ANALELE UNIVERSITĂȚII DIN CRAIOVA
SERIA MATEMATICĂ, FIZICĂ-CHIMIE, VOL. XV (1987), 33–38

ITERATIVE PROCESSES FOR Z_2^n

BY VALENTIN BOJU AND LOUIS FUNAR

In the last fifty years the study of iterative processes suffered an important development. Begining with the paper of Moshe Lotan [1] upon now, a lot of articles consecrated to this subject has been published. A retrospective look is given in the expository paper of J.J. te Riele [2] about iteration of number theoretic functions, where it can be found many bibliographical refferences. This great interest in the field of iterative processes motivates our paper.

Let G be a graph with labelled vertices from 1 to n, and the set of edges E. It induces a transformation $l_G : Z_p^n \to Z_p^n$ in the following manner; For $X \in Z_p^n$ let X_i denote the i-th component in the standard basis $e_1, \ldots, e_n$. Then l_G is defined by:

$$(l_G X)_i = \sum_{(i,j) \in E} X_j \tag{1}$$

We consider in our paper that $p = 2$.

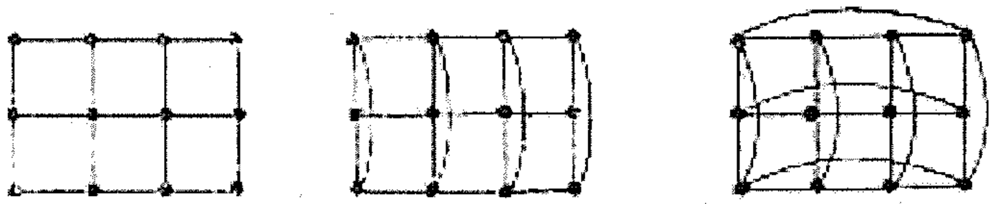
Definition. 1 *The graph G is p-nilpotent if exists k such that for every $X \in Z_p^n$ we have:*

$$\underbrace{l_G \circ l_G \circ \ldots \circ l_G}_{k} (x) = 0 \tag{2}$$

(0 denote the null element of Z_p^n).

We shall give a characterization of 2-nilpotent $\Pi_{n,k}$, $T_{n,k}$, $C_{n,k}$ graphs.

If we paint a table $(n \times k)$ in the plane, on the torus or on the cilinder the corresponding graphs induced by the relations between neighbours are denoted $\Pi_{n,k}$, $T_{n,k}$, $C_{n,k}$ respectively.



$\Pi_{3,4}$ $C_{3,4}$ $T_{3,4}$

Figure

We must study 2-nilpotence for theses graphs.

Received, June 25, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*): 05 C 10; 05 C 50.

Theorem 1. (i) *$T_{n,k}$ is 2-nilpotent iff $T_{n,1}$ and $T_{k,1}$ are 2-nilpotent.*
(ii) *$C_{n,k}$ is 2-nilpotent iff $\Pi_{n,1}$ and $T_{k,1}$ are 2-nilpotent.*
(iii) *$\Pi_{n,k}$ is 2-nilpotent iff $\Pi_{n,1}$ and $\Pi_{k,1}$ are 2-nilpotent.*

Proof. We identify $Z_2^{n\cdot k}$ with the set of matrices $M_{n,k}(Z_2)$, and denote with same letters $t_{T_{n,k}}, t_{C_{n,k}}, t_{\Pi_{n,k}}$ the induced transformations. Let :

$$E_n = \left|\begin{array}{l} 010 \;\ldots\; 0 \\ 101 \;\ldots\; 0 \\ 0101\ldots\; 0 \\ \cdot \\ \cdot \\ \cdot \\ 0000\ldots101 \\ 00 \;\ldots 010 \end{array}\right| , \quad E_n \in M_{n,n}(Z_2)$$

$$D_n = \left|\begin{array}{lll} 0100 & \ldots & 001 \\ 1010 & \ldots & 000 \\ 0101 & \ldots & 000 \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \\ 0000 & \ldots & 101 \\ 1000 & \ldots & 010 \end{array}\right| , \quad D_n \in M_{n,n}(Z_2)$$

By direct computation it follows ;

$$
\begin{array}{lll}
 & t_{\Pi_{n,k}}X \equiv E_nX + XE_k & (M_{n,k}(Z_2)) \\
(6) & t_{C_{n,k}}X \equiv E_nX + XD_k & (M_{n,k}(Z_2)) \\
 & t_{T_{n,k}}X \equiv D_nX + XD_k & (M_{n,k}(Z_2))
\end{array}
$$

Proposition 1 (i) *$\Pi_{n,k}$ 2-nilpotent iff there exists q such that ;*

(7) $$E_n^{2^q}X \equiv XE_k^{2^q} \text{ for every } X \in M_{n,k}(Z_2)$$

(ii) *$C_{n,k}$ is 2-nilpotent iff there exists q such that*

$$E_n^{2^q}X \equiv XD_k^{2^q} \text{ for every } X \in M_{n,k}(Z_2)$$

(iii) *$T_{n,k}$ is 2-nilpotent iff there exists q such that ;*

$$D_n^{2^q}X \equiv XD_k^{2^q} \text{ for every } X \in M_{n,k}(Z_2).$$

Proof (i). We have :

$t^2_{\Pi_{n,k}}X \equiv E_n(E_nX + XE_k) + (E_nX + XE_k)E_k \equiv E_n^2X + XE_k^2 (M_{n,k}(Z_2))$ and by induction ;

$$t^{2^q}_{\Pi_{n,k}}X \equiv E_n^{2^q}X + XE_k^{2^q}$$

If there exists h such that $t^{h}_{\Pi_{n,k}} X \equiv 0(M_{n,k}(Z_2))$ then, for every q, $2^q > h$

$$t^{2^q}_{\Pi_{n,k}} X \equiv 0(M_{n,k}(Z_2)) \text{ so it follows (7)}$$

Conversely if we have (7) then $t^{2^q}_{\Pi_{n,k}} X \equiv 0(M_{n,k}(Z_2))$ for every $X \in M_{n,k}(Z_2)$ so $\Pi_{n,k}$ is 2-nilpotent.

In the same manner (ii) and (iii) are proved.

Lemma 2. *If $R \in M_{n,n}(Z_2)$, $S \in M_{k,k}(Z_2)$ such that for every*

$$X \in M_{n,k}(Z_2) \text{ we have}$$

$$RX \equiv XS$$

then there exist β such that $R = \beta I_n$, $S = \beta I_k$ (A Schur type lemma [3]).

Proof. Let $R = (r_{ij})_{1 \le i,j \le n}$, $S = (S_{ij})_{1 \le i,j \le k}$

$$\text{Put } X = \begin{Vmatrix} 00 \ldots 0 \\ 00 \ldots 0 \\ 11 \ldots 1 \\ 00 \ldots 0 \\ \vdots \\ 00 \ldots 0 \end{Vmatrix} \Big\} i$$

Then :

$$r_{ii} = S_{11} + s_{21} + \ldots + s_{k1} = s_{12} + s_{22} + \ldots + s_{k2} = \ldots = s_{1k} +$$
$$+ s_{2k} + \ldots + s_{kk} \text{ and } r_{ji} = 0, \text{ if } j \neq i.$$

It follows, that $R = \beta I_n$ if $i \in \{1, \ldots, n\}$. In the same manner $S = \gamma I_k$, and because $RX = XS$ it follows $\beta = \gamma$ which proves lemma.

From Proposition 1 and Lemma 2 it follows;

Proposition 3. (i) *$\Pi_{n,k}$ is 2-nilpotent iff there exists q such that;*

(8) $$E^{2^q}_n = \beta I_n(M_{n,n}(Z_2)), \quad E^{2^q}_k = \beta I_k(M_{k,k}(Z_2))$$

(ii) *$C_{n,k}$ is 2-nilpotent iff there exists q such that*

$$E^{2^q}_n = \beta I_n(M_{n,n}(Z_2)), \quad D^{2^q}_k = \beta I_k(M_{k,k}(Z_2))$$

(iii) *$T_{n,k}$ is 2-nilpotent iff there exists q such that*

$$D^{2^q}_n = \beta I_n(M_{n,n}(Z_2)), \quad D^{2^q}_k = \beta I_k(M_{k,k}(Z_2))$$

From Proposition 3 it results that Theorem 1 holds.

Proposition 4. *$T_{n,1}$ is 2-nilpotent iff $n = 2^d$, $d \in Z_+$.*

Proof. We consider $a \in Z_2^n$ and denote by

$$a^{(i)} = (\underbrace{r_{n,1} 0 \ldots 0 r_{n,1}}_{i}) \qquad (9)$$

We prove by induction that $a_{i+2^{k-1}} + a_{i-2^{k-1}} \equiv a_i^{(2^k)}$ (mod 2) For $k = 1$ this is obvious. Also,

$$a_i^{(2^{k+1})} \equiv (a^{(2^k)})_i^{(2^k)} \equiv a_{i+2^{k-1}}^{(2^k)} + a_{i-2^{k-1}}^{(2^k)}$$

$$a_{i+2^k} + a_i + a_i + a_{i-2^k} \equiv a_{i+2^k} + a_{i-2^k} \pmod 2$$

If $n = 2^d$ then for $k = d$

$$a_i^{(2^d)} \equiv a_{i-2^{d-1}} + a_{i+2^{d-1}} \equiv 0 \pmod 2$$

so :

$$\underbrace{t_{T_{n,1}} \circ \ldots \circ T_{n,1}}_{n=2^d \text{ ways}}(a) = 0$$

for every $a \in M_{n,1}(Z_2)$. Also this follows from [4]. For the converse, let n odd, and k be the smallest integere such that $a_i^{(k)} \equiv 0$ (mod 2), $i \in \{1, \ldots n\}$ so the $a_i^{(k-1)} \equiv 1$ (mod 2) if $k \geq 3$ one deduces that $a_{i+1}^{(k-2)} + a_{i-1}^{(k-2)} \equiv 1$ (mod 2) so :

$$n = \sum_{i=1}^{n} (a_{i+1}^{(k-2)} + a_{i-1}^{(k-2)}) \equiv 2 \sum_{i=1}^{n} a_i \equiv 0 \pmod 2$$

which is false because n is odd. Let now $n = 2^m h$, h odd $h > 1$, and $\tilde{\tilde{a}}_s =$ $= (a_s, a_{s+2^m}, \ldots, a_{s+(h-1)2^m})$. Then

$$\underbrace{(t_{T_{n,1}} \circ \ldots \circ t_{T_{n,1}} a)_s}_{2^{m+1}} = t_{T_{h,1}} \tilde{\tilde{a}}_s$$

since :

$$a_{i-2m}^{(2^{m+1})} \equiv a_i + a_{i+2^{m+1}} \pmod 2$$

Because h is odd, if $\tilde{\tilde{a}}_s \notin \{(0, 0, \ldots, 0), (1, \ldots, 1)\}$ as bellow, it can be proved, $\underbrace{t_{T_{h,1}} \circ \ldots \circ t_{T_{h,1}}}_{m \text{ ways}} \tilde{\tilde{a}}_s \not\equiv 0$ $(M_{h,1}(Z_2))$ for any m, which implies :

$t_{T_{n,1}} \circ \ldots \circ t_{T_{n,1}} a \equiv 0$ $(M_{n,1}(Z_2))$ iff $\tilde{\tilde{a}}_s \in \{(0, 0, \ldots, 0), (1, \ldots, 1)\}$ for every s. So $T_{n,1}$ is 2-nilpotent iff $n = 2^d$.

Proposition 5. *$\Pi_{n,1}$ is 2-nilpotent iff $n = 2^d - 1$, $d \in Z_1^*$.*

Proof : Let $a \in Z_2^n$, $fa \in Z_2^{n+1}$ such that

$$(fa)_i = \begin{cases} a_i, & \text{if } i \leq n \\ 0, & \text{if } i = n+1 \end{cases}$$

if $n + 1 \neq 2^d$, let $n + 1 = 2^d h$, $h > 1$, h odd, and let $a \in Z_2^n$ suchthat

$$a_{n-1-i} = a_i,\ n \geq 3$$

$$a_{2^d} \equiv 1 \pmod 2, \text{ and } a \neq (1, 1, \ldots, 1).$$

From this relations it follows that

$$\Psi^{m}_{n+1,1}(fa) \in f(Z_2^n) \text{ for every } m \text{ and also, } f(\Pi^m_{n,1}a) = \Psi^m_{n+1,1}(fa).$$

But from Proposition 4 because $(fa)_2 d \notin \{(0, 0, \ldots, 0), (1, \ldots, 1)\}$ it follows : $\Psi^m_{n+1,1}(fa) \neq 0$ for any m so $\Pi^m_{n,1}a \neq 0$ for any m. So $n+1$ must be a power of 2, or $n = 2$ in which case it is easy to verify that $\Pi_{n,1}$ is not 2-nilpotent.

Let $\Omega_{k,n}$ be a square matrix of order n with elements a_{ij} defined as follows :

$$\text{If } k < \frac{n-3}{2}:$$

$a_{1,k+1} = a_{1,k+2} = 1$, $a_{2,1} = a_{2,k+3} = 1$, $a_{k+1,2} = a_{k+1,2k+3} = 1$, $a_{k+2,2k+2}$ $a_{k+2,2k+4} = 1$, $a_{k+3,2} = a_{k+3,2k+5} = 1, \ldots, a_{n-k-2,n-2k-4} = a_{n-k-2,n} = 1$, $a_{n-k-1,n-2k-3} = 1$, $a_{n-k,n-2k-2} = a_{n-k,n} = 1, \ldots, a_{n,n-k-2} = a_{n,n-k} = 1$, $a_{ij} = 0$ *for other values of* (i, j).

$$\text{If } k > \frac{n-3}{2},\ \Omega_{k,n} = \Omega^T_{n-k-3,n} \text{ and if } k = \frac{n-3}{2}$$

$a_{i,k+2-i} = a_{i,k+1+i} = 1$, *for* $i \leq k+1$, $a_{i,i-k-1} = a_{i,k+1-i} = 1$, *for* $i \geq k+3$,

$$a_{ij} = 0 \text{ for other values of } (i, j).$$

Then a direct computation give us :

$$\Omega^2_{k,n} \equiv \Omega_{\min(2k+2,\, 2n-2k-8),\, n}(M_{n,n}(Z_2))$$

$$\text{for } n \neq 2k+3 \text{ and } \Omega^2_{k,2k+3} \equiv 0(M_{n,n}(Z_2)).$$

But we have :

$$E^2_n = \Omega_{0,n} \text{ so } E^{2^d}_n = \Omega_{a_d,n}(M_{n,n}(Z_2)), \text{ where } a_1 = 0,\ a_{d+1} = \min(2a_d + 2, 2n - 2a_d - 4).$$

But for $n = 2^d$ we have $a_d(n) = 2^{d-1} - 2$, $n = 2a_d - 3$ which imply

$$E^{2^{d-1}}_n \equiv 0(M_{n,n}(Z_2))$$

so after Proposition 3 $\Pi_{n,1}$ is 2-nilpotent.
Theorem 1 and Propositions 4, 5 give us.

Theorem 2. (i) $\Pi_{n,k}$ *is 2-nilpotent iff* $n = 2^d - 1$, $k = 2^f - 1$, $d, f \in Z_+$,
(ii) $C_{n,k}$ *is 2-nilpotent iff* $n = 2^d - 1$, $k = 2^f$, $d \in Z_+$, $f \in Z_+$.
(iii) $T_{n,k}$ *is 2-nilpotent iff* $n = 2^d$, $k = 2^f$, $d, f \in Z_+$.

Corollary. *Let the sequence* $a_k(n)$, *defined by* $a_1(n) = 0$,

$$a_{k+1}(n) = \min(2a_k(n) + 2,\ 2n - 2a_k(n) - 4)$$

Then there exists k *such that* $a_k(n) = n - 1$ *iff* $n = 2^d - 1$, $d \in Z_+$.

3. Unsolved problems

There are a lot of questions which naturally arise when we study the p-nilpotent graphs.

We enumerate some of them without comments.

Problem 1. For what g, p there exist p-nilpotent graphs of genus g?

Problem 2. If $p \geq 3$, for what n, there exist p-nilpotent graphs with n vertices?

Problem 3. For what m, n the graphs $\Pi_{m,n}$, $C_{m,n}$, $T_{m,n}$ are each p-nilpotent? ($p \geq 3$).

BIBLIOGRAPHY

[1] Moshe Lotan, *A problem in difference sets*, Am. Math Monthly 56 (1949), 535—541.
[2] H.J.J. te Riele, *Iteration of number theoretic functions* Nieuw Archief voor Wiskunde (4), v. 1 (1983), 345—360.
[3] J.-Pierre Serre, *Représentations linéaires des groupes finis*, Hermann Paris, 1971.
[4] M. Burmester, R. Forcade, E. Jacobs, *Circles of numbers*, Glasgow Math. Journal, v. 19 p. 2, July 1978, p. 115—121.

VALENTIN BOJU
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CRAIOVA
AL. I. CUZA, NR. 13
R-1100 CRAIOVA
ROMANIA

LOUIS FUNAR
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BUCHAREST
ACADEMIEI, NR. 14,
R-70109 BUCHAREST
ROMANIA