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GENERALIZED AREA IN MINKOWSKIAN SPACES

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It is only a one way to talk about the length of a curve in a Banach space, while the concept of area (especially the one given by geometric methods) is a subject not exhausted even in Euclidean space. See [4], [11].

For $\| \ \|$ a norm on $\mathbb{R}^n$ we denote by K the closed unit ball and by S the unit sphere associated to $\| \ \|$; K is a symmetric convex body and the norm $\| \ \|$ is related to K by $\|X\| = \inf\{t \geq 0;\ X \in tK\}$.

Let $\mathcal{M}$ be the class of all hypersurfaces M in $\mathbb{R}^n$ having area denoted by a(M) with respect to the Lebesgue measure on $\mathbb{R}^n$.

Our purpose is to give an intrinsic definition for area a(M,K) of a hypersurface $M \in \mathcal{M}$ with respect to an arbitrary fixed norm $\| \ \|$ on $\mathbb{R}^n$. We have a (M,K) = a(M) in the Euclidean case. We shall call a (M,K) the K-area of M.

First for a polyhedral hypersurface T having T_i as (n-1) dimensional faces we set

$$(1)\quad a(T,K) = W_{n-1} \sum_i \frac{a(T_i)}{a(K(T_i))}$$

where $K(T_i)$ is the central section of K, parallel with T_i, and W_{n-1} is the volume of unit ball in Euclidean $\mathbb{R}^{n-1}$.

For a C^1 - hypersurface M having volume form $d\sigma$, we denote by n: M $\to \mathbb{R}$ the function defined by

$n(x) = a(K(T_x M))$

where $T_x M$ is the $(n-1)$ - plane tangent to M. Then we define

$$(2) \quad a(M,K) = W_{n-1} \int \frac{d\sigma}{n} .$$

We remark that formulas (1) and (2) agree for M a polyhedral hypersurface. In fact,

$$a(M,K) = \sum_i a(M_i, K)$$

for every decomposition $M = \bigcup M_i$ into smooth pieces M_i such that $a(M_i \cap M_j) = 0$ for $i \neq j$.

We say that two smooth hypersurfaces M_0 and M_1 are ε - neighbours ($\varepsilon \in \mathbb{R}_+^*$) if there exists an application $\Phi : R^n \times [0,1] \to R^n$ (called ambient isotopy), having the following properties:

$AI_1)$ $\Phi(*,0) = id_{\mathbb{R}^n}$;

$AI_2)$ $\Phi(*,1)$ is a bijection of M_0 and M_1;

$AI_3)$ the length of $\Phi(*,t): [0,1] \to \mathbb{R}^n$ is less than ε ;

$AI_4)$ $\| I - d_x \Phi(x,t) \| < \varepsilon$, $\forall x \in M_0$, $t \in [0,1]$.

The concept of ε - neighbours can be extended to the case of piecewise smooth hypersurfaces by imposing for the ambient isotopy Φ to be of class C^1 in a neighbourhood of $(M_0 \cup M_1) \times [0,1]$ and $AI_4)$ to work for almost all points $(x,t) \in M_0$, $x \in [0,1]$.

The space $\mathcal{H}$ of all piecewise smooth compact hypersurfaces in $\mathbb{R}^n$ is endowed with the topology having as a base the sets

$\{M \mid M$ and M_0 are ε - neighbours $\}$

for $M_0 \in \mathcal{H}$, $\varepsilon \in \mathbb{R}_+^*$. Of course, this topology is associated to a metric δ.

THEOREM. If $P_n \to P$ in $\mathcal{H}$ then

$$a(P_n, K) \to a(P,K).$$

Sketch of proof. We shall show only the fact that given two smooth hypersurfaces H_1 and H_2 that are ε - neighbours then

$$|a(H_1, K) - a(H_2,K)| < C\varepsilon$$

for C a real number depending on H_1, K. According to (1) and (2) (eventually cutting H_j into small pieces) we have to prove that

$$(3) \qquad \left| \int_{H_1} \frac{d\sigma_1}{n_1} - \int_{H_2} \frac{d\sigma_2}{n_2} \right| < C\varepsilon.$$

From hypothesis

$$(4) \qquad | n_1(x) - n_2(\varphi(x,1))| < C_1\varepsilon.$$

If $N_{1,x}$ is the unit normal at H_1 in x, if defined, then the condition (AI_4) implies

$$(5) \quad \langle N_{1,x}, N_2, \varphi(x,1) \rangle < \varepsilon.$$

Because the n_i's are continuous and $a(x^\perp \cap K)$ is also continuous (as a function $S^{n-1} \to R^+$), then (3) holds. From (AI_3), (AI_4), and (5) we obtain

$$(6) \qquad \left| \int_{H_1} d\sigma_1 - \int_{H_2} d\sigma_2 \right| < C_2\varepsilon$$

(H_i are compact hypersurfaces). Because $n_i(x) > C_3 > 0$ the inequalities (4) and (6) give (3). The theorem will follow because P_n and P are ε_n- neighbours where $\varepsilon_n \to 0$. We observe that a convex hypersurface is almost smooth, and two ε- neighbours convex hypersurfaces are $c\varepsilon$ - neighbours for a suitable constant c depending only of the geometry of the hypersurfaces. Consequently:

COROLLARY. If a sequence of convex hypersurfaces P_n is simply convergent to a convex hypersurface P then

$$a(P_n, K) \to a(P, K).$$

Upon a theorem of Busemann, characterizing the isometries of a finite dimensional Banach space, it follows that the K-area so defined a(M, K) is invariant under the motions of the Minkowski space. This enables us to say that the measure introduced above is natural.

We mention that for higher order measures over hypersurfaces a similar result still holds by giving a structure of metric space $(\mathcal{H}, \delta_p)$ which is compatible with the continuity of them. This process is an analogue of the contruction of Sobolev spaces so we name the spaces $(\mathcal{H}, \delta_p)$ Sobolev spaces associated to a given measure. This will be treated in a further paper.

We see that the function $a(K \cap x^{\perp})$ satisfies for a symmetric convex body K the conditions imposed in Minkowski's problem [6, 8, 9, 10] so there exists a convex symmetric body K^* having the element of area given locally by $a(K \cap x^{\perp})\, d\sigma$, where $d\sigma$ is the area form on S^{n-1}.

Then the standard Cauchy formula [2, 7, 12] enables us to write

$$(7) \quad a(M,K) = \frac{1}{W_{n-1}} \int_{K^*} a(pr_x M, K)\, d\sigma_{K^*}$$

where $pr_x\, M$ is the projection of M onto the tangent plane at x to K. If we take the maximal cross-section D(M.K) = $= \sup a(pr_x M, K)$, and the minimal cross-section W(M,K) = $= \inf a(pr_x M, K)$ then is straight forward:

$$(8) \qquad \frac{a(K^*)}{W_{n-1}} W(M,K) \leq a(M,K) \leq \frac{a(K^*)}{W_{n-1}} D(M,K).$$

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