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HOMOLOGY OF $P(w_0, w_1, w_2)$

BY

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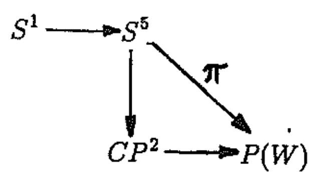
Let $w = (w_0, \ldots, w_n)$ be a set of positive integers and denote by S the polynominal ring $\mathbf{C}[x_0, \ldots, x_n]$ graded by $\deg x_i = w_i$, $i = \overline{0, n}$. Then the projective variety Proj $(S) = P(w)$ is called the weighted projective space of specified weights. For general w, $P(w)$ is a singular space. Its singularities are quotient ones, whose germs are isomorphic to $(\mathbf{C}^n/H, 0)$ where $H \subset$ GL (n, C) is a small abelian group, and whose corresponding links are generalized lens spaces [2, 3, 4]. It is known that the rational cohomology groups of $P(w)$ and $\mathbf{C}P^n$ agree [6] as graded abelian groups.

Our main result is that $H_*(P(w))$ is torsion free for $n = 2$. This seems to have some significance when we folow the lines developed in[1] in studying the quasi -homogeneous singularities. We shall give a cell decomposition for $P(w)$ in same manner as Dold [5] has done for lens spaces. Then an elementary combinatorial computation will answer our question.

We restrict our attention to the case $n = 2$ and remember that an alternative way to describe $P(w)$ is as a quotient of CP^2 under the action of $G = Z/w_0Z \times \ldots \times Z/w_2Z$

$$(k_0, k_1, k_2)(z_0, z_1, z_2) = (\xi_{w_0}^{k_0}, \xi_{w_1}^{k_1}, \xi_{w_2}^{k_2}),$$

where ξ_j is a primitive root of unity of order j. Now consider the Hopf fibration



and set π for the composition of the canonical projections. Our aim is to give a G - equivariant cell decomposition of $\mathbf{C}P^2$. We shall define a decomposition of S^5 satisfying :

i) the decomposition is G-equivariant;
ii) any set of the decomposition is stable with respect to the action of S^1 coming from the Hopf fibration;

iii) it induces a cell decomposition of $\mathbf{C}P^2$, hence any set of the decomposition is topologically $S^1 \times$ cell.

Now, below, P^j consists of sets of dimension $j+1$. Since the action of S^1 is fixed point free P^{-1} must be void. Otherwise

(1) $P^0 = \{e_j^0, j \in \{0,1,2\}\}$ where $e_j^0 = \{z_i = 0 \text{ for } i \neq j\}$

(2) $P^1 = \{e_r^1(j), j \in \{0,1,2\}, r \in \{0,1,\dots A-1\}\}$ where $A = \ell\text{cm}\,(w_0, w_1, w_2)$ *i.e.* A is the least common multiple of the weights, and $e_r^1(2) = \{\arg z_0 - \arg z_1 = 2\pi r/A, z_2 = 0\}$, the others being obtained by cyclic permutations

(3) $P^2 = \{b_{rpq}, c_r(j); r,p,q \in \{0,1,\dots A-1\}, r+p+q = 0(\bmod A), j \in \{0,1,2\}\}$ where $b_{rpq} = \{\arg z_0 - \arg z_1 = 2\pi r/A, \arg z_1 - \arg z_2 = 2\pi p/A, \arg z_2 - \arg z_0 = 2\pi q/A\}$ $c_r(2) = \{\arg z_0 - \arg z_1 \in (2\pi r/A, 2\pi(r+1)/A), z_2 = 0\}$, and the others are obtained by cyclic permutations

(4) $P^3 = \{e_{rpq}^3(j), r,p,q \in \{0,1,\dots,A-1\}, r+p+q+1 = 0(\bmod A), j \in \{0,1,2,\}\}$, where $e_{rpq}^3 = \{\arg z_0 - \arg z_1 = 2\pi r/A, \arg z_1 - \arg z_2 \in \in (2\pi p/A, 2\pi(p+1)/A), \arg z_2 - \arg z_0 \in (2\pi q/A, 2\pi(q+1)/A)\}$ and the others are obtained by cyclic permutations

(5) $P^4 = \{e_{rpq}^4, r,p,q \in \{0,1,\dots,A-1\}, r+p+q(\bmod A) \in \{-1,-2\}\}$ where
$e_{rpq}^4 = \{\arg z_0 - \arg z_1 \in (2\pi r/A, 2\pi(r+1)/A),$
$\arg z_1 - \arg z_2 \in (2\pi p/A, 2\pi(p+1)/A),$
$\arg z_2 - \arg z_0 \in (2\pi q/A, 2\pi(q+1)/A)\}.$

Let now make some notations

$$A/w_i = A_i, i \in \{0,1,2\} \quad d_i = \ell\text{cm}\,(A_{i+1}, A_{i+2})$$

(cyclic numerotated),

$$d = \ell\text{cm}\,(a_0, A_1, A_2),$$
$$N = A^3/Bd, B = w_0 w_1 w_2,$$
$$P = A^2/w_0 w_1 d_2 + A^2/w_1 w_2 d_0 + A^2/w_2 w_0 d_1.$$

We consider the following actions of G on $\mathbf{Z}/A\mathbf{Z}$

(i) $(k,1,m)(r) = r + kA_0 - 1A_1,$

(ii) $(k,1,m)(p) = p + 1A_1 - mA_2,$

(iii) $(k,1,m)(q) = q + mA_2 - kA_0.$

The orbits of theses actions are denoted by $(r)_2$, $(p)_1$, $(q)_0$ eventually the indices omitted, if are understood.

Let consider the action of G on $(\mathbf{Z}/A\mathbf{Z})^3$ obtained by summing the preceding ones i. e.

(iv) $(k,1,m)(r,p,q) = (r + kA_0 - 1A_1, p + 1A_1 - mA_2, q + mA_2 - kA_0)$ whose orbits we denote by (r,p,q). If $S_m = \{(r,p,q); r+p+q = -m(\bmod A)\}$, then S_m are G–invariant. If we denote by K^i the projection under π of P^i we obtain a cell decomposition of $P(w)$, namely

$$K^0 = \{e_i^0, i \in 0,1,2\}, \text{ where } e_i^0 = \pi(\tilde{e}_i^0);$$

$$K^1 = \{e^1_{(r)_i}(i), i \in \{0,1,2\}\}, \text{ where } e^1_{(r)_i}(i) = \pi(\tilde{e}_r(i));$$

$$K^2 = \{c_{(r)_0}(i), b_{(rpq)}, i \in \{0,1,2\}, (r,p,q) \in S_0\}, \text{ where}$$
$$c_{(r)_i}(i) = \pi(c_r(i)) \text{ and } b_{(rpq)} = \pi(b_{rpq});$$

$$K^3 = \{e^3_{(rpq)}(i), i \in \{0,1,2\}, (r,p,q) \in\} S_1\}, \text{ where } e^3_{(rpq)}(i) = \pi(e^3_{rpq}(i));$$

$$K^4 = \{e^4_{(rpq)}, (r,p,q)\ in S_1 \cup S_2\}, \text{ where } e^4_{(rpq)} = \pi(e^4_{rpq}).$$

The map π being cellular we can compute how acts on the generators of chain groups

(6) $$\partial e^0_i = 0,$$

(7) $$\partial e^1(r)_i(i) = e^0_{i-1} - e^0_{i+1},$$

(8) $$\partial c_{(r)_i}(i) = e^1_{(r)_i}(i) - e^1_{(r+1)_i}(i),$$

(9) $$\partial b_{(rpq)} = e^1_{(r)_2}(2) + e^1_{(p)_0}(0) + e^1_{(q)_1}(1)$$

with the mention that the actions $(i - iv)$ are compatible i.e. if

$$(rpq) = (\tilde{r}\tilde{p}\tilde{q}) \text{ then } (r)_2 = (\tilde{r})_2, (p)_0 = (\tilde{p})_0, (q)_1 = (\tilde{q})_1 \text{ so (9)}$$

has sense,

(10) $$\partial e^3_{(rpq)}(2) = c_{(p)_0}(0) - c_{(q)_1}(1) + b_{(rp+1q)} - b_{(rpq+1)}$$

and the others are obtatined by cyclic permutations,

(11) $$\partial e^4_{(rpq)} = e^3_{(rpq)}(1) + e^3_{(rpq)}(2) + e^3_{(rpq)}(0), \quad \text{if } (rpq) \in S_1$$

(12) $$\partial e^4_{(rpq)} = e^3_{(r+1\,pq)}(2) + e^3_{(rp+1\,q)}(0) + e^3_{(rpq+1)}(1) \quad \text{if } (r,p,q) \in S_2.$$

Now let C_j be the group of Z - chains of dimension j determined by the simplicial complex $K = \bigoplus_0^3 K^i$. The graded differential complex $C = \bigoplus_0^3 C_j$ is isomorphic with the following one

$$0 \to Z^{2N} \xrightarrow{D} Z^{3N} \xrightarrow{A} Z^{N+P} \xrightarrow{B} Z^P \xrightarrow{C} Z^3 \to 0$$

where A, B, C, D are suitable matrices, which acts by left composition and correspond to ∂ in the standard basi of C. Now by general arguments $H_0(P(w), \mathbf{Z}) = \mathbf{Z}$, $\pi_1(P(w)) = 0$ so $H_1(P(w), \mathbf{Z}) = 0$, and $H_4(P(w)) = \mathbf{Z}$. We need only look at H_2 and H_3. For a $p \times q$ matrix with integer coefficients we denote by $J(m) = \min \{a; a \in \mathbf{Z}^+$ and there exists a $k \times k$ submatrix W, where $k =$ rank m such that $\det W = a\}$.

Lemma. *We have the followings*

$$rank\ B = P - 2,$$

$$dim_Q\ ker\ (B + f) \otimes_{\mathbf{Z}} \mathbf{Q} = N + 1,$$

$$rank\ A = N + 1,$$

$$J(A) = 1,$$

$$rank\ D = 2N - 1,$$

$$j(D) = 1,$$

where $f: Z^{N+P} \to Z$ *is induced by*

$$f\left(\sum_{(rpq)} n_{(rpq)} b_{(rpq)} + \sum_{i,(r)_i} v_{(r)_i}, ic_{(r)_i}(i)\right) = \sum_{i,(r)_i} v_{(r)_i}.$$

Observe that if lemma is proved then $\text{im}\, A \otimes_{\mathbf{Z}} \mathbf{Q} \subset \ker\,(B + f) \otimes_{\mathbf{Z}} \mathbf{Q}$ and because have the same dimension, these spaces coincide. Also if A_0 is a $N+1 \times N+1$ minor of A such that $\det A_0 = J(A) = 1$, then A_0 is $\mathbf{Z}$-invertible and means of a_0^{-1} we obtain an isomorphism between $\ker\,(B + f)$ and $\text{im}\, A$; but f factors to an isomorphism f': $\ker B/\ \text{im}\, A \to \mathbf{Z}$, because $\text{rank}\,(B+f) = \text{rank}\, B + 1$. Thus

$$H_2(P(w), \mathbf{Z}) = \mathbf{Z}$$

and similar calculus shows that

$$H_3(P(w), \mathbf{Z}) = 0.$$

Hence it follows

Theorem. *The weighted projective spaces of dimension 2 have the same integer homology to those of* $\mathbf{C}P^2 = P(1,1,1)$.

Now the proof of the lemma is a consequence of the combinatorial description of the matrices A, B, D. So we have

$$A = \begin{bmatrix} T_2 & -T_2 & 0 \\ T_1 & 0 & T_1 \\ 0 & T_3 & -T_3 \\ I_N - Q_1 & I_N - Q_2 & I_N - Q_3 \end{bmatrix}$$

where T_1 is the $\sharp(\mathbf{Z}/A\mathbf{Z})/G(i) \times N$ matrix whose entries are

$$t_{(r)_2,(uvw)} = \begin{cases} 1, & \text{if there exists } p, q, s \text{ with } (p)_2 = (r)_2 \text{ and } (pqs) = (uvw) \\ 0, & \text{elsewhere,} \end{cases}$$

and T_2, T_3 are deduced by analogy from this, I_N is the $N \times N$ matrix and Q_1, Q_2, Q_3 are circulant matrices corresponding to the permutations of $1, 2, \ldots, N$ induced by

$$(r + 1pq) \xrightarrow{q_1} (rp + 1q), (rp + 1q) \xrightarrow{q_2} (rpq + 1), (rpq + 1) \xrightarrow{q_3} (r + 1pq),$$

which can be written as $(Q_1)_{uv} = \partial_{u,q_i}(v)$. From arithmetic considerations order $(q_i) = N$ and by bordering the minor $I_N - Q_2$ by one stratum is obtained a $N + 1 \times N + 1$ matrix with unit determinant so rank $A \geq N + 1$. Looking to B is easy to see that

$$B = \begin{bmatrix} T_1 & K_1 & 0 & 0 \\ T_2 & 0 & K_2 & 0 \\ T_3 & 0 & 0 & K_3 \end{bmatrix}$$

where K_1 is a $\sharp(\mathbf{Z}/A\mathbf{Z})/G(i)$ square matrix

$$K_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \ldots & 0 & 0 \\ . & . & . & . & . & \ldots & . & . \\ -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

and similarly for K_2, K_3. Then rank $B = P - 2$ and because $H_2(P(w), \mathbf{Q}) = \mathbf{Q}$ we have im $A \otimes_{\mathbf{Z}} \mathbf{Q} \subset$ ker $(B + f) \otimes_{\mathbf{Z}} \mathbf{Q}$ so rank $A = N + 1$ and $J(A) = 1$.

Also D as the form

$$D = \begin{bmatrix} I_N & -R_1 \\ I_N & -R_2 \\ I_N & -R_3 \end{bmatrix}$$

where R_1, R_2, R_3 are the square matrices corresponding to the following permutations of $1, 2, \ldots, N$

$$S_2/G \ni (rpq) \xrightarrow{t_1} (r - 1pq) \in S_1/G$$

and the analoug ones. We have order $(t_i) = N$ and then the $2N - 1 \times 2N - 1$ principal minor of D is of determinant 1. Because $H_3(P(w), \mathbf{Q}) = 0$ so im $D \otimes_{\mathbf{Z}} \mathbf{Q} \subset$ ker $A \otimes_{\mathbf{Z}} \mathbf{Q}$ then $J(D) = 1$ which finishes the proof of the lemma.

Remark. Using other methods the author has proved that in fact theorem holds in any dimension. This will be explained in a further paper.

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