

5.1. The Pressley-Segal extension. Let \mathcal{H} be a polarized separable Hilbert space as above, that is, the orthogonal sum of two separable Hilbert spaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The restricted linear group $\text{GL}_{\text{res}}(\mathcal{H})$ (see [25, 27]) is the Banach-Lie group of operators in $\text{GL}(\mathcal{H})$ whose block decomposition $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is such that b and c are Hilbert-Schmidt operators. Moreover, the invertibility of A implies that a is Fredholm in \mathcal{H}_+ , and has an index $\text{ind}(a) \in \mathbb{Z}$. This gives a homomorphism $\text{ind} : \text{GL}_{\text{res}}(\mathcal{H}) \rightarrow \mathbb{Z}$, that induces an isomorphism $\pi_0(\text{GL}_{\text{res}}(\mathcal{H})) \cong \mathbb{Z}$. Denote by $\text{GL}_{\text{res}}^0(\mathcal{H})$ the connected component of the identity. Then $\text{GL}_{\text{res}}^0(\mathcal{H})$ is a perfect group (cf. [6], §5.4).

Proposition 5.1. *The restricted symplectic group $\text{Sp}_{\text{res}}(\mathcal{H}_r)$ embeds into the restricted linear group $\text{GL}_{\text{res}}^0(\mathcal{H})$. It is given the induced topology.*

Proof. Let $\begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix}$ be in $\text{Sp}_{\text{res}}(\mathcal{H}_r)$. Since Ψ is a Hilbert-Schmidt operator, $K = {}^t\Psi\bar{\Psi}$ is trace-class, hence compact. Then ${}^t\bar{\Phi}\Phi = 1 + K \geq 1$, hence ${}^t\bar{\Phi}\Phi \geq 1$ is injective, and the Fredholm alternative implies it is invertible. In particular, Φ itself is invertible, and has null index. \square

Pressley-Segal’s extension of the restricted linear group. Let $\mathcal{L}_1(\mathcal{H}_+)$ denote the ideal of trace-class operators of \mathcal{H}_+ . It is a Banach algebra for the norm $\|b\|_1 = \text{Tr}(\sqrt{b^*b})$, where Tr denotes the trace form. We say that an invertible operator q of \mathcal{H}_+ has a determinant if $q - id_{\mathcal{H}_+} = Q$ is trace-class. Its determinant is the complex number $\det(q) = \sum_{i=0}^{+\infty} \text{Tr}(\wedge^i Q)$, where $\wedge^i Q$ is the operator of the Hilbert space $\wedge^i \mathcal{H}_+$ induced by Q (cf. [28]).

Denote by \mathfrak{T} the subgroup of $\text{GL}(\mathcal{H}_+)$ consisting of operators which have a determinant, and by \mathfrak{T}_1 the kernel of the morphism $\det : \mathfrak{T} \rightarrow \mathbb{C}^*$.

Let \mathfrak{E} be the subgroup of $\text{GL}_{\text{res}}(\mathcal{H}) \times \text{GL}(\mathcal{H}_+)$ consisting of pairs (A, q) such that $a - q$ is trace-class. Then $\text{ind}(a) = \text{ind}(q + (a - q)) = \text{ind}(q) = 0$, so that A belongs to $\text{GL}_{\text{res}}^0(\mathcal{H})$. There is a short exact sequence

$$1 \rightarrow \mathfrak{T} \xrightarrow{i} \mathfrak{E} \xrightarrow{p} \text{GL}_{\text{res}}^0(\mathcal{H}) \rightarrow 1$$

called the *Pressley-Segal extension*. Here $p(A, q) = A$, and $i(q) = (\mathbf{1}_{\mathcal{H}}, q)$. It induces the central extension

$$1 \rightarrow \frac{\mathfrak{T}}{\mathfrak{T}_1} \cong \mathbb{C}^* \longrightarrow \frac{\mathfrak{E}}{\mathfrak{T}_1} \longrightarrow \mathrm{GL}_{\mathrm{res}}^0(\mathcal{H}) \rightarrow 1.$$

The corresponding cohomology class in $H^2(\mathrm{GL}_{\mathrm{res}}^0(\mathcal{H}), \mathbb{C}^*)$ is denoted by PS , and called the *Pressley-Segal class* of the restricted linear group.

The Pressley-Segal class and the universal first Chern class. For G a Banach-Lie group, set $Ext(G, \mathbb{C}^*)$ for the set of equivalence classes of central extensions of G by \mathbb{C}^* , which are locally trivial fibrations (see [26]). Note that $Ext(G, \mathbb{C}^*)$ must not be confused with the group of continuous cohomology $H_{\mathrm{cont}}^2(G, \mathbb{C}^*)$, since the latter only classifies the topologically split central extensions. One introduces two maps $H^2(G, \mathbb{C}^*) \xleftarrow{\delta} Ext(G, \mathbb{C}^*) \xrightarrow{\tau} H_{\mathrm{top}}^2(G, \mathbb{Z})$. The map δ associates to an extension E in $Ext(G, \mathbb{C}^*)$ its cohomology class in the Eilenberg-McLane cohomology of G . The map τ is the composition of $Ext(G, \mathbb{C}^*) \rightarrow [G, B\mathbb{C}^* = K(\mathbb{Z}, 2)]$, which sends an extension to the homotopy class of its classifying map $G \rightarrow B\mathbb{C}^*$, with the isomorphism $[G, K(\mathbb{Z}, 2)] \cong H_{\mathrm{top}}^2(G, \mathbb{Z})$.

Let us apply this formalism to $G = \mathrm{GL}_{\mathrm{res}}^0(\mathcal{H})$ and the central Pressley-Segal extension, viewed as an element $\mathcal{P}S \in Ext(\mathrm{GL}_{\mathrm{res}}^0(\mathcal{H}), \mathbb{C}^*)$. Then $\delta(\mathcal{P}S) = PS$. The point is that $\mathrm{GL}_{\mathrm{res}}^0(\mathcal{H})$ is a homotopic model of the classifying space BU . In fact \mathfrak{E} is contractible (see [25], 6.6.2) and \mathfrak{T} is homotopically equivalent to U (see [23]), hence the claim. It follows that the fibration $\mathcal{P}S$ corresponds to the universal first Chern class, that is, $\tau(\mathcal{P}S) = c_1(BU) \in H^2(BU, \mathbb{Z})$.

5.2. Cocycles on $\mathrm{GL}_{\mathrm{res}}^0(\mathcal{H})$, $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r)$ and \mathcal{M} .

Lemma 5.1. *The class $i^*(PS)$ in $H^2(\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r), \mathbb{C}^*)$ is represented by the cocycle*

$$C_1(g, g') = \det(\Phi(g)\Phi(g')\Phi(gg')^{-1}).$$

Proof. Let \mathcal{V} be the open subset of $\mathrm{GL}_{\mathrm{res}}^0(\mathcal{H})$ consisting of operators A such that a is invertible. It is known that the central Pressley-Segal extensions splits over \mathcal{V} , since it has the section $\sigma : \mathcal{V} \rightarrow \mathfrak{E}$, $A \mapsto (A, a)$. In particular, there is a local cocycle for PS given by the formula ([25], 6.6.4): $C(A, A') = \det(1 + aa'a''^{-1})$, for $A, A' \in \mathcal{V}$, where a'' is the first block of $A \cdot A'$. It suffices now to observe that $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r)$ embeds into \mathcal{V} . □

In order to prove Proposition 5.2 below, we need the following result that contrasts sharply with the finite dimensional case:

Lemma 5.2. *The restricted symplectic group $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r)$ is contractible.*

Proof. Denote by Z the set of symmetric Hilbert-Schmidt operators $\mathcal{H}_- \rightarrow \mathcal{H}_+$ with norm < 1 . Clearly, Z is a contractible subspace of the Banach space of Hilbert-Schmidt operators. The group $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r)$ acts transitively and continuously (see [22], p. 177) on Z by means of

$$g(S) = (\Phi(g)S + \Psi(g))(\overline{\Psi(g)}S + \overline{\Phi(g)})^{-1} \in Z, \text{ for } g \in \mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r), S \in Z.$$

The stabilizer of $S = 0$ is the group of matrices $\begin{pmatrix} \Phi & 0 \\ 0 & \overline{\Phi} \end{pmatrix}$ such that Φ is unitary in \mathcal{H}_+ . Thus, it is isomorphic to $\mathcal{U}(\mathcal{H}_+)$. By a result of Kuiper ([17]), $\mathcal{U}(\mathcal{H}_+)$ is contractible. The claim is now a consequence of the contractibility of $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r)/\mathcal{U}(\mathcal{H}_+) \cong Z$. \square

Proposition 5.2. *For each integer $n \in \mathbb{Z}$, there is a well-defined continuous cocycle C_n defined on $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r)$, with values in \mathbb{C}^* , such that*

$$C_n(g, g') = \det \left((\Phi(g)\Phi(g')\Phi(gg')^{-1})^{\frac{1}{n}} \right).$$

Moreover, $\frac{C_n}{|C_n|}$ may be lifted to a real cocycle $\widehat{\zeta}_n : \mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r) \times \mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r) \rightarrow \mathbb{R}$ such that

$$\frac{C_n(g, g')}{|C_n(g, g')|} = e^{2i\pi \widehat{\zeta}_n(g, g')}, \text{ for all } g, g' \in \mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r).$$

The restriction ζ_1 of $\widehat{\zeta}_1$ to $\mathrm{Sp}(2\infty, \mathbb{R})$ defines an integral cohomology class $[\zeta_1] \in H^2(\mathrm{Sp}(2\infty, \mathbb{R}), \mathbb{Z})$.

Proof. In fact, $\Phi(g)^{-1}\Phi(g')^{-1}\Phi(gg') = 1 + (\Phi(g')^{-1}\Phi(g)^{-1}\Psi(g)\overline{\Psi(g')})$. But, according to ([22], p. 168) we have

$$\|\Phi^{-1}(g)\Psi(g)\| < 1 \text{ and } \|\overline{\Psi(g')}\Phi(g')^{-1}\| < 1.$$

Thus, there is a non-ambiguous definition of $(\Phi^{-1}(g)\Phi(gg')\Phi^{-1}(g'))^{\frac{1}{n}}$ given by an absolutely convergent series.

The existence of ζ_n is now an immediate consequence of the preceding lemma.

The map $\ell : g \in \mathrm{Sp}(2\infty, \mathbb{R}) \mapsto \ell(g) = \frac{\det(\Phi(g))}{|\det(\Phi(g))|}$ is well-defined, so that the cocycle

$$(g, g') \in \mathrm{Sp}(2\infty, \mathbb{R}) \times \mathrm{Sp}(2\infty, \mathbb{R}) \mapsto e^{2i\pi \zeta_1(g, g')}$$

is the coboundary of ℓ . This proves that the cohomology class of ζ_1 restricted to $\mathrm{Sp}(2\infty, \mathbb{R})$ is integral. \square

Remark 5.1. 1. The restrictions of the real cocycles ζ_n on the finite dimensional Lie group $\mathrm{Sp}(2g, \mathbb{R})$ are those constructed by Dupont-Guichardet-Wigner (see [12]). In fact, the authors of [12] proved that the cohomology class of the restriction of ζ_1 to $\mathrm{Sp}(2g, \mathbb{R})$ is integral, and is the image in $H^2(\mathrm{Sp}(2g, \mathbb{R}), \mathbb{R})$ of the generator of $H^2_{\mathrm{bor}}(\mathrm{Sp}(2g, \mathbb{R}), \mathbb{Z}) = \mathbb{Z}$, the second group of borelian cohomology of $\mathrm{Sp}(2g, \mathbb{R})$. They prove also that it is the image of the first Chern class $c_1(BU(g, \mathbb{C}))$ by the composition of maps

$$\begin{aligned} H^2(BU(g, \mathbb{C}), \mathbb{Z}) &\approx H^2(B\mathrm{Sp}(2g, \mathbb{R}), \mathbb{Z}) \rightarrow H^2(B\mathrm{Sp}(2g, \mathbb{R})^\delta, \mathbb{Z}) \\ &\approx H^2(\mathrm{Sp}(2g, \mathbb{R}), \mathbb{Z}) \rightarrow H^2(\mathrm{Sp}(2g, \mathbb{R}), \mathbb{R}), \end{aligned}$$

where $B\mathrm{Sp}(2g, \mathbb{R})^\delta$ is the classifying space of $\mathrm{Sp}(2g, \mathbb{R})$ as a discrete group.

2. The remark above implies that the map

$$H^*(BU, \mathbb{Z}) \approx H^*(B\mathrm{Sp}(2\infty, \mathbb{R}), \mathbb{Z}) \rightarrow H^*(\mathrm{Sp}(2\infty, \mathbb{R}), \mathbb{Z})$$

sends the first universal Chern class $c_1(BU)$ onto $[\zeta_1] \in H^2(\mathrm{Sp}(2\infty, \mathbb{R}), \mathbb{Z})$. Further, the symplectic representation $\rho : P\mathcal{M} \rightarrow \mathrm{Sp}(2\infty, \mathbb{R})$ maps $[\zeta_1]$ onto the generator $c_1(P\mathcal{M})$ of $H^2(P\mathcal{M}, \mathbb{Z})$.

3. According to ([22], Theorem 6.2.3), the Berezin-Segal-Shale-Weil cocycle is the complex conjugate of the cocycle $C_{-\frac{1}{2}}$.

Theorem 5.1. *Let $[\hat{\zeta}_1] \in H^2(\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r), \mathbb{R})$ be the cohomology class of $\hat{\zeta}_1$. The pull-back of $[\hat{\zeta}_1]$ in $H^2(\mathcal{M}, \mathbb{R})$ by the representation $\hat{\rho}$ of Theorem 1.3 is integral, and is the natural image of the generator $c_1(\mathcal{M})$ of $H^2(\mathcal{M}, \mathbb{Z})$ in $H^2(\mathcal{M}, \mathbb{R})$.*

Proof. Let $\iota : P\mathcal{M} \rightarrow \mathcal{M}$ and $j : \mathrm{Sp}(2\infty, \mathbb{R}) \rightarrow \mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r)$ be the natural embeddings. Plainly, $j \circ \rho = \hat{\rho} \circ \iota$. Since $j^* : H^2(\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_r), \mathbb{R}) \rightarrow H^2(\mathrm{Sp}(2\infty, \mathbb{R}), \mathbb{R})$ maps $[\hat{\zeta}_1]$ onto $[\zeta_1]$, one has $\iota^*(\hat{\rho}^*[\hat{\zeta}_1]) = \rho^*[\zeta_1]$. Let us denote by $\bar{c}_1(P\mathcal{M})$ (respectively $\bar{c}_1(\mathcal{M})$) the image of $c_1(P\mathcal{M})$ (respectively $c_1(\mathcal{M})$) in $H^2(P\mathcal{M}, \mathbb{R})$ (respectively $H^2(\mathcal{M}, \mathbb{R})$). According to Remark 5.1, 2., $\rho^*[\zeta_1] = \bar{c}_1(P\mathcal{M})$. By Proposition 3.1, $\iota^*(\bar{c}_1(P\mathcal{M})) = \bar{c}_1(\mathcal{M})$, hence $\hat{\rho}^*[\hat{\zeta}_1] = \bar{c}_1(\mathcal{M})$. \square

Acknowledgements. The authors are indebted to Vlad Sergiescu for enlightening discussions and particularly for suggesting the existence of a connection between the first universal Chern class of \mathcal{M} and the Pressley-Segal class. They are thankful to the referees for suggestions improving the exposition.

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Communicated by Y. Kawahigashi