On proper homotopy type and simple connectivity at infinity for open 3-manifolds

Louis Funar

Institut Fourier BP 74, UFR Mathématiques, Univ.Grenoble I

38402 Saint-Martin-d'Hères Cedex, France

e-mail: funar@fourier.ujf-grenoble.fr

October 26, 1998

Abstract

The main result of this note is that a contractible open 3-manifold W^3, which has the same proper homotopy type as a geometrically simply connected simplicial complex P, is simply connected at infinity. This generalizes a theorem proved by Poénaru in [7].

AMS MOS Subj.Classification(1991): 57 M 50, 57 M 10, 57 M 30.

Keywords and phrases: Proper homotopy, geometric simple connected, simple connected at infinity, Φ/Ψ-theory.

1 Introduction

The startpoint of this paper is the following theorem of V.Poénaru ([7]):

Theorem 1.1 *Let W^3 be an open contractible 3-manifold such that the product $W^3 \times D^n$ with the closed n-ball has no 1-handles. Then W^3 is simply connected at infinity.*

We recall that an open contractible 3-manifold W^3 is *simply connected at infinity (s.c.i.)*, and we write also $\pi_1^\infty(W) = 0$), if for any compact set K_1, there exists another compact set K_2, with $K_1 \subset K_2 \subset W^3$, such that any loop in $W^3 - K_2$ is null-homotopic in $W^3 - K_1$. The definition of a non-compact manifold with boundary and without 1-handles is given in [7, 10]. The translation in polyhedral language of this condition is the geometric simple connectivity. A locally finite simplicial complex P is *geometrically simply connected (g.s.c.)* if there exists an exhaustion $Z_0 \subset Z_1 \subset Z_2 \subset ...Z_n \subset ...$ of P by finite sub-complexes with all Z_n being connected and simply connected.

Poénaru in ([7], Remark C, p. 432) claimed that it might be possible to have a connection between the simple homotopy type and π_1^∞ in dimension 3. The natural conjecture would be that W^3 is simple homotopy equivalent to a g.s.c. simplicial complex P if and only if W^3 is simply connected at infinity. Using Chapman's stabilization lemma [1] this corresponds to replacing the ball D^n by the Hilbert cube in the statement of the previous theorem. Our main result establishes a stronger claim without requiring the homotopy equivalence to be simple:

Theorem 1.2 *An open contractible 3-manifold which is proper homotopy equivalent to a locally finite g.s.c. simplicial complex is s.c.i.*

This result was proved using standard 3-dimensional techniques in [2]. Our aim is to give here a proof along the lines of [7].

Notice that there exist manifolds W^n in every dimension $n \geq 4$ (e.g. the Poénaru-Mazur manifolds, see [5, 3]), such that $W \times D^k$ is g.s.c. for some k, but $W \times D^k$ (and henceforth W) is not s.c.i., so our main theorem is a purely 3-dimensional result, which cannot be extended in higher dimensions. Remark that the properness condition is essential above. In fact, for any Whitehead-type manifold W^3 the product $W^3 \times int(D^{n-3})$ has no 1-handles ([4]) for large enough n, since its homeomorphism class depends only on the homotopy type of W^3.

Acknowledgements: I'm indebted to Valentin Poénaru for proposing me this problem, and for the numerous suggestions and remarks as well as Ross Geoghegan, Frank Quinn and Larry Siebenmann for helpful conversations and advice.

2 The plan of the proof

Definition 2.1 *An enlargement of the 3-manifold M^3 is a locally finite simplicial complex X which admits M^3 as a proper strong deformation retract, and whose 3-skeleton ske^3X of X is strongly connected, i.e. any two 3-simplexes are connected by a chain of 3-simplexes, the consecutive ones having a common face.*

We reduce the theorem to the following

Proposition 2.1 *Let W^3 be an open contractible 3-manifold, and X^n be a finite dimensional enlargement of W^3. If X^n is g.s.c. then W^3 is s.c.i.*

Actually, we will prove a stronger statement, that there exists an exhaustion $Z_0 \subset Z_1 \subset Z_2 \subset ...Z_n \subset ...$ of W^3 by compact connected and simply connected sub-manifolds Z_n. In dimension 3, this condition implies that $\pi_1^\infty(W^3) = 0$.

Proof of the Theorem 1.2 assuming Proposition 2.1. If the 3-manifold W^3 and X^n are proper homotopy equivalent then there exists a locally finite simplicial complex Y^m such that both X and M^3 are proper strong deformation retracts (e.g. the mapping cylinder of a homotopy equivalence). Possibly replacing Y^m by a product with a closed ball we can assume that its 3-skeleton is strongly connected, hence Y^m is an enlargement of W^3. Assume now that X^n is g.s.c. It follows that Y^m is also g.s.c. and the proposition above implies that W^3 is s.c.i. as claimed. □

Proof of Proposition 2.1: The idea comes from the series of papers [7, 8, 9, 10]. The main arguments are contained in the following three lemmas. In order to make this paper self-contained we added an appendix on the Φ/Ψ-theory (developed in [6]) at the end. We will use the following notation: if $h : A \longrightarrow B$ is a map and $n \in \mathbf{Z}_+$, we will denote by $M_n(h) \subset A$ the set of $x \in A$ which are such that $card(f^{-1}(f(x)) \geq n$. We also write $M^2(h) \subset A \times A$ for the set of pairs $(x, y) \in A \times A$ with $x \neq y$ and $h(x) = h(y)$.

Lemma 2.1 *Let X^3, M^3 be two simply-connected manifolds, K be a connected compact set, such that X^3 is compact, connected with non-void boundary and M^3 is closed without boundary. Assume we have a commutative diagram*

$$\begin{array}{ccc} K & \stackrel{g}{\hookrightarrow} & int(X^3) \subset X^3 \\ & f \searrow & \downarrow F \\ & & M^3 \end{array}$$

fulfilling the conditions:

1. *f and g are embeddings.*

2. *F is a smooth generic immersion.*
3. *$gK \cap M_2(F) = \emptyset$.*

Then fK can be engulfed in a smooth connected and simply connected sub-manifold Y^3 of M^3.

For the proof of this Dehn-type lemma see ([7], p.433-439).

The hypothesis gives a proper PL embedding $i : M^3 \to X^n$ and a proper surjection $\pi : X^n \to M^3$ such that $\pi \circ i = 1$.

Lemma 2.2 *There exists a triangulation τ_W of W^3 and a subdivision τ_X of X^n such that:*

1. *$i : \tau_W \hookrightarrow \tau_X$ is a simplicial embedding, identifying τ_W to a sub-complex of τ_X.*
2. *τ_X is g.s.c.*
3. *there is some subdivision θ of the 3-dimensional skeleton of τ_X and a map $\lambda : \theta \longrightarrow \tau_W$ such that λ is proper simplicial and non-degenerate, and $\lambda \circ i = id$.*

This lemma does not use the strong connectivity of the 3-skeleton, but only the strong deformation retract properties. Notice that this lemma implies that θ is a enlargement of τ_W, but only when the natural projection map is replaced by λ (so that all the maps become simplicial). The proof will be given in the next section.

It follows that θ is g.s.c. from ([7] Lemma 5.1): thus, there exists a sequence of finite simply connected sub-complexes $Z_0 \subset Z_1 \subset Z_2 \subset ... \subset \theta$ exhausting θ. Set $\lambda^\infty = \lambda \mid_\theta$, $\lambda^j = \lambda \mid_{Z_j}$, and also $\Psi_j = \Psi(\lambda^j)$, $\Phi_j = \Phi(\lambda^j)$, $j = 1, 2, ..., \infty$. The equivalence relations Φ and Ψ were introduced in [6] and all the definitions are included in the appendix. Recall that for $j < \infty$ we have $\Phi_j = \Phi_\infty \mid_{Z_j}$, but in general we have only an inclusion $\Psi_j \subset \Psi_\infty \mid_{Z_j}$.

Lemma 2.3 *The equality $\Phi_\infty(\lambda) = \Psi_\infty(\lambda)$ holds.*

Proof of Proposition 2.1 assuming the Lemmas. The conclusion of Proposition B from [7] remains true in our situation, so that for any k, there exists some number $N(k) > k$ fulfilling:

$$\Psi_{N(k)} \mid_{Z_k} = \Phi_k.$$

Fix further a connected compact $K \subset W^3$. Then there is some m for which $\lambda^m Z_m \supset K$ holds, and therefore we can find some (sufficiently large) n satisfying $(\lambda^\infty)^{-1}(\lambda^m Z_m) \subset Z_n$ (both assertions follow from a compactness argument).

If $(x_1, x_2) \in M^2(\lambda^\infty)$ and $x_1 \in i(K)$ then necessarily $x_2 \in Z_n$. Furthermore we have the following diagram of maps:

$$i(K) \subset Z_n/\Phi_n = Z_n/\Psi_N(n) \subset Z_{N(n)}/\Psi_{N(n)} \stackrel{\lambda^{N(n)}}{\to} W^3.$$

Since the map $\lambda^{N(n)}$ is an immersion and no double point of $\lambda^{N(n)}$ can involve Z_n (as a consequence of the relation $\Psi_{N(k)} \mid_{Z_k} = \Phi_k$, which was previously obtained) we derive that

$$K \cap M_2(\lambda^{N(n)}) = \emptyset.$$

From Lemma 3.1 of [7] we have $\pi_1(Z_{N(n)}/\Psi_{N(n)}) = 0$. Therefore the diagram below

$$\begin{array}{ccc} K & \stackrel{g}{\hookrightarrow} & Z_{N(n)}/\Psi_{N(n)} \\ & f \searrow & \downarrow \lambda^{N(n)} \\ & & W^3 \end{array}$$

has all the properties required in the Dehn-type lemma except that $Z_{N(n)}/\Psi_{N(n)}$ is a simplicial complex. But as already noticed in ([7] p.444), we can replace it by a smooth regular neighborhood of $Z_{N(n)}/\Psi_{N(n)}$, generically immersed in W^3. Thus the compact K can be engulfed in a simply connected compact submanifold of W^3. Once we know this for any connected compact it follows automatically for any compact subset of W^3. Therefore W^3 is simply connected at infinity. □

3 The proof of lemma 2.2

By a suitable subdivision of the initial triangulations τ_X^0, τ_W^0 of X and W we can suppose that both $\pi : \tau_X^1 \longrightarrow \tau_W^1$ and $i : \tau_W^1 \hookrightarrow \tau_X^1$ are simplicial. Remark that τ_X^1 is g.s.c. by lemma 5.1. from [7]. It remains to prove that π can be replaced by another map f which is simplicial and whose restriction to the 3-skeleton of some subdivision τ_X^2 is non-degenerate. The image of the latter is some subdivision $\tau_W^2 < \tau_W^1$. Denote $ske^3\tau_X^1$ by t and τ_W^1 by τ, for simplicity.

Notice that W^3 has a non-complete flat Riemannian structure, hence τ_W^1 can be realized as an affine triangulation of W^3 because the geodesics are unique.

Since π is proper then for any (closed) 3-simplex $\sigma \subset \tau$, the preimage $\pi^{-1}(\sigma) = \cup_i \sigma_i$ is a finite union of 3-simplexes of t. We choose an arbitrary simplex σ at the beginning. Among the preimage simplexes σ_i's there is one, which we denote by $i(\sigma)$, such that the restriction of π to $i(\sigma)$ is an isomorphism on the image. Set $V(\sigma)$ for the union of the set of vertices of all σ_i which do not appear as vertices of $i(\sigma)$, and order them arbitrarily $V(\sigma) = \{v_i, i \geq 4\}$. Afterwards we label $\{v_1, ..., v_4\}$ the vertices of $i(\sigma)$.

Choose some set of points (in a generic position) $A^0(\sigma) \subset int(\sigma) \subset W^3$ which are in one-to-one correspondence with $V(\sigma)$. We denote the points of $A^0(\sigma)$ as $\{x_i, i \geq 4\}$ and the vertices of σ by $\{x_1, ..., x_4\}$. We suppose that the projection of v_i is x_i for $i = 1, 2, 3, 4$. Consider now the set $A^k(\sigma)$ of those k-dimensional simplexes whose vertices are from $A^0(\sigma)$, which are realized as affine simplexes in W^3, and are related to the simplexes from $\pi^{-1}(\sigma)$ in the following way:

$$A^k(\sigma) = \{[x_{i_0}, x_{i_1}, ..., x_{i_k}] \subset \sigma; \text{ such that } [v_{i_0}, v_{i_1}, ..., v_{i_k}] \text{ is a simplex in } \pi^{-1}(\sigma)\}.$$

Here $[y_0, y_1, ..., y_k]$ denotes the simplex having the vertices y_i. Notice that the affine simplex $[x_{i_0}, x_{i_1}, ..., x_{i_k}]$ is uniquely determined by its vertices in W^3, because W^3 has an affine structure.

We assume now that all simplexes in τ are sufficiently small to be convex with respect to the affine structure. Let $A_1^*(\sigma)$ be the closure of $A^*(\sigma) \cup \{\sigma\} \cup \emptyset$ with respect to the intersection operation: this means that:

1. once σ_1 and σ_2 are in $A_1^*(\sigma)$, their intersection $\sigma_1 \cap \sigma_2$ must belong to $A_1^*(\sigma)$, too.
2. $A^*(\sigma) \cup \{\sigma\} \cup \emptyset$ is a subset of $A_1^*(\sigma)$.
3. $A_1^*(\sigma)$ is the smallest collection fulfilling the previous two conditions.

Set also $A_{11}^*(\sigma)$ for the closure of $A_1^*(\sigma)$ with respect to the face-boundary operator ∂, which is extended canonically to convex cells. The face-boundary operator associates to a cell c the collection of faces of the boundary. We refine further the complex $A_{11}^*(\sigma)$ to the collection $A_2^*(\sigma)$ with the property that the interiors of the cells form a partition, as follows: set c_i for a maximal set of open cells in $A_{11}^*(\sigma)$, all of them being contained inside some other cell c. Then remove c and add $c - \cup_i c_i$ as a new cell. It is possible that concave cells have been introduced this way. Consider now the map $f : \pi^{-1}(\sigma) \longrightarrow \sigma$, which is the extension by linearity of the map defined on vertices by $f(v_i) = x_i$ for all i. Set $C^*(\sigma) = f^{-1}(A_2^*(\sigma))$. Then $C^*(\sigma)$ is a cellular complex and f is a non-degenerate cellular map.

We wish to pass now to a global picture, from the simplex σ to the whole complex τ. We choose an enumeration of all 3-simplexes of τ, say $\sigma_1, \sigma_2, \sigma_3, ..., \sigma_k, ...$ with the property that, for each compact K, there exists some integer $m = m(K)$ such that $\cup_{i=1}^{m}\sigma_i \supset K$. We built up now, inductively, the global complex associated to this exhaustion. At the beginning (for σ_1) we start with the cell decomposition defined above $A_2^*(\sigma_1)$.

Assume we constructed refinements of $\pi^{-1}(\cup_{i=1}^{n}\sigma_i)$ and of $\cup_{i=1}^{n}\sigma_i$, and a cellular map f replacing the projection π, between the refined cell complexes. We look now for the new vertices which appear in $\pi^{-1}(\sigma_{n+1})$. There are some of the vertices of $\pi^{-1}(\sigma_{n+1})$, namely those which are also vertices of $\pi^{-1}(\cup_{i=1}^{n}\sigma_i)$, which have been taken already into account at the previous stages of the construction. In fact the vertex v has been considered (i.e. at an earlier stage, a value $x = f(v) \in \cup_{i=1}^{n}\sigma_i$ has been associated to v) at the k^{th} step, where k is the smallest integer such that v is the vertex of $\pi^{-1}(\cup_{i=1}^{k}\sigma_i)$. Define therefore $V^0(\sigma_{n+1})$ be the set of vertices of $\pi^{-1}(\cup_{i=1}^{n+1}\sigma_i)$ which have not been considered before. Choose, as in the first step, a set of points $A^0(\sigma_{n+1})$ inside the simplex σ_{n+1} so that the vertices of $i(\sigma_{n+1})$ are in one-to-one correspondence with the vertices of σ_{n+1} and the other points are lying the interior of σ_{n+1}. We assume that the vertices in $V^0(\sigma_{n+1}) - i(\sigma_{n+1})$ are in bijection with the interior points. The restriction of f to vertices can be naturally extended now from $\pi^{-1}(\cup_{i=1}^{n}\sigma_i)$ to $\pi^{-1}(\cup_{i=1}^{n+1}\sigma_i)$, say $f(v_i) = x_i$, for all i. Remark that this procedure is highly non canonical but it is well enough for our purposes.

The global complex B_0^*, which depends on the enumeration we chose, is therefore given by:

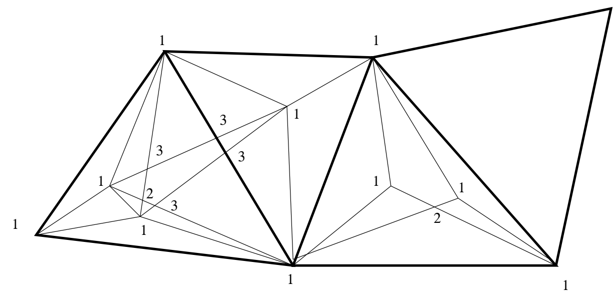
$$B_0^k = \{[x_{i_0}, x_{i_1}, ..., x_{i_k}]; x_{i_j} \in \cup_{j=1}^{n+1} A^0(\sigma_j) \text{ such that } [v_{i_0}, v_{i_1}, ..., v_{i_k}] \text{ is a simplex in } t.\}$$

Here all simplexes in W^3 are affine simplexes. Remark we have specified only the first generation vertices from A^0, not from A_2^0. Consider now the closure B_1^* of B_0^* with respect to the intersection and B_2^* be the closure of B_1^* with respect to the face-boundary operator. An easy remark is that B_2^* is closed also for the intersection. We saw before how to refine B_2^* by adding the complementary of unions of cells (and removing the cells which contain them), in such a way that the formal interiors form a partition: if $x_i \subset y$ are k-cells in B_2^k then we want that the complementary $cl(y - \cup_i x_i)$ be also an union of cells. In the first step we considered such maximal families $\{x_i\}$ inside a fixed cell y, added the complementary, as a new cell, and removed the cell y from our collection. But some of the new cells arising this way, are not convex. Observe that all of them are polyhedra whose edges are geodesics and the faces are flats in W^3. A polyhedron with geodesic edges, and affine faces in an affine manifold can be partitioned into convex polyhedra, possibly introducing new vertices, as intersection among flats spanned by the vertices. These can be lifted upside, and the initial triangulation can be refined to include the new vertices. Eventually the downside cells will be convex. Thus we can suppose, without loss of generality, that there are no vertices to add and the partition has convex components.

We obtained another complex, say B_3^*, which is closed to intersection, to the face-boundary operator and is made of convex cells. Now the map f extends to B_3 in the obvious way.

This completes the induction step, and so we obtain a cellular structure of W^3, which can further be refined to a g.s.c. simplicial decomposition B^*. Now the map f gets a map $f : D^* = f^{-1}(B^*) \longrightarrow B^*$ which is simplicial. The pull-back complex D^* is a cellular complex, finer than t, and we will show that it is a simplicial complex.

Example 3.1 *The vertices we added to our initial triangulations are therefore of 3 generations, as shown in the picture 1. These corresponds to A^0, to $A_1^0 = A_2^0$, and B_1^0. An example of how f looks like is given in picture 4: here τ is the union of 2 simplexes of dimension 2, and t is the 2-skeleton of $\partial[v_1, v_2, x_3, v_5] \cup \partial[v_4, v_2, x_3, v_6] \cup \partial[v_6, v_2, x_3, v_5]$. There are two new vertices of first generation figured in A^* and a new*



1 points of the first generation

2 points of the second generation

3 points of the third generation

Figure 1: The vertices

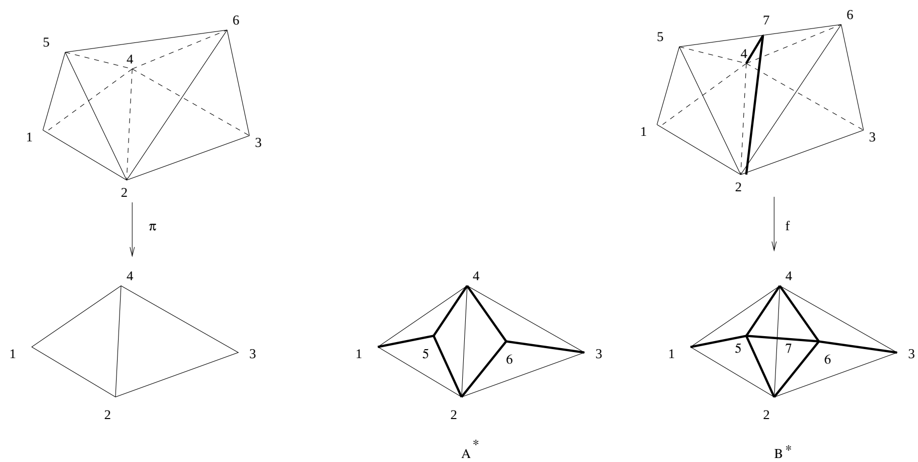


Figure 2: The general picture

vertex x_7 *when we pass to* B^*. *The preimage cellular complex and the modified map* f *corresponds to the cone over the subdivision of the edge* $[v_5, v_6]$.

Lemma 3.1 *The map* f *is non-degenerate and simplicial.*

Proof: These features were achieved directly by construction. It suffices to understand how D^* is obtained from t, and that D^* is indeed a simplicial complex. The new vertices in $D^0 - t^0$ come from intersections points of 2-flats in W. For a generic choice of $A^0(\sigma_j)$ the 2-flats are in general position, and there are only 1-dimensional intersections. Since f was made cellular the local model around a singularity of f is a folding. This means that we have two simplexes σ_1^3 and σ_2^3 in D^* having a common 2-face which are projected down by f onto the union of two 3-simplexes $f\sigma_1^3$ and $f\sigma_2^3$. The last two have a common face and $f\sigma_1^3 \cap f\sigma_2^3 = \sigma$ is a simplex with a new vertex o which is the intersection of three singular lines ox_i, $i = 1, 3$. The double point o has two preimages, $o_i \in \sigma_i^3$ and we have also the preimages of double lines which are $o_i v_j$, $i = 1, 2$ and $j = 1, 3$. We must add to our decomposition the edge ox_4, and this yields a decomposition of $f\sigma_1^3$ into 3 simplexes while $f\sigma_2^3$ is cut into two tetrahedra. The preimage decomposition is a decomposition into

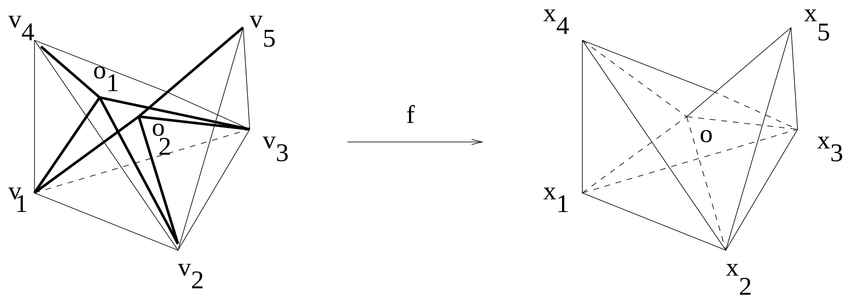


Figure 3: The local picture around a singularity

tetrahedra of $\sigma_1^3 \cup \sigma_2^3$, which is the pull-back of the partition into tetrahedra from downside and it is not necessary to add any other edges or vertices.

Further f is locally an etale map around a non-singular point. It follows that D^* is in fact a simplicial complex finer than t, and f is non-degenerate and simplicial as we wanted. $\Box$

Lemma 3.2 *The induced map $f : X \longrightarrow W^3$ is proper.*

Proof: Observe first that all objects $\bigcup_{\sigma^3 \in \tau} C^*(\sigma)$, $\bigcup_{\sigma^3 \in \tau} A_2^*(\sigma)$ are locally finite. Therefore the new vertices of $\bigcup_{j, \sigma_j^3 \in \tau} A_2^0(\sigma_j)$ are not accumulating in W^3 except at infinity. Since everything takes place in some small convex region we derive that the edges in $\bigcup_{\sigma^3 \in \tau} A_2^1(\sigma)$ (which are viewed as geodesics in W^3) do not have accumulating points either. Notice that the geodesics are unique in the flat structure of W^3. Since the simplexes are affine we derive that no k-simplexes (from those whose vertices are in $\bigcup_{j, \sigma_j^3 \in \tau} A_2^0(\sigma_j)$) have accumulating points.

It remains to look at the edges introduced at the second step. Assume that in the induction process, when we pass from the stage n to $n+1$ we have to add some new edge. The image by π of such an edge $e \in B_2^1$ is either one edge of τ or else a vertex of τ. The second case corresponds to the following situation: we have two vertices v_1 and v_2 having the same image x by π. These two may appear either at the same stage of the enumeration (so that their perturbed images by f will belong to the same simplex), or else in different places. But then the images are sitting inside two simplexes, say σ_1 and σ_2, having the vertex x in common. The first case leads to the following: the images are sitting in σ_1 and σ_2 , such that there exists a 1-dimensional simplex e having one endpoint on σ_1 and the other one on σ_2. The other edges in $B_3^1 - B_2^1$ were added inside a convex cell, in order to complete the partition into a partition with smaller convex cells.

We claim now that the new edges cannot be too long: in fact, by the triangle inequality, the length of a new edge in a compact ball R is at most 3 times the longest (old) edge in that ball. We used compacts because all the choices we made were local, and the upper bound on the edge length is uniform (in [7] the initial triangulation is chosen with simplexes which become smaller and smaller when the distance from a fixed point goes to infinity). This argument shows that the new edges have not accumulating points except at infinity. For a generic choice of $A^0(\sigma_j)$ the affine k-simplexes are in general position. Since the edges are not accumulating somewhere, the k-simplexes are not accumulating either. This proves that f is proper. $\Box$

4 The proof of lemma 2.3

We consider now the central object in this section, namely the canonical diagram

$$\begin{tikzcd} \theta \arrow[r] \arrow[dr, "\lambda^\infty"'] & \theta/\Psi_\infty \arrow[d, "\overline{\lambda^\infty}"] \\ & W^3 \end{tikzcd}$$

The map $\overline{\lambda^\infty}$, which we obtained after factorization, is known to be an immersion from the definition of Ψ.

Lemma 4.1 *The map $\overline{\lambda^\infty}$ is a simplicial isomorphism between θ/Ψ_∞ and τ_W.*

Proof: Consider the sub-complex $i(\tau_W) \subset \theta$. There is an induced map $\iota : i(\tau_W)/\Psi_\infty \longrightarrow \theta/\Psi_\infty$ and we have the following commutative diagram

$$\begin{tikzcd} i(\tau_W)/\Psi_\infty \arrow[r, "\iota"] & \theta/\Psi_\infty \arrow[r, "\overline{\lambda^\infty}"] & \tau_W \\ i(\tau_W) \arrow[u] \arrow[r, hook] & \theta \arrow[u] \arrow[ur, "\lambda^\infty"'] & \\ \tau_W \arrow[u] \arrow[ur, "i"'] & & \end{tikzcd}$$

Then we have the following:
Claim 1: The map $\alpha = \overline{\lambda^\infty} \circ \iota : i(\tau_W)/\Psi_\infty \longrightarrow \tau_W$ is a simplicial isomorphism.
Proof: In fact the map α is

- *surjective* since $\alpha(i(\tau_W)/\Psi_\infty) = \overline{\lambda^\infty} \circ i(\tau_W) = \lambda^\infty \circ i(\tau_W) = \tau_W$.
- *simplicial* as a composition of simplicial maps.
- an *immersion* because $\Psi_\infty(\lambda^\infty \circ i) \subset \Psi_\infty$; this may be rephrased by saying that, once we kill all the singularities, then a fortiori the singularities lying in $i(\tau_W)$ are killed.
- *injective* because the composition $\alpha \circ \beta = id$, where β is the vertical map in the diagram going from τ_W to $i(\tau_W)/\Psi_\infty$. $\square$

Claim 2: Consider the simplicial complex (or cellulation) τ which has a strongly connected 3-skeleton. Assume that we pass from τ to another complex τ' by using one of the following transformations:

1. by subdivisions (or respectively, by a proper family of bisections).
2. we replace τ by $ske^3\tau$.
3. assume that f is a non-degenerate simplicial map, and $\tau' = \tau/\Psi(f)$.

Then τ' has strongly connected 3-skeleton, too.
Proof: Obvious.$\square$
Claim 3: The map $\iota : i(\tau_W)/\Psi_\infty \longrightarrow \theta/\Psi_\infty$ is surjective.
Proof: Assume the contrary holds. Then, for some 3-simplex $\sigma \subset \theta/\Psi_\infty$ we will have $int(\sigma) \cap Im(\iota) = \emptyset$. But we know that θ is strongly connected henceforth θ/Ψ_∞ is strongly connected, so that any two 3-simplexes can be joined by a continuous chain of 3-simplexes. This follows from the previous claim. Notice that this is the only place where the third condition in the definition of the enlargement is used. It follows that there exists some σ with $int(\sigma) \cap Im(\iota) = \emptyset \neq \sigma \cap Im(\iota)$. But we have seen above that $\overline{\lambda^\infty}(\iota \circ i(\tau_W)/\Psi_\infty) = \tau_W$, so that any point $z \in \partial\sigma \cap Im(\iota)$ would be singular for $\overline{\lambda^\infty}$. But this is a contradiction because $\overline{\lambda^\infty}$ is an immersion. $\square$

Now ι is obviously injective hence $\overline{\lambda^{\infty}}$ is injective so that it is an isomorphism. This ends the proof of the lemma 4.1. □

The final argument is by now standard (see [7]): We have two bijections $\theta/\Phi_{\infty} \stackrel{\sim}{\to} \tau_W$ (from the definition the quotient by Φ_{∞} is the image) and $\theta/\Psi_{\infty} \stackrel{\sim}{\to} \tau_W$. But we have also an inclusion among the two relations which induces a map $\theta/\Psi_{\infty} \stackrel{\sim}{\to} \theta/\Phi_{\infty}$, hence $\Phi_{\infty} = \Psi_{\infty}$. □

5 Appendix: the Φ/Ψ-theory

For the sake of completeness we recall here some of the basic tools of this paper, which were originally introduced and used by Poénaru in [6, 7].

Let $f : P \longrightarrow M^3$ be a non-degenerate simplicial map between the locally finite simplicial complex P and the 3-manifold M. The *equivalence relation defined by* f is $\Phi(f) \subset P \times P$ given by

$$(x, y) \in \Phi(f) \text{ iff } fx = fy.$$

It is clear that $P/\Phi(f)$ is just the image fP.

The other relation, $\Psi(f)$ is introduced in order to see whether it is possible to exhaust all singularities of fP by folding maps, and it is also called the equivalence relation which is commanded by the singularities of f. A *folding* map corresponds to the following situation: if $x \in \sigma_1$, and $y \in \sigma_2$ are two points of P lying on the simplexes σ_i of same dimension, if $fx = fy$ and $f\sigma_1 = f\sigma_2$ then we wish to identify firstly $f\sigma_1$ to $f\sigma_2$. When we pass to such a quotient (by a folding) the induced map remains simplicial.

The equivalence relation $\Psi(f) \subset \Phi(f)$ is completely characterized by the following two properties:

- If $\bar{f}$ denotes the induced map $P/\Psi(f) \longrightarrow M^3$ then, $\bar{f}$ is an immersion (i.e. it has no singularities). The point z is singular for f if the restriction of f to the star of z is not immersive. Alternatively, there exist two distinct simplexes σ_1 and σ_2 such that $z \in \sigma_1 \cap \sigma_2$ and $f(\sigma_1) = f(\sigma_2)$.
- There is no $R \subset \Phi(f)$, equivalence relation, smaller than $\Psi(f)$ having the first property. Thus, $\Psi(f)$ is the smallest equivalence relation compatible with f which kills all the singularities.

Furthermore the projection map $\pi : P \longrightarrow P/\Psi(f)$ induces a surjection on fundamental groups $\pi_* : \pi_1(P) \longrightarrow \pi_1(P/\Psi(f))$. In particular if P is simply connected then $P/\Psi(f)$ is simply connected too. Remark that also the strongly connectivity of the 3-skeleton is preserved when passing from P to $P/\Psi(f)$.

Roughly speaking, the construction of $P/\Psi(f)$ is given by considering the quotients, obtained recurrently, by all foldings commanded by the singular points of f. In this way all singularities will disappear, one after the other, and no new others are created. Specifically, let z be a singular point and σ_i two simplexes containing z, having the same dimension and the same image by f. Consider the quotient P' of P, obtained by identifying σ_1 to σ_2. The map f induces a simplicial non-degenerate map $f' : P' \longrightarrow M^3$. If f' is not an immersion it has a singular point, say $z' \in P'$, and therefore some simplexes σ_i', as above. We consider next the quotient P'' of P' commanded by the singular point z' and so on. If P is a finite simplicial complex this process stops when we get an immersion $f^{(n)} : P^{(n)} \longrightarrow M^3$. The quotient $P^{(n)}$ is in this case $P/\Psi(f)$. If P is not finite, we need a transfinite recurrence to construct the analogous immersion.

References

[1] T.A Chapman. Lectures on Hilbert cube manifolds. *CBMS Series in Mathematics*, no.28, 1976.

[2] L. Funar and T.L. Thickstun. On open 3-manifolds proper homotopy equivalent to geometrically simply connected polyhedra. preprint, 1998.

[3] B. Mazur. A note on some contractible 4-manifolds. *Ann. of Math.*, 73:221-228, 1961.

[4] B. Mazur. Differential topology from the point of view of simple homotopy theory. *Publ.IHES*, 15:1–93, 1963.

[5] V. Poénaru. La décomposition de l'hypercube en produit topologique. *Bull.Soc.Math.France*, 88:113-129, 1960.

[6] V. Poénaru. On the equivalence relation forced by the singularities of a non-degenerate simplicial map. *Duke Math. J.*, 63:421–429, 1991.

[7] V. Poénaru. Killing handles of index one stably and π_1^∞. *Duke Math. J.*, 63:431–447, 1991.

[8] V. Poénaru. A general finiteness theorem in geometric group theory. *Prépublication Orsay*, 92-33, 1992.

[9] V. Poénaru. Representations of open simply-connected 3-manifolds; a finiteness result. *Prépublication Orsay*, 92-69, 1992.

[10] V. Poénaru. π_1^∞ and infinite simple homotopy type in dimension three. A research announcement. *Preprint IHES*, 95/6, 1995.