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Surface cubications mod flips

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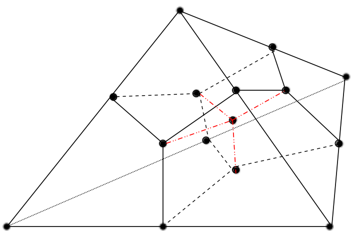
Abstract. Let Σ be a compact surface. We prove that the set of marked surface cubications modulo flips, up to isotopy, is in one-to-one correspondence with $\mathbb{Z}/2\mathbb{Z} \oplus H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$.

1. Introduction and statements

Cubical complexes and marked cubications. A *cubical complex* is a finite dimensional complex C consisting of Euclidean cubes, such that the intersection of two of its cubes is a finite union of cubes from C, once a cube is in C then all its faces belong to C and each point has a neighborhood intersecting only finitely many cubes of C. A *cubication* of a topological manifold is a cubical complex that is homeomorphic to the manifold. If the manifold is a PL manifold then one requires that the cubication be combinatorial and compatible with the PL structure. Our definition of cubication is slightly more general than the usual one, because we do not require that the intersection of two cubes consists of a single cube but only a *finite union* of cubes.

The study of simplicial complexes and manifold triangulations lay at the core of combinatorial topology. Cubical complexes and cubications might offer an alternative approach since, despite their similarities, they present some new features.

Any triangulated manifold admits a cubication, since we can decompose an n-dimensional simplex Δ^n into $n+1$ cubes of dimension n. For $k = 1$ to n we adjoin, inductively, the barycenter of each k-simplex in Δ^n and join it with the barycenters of its faces. This way we obtain the one-skeleton of a cubical complex, as shown in the figure below for $n = 3$.



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Thus, roughly speaking, working with simplicial complexes is equivalent to working with cubical complexes, from topological viewpoint. However, we will show that cubications encode additional topological information.

It will be more convenient in the sequel to work with marked cubications instead of cubications. A *marked* cubication of the manifold M consists of a couple (C, φ), where C is a cubication and $\varphi : |C| \longrightarrow M$ is a PL homeomorphism (called the *marking*) of its subjacent space $|C|$ onto M. The marked cubications (C, φ) and (C', φ') are said to be *isotopic* if there exists a combinatorial isomorphism $j : C \to C'$ between the two cubical complexes and a PL homeomorphism Φ of M such that $\Phi \circ \varphi = \varphi' \circ J$, where $J : |C| \to |C'|$ is the PL homeomorphism induced by j, and both J and Φ are isotopic to identity. The isotopy class of the image by φ of the skeleton of C in M determines the isotopy class of the marked cubulation (C, φ). Thus, marked cubications underlying a given cubication C are acted upon transitively by the mapping class group of M.

Bi-stellar moves. We will consider below PL manifolds, i.e. topological manifolds endowed with triangulations (called combinatorial) for which the link of each vertex is PL homeomorphic to the boundary of the simplex. Recall that two simplicial complexes are PL homeomorphic if they admit combinatorially isomorphic subdivisions. There exist topological manifolds which have several PL structures and it is still unknown whether all topological manifolds have triangulations (i.e. whether they are homeomorphic to simplicial complexes), without requiring them to be combinatorial.

It is not easy to decide whether two given triangulations define or not the same PL structure. One difficulty is that one has to work with arbitrary subdivisions and there are infinitely many distinct combinatorial types of such. In the early 1960s one looked upon a more convenient set of transformations permitting to connect PL equivalent triangulations of a given manifold. The simplest proposal was the so-called *bi-stellar* moves which are defined for n-dimensional complexes, as follows: we excise B and replace it by B', where B and B' are complementary balls that are unions of simplexes in the boundary $\partial\Delta^{n+1}$ of the standard $(n+1)$-simplex. It is obvious that such transformations do not change the PL homeomorphism type of the complex. Moreover, U. Pachner ([26,27]) proved in 1990 that conversely, any two PL triangulations of a PL manifold (i.e. the two triangulations define the same PL structure) can be connected by a sequence of bi-stellar moves. One far reaching application of Pachner's theorem was the construction of the Turaev–Viro quantum invariants (see [31]) for three-manifolds.

Habegger's problem on cubical decompositions. It is natural to wonder whether a similar result holds for cubical decompositions, as well. The cubical decompositions that we consider will be PL decompositions that define the same PL structure of the manifold.

Specifically, N. Habegger asked ([17], Problem 5.13) the following:

Problem 1. Suppose that we have two PL cubications of the same PL manifold. Are they related by the following set of moves: excise B and replace it by B', where B and B' are complementary balls (union of n-cubes) in the boundary of the standard $(n+1)$-cube?

These moves have been called *cubical* or *bubble* moves in [10,11], and (cubical) *flips* in [4]. Notice that the flips did already appear in the mathematical polytope literature ([5,37]).

The problem above was addressed in ([10,11]), where we show that, in general, there are topological obstructions for two cubications being flip equivalent.

Notice that acting by cubical flips one can create cubications where cubes have several faces in common or pairs of faces of the same cube are identified. Thus we are forced to allow this greater degree of generality in our definition of cubical complexes.

Related work on cubications. In the meantime this and related problems have been approached by several people working in computer science or combinatorics of polytopes (see [4,7,9,20,29]). Notice also that the two-dimensional case of the sphere S^2 was actually solved earlier by Thurston (see [30]). Observe that there are several terms in the literature describing the same object. For instance the cubical decompositions of surfaces are also called *quadrangulations* ([23–25]) or *quad surface meshes*, while three-dimensional cubical complexes are called *hex meshes* in the computer science papers (e.g. [4]). We used the term *cubulation* in [10,11].

Remark that there is some related work that has been done by Nakamoto (see [23–25]) concerning the equivalence of cubications *of the same order* by means of two transformations (that preserve the number of vertices): the *diagonal slide*, in which one exchanges one diameter of a hexagon for another, and the *diagonal rotation*, in which the neighbors of a vertex of degree two inside a quadrilateral are switched. In particular, it was proved that any two cubications of a closed orientable surface can be transformed into each other, up to isotopy, by diagonal slides and diagonal rotations if they have the same (and sufficiently large) number of vertices and if their one-skeleta define the same mod two homology classes. Moreover, one can do this while preserving the simplicity of the cubication (i.e. not allowing double edges).

Immersions and cobordisms. Let M be a n-dimensional manifold. Consider the set of immersions $f : F \to M$ with F a closed $(n-1)$-manifold. Impose on it the following equivalence relation: (F, f) is *cobordant* to (F', f') if there exist a cobordism X between F and F', that is, a compact n-manifold X with boundary $F \sqcup F'$, and an immersion $\Phi : X \to M \times I$, transverse to the boundary, such that $\Phi_{|F} = f \times \{0\}$ and $\Phi_{|F'} = f' \times \{1\}$.

Once the manifold M is fixed, the set $N(M)$ of cobordism classes of codimension-one immersions in M is an abelian group with the composition law given by disjoint union.

Cubications versus immersions' conjecture. Our approach in [10] to the flip equivalence problem aimed at finding a general solution in terms of some algebraic topological invariants. Specifically, we stated (and proved half of) the following conjecture:

Conjecture 1. The set of marked cubical decompositions of the closed manifold M^n modulo cubical flips is in bijection with the elements of the cobordism group of codimension one immersions into M^n.

The solution of this conjecture would lead to a quite satisfactory answer to the problem of Habegger.

Notice that, when a cubical move is performed on the cubication C endowed with a marking, there is a natural marking induced for the flipped cubication. Thus it makes sense to consider the set of marked cubications mod flips.

We proved in [10] the existence of a surjective map between the two sets.

Digression on smooth versus PL category. There might be several possible interpretations for the conjecture above. We can work, for instance, in the PL category and thus cubications, immersions and cobordisms are supposed PL. Generally speaking there is little known about the cobordism group of PL immersions and their associated Thom spaces, in comparison with the large literature on smooth immersions. However, in the specific case of codimension-one immersions we are able to compare the relevant bordism groups. If M is a compact n-dimensional manifold then the bordism group $N_k(M)^{\rm PL}$ of PL codimension-k immersions up to PL cobordisms is given in homotopy theoretical terms by the formula:

$$N_k(M)^{\rm PL} = [M, \Omega^\infty S^\infty MPL(k)]$$

where $PL(k)$ is the semi-simplicial group of PL germs of maps on $\mathbb{R}^k$, $MPL(k)$ is a suitable Thom space associated to it (see e.g. [36]), Ω denotes the loop space and S denotes the reduced suspension, while $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$. This follows along the same lines as the results of Wells ([33]), where is considered only the smooth case, by using instead of the classical Smale–Hirsch theory on smooth immersions the Haefliger–Poenaru classification of combinatorial immersions from ([14]).

On the other hand, when M is smooth, the bordism group $N_k(M)$ of codimension-k smooth immersions is given by the similar formula from ([33]):

$$N_k(M) = [M, \Omega^\infty S^\infty MO(k)]$$

where $MO(k)$ is the Thom space associated to the orthogonal group $O(k)$.

From the general results of Kuiper and Lashof ([18]) concerning the unstable homotopy type of $PL(k)$ one obtains that the natural inclusion map $O(1) \hookrightarrow PL(1)$ induces a weak homotopy equivalence $MO(1) \to MPL(1)$. This result was improved later by Akiba, Scott and Morlet ([1,21,28]) to weak homotopy equivalences $MO(k) \to MPL(k)$ for all $k \leq 3$.

This shows that codimension-one immersions of PL manifolds into a smooth manifold are PL cobordant to a smooth immersion and also that the existence of a PL cobordism between two smooth immersions implies the existence of a smooth cobordism. Consequently, when the manifold M is smooth, we can use either PL or smooth immersions and bordisms, as the associated groups are naturally isomorphic.

Smooth cubications. However, in the DIFF category it is appropriate to consider only those cubications which are smooth. Smooth cubications are defined following Whitehead's definition of smooth triangulations from ([34]), but we have to change it slightly in order to apply to the more general cubical complexes considered here.

Let M be a smooth manifold and C a cubical complex. A map $f : |C| \to M$ is called *smooth* if the restriction of f to each cube of C is smooth. Moreover, f is *non-degenerate* if all these restrictions are of maximal rank. Finally $f : |C| \to M$ is a *smooth cubication* of M if f is a non-degenerate homeomorphism onto M. According to ([22], Theorem 8.4) this definition is equivalent to Whitehead's one, when applied to simplicial complexes.

In small dimensions (e.g. when the dimension is at most 3) the PL and DIFF categories are equivalent. In particular, we can assume from now on that we are working in the DIFF category and all objects are smooth, unless the opposite is explicitly stated.

Computations of the cobordism group of immersions. Finding the cobordism group $N(M^n)$ of (smooth) codimension-one immersions into the n-manifold M^n was reduced to a homotopy problem by the results of [32,33], as explained above. However, these techniques seem awkward to apply when one is looking for effective results. The group $N(S^n)$ of codimension-one immersions in the n-sphere, up to cobordism, is the nth stable homotopy group of $\mathbb{RP}^\infty$ (since the Thom space $MO(1)$ is homotopy equivalent to $\mathbb{RP}^\infty$) and it was computed by Liulevicius ([19]) for $n \leq 9$ as follows:

n	1	2	3	4	5	6	7	8	9
$N(S^n)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$

It is known that, if M^2 denotes a closed surface, then:

$$N(M^2) \cong H_1(M^2, \mathbb{Z}/2\mathbb{Z}) \oplus H_2(M^2, \mathbb{Z}/2\mathbb{Z}).$$

Using geometric methods Benedetti and Silhol ([3]) and further Gini ([13]) proved that, if M^3 is a three-manifold, then

$$N(M^3) \cong H_1(M^3, \mathbb{Z}/2\mathbb{Z}) \oplus H_2(M^3, \mathbb{Z}/2\mathbb{Z}) \oplus H_3(M^3, \mathbb{Z}/8\mathbb{Z})$$

the right side groups being endowed with a twisted product. The result has been extended to higher dimensional manifolds in [12].

Manifolds with boundary. Habegger's problem from above makes sense also for PL cubications of manifolds with boundary. The question is whether two cubications that induce the same cubication on the boundary are flip equivalent.

Let M be a compact n-dimensional manifold with boundary ∂M. We will consider then the *proper* immersions $f : F \to M$ with F a compact $(n-1)$-manifold with boundary ∂F. This means that ∂M is transversal to f and $f^{-1}(\partial M \cap f(F)) = \partial F$.

In order to define the cobordism equivalence for proper immersions we need to introduce more general immersions and manifolds. The compact n-manifold X is a manifold with *corners* if X is a PL manifold whose boundary ∂X has a splitting $\partial X = F \cup \partial F \times [0, 1] \cup F'$, where $\partial F = \partial F'$, and F, F' are manifolds with boundary. One says that $F \cup F'$ is the *horizontal boundary* $\partial_H X$, $\partial F \times [0, 1]$ is the *vertical boundary* $\partial_V X$ and their intersection $\partial F \times \{0, 1\}$, is the *corners* set.

Observe now that $M \times [0, 1]$ is naturally a manifold with corners if M has boundary, by using the splitting $\partial(M \times [0, 1]) = M \times \{0\} \cup \partial M \times [0, 1] \cup M \times \{1\}$. Let X be as above. One defines then an immersion $\Phi : X \to M \times [0, 1]$ to be an immersion of manifolds with corners if

1. Φ is proper and preserves the boundary type, by sending the horizontal (resp. vertical) part into the horizontal (resp. vertical) boundary. Moreover, Φ is transversal to the boundary.
2. The restriction to the vertical part $\Phi : \partial F \times [0,1] \to \partial M \times [0,1]$ is a product i.e. it is of the form $\Phi(x,t) = (\Phi(x,0), t)$.

We say that the proper immersion (F, f) is *cobordant* to (F', f') if there exist a cobordism X between F and F' (and thus $\partial F = \partial F'$) which is a manifold with corners and a proper immersion of manifolds with corners $\Phi : X \to M \times I$, such that $\Phi_{|F} = f \times \{0\}$ and $\Phi_{|F'} = f' \times \{1\}$.

The set of cobordism classes of immersions of codimension one manifolds with boundary into a given manifold M^n with prescribed boundary immersion can be computed using the methods of [12].

The main result. The aim of this paper is to solve the extension of the cubications versus immersions conjecture in the case of compact surfaces, possibly with boundary.

Theorem 1.1. *The set of marked cubications of the compact surface* Σ *with prescribed boundary mod cubical flips is in one to one correspondence with the elements of* $\mathbb{Z}/2\mathbb{Z} \oplus H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$.

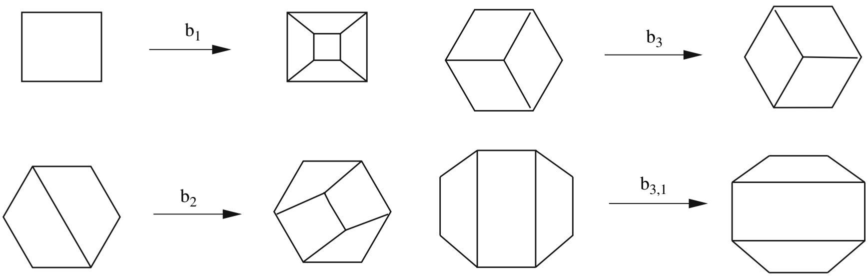
The proof of this theorem, although elementary, uses some methods from geometric topology and Morse theory.

Remark 1.1. One can identify a marked cubication with an embedding of a connected graph in the surface, whose complementary is made of squares. The theorem says that any two graphs like that are related by a sequence of cubical flips and an isotopy of the surface.

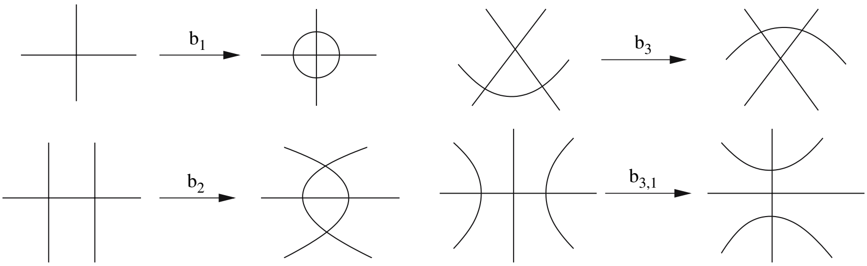
2. Outline of the Proof

Immersions associated to cubications. We associate to each marked cubication C of the n-dimensional manifold M a codimension-one generic immersion $\varphi_C : N_C \longrightarrow M$ (the cubical complex N_C is also called the derivative complex in [2]) of a manifold N_C having one dimension less than M. Here is the construction. Each cube is divided into 2^n equal cubes by n hyperplanes which we call sections. When gluing together cubes in a cubical complex the sections are glued accordingly. Then the union of the hyperplane sections form the image $\varphi_C(N_C)$ of a codimension-one generic immersion. The cubulated manifold N_C is constructed as follows: consider the disjoint union of a set of $(n-1)$-cubes which is in bijection with the set of all sections, then glue together two $(n-1)$-cubes if their corresponding sections are adjacent in M. The immersion φ_C is tautological: it sends a cube of N_C into the corresponding section. If the cubication C is smooth then N_C has a smooth structure and the immersion φ_C can be made smooth by means of a small isotopy. This connection between cubications and immersions appeared independently in [2,10] but this was presumably known to specialists long time ago (see e.g. [30]).

Surface cubications and admissible immersions. The case of the surfaces is even simpler to understand. The immersion $\varphi_C(N_C)$ is obtained by drawing arcs connecting the opposite sides for each square of the cubication C and N_C is a disjoint union of several circles. The immersions which arise from cubications are required to some mild restrictions. First the immersion is normal (or with normal crossings), since it has only transversal double points. All immersions encountered below will be normal crossings immersions. Since we can travel from one square of C to any other square of C by paths crossing the edges of C it follows that the image of the immersion $\varphi_C(N_C)$ should be connected. On the other hand, by cutting the surface Σ along the arcs of $\varphi_C(N_C)$ we get a number of polygonal disks. An immersion having these two properties was called *admissible* in [10]. Further we have a converse for the construction given above. If j is an admissible immersion of circles in the surface Σ then j is φ_C for some cubication C of Σ. The abstract complex C is the dual of the partition of Σ into polygonal disks by means of the arcs of j. Since j can have at most double points it follows that C is made of squares.
Cubical flips on surfaces. There are four different flips (and their inverses) on a surface, that we denoted by b_1, b_2, b_3 and $b_{3,1}$. They are pictured below.



In particular, we have the flips denoted by the same letters that act on immersions of curves on surfaces. These transformations are *local moves* in the sense that they change just a small part of the immersion that lives in a disk, leaving the immersion unchanged outside this disk. Specifically, here are the flip actions.



The invariant of cubications. We associate to each proper immersion $\alpha : L^1 \to \Sigma$ of a disjoint union of circles and intervals L^1, two independent invariants, as follows.

The image $\alpha(L^1) \subset \Sigma$ is a union of curves on Σ and it can be viewed as a singular one-cycle of Σ. Notice that circles lay in the interior of Σ while intervals are properly immersed and thus their endpoints lay on the boundary. We set then

$$j_1(\alpha) = [\alpha(L^1)] \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z}).$$

Further we denote by $j_2(\alpha) \in \mathbb{Z}/2\mathbb{Z}$ the number of double points of $\alpha(L^1)$ mod two, and eventually

$$j_*(\alpha) = (j_1(\alpha), j_2(\alpha)) \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

Further we are able to define the invariant associated to cubications by means of the formula:

$$j(C) = j_*(\varphi_C) \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

Remark that we don't need to know that j factors through the cobordism group $N(\Sigma)$ in order to define the invariant. Observe also that the boundary cubication is the disjoint union of polygons corresponding to the boundary circles and thus the numbers of edges determines completely their combinatorial type. The main theorem above is a consequence of the following more precise statement:

Theorem 2.1. *Two marked cubications C_0 and C_1 of the compact surface Σ are flip equivalent if and only if $j(C_0) = j(C_1)$ and their boundaries agree.*

Remark 2.1. In the case of closed orientable surfaces our result is a consequence of the Nakamoto–Ota theorem ([25]). In fact, we will prove in Sect. 5 that the diagonal transformations introduced by Nakamoto can be written as products of cubical flips. However, their method could not be used to cover the case where the surface is non-orientable or has boundary. Remark, however, that a weaker result holds true for diagonal transformations on arbitrary closed surfaces (see [23,24]), in which one replaced marked cubications up to isotopy by marked cubications up to homeomorphism.

Remark 2.2. Our methods are not combinatorial, as was the case of the sphere (see [30,10,4]), since one uses in an essential manner the identification of $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$ with $N(\Sigma)$, which is of topological nature. The main interest in developing the topological proof below is that one can give an unifying treatment of all surfaces and the hope that these arguments might be generalized to higher dimensions. However, it would be interesting to find a direct combinatorial proof that provides an algorithm which gives explicitly a sequence of flips connecting two cubications. Such an algorithm can be obtained in the closed orientable case by using the diagonal slides.

Consider two cubications C_0 and C_1 having the same invariants. The first step in the proof of Theorem 2.1 is to return to the language of immersions and show that:

Proposition 2.1. *If C_0 and C_1 have the same boundary and $j(C_0) = j(C_1)$ then φ_{C_0} and φ_{C_1} are cobordant immersions.*

The second step is to use the existence of a cobordism in order to produce flips and prove that:

Proposition 2.2. *If φ_0 and φ_1 are admissible immersions which are cobordant then there exists a sequence of flips which connects them.*

These two propositions will end the proof of the Theorem 2.1.

3. Cobordant immersions are flip equivalent

3.1. Flips, saddle and X-transformations relating cobordant immersions

Connecting the immersions by means of maps with higher singularities. This section is devoted to the proof of Proposition 2.2. Consider thus a proper immersion $\varphi : F \to \Sigma \times [0, 1]$ of a surface F which is a cobordism between the immersions φ_0 and φ_1.

The image $\varphi(F)$ is an immersed surface having therefore a set of finitely many triples points that we denote by $S_3(\varphi(F))$. The set $S_2(\varphi(F))$ of double points of φ (at the target) form a one-dimensional manifold, whose closure contains the triple points. We have then a stratification of $\varphi(F)$ by manifolds

$$\varphi(F) = R(\varphi(F)) \cup S_2(\varphi(F)) \cup S_3(\varphi(F))$$

where $R(\varphi(F))$ is the set of non-singular (or regular) points.

Our aim is to analyze the critical points of the restriction of the height function to $\varphi(F)$, by taking into account the singularities of $\varphi(F)$. In order to define critical points properly we need more terminology. Note that $R(\varphi(F))$ and $S_2(\varphi(F))$ are subsets of $\varphi(F)$ which might cause some troubles because critical points on the closure of a stratum might belong to another stratum.

Any point $p \in \varphi(F)$ has an open neighborhood that is diffeomorphic to one coordinate plane, the union of two coordinate planes or the union of the three coordinate planes in $\mathbb{R}^3$, depending on whether $p \in R(\varphi(F))$, $p \in S_2(\varphi(F))$ or $p \in S_3(\varphi(F))$. The images of coordinate planes by this diffeomorphism are called the *leaves* of $\varphi(F)$ around p. Actually the leaves are well-defined only in a small neighborhood. A point $p \in \varphi(F)$ will be called *critical* for h if p is critical either for the restriction of h to some leaf containing p, or for $h|_{S_2(\varphi(F))}$, or else $p \in S_3(\varphi(F))$. Moreover, by using a small perturbation of φ that is identity on the boundary, we can assume that the restriction of h to the leaves is also a Morse function.

A consequence of the Morse theory is the following. If the interval $[t_1, t_2]$ does not contain any critical value for h then there exists a diffeomorphism of $\Sigma \times \{t_1\}$ into $\Sigma \times \{t_2\}$ that sends $\varphi(F) \cap h^{-1}(t_1)$ on $\varphi(F) \cap h^{-1}(t_2)$. Thus changes in the topology of the slice $\varphi(F) \cap h^{-1}(t)$ arise only at critical t.

A critical point p will be said to be *unstable* if there are at least two leaves around p and the restriction of h to some leaf has a critical point at p. The other critical points are called *stable*.

Lemma 3.1. *One can perturb slightly φ by using an arbitrary small isotopy that is identity on the boundary such that h has only stable critical points.*

Proof. Let p be an unstable critical point and α be a leaf that is critical for the restriction of h at p. This is equivalent to the fact that the gradient of h (which is nowhere zero since h is regular) is orthogonal to the tangent plane at α. Since the immersion is normal crossings any other leaf β around p should be transverse to α and thus the restriction of h to that leaf is non-critical at p.

Use now a small perturbation of the extra leaf around p by an isotopy that moves the intersection arc $\alpha \cap \beta$ off p. The new intersection point is not anymore critical for the restriction of h at α. □

There are the following situations when the values are not regular:

1. The slice $\Sigma \times \{t\}$ passes thru a triple point p and the restriction of h to all leaves is regular.
2. The slice $\Sigma \times \{t\}$ contains a critical point p of the restriction $h|_{S_2(\varphi(F))}$, to the double points locus. Let α and β denote the two leaves around p. The restriction of h to the two leaves is regular.
3. If the slice $\Sigma \times \{t\}$ contains a critical point of the height restriction $h|_{R(\varphi(F))}$, to the regular locus.

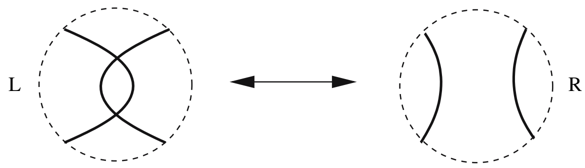
Passing a triple point. We can assume by general position arguments that the (finitely many) triple points have distinct heights and thus the slice $\Sigma \times \{t_c\}$ contains precisely one triple point p and no other critical point.

Proposition 3.1. *When crossing a critical value t_c corresponding to a stable triple point the image of the sliced immersion $\varphi(F) \cap \Sigma \times \{t\}$ changes according to a flip b_3.*

Proof. Consider a coordinates chart (V, ν) on $\Sigma \times [0, 1]$ containing the point p. We assume that V is diffeomorphic to $\mathbb{R}^3$ and the diffeomorphism ν sends $\varphi(F)$ into Π, where Π denotes the union of the three coordinates planes in $\mathbb{R}^3$. Denote by $H : \mathbb{R}^3 \to \mathbb{R}$ the function h expressed in these coordinates. Notice that H is regular and one can assume that $H(0) = 0$. The level hypersurface $\mathcal{H} = H^{-1}(0)$ corresponds to a neighborhood of p into the critical slice. Moreover, one knows that H has no critical points when restricted to the leaves around p (i.e. the coordinate planes) which amounts to say that the gradient $\text{grad}_0 H$ is not orthogonal to any coordinate plane. Let $T_0\mathcal{H}$ be the tangent space at $\mathcal{H}$ at the origin. There is only one generic position of a plane through O with respect to Π. Translating the plane along the direction of $\text{grad}_0 H$ yields the desired bifurcation. □

Critical points on the double points locus. The double locus $S_2(\varphi(F))$ is a 1-manifold and each of its points has a coordinate chart in which $\varphi(F)$ is sent into the union of the first two coordinates planes in $\mathbb{R}^3$. The critical points correspond to local extrema of h when restricted to the double line.

Proposition 3.2. *When crossing a critical value t_c corresponding to a stable critical point of the restriction of h to $S_2(\varphi(F))$ the image of the sliced immersion $\varphi(F) \cap \Sigma \times \{t\}$ is transformed according to the following picture*:



This means that we replace the left diagram L by the right diagram R, leaving the part of the immersion unchanged outside the small ball figured above.

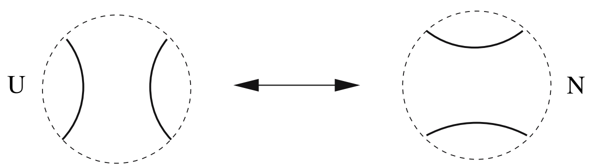
Proof. The proof from above applies with minor modifications. However, here the singularity consists of a line of double points and thus the intersection of the two planes with the boundary sphere of a small disk centered at the critical point is the union of two great circles instead of three. Since there is an extremum on the double line there is one half of the sphere that contains no double points. Thus the only transformation possible is that from the picture. □

The local transformation of an immersion (and its inverse) occurring in the proposition above is called an *X-transformation*.

Critical points on the regular strata. The height function $h|_{R(\varphi(F)}$ is a Morse function on a surface and thus it has critical points of three types: maximum, minimum and saddle (index one).

Passing thru a minimum point amounts to adjoin a small embedded sphere to the immersion, while a maximum point contributes with deleting a small sphere. We will speak about creation/annihilation of a circle.

When passing thru a saddle point the sliced immersion $\varphi(F) \cap \Sigma \times \{t\}$ is subject to the following familiar change:



This transformation will be called *saddle transformation*.

3.2. Stabilizations

In order to establish the claim of Proposition 2.2 it would suffice to show that any saddle move, X-move or creation/annihilation can be actually realized by means of flips. As stated, this cannot be true, but a slight modification of this statement will hold true. One reason is that creating a new circle destroys the connectivity of the graph, while flips preserve it. Recall, however, that our immersions were supposed to be admissible. In particular, in dimension 2 one has a connected graph whose complementary is a union of open disks on the surface.

We will introduce now another operation on immersions (or, equivalently on graphs on surfaces) that will be called *stabilization*. Let $\varphi : \sqcup S^1 \to \Sigma$ be an immersion of circles and λ a simple arc in Σ that joins two points of the image of φ and is transversal to it. Let $u(\lambda)$ be the embedding of a circle into Σ as the boundary of a small regular neighborhood of λ. Then the union of φ and $u(\lambda)$ define an immersion that we denote by $\varphi *_\lambda u$. One allows also the particular case when λ is trivial and the two points coincide, denoted $\varphi * u$. We will also denote by $\varphi \sqcup u$ the disjoint union of φ with a small trivially embedded circle.

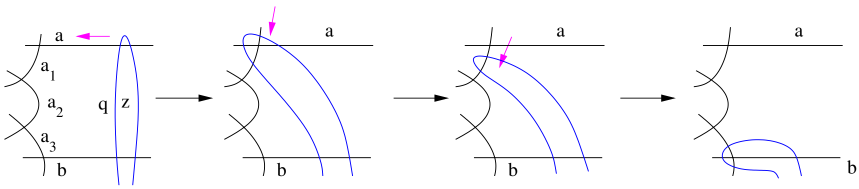
The main result of this subsection is:

Proposition 3.3. *If φ is admissible then $\varphi *_\lambda u$ is flip equivalent to φ.*

Proof. Assume that λ intersects the image of φ at the points $p_1, p_2, \ldots, p_k$, where $k \geq 2$. We use induction on k. Let λ' be the sub-arc of λ joining the last two points p_{k-1} and p_k and λ" its complementary arc. Then $u(\lambda)$ is the connect sum of $u(\lambda')$ and $u(\lambda")$. Let z be the arc which is common to $u(\lambda')$ and $u(\lambda)$. We will use flips whose action is trivial outside the arc z.

The circle $u(\lambda')$ intersects two arcs a and b of the image of φ in two points that determine an arc q of $u(\lambda)$ which is free of other intersection points. The arcs a and b are connected by some chain of arcs of φ, namely $a_1, a_2, \ldots, a_m$, since φ is admissible. Moreover, the arcs a_i, a, b and q are bounding a face of the complementary of the immersion. Since φ was supposed admissible the simple arc q subdivides a topological disk into two pieces which should be again disks. Thus the polygon determined by a_i, a, b and q is a topological disk Q.

Now, we can slide the intersection part between $z \cap a$ along the path $a_1, a_2, \ldots, a_m$, b until z will intersect both a_m and b as below:



This can be realized by flips that acts non-trivially only on the arc z and are identity outside.

If $k = 2$ then $u(\lambda)$ is obtained from z by joining its extremities. Then the inverse flip move b_1 destroys the circle $u(\lambda)$.

If $k > 2$ then use the inverse move b_2. The result of this sequence of flips is that $u(\lambda)$ is transformed into $u(\lambda")$ and λ" has $k - 1$ intersection points with the image of φ. The claim follows then by induction. ◻

3.3. Saddle transformations

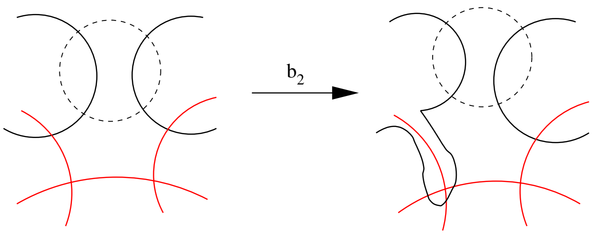
Consider an immersion ψ containing a small ball where it coincides with the diagram U, the left hand side in the picture of the saddle move. We denote by $S(\psi)$ the immersion obtained by a saddle move from ψ, namely by excising U and gluing back N.

Lemma 3.2. *Assume that the immersion ψ is admissible. Then ψ is flip equivalent to some stabilization $S(\psi) *_\lambda u$ of $S(\psi)$.*

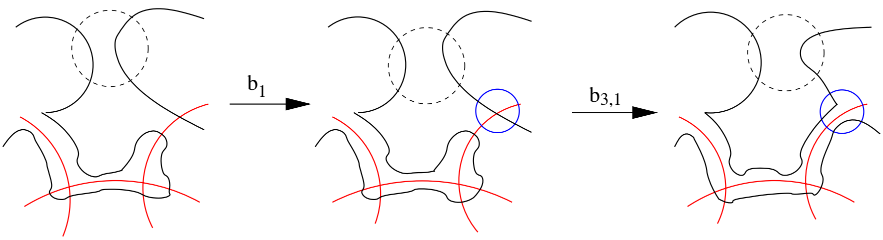
Remark 3.1. The fact that in general ψ is equivalent only with the stabilization of $S(\psi)$ is not unexpected. In fact saddle move might destroy the connectivity of the image, and for that reason one should add a new circle $u(\lambda)$ in order to restore it.

Proof. It is essential that ψ is admissible and thus the image is connected. In particular, there are several arcs of ψ which join the left arc of U to the right arc of U. These intermediary arcs are outside the small ball. However, we can choose a specific family of arcs, namely those bounding a face (i.e. a topological disk) in the complement of the image of ψ.

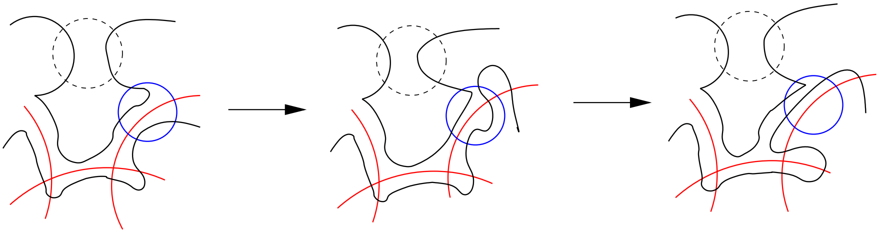
We use inductively the move b_2 in order to push and slide the left arc of U across each intermediary arc, as it is shown below for the case of the first arc.



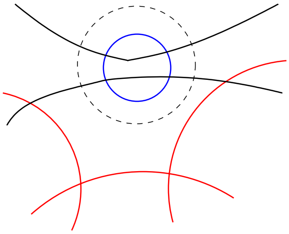
We do that until the new position of the left arc intersects the last intermediary arc. Use b_1 to create a small circle centered at the intersection point between the last intermediary arc and the right arc of U. The next step is to use the move $b_{3,1}$ once. Its support is centered at the intersection between the last intermediary arc and the small circle created at the previous step. We obtain the configuration below:



We realized half of the saddle move, namely the upper arc from N. However, the bottom arc intersects all intermediary arcs used above. We will further use the inverse moves b_2 in order to slide the bottom arc over the intermediary arcs, this time from the right side back to the left. We first use the intersections with the additional small circle and then go along the intermediary arcs as follows:



At the end we transformed the bottom arc into another arc which goes along the path of intermediary arcs but lays on its upper side. Further the small additional circle can be slid outside the last intermediary arc. The face determined by the intermediary arcs is topologically a disk, since the immersion is admissible. Thus both arcs (upper and lower) can be isotoped to the position that they have in the N diagram. We obtained thus the following configuration:



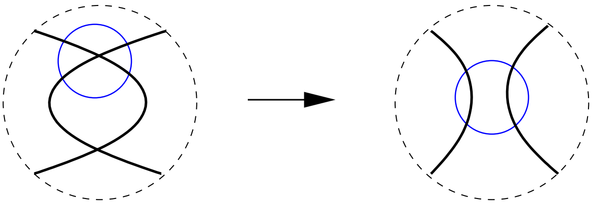
This is the stabilization $S(\psi) *_\lambda u$ where we used the arc λ that connects the two arcs in the diagram N. □

3.4. X transformations

We denote again by ψ an immersion containing in a small ball the diagram L and by $X(\psi)$ the result of the X transformation.

Lemma 3.3. *If ψ is admissible then ψ is flip equivalent to $X(\psi) *_\lambda u$.*

Proof. We use b_1 to create a small circle centered at one intersection point of the arcs in L. Using next the move b_2 we can slide the two arcs from L far apart and obtain $X(\psi) *_\lambda u$.



□

Consider now the immersion ψ that contains the diagram R so that we can apply the inverse move X^{-1} to the immersion ψ. In this case we have

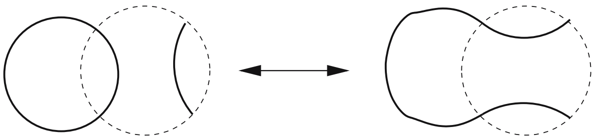
Lemma 3.4. *If ψ is admissible then $X^{-1}(\psi)$ is flip equivalent to ψ.*

Proof. The figure above shows that $X^{-1}(\psi)$ is flip equivalent to $\psi *_\lambda u$. If ψ is admissible then ψ is equivalent to $\psi *_\lambda u$ by Lemma 3.3. □

3.5. Proof of Proposition 2.2

Consider now two admissible immersions φ and ψ which are cobordant. According to the previous description of cobordisms there exists a sequence of moves $A_1, A_2, \ldots, A_{n-1}$ transforming φ into ψ which are either flips, saddle transformations, X moves or their inverses, or creation/annihilation moves. Let the sequence of immersions so obtained be $\varphi = \varphi_1, \varphi_2, \ldots, \varphi_n = \psi$.

The problem we face is that some of the φ_j might have disconnected image and are therefore not admissible. First, we can dispose of the creation/annihilation of circles, by using instead saddle moves, as below:



The key ingredient is the existence of an admissible modification of the sequence above. Let us introduce first more terminology. Assume that we have a sequence of moves A_j that might be saddle, X-transformations of flips. The move A_j replaces the part of the image of some immersion φ_j contained in a disk D_j by a different graph, while keeping the complementary unchanged. We define now an *extended move* $\overline{A}_j$ that acts on some stabilization $\overline{\varphi_j}$ of φ_j that uses the arcs λ_i, as follows. The extended move $\overline{A_j}$ is assumed to act exactly in the same way as A_j, namely it replaces the part of the image of φ_j contained in the disk D_j by the corresponding graph (as prescribed by A_j), while keeping untouched both the complementary of the ball and also the new circles $u(\lambda_i)$. The extended moves $\overline{A}_j$ are not anymore usual saddles, X-moves etc unless the disk D_j avoids the arcs λ_i. Notice also that $\overline{A}_j\overline{\varphi}_j$ might be non-admissible.

Lemma 3.5. *There exists a sequence of immersions $\overline{\varphi}_j$ and extended moves $\overline{A}_j$ with the following properties*:

1. *$\overline{\varphi}_j$ is a stabilization of φ_j, of the form $\overline{\varphi}_j = \varphi_j *_{\lambda_1} u *_{\lambda_2} u \cdots *_{\lambda_{j-1}} u$. Thus φ_j is obtained by φ_j by adding several circles of type $u(\lambda_i)$, where the arcs λ_i might intersect each other.*
2. *The immersions $\overline{\varphi}_j$ are admissible and $\overline{\varphi}_j$ and $\overline{\varphi}_{j+1}$ are flip equivalent.*

Proof. We use a recurrence on the length of the sequence. By the definition of the extended moves we have:

$$\overline{A}_j\overline{\varphi}_j = \varphi_{j+1} *_{\lambda_1} u *_{\lambda_2} u \cdots *_{\lambda_{j-1}} u, \text{ and } \overline{\varphi}_{j+1} = (\overline{A}_j\overline{\varphi}_j) *_{\lambda_j} u.$$

We set $\overline{\varphi}_1 = \varphi_1$. Then $A_1 \in \{X, X^{-1}, S, b_3\}$. We further define

$$\overline{\varphi_2} = \begin{cases} \varphi_2, & \text{if } A_1 \in \{X^{-1}, b_3\} \\ \varphi_2 *_{\lambda_1} u, & \text{otherwise} \end{cases}$$

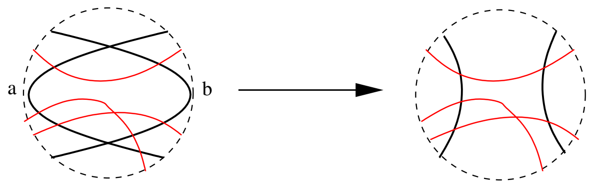
where the arc λ_1 is that furnished by the Lemmas 3.2 and 3.3. According to these lemmas $\overline{\varphi_2}$ is flip equivalent to $\overline{\varphi_1}$ and both are admissible. Moreover, $\overline{A_1} = A_1$.

Assume now that $\overline{\varphi_j}$ and $\overline{A_j}$ are defined for $j \leq k$. If $A_k \in \{b_3, X^{-1}\}$ then we set

$$\overline{\varphi_{k+1}} = \overline{A_k}\overline{\varphi_k}$$

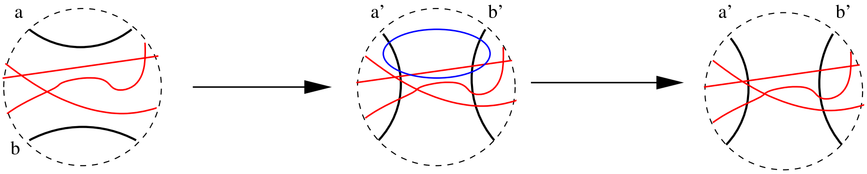
In the other two cases we have to analyze the picture inside the disk D_k. We set first $\Lambda = D_k \cap (\lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_{k-1})$ and call its components the special arcs.

1. Assume that $A_k = X$. We can take for D_k a very tiny regular neighborhood of the two arcs a and b of φ joining the double points in the diagram U and thus all arcs λ_j entering D_k should intersect these arcs. Another useful observation is that any sub-arc of a special arc that has two consecutive intersection points with a can be moved off D_k by means of b_2 moves. Thus either there are no special arcs within D_k or else there exists arcs that cross both arcs a and b.
   (a) If $\Lambda = \emptyset$ then we set $\overline{\varphi_{k+1}} = X(\overline{\varphi_k}) *_{\lambda_k} u$ where λ_k is the arc given by Lemma 3.3. That lemma shows that $\overline{\varphi_{k+1}}$ is flip equivalent to $\overline{\varphi_k}$.
   (b) Otherwise we define $\overline{\varphi_{k+1}} = \overline{A_k}(\overline{\varphi_k})$, and it suffices to show the flip equivalence with $\overline{\varphi_k}$. From above one knows that $a \cup b \cup \Lambda$ is connected. We can use the moves b_3 in order to move the arc a across the vertices of the diagram Λ and the moves b_2 in order to move the arc a along special arcs without vertices on it i.e. arcs which are connected components of Λ.



2. Consider now that $A_k = S$. Let a and b denote the arcs from the diagram U and a', b' the arcs from the diagram N.
   (a) If $\Lambda = \emptyset$ then we set $\overline{\varphi_{k+1}} = S(\overline{\varphi_k}) *_{\lambda_k} u$ where λ_k is the arc given by Lemma 3.2. That lemma shows that $\overline{\varphi_{k+1}}$ is flip equivalent to $\overline{\varphi_k}$.

(b) The move S takes place in a very tiny neighborhood of an arc that joins two points, one from a and the other from b. We call it the *core* and it corresponds to the 1-handle to be added to the immersion by the saddle move. Thus we can discard (by using an isotopy) all special arcs except those that intersect the core. Moreover, one can suppose that there is at least one such special arc. We define then $\overline{\varphi_{k+1}} = \overline{A_k}(\overline{\varphi_k})$, and it suffices to show the flip equivalence with $\overline{\varphi_k}$. Lemma 3.2 shows that $\overline{\varphi_k}$ is flip equivalent to $\overline{A_k}(\overline{\varphi_k}) *_{\lambda_k} u$, where λ_k is the arc that joins a' to b'. One has to notice that $\Lambda \cup a' \cup b'$ is connected since the special arcs intersect the core. Thus a suitable application of the moves b_3, slidings across the special arcs and an inverse b_1 move will get rid of the extra circle $u(\lambda_k)$.



□

Now the lemma ends the proof of the Proposition 2.2 immediately. In fact one knows that the last immersion φ_n is admissible, by hypothesis. Then Lemma 3.3 stated that $\overline{\varphi_n} = \varphi_n *_{\lambda_1} u *_{\lambda_2} u * \cdots *_{\lambda_{n-1}} u$ is flip equivalent to φ_n. On the other hand φ_1 is equivalent to $\overline{\varphi_j}$ for all j, and we are done.

4. The cobordism group of immersions in a surface

Although the computation of $N(\Sigma)$ for closed surfaces is folklore we didn't find a reference addressing precisely this issue. Related results of similar nature are recorded in [6,8]. This also follows from our computations of the cobordism groups of codimension one immersions for manifolds of small dimensions, from [12]. The proof that we give below is elementary in that it uses only basic methods and results from three-dimensional topology. The Proposition 2.1 can be reformulated as follows:

Proposition 4.1. *Two, one-dimensional immersions in a compact surface are cobordant if and only if they have the same boundary and their j_* invariants agree. In particular the map j_* factors to an isomorphism $j_* : N(\Sigma) \to H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$.*

Proof. Assume that $\varphi_i : L_i^1 \to \Sigma$, $i = 0, 1$ are proper immersions of two disjoint unions of circles and intervals, L_0^1 and L_1^1, having the same boundary and invariants. Thus the number of intervals within L_i^1 is the same for both values of i, and their images by the respective immersions intersect $\partial\Sigma$ in the same set of points.

Since $j_1(\varphi_0) = j_1(\varphi_1)$ there exists a singular 2-cycle of the pair $(\Sigma, \partial\Sigma)$ with boundary $\varphi_1(L_0^1) - \varphi_2(L_1^1)$. This means that there exists a surface with corners

F having boundary $\partial F = L_0^1 - L_1^1$ and a (singular) map $f : F \to \Sigma$ such that $f|_{\partial F} = \varphi_0 \sqcup \varphi_1$. The corners set is the union of boundary points of intervals in $\varphi_i(L_i^1)$.

Let $h : F \to [0,1]$ be a proper smooth function having $h^{-1}(0) = L_0^1$ and $h^{-1}(1) = L_1^1$. We can lift f to a map $\phi : F \to \Sigma \times [0,1]$ by means of the formula

$$\phi(x) = (f(x), h(x))$$

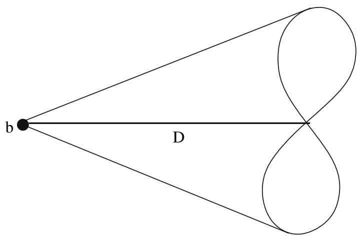
Then ϕ is a proper map i.e. it sends the boundary into the boundary.

Let us consider first the case when the surface Σ has no boundary and thus there are no intervals among the components of L_i^1. Thus F is a surface with boundary.

Lemma 4.1. *Let $\phi : F \to M^3$ be a proper map of a surface into the connected 3-manifold M^3, whose restriction to the boundary $\partial\phi : \partial F \to \partial M^3$ is an immersion. Then there exists an immersion $\phi' : F' \to M^3$ of a possibly different surface F' such that $\partial F = \partial F'$ and $\partial\phi' = \partial\phi$ if and only if $j_2(\partial\phi) = 0$.*

Proof. The proof is a variation of that given by Hass and Hughes ([15]) in the case when F is closed and thus the invariant j_2 trivially vanishes. The main arguments below appeared also in Whitehead's paper ([35], proof of Theorem 4.1).

We use a classical result of Whitney which states that any smooth map $\phi : F \to M$ which is transverse to the boundary is homotopic rel boundary to a proper general position map having only *simple branch points*. A simple branch point in the image is a point having a neighborhood homeomorphic to Whitney's umbrella, namely the cone over the bouquet of two circles (usually known as the figure eight), as pictured below:



According to Thom this is the only generic local singularity of smooth germs of maps $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$.

In particular, from any branch point b of $\phi(F)$ emerges a line of double points, denoted by D in the figure above. Thus the map is now an immersion everywhere except at the (finitely many) branch points.

The singularities of the new map (that we denote by the same letter ϕ) consist of (see also [16], p.41-42):

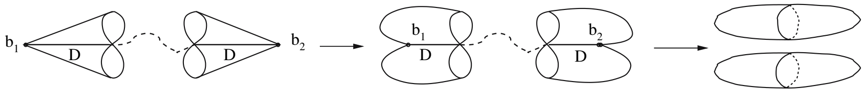
- Finitely many triple points, each one having three points in the preimage.
- Finitely many branch points, each one having one point in the preimage.
- Curves of double points, which are points where the surface $\phi(F)$ has normal crossing self-intersections. These curves are of two types:

- Type I curves that are either closed simple loops or joining two boundary double points of $\partial\phi$. These curves intersect transversally at triple points.
- Type II curves that are either segments which join two distinct branch points or else a branch point to some boundary double point of $\partial\phi$.

If the map ϕ is an immersion then there are no branch points and thus the double points of the restriction $\partial\phi$ are paired by means of the type I curves, and thus there is an even number of such double points. This establishes the necessity of our condition.

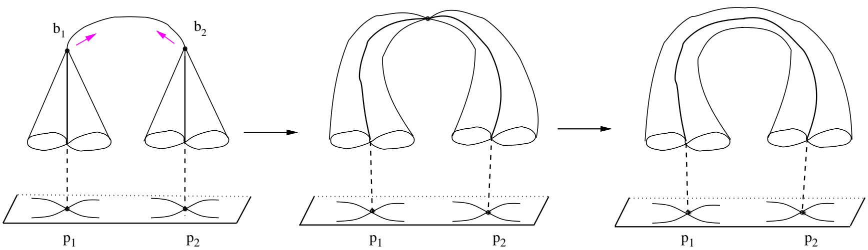
Conversely, if there is an even number of such double points, then we have a number of branch points paired together by means of type II curves and also an even number of pairs (b_j, p_j) where b_j is a branch points and p_j is a boundary double point.

Let b_1, b_2 being two branch points connected by a curve of type II. There exists a standard way to modify ϕ by means of a homotopy so that the branch points are pushed one end towards the other (see also [15]):



This is equivalent to cutting the surface $\varphi(F)$ along the connecting double curve. The domain of the map will change accordingly but we can further add a small tube between the components separated by the cut, in order to recover the same surface F.

The other case is when we have a couple (b_1, p_1) and (b_2, p_2) of branch points paired with boundary double points. Choose now a segment embedded in M that joins b_1 to b_2 and avoids the triple points and other branch points. We will push again the branch points towards one another along this segment until they collide and then disappear, as in the figure below:



In particular ϕ was changed into an immersion ϕ' of a surface F', which is obtained from F by adding a one-handle.

Eventually we can slightly perturb the immersion in order to become normal crossings immersions, since the later are generic. □

The lemma settles our claim, because we know that $j_2(\phi) = j_2(\varphi_0) - j_2(\varphi_1) = 0$ and thus there exists an immersion $\Phi : F \to \Sigma \times [0, 1]$ extending the immersions $\varphi_0 \sqcup \varphi_1$ on the boundary. □

Remark 4.1. It is considerably more difficult to get control on the genus of F' out of the data ϕ, F, M. The typical result is the celebrated Dehn lemma ([16]).

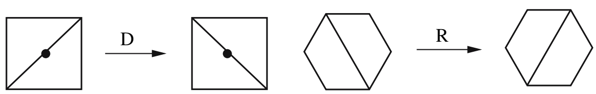
Consider now the general case of surfaces Σ with nonempty boundary, which is more difficult only at the terminology level. The main difference is that now $\Sigma \times [0, 1]$ is a manifold with corners. Then the map $\varphi : F \to \Sigma \times [0, 1]$ is a proper map between manifolds with corners that respects the horizontal part of the boundary and it is a product on the vertical boundary.

Lemma 4.2. *Let $\phi : F \to M^3$ be a proper map of a surface with corners into the connected 3-manifold M^3, whose restriction to the boundary $\partial\phi : \partial F \to \partial M^3$ is an immersion which respects the horizontal part of the boundary and it is a product on the vertical boundary. Then there exists an immersion between manifolds with corners $\phi' : F' \to M^3$ of a possibly different surface with corners F' such that $\partial F = \partial F'$ and $\partial\phi' = \partial\phi$ if and only if $j_2(\partial\phi) = 0$.*

Proof. The proof from above applies with only minor modifications. In fact the lines of double points of ϕ in M^3 are disjoint from the vertical boundary $\partial_V M$ because the immersion is proper and its restriction $\partial\phi$ is a product on the vertical boundary. □

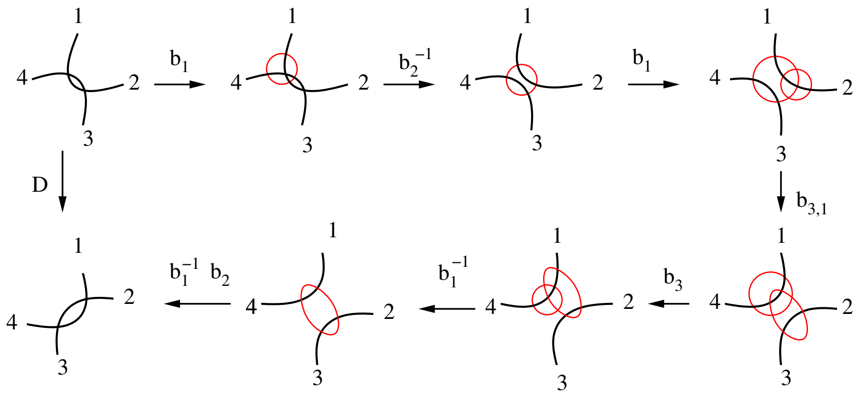
5. Diagonal transformations

Nakamoto and Ota (see [23–25]) considered the problem of moves acting on the cubications of surfaces but they used a different family of transformations that they called diagonal transformations. Specifically, these are the diagonal slide R (or rotation of order three) and the diagonal rotation D (or order two rotation) drawn below:

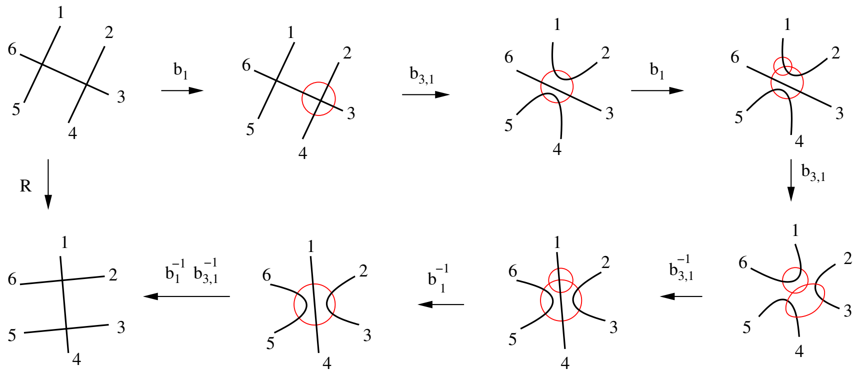


Proposition 5.1. *The diagonal transformations are products of cubical flips.*

Proof. There is an immediate corollary of the main result for surfaces with boundary since the immersions associated to these cubications of the disk have the same number of double points and boundary points. However, there exists an elementary pictorial proof, in which we decompose D into flips as below:



and further the order three rotation move R:



□

Nakamoto and Ota proved in [25] that there exists a natural number n_g such that any two cubications of a closed orientable surface of genus g having $n \geq n_g$ vertices are equivalent by means of diagonal transformations, and moreover this can be realized among cubications without double edges. In particular, our main result is a consequence of the theorem of Nakamoto–Ota, in the case of closed orientable surfaces.

Corollary 5.1. *Let C_0 and C_1 be marked cubications of a closed orientable surface Σ that have the same invariants. Then there exists a sequence of flips relating them with the following properties*:

1. *Start with a number of moves b_1 and b_2 such that C_0 and C_1 have the same number of vertices, which is larger than n_g.*
2. *Use then diagonal transformations to connect the two cubications as in* [25]. *Decompose further the diagonal transformations into cubical flips as in the previous two pictures.*

Thus, provided that the numbers of vertices of C_i are large enough, there exists a sequence of flips that relating them among cubications having no more than six extra vertices. In particular, the problem of finding a connecting sequence of flips is algorithmically solvable.

Remark 5.1. Although the methods of [25] do not work for non-orientable surfaces or surfaces with boundary, we expect that a result similar to the corollary above holds true in the general case.

Acknowledgments. We are grateful to the referee for many valuable comments and suggestions leading to a better presentation and the simplification of some proofs.

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