Mathematical relations between “Deterministic classical Chaos” and “Quantum Chaos”. via Ruelle resonances.

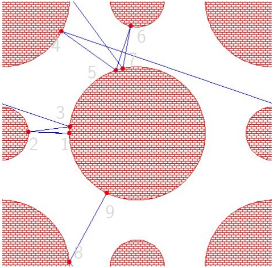
Frédéric Faure, institut Fourier, Grenoble.
Collab.: Masato Tsujii (Kyushu Univ.), Johannes Sjöstrand (Dijon).

July 2013, SFP Marseille.

Deterministic Chaos

- **Deterministic** dynamical system with **sensitivity to initial conditions**.
  $\Longrightarrow$ A typical individual trajectory has **unpredictable behavior**, confused, disordered (= chaotic).

- **Example in Sinaï billiard.** Observe 1 ball: it has deterministic but unpredictable behavior. Why?



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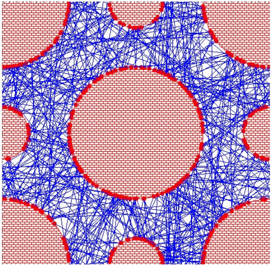

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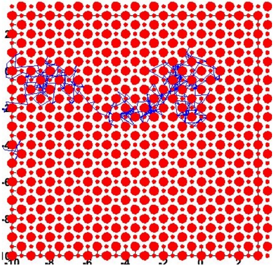

Motion on the cover is like a random walk.

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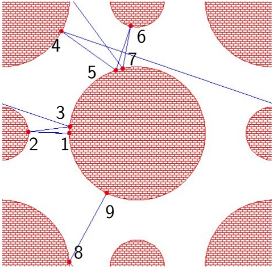

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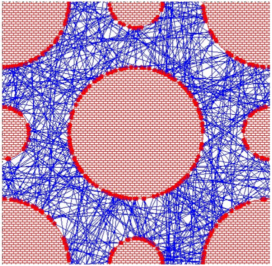

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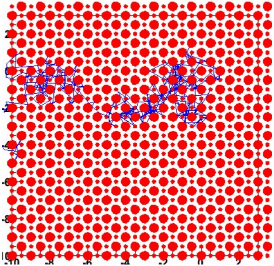

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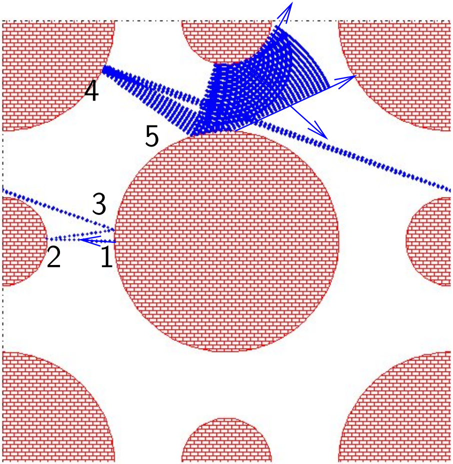
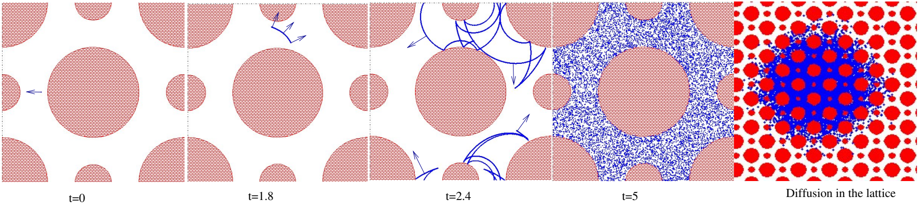
Deterministic Chaos (2)

Heuristic explanation: observe one ball with initial uncertainty $\Delta y = 10^{-4}$: the uncertainty increases exponentially, hence the behavior may differ after a short time.



Deterministic Chaos (3)

Observe $N = 10^4$ independent balls with similar initial conditions $\Delta y = 10^{-4}$: the distribution converges towards equilibrium and diffuse in the lattice. We observe a **predictible** but **irreversible** "effective evolution" of the probability distribution.



See videos 1,2,3.

- **Questions:** describe this "effective evolution of the probability distribution": convergence to equilibrium and fluctuations around it? Explain the limiting Gaussian diffusion? How to express and prove this?

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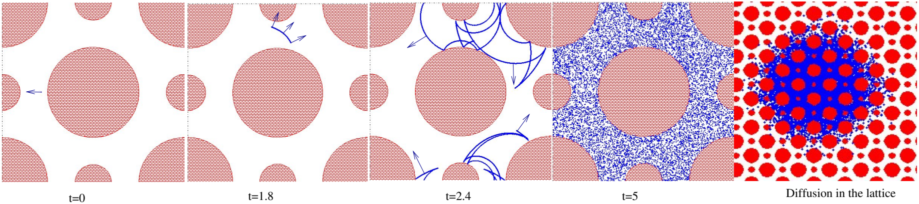

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Mathematical model of deterministic chaos

Definitions

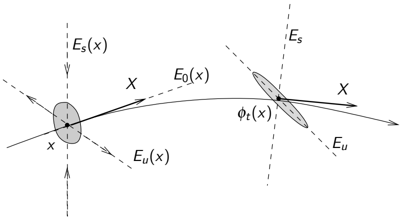
A **contact Anosov dynamics** is a vector field X on a closed manifold M that generates a flow $\phi_t : M \to M$, $t \in \mathbb{R}$ such that

$$TM = \mathbb{R}X \oplus E_{stable} \oplus E_{unstable}$$

$\exists C > 0, \lambda > 0, \forall t \geq 0$,

$$\left\| (D\phi_t)_{|E_s} \right\| \leq Ce^{-\lambda t}, \quad \left\| (D\phi_{-t})_{|E_u} \right\| \leq Ce^{-\lambda t},$$

and the distribution $E_u \oplus E_s$ is maximally non integrable (i.e. contact).



Mathematical model of deterministic chaos (2)

Theorem ([Anosov 1950])

*If $\mathscr{M}$ is a closed Riemannian manifold with negative sectional curvature, the **geodesic flow** on $M = T_1^*\mathscr{M}$ (:the energy shell) is a "**contact Anosov flow**".*

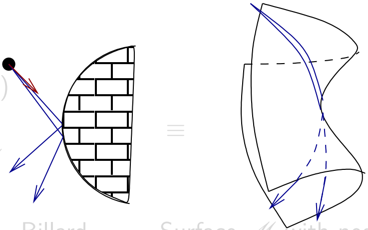
Remark:[Sinaï 1970] a Sinaï billiard is a "non smooth limit" of a geodesic flow:

position $q = (q_1, q_2)$

momentum $p = (p_1, p_2)$

with $|p| = 1$

$x = (q, p) \in M = T_1^*\mathscr{M}$



Billard $\equiv$ Surface $\mathscr{M}$ with negative curvature

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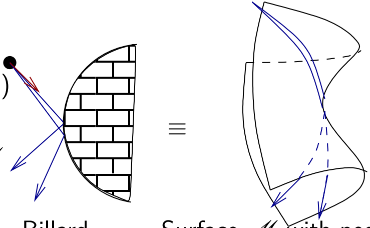
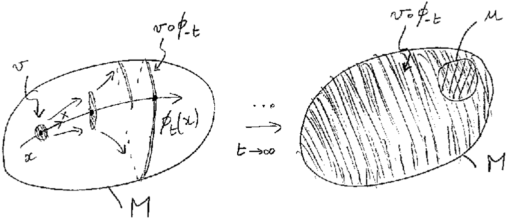


Billard Surface $\mathscr{M}$ with negative curvature

Dynamical correlation functions and mixing

For $u \in C^{\infty}(M)$, $v \in C^{\infty}(M)$, the **correlation function** at time $t \in \mathbb{R}$ is:

$$C_{u,v}(t) := \int_M \overline{u}(x).v(\phi_{-t}(x))\,dx$$



Theorem (Anosov 60, Liverani 04. "Mixing")

$\exists \alpha > 0, \exists C > 0, \forall t \geq 0$

$$\left| \int_M \overline{u}(x).v(\phi_{-t}(x))\,dx - \int \overline{u}(x)\,dx.\int v(x)\,dx \right| \leq C.e^{-\alpha t}$$

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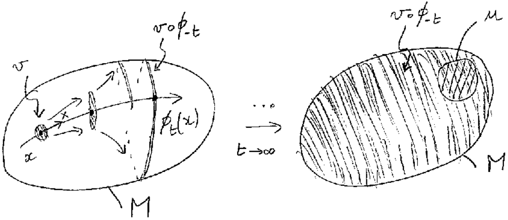

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(*) Central limit theorem

Question: in the Sinaï billiard, show that the distribution of positions $q_1(t)$ diffuse like a Gaussian with width $\simeq D.\sqrt{t}$?

Theorem ([Chernov 90' and others] "C.L.T. for contact Anosov flow")

If $v \in C^\infty(M)$ with $\langle v \rangle_M := \int v(x)\, dx = 0$, let

$$v_t(x) := \int_0^t v(\phi_{-s}(x))\, ds$$

*then $\frac{1}{\sqrt{t}} v_t(x)$ "**distributes as a Gaussian**" w.r.t. dx, i.e. $\forall \chi \in C_0^\infty(\mathbb{R})$,*

$$\int \chi\left(\frac{1}{\sqrt{t}} v_t(x)\right) dx \underset{t\to\infty}{\longrightarrow} C. \int \chi(X)\, e^{-\frac{X^2}{2D}}\, dX$$

*with "**diffusion coef.**" $D = \langle v^2 \rangle_M + 2\int_0^\infty \langle v.v \circ \phi_{-t} \rangle_M\, dt$.*

Example: in the Sinaï billiard, with $v(x) = p_1 = \frac{dq_1}{dt}$, then $v_t(x) := \int_0^t \frac{dq_1}{ds} ds = q_1(x(t)) - q_1(x)$ "diffuses" with this Gaussian law.

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Ruelle resonances

Objective: describe the "**irreversible effective dynamics**", looking for **the discrete spectrum of the "transfer operator".**

Ideas of D. Ruelle, R. Bowen 70' ... P. Cvitanovic, P.Gaspard and others in physics ..(using Markov partitions).

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Definition

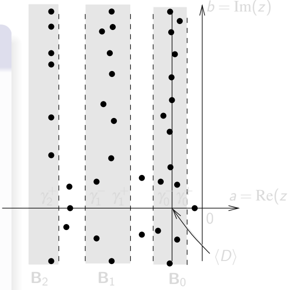
If X is a **contact Anosov vector field**, $V \in C^{\infty}(M)$ is a "**potential**", $u \in C^{\infty}(M)$, $t \in \mathbb{R}$,

$$(\mathscr{L}_t u)(x) := \left(e^{t(-X+V)}u\right)(x) = e^{V_t(x)}u(\phi_{-t}(x)) \quad : \text{transfer operator}$$

Theorem ((1):[Liverani 07, F.-Sjöstrand 08], (2):[F.-Tsujii 13])

1. $\forall C > 0$, $\exists$ ***anisotropic Sobolev space*** $\mathscr{H}_C$, $C^{\infty}(M) \subset \mathscr{H}_C \subset \mathscr{D}'(M)$ *s.t.* $(-X+V)$ *has an intrinsic discrete spectrum on* $\mathrm{Re}(z) > -C$, *called* ***Ruelle resonances.***

   Rem: an eigenvector behaves like $\mathscr{L}_t u = e^{t(a+ib)}u$.
2. *Spectrum in* ***vertical bands****.* $\gamma_0^{\pm} = \lim_{t\to\infty} \max_x / \min_x \left(\frac{1}{t}D_t(x)\right)$, *with "****damping function****":* $D(x) = V(x) - \frac{1}{2}\mathrm{div}X_{E_u}$.



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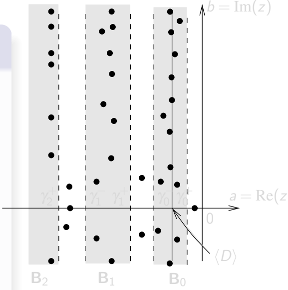

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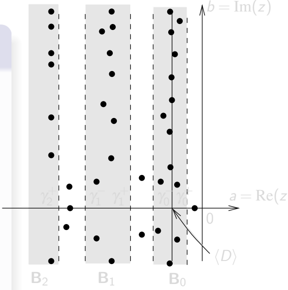

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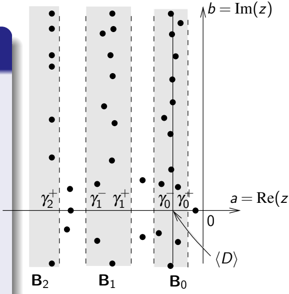

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- The choice $V(x) = \frac{1}{2}\mathrm{div} X_{E_u(x)} > 0$ gives $\gamma_0^{\pm} = 0$: Spectrum accumulates on the axis $i\mathbb{R}$.

Special case of constant negative curvature:
The Poincaré disk $\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}$ has metric $ds^2 = \frac{1}{(1-|z|^2)}(dx^2 + dy^2)$ giving constant negative curvature.
On a compact surface $\Gamma \backslash \mathbb{D}^2$ the geodesic flow is Anosov and contact.
From Selberg formula (1950) and representation theory of $\mathrm{SL}_2\mathbb{R}$, the Ruelle resonances are

$$z_{k,l} = -\frac{1}{2} - k \pm i\sqrt{\mu_l - \frac{1}{4}}, \quad k \geq 0$$

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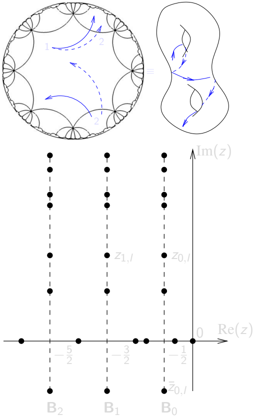

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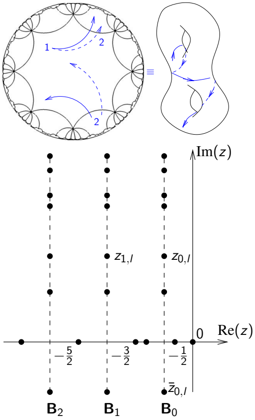

(a)

Consequence for dynamical correlation functions

Theorem ([F.-Tsujii 13])

For a ***general contact Anosov vector field*** X*, with* $\mathscr{L}_t = \exp(t(-X+V))$*, if* $\gamma_1^+ < \gamma_0^-$*, then* $\forall u, v \in C^\infty(M)$*,*

$$\langle v|\mathscr{L}_t u\rangle = \sum_{z_j = a_j + ib_j, a_j \geq \gamma_1^+ + \varepsilon} e^{t(a_j + ib_j)} \langle v| \underbrace{\Pi_j}_{spec.projec.} u\rangle + O\left(e^{\mathrm{Re}(\gamma_1^+ + \varepsilon)t}\right)$$

$$\underset{V=0, t\to\infty}{\longrightarrow} \int \overline{v}(x)\, dx. \int u(x)\, dx + O\left(e^{\mathrm{Re}(z_1)t}\right) \quad \text{:mixing}$$

- So the **fluctuations around the equilibrium** are given by an effective linear dynamics or **"emergent dynamics" governed by the Ruelle spectrum.**
- In case of constant curvature < 0, the emergent dynamics is (conjugated to) the "**damped wave equation**" $\varphi(t) = e^{-t/2} e^{it\sqrt{\Delta - \frac{1}{4}}} \varphi(0)$.
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- So the **fluctuations around the equilibrium** are given by an effective linear dynamics or **"emergent dynamics" governed by the Ruelle spectrum.**
- In case of constant curvature < 0, the emergent dynamics is (conjugated to) the "**damped wave equation**" $\varphi(t) = e^{-t/2} e^{it\sqrt{\Delta - \frac{1}{4}}} \varphi(0)$.
- **Question:** is it true in general that the emergent dynamics is a model of "**quantum chaos**"?

Consequence for dynamical correlation functions

Theorem ([F.-Tsujii 13])

For a ***general contact Anosov vector field*** *X, with $\mathscr{L}_t = \exp(t(-X+V))$, if $\gamma_1^+ < \gamma_0^-$, then $\forall u, v \in C^\infty(M)$,*

$$\langle v|\mathscr{L}_t u\rangle = \sum_{z_j = a_j + ib_j, a_j \geq \gamma_1^+ + \varepsilon} e^{t(a_j + ib_j)} \langle v| \underbrace{\Pi_j}_{spec.projec.} u\rangle + O\left(e^{\mathrm{Re}(\gamma_1^+ + \varepsilon)t}\right)$$

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Formula of Balian-Bloch-Gutzwiller [69]

Start from the **Atiyah-Bott trace formula** (66):

$$\mathrm{Tr}^{\flat}\left(\mathscr{L}_t\right) = \int_M e^{\int^t V}.\delta\left(x - \phi_{-t}(x)\right) dx = \sum_{\gamma:o.p.} |\gamma| \sum_{n\geq 1} \frac{e^{\int^t V}.\delta\left(t - n|\gamma|\right)}{\left|\det\left(1 - D_\gamma \phi\right)\right|}$$

and get:

Theorem ([F.Tsujii 13])

*"**Gutzwiller trace formula**":*

$$\mathrm{Tr}^{\flat}\left(\mathscr{L}_{t|1st\ \ band}\right) = \sum_{z_j = a_j + ib_j, a_j \geq \gamma_1^+ + \varepsilon} e^{t\left(a_j + ib_j\right)}$$

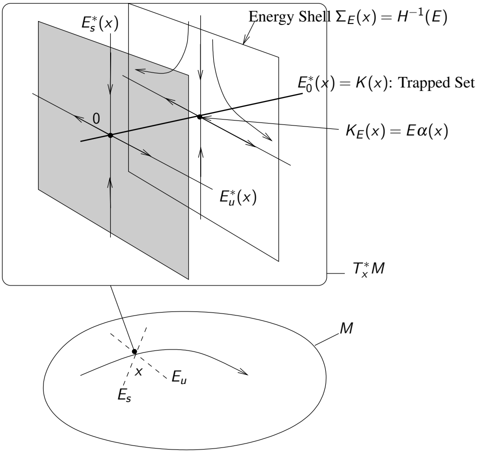
$$= \sum_{\gamma:o.p.} |\gamma| \sum_{n\geq 1} \frac{\delta\left(t - n|\gamma|\right) e^{\int^t D}}{\left|\det\left(1 - D_\gamma \phi\right)\right|^{1/2}} + O\left(e^{\left(\gamma_1^+ + \varepsilon\right)t}\right)$$

which shows that the "emergent dynamics" is governed by "a quantum operator" of "quantum chaos" (it gives Selberg trace formula in cste curvature).

- This has been conjectured in physics in 90' with "semiclassical zeta functions" (Voros, Vattay, ...).

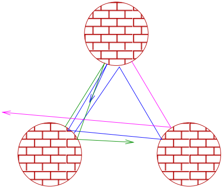
Geometric ideas of the proof

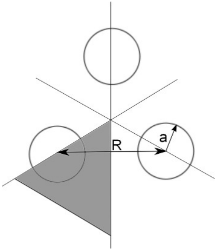
We study the **transfer operator** $\mathscr{L}_t = \exp(t(V - X)) : C^\infty(M) \to C^\infty(M)$ in phase space T^*M (recall that $M = T_1^*\mathscr{M}$), using "**semiclassical analysis**", and "**quantum scattering theory in phase space**" of B. Helffer J.Sjöstrand 86. The resonances states "live on the trapped set K" which is **symplectic**.

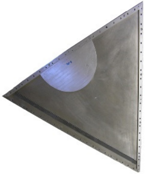


(*) Open systems

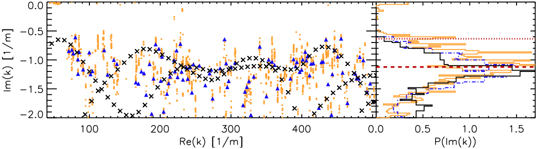
There are similar results for open systems with hyperbolic dynamics on the trapped set, which is a **cantor set**:







Experiments with microwaves [S.Barkhofen T.Weich et al. PRL 2013] . They measure spectrum of “quantum resonances”, $\Delta\varphi = k^2\varphi$, $k \in \mathbb{C}$.



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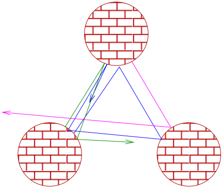

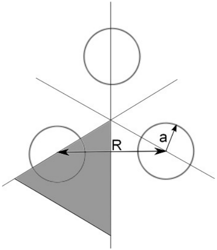

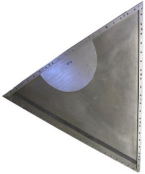

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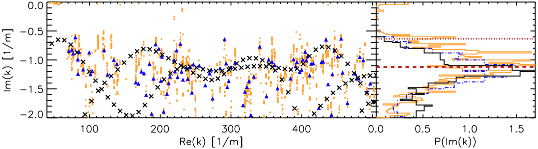
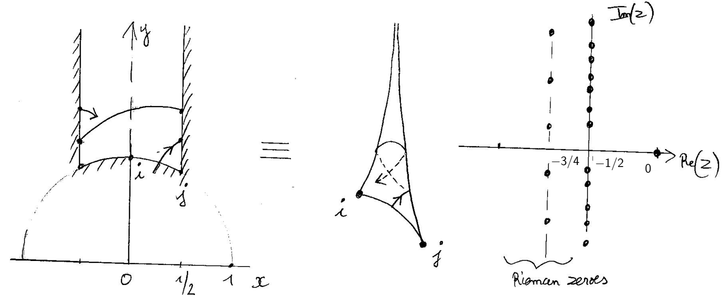


(*) Quantum resonances of the modular surface $\mathscr{S} = \mathrm{SL}_2\mathbb{Z}\backslash\mathbb{H}^2$

Quantum resonances of $\Delta\psi = -z(z+1)\psi$ on the **modular surface** $\mathscr{S} = \mathrm{SL}_2\mathbb{Z}\backslash\mathbb{H}^2$:



The **quantum resonances** are $z_j = \frac{1}{2}s_j - 1$ where s_j are the **"non trivial" zeroes** of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1}$$

The **Riemann hypothesis** (1859) is that $\mathrm{Re}\left(s_j\right) = \frac{1}{2}$ and implies that

$$\forall \varepsilon > 0, \quad \sharp\{\text{primes } p \leq x\} = \int_2^x \frac{du}{\log u} + O\left(x^{\frac{1}{2}+\varepsilon}\right), \quad \text{as } x \to \infty$$

Conclusion and open questions

- For contact Anosov flows, the **fluctuations of probability around the equilibrium are governed by an "effective quantum dynamics"** (quantum chaos emerges). Is there a physical meaning?
- Conjectures of **Random Matrix Theory**, **Unique Quantum ergodicity** and **scars**, for the **Ruelle resonances** and Ruelle spectrum? may be **more tractable**?
- Ruelle spectrum for **more general dynamical systems** (than Anosov)? is there still a **relation with quantum spectrum** or "effective quantum operator"?

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