# Semiclassical approach for the Ruelle-Pollicott spectrum of hyperbolic dynamics

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**Abstract**

Uniformly hyperbolic dynamics (Axiom A) have "sensitivity to initial conditions" and manifest "deterministic chaotic behavior", e.g. mixing, statistical properties etc. In the 70', David Ruelle, Rufus Bowen and others have introduced a functional and spectral approach in order to study these dynamics which consists in describing the evolution not of individual trajectories but of functions, and observing the convergence towards equilibrium in the sense of distribution. This approach has progressed and these last years, it has been shown by V. Baladi, C. Liverani, M. Tsujii and others that this evolution operator ("transfer operator") has a discrete spectrum, called "Ruelle-Pollicott resonances" which describes the effective convergence and fluctuations towards equilibrium.

Due to hyperbolicity, the chaotic dynamics sends the information towards small scales (high Fourier modes) and technically it is convenient to use "semiclassical analysis" which permits to treat fast oscillating functions. More precisely it is appropriate to consider the dynamics lifted in the cotangent space $T^*M$ of the initial manifold $M$ (this is an Hamiltonian flow). We observe that at fixed energy, this lifted dynamics has a relatively compact non-wandering set called the trapped set and that this lifted dynamics on $T^*M$ scatters on this trapped set. Then the existence and properties of the Ruelle-Pollicott spectrum enters in a more general theory of semiclassical analysis developed in the 80' by B. Helffer and J. Sjöstrand called "quantum scattering on phase space".

We will present different models of hyperbolic dynamics and their Ruelle-Pollicott spectrum using this semi-classical approach, in particular the geodesic flow on (non necessary constant) negative curvature surface $\mathcal{M}$. In that case the flow is on $M = T_1^*\mathcal{M}$, the unit cotangent bundle of $\mathcal{M}$. Using the trace formula of Atiyah-Bott, the spectrum is related to the set of periodic orbits.

We will also explain some recent results, that in the case of Contact Anosov flow, the Ruelle-Pollicott spectrum of the generator has a structure in vertical bands. This band spectrum gives an asymptotic expansion for dynamical correlation functions. Physically the interpretation is the emergence of a quantum dynamics from the classical fluctuations. This makes a connection with the field of quantum chaos and suggests many open questions.

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# Contents

# 1 Introduction

These are lectures notes for the summer school 13-17 May 2013 at ROMA, "Geometric, analytic and probabilistic approaches to dynamics in negative curvature". We review and present the main ideas of some results that have been presented in other papers given in the references.

In these lecture notes, we present the use of semiclassical analysis for the study of hyperbolic dynamics. This approach is particularly useful in the case where the dynamics has neutral direction(s) like extensions of expanding maps, hyperbolic maps or Anosov flows.

In this approach we study the transfer operator associated to the dynamics and its spectral properties. The objective is to describe the discrete spectrum of the transfer operator, called "Ruelle-Pollicott resonances" and its importance to express the exponential time decay of correlation functions. This discrete spectrum (together with eigenvectors) is also useful to obtain further results for the dynamics as statistical results (central limit theorem, large deviations, linear response theory...), and to obtain estimates for counting of periodic orbits in the case of flow.

### The general idea behind the semiclassical approach

1. Consider a smooth diffeomorphism $f : M \to M$ on a smooth manifold $M$ (or a flow $f^t = \exp(tX) : M \to M$, $t \in \mathbb{R}$ generated by a vector field $X$). In the 70', David Ruelle, Rufus Bowen and others have suggested to consider **evoluimportion of functions** (respect. probability measures) with the pull back operator also called the transfer operator $\mathcal{L}^t \varphi = \varphi \circ f^{-t}$ (respect. its adjoint $\mathcal{L}^{t*}$) **instead of evolution of individual trajectories** $x(t) = f^t(x)$. This functional approach is useful for chaotic dynamical systems for which individual trajectories have unpredictable behavior, whereas a smooth density may converge towards equilibrium in a predictable manner[1]. Remark that this description is not reductive because taking $\varphi = \delta_x$ a Dirac measure at point $x$, one recovers the individual trajectory. See figure 1.1.

2. By linearity of the transfer operator $\mathcal{L}^t$, **a function (or distribution) on $M$ can be decomposed as a superposition of "elementary wave packets"**[2] $\varphi_{x,\xi}$: this is a function with parameters $(x, \xi)$ which has small support around $x \in M$ in space and whose Fourier transform (in local chart) also decay very fast outside some value $\xi \in T_x^* M$ in Fourier space [3]. Geometrically $(x, \xi) \in T^*M$ is a point on **the cotangent space**. A fundamental observation is that the time evolution of this

[1] Rem: this is somehow the weather is "predicted" by computer simulations from different initial conditions

[2] In signal theory and analysis this decomposition corresponds to wavelet transform or F.B.I. transform. In quantum physics an elementary wave packets is also called a "quantum".

[3] Fourier transform of $\varphi$ is written $(\mathcal{F}\varphi)(\xi) = \frac{1}{(2\pi)^n} \int e^{-i\xi \cdot x} \varphi(x)\, dx$.

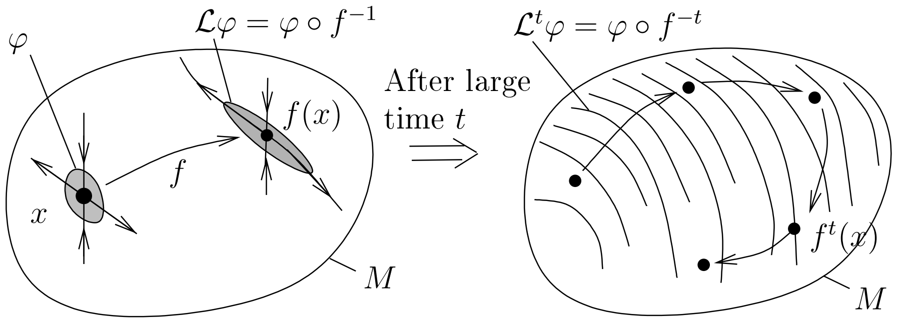


Figure 1.1: An (hyperbolic) map $f$ defines the evolution of a point $x \in M$ by $f^t(x)$ and evolution of a function $\varphi(x)$ by $\mathcal{L}^t\varphi = \varphi \circ f^{-t}$. The support of $\mathcal{L}^t\varphi$ spreads and folds after large time $t$.

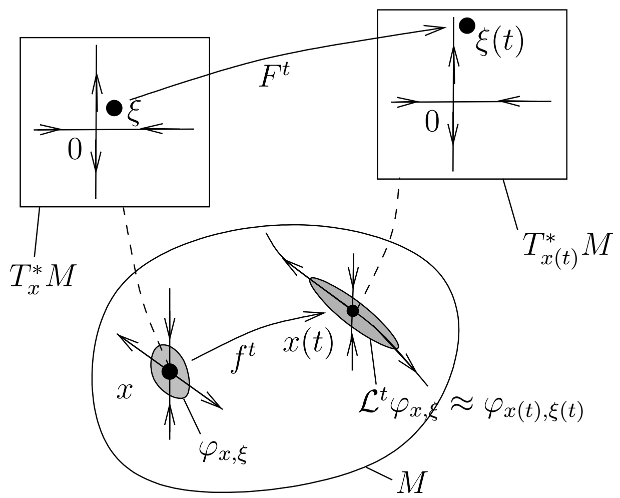


Figure 1.2: Evolution of a wave packet.

wave packet $\mathcal{L}_t\varphi_{x,\xi}$ after finite time $t$, **remains a wave packet** with new parameters $(x(t), \xi(t)) = F^t(x,\xi) \in T^*M$ which follow the canonical lift $F : T^*M \to T^*M$ of the map $f : M \to M$. See figure 1.2.

3. We therefore study the dynamics of the lift map $F^t : T^*M \to T^*M$. In the case of hyperbolic (Anosov) dynamics every point $(x(t), \xi(t))$ escape towards infinity $|\xi(t)| \to \infty$ as $t \to \pm\infty$, except if $(x(0), \xi(0)) \in K := \{(x,\xi), \xi = 0\}$, the zero section, called the "**trapped set**". A consequence is the **decay of correlation functions** $(\varphi_{x',\xi'}, \mathcal{L}_t\varphi_{x,\xi})$ as $t \to \infty$ (intuitively only the constant function with $\xi = 0$ component survives). From the uncertainty principle in phase space $T^*M$ this also implies that the transfer operator has **discrete spectrum** in some functional spaces "adapted" to the dynamics (so called Ruelle-Pollicott resonances). Here "adapted" means that the norm of this functional space has the ability to "truncate" the high

frequencies. The limit of high frequencies $|\xi| \gg 1$ is called **the semiclassical limit.** Technically we will use semiclassical analysis and "quantum scattering theory" developed by Helffer-Sjöstrand and others in the 80's [HS86] with "**escape functions**" (or Lyapounov function in phase space) in order to define these "**anisotropic Sobolev spaces**".

4. In the case of partially hyperbolic dynamics, e.g. Anosov vector field, then $|\xi(t)| \to \infty$ outside a "trapped set" $K \subset T^*M$ (or non wandering set) which is non compact. Geometrical properties of the trapped set $K$ gives some more refined properties of the Ruelle-Pollicott spectrum of resonances, and also properties of the eigenspaces. For example its fractal dimension gives an (upper bound) estimate for the **density of Ruelle resonances**. If $K \subset T^*M$ is a symplectic submanifold this implies an **asymptotic spectral gap**, a **band structure** for the Ruelle spectrum, etc.

In order to present this approach we will consider different models. These models are very similar and the elaboration is increasing from one to the next. In particular we will present recent results for

1. **"U(1) extension of Anosov diffeomorphism preserving a contact form"** [Fau07a, FT15]. This model is also called **prequantum Anosov map.** It can be considered as a simplified model of a contact Anosov flow: there is a neutral direction for the dynamics and a contact one form that is preserved. This allows to obtain precise information on the Ruelle-Pollicott spectrum in the semiclassical limit of high frequencies along the neutral direction. In particular we will show that the spectrum has some **band structures** and obtain the "Weyl law" giving the number of resonances in each band. We will also show that surprisingly the correlation functions have some "**quantum behavior**". We will discuss the fact that these results propose a direct bridge between the study of Ruelle-Pollicott resonances in dynamics and questions in "quantum chaos" or "wave chaos". Using the Atiyah-Bott trace formula, we will relate the spectrum with the periodic orbits.

2. **"Contact Anosov flow"** [FS11, FT13, FT16]. This dynamical model can be considered as the analogous of the previous model in case of continuous time. This model is interesting in geometry because it includes the case of geodesic flow on a Riemannian manifold $\mathcal{M}$ with negative (sectional) curvature. In that case the flow takes place on the unit (co)tangent bundle $M = T_1^*\mathcal{M}$. We will show that all the results obtained for the previous model are also true here and concern the spectrum of the generator of the flow (the vector field). We will discuss the relation with the spectrum of the Laplacian operator $\Delta$ on $\mathcal{M}$. We will express these results using zeta functions.

Sections or paragraphs marked with (*) can be skipped for a first lecture.

### Some general references (books or reviews)

- On dynamical systems: [BS02, KH95, Bal00]

- On semiclassical analysis: [Tay96b, **?**, Mar02, GS94]
- On quantum chaos: [Gut91][Non08]

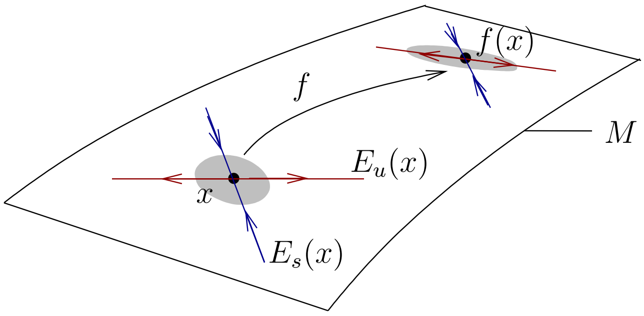


Figure 2.1: An Anosov map $f$

# 2 Hyperbolic dynamics

## 2.1 Anosov maps

**Definition 2.1.** On a $C^\infty$ closed connected manifold $M$, a $C^\infty$ diffeomorphism $f : M \to M$ is **Anosov** if there exists a Riemannian metric $g$ on $M$, an $f-$invariant continuous decomposition of $TM$:

$$T_x M = E_u(x) \oplus E_s(x), \quad \forall x \in M, \tag{2.1}$$

a constant $\lambda > 1$ such that for every $x \in M$,

$$\forall v_s \in E_s(x), \quad \|D_x f(v_s)\|_g \leq \frac{1}{\lambda} \|v_s\|_g \tag{2.2}$$

$$\forall v_u \in E_u(x), \quad \left\|D_x f^{-1}(v_u)\right\|_g \leq \frac{1}{\lambda} \|v_u\|_g .$$

We call $E_u(x)$ the **unstable subspace** and $E_s(x)$ **the stable subspace.**

### 2.1.1 Example "Hyperbolic automorphism on the torus".

$$f : \begin{cases} \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d & \to \mathbb{T}^d \\ x & \to Mx \mod \mathbb{Z}^d \end{cases} \tag{2.3}$$

with $M \in SL_d(\mathbb{Z})$ hyperbolic , i.e. every eigenvalues $\lambda$ satisfy $|\lambda| \neq 1, 0$.

*Remark* 2.2.

- $f$ in (2.3) is well defined because if $n \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$ then

$$M(x+n) = Mx + \underbrace{Mn}_{\in \mathbb{Z}^d} = Mx \text{ mod } \mathbb{Z}^d$$

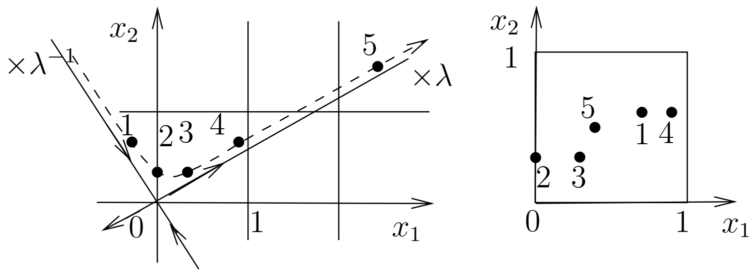


Figure 2.2: Trajectory of an initial point $(-0.3, 0.6)$ under the cat map, on $\mathbb{R}^2$ (there the trajectory is on an hyperbola) and on $\mathbb{T}^2$. After restriction by modulo 1, the trajectory is "chaotic".

- $f$ is invertible on $\mathbb{T}^d$ and $f^{-1}(x) = M^{-1}x$ with $M^{-1} \in SL_d(\mathbb{Z})$.
- The simplest example of (2.3) is the "**cat map**" on $\mathbb{T}^2$[AA67],

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \lambda = \lambda_u = \frac{3+\sqrt{5}}{2} \simeq 2.6 > 1, \quad \lambda_s = \lambda^{-1} < 1. \tag{2.4}$$

### 2.1.2 General properties of Anosov diffeomorphism

- In general, the maps $x \in M \to E_u(x), E_s(x)$ are not $C^\infty$ but only Hölder continuous with some exponent $0 < \beta \leq 1$. (This is similar to the Weierstrass function).
- (*) It is conjectured that $M$ is an infranil manifold. Ex: $M = \mathbb{T}^d$ is a torus.

**Proposition 2.3.** *[KH95](*)* "**Structural stability**". *If $f : M \to M$ is Anosov there exists $\varepsilon > 0$ such that for any $g : M \to M$ such that $\|g - \mathrm{Id}\|_{C^1} \leq \varepsilon$ then*

1. *$g \circ f$ is Anosov.*
2. *There exists an homeomorphism $h : M \to M$ (Hölder continuous) such that we have a commutative diagram:*

$$\begin{array}{ccc} M & \xrightarrow{g\circ f} & M \\ \uparrow h & & \uparrow h \\ M & \xrightarrow{f} & M \end{array}$$

*Proof.* See [KH95]. The proof uses a description in terms on cones. $\square$

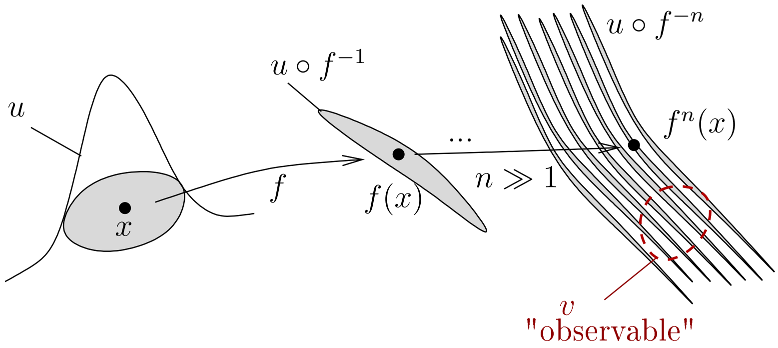


Figure 2.3: The correlation function $C_{v,u}(n) := \int_M v.\left(u \circ f^{-n}\right) dx$ represents the evolved function $u \circ f^{-n}$ tested against an "observable" function $v$.

**Theorem 2.4.** *(Anosov) If $f : M \to M$ is Anosov and preserves a smooth measure $dx$ on $M$ then $f$ is* ***exponentially mixing****: $\exists \alpha > 0, \forall u, v \in C^\infty(M)$, for $n \to +\infty$,*

$$\left| \underbrace{\int_M v.\left(u \circ f^{-n}\right) dx}_{C_{v,u}(n)} - \int v dx. \int u dx \right| = O\left(e^{-\alpha n}\right) \tag{2.5}$$

In the last equation, the term

$$C_{v,u}(n) := \int_M v.\left(u \circ f^{-n}\right) dx \tag{2.6}$$

is called a **correlation function**.

*Remark* 2.5. Mixing means "loss of information" because for $n \to \infty$, $u \circ f^{-n}$ normalized by $\left(\int u dx\right)^{-1}$ converges in the sense of distribution towards the measure $dx$. See figure 2.3.

*Proof.* This will be obtained in (3.27) as a consequence of Theorem 3.18, using semiclassical analysis. (From [FRS08]). □

*Remark* 2.6. For *linear* Anosov map on $\mathbb{T}^d$, Eq.(2.3), the proof of exponentially mixing is easy and is true for any $\alpha > 0$. Let $k, l \in \mathbb{Z}^d$, let $\varphi_k(x) := \exp(i2\pi k.x)$ be a **Fourier mode.** Then

$$\begin{aligned} \int_{\mathbb{T}^d} \left(\varphi_k \circ f^{-n}\right) .\overline{\varphi}_l dx &= \int \exp\left(i2\pi\left(k.M^{-n}x - l.x\right)\right) dx \\ &= \int \exp\left(i2\pi\left({}^tM^{-n}k - l\right).x\right) dx = \delta_{{}^tM^{-n}k = l} \end{aligned} \tag{2.7}$$

But if $k \neq 0$ then $|{}^t M^{-n} k| \to \infty$ as $n \to +\infty$ because $M$ is hyperbolic. So (2.7) vanishes for $n$ large enough. Finally smooth functions $u, v$ have Fourier components which decay fast and one deduces (2.5) for any $\alpha > 0$.

**Proposition 2.7.** *(*) $f$ Anosov is* ***ergodic****: $\forall u, v \in C^{\infty}(M)$,*

$$\frac{1}{n}\sum_{k=0}^{n-1}\int v.\left(u \circ f^{-k}\right) dx \underset{n\to\infty}{\longrightarrow} \int v dx. \int u dx \tag{2.8}$$

*Proof.* Using Cesaro's Theorem, one sees that mixing (2.5) implies ergodicity (2.8). □

*Remark* 2.8. (*) Ergodicity means that the "time average" of $v$ i.e $\frac{1}{n}\sum_{k=0}^{n-1}\left(u \circ f^{-k}\right)$ normalized by $\left(\int u dx\right)^{-1}$ converges (in the sense of distribution) towards the measure $dx$.

*Remark* 2.9. Exponentially mixing (2.5) implies some **statistical properties** such as the central limit theorem for time average of functions etc.

## 2.2 Prequantum Anosov maps

We introduce now "prequantum Anosov map": it is a $U(1)$ extension of an Anosov diffeomorphism $f$ preserving a contact form. This corresponds to the "**geometric prequantization**" following Souriau-Kostant-Kirillov (70'), Zelditch (05)[Zel05].

We will suppose that $(M, \omega)$ is a **symplectic manifold** and $f : M \to M$ is an Anosov map preserving $\omega$:

$$f^{*}\omega = \omega \tag{2.9}$$

i.e. $f$ is symplectic. Then $\dim M = 2d$ is even and $f$ preserves the non degenerate volume form $dx = \omega^{\wedge d}$ of degree $2d$.

**Example 2.10.** As (2.3) but with $f \in Sp_{2d}(\mathbb{Z}) : \mathbb{T}^{2d} \to \mathbb{T}^{2d}$ symplectic and hyperbolic. The linear cat map (2.4) is symplectic for $\omega = dq \wedge dp$ with coordinates $(q, p) \in \mathbb{R}^2$.

*Remark* 2.11. For every $x \in M$, $(T_x M, \omega)$ is a symplectic linear space (by definition) and $E_u(x), E_s(x) \subset T_x M$ given by (2.1) are **Lagrangian** linear subspaces hence

$$\dim E_u(x) = \dim E_s(x) = d$$

*Proof.* If $u_s, v_s \in E_s(x)$ then

$$\omega(u_s, v_s) \underset{(2.9)}{=} \omega\left(D_x f^n(u_s), D_x f^n(v_s)\right) \underset{n\to\infty,(2.2)}{\to} 0$$

Similarly for $E_u(x)$ with $D_x f^{-n}$. □

**Assumption** 1 : The cohomology class $[\omega] \in H^2(M,\mathbb{R})$ represented by the symplectic form $\omega$ is integral, that is, $[\omega] \in H^2(M,\mathbb{Z})$.

**Assumption** 2 : (a): $H_1(M,\mathbb{Z}) \hookrightarrow H_1(M,\mathbb{R})$ is injective (i.e. no torsion part) and (b): 1 is not an eigenvalue of the linear map $f_* : H_1(M,\mathbb{R}) \to H_1(M,\mathbb{R})$ induced by $f : M \to M$.

*Remark* 2.12. Assumption 1 is true for the cat map because $\int_{\mathbb{T}^2} \omega = \int_{\mathbb{T}^2} dq \wedge dp = 1 \in \mathbb{Z}$. Assumption 2-(b) is conjectured to be true for every Anosov map.

**Theorem 2.13.** *[FT15]With Assumption 1, there exists a $U(1)-$principal bundle $\pi : P \to M$ with connection one form $A \in C^\infty(P;\Lambda^1 \otimes i\mathbb{R})$with curvature $\Theta = dA = -i(2\pi)(\pi^*\omega)$.*
*With Assumption 2, we can choose the connection $A$ above such that there exists a map $\tilde{f} : P \to P$ called **prequantum map** such that*

1. *The following diagram commutes*

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & P \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array} \tag{2.10}$$

2. *"Equivariance" with respect to the action of $e^{i\theta} \in U(1)$:*

$$\forall p \in P, \forall \theta \in \mathbb{R}, \quad \tilde{f}\left(e^{i\theta}p\right) = e^{i\theta}\tilde{f}(p). \tag{2.11}$$

3. *$\tilde{f}$ preserves the connection*

$$\tilde{f}^* A = A \tag{2.12}$$

*Proof of Theorem 2.13 .* See [FT15]. □

*Remark* 2.14. At every point $p \in P$, $(\mathrm{Ker}A)(p) = \tilde{E}_u(p) \oplus \tilde{E}_s(p)$ is the strong distribution of stable/unstable directions of the map $\tilde{f}$. We recall the interpretation of the curvature two form $\Theta$ as an infinitesimal holonomy ([Tay96b, (6.22)p.506])). The fact that $\omega$ is symplectic here means that the distribution $\tilde{E}_u \oplus \tilde{E}_s$ is maximally "non integrable".

$\alpha = \frac{i}{2\pi}A$ is a **contact one form** on $P$ preserved by $\tilde{f}$ because

$$\mu_P = \frac{1}{d!}\alpha \wedge (d\alpha)^d = \frac{1}{d!}\left(\frac{1}{2\pi}d\theta\right) \wedge \omega^d$$

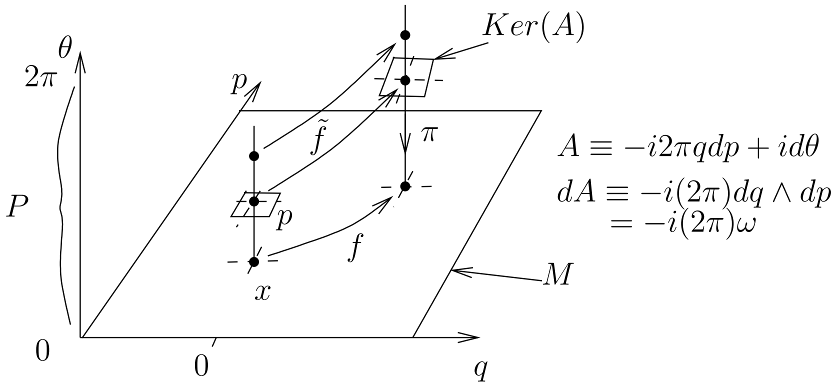


Figure 2.4: A picture of the prequantum bundle $P \to M$ in the case of $M = \mathbb{T}^2$, e.g. for the "cat map" (2.4), with connection one form $A$ and the prequantum map $\tilde{f} : P \to P$ which is a lift of $f : M \to M$. A fiber $P_x \equiv U(1)$ over $x \in M$ is represented here as a segment $\theta \in [0, 2\pi[$. The plane at a point $p$ represents the horizontal space $H_p P = \mathrm{Ker}(A_p)$ which is preserved by $\tilde{f}$. These plane form a non integrable distribution with curvature given by the symplectic form $\omega$.

is a non degenerate $(2d+1)$ volume form on $P$ preserved by $\tilde{f}$.

*Remark* 2.15. $\tilde{f}$ is a "partially hyperbolic map" with neutral direction $\theta$, preserving a contact one form $\alpha = \frac{i}{2\pi} A$. Then $\tilde{f}$ is exponentially mixing (see Definition (2.5)), but this is not obvious. This is a result of D. Dolgopyat [Dol02]. We will obtain this in remark 3.39 page 42.

*Remark* 2.16. (*) If $x = f^n(x)$ with $n \geq 1$, i.e. $x$ is a periodic point of $f$, then for any $p \in P_x = \pi^{-1}(x)$,

$$\tilde{f}^n(p) = e^{i2\pi S_{n,x}} p \tag{2.13}$$

with some phase $S_{n,x} \in \mathbb{R}/\mathbb{Z}$ called the **action** of the periodic point. This will appear in Trace formula in Section 4.

## 2.3 Anosov vector field

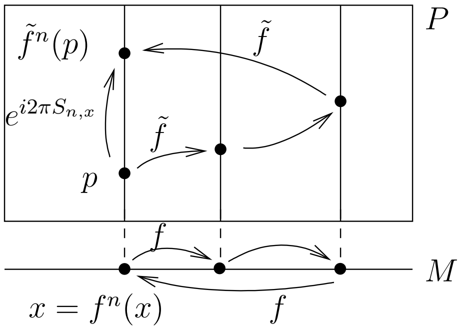


Figure 2.5: Action of a periodic point $x = f^n(x)$.

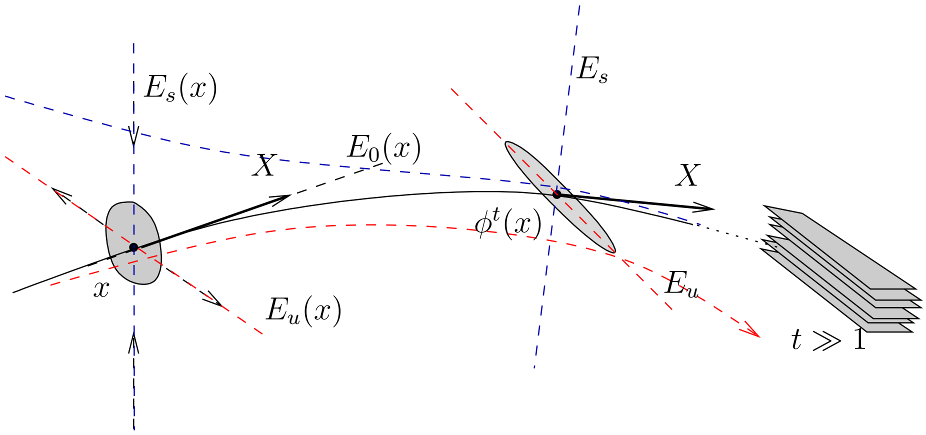


Figure 2.6: Anosov flow.

**Definition 2.17.** On a $C^\infty$ manifold $M$, a smooth vector field $X$ is **Anosov** if its flow $\phi_t = e^{-tX}$,$t \in \mathbb{R}$ satisfies

1. $\forall x \in M$, we have an $\phi_t$ invariant and continuous decomposition:

$$T_x M = E_u(x) \oplus E_s(u) \oplus \underbrace{E_0(x)}_{\mathbb{R}X} \tag{2.14}$$

2. There exists a metric $g$ on $M$, $\exists \gamma > 0, C > 0$, $\forall x \in M$,$\forall t \geq 0$,

$$\forall v_s \in E_s(x), \quad \|D_x \phi_t(v_s)\|_g \leq Ce^{-t\gamma} \|v_s\|_g \tag{2.15}$$

$$\forall v_u \in E_u(x), \quad \|D_x \phi_{-t}(v_u)\|_g \leq Ce^{-t\gamma} \|v_u\|_g .$$

*Remark* 2.18. In general the maps $x \in M \to E_u(x), E_s(x)$ are not smooth. They are only

Hölder continuous with some exponent $0 < \beta \leq 1$.

**Definition 2.19.** We define the **Anosov one form** $\alpha$ on $M$ by: $\forall x \in M$,

$$\alpha\left(E_u\left(x\right) \oplus E_s\left(x\right)\right) = 0, \qquad \alpha\left(X\right) = 1. \tag{2.16}$$

In general $\alpha\left(x\right)$ is Hölder continuous with respect to $x \in M$. It is preserved by the flow: its Lie derivative is (in the sense of distributions) $\mathcal{L}_X \alpha = 0$. Conversely there is a unique one form $\alpha$such that $\mathcal{L}_X \alpha = 0$ and $\alpha\left(X\right) = 1$.

**Definition 2.20.** $\left(\phi_t\right)_{t\in\mathbb{R}}$ is a **contact Anosov flow** if $\alpha$ is $C^\infty$ and a contact one form, i.e. if $\omega = d\alpha_{|E_u(x)\oplus E_s(x)}$ is non degenerate (i.e. symplectic) or equivalently $dx = \alpha \wedge \left(d\alpha\right)^d$ is an invariant smooth volume form on $M$, with $d = \mathrm{dim} E_u\left(x\right) = \mathrm{dim} E_s\left(x\right)$ (Lagrangian subspaces, see rem. 2.11 ). Then $\mathrm{dim} M = 2d + 1$.

*Remark* 2.21. That the flow is contact means that the distribution of hyperspaces $E_u\left(x\right) \oplus E_s\left(x\right)$ is maximally non integrable, this is similar to figure (2.4).

**Example 2.22. "geodesic flow with negative curvature".** Let $\mathcal{M}$ be a smooth compact Riemannian manifold.

- The cotangent space $T^*\mathcal{M}$ has a canonical one form called the **Liouville one form** given by $\alpha = -\sum_{j=1}^n p^j dq^j$ in canonical coordinates ($q^j$ are coordinates on $\mathcal{M}$ and $p^j$ on $T_q^*\mathcal{M}$) [Sal01]. The canonical symplectic form on $T^*\mathcal{M}$ is given by

$$\omega := \sum_j dq^j \wedge dp^j = d\alpha$$

- On the cotangent space $T^*\mathcal{M}$, the Hamiltonian function $H\left(q, p\right) := \left\|p\right\|_g$ (with $p \in T_q\mathcal{M}$) defines a Hamiltonian vector field $X$ by $\omega\left(X, .\right) = dH$ whose flow is called the **geodesic flow**. The energy level of energy 1 is the unit cotangent bundle $H^{-1}\left(1\right) = T_1^*\mathcal{M}$. The Hamiltonian flow preserves $\omega$ but also the one form $\alpha$ because $H\left(q, p\right)$ is homogeneous[4] in $p$. Therefore the geodesic flow is a contact flow on $M = T_1^*\mathcal{M}$ preserving $\alpha$. The Anosov one form is $-\alpha$.

---

4

*Proof.* Let

$$\mathscr{E} := \sum_j p^j \frac{\partial}{\partial p^j}$$

- In the case where $\mathcal{M}$ has **negative sectional curvature** it is known that the geodesic flow is Anosov. This is therefore a **contact Anosov flow** on $M = T_1^*\mathcal{M}$. One has $\dim M = 2\dim\mathcal{M} - 1 = 2d + 1$. Therefore $n = \dim\mathcal{M} = d + 1$.

**Example 2.23.** A particular example is when $\mathcal{M}$ is a **homogeneous manifold**: $\mathcal{M} = \Gamma\backslash\mathrm{SO}(1,n)/SO(n) = \Gamma\backslash\mathbb{H}^n$ where $\Gamma$ is a discrete co-compact subgroup and $\mathbb{H}^n$ is the hyperbolic space of dimension $n$. The simplest case is when $\mathcal{M}$ is a surface ($n = \dim\mathcal{M} = 2$): one has $\mathrm{SO}(2,1) \equiv \mathrm{SL}_2(\mathbb{R})$. This case is explained in details below.

The following proposition shows how to obtain other contact (Anosov) vector field from a given one by "re-parametrization".

**Proposition 2.24.** *(*) If $X_0$ is a contact Anosov vector field with contact one form $\alpha_0$, let $\beta$ a closed one form on $M$ such that$|\beta(X_0)| < \alpha_0(X_0) = 1$ then*

$$X = \frac{1}{1+\beta(X_0)}X_0$$

*is a also a contact Anosov vector field for the contact one form*

$$\alpha = \alpha_0 + \beta$$

*Proof.* We have $d\alpha = d\alpha_0$ and

$$\alpha(X) = \frac{1}{1+\beta(X_0)}(\alpha_0(X_0) + \beta(X_0)) = 1$$

---

be the **canonical Euler vector field** on $T^*\mathcal{M}$ (it preserves fibers, it is canonically defined in any vector space). $\mathscr{E}$ generates the flow of "scaling":

$$S_\lambda : (q,p) \in T^*\mathcal{M} \to (q, e^\lambda p) \in T^*\mathcal{M}, \qquad \lambda \in \mathbb{R} \tag{2.17}$$

We have $\mathscr{E}(H) = H$ because $H$ is homogeneous of degree 1 in $p$. We have $\iota_{\mathscr{E}}\alpha = 0$, $\iota_{\mathscr{E}}\omega = \alpha$ and $\mathcal{L}_{\mathscr{E}}\alpha = \alpha$, $\mathcal{L}_{\mathscr{E}}\omega = \omega$. The Hamiltonian vector field $X_N$ is the associated **Reeb vector field**, i.e. it is uniquely defined by

$$\alpha(X) = -H = -1, \quad (d\alpha)(X) = 0. \tag{2.18}$$

In particular $X$ preserves $\alpha$, i.e. $\mathcal{L}_X\alpha = 0$, i.e. it is a **contact vector field**. Indeed: we have on $T_E^*\mathcal{M}$

$$\alpha(X) = \iota_X\alpha = \iota_X(\iota_{\mathscr{E}}\omega) = -\omega(X,\mathscr{E}) = -\mathscr{E}(H) = -H = -E$$

Also

$$(d\alpha)(X) = \omega(X,.) = dH = 0, \qquad \text{on } \Sigma_E.$$

Then on $T^*\mathcal{M}$

$$\mathcal{L}_X\alpha = d(\iota_X\alpha) + \iota_X d\alpha = -d(H) + dH = 0$$

□

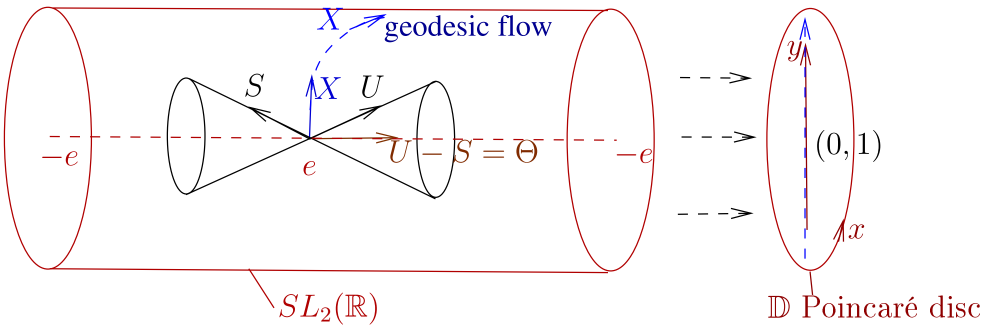


Figure 2.7: Geodesic flow the Poincaré disc is generated by $X \in sl_2\mathbb{R}$.

and

$$\mathcal{L}_X\alpha = \iota_X d\alpha + d(\iota_X\alpha) = \frac{1}{1+\beta(X_0)}\iota_{X_0}d\alpha_0 = \frac{1}{1+\beta(X_0)}\mathcal{L}_{X_0}\alpha_0 = 0$$

□

*Remark* 2.25. (*) P. Foulon and B. Hasselblatt [FH13] have shown that even in 3 dimension there are numerous contact Anosov flow that are not topologically orbit equivalent to geodesic flows.

### 2.3.1 Example of a contact Anosov flow: "geodesic flow on a constant negative curvature surface".

We present here a standard example of contact Anosov flow, the geodesic flow on Riemann surface $\mathcal{M} = \Gamma \setminus (SL_2\mathbb{R}/SO_2)$ where $\Gamma < SL_2\mathbb{R}$ is a co-compact discrete subgroup. This example is a particular case of the example 2.22 above. We present it in details, because we will use it later on in Section 3.5.1.

From Iwasawa decomposition, a matrix $g \in SL_2\mathbb{R}$ can be written[5]

$$g = \begin{pmatrix} y^{1/2} & x \\ 0 & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad x \in \mathbb{R}, y > 0, \theta \in SO_2.$$

Hence with $z = x + iy \in \mathbb{H}^2$ in the Poincaré half plane, we have the homeomorphism $SL_2\mathbb{R} \equiv \mathbb{H}^2 \times SO_2$.

A basis of the Lie algebra $sl_2\mathbb{R} \equiv T_e(SL_2\mathbb{R})$ is[6]

$$X = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

[5] Recall that $g \in \mathrm{SL}_2\mathbb{R} \Leftrightarrow \det g = 1$.

[6] Because $a \in sl_2\mathbb{R} \Leftrightarrow \det(e^a) = e^{\mathrm{Tr}a} = 1 \Leftrightarrow \mathrm{Tr}a = 0$.

and satisfies

$$[X,U]=U,\quad [X,S]=-S,\quad [U,S]=2X. \tag{2.19}$$

These tangent vector $X,U,S$ can be extended as left invariant vector fields on $SL_2\mathbb{R}$ by $X=g.X_e$ etc. . Then the vector field $X$ generates the flow $\phi_t=e^{-tX}$. It is given by the right action[7] of $e^{-tX_e}$: $\phi_t(g):=g.e^{-tX_e}$and taking any left invariant metric $g$ on $SL_2\mathbb{R}$ we have

$$\|D\phi_t(U)\|_g=\left\|U.e^{-tX}\right\|_g \underbrace{=}_{\|\cdot\|_g left-inv.} \left\|e^{tX}.U.e^{-tX}\right\|_g=\left\|e^{t[X,.]}U\right\|_g \underset{(2.19)}{=} e^t\|U\|_g \tag{2.20}$$

According to (2.15), this shows that $U$ spans the unstable direction $E_u(g)$ with $\lambda=e>1$. Similarly we get $\|D\phi_t(S)\|_g=e^{-t}\|S\|_g$ and $S$ spans $E_s(g)$. Therefore, if $\Gamma<SL_2\mathbb{R}$ is a discrete co-compact subgroup then $M:=\Gamma\backslash SL_2\mathbb{R}$ is a compact manifold and $X$ is a smooth contact Anosov vector field on $M$ with $E_u=\mathbb{R}U,\quad E_s=\mathbb{R}S,\quad E_0=\mathbb{R}X$. The property of contact comes from the last commutator $[U,S]=2X$. More precisely the Anosov one form $\alpha$, Eq. (2.16) is given by

$$\alpha=\frac{1}{2}K(X,.)$$

where $K=2X^*\otimes X^*+4(U^*\otimes S^*+S^*\otimes U^*)$ is the Killing metric on $SL_2\mathbb{R}$. To show this, observe that $\alpha(X)=1$ and

$$\begin{aligned}(d\alpha)(S,U)&=U(\alpha(S))+S(\alpha(U))-\alpha([S,U])=-\frac{1}{2}K(X,[S,U])\\&=K(X,X)=1\end{aligned}$$

hence $d\alpha$ is symplectic on $E_u\oplus E_s=\mathrm{Span}(U,S)$.

If $(-\mathrm{Id})\in\Gamma$ it is known that this flow can be identified with the geodesic flow on the Riemann surface $\mathcal{M}=\Gamma\setminus(SL_2\mathbb{R}/SO_2)=\Gamma\backslash\mathbb{H}^2$ which has constant negative curvature $\kappa=-1$ and that $M\equiv T_1^*\mathcal{M}$.

*Remark* 2.26. In fact $SL_2\mathbb{R}\equiv SO_{1,2}$ and similarly in higher dimension, some left invariant vector field on $M:=\Gamma\backslash SO_{1,n}/SO_{n-1}$ are contact Anosov vector field and can be interpreted as the geodesic flow on a compact hyperbolic manifold $\mathcal{N}=\Gamma\backslash\mathbb{H}^n=M/SO_n$. The left invariant vector field on $\Gamma\backslash SO_{1,n}$ generates the frame flow.

[7] indeed $\frac{d\phi_t}{dt}_{/t=0}=-g.X_e=-X$.

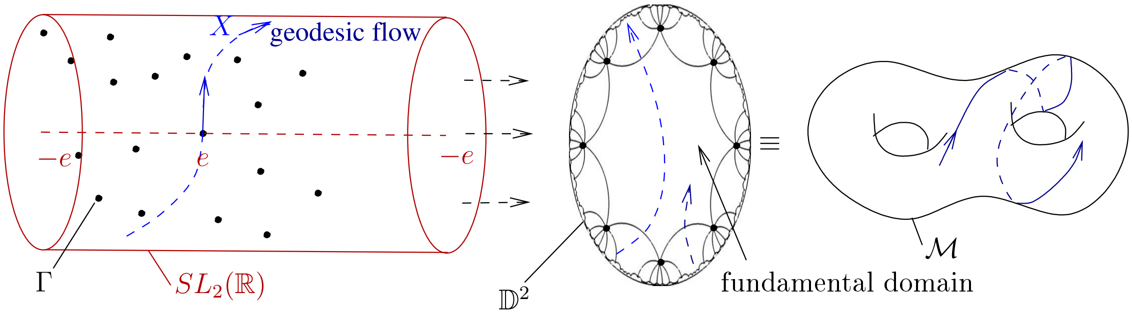


Figure 2.8: Geodesic flow on a surface $\mathcal{M}$ with constant negative curvature.

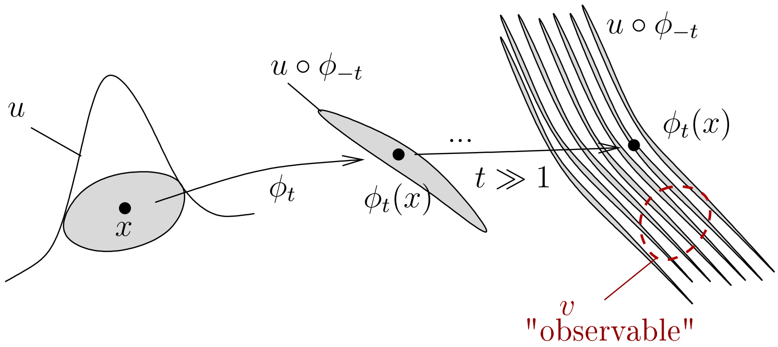


Figure 2.9: Exponential mixing from the correlation function $C_{v,u}(t) = \int_M v.(u \circ \phi_{-t})\, dx$.

### 2.3.2 (*) General properties of contact Anosov flows

**Theorem 2.27.** *(*) A contact Anosov flow is* ***exponentially mixing:*** $\exists \alpha > 0$, $\forall u, v \in C^\infty(M)$, *for* $t \to \infty$ *one has*

$$\left| \underbrace{\int_M v.(u \circ \phi_{-t})\, dx}_{C_{v,u}(t)} - \int v dx. \int u dx \right| = O\left(e^{-\alpha t}\right) \tag{2.21}$$

*The term* $C_{v,u}(t)$ *above is called a* ***correlation function****.*

See figure 2.9.

*Remark* 2.28. (*) Mixing implies ergodicity. This is the same definition and same proof as in (2.8).

Usually the term "correlation function" is for the whole difference $\int_M v.\left(u \circ \phi_{-t}\right) dx - \int v dx. \int u dx$.

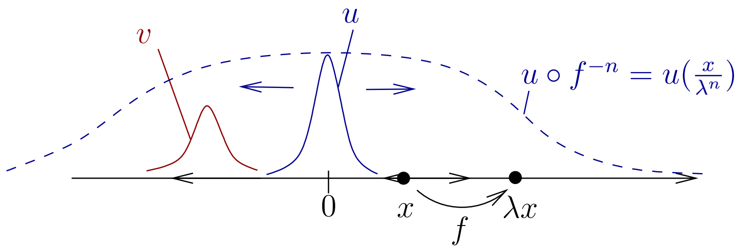


Figure 3.1: Illustration of the correlation function (3.2).

# 3 Transfer operators and their discrete Ruelle-Pollicott spectrum

Before considering the Ruelle spectrum of Anosov dynamics, the following Section introduces the techniques on a very simple example. This simple example (extended in $\mathbb{R}^d$) will also be important later on in the proof of th. 3.34 and 3.54 because it will serve as a universal "normal form".

## 3.1 Ruelle spectrum for a basic model of expanding map

Let $\lambda > 1$ and consider the expanding map:

$$f : \begin{cases} \mathbb{R} & \rightarrow \mathbb{R} \\ x & \rightarrow \lambda x \end{cases} \tag{3.1}$$

### 3.1.1 Transfer operator

Let $u, v \in \mathcal{S}(\mathbb{R})$. The **time correlation function** (2.6) is for $n \geq 1$,

$$C_{v,u}(n) := \int_{\mathbb{R}} \overline{v}.\left(u \circ f^{-n}\right) dx = \int \overline{v(x)}.u\left(\frac{x}{\lambda^n}\right) dx \underset{n \to +\infty}{\rightarrow} \left(\int \overline{v} dx\right).u(0) \tag{3.2}$$

See figure 3.1.

Let us write $\langle v|u\rangle_{L^2} := \int_{\mathbb{R}} \overline{v}.udx$ for the $L^2$scalar product. Let us define the **transfer operator**

$$\left(\hat{F}u\right)(x) := \left(u \circ f^{-1}\right)(x) = u\left(\frac{x}{\lambda}\right) \tag{3.3}$$

which is useful to express the correlation function:

$$C_{v,u}(n) = \int_{\mathbb{R}} \overline{v}.u \circ f^{-n} dx = \langle v|\hat{F}^n u\rangle_{L^2}$$

*Remark* 3.1. The dual operator $\hat{F}^*$ defined by $\langle u|\hat{F}^* v\rangle = \langle \hat{F} u|v\rangle$ is given by[8]

$$\left(\hat{F}^* v\right)(y) = \lambda.v(\lambda y) \tag{3.4}$$

Taking $u = 1$ in $\langle u|\hat{F}^* v\rangle = \langle \hat{F} u|v\rangle$ gives that $\int \left(\hat{F}^* v\right)(x)\,dx = \int v(x)\,dx$. Hence $\hat{F}^*$ preserves probability measures. It is called the **Perron-Frobenius operator** or **Ruelle operator**.

### 3.1.2 Asymptotic expansion

In this subsection we perform heuristic (non rigorous) computation in order to motivate the next Section where these computations will be put in rigorous statements. The objective is to show the appearance and meaning of Ruelle spectrum of resonances. From Taylor formula (we don't care about the reminder for the moment) one has

$$u\left(\frac{x}{\lambda^n}\right) = \sum_{k\geq 0} \frac{x^k}{k!\lambda^{kn}} u^{(k)}(0)$$

Let $\delta^{(k)}$ be the $k$-th derivative of the Dirac distribution. Then

$$\begin{aligned} C_{v,u}(n) &= \int \overline{v(x)}.u\left(\frac{x}{\lambda^n}\right) dx \\ &= \sum_{k\geq 0} \frac{1}{k!\lambda^{kn}} \left(\int x^k \overline{v(x)} dx\right).u^{(k)}(0) \\ &= \sum_{k\geq 0} \frac{1}{\lambda^{kn}} \langle v|x^k\rangle\langle \frac{1}{k!}\delta^{(k)}|u\rangle \qquad (3.5) \\ &= \left(\int \overline{v} dx\right).u(0) + O\left(\frac{1}{\lambda^n}\right) \qquad (3.6) \end{aligned}$$

We have[9] for $k, l \geq 0$

$$\langle \frac{1}{k!}\delta^{(k)}|x^l\rangle = \delta_{k=l}. \tag{3.7}$$

Let[10]

$$\Pi_k := |x^k\rangle\langle \frac{1}{k!}\delta^{(k)}| \tag{3.8}$$

be a rank one operator. Then (3.7) implies that

$$\Pi_k \circ \Pi_l = \delta_{k=l}.\Pi_k$$

[8] proof: with the change of variable $y = \frac{x}{\lambda}$, we write $\langle u|\hat{F}^* v\rangle = \langle \hat{F} u|v\rangle = \int \overline{u\left(\frac{x}{\lambda}\right)} v(x)\,dx = \int \overline{u(y)} v(\lambda x)\,\lambda dy$ hence $\left(\hat{F}^* v\right)(y) = \lambda.v(\lambda y)$.

[9] Because $\left(\frac{d^k x^l}{dx^k}\right)(0) = 0$ if $k \neq l$ and $= k!$ if $k = l$.

[10] $|x^k\rangle\langle \frac{1}{k!}\delta^{(k)}|$ is a notation (called "Dirac notation" in physics) for the rank one operator $x^k\langle \frac{1}{k!}\delta^{(k)}|.\rangle$.

i.e. $(\Pi_k)_k$ is a family of rank one projectors and the Taylor expansion (3.5) writes:

$$C_{v,u}(n) = \langle v|\hat{F}^n u\rangle = \sum_{k\geq 0} \frac{1}{(\lambda^k)^n} \langle v|\Pi_k u\rangle \tag{3.9}$$

**Question:** Formally this suggests the following spectral decomposition for the transfer operator $\hat{F}$:

$$"\hat{F} = \sum_{k\geq 0} \lambda^{-k}\Pi_k", \qquad \hat{F}x^k = \lambda^{-k}x^k \tag{3.10}$$

i.e. $\lambda_k = \lambda^{-k}$ should be "simple eigenvalues" and $\Pi_k$ associated "spectral projector"; but in which space?

Notice that this statement can not be true in the Hilbert space $L^2(\mathbb{R})$ because the distributions $x^k, \delta^{(k)}$ do not belong to it. The aim is to find an Hilbert space of distributions containing $\mathcal{S}(\mathbb{R})$ where the statement (3.10) holds true. We will have to consider Hilbert spaces as subspace of distributions. Notice first that the operator $\hat{F}$ defined in (3.3) can be extended by duality[11] to distributions $\hat{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$.

*Remark* 3.2. The expanding map $f$ in (3.1) is the time one flow $f = \phi_{t=1}$ generated by the vector field

$$X = \gamma x \frac{d}{dx} \tag{3.12}$$

on $\mathbb{R}$ with $e^\gamma = \lambda > 1$. The transfer operator can be written in terms of the generator $X$:

$$\hat{F} = e^{-X}$$

*Remark* 3.3. In $L^2(\mathbb{R})$ the operator $\frac{1}{\sqrt{\lambda}}\hat{F} = \frac{1}{\sqrt{\lambda}}e^{-X}$ is unitary and has continuous spectrum on the unit circle. Correspondingly the operator $i\left(X + \frac{\gamma}{2}\right)$ is selfadjoint in $L^2(\mathbb{R})$ and has continuous spectrum on $\mathbb{R}$. But as said above, we will not consider the Hilbert space $L^2(\mathbb{R})$.

### 3.1.3 Ruelle spectrum

[11] if $\alpha \in \mathcal{S}'(\mathbb{R})$, $\hat{F}\alpha$ is defined by

$$\forall u \in \mathcal{S}(\mathbb{R}), \quad \hat{F}(\alpha)(\overline{u}) = \langle u|\hat{F}\alpha\rangle = \langle \hat{F}^* u|\alpha\rangle = \alpha\left(\overline{\hat{F}_\nu^*(u)}\right). \tag{3.11}$$

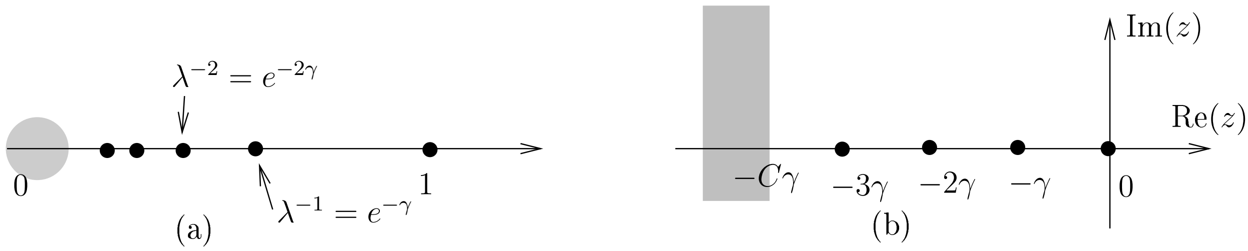


Figure 3.2: (a) Spectrum of $\hat{F} = e^{-X} : \mathcal{H}_C \to \mathcal{H}_C$. (b) spectrum of its generator $(-X) : \mathcal{H}_C \to \mathcal{H}_C$.

**Theorem 3.4.** *[FT15, Prop. 4.19]For any $C > 0$, there exists a Hilbert space $\mathcal{H}_C$ (an "anisotropic Sobolev space" defined below)*

$$\mathcal{S}(\mathbb{R}) \subset \mathcal{H}_C \subset \mathcal{S}'(\mathbb{R})$$

*such that the operator (3.3): $\hat{F} : \mathcal{H}_C \to \mathcal{H}_C$ is bounded and has essential spectral radius $r_{ess} = \mathrm{cste} \cdot \lambda^{-C}$ $\left(\underset{C\to+\infty}{\to} 0\right)$. The eigenvalues outside $r_{ess}$ are $\lambda_k = \lambda^{-k}$ with $k \in \mathbb{N}$ and their spectral projector are $\Pi_k : \mathcal{H}_C \to \mathcal{H}_C$, given by Eq.(3.8). These eigenvalues $(\lambda_k)_{k\geq 0}$ are called **Ruelle-Pollicott** resonances. The generator $-X : \mathcal{H}_C \to \mathcal{H}_C$ in (3.12) has discrete spectrum on $\mathrm{Re}(z) > -C\gamma + \mathrm{cste}$ $\left(\underset{C\to+\infty}{\to} -\infty\right)$ and has eigenvalues $-k\gamma$, $k \in \mathbb{N}$.*

A consequence is an expansion of correlation functions $C_{v,u}(n) = \langle v|\hat{F}^n u\rangle$ as (3.5),(3.9) but with a controlled remainder:

**Corollary 3.5.** For any $K \geq 0$, there exists $C_K > 0$, such that for any $u, v \in \mathcal{S}(\mathbb{R})$,

$$\left| \langle v|\hat{F}^n u\rangle - \sum_{k=0}^{K} \frac{1}{(\lambda^k)^n} \langle v|\Pi_k u\rangle \right| \leq C_K \|v\|_{\mathcal{H}'_C} \|u\|_{\mathcal{H}_C} \frac{1}{(\lambda^{K+1})^n}$$

*Proof.* (*) Let $K \geq 0$. Let $C$ large enough so that from Theorem 3.4 $r_{ess} < \frac{1}{(\lambda^{K+1})}$. Let

$$\hat{F} = \hat{K} + \hat{R}$$

be a spectral decomposition in the space $\mathcal{H}_C$ with $r_{ess} < r_{spec.}\left(\hat{R}\right) < \frac{1}{(\lambda^{K+1})}$ and $\hat{K} = \sum_{k=0}^{K} \frac{1}{\lambda^k} \Pi_k$. Then $\hat{F}^n = \hat{K}^n + \hat{R}^n$ and

$$\langle v|\hat{F}^n u\rangle = \sum_{k=0}^{K} \frac{1}{(\lambda^k)^n} \langle v|\Pi_k u\rangle + \langle v|\hat{R}^n u\rangle$$

We have $\left|\langle v|\hat{R}^n u\rangle\right| \leq \|v\|_{\mathcal{H}'_C} \|u\|_{\mathcal{H}_C} \left\|\hat{R}^n\right\|_{\mathcal{H}_C}$ and $\left\|\hat{R}^n\right\|_{\mathcal{H}_C}^{1/n} \underset{n\to\infty}{\rightarrow} r_{spec.}\left(\hat{R}\right) < \frac{1}{(\lambda^{K+1})}$ so $\left\|\hat{R}^n\right\|_{\mathcal{H}_C} \leq C_K \frac{1}{(\lambda^{K+1})^n}$. □

### 3.1.4 Arguments of proof of Theorem 3.4

We will prove that $\hat{F} : \mathcal{H}_C \to \mathcal{H}_C$ has discrete spectrum. The proof presented below relies on a semiclassical approach, and is close to the proof of Theorem 1 in [FRS08]. It also similar in spirit to the "quantum scattering theory in phase space" of by B. Helffer, J. Sjöstrand 80' [HS86]. The same strategy will be used for Anosov maps in section 3.2, 3.3 and Anosov flows in Section 3.4. The proof uses the "semiclassical theory of PDO" (cf appendix) and the idea behind is decomposition in wavepackets as explained in the introduction. The proof in [FT15] is closer to this idea.

Before let us give some important remarks.

*Remark* 3.6. The transfer operator is $\left(\hat{F}u\right)(x) = u\left(\frac{1}{\lambda}x\right)$. Let us consider the Fourier transform

$$\tilde{u}(\xi) := \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} u(x)\, dx$$

Then[12]

$$\widetilde{\left(\hat{F}u\right)}(\xi) = \lambda \tilde{u}(\lambda\xi)$$

Geometrically $(x, \xi)$ are coordinates on the cotangent space $T^*\mathbb{R} \equiv \mathbb{R}^2$. This shows that $u$ et $\tilde{u}$ are "transported" by the following canonical map $F : T^*\mathbb{R} \to T^*\mathbb{R}$ in the cotangent space $T^*\mathbb{R}$:

$$F : (x, \xi) \to \left(\lambda x, \lambda^{-1}\xi\right) \tag{3.13}$$

The map $F$ is the canonical lift of the map $f : \mathbb{R} \to \mathbb{R}$. See figure 3.3.

We observe that the map $F$ has a **trapped set** (or non wandering set) $K = (0, 0)$ **compact** in $T^*\mathbb{R}$, in the precise sense that

$$K := \{(x, \xi), \exists C \Subset T^*M \text{ compact}, \forall n \in \mathbb{Z}, F^n(x, \xi) \in C\} = \{(0, 0)\}$$

*Remark* 3.7. The dynamics of the map $F$ in $\mathbb{R}^2 \equiv T^*\mathbb{R}$ looks like **"scattering" on the trapped set** $K$.

*Remark* 3.8. In the cotangent space $T^*\mathbb{R}$, the wave front (see definition (A.13)) of the distribution $x^k$ which enter in the spectral projector (3.8) is the line $E_u = \{(x, \xi), x \in \mathbb{R}, \xi = 0\}$ and the wavefront set of $\delta^{(k)}$ is the line $E_s = \{(x, \xi), x = 0, \xi \in \mathbb{R}\}$. They are respectively the unstable/stable manifolds for the trapped set $K$ of the canonical map $F$.

[12]proof: $\widetilde{\left(\hat{F}u\right)}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} u\left(\frac{1}{\lambda}x\right) dx = \lambda \frac{1}{\sqrt{2\pi}} \int e^{-i\xi\lambda y} u(y)\, dy = \lambda\tilde{u}(\lambda\xi)$.

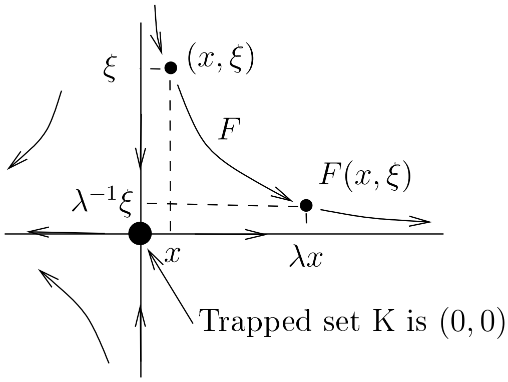


Figure 3.3: The canonical map $F$, eq.(3.13)

Write

$$z := (x,\xi) \in \mathbb{R}^2$$

Let $C > 0$ and the $C^\infty$ function

$$A_C(z) := \langle z\rangle^{m(z)} \tag{3.14}$$

called **Lyapounov function** of **escape function**, where $\langle z\rangle := \sqrt{1+|z|^2}$ and $m(z) \in C^\infty(\mathbb{R}^2)$ called the **order function** is homogeneous of degree 0 on $|z| \geq 1$ (that is $m(\lambda z) = m(z)$ for $|z| \geq 1$, $\lambda \geq 1$) and such that $m(z) = +C$ in a conical vicinity of the stable axis $x = 0$ and $m(z) = -C$ in a conical vicinity of the unstable axis $\xi = 0$. And $m(z)$ decreases between these two directions so that

$$m(F(z)) \leq m(z), \forall |z| \geq 1. \tag{3.15}$$

Along the stable direction one has $|z| \sim |\xi| \gg 1$ and from (3.14) and (3.13) one has

$$A_C(z) \sim |\xi|^C, \qquad \frac{A_C(F(z))}{A_C(z)} \simeq \frac{|\lambda^{-1}\xi|^C}{|\xi|^C} \simeq \lambda^{-C} \ll 1$$

Similarly along the unstable drection, one has $|z| \sim |x| \gg 1$ and

$$A_C(z) \sim |x|^{-C}, \qquad \frac{A_C(F(z))}{A_C(z)} \simeq \frac{|\lambda x|^{-C}}{|x|^{-C}} \simeq \lambda^{-C} \ll 1$$

One can check in fact that in every direction and for $|z| \gg 1$ one has

$$\frac{A_C(F(z))}{A_C(z)} \lesssim \lambda^{-C} \ll 1 \tag{3.16}$$

*Remark* 3.9. (*) The function $m(z) \in S^0(\mathbb{R}^2)$ is a symbol according to (A.3) and the function $A_C \in S_\rho^{m(z)}$ is a symbol with variable order $m(z)$ according to (A.4), with any $0 < \rho < 1$.

Let us define the pseudodifferential operator (PDO) $\mathrm{Op}\left(A_C\right) : \mathcal{S}\left(\mathbb{R}\right) \to \mathcal{S}\left(\mathbb{R}\right)$ by ordinary quantization (see Appendix A)

$$\left(\mathrm{Op}\left(A_C\right) u\right)(x) := \frac{1}{2\pi} \int e^{i\xi x} A_C\left(x, \xi\right) e^{-i\xi y} u\left(y\right) d\xi dy$$

(it can be modified by a subleading PDO, i.e. with lower order, so that it becomes selfadjoint and invertible). Then in $L^2\left(\mathbb{R}\right)$, let us consider the operator obtained by conjugation:

$$\hat{Q} := \mathrm{Op}\left(A_C\right) \circ \hat{F} \circ \mathrm{Op}\left(A_C\right)^{-1}$$

From **Egorov Theorem** we have that

$$\mathrm{Op}\left(A_C\right) \circ \hat{F} \circ \mathrm{Op}\left(A_C\right)^{-1} = \hat{F} \circ \left(\mathrm{Op}\left(A_C \circ F\right) + O\left(\mathrm{Op}\left(S^{m\circ F - \rho}\right)\right)\right) \circ \mathrm{Op}\left(A_C\right)^{-1}$$

where $\left(A_C \circ F\right) \in S^{m\circ F}$, the notation $O\left(\mathrm{Op}\left(S^{m'}\right)\right)$ means a term which belongs to $\mathrm{Op}\left(S^{m'}\right)$ and for any $1/2 < \rho < 1$. The **Theorem of composition of PDO** (see Appendix A) gives that

$$\mathrm{Op}\left(A_C \circ F\right) \circ \mathrm{Op}\left(A_C\right)^{-1} = \mathrm{Op}\left(\frac{A_C \circ F}{A_C}\right) + O\left(\mathrm{Op}\left(S^{m\circ F - m - \rho}\right)\right)$$

where

$$\frac{A_C \circ F}{A_C} \in S^{m\circ F - m} \subset S^0$$

The last inclusion is because $m \circ F - m \leq 0$ from (3.15). In conclusion we have that

$$\hat{Q} = \hat{F} \circ \left(\mathrm{Op}\left(\frac{A_C \circ F}{A_C}\right) + O\left(\mathrm{Op}\left(S^{-\rho}\right)\right)\right)$$

The **theorem of $L^2$-continuity** gives that for norm operator

$$\left\|\mathrm{Op}\left(\frac{A_C \circ F}{A_C}\right) + O\left(\mathrm{Op}\left(S^{-\infty}\right)\right)\right\| \leq \limsup_{(x,\xi)} \left|\frac{A_C \circ F}{A_C}\left(x, \xi\right)\right| \underset{(3.16)}{\leq} \lambda^{-C}$$

Since $\hat{F}$ is bounded on $L^2\left(\mathbb{R}\right)$ we have that $\left\|\hat{Q} + O\left(\mathrm{Op}\left(S^{-\rho}\right)\right)\right\| \leq \left\|\hat{F}\right\|_{L^2(\mathbb{R})} \cdot \lambda^{-C}$. Finally an operator $\hat{K} \in \mathrm{Op}\left(S^{-\rho}\right)$ with $\rho > 0$ is compact hence

$$\hat{Q} = \hat{K} + \hat{R}$$

with $\left\|\hat{R}\right\| \leq \mathrm{cste}.\lambda^{-C}$ and $\hat{K}$ a **compact operator**. From the commutative diagram

$$\begin{array}{ccc} L^2\left(\mathbb{R}\right) & \xrightarrow{\hat{Q}} & L^2\left(\mathbb{R}\right) \\ \mathrm{Op}\left(A_C\right)\uparrow & & \mathrm{Op}\left(A_C\right)\uparrow \\ \mathcal{H}_C & \xrightarrow{\hat{F}} & \mathcal{H}_C \end{array} \tag{3.17}$$

one has the same result for $\hat{F}$ in the space

$$\mathcal{H}_C := \mathrm{Op}\left(A_C\right)^{-1}\left(L^2\left(\mathbb{R}\right)\right)$$

with norm $\|u\|_{\mathcal{H}_C} := \|\mathrm{Op}\left(A_C\right)u\|_{L^2}$. The space $\mathcal{H}_C$ is called[13] **anisotropic Sobolev space.** Notice that $\mathcal{H}_C$ contains regular (smooth) functions but that may grows in $x$. So $x^k \in \mathcal{H}_C$ for $k \leq C$, but $\delta^{(k)} \notin \mathcal{H}_C$. For the dual space $\mathcal{H}'_C = \mathrm{Op}\left(A_C\right)\left(L^2\left(\mathbb{R}\right)\right) = \mathcal{H}_{-C}$ this is the opposite: $\delta^{(k)} \in \mathcal{H}_{-C}$. As a result, the operator $\Pi_k$ is bounded in $\mathcal{H}_C \to \mathcal{H}_C$.

*Remark* 3.10. (*) The dual operator (3.4) (or Perron Frobenius operator)

$$\hat{F}^* : \begin{cases} \mathcal{H}_{-C} & \to \mathcal{H}_{-C} \\ v & \to \lambda.v\left(\lambda x\right) \end{cases}$$

has the same spectrum $\lambda^{-k}$, $k \geq 0$. (conjugate spectrum, but the spectrum is real).

*Remark* 3.11. In a finite dimensional vector space a conjugation like (3.17) does not change the spectrum of the operator. In our case, with infinite dimension, the essential spectrum is moved away, and reveals discrete (Ruelle) spectrum that is “robust and intrinsic”.

### 3.1.5 Ruelle spectrum for expanding map in $\mathbb{R}^d$

Theorem 3.4 can be easily generalized for an expanding linear map on $\mathbb{R}^d$ with any $d \geq 1$. We will use this later.

Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear invertible expanding map satisfying $\|A^{-1}\| \leq 1/\lambda$ for some $\lambda > 1$. Let

$$\mathcal{L}_A : \begin{cases} \mathcal{S}\left(\mathbb{R}^d\right) & \to \mathcal{S}\left(\mathbb{R}^d\right) \\ u & \to u \circ A^{-1} \end{cases} \tag{3.18}$$

be the associated transfer operator. For $k \in \mathbb{N}$, let[14] $\mathrm{Polynom}^{(k)} := \mathrm{Span}\left\{x^\alpha, \alpha \in \mathbb{N}^d, |\alpha| = k\right\}$ be the space of homogeneous polynomial on $\mathbb{R}^d$ of degree $k$.

$$\dim\left(\mathrm{Polynom}^{(k)}\right) = \begin{pmatrix} d+k-1 \\ d-1 \end{pmatrix} = \frac{(d+k-1)!}{(d-1)!k!}$$

Then we consider the finite rank operator

$$\Pi_k : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathrm{Polynom}^{(k)}, \qquad \left(\Pi_k u\right)(x) = \sum_{\alpha \in \mathbb{N}^d, |\alpha| = k} \frac{\partial^\alpha u(0)}{\alpha!} \cdot x^\alpha. \tag{3.19}$$

[13] Recall that the usual Sobolev space with constant order $m \in \mathbb{R}$ is defined by[Tay96a] $H^m\left(\mathbb{R}\right) := \left(\mathrm{Op}\left(\langle\xi\rangle^m\right)\right)^{-1}\left(L^2\left(\mathbb{R}\right)\right)$.

[14] for a multi index $\alpha \in \mathbb{N}^d$, $\alpha = (\alpha_1, \ldots \alpha_d)$, we write $|\alpha| = \alpha_1 + \ldots + \alpha_d$.

This is a projector which extracts the terms of degree $k$ in the Taylor expansion.We have the following relations

$$\Pi_j \circ \Pi_k = \delta_{j=k} \Pi_k \tag{3.20}$$

and

$$[\Pi_k, \mathcal{L}_A] = 0. \tag{3.21}$$

Let us prepare some notations. For a linear invertible map $L$ we will use the notation

$$\|L\|_{\max} := \|L\|, \qquad \|L\|_{\min} := \left\|L^{-1}\right\|^{-1}. \tag{3.22}$$

**Theorem 3.12.** *[FT15, Prop. 4.19]For any $C > 0$, there exists a Hilbert space $\mathcal{H}_C$ (an "anisotropic Sobolev space")*

$$\mathcal{S}\left(\mathbb{R}^d\right) \subset \mathcal{H}_C \subset \mathcal{S}'\left(\mathbb{R}^d\right)$$

*such that the operator (3.18): $\mathcal{L}_A : \mathcal{H}_C \to \mathcal{H}_C$ is bounded and has essential spectral radius $r_{ess} = cste.\lambda^{-C}$ $(\underset{C\to+\infty}{\to} 0)$. For $K \leq C - 2d$, there is a decomposition preserved by $\mathcal{L}_A$:*

$$\mathcal{H}_C = \left(\bigoplus_{k=0}^{K} \mathrm{Polynom}^{(k)}\right) \oplus \tilde{\mathcal{H}}$$

*such that*

1. *$\exists C_0$, for any $0 \leq k \leq K$ and $0 \neq u \in \mathrm{Polynom}^{(k)}$, we have for any $n \geq 1$,*

$$C_0^{-1} \|A^n\|_{\max}^{-k} \leq \frac{\|\mathcal{L}_{A^n} u\|_{\mathcal{H}_C}}{\|u\|_{\mathcal{H}_C}} \leq C_0 \|A^n\|_{\min}^{-k} \tag{3.23}$$

2. *The operator norm of the restriction of $\mathcal{L}_A$ to $\widetilde{\mathcal{H}}$ is bounded by*

$$C_0 \max\{\|A^n\|_{\min}^{-(K+1)}, \|A^n\|_{\min}^{-C} \cdot |\det A^n|\}. \tag{3.24}$$

*Remark* 3.13. (*) Theorem 3.12 implies that the spectrum of the transfer operator $\mathcal{L}_A$ in the Hilbert space $\mathcal{H}_C$ is discrete outside the radius $r_{ess}$. The eigenvalues outside this radius are given by the action of $\mathcal{L}_A$ in the finite dimensional space $\mathrm{Polynom}^{(k)}$. These eigenvalues can be computed explicitly from the Jordan block decomposition of $A$. In particular if $A = \mathrm{Diag}\,(a_1, \ldots a_d)$ is diagonal then the monomials $x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ are obviously eigenvectors of $\mathcal{L}_A$ with respective eigenvalues $\prod_j a_j^{-\alpha_j}$.

## 3.2 Ruelle Spectrum of Anosov map

Let $f : M \to M$ be an Anosov map as in Definition (2.1).

**Definition 3.14.** Let $V \in C^{\infty}(M)$ real valued, called potential. The transfer operator is

$$\hat{F} : \begin{cases} C^{\infty}(M) & \to C^{\infty}(M) \\ u & \to e^{V} \cdot (u \circ f^{-1}) \end{cases} \tag{3.25}$$

*Remark* 3.15. The choice $u \circ f^{-1}$ instead of $u \circ f$ is such that $f$ maps $\mathrm{supp}(u)$ to $\mathrm{supp}\left(\hat{F}u\right)$.

*Remark* 3.16. (*)The $L^2$ adjoint operator is given by

$$\left(\hat{F}^{*}v\right)(y) = \frac{e^{V\circ f}}{|\mathrm{det}Df|}(v \circ f)$$

and called **Perron-Frobenius operator**. It transports densities and preserves probabilities if $V = 0$:

$$\int_M \left(\hat{F}^{*}v\right) dy = \langle 1|\hat{F}^{*}v\rangle = \langle \hat{F}1|v\rangle = \int v dy$$

*Remark* 3.17. By duality the transfer operator can be extended to distributions: $\hat{F} : \mathcal{D}'(M) \to \mathcal{D}'(M)$.

Let $T^{*}M = E_s^{*} \oplus E_u^{*}$ be the decomposition dual to Eq.(2.1), i.e. $E_s^{*}(E_s) = 0$ and $E_u^{*}(E_u) = 0$.

**Theorem 3.18.** *[Rug92, BKL02, BT07, FRS08]"**Discrete spectrum**". For any $C > 0$, there exists an anisotropic Sobolev space $\mathcal{H}_C$:*

$$C^{\infty}(M) \subset \mathcal{H}_C \subset \mathcal{D}'(M)$$

*with variable order function $m \in C^{\infty}(T^{*}M)$ with $m(x,\xi) = \pm C$ along $E_{u/s}^{*}$such that*

$$\hat{F} : \mathcal{H}_C \to \mathcal{H}_C$$

*is bounded and has essential spectral radius $r_{ess} = O(1).\lambda^{-C}$ $\left(\underset{C\to+\infty}{\to} 0\right)$. The eigenvalues (and eigenspaces) outside $r_{ess}$ do not depend on $m$ and are called **Ruelle-Pollicott resonances.** The space $\mathcal{H}_C$ does not depend on $V$. The wavefront set of the eigendistributions is contained in $E_u^{*}$.*

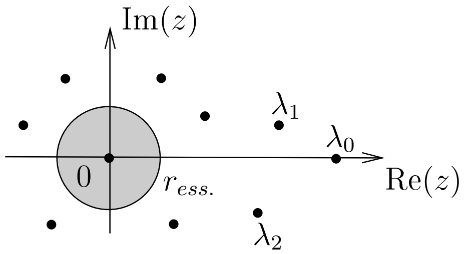


Figure 3.4: Ruelle Pollicott resonances of $\hat{F}$.

See figure 3.4.

*Remark* 3.19. We will denote Res $\left(\hat{F}\right)$ the set of Ruelle-Pollicott resonances (eigenvalues). The only obvious eigenvalue is for the case $V = 0$: it is $\lambda_0 = 1$ with eigenfunction $u_0 = 1$.

*Remark* 3.20. For the hyperbolic automorphism on the torus (2.3), with $V = 0$, the Ruelle spectrum is only Res $\left(\hat{F}\right) = \{1\}$. To show this, use (2.7) and (3.26).

*Remark* 3.21. The Ruelle spectrum describes asymptotic of time correlation functions (2.6): for $V = 0$ in (3.25), one has for $u, v \in C^\infty(M)$ and any $\varepsilon > 0$,

$$\begin{aligned} C_{v,u}(n) &\underset{(2.6)}{=} \int v.u \circ f^{-n} dx \underset{(3.25)}{=} \langle v | \hat{F}^n u \rangle \\ &= \sum_{\lambda_j \in \mathrm{Res}\left(\hat{F}\right), |\lambda_j| \geq \varepsilon} \langle v | \left(\hat{F}^n \Pi_j\right) u \rangle + \|u\|_{\mathcal{H}_C} . \|v\|_{\mathcal{H}_{-C}} .O(\varepsilon^n) \end{aligned} \tag{3.26}$$

where $\Pi_j$ denotes the finite rank spectral projector $\hat{F}$ associated to the eigenvalue $\lambda_j$. $\mathcal{H}_{-C}$ is the space dual to $\mathcal{H}_C$ (precisely defined with the order function $-m(x,\xi)$ instead of $+m(x,\xi)$).

**Proposition 3.22.** *(Anosov) If $f : M \to M$ is an Anosov diffeomorphism preserving a smooth measure $dx$, then for any real valued potential $V$, there is a simple "leading" eigenvalue $\lambda_0 > 0$ in the sense that the other ones are $\lambda_j \in \mathbb{C}$ with $|\lambda_j| < \lambda_0$ as in figure 3.4.*

*Remark* 3.23. In the particular case $V=0$ then $\lambda_0=1$, $\Pi_0=|1\rangle\langle 1|$ and $|\lambda_1|<1$. Then (3.26) gives that for any $\varepsilon>|\lambda_1|$:

$$\begin{aligned} C_{v,u}(n) &= \int v.u\circ f^{-n}dx = \langle v|1\rangle\langle 1|u\rangle + O(\varepsilon^n) \\ &= \int \overline{v}dx.\int udx + O(\varepsilon^n) \end{aligned} \tag{3.27}$$

This proves the exponential mixing (2.5).

### 3.2.1 Proof of Theorem 3.18

This proof from [FRS08] uses Semiclassical analysis. The proof is very similar to the proof of Theorem 3.4 given above.

The transfer operator (3.25) is a Fourier integral operator. Its canonical map is

$$F:\begin{cases} T^*M & \rightarrow T^*M \\ (x,\xi) & \rightarrow (x',\xi') = \left(f(x), {}^tDf_x^{-1}.\xi\right) \end{cases} \tag{3.28}$$

$F$ is the canonical lift of $f:M\rightarrow M$ on the cotangent bundle $T^*M$. See figure 3.5.

**Heuristic interpretation of the canonical map $F$ from the expression of the transfer operator $\hat{F}$ (3.25):**

- If $u\in C^\infty(M)$ with support $\mathrm{supp}(u)$ then $\hat{F}u$ as support $f(\mathrm{supp}(u))$. This explains that $x'=f(x)$ in (3.28).
- If on some local chart $u(x)=e^{i\xi.x}$ with some $|\xi|\gg 1$, i.e. $u$ is a "fast oscillating function", then $\left(\hat{F}u\right)(y)=e^Ve^{i\xi.f^{-1}(y)}$. Put $y=f(x)+y'$ with $|y'|\ll 1$, so $f^{-1}(y)=x+Df_y^{-1}.y'+o(|y'|)$ (by Taylor) so

  $$\left(\hat{F}u\right)(y)\simeq e^Ve^{i\xi.\left(x+Df_y^{-1}.y\right)} = C.e^{i\left({}^tDf^{-1}\xi\right).y} = C.e^{i\xi'.y}$$

  with $\xi'={}^tDf^{-1}\xi$. We have obtained (3.28).

The **trapped set** (or non wandering set) of the map $F:T^*M\rightarrow T^*M$ is the zero section

$$K=\{(x,\xi)\in T^*M, x\in M, \xi=0\}. \tag{3.29}$$

For $\rho\in K$ we let

$$E_u^*(\rho):=\left\{v\in T_\rho(T^*M), \left|DF_\rho^{-n}(v)\right|\underset{n\rightarrow+\infty}{\rightarrow}0\right\}$$

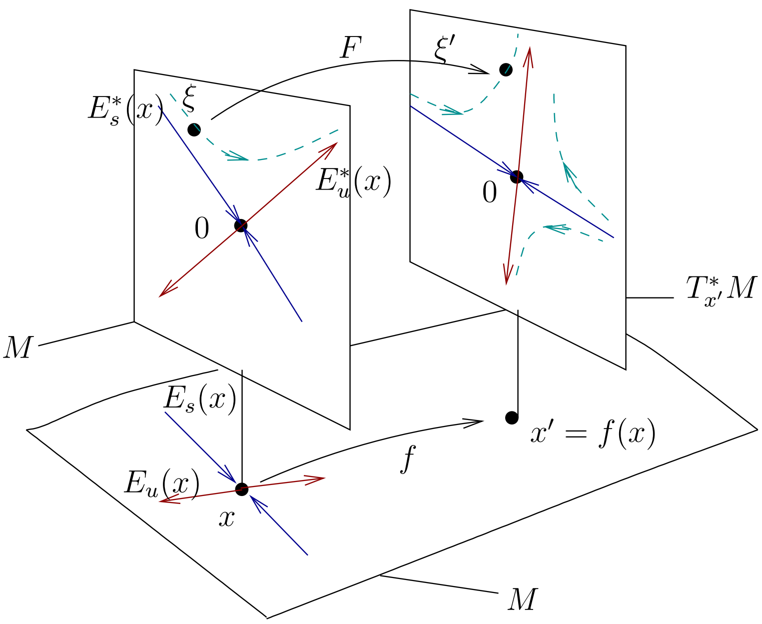


Figure 3.5: The canonical map $F$, eq.(3.28).

$$E_s^*(\rho) := \left\{ v \in T_\rho(T^*M), \left| DF_\rho^n(v) \right| \underset{n\to+\infty}{\to} 0 \right\}$$

We define an **escape function** with **variable order** $m \in C^\infty(T^*M)$ so that

$$\left(\frac{A_m \circ F}{A_m}\right)(x,\xi) < C.\lambda^{-C} \ll 1 \quad \text{for } |\xi| \gg 1, \tag{3.30}$$

and such that $A_m \in S_\rho^m$ is a "good symbol" (see definition A.2). For this we choose

$$A_m(x,\xi) := \langle \xi \rangle^{m(x,\xi)} \tag{3.31}$$

with

$$\begin{aligned} m(x,\xi) &= C \gg 0 \quad \text{along } E_s^* \\ m(x,\xi) &= -C \ll 0 \quad \text{along } E_u^* \end{aligned}$$

Define the following **pseudo-differential operator** using local coordinates

$$\hat{A}_m u := \mathrm{Op}(A_m)\, u := \frac{1}{(2\pi)^{2d}} \int e^{i\xi.x} A_m(x,\xi)\, e^{-i\xi.y} u(y)\, dy d\xi$$

and the **anisotropic Sobolev space:**

$$\mathcal{H}_C := \hat{A}_m^{-1}\left(L^2(M)\right)$$

Then one has a commutative diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}) & \overset{\hat{Q}:=\hat{A}_m\hat{F}\hat{A}_m^{-1}}{\longrightarrow} & L^2(\mathbb{R}) \\ \hat{A}_m \uparrow & & \hat{A}_m \uparrow \\ \mathcal{H}_C & \overset{\hat{F}}{\longrightarrow} & \mathcal{H}_C \end{array}$$

Then

$$\hat{Q} := \mathrm{Op}\left(A_m\right)\circ\hat{F}\circ\mathrm{Op}\left(A_m\right)^{-1} = \hat{F}\circ\mathrm{Op}\left(A_m\circ F\right)\circ\mathrm{Op}\left(A_m\right)^{-1} + l.o.t. = \hat{F}\circ\mathrm{Op}\left(\frac{A_m\circ F}{A_m}\right) + l.o.t$$

From $L^2$ continuity theorem and (3.30), on has

$$\mathrm{Op}\left(\frac{A_m\circ F}{A_m}\right) = \hat{K} + \hat{R}$$

with $\left\|\hat{R}\right\| \leq c.\lambda^{-C}$ and $\hat{K}$ a **compact operator** (smoothing). The same decomposition holds for $\hat{Q}: L^2 \to L^2$ and $\hat{F}: \mathcal{H}_C \to \mathcal{H}_C$.

### 3.2.2 The Atiyah-Bott trace formula

**Definition 3.24.** The **flat trace** of the transfer operator (3.25) is

$$\mathrm{Tr}^\flat \hat{F} := \int_M K\left(x, x\right) dx \tag{3.32}$$

where $K\left(x, y\right) dy$ is the Schwartz kernel of $\hat{F}$.

*Remark* 3.25. We recall that the Schwartz kernel of $\hat{F}$ is defined by $\left(\hat{F}u\right)(x) = \int K\left(x, y\right) u\left(y\right) dy$. It is a current. More generally the flat trace can be defined for a vector bundle map $B: E \to E$ lifting a diffeomorphism $f: M \to M$ on a vector bundle $E \to M$, such that all fixed points of $f$ are hyperbolic.

**Proposition 3.26.** *[AB67]For any $n \geq 1$, the **Atiyah-Bott trace formula** is*

$$\mathrm{Tr}^\flat\left(\hat{F}^n\right) = \sum_{x=f^n(x)} \frac{e^{V_n(x)}}{\left|\det\left(1 - Df_x^{-n}\right)\right|} \tag{3.33}$$

*where*

$$V_n\left(x\right) = \sum_{k=0}^{n-1} V\left(f^{-k}\left(x\right)\right) \tag{3.34}$$

*Remark* 3.27. In (3.33), this is a finite sum over periodic points.

*Proof.* (*) (Atiyah-Bott 1965 [AB66, AB67]. From (3.25), and denoting $\delta_y(x) = \delta(y-x)$ the Dirac distribution at $y$, the Schwartz kernel of $\hat{F}^n$ is

$$\begin{aligned} K_n(x,y) &= \left(\hat{F}\delta_y\right)(x) = \delta_y\left(f^{-n}(x)\right)e^{V_n(x)} \\ &= \delta\left(y - f^{-n}(x)\right)e^{V_n(x)} \end{aligned}$$

From (3.32), one has (using the change of variable $y = x - f^{-n}(x)$ in the second line)

$$\begin{aligned} \mathrm{Tr}^\flat\left(\hat{F}^n\right) &= \int_M \delta\left(x - f^{-n}(x)\right)e^{V_n(x)}dx \\ &= \sum_{x=f^n(x)} \frac{e^{V_n(x)}}{\left|\det\left(1 - Df_x^{-n}\right)\right|} \end{aligned}$$

□

*Remark* 3.28. If $f$ preserves $dx$ then $\left|\det\left(Df_x^n\right)\right| = 1$ so

$$\mathrm{Tr}^\flat\left(\hat{F}^n\right) = \sum_{x=f^n(x)} \frac{e^{V_n(x)}}{\left|\det\left(1 - Df_x^n\right)\right|}$$

**Lemma 3.29.** *[BT06]***"Flat trace and spectrum".** *For any* $\varepsilon > 0$,

$$\mathrm{Tr}^\flat\left(\hat{F}^n\right) = \sum_{\lambda_j\in\mathrm{Res}\left(\hat{F}\right),|\lambda_j|\geq\varepsilon} \lambda_j^n + O(1).\varepsilon^n \tag{3.35}$$

$$\underset{(3.33)}{=} \sum_{x=f^n(x)} \frac{e^{V_n(x)}}{\left|\det\left(1 - Df_x^{-n}\right)\right|} \tag{3.36}$$

*Proof.* From [BT06] (see also [FT15, chap.11]) we decompose

$$\hat{F}^n = \hat{F}_0^n + \hat{F}_1^n$$

where $\hat{F}_0 = \sum_{\lambda_j,|\lambda_j|\geq\varepsilon} \hat{F}_0\Pi_j$ is the finite rank spectral component of $\hat{F}$, $\hat{F}_1 = \hat{F} - \hat{F}_0$ so $\left[\hat{F}_0, \hat{F}_1\right] = 0$. One has $\left\|\hat{F}_1^n\right\| \leq O(1).\varepsilon^n$ and prove that

$$\left|\mathrm{Tr}^\flat\left(\hat{F}_1^n\right)\right| \leq O(1).\left\|\hat{F}_1^n\right\| \leq O(1)\,\varepsilon^n$$

□

**Consequences** As in Prop 3.22 let $\lambda_0 > 0$ be the leading eigenvalue and $|\lambda_1| < \lambda_0$ the next one. One has for any $\varepsilon > 0$,

$$\begin{aligned}\mathrm{Tr}^\flat\left(\hat{F}^n\right) &\underset{(3.35)}{=} \lambda_0^n + O(1)\left(|\lambda_1| + \varepsilon\right)^n \\ &\underset{(3.33)}{=} \sum_{x=f^n(x)} \frac{e^{V_n(x)}}{\left|\det\left(1 - Df_x^{-n}\right)\right|}\end{aligned}$$

so for $n \gg 1$,

$$\log\lambda_0 = \frac{1}{n}\log\left(\sum_{x=f^n(x)} \frac{e^{V_n(x)}}{\left|\det\left(1 - Df_x^{-n}\right)\right|}\right) + O(1)\left(\frac{(|\lambda_1| + \varepsilon)}{\lambda_0}\right)^n$$

For a function $\varphi \in C(M)$ let

$$\Pr(\varphi) := \lim_{n\to\infty} \frac{1}{n}\log\left(\sum_{x=f^n(x)} e^{\varphi_n(x)}\right) \tag{3.37}$$

called **the topological pressure** of $\varphi$ with $\varphi_n := \sum_{k=0}^{n-1}\varphi\left(f^{-k}(x)\right)$. Using other transfer operators and because

$$\left|\det\left(1 - Df_x^{-n}\right)\right|^{-1} \underset{n\to\infty}{\sim} \left|\det Df^{-n}_{|E_s(x)}\right|^{-1} = e^{-J_n(x)}$$

with "**the unstable Jacobian**"[15]

$$J(x) := \log\left|\det Df^{-1}_{|E_s(x)}\right| \tag{3.38}$$

on can show that

**Proposition 3.30.** *One has*

$$\log\lambda_0 = \Pr(V - J) \tag{3.39}$$

- *In particular in the case $V = 0$, we have $\lambda_0 = 1$ from remark 3.23 so (3.39) gives* $\Pr(-J) = 0$.
- *In the particular case $V = J$, $\lambda_0 = \Pr(0) =: h_{top}$ is called the* ***topological entropy.*** *From def. (3.37), $h_{\mathrm{top}}$ gives the exponential rate for the number of periodic points:* $\sharp\left\{x = f^n(x)\right\} \underset{n\to\infty}{\sim} e^{(h_{top}+o(1))n}$.

[15] Notice that if $\det Df = 1$ then $\det Df^{-1}_{|E_s(x)} = \det Df_{|E_u(x)}$.

## 3.3 Ruelle band spectrum for prequantum Anosov maps

Consider the prequantum map $\tilde{f} : P \to P$ defined in (2.10). We follow Section 3.2.

**Definition 3.31.** Let $V \in C^{\infty}(M)$ real valued, called **potential**. The prequantum transfer operator is

$$\hat{F} : \begin{cases} C^{\infty}(P) & \to C^{\infty}(P) \\ u & \to e^{V\circ\pi} \cdot \left(u \circ \tilde{f}^{-1}\right) \end{cases} \tag{3.40}$$

It preserves the Fourier $N-$mode space for every $N \in \mathbb{Z}$:

$$C_N^{\infty}(P) := \left\{u \in C^{\infty}(P), \forall p \in P, \forall e^{i\theta} \in U(1), \quad u\left(e^{i\theta}p\right) = e^{iN\theta}u(p)\right\} \tag{3.41}$$

We denote

$$\hat{F}_N := \hat{F}_{|C_N^{\infty}(P)} \tag{3.42}$$

and for $N \neq 0$ we put

$$\hbar := \frac{1}{2\pi N} \tag{3.43}$$

*Remark* 3.32. From (3.43) we have $e^{iN\theta} = e^{i\theta/(2\pi\hbar)}$. This notation is used in quantum mechanics. So the space $C_N^{\infty}(P)$ contains functions which oscillates fast along the direction $\frac{\partial}{\partial\theta}$ as $N \to \infty$. For that reason, the limit $N \to \infty$ or $\hbar \to 0$ is called the **semiclassical limit**.

(*) In the theory of associated vector bundles it is shown that[Tay96b]

$$C_N^{\infty}(P) \equiv C^{\infty}\left(M, L^{\otimes N}\right)$$

the space of sections of a complex line bundle power $N$, where $L \to M$ is the associated complex line bundle usually called "**prequantum line bundle**". For simplicity it is equivalent to work with (3.41).

*Remark* 3.33. Theorem 3.18 extends to transfer operators acting on vector bundles. So it applies for the operator $\hat{F}_N$, for any $N$. Consequently the operator $\hat{F}_N$ has discrete spectrum of Ruelle resonances $\mathrm{Res}\left(\hat{F}_N\right)$.

We define the special "potential of reference"

$$V_0(x) := \frac{1}{2}\log\left|\det Df_{f^{-1}(x)}|_{E_u(f^{-1}(x))}\right| \tag{3.44}$$

Notice that the unstable foliation $E_u(x)$ is not smooth in $x$ in general which implies that the function $V_0$ is Hölder continuous but not smooth in $x$. We then consider the difference

$$D := V - V_0 \quad \in C^{\beta}(M) \tag{3.45}$$

which is also a Hölder continuous function on $M$ and that will be called the "**effective damping function**". It will appear in many results below. Finally we denote by

$$D_n(x) := \sum_{j=1}^{n} D\left(f^j(x)\right) \tag{3.46}$$

the Birkhoff sum of the damping function. Recall (3.22) for the definition of $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$.

**Theorem 3.34.** *[FT15]* "**Band structure**". *For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, $N_\varepsilon \geq 1$ such that for any $N \geq N_\varepsilon$*

1. *the Ruelle-Pollicott resonances of $\hat{F}_N$ is contained in a small neighborhood of the union of annuli $\left(\mathcal{A}_k := \{r_k^- \leq |z| \leq r_k^+\}\right)_{k\geq 0}$:*

$$Res\left(\hat{F}_N\right) \subset \bigcup_{k\geq 0} \underbrace{\left\{r_k^- - \varepsilon \leq |z| \leq r_k^+ + \varepsilon\right\}}_{\varepsilon\text{-neighborhood of } \mathcal{A}_k} \tag{3.47}$$

*with*

$$\begin{aligned} r_k^- &:= \lim_{n\to\infty} \inf_{x\in M} \left(e^{\frac{1}{n}D_n(x)} \|Df_x^n|_{E_u}\|_{\max}^{-k/n}\right), \\ r_k^+ &:= \lim_{n\to\infty} {}_{n\to\infty} \sup_{x\in M} \left(e^{\frac{1}{n}D_n(x)} \|Df_x^n|_{E_u}\|_{\min}^{-k/n}\right) \end{aligned} \tag{3.48}$$

2. *Suppose that $r_k^+ < r_{k-1}^-$ for some $k \geq 1$. For any $z \in \mathbb{C}$ such that $r_k^+ + \varepsilon < |z| < r_{k-1}^- - \varepsilon$, i.e. such that $z$ is in a "gap", the resolvent of $\hat{F}_N$ on $\mathcal{H}_N^r(P)$ is controlled uniformly with respect to $N$:*

$$\left\| \left(z - \hat{F}_N\right)^{-1} \right\| \leq C_\varepsilon \tag{3.49}$$

*This is also true for $|z| > r_0^+ + \varepsilon$.*

3. *If $r_1^+ < r_0^-$, i.e. if the outmost annulus $\mathcal{A}_0$ is isolated from other annuli, then the number of resonances in its neighborhood satisfies the estimate called "**Weyl formula**"*

$$\sharp\left\{Res\left(\hat{F}_N\right) \bigcap \left\{r_0^- - \varepsilon \leq |z| \leq r_0^+ + \varepsilon\right\}\right\} = N^d \mathrm{Vol}_\omega(M)\left(1 + O\left(N^{-1}\right)\right). \tag{3.50}$$

*with $Vol_\omega(M) := \int_M \frac{1}{d!}\omega^{\wedge d}$ being the symplectic volume of $M$ and $\delta > 0$. Moreover in the limit $N \to \infty$, most of these resonances **concentrate and equidistribute** on the circle of radius*

$$R := e^{\langle D\rangle}, \quad \text{with } \langle D\rangle := \frac{1}{Vol_\omega(M)} \int_M D(x)\, dx \tag{3.51}$$

*Remark* 3.35.

1. Since $\|Df_x^n|_{E_u}\|_{\max}^{1/n} \geq \|Df_x^n|_{E_u}\|_{\min}^{1/n} > \lambda > 1$, from (2.2), we have obviously $r_k^- \leq r_k^+$, $r_{k+1}^- < r_k^-$ and $r_{k+1}^+ < r_k^+$ for every $k \geq 0$. However we don't always have $r_{k+1}^+ < r_k^-$ therefore the annuli $\mathcal{A}_k$ may intersect each other.

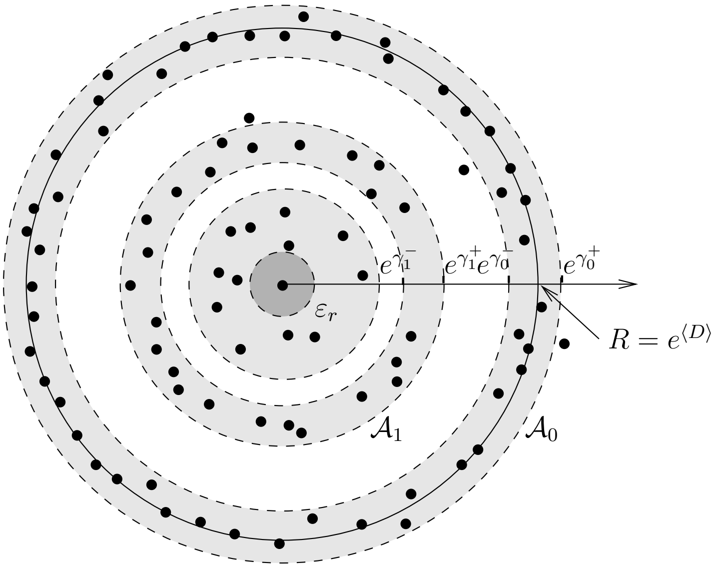


Figure 3.6: Figure for Theorem 3.34.

2. In the case $V = 0$, one has $r_0^+ < 1$ so one can deduce exponential mixing for the prequantum map $\tilde{f}$, see remark 2.15.

3. It is tempting to take the potential $V = V_0$ defined in (3.44) which would indeed give $D = 0$ hence $r_0^+ = r_0^- = 1$ in (3.48). In that case the external band $\mathcal{A}_0$ would be the unit circle, separated from the internal band $\mathcal{A}_1$ by a spectral gap $r_1^+$ given by

$$r_1^+ = \lim_{n\to\infty} {}_{n\to\infty} \sup_{x\in M} \left( \| Df_x^n|_{E_u} \|_{\min}^{-1/n} \right) < \frac{1}{\lambda} < 1$$

   See figure 3.7. However Theorem 3.34 does not apply in this case because the function $V_0$ is not smooth in $x$ as required. In [FT15] it is shown how to generalize the result to this case using an extension of the transfer operator to the Grassmanian bundle.

4. In the simple case of a linear hyperbolic map on the torus $\mathbb{T}^2$, i.e. example (2.4) with $V(x) = 0$, then $r_k^+ = r_k^- = \lambda^{-k-\frac{1}{2}}$, with $\lambda = Df_{0/E_u} = \frac{3+\sqrt{5}}{2} \simeq 2.6$ (constant), i.e. each annulus $\mathcal{A}_k$ is a circle. In this case Theorem 3.34 has been obtained in [Fau07a, fig.1-b]. If one chooses $V(x) = \frac{1}{2}\log|\mathrm{det} Df_x|_{E_u}| = \frac{1}{2}\log\lambda$ the external band $\mathcal{A}_0$ is the unit circle and it is shown in [Fau07a] that the Ruelle-Pollicott resonances on the external band coincide with the spectrum of the quantized map called the "quantum cat map".

5. There is a conjecture of Pollicott and Dolgopyat [DP98] for a better estimate of $r_0^+$ in (3.47) in terms of the pressure (3.37) and $J$ in (3.38):

$$\log r_0^+ = \frac{1}{2}\mathrm{Pr}\,(2V - 2J)$$

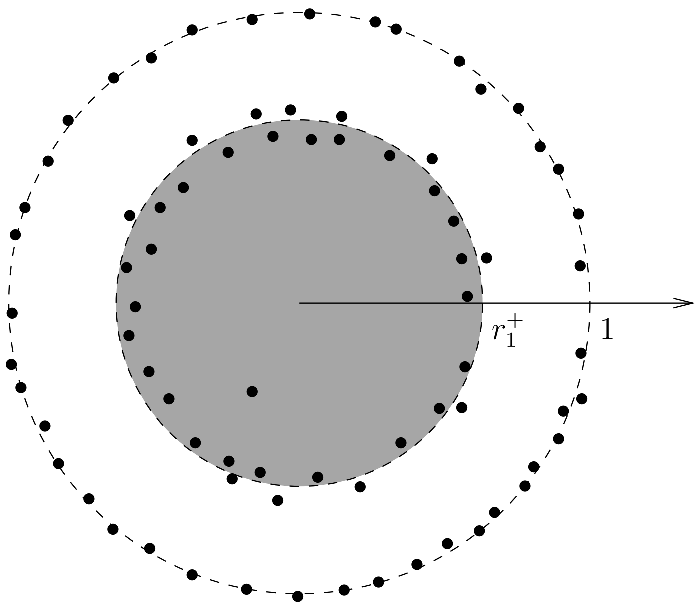


Figure 3.7: With the particular potential $V_0 = \frac{1}{2}\log\left|\det\, Df_x|_{E_u(x)}\right|$ the external spectrum of the transfer operator $\hat{F}_N$ concentrates uniformly on the unit circle as $N = 1/(2\pi\hbar) \to \infty$. (We have not represented here the structure of the internal bands inside the disc of radius $r_1^+$).

**Definition 3.36.** Suppose $r_1^+ < r_0^-$ (isolated external band). Let $\varepsilon > 0$, and $N_\varepsilon \geq 1$ given by Theorem 3.34. Let $\Pi_\hbar$ be the spectral projector on the external band $\mathcal{A}_0$ which is finite rank from (3.50). Let

$$\mathcal{H}_\hbar := \mathrm{Im}\left(\Pi_\hbar\right) \tag{3.52}$$

that we call the **"quantum space"** which is finite dimensional and let

$$\hat{\mathcal{F}}_\hbar : \mathcal{H}_\hbar \to \mathcal{H}_\hbar \tag{3.53}$$

be the finite dimensional spectral restriction of $\hat{F}_N$. We call $\hat{\mathcal{F}}_\hbar$ the **"quantum operator".**

In fact, for every $N$ we define $\Pi_N$ as the spectral projector $|z| > r_1^+ + \varepsilon$ and put $\hat{\mathcal{F}}_N := \hat{F}_N \Pi_N$. In particular for $N \geq N_\varepsilon$ $\Pi_N = \Pi_\hbar$ and $\hat{\mathcal{F}}_N = \hat{\mathcal{F}}_\hbar$.

**Theorem 3.37.** ***"correlation functions and interpretation". [FT15]*** *With the same setting as in the previous definition, for any $u, v \in C^{\infty}(P)$, and for $n \to \infty$, one has*

$$\underbrace{\left(v, \hat{F}^n u\right)_{L^2}}_{\text{"classical"}} = \sum_N \underbrace{\left(v_N, \hat{\mathcal{F}}_N^n u_N\right)}_{\text{"quantum"}} + O\left(\left(r_1^+ + \varepsilon\right)^n\right) \tag{3.54}$$

*where $u_N, v_N \in C_N^{\infty}(P)$ are the Fourier components of the functions $u$ and $v$. In the right hand side of (3.54), the sum is infinite but convergent.*

*Remark* 3.38. Eq.(3.54) has a nice interpretation: the classical correlation functions $\left(v, \hat{F}^n u\right)$ are governed by the quantum correlation functions $\left(v_N, \hat{\mathcal{F}}_N^n u_N\right)$ for large time, or equivalently the **"quantum dynamics emerge dynamically from the classical dynamics"**.

*Remark* 3.39. It is known that for $n \to \infty$,

$$\left(v, \hat{F}^n u\right) = \lambda_0^n \left(v, \Pi_{\lambda_0} u\right) + O\left(|\lambda_1|^n\right)$$

where $\lambda_0 > 0$ is the leading and simple eigenvalue of $\widetilde{F}$ (in the space $\mathcal{H}_{N=0}^r$) and $\lambda_1$ is the second eigenvalue with $|\lambda_1| < \lambda_0$. The case $V = 0$ for which $\lambda_0 = 1$ gives that the map $\tilde{f} : P \to P$ is mixing with exponential decay of correlations.

*Remark* 3.40. (*) In [FT15] we show that $\hat{\mathcal{F}}_\hbar$ is a valuable quantization of the symplectic map $f$ but different from usual "**geometric quantization**".

### 3.3.1 Proof of Theorem 3.34

The idea is the same as in the proof in Section 3.2.1 page 32, but we use now $\hbar-$semiclassical analysis with $\hbar := 1/(2\pi N) \ll 1$.

We consider charts $U_\alpha \subset M$ and local trivializations of the bundle $P$: $\tau_\alpha : U_\alpha \subset M \to P$, i.e. diffeomorphism

$$T_\alpha : \begin{cases} U_\alpha \times \mathbf{U}(1) & \to \pi^{-1}(U_\alpha) \\ \left(x, e^{i\theta}\right) & \to e^{i\theta}\tau_\alpha(x) \end{cases} \tag{3.55}$$

Consequently the pull-back of the connection $A$ on $P$ by the trivialization map (3.55) is written as

$$T_\alpha^* A = id\theta - i2\pi\eta_\alpha \tag{3.56}$$

where $\eta_\alpha \in C^{\infty}(U_\alpha, \Lambda^1)$ is a one-form on $U_\alpha$ which depends on the choice of the local section $\tau_\alpha$. We have

$$\omega = d\eta_\alpha. \tag{3.57}$$

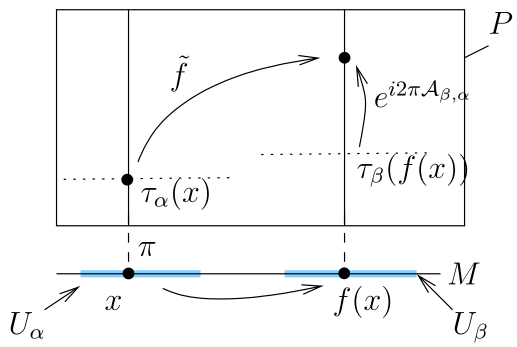


Figure 3.8: Illustrates the expression (3.58) of the prequantum map $\tilde{f}$ with respect to local trivialization. It is characterized by the action function $\mathcal{A}_{\beta,\alpha}(x)$.

**Lemma 3.41.** ***"Local expression of the prequantum map*** $\tilde{f}$***".*** *Suppose that* $V \subset U_\alpha \cap f^{-1}(U_\beta)$ *is a simply connected open set. We have*

$$\tilde{f}(\tau_\alpha(x)) = e^{i2\pi\mathcal{A}_{\beta,\alpha}(x)}\tau_\beta(f(x)) \tag{3.58}$$

*with the "action function" given by*

$$\mathcal{A}_{\beta,\alpha}(x) = \int_{f(\gamma)} \eta_\beta - \int_\gamma \eta_\alpha + c(x_0) = \int_\gamma (f^*(\eta_\beta) - \eta_\alpha) + c(x_0). \tag{3.59}$$

*In the last integral,* $x_0 \in V$ *is any point of reference,* $\gamma \subset V$ *is a path from* $x_0$ *to* $x$ *and* $c(x_0)$ *does not depend on* $x$*. See figure 3.8.*

**Lemma 3.42.** ***"Local expression of*** $\hat{F}_N$ ***"*** *(37). Let* $u \in C_N^\infty(P)$ *and* $u' := \hat{F}_N u \in C_N^\infty(P)$*. Let the respective associated functions be* $u_\alpha = u \circ \tau_\alpha$ *and* $u'_\alpha = u' \circ \tau_\alpha$ *for any indices* $\alpha$*. Then*

$$u'_\beta = e^V \cdot e^{-i2\pi N\mathcal{A}_{\beta,\alpha}\circ f^{-1}}\left(u_\alpha \circ f^{-1}\right) \tag{3.60}$$

**Proposition 3.43.** *$\hat{F}_N$ is a $\hbar$-*Fourier Integral Operator*. Its local canonical map is*

$$F_{\alpha,\beta}:\begin{cases}T^*U_\alpha & \to T^*U_\beta\\ (x,\xi) & \to (x',\xi')=\left(f(x),{}^t\left(Df_{x'}^{-1}\right)(\xi+\eta_\alpha(x))-\eta_\beta(x')\right)\end{cases}\tag{3.61}$$

*where $x\in U_\alpha$, $f(x)\in U_\beta$ and $\xi\in T_x^*U_\alpha$. The map $F_{\alpha,\beta}$ preserves the canonical symplectic structure*

$$\Omega:=\sum_{j=1}^{2d}dx_j\wedge d\xi_j\tag{3.62}$$

*Proof.* This comes from (3.60). See explanation of (3.28). There is a new term in (3.60): the multiplication operator by a "fast oscillating phase" (recall that $\hbar\ll 1$):

$$\hat{F}_2:\quad u(x)\to u'(x)=e^{iS(x)/\hbar}u(x)$$

with $S(x)=-\mathcal{A}_{\beta\alpha}\circ f^{-1}=\int_{f^{-1}(\gamma)}\eta_\alpha-\int_\gamma\eta_\beta-c(x_0)$. If $u(x)=e^{\frac{i}{\hbar}\xi.x}$ then it is transformed to

$$u'(y)=\left(\hat{F}_2u\right)(y)=e^{\frac{i}{\hbar}(\xi.y+S(y))}$$

and for $y=x+y'$ with $|y'|\ll 1$, we have

$$u'(y)\simeq Ce^{\frac{i}{\hbar}(\xi.y+dS\cdot y)}=Ce^{\frac{i}{\hbar}\xi'\cdot y}$$

with $\xi'=\xi+dS$, $C=e^{\frac{i}{\hbar}(S(x)-dS_x.x)}$ and $dS=f^{-1*}\eta_\alpha-\eta_\beta$. This gives (3.61). □

**Lemma 3.44.** *With the following change of variable*

$$(x,\xi)\in T^*U_\alpha\to(x,\zeta)=(x,\xi+\eta_\alpha(x))\in T^*M,\tag{3.63}$$

*the **canonical map** (3.61) get the simpler and global expression*

$$F:\begin{cases}T^*M & \to T^*M\\ (x,\zeta) & \to (x',\zeta')=\left(f(x),{}^t\left(Df_{x'}^{-1}\right)\zeta\right)\end{cases}.\tag{3.64}$$

*similar to (3.28), but the symplectic form $\Omega$in (3.62) preserved by $F$ is:*

$$\Omega=\sum_{j=1}^{2d}(dx_j\wedge d\zeta_j)+\tilde{\pi}^*(\omega).\tag{3.65}$$

*with the canonical projection map $\tilde{\pi}:T^*M\to M$.*

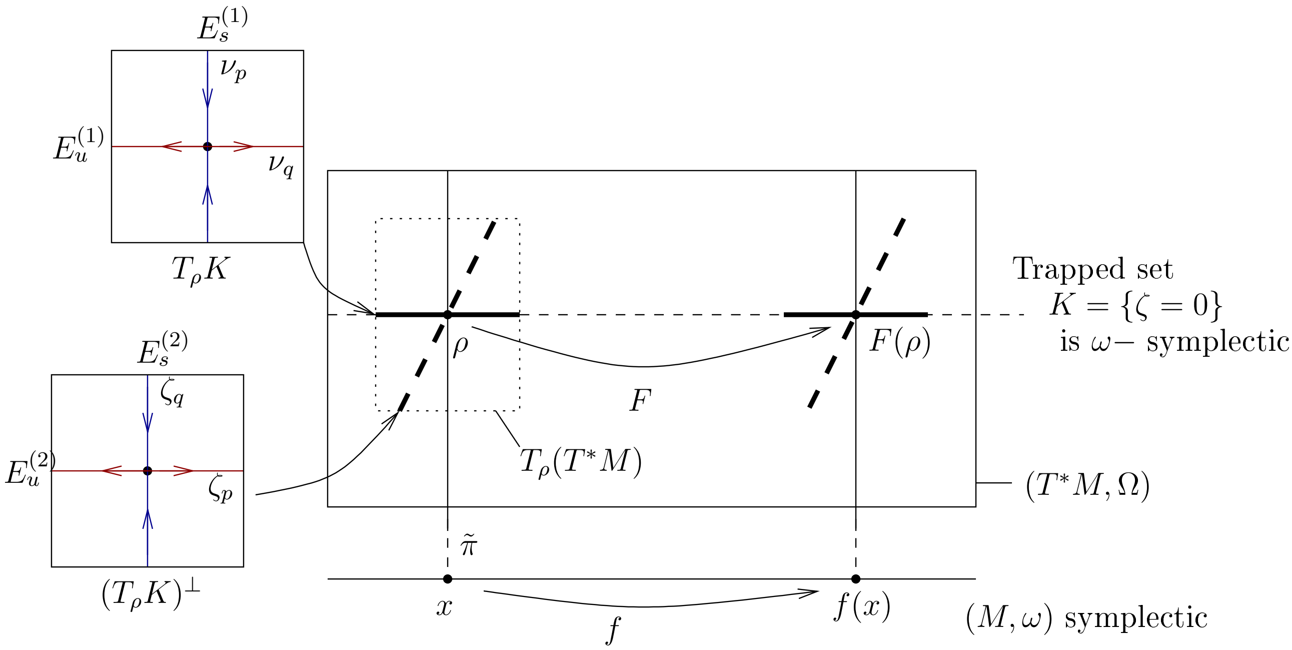


Figure 3.9: The decompositions of the tangent space $T_\rho(T^*M)$.

So as in (3.29), the trapped set is the zero section

$$K = \{(x, \xi) \in T^*M, x \in M, \xi = 0\} \subset T^*M \tag{3.66}$$

Here $(K, \Omega) \equiv (M, \omega)$ is a symplectic submanifold.

For every $\rho \in K$, we can decompose $\Omega$orthogonally:

$$T_\rho \left(T^*M\right) = T_\rho K \overset{\perp_\Omega}{\bigoplus} \left(T_\rho K\right)^{\perp_\Omega} \tag{3.67}$$

Moreover

$$T_\rho K = \underbrace{E_u^{(1)} \oplus E_s^{(1)}}_{2d}, \qquad \left(T_\rho K\right)^\perp = \underbrace{E_u^{(2)} \oplus E_s^{(2)}}_{2d}$$

with

$$E_u^{(1)} := T_\rho K \cap E_u^* \left(\rho\right)$$

etc

With respect to the decomposition (3.67), the canonical map $F$ is within the linear approximation

$$
\begin{array}{ccccc}
T_\rho (T^*M) & = & \underbrace{E_u^{(1)}(\rho) \oplus E_s^{(1)}(\rho)}_{T_\rho K} & \overset{\perp}{\oplus} & \underbrace{E_u^{(2)}(\rho) \oplus E_s^{(2)}(\rho)}_{(T_\rho K)^\perp} \\
D\Phi \downarrow & & \downarrow & & \downarrow \\
T^*\mathbb{R}^{2d}_{(q,p)} & = & \underbrace{\left(\mathbb{R}^d_{\nu_q} \oplus \mathbb{R}^d_{\nu_p}\right)}_{T^*\mathbb{R}^d_{\nu_q}} & \overset{\perp}{\oplus} & \underbrace{\left(\mathbb{R}^d_{\zeta_p} \oplus \mathbb{R}^d_{\zeta_q}\right)}_{T^*\mathbb{R}^d_{\zeta_p}}
\end{array}
$$

With respect to these coordinates the differential of the canonical map $DF_\rho : T_\rho (T^*M) \to T_{F(\rho)} (T^*M)$ is expressed as

$$
D\Phi \circ DF_\rho \circ D\Phi^{-1} = F^{(1)} \oplus F^{(2)}, \qquad F^{(1)} \equiv \begin{pmatrix} A_x & 0 \\ 0 & {}^tA_x^{-1} \end{pmatrix}, \quad F^{(2)} \equiv \begin{pmatrix} A_x & 0 \\ 0 & {}^tA_x^{-1} \end{pmatrix} \tag{3.68}
$$

where

$$
A_x \equiv Df\,|_{E_u(x)}: \mathbb{R}^d \to \mathbb{R}^d \tag{3.69}
$$

is an expanding linear map. $\|A_x\|_{\min} \geq \lambda > 1$.

At the level of operators, we perform a decomposition similar to (3.67) and obtain a microlocal decomposition of the transfer operator $\hat{F}_N$ as a tensor product $\hat{F}_{N|T_\rho K} \otimes \hat{F}_{N|(T_\rho K)^\perp}$. Precisely we obtain correspondingly to (3.68) above

$$
\hat{F}_N \equiv e^V \cdot \mathcal{L}_A \otimes \mathcal{L}_{{}^tA^{-1}} \tag{3.70}
$$

with

$$
\mathcal{L}_A u := u \circ A^{-1} \text{ on } C_0^\infty \left(\mathbb{R}^d\right)
$$
$$
\mathcal{L}_{{}^tA^{-1}} u := u \circ^t A \text{ on } C_0^\infty \left(\mathbb{R}^d\right)
$$

We observe that

- $|\mathrm{det}A|^{-1/2} \mathcal{L}_A$ is unitary on $L^2\left(\mathbb{R}^d\right)$
- From model in Theorem 3.12, we have shown that in an anisotropic Sobolev space, $\mathcal{L}_A$ has discrete Ruelle spectrum in bands indexed by $k \geq 0$ and given by:

  $$
  \|A\|_{\max}^{-k} \leq |z_k| \leq \|A\|_{\min}^{-k}
  $$

  and that corresponding eigenspace are homogeneous polynomials of degree $k$. We observe that the adjoint operator is $\mathcal{L}_A^* = |\mathrm{det}A| \,.\mathcal{L}_{A^{-1}}$. The spectrum of $\mathcal{L}_A^*$ is the conjugate of that of $\mathcal{L}_A$. We have $\mathcal{L}_{{}^tA^{-1}} = \frac{1}{|\mathrm{det}A|} \cdot \mathcal{L}_{{}^tA}^*$ and deduce that $\mathcal{L}_{{}^tA^{-1}}$ has a discrete Ruelle spectrum in bands indexed by $k \geq 0$ and given by:

  $$
  |\mathrm{det}A|^{-1} \cdot \|A\|_{\max}^{-k} \leq |z_k| \leq |\mathrm{det}A|^{-1} \cdot \|A\|_{\min}^{-k} \tag{3.71}
  $$

Therefore we prefer to write (3.70) as

$$\hat{F}_N = e^V \cdot \left( \underbrace{|\mathrm{det}A|^{-1/2} \cdot \mathcal{L}_A}_{\text{unitary}} \right) \otimes \left( \underbrace{|\mathrm{det}A|^{1/2} \cdot \mathcal{L}_{{}^tA^{-1}}}_{discrete\,spectrum} \right)$$

and from (3.71) the discrete spectrum of $|\mathrm{det}A|^{1/2} \cdot \mathcal{L}_{{}^tA^{-1}}$ is

$$|\mathrm{det}A|^{-1/2} \cdot \|A\|_{\max}^{-k} \leq |z_k| \leq |\mathrm{det}A|^{-1/2} \cdot \|A\|_{\min}^{-k}$$

From this microlocal description we obtain that for a given $k$ (this will correspond to the $k$-th band), the transfer operator $\hat{F}_N$ has "local norm max/min" bounded by

$$e^{\Gamma_k^{\pm}(x)} = e^V \cdot |\mathrm{det}A|^{-1/2} \cdot \|A\|_{\max/\min}^{-k}$$

From (3.69) and (3.45) this gives

$$\begin{aligned} \Gamma_k^{\pm}(x) &= V + \log |\mathrm{det}A|^{-1/2} - k \log \|A\|_{\max/\min} \\ &= V - \frac{1}{2} \log \left|\det\, Df_x|_{E_u(x)}\right| - k \log \left\| Df_{|E_u(x)} \right\|_{\max/\min} \\ &= D(x) - k \log \left\| Df_{|E_u(x)} \right\|_{\max/\min} \end{aligned} \tag{3.72}$$

For the operator $\hat{F}_N^n$ we have similarly that it has "local norm max/min" bounded by $e^{\Gamma_k^{\pm}(x,n)}$ with

$$\Gamma_k^{\pm}(x, n) = D_n(x) - k \log \left\| Df^n_{|E_u(x)} \right\|_{\max/\min} \tag{3.73}$$

From the previous local description, we can construct explicitly some approximate local spectral projectors $\Pi_k$ for every value of $k$, and patching these locals expression together we get global spectral operators for each band (under pitching conditions). We deduce that the spectrum is contained in bands $\mathbf{B}_k$ limited by $\gamma_k^- \leq \log|z| \leq \gamma_k^+$ (image of the projector $\Pi_k$) with

$$\gamma_k^+ = \limsup_{n\to\infty} \left( \sup_x \frac{1}{n} \Gamma_k^+(x, n) \right), \quad \gamma_k^+ = \liminf_{n\to\infty} \left( \inf_x \frac{1}{n} \Gamma_k^-(x, n) \right)$$

Then (3.73) gives expressions (3.48) of the Theorem.

The proof of the Weyl law is similar to the proof of J.Sjöstrand about the damped wave equation [Sjö00] but needs more arguments. The accumulation of resonances on the value $\exp\langle D\rangle$ uses the ergodicity property and is also similar to the spectral results obtained in [Sjö00] for the damped wave equation.

In [FT15] the proof needs more arguments because one has to show that non linear corrections are negligible.

## 3.4 Ruelle spectrum for Anosov vector fields

We suppose that $X$ is an Anosov vector field on a smooth closed manifold $M$. Let $V \in C^{\infty}(M)$ be a smooth function called "**potential function**".

**Definition 3.45.** The **transfer operator** is the group of operators

$$\hat{F}_t : \begin{cases} C^{\infty}(M) & \to C^{\infty}(M) \\ v & \to e^{tA}v \end{cases}, \quad t \geq 0$$

with the generator

$$A := -X + V \tag{3.74}$$

which is a first order differential operator (in local coordinates $A = -\sum_j X^j \frac{\partial}{\partial x^j} + V(x)$).

*Remark* 3.46.

- Since $X$ generates the flow $\phi_t$ we can write[16] $\hat{F}_t v = \left(e^{\int_0^t V \circ \phi_{-s} ds}\right) v(\phi_{-t}(x))$, hence $\hat{F}_t$ acts as transport of functions by the flow with multiplication by exponential of the function $V$ averaged along the trajectory.
- In the case $V = 0$, the operator $\hat{F}_t$ is useful in order to express "time correlation functions" between $u, v \in C^{\infty}(M)$, $t \in \mathbb{R}$:

  $$C_{u,v}(t) := \int_M u \cdot (v \circ \phi_{-t})\, dx = \langle u, \hat{F}_t v \rangle_{L^2} \tag{3.75}$$

  The study of these time correlation functions permits to establish the mixing properties and other statistical properties of the dynamics of the Anosov flow.
- In the particular case $V = 0$, $u = cste$ is an obvious eigenfunction of $A = -X$ with eigenvalue $z_0 = 0$.
- If $dx$ is a smooth measure preserved by the flow (this is the case for a contact Anosov flow) then $\mathrm{div}X = 0$ and in the case $V = 0$, we have that $\hat{F}_t$ is unitary in $L^2(M, dx)$ and $iA = (iA)^*$is self-adjoint and has essential spectrum on the imaginary axis $\mathrm{Re}z = 0$, that is useless. In the next theorem we consider more interesting functional spaces where the operator $A$ has discrete spectrum but is non self-adjoint.

By duality, we extend $A : C^{\infty}(M) \to C^{\infty}(M)$ to $A : \mathcal{D}'(M) \to \mathcal{D}'(M)$.

[16] to prove this we derive the right hand side $B(x,t) = \left(e^{\int_0^t V \circ \phi_{-s} ds}\right) v(\phi_{-t}(x))$ giving $\frac{\partial B}{\partial t} = (V - X)B = AB$. On the other hand $\frac{\partial}{\partial t}\left(\hat{F}_t v\right) = A\left(\hat{F}_t v\right)$ also. Unicity of the solution gives that $B = \hat{F}_t v$.

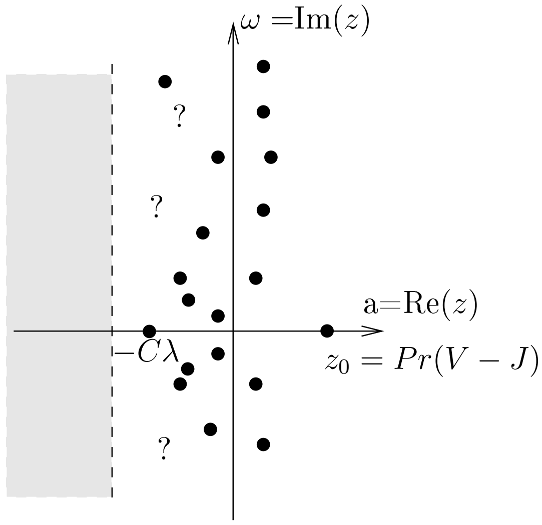


Figure 3.10: Illustration of Theorem 3.47. The spectrum of $A = -X + V$ is discrete on $\mathrm{Re}(z) > -C\lambda$ in space $\mathcal{H}_C$ (for any $C > 0$) but it does not give existence of eigenvalues. That's why we put the sign "?".

**Theorem 3.47.** ***"discrete spectrum"***.*[BL07][FS11]. If $X$ is an Anosov vector field and $V \in C^\infty(M)$ then for every $C > 0$, there exists a Hilbert space $\mathcal{H}_C$ called "**anisotropic Sobolev space**" with $C^\infty(M) \subset \mathcal{H}_C \subset \mathcal{D}'(M)$, such that*

$$A = -X + V \qquad : \mathcal{H}_C \to \mathcal{H}_C$$

*has discrete spectrum on the domain $\mathrm{Re}(z) > -C\lambda$, called **Ruelle-Pollicott resonances**, independent on the choice of $\mathcal{H}_C$.*
*We have an upper bound for the density of resonances : for every $\beta > 0$, in the limit $b \to +\infty$ we have*

$$\sharp\{z \in \mathrm{Res}(A),\ |\mathrm{Im}(z) - b| \leq \sqrt{b},\ \mathrm{Re}(z) > -\beta\} \leq o(b^{n-1/2}), \tag{3.76}$$

*with $n = \dim M$.*

*Remark* 3.48. Concerning the meaning of these eigenvalues, notice that with the choice $V = 0$, if $(-X)v = zv$, $v$ is an invariant distribution with eigenvalue $z = -a + ib \in \mathbb{C}$, then $v \circ \phi_{-t} = e^{-tX}v = e^{-at}e^{ibt}v$, i.e. $a = -\mathrm{Re}(z)$ contributes as a damping factor and $b = \mathrm{Im}(z)$ as a frequency in time correlation function (3.75). See Theorem 3.57 below for a precise statement. Notice also the symmetry of the spectrum under complex conjugation that $Av = zv$ implies $A\overline{v} = \overline{zv}$.

*Remark* 3.49. (*) The term "**resonance**" comes from quantum physics where an (elementary or composed) particle usually decay towards other particles. It is modeled by a "resonance", i.e. a quantum state which an eigenvector of the Hamiltonian operator and an eigenvalue $z = -a + ib \in \mathbb{C}$ which behaves as $e^{zt} = e^{-at}e^{ibt}$. The imaginary part of $z$ is written $b = \frac{E}{\hbar}$ with the energy $E = mc^2$ related to the mass $m = \frac{\hbar}{c^2}b$ of the particle. The real part gives $e^{-at} = e^{-t/\tau}$ with $\tau = 1/a$ the "mean life time" of the particle. For example the neutron has $\tau \simeq 15$mn (very long) and $E = 940$GeV. In nuclear physics, the mean life time of resonances $\tau$ is usually of order $10^{-22}$s.

(*) See on a movie (`http://www-fourier.ujf-grenoble.fr/~faure/articles`): the spectrum of the partially expanding map

$$(x, y) \to (2x \bmod 1, y + \sin 2\pi x) \in S^1 \times \mathbb{R}.$$

In theorem 3.47 the last result gives an upper bound for the number of resonances. The difficulty of giving a lower bound is common in problems which involves "non normal operators" [TE05] (here $A$ is non normal in $\mathcal{H}_C$). This is due to the fact that for non normal operators, the spectrum may be very unstable with respect to perturbation. The simplest example to have in mind is the following $N \times N$ matrix with parameter $\varepsilon \in \mathbb{R}$:

$$M_\varepsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \\ & & \ddots & 1 \\ \varepsilon & & & 0 \end{pmatrix}$$

For $\varepsilon = 0$ the spectrum is 0 with multiplicity $N$. For $\varepsilon > 0$ is it easy to check that there are $N$ eigenvalues on the circle of radius $r_{\varepsilon,N} = \varepsilon^{1/N}$. So for $\varepsilon = 10^{-10}$, and $N = 10$ the radius is $r = 0.1$.

### 3.4.1 Sketch of proof of Theorem 3.47

This proof (taken from [FS11]) uses semiclassical analysis. Let us consider the differential operator

$$P := iA \underset{(3.74)}{=} -iX + iV \tag{3.77}$$

On the cotangent space $T^*M$ we denote $x \in M$ and $\xi \in T_x^*M$. The principal symbol of $P$ is the function $p \in C^\infty(T^*M)$ given by (see (A.6) or [Tay96b, p.2])

$$p(x, \xi) = X_x(\xi). \tag{3.78}$$

The function $p$ defines a Hamiltonian vector field $\mathbf{X}$ on $T^*M$ by

$$\Omega(\mathbf{X}, .) = dp$$

with $\Omega = \sum_j dx^j \wedge d\xi^j$ being the canonical symplectic form. In fact $\mathbf{X}$ is the canonical lift of $X$ on the cotangent space. Its flow

$$\Phi_t = e^{-t\mathbf{X}} \tag{3.79}$$

is a lift of $\phi_t : M \to M$ and acts lineary in the fibers $\Phi_t : T_x^* M \to T_{\phi_t(x)}^* M$. It preserves the decomposition of the cotangent bundle

$$T_x^* M = E_u^* (x) \oplus E_s^* (x) \oplus E_0^* (x)$$

defined as the dual decomposition of the tangent space (2.14) by

$$E_u^* (E_u \oplus E_0) = 0, \quad E_s^* (E_s \oplus E_0) = 0, \quad E_0^* (E_u \oplus E_s) = 0.$$

From definition (2.16), we have that $E_0^* = \mathbb{R}\alpha$. For a point $(x, \xi) \in T^*M$ we can consider $\mathcal{E} = p(x, \xi) = X_x(\xi)$ as the component of $\xi$ along the axis $E_0^*(x)$, called the energy and preserved by the flow. The energy level is $\Sigma_{\mathcal{E}} := p^{-1}(\mathcal{E})$. From (3.78) and (2.16), $\Sigma_{\mathcal{E}}$ is an affine subbundle of $T^*M$ given by

$$\Sigma_{\mathcal{E}} = p^{-1}(\mathcal{E}) = (\mathcal{E} \cdot \alpha) + (E_u^* \oplus E_s^*).$$

By duality, for $t > 0$, the map $\Phi_t : E_u^*(x) \to E_u^*(\phi_t(x))$ is expanding and $\Phi_t : E_s^*(x) \to E_s^*(\phi_t(x))$ is contracting. See figure 3.11.

The **trapped set** (or non wandering set) of the flow $\Phi_t$ is defined as the set of point who do not escape to infinity in the past or future:

$$K := \{(x, \xi) \in T^*M, \exists C \Subset T^*M \text{ compact}, \forall t \in \mathbb{R}, \Phi_t(x, \xi) \in C\} \subset T^*M$$

From the previous description we have that the trapped set is the rank one subundle $E_0^*$:

$$K = E_0^*, \qquad \dim K = \dim M + 1.$$

For an arbitrary large constant $C > 0$, we construct an **escape function** $a(x, \xi)$ on $T^*M$ such that[17] far from the trapped set $K$ one has:

$$\mathbf{X}(a) \ll -C.\lambda$$

Then let us consider the conjugated operator

$$\tilde{P} := e^{\mathrm{Op}(a)} P e^{-\mathrm{Op}(a)} = P + [\mathrm{Op}(a), P] + \dots \tag{3.80}$$

From (A.10) its symbol is

$$\begin{aligned} \tilde{p}(x, \xi) &= p(x, \xi) - i\{a, p\} + iV + O\left(S^{-1+0}\right) \\ &= X(\xi) + i\mathbf{X}(a) + iV + O\left(S^{-1+0}\right) \end{aligned} \tag{3.81}$$

Let $D \subset \mathbb{C}$ a compact domain of the spectral plane. If $C > 0$ is large enough then $\tilde{p}^{-1}(D)$ is a compact subset of $T^*M$. See figure 3.12.

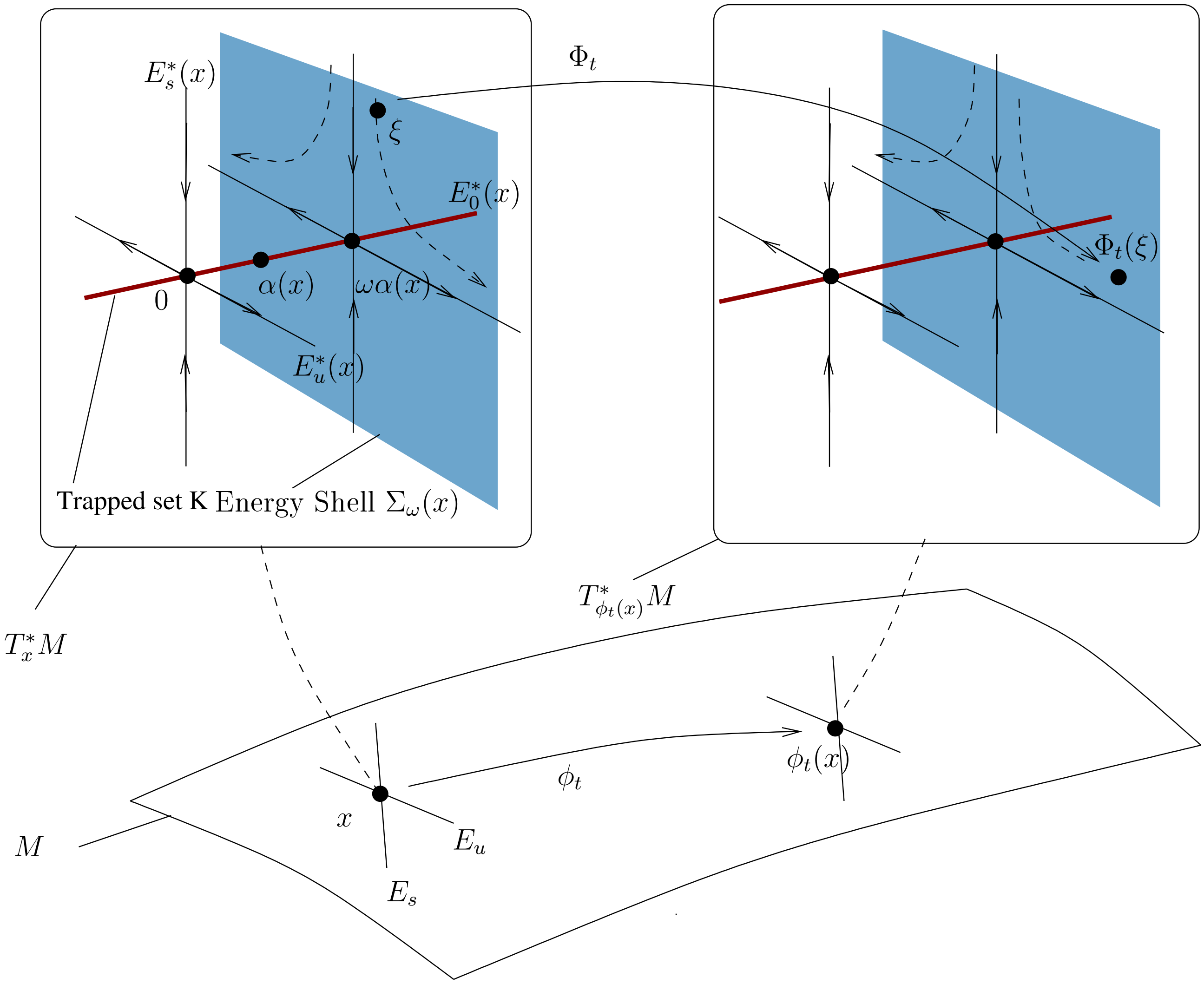


Figure 3.11: Picture of the flow $\Phi_t$ in the cotangent space $T^*M$.

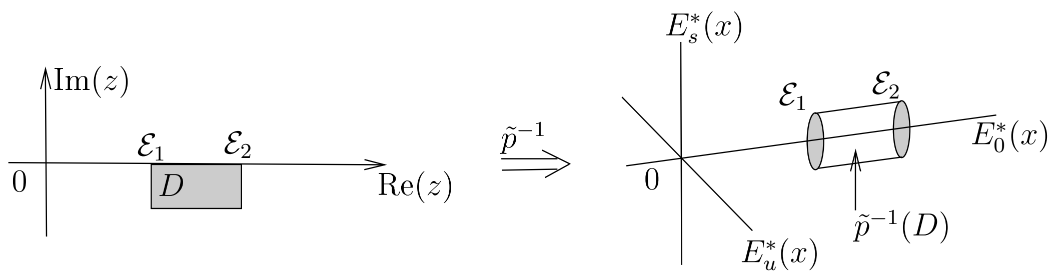


Figure 3.12: t $\tilde{p}^{-1}(D) \subset T^*M$ compact implies that $\tilde{P} = \mathrm{Op}(\tilde{p})$ has discrete spectrum on $D$.

As a consequence

$$\tilde{P} : L^2(M) \to L^2(M) \tag{3.82}$$

has discrete spectrum[18] on the domain $D$. Let

$$\mathcal{H}_C := e^{-\mathrm{Op}(a)} L^2(M)$$

be the **anisotropic Sobolev space.** Equivalently, from (3.80) and (3.77), (3.82) gives

$$P \quad : \mathcal{H}_C \to \mathcal{H}_C, \qquad A = -iP \quad : \mathcal{H}_C \to \mathcal{H}_C$$

have discrete spectrum respectively on the domain $D$ and $-iD$.

The Weyl upper bound is obtained by computing the symplectic volume of $\tilde{p}^{-1}(D)$.

## 3.5 Ruelle band spectrum for contact Anosov vector fields

We present here the result announced in [FT13].

*Remark* 3.50. Recently there appeared fews papers where the authors obtain results for contact Anosov flows using this semiclassical approach: spectral gap estimate and decay of correlation [NZ15], Weyl law upper bound [DDZ12] and meromorphic properties of the dynamical zeta function [DZ13]. We would like to mention also a closely related work: in [Dya13], for a problem concerning decay of waves around black holes, S. Dyatlov show that the spectrum of resonances has a band structure similar to what is observed for contact Anosov flows. In fact these two problems are very similar in the sense that in both cases the trapped set is symplectic and normally hyperbolic. This geometric property is the main reason for the existence of a band structure. However in [Dya13], some regularity of the hyperbolic foliation is required and that regularity is not present for contact Anosov flows.

[17] Precisely we choose $e^{a(x,\xi)} = \langle \xi \rangle^{m(x,\xi)}$ i.e. $a(x,\xi) = m(x,\xi) \log \langle \xi \rangle$ with $m(x,\xi) = \pm C$ along the stable/unstable directions $E^*_{s,u}(x)$ respectively. Hyperbolicity assumption gives that $\mathbf{X}\left(\xi_{s/u}\right) = \mp \lambda . \xi_{s/u}$ hence $\mathbf{X}(a) = m.X\left(\log\left(\xi_{s/u}\right)\right) = -C\lambda$.

[18] To show the general statement used here that $\tilde{p}^{-1}(D) \subset T^*M$ is compact implies that $\tilde{P} = \mathrm{Op}(\tilde{p}) : L^2(M) \to L^2(M)$ has discrete spectrum on $D$ we use the resolvent as follows: let $z_0 \in D$. From "**semiclassical functional calculus**"[GS94, DS99], $R_{\tilde{P}}(z_0) := \left(z_0 - \tilde{P}\right)^{-1}$ is a PDO with symbol $r_{\tilde{p}}(z_0) = (z_0 - \tilde{p})^{-1}$. From (3.81) on can write

$$r_{\tilde{p}}(z_0) = r_{\tilde{p}-K}(z_0) + r_{\tilde{p}-K}(z_0) K r_{\tilde{p}}(z_0)$$

where the first term of the right is bounded so that $\mathrm{Op}\left(\mathrm{r}_{\tilde{\mathrm{p}}-\mathrm{K}}(\mathrm{z}_0)\right)$ has a small norm and the second term decay so that $\mathrm{Op}(//)$ is compact. With this kind of argument, we deduce that $\tilde{P}$ has discrete spectrum in $D$.

### 3.5.1 Case of geodesic flow on constant curvature surface

In Section 2.3.1 we have observed that there is a contact Anosov flow $X$ on $\Gamma\backslash SL_2\mathbb{R}$ corresponding to the geodesic flow on $\Gamma\backslash\mathbb{H}^2$.

Using representation theory, it is known that the Ruelle-Pollicott spectrum of the operator $(-X)$ coincides with the zeros of the dynamical Fredholm determinant. This dynamical Fredholm determinant is expressed as the product of the Selberg zeta functions and gives the following result; see figure 3.14(a). We refer to [FT16] for further details.

**Proposition 3.51.** *If $X$ is the geodesic flow on an hyperbolic surface $\mathcal{S}=\Gamma\backslash\mathbb{H}^2$ then the Ruelle-Pollicott eigenvalues $z$ of $(-X)$, i.e. giving $(-X)\,u=z\cdot u$ with $u\in\mathcal{H}_C$, are of the form*

$$z_{k,l}=-\frac{1}{2}-k\pm i\sqrt{\mu_l-\frac{1}{4}} \tag{3.83}$$

*where $k\in\mathbb{N}$ and $(\mu_l)_{l\in\mathbb{N}}\in\mathbb{R}^+$ are the discrete eigenvalues of the hyperbolic Laplacian $\Delta=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ on the surface $\mathcal{S}=\Gamma\backslash\mathbb{H}^2$. There are also $z_n=-n$ with $n\in\mathbb{N}^*$. Each set $(z_{k,l})_l$ with fixed $k$ will be called the line $\mathbf{B}_k$. The "Weyl law" for $\Delta$ gives the density of eigenvalues on each vertical line $\mathbf{B}_k$, for $b\to\infty$,*

$$\sharp\left\{z_{k,l},\ b<\operatorname{Im}(z_{k,l})<b+1\right\}\sim|b|\,\frac{\mathcal{A}}{2\pi} \tag{3.84}$$

*where $\mathcal{A}$ is the area of $\mathcal{S}$.*

*Proof.* For the proof we can use representation theory: it is known that the Ruelle-Pollicott spectrum of the operator $(-X)$ coincides with the zeros of the dynamical Fredholm determinant. This dynamical Fredholm determinant is expressed as the product of the Selberg zeta functions.

Here is an argument that Ruelle resonances are related to the spectrum of the Laplacian and comes by bands. Suppose that $(-X)\,u=zu$ is a Ruelle-Pollicott eigenvector. From (2.19) we deduce that:

$$(-X)\,(Uu)=(-UX+U)\,u=(z+1)\,(Uu)\,,$$

$$(-X)\,(Su)=(-SX-S)\,u=(z-1)\,(Su)$$

This gives a family of other eigenvalues $z+k$, $k\in\mathbb{Z}$. But the condition that the spectrum is in the domain $\operatorname{Re}(z)\leq 0$ implies that there exists $k\geq 0$ such that $U^{k+1}u=0, U^k u\neq 0$. We say that $u\in\mathbf{B}_k$ belongs to the band $k$. Notice also that if $u\in\mathbf{B}_0$ i.e. $Uu=0$ then using the **Casimir operator** $\triangle=-X^2-\frac{1}{2}SU-\frac{1}{2}US$ of $SL_2\mathbb{R}$ we have

$$\triangle u=\left(-X^2-\frac{1}{2}SU-\frac{1}{2}US\right)u\underset{(2.19)}{=}\left(-X^2+X-SU\right)u=-z\,(z+1)\,u=\mu u$$

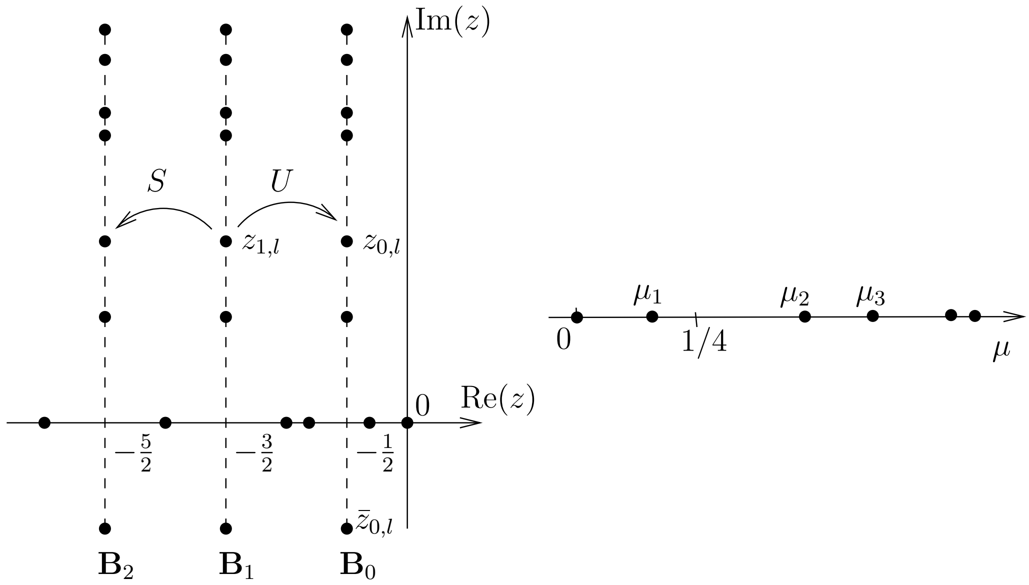


Figure 3.13: Ruelle Pollicott resonances for the geodesic flow on a hyperbolic surface.

Let $\langle u \rangle_{SO_2} \in \mathcal{D}'(\mathcal{M})$ be the distribution $u$ averaged by the action of $SO_2$. We suppose that $\langle u \rangle_{SO_2} \neq 0$. It is shown in [FS11] that the wavefront of $u$ is included in the unstable manifold $E_u^* \subset T^*M$. Using an argument of Hörmander, since $E_u^*$ is not contained in the kernel of $\Theta = S - U$ the generator of $SO_2$, then this wavefront is killed by the action of $SO_2$ and $\langle u \rangle_{SO_2} \in C^\infty(\mathcal{M})$ is in fact a smooth function on the surface $\mathcal{M} = \Gamma \setminus (SL_2\mathbb{R}/SO_2)$. Moreover since $\Delta$ commutes with the action of $SO_2$, we still have that $\triangle \langle u \rangle_{SO_2} = \mu \langle u \rangle_{SO_2}$ with $\Delta \equiv -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ being the hyperbolic Laplacian. $\Delta$ being elliptic on $\mathcal{M}$ also implies that $\langle u \rangle_{SO_2}$ is smooth. From spectral theory in $L^2(\mathcal{M})$, $\Delta$ is a positive self-adjoint operator and has discrete real and positive eigenvalues $\mu_l = -z(z+1) \geq 0$. Therefore the Ruelle eigenvalue is

$$z = -\frac{1}{2} \pm i\sqrt{\mu_l - \frac{1}{4}}$$

We deduce the other Ruelle eigenvalues by the shift $z - k$, $k \in \mathbb{N}$. The Weyl law for the Laplacian gives (3.84). Using Representation theory we can show that they are no other eigenvalues; i.e. that $\mathrm{Im}z \neq 0$ implies that $\langle u \rangle_{SO_2} \neq 0$ [FF03]. See figure 3.13. $\square$

### 3.5.2 General case

Proposition 3.51 above shows that the Ruelle-Pollicott spectrum for the geodesic flow on constant negative surface has the structure of vertical lines $\mathbf{B}_k$ at $\mathrm{Re}z = -\frac{1}{2} - k$. In each line the eigenvalues are in correspondence with the eigenvalues of the Laplacian $\Delta$. We address now the question if this structure persists somehow for geodesic flow on manifolds with negative (variable) sectional curvature and more generally for any contact Anosov flow.

We consider here an contact Anosov vector field $X$ on a smooth closed manifold $M$ and a smooth potential function $V \in C^{\infty}(M)$.

*Remark* 3.52. "Concerning the leading eigenvalue". Similarly to (3.39) above, we can show that for contact Anosov flow the Ruelle spectrum has a leading real eigenvalue $z_0 \in \mathbb{R}$ (i.e. other eigenvalues are $\mathrm{Re}(z_j) < z_0$) given by

$$z_0 = \mathrm{Pr}(V - J)$$

where $J = \mathrm{div} X_{|E_u}$ is the "**unstable Jacobian**"[19] and for a function $\varphi \in C(M)$,

$$\mathrm{Pr}(\varphi) := \lim_{t\to\infty} \frac{1}{t} \log\left( \sum_{\gamma, |\gamma| \leq t} \exp\left( \int_0^t \varphi \right)(\gamma) \right)$$

is called the **topological pressure**.

We introduce now the following function called **"potential of reference"** that will play an important role

$$V_0(x) := \frac{1}{2} \mathrm{div} X_{|E_u} = \frac{1}{2} J. \tag{3.86}$$

*Remark* 3.53. From (2.15) we have $V_0(x) \geq \frac{1}{2} d \cdot \gamma$. Since $E_u(x)$ is only Hölder in $x$ so is $V_0(x)$.

We will also consider the difference

$$D(x) := V(x) - V_0(x) \tag{3.87}$$

and called it the "**effective damping function**". For simplicity we will write:

$$\left( \int_0^t D \right)(x) := \int_0^t (D \circ \phi_{-s})(x)\, ds, \quad x \in M,$$

for the **Birkhoff sum** of $D$ along trajectories. Finally we recall the notation $\|L\|_{min,max}$ in (3.22) for an invertible linear operator. The following theorem is similar to Theorem 3.34 that was for prequantum maps.

[19]Let $\mu_g$ be the induced Riemann volume form on $E_u(x)$ defined from the choice of a metric $g$ on $M$. As the usual definition in differential geometry [Tay96a, p.125], for tangent vectors $u_1, \dots u_d \in E_u(x)$, $\mathrm{div} X_{|E_u}$ measures the rate of change of the volume of $E_u$ and is defined by

$$\left( \mathrm{div} X_{|E_u}(x) \right) \cdot \mu_g(u_1, \dots u_d) = \lim_{t\to 0} \frac{1}{t} \left( \mu_g(D\phi_t(u_1), \dots, D\phi_t(u_d)) - \mu_g(u_1, \dots u_d) \right)$$

Equivalently we can write that

$$\mathrm{div} X_{|E_u}(x) = \frac{d}{dt} \left( \det (D\phi_t)_{|E_u} \right)_{t=0} \tag{3.85}$$

.

**Theorem 3.54.** *[FT16]"**asymptotic band structure**". If $X$ is a contact Anosov vector field on $M$ and $V \in C^{\infty}(M)$ then for every $C > 0$, there exists an Hilbert space $\mathcal{H}_C$ with $C^{\infty}(M) \subset \mathcal{H}_C \subset \mathcal{D}'(M)$, such that for any $\varepsilon > 0$, the Ruelle-Pollicott eigenvalues $(z_j)_j \in \mathbb{C}$ of the operator $A = -X + V : \mathcal{H}_C \to \mathcal{H}_C$ on the domain $\mathrm{Re}(z) > -C\lambda$ are contained, up to finitely many exceptions, in the union of finitely many bands*

$$z \in \bigcup_{k\geq 0} \underbrace{\left[\gamma_k^- - \varepsilon, \gamma_k^+ + \varepsilon\right] \times i\mathbb{R}}_{\text{Band } \mathbf{B}_k}$$

*with for $k \geq 0$,*

$$\gamma_k^+ = \lim_{t\to\infty} \sup_x \frac{1}{t} \left( \left( \int_0^t D \right)(x) - k \log \left\| D\phi_t(x)_{/E_u} \right\|_{min} \right), \tag{3.88}$$

$$\gamma_k^- = \lim_{t\to\infty} \inf_x \frac{1}{t} \left( \left( \int_0^t D \right)(x) - k \log \left\| D\phi_t(x)_{/E_u} \right\|_{max} \right) \tag{3.89}$$

*and where $D = V - V_0$ is the damping function (3.87). In the gaps (i.e. between the bands) the norm of the resolvent is controlled: there exists $c > 0$ such that for every $z \notin \bigcup_{k\geq 0} \mathbf{B}_k$ with $|\mathrm{Im}(z)| > c$*

$$\left\|(z - A)^{-1}\right\| \leq c. \tag{3.90}$$

*For some $k \geq 0$, if the band $\mathbf{B}_k$ is "isolated", i.e. $\gamma_{k+1}^+ < \gamma_k^-$ and $\gamma_k^+ < \gamma_{k-1}^-$ (this last condition is for $k \geq 1$) then the number of resonances in $\mathbf{B}_k$ obeys a "**Weyl law**": $\forall b > c$,*

$$\frac{1}{c}|b|^d < \frac{1}{|b|^{\varepsilon}} \cdot \sharp\left\{z_j \in \mathbf{B}_k, b < \mathrm{Im}(z_j) < b + b^{\varepsilon}\right\} < c|b|^d \tag{3.91}$$

*with $\dim M = 2d + 1$. The upper bound holds without the condition that $\mathbf{B}_k$ is isolated. If the external band $\mathbf{B}_0$ is isolated i.e. $\gamma_1^+ < \gamma_0^-$, then most of the resonances accumulate on the vertical line*

$$\mathrm{Re}(z) = \langle D\rangle := \frac{1}{\mathrm{Vol}(M)} \int_M D(x)\,dx$$

*in the precise sense that*

$$\frac{1}{\sharp\mathcal{B}_b} \sum_{z_i\in\mathcal{B}_b} |\mathrm{Re}(z_i) - \langle D\rangle| \underset{b\to\infty}{\longrightarrow} 0, \qquad \textit{with } \mathcal{B}_b := \{z_i \in \mathbf{B}_0, |\mathrm{Im}(z_i)| < b\}. \tag{3.92}$$

*Remark* 3.55. In 2009 M. Tsujii has obtained $\gamma_0^+$ in [Tsu10, Tsu12]. He also obtained the estimate (3.90) for $\mathrm{Re}(z) \geq \gamma_0^+ + \varepsilon$.

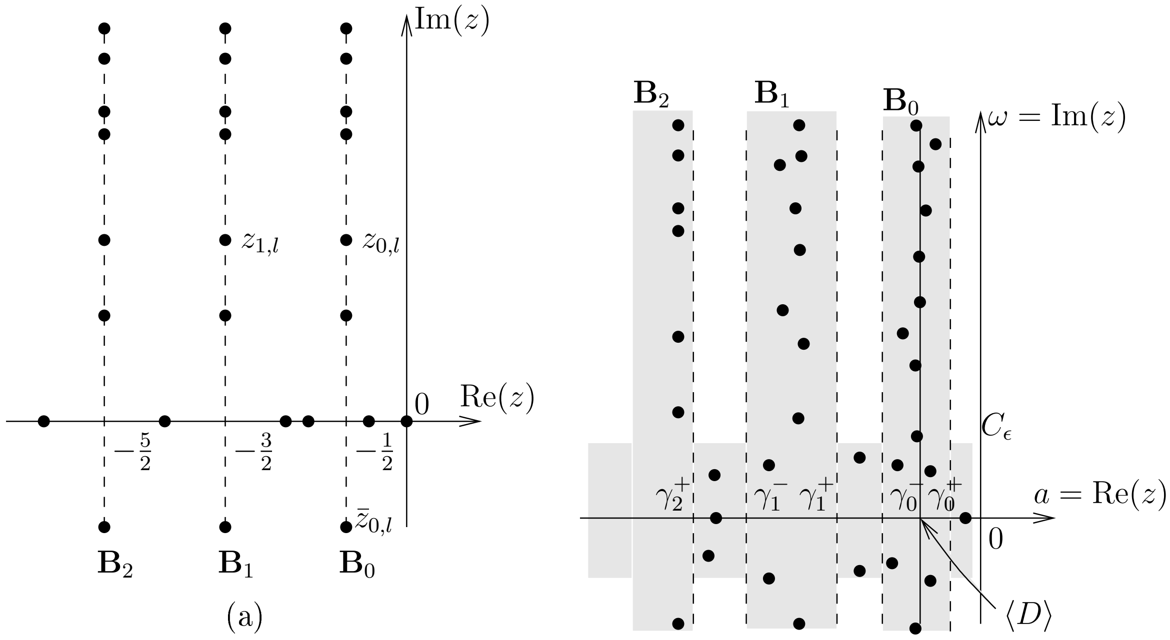


Figure 3.14: (a) For an hyperbolic surface $\mathcal{S} = \Gamma\backslash\mathbb{H}^2$, the Ruelle-Pollicott spectrum of the geodesic vector field $-X$ given by Proposition 3.51. It is related to the eigenvalues of the Laplacian by (3.83). (b) For a general contact Anosov flow, the spectrum of $A = -X + V$ and its asymptotic band structure given by Theorems 3.54.

*Remark* 3.56. For a general contact Anosov vector field it is possible to choose the potential $V = V_0$ (although it is non smooth)[FT16], giving $\gamma_0^+ = \gamma_0^- = 0$, i.e. the first band is reduced to the imaginary axis and is isolated from the second band by a gap, $\gamma_1^+ < 0$.

### 3.5.3 Consequence for correlation functions expansion

We mentioned the usefulness of dynamical correlation functions in (3.75). Let $\Pi_j$ denotes the finite rank spectral projector associated to the eigenvalue $z_j$. The following Theorem provides an expansion of correlation functions over the spectrum of resonances of the first band $\mathbf{B}_0$. This is an infinite sum.

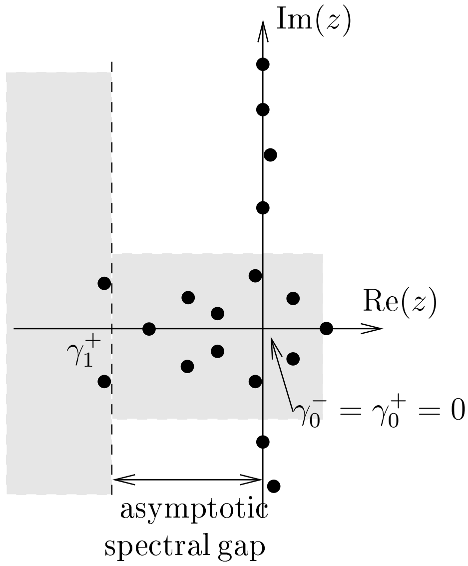


Figure 3.15: Ruelle-Pollicott spectrum for a general Contact Anosov flow and with potential $V_0 = \frac{1}{2}\mathrm{div}X_{|E_u}$.

**Theorem 3.57.** *[FT13]Suppose that $\gamma_1^+ < \gamma_0^-$. Then for any $\varepsilon > 0$,$\exists C_\varepsilon > 0$, any $u, v \in C^\infty(M)$ and $t \geq 0$,*

$$\left| \langle u, \hat{F}_t v \rangle_{L^2} - \sum_{z_j, \mathrm{Re}(z_j) \geq \gamma_1^+ + \varepsilon} \langle u, \hat{F}_t \Pi_j v \rangle \right| \leq C_\varepsilon \cdot \|u\|_{\mathcal{H}'_C} \|v\|_{\mathcal{H}_C} e^{\left(\gamma_1^+ + \varepsilon\right)t}. \tag{3.93}$$

*The infinite sum above converges because for arbitrary large $m \geq 0$ there exists $C_{m,\varepsilon}(u,v) \geq 0$ such that $\left|\langle u, \hat{F}_t \Pi_j v \rangle\right| \leq C_{m,\varepsilon}(u,v) \cdot |\mathrm{Im}(z_j)|^{-m} \cdot e^{\left(\gamma_0^+ + \varepsilon\right)t}$.*

*Remark* 3.58. Eq.(3.93) is a refinement of decay of correlation results of Dolgopyat [Dol98], Liverani [Liv04], Tsujii [Tsu10, Tsu12, Cor.1.2] and Nonnenmacher-Zworski [NZ15, Cor.5] where their expansion is a finite sum over one or a finite number of leading resonances.

*Remark* 3.59. In the case of simple eigenvalues $z_j = -a_j + ib_j$ then $\Pi_j$ is a rank one projector and $\langle u, \hat{F}_t \Pi_j v \rangle = e^{-a_j t} e^{ib_j t} \langle u, \Pi_j v \rangle$.

*Remark* 3.60. As we did in (3.54), we call the second term of (3.93), the **quantum correlation function**.

### 3.5.4 Proof of Theorem 3.54

The band structure and all related results presented in Theorem 3.54 have already been proven for the spectrum of Anosov prequantum map in [FT15] and presented in Theorem 3.34. An Anosov prequantum map $\tilde{f}: P \to P$ is an equivariant lift of an Anosov diffeomorphism $f: M \to M$ on a principal bundle $U(1) \to P \to M$ such that $\tilde{f}$ preserves a contact one form $\alpha$ (a connection on $P$). Therefore $\tilde{f}: P \to P$ is very similar to the contact Anosov flow $\phi_t : M \to M$ considered here, that also preserves a contact one form $\alpha$. Our proof of Theorem 3.54 is directly adapted from the proof given in [FT15] and presented in Section 3.3.1. We refer to this paper for more precisions on the proof and we use the same notations below. The techniques rely on semiclassical analysis adapted to the geometry of the contact Anosov flow lifted in the cotangent space $T^*M$. In the limit $|\mathrm{Im}z| \to \infty$ of large frequencies under study, the semiclassical parameter is written $\hbar := 1/|\mathrm{Im}z|$. We now sketch the main steps of the proof.

The proof is very similar to that of Theorem 3.34. Recall that $\mathrm{dim}M = 2d+1$, so $\mathrm{dim}T^*M = 2(2d+1)$.

**Global geometrical description.** $A = -X + V$ is a differential operator. Its principal symbol is the function $\sigma(A)(x,\xi) = X_x(\xi)$ on phase space $T^*M$ (the cotangent bundle). It generates an Hamiltonian flow which is simply the canonical lift of the flow $\phi_t$ on $M$. See figure 3.11. Due to Anosov hypothesis on the flow, the non-wandering set of the Hamiltonian flow is the continuous sub-bundle $K = \mathbb{R}\alpha \subset T^*M$ where $\alpha$ is the Anosov one form. $K$ is **normally hyperbolic**. This analysis has already been used in [FS11] for the semiclassical analysis of Anosov flow (not necessary contact). With the additional hypothesis that $\alpha$ is a smooth contact one form, the following Lemma shows that the trapped set $K\setminus\{0\}$ is a smooth symplectic submanifold of $T^*M$ (usually called the symplectization of the contact one form $\alpha$).

**Lemma 3.61.** *[Arn76]The trapped set $K\setminus\{0\} = (\mathbb{R}\alpha)\setminus\{0\}$ is a symplectic submanifold of $T^*M$ of dimension $\mathrm{dim} = 2(d+1)$, called the **symplectization** of the contact one form $\alpha$.*

*Proof.* Denote $\pi : T^*M \to M$ the projection map. A point on the trapped set $K \subset T^*M$ can be written $\xi = \omega \cdot \alpha(x)$ with $\omega \in \mathbb{R}$ and $x \in M$. The Liouville one form on $T^*M$ at point $\xi = \omega \cdot \alpha(x) \in K$ is

$$\sum_{j=1}^{2d+1} \xi^j dx^j \equiv \omega \cdot \pi^*(\alpha)$$

For simplicity we write the previous equation $\xi dx = \omega \cdot \alpha$. Then $d(\xi dx) = d(\omega\alpha) = d\omega \wedge \alpha + \omega d\alpha$ is a 2 form on $K$ giving the following volume form on $K\setminus\{0\}$:

$$(d(\omega\alpha))^{d+1} = (d+1)\,\omega^d \cdot d\omega \wedge \alpha \wedge (d\alpha)^d$$

which is non degenerate on $K\backslash\{0\}$ since $\alpha\wedge(d\alpha)^d$ is supposed to be non degenerated on $M$. In other words the canonical two form $\Omega=\sum_{j=1}^{2d+1}dx^j\wedge d\xi^j=-d(\xi dx)$ restricted to $K\backslash\{0\}$ is symplectic. □

**Differential of the flow on the trapped set.** Let $\rho=(x,\xi)\in K$ be a point on the trapped set. We suppose $X_x(\xi)>0$ and we let $\omega:=\hbar^{-1}:=X_x(\xi)$ be its "energy". Let $\Omega=\sum_j dx^j\wedge d\xi^j$ be the canonical symplectic form on $T^*M$ and consider the $\Omega$-orthogonal splitting of the tangent space at $\rho\in K$:

$$T_\rho(T^*M)=T_\rho K\overset{\perp}{\bigoplus}(T_\rho K)^\perp \tag{3.94}$$

that we can decompose further according to their (un)stability:

$$T_\rho K=\left(\underbrace{E_u^{(1)}\oplus E_s^{(1)}}_{E^{(1)},\dim=2d}\right)\overset{\perp}{\bigoplus}\left(\underbrace{E_0\oplus E_0^*}_{E^{(0)},\dim=2}\right) \tag{3.95}$$

$$(T_\rho K)^\perp=\underbrace{E_u^{(2)}\oplus E_s^{(2)}}_{E^{(2)},\dim=2d} \tag{3.96}$$

with $E_{u,s}^{(1)}:=(D\pi)^{-1}(E_{u,s})\cap T_\rho K$, $E_{u,s}^{(2)}:=(D\pi)^{-1}(E_{u,s})\cap(T_\rho K)^\perp$ and $E_0:=(D\pi)^{-1}(\mathbb{R}X)\cap T_\rho K$, $E_0^*=\mathbb{R}\alpha$. We have written their dimension below. Notice that in the obtained decomposition

$$T_\rho(T^*M)=E^{(1)}\overset{\perp}{\bigoplus}E^{(0)}\overset{\perp}{\bigoplus}E^{(2)}$$

each component is symplectic and each subcomponent $E_{u,s}^{(j)}$ is Lagrangian. This decomposition is preserved by the differential of the lifted flow (3.79), $\Phi_t=e^{-t\mathbf{X}}:T^*M\to T^*M$ so we write accordingly

$$D\Phi_t\equiv D\Phi_t^{(1)}\overset{\perp}{\bigoplus}D\Phi_t^{(0)}\overset{\perp}{\bigoplus}D\Phi_t^{(2)}. \tag{3.97}$$

with

$$D\Phi_t^{(1)}\equiv D\Phi_t^{(2)}\equiv\begin{pmatrix}L_x & 0\\ 0 & {}^tL_x^{-1}\end{pmatrix}:\mathbb{R}^{2d}\to\mathbb{R}^{2d}$$

is linear symplectic and hyperbolic with

$$L_x:=(D\phi_t)_{|E_u(x)} \tag{3.98}$$

being a linear expanding map: we have $\|L_x\|_{\min}>e^{t\gamma}>1$. concerning the neutral part we have $D\Phi_t^{(0)}=\mathrm{Id}_{|\mathbb{R}^2}$.

**Partition of unity.** We choose an energy $\mathcal{E} = \frac{1}{\hbar} \gg 1$. We decompose functions on the manifold using a microlocal partition of unity of size $\hbar^{1/2-\varepsilon}$ with some $1/2 > \varepsilon > 0$, that is refined as $\hbar \to 0$. In each chart we use a canonical change of variables adapted to the decomposition (3.94) and construct an escape function adapted to the local splitting $E_u^{(2)} \oplus E_s^{(2)}$ above. This escape function has "strong damping effect" outside a vicinity of size $O\left(\hbar^{1/2}\right)$ of the trapped set $K$. We use this to define the anisotropic Sobolev space $\mathcal{H}_C$. At the level of operators, we perform a decomposition similar to (3.94) and obtain a microlocal decomposition of the transfer operator $\hat{F}_t = e^{tA}$ as a tensor product $\hat{F}_{t|T_\rho K} \otimes \hat{F}_{t|(T_\rho K)^\perp}$. Precisely we obtain correspondingly to (3.97) above

$$\hat{F}_t = e^{tA} \underset{\text{microloc.}}{\equiv} e^{\int_0^t V} \cdot \mathcal{L}_L \otimes e^{-i\mathcal{E}t}\mathrm{Id}_{\mathbb{R}} \otimes \mathcal{L}_{{}^tL^{-1}} \tag{3.99}$$

with

$$\mathcal{L}_L u := u \circ L^{-1} \text{ on } C_0^\infty\left(\mathbb{R}^d\right)$$

$$\mathcal{L}_{{}^tL^{-1}} u := u \circ^t L \text{ on } C_0^\infty\left(\mathbb{R}^d\right)$$

and $\underset{\text{microloc.}}{\equiv}$ means after multiplication of some cutoff function defining a partition of unity, and up to conjugation by some unitary (Fourier integral operators, F.I.O) operators. We observe that

- $|\mathrm{det}L|^{-1/2}\,\mathcal{L}_L$ is unitary on $L^2\left(\mathbb{R}^d\right)$
- From model in Theorem 3.12, we have shown that in some anisotropic Sobolev space, $\mathcal{L}_L$ has discrete Ruelle spectrum in bands indexed by $k \geq 0$ and that:

$$C_0^{-1}\,\|L\|_{\max}^{-k} \leq \frac{\|\mathcal{L}_L u\|_{\mathcal{H}_C}}{\|u\|_{\mathcal{H}_C}} \leq C_0\,\|L\|_{\min}^{-k} \tag{3.100}$$

  and the corresponding group of eigenspaces are **homogeneous polynomials** on $\mathbb{R}^d$ of degree $k$. We observe that the adjoint operator is $\mathcal{L}_L^* = |\mathrm{det}L|\,.\mathcal{L}_{L^{-1}}$. For the adjoint $\mathcal{L}_L^*$ we have similar bounds. We have $\mathcal{L}_{{}^tL^{-1}} = \frac{1}{|\mathrm{det}L|} \cdot \mathcal{L}_{{}^tL}^*$ and deduce that $\mathcal{L}_{{}^tL^{-1}}$ has also a discrete Ruelle spectrum in bands indexed by $k \geq 0$ and similar bounds but with the additional factor $|\mathrm{det}L|^{-1}$. Therefore we prefer to write (3.99) as

$$e^{tA} = e^{\int_0^t V} \cdot \left(\underbrace{|\mathrm{det}L|^{-1/2} \cdot \mathcal{L}_L}_{\text{unitary}}\right) \otimes e^{-i\mathcal{E}t}\mathrm{Id}_{\mathbb{R}} \otimes \left(\underbrace{|\mathrm{det}L|^{1/2} \cdot \mathcal{L}_{{}^tL^{-1}}}_{\textit{discrete bands}}\right)$$

  and from (3.100) the discrete spectrum of $|\mathrm{det}L|^{1/2} \cdot \mathcal{L}_{{}^tL^{-1}}$ is in bands with the bounds

$$C_0^{-1}\,|\mathrm{det}L|^{-1/2} \cdot \|L\|_{\max}^{-k} \leq \frac{\|\,|\mathrm{det}L|^{1/2} \cdot \mathcal{L}_{{}^tL^{-1}} u\|_{\mathcal{H}_C}}{\|u\|_{\mathcal{H}_C}} \leq C_0\,|\mathrm{det}L|^{-1/2} \cdot \|L\|_{\min}^{-k}$$

From this microlocal description we obtain that for given $k$, the transfer operator $e^{tA}$ has "local norm max/min" bounded by

$$e^{\Gamma_k^{\pm}(x,t)} \asymp e^{\int_0^t V} \cdot |\mathrm{det} L|^{-1/2} \cdot \|L\|_{\max/\min}^{-k}$$

From (3.98) and (3.45) this gives

$$\begin{aligned}\Gamma_k^{\pm}(x,t) &= \int_0^t V - \frac{1}{2}\log\left|\mathrm{det}\phi_{t|_{E_u(x)}}\right|^{-1/2} - k\log\left\|D\phi_{t|_{E_u(x)}}\right\|_{\max/\min} + O(1) \\ &= \int_0^t D - k\log\left\|D\phi_{t|_{E_u(x)}}\right\|_{\max/\min} + O(1) \qquad (3.101)\end{aligned}$$

From the previous local description, we can construct explicitly some approximate local spectral projectors $\Pi_k$ for every value of $k$, and patching these locals expressions together we get global spectral operators for each band (under pitching conditions). For the generator $A$ of $e^{tA}$ we deduce that the spectrum is contained in bands $\mathbf{B}_k$ limited by $\gamma_k^- \leq \mathrm{Re}(z) \leq \gamma_k^+$ (image of the projector $\Pi_k$) with

$$\gamma_k^+ = \limsup_{t\to\infty}\left(\sup_x \frac{1}{t}\Gamma_k^+(x,t)\right), \quad \gamma_k^+ = \liminf_{t\to\infty}\left(\inf_x \frac{1}{t}\Gamma_k^-(x,t)\right)$$

Then (3.101) gives expressions (3.88) of the Theorem.

The proof of the Weyl law (3.91) is similar to the proof of J.Sjöstrand about the damped wave equation [Sjö00] but needs more arguments. The accumulation of resonances on the value $\langle D\rangle$ given by the spatial average of the damping function, Eq.(3.92), uses the ergodicity property of the Anosov flow and is also similar to the spectral results obtained in [Sjö00] for the damped wave equation.

# 4 Trace formula and zeta functions

We have already presented the Atiyah Bott trace formula in Section 3.2.2. This "simple formula" is at the basis for exact relations between the Ruelle spectrum and periodic orbits of the dynamics. We have saw such a relation in (3.35) for Anosov maps.

In this Section we want to present more precisely what this relation gives when there is a band structure in the Ruelle spectrum. This is the case for prequantum Anosov maps or contact Anosov flows. A consequence of this will be some refined counting formula for periodic orbits.

## 4.1 Gutzwiller trace formula for Anosov prequantum map

In this Section we consider the prequantum transfer operators $\hat{F}_N$ defined in (3.42). We assume the condition $r_1^+ < r_0^-$. (This condition holds if we consider the potential of reference $V = V_0$) As in (3.52), let $\Pi_\hbar : \mathcal{H}_N^r \to \mathcal{H}_N^r$ be the spectral projector for the external band and let $\mathcal{H}_\hbar$ be its image called quantum space. Let $\hat{\mathcal{F}}_\hbar : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$ be the restriction of $\hat{F}_N$ to $\mathcal{H}_\hbar$.

**Theorem 4.1.** ***[FT15]"Gutzwiller trace formula for large time".*** *Let $\varepsilon > 0$. For any $\hbar = 1/(2\pi N)$ small enough, in the limit $n \to \infty$, we have*

$$\left| Tr\left(\hat{\mathcal{F}}_\hbar^n\right) - \sum_{x=f^n(x)} \frac{e^{D_n(x)} e^{iS_{n,x}/\hbar}}{\sqrt{|Det(1 - Df_x^n)|}} \right| < CN^d (r_1^+ + \varepsilon)^n \tag{4.1}$$

*where $e^{i2\pi S_{n,x}}$ is the **action** of a periodic point defined in (2.13) and $D_n$ is the Birkhoff sum (3.46) of the effective damping function $D(x) = V(x) - V_0(x)$.*

### 4.1.1 The question of existence of a "natural quantization"

The following problem is a recurrent question in mathematics and physics in the field of quantum chaos, since the discovery of the Gutzwiller trace formula. For simplicity of the discussion we consider $V = V_0$ i.e. no effective damping, as in Figure 3.7.

**Problem 4.2.** Does there exists a sequence $\hbar_j > 0$, $\hbar_j \to 0$ with $j \to \infty$, such that for every $\hbar = \hbar_j$,

1. there exists a space $\mathcal{H}_\hbar$ of **finite dimension**, an operator $\hat{\mathcal{F}}_\hbar : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$ which is **quasi unitary** in the sense that there exists $\varepsilon_\hbar \geq 0$ with $\varepsilon_{\hbar_j} \to 0$, with $j \to \infty$ and

$$\forall u \in \mathcal{H}_\hbar, (1-\varepsilon_\hbar)\|u\| \leq \left\|\hat{\mathcal{F}}_\hbar u\right\| \leq (1+\varepsilon_\hbar)\|u\| \tag{4.2}$$

2. The operator $\hat{\mathcal{F}}_\hbar$ satisfies the **asymptotic Gutzwiller Trace formula** for large time; i.e. there exists $0 < \theta < 1$ independent on $\hbar$ and some $C_\hbar > 0$ which may depend on $\hbar$, such that for $\hbar$ small enough (such that $\theta < 1 - \varepsilon_\hbar$):

$$\forall n \in \mathbb{N}, \quad \left| \mathrm{Tr}\left(\hat{\mathcal{F}}_\hbar^n\right) - \sum_{x=f^n(x)} \frac{e^{iS_{x,n}/\hbar}}{\sqrt{|\det(1-Df_x^n)|}} \right| \leq C_\hbar \theta^n \tag{4.3}$$

Let us notice first that Theorem 4.1 (for the case $V = V_0$) provides a solution to Problem 4.2: this is the quantum operator $\hat{\mathcal{F}}_h : \mathcal{H}_h \to \mathcal{H}_h$ defined in (3.53) obtained with the choice of potential $\tilde{V} = \tilde{V}_0$, giving $V = V_0$. Indeed (4.2) holds true and (4.3) holds true from (4.1) and because $\theta := r_1^+ + \varepsilon < 1$.

Some importance of the Gutzwiller trace formula (4.3) comes from the following property which shows uniqueness of the solution to the problem:

**Proposition 4.3.** *If $\hat{\mathcal{F}}_\hbar : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$ is a solution of Problem 4.2 then the spectrum of $\hat{\mathcal{F}}_\hbar$ is **uniquely defined** (with multiplicities). In particular dim$(\mathcal{H}_\hbar)$ is uniquely defined.*

*Proof.* This is consequence of the following lemma.

**Lemma 4.4.** *If $A, B$ are matrices and for any $n \in \mathbb{N}$, $|Tr(A^n) - Tr(B^n)| < C\theta^n$ with some $C > 0$, $\theta \geq 0$ then $A$ and $B$ have the same spectrum with same multiplicities on the spectral domain $|z| > \theta$.*

*Proof of Lemma 4.4.* From the formula[20]:

$$\det(1-\mu A) = \exp\left(-\sum_{n\geq 1} \frac{\mu^n}{n} \mathrm{Tr}(A^n)\right)$$

[20]This formula is easily proved by using eigenvalues $\lambda_j$ of $A$ and the Taylor series of $\log(1-x) =$

The sum on the right is convergent if $1/|\mu| > \|A\|$. Notice that we have (with multiplicities): $\mu$ is a zero of $d_A(\mu) = \det(1-\mu A)$ if and only if $z = \frac{1}{\mu}$ is a (generalized) eigenvalue of $A$. Using the formula we get that if $1/|\mu| > \theta$ then

$$\begin{aligned}\left|\frac{\det(1-\mu A)}{\det(1-\mu B)}\right| &\leq \exp\left(\sum_{n\geq 1}\frac{|\mu|^n}{n}\left|\mathrm{Tr}(A^n) - \mathrm{Tr}(B^n)\right|\right)\\ &< \exp\left(C\sum_{n\geq 1}\frac{(|\mu|\theta)^n}{n}\right) = (1-\theta|\mu|)^{-C} =: \mathcal{B}\end{aligned}$$

Similarly $\left|\frac{\det(1-\mu A)}{\det(1-\mu B)}\right| > \frac{1}{\mathcal{B}}$, hence $d_A(\mu)$ and $d_B(\mu)$ have the same zeroes on $1/|\mu| > \theta$. Equivalently $A$ and $B$ have the same spectrum on $|z| > \theta$. □

If $\hat{G}_\hbar$ is another solution of the problem 4.2 then (4.3) implies that $\left|\mathrm{Tr}\left(\hat{\mathcal{F}}_\hbar^n\right) - \mathrm{Tr}\left(\hat{G}_\hbar^n\right)\right| \leq 2C\theta^n$ and Lemma 4.4 tells us that $\hat{G}_\hbar$ and $\hat{\mathcal{F}}_\hbar$ have the same spectrum on $|z| > \theta$. But by hypothesis (4.2) their spectrum is in $|z| > 1-\varepsilon_\hbar > \theta$. Therefore all their spectrum coincides. This finishes the proof of Proposition 4.3. □

*Remark* 4.5. Previous results in the literature concerning the "semiclassical Gutzwiller formula" for "quantum maps" do not provide an answer to the problem 4.2 above. We explain why. For any reasonable quantization of the Anosov map $f: M \to M$, e.g. the Weyl quantization or geometric quantization, one obtains a family of unitary operators $\hat{\mathcal{F}}_\hbar : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$ acting in some finite dimensional (family of) Hilbert spaces. So this answer to (4.2). Using semiclassical analysis it is possible to show a Gutzwiller formula like (4.3) but with an error term on the right hand side of the form $\mathcal{O}(\hbar\theta^n)$ with $\theta = e^{h_{top}/2} > 1$ where $h_{top} > 0$ is the topological entropy which represents the exponential growing number of periodic orbits ([Fau07b] and references therein). Using more refined semiclassical analysis at higher orders, the error can be made

$$\mathcal{O}\left(\hbar^M\theta^n\right) \tag{4.4}$$

with any $M > 0$ [Fau07b], but nevertheless one has a total error which gets large after the so-called **Ehrenfest time**: $n \gg M\frac{\log(1/\hbar)}{\lambda_0}$. So all these results obtained from any quantization scheme do not provide an answer to the problem 4.2. We may regard the operator in (3.53) as the only "quantization procedure" for which (4.3) holds true. For that reason we may call it a **natural quantization** of the Anosov map $f$.

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$-\sum_{n\geq 1}\frac{x^n}{n}$ which converges for $|x| < 1$:

$$\begin{aligned}\det(1-\mu A) &= \prod_j(1-\mu\lambda_j) = \exp\left(\sum_j \log(1-\mu\lambda_j)\right)\\ &= \exp\left(-\sum_j\sum_{n\geq 1}\frac{(\mu\lambda_j)^n}{n}\right) = \exp\left(-\sum_{n\geq 1}\frac{\mu^n}{n}\mathrm{Tr}(A^n)\right)\end{aligned}$$

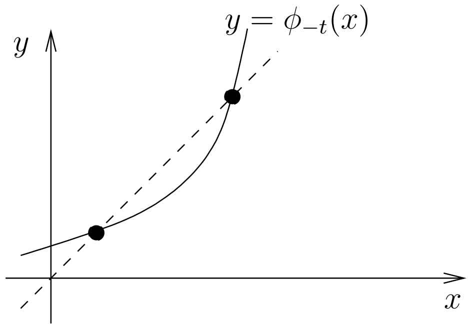


Figure 4.1: Graph of the flow.

## 4.2 Gutzwiller trace formula for contact Anosov flows

These results are in a work in preparation [FT16]. These results are transposition of the results of Section 4.1 in the case of contact Anosov flow.

We write the transfer operator as

$$\begin{aligned}\left(\hat{F}_t v\right)(x) &= \left(e^{tA} v\right)(x) = e^{\left(\int^t V\right)(x)}.\left(v\left(\phi_{-t}(x)\right)\right) \\ &= \int_M K_t(x,y)\, v(y)\, dy\end{aligned}$$

with the distributional Schwartz kernel given by $K_t(x,y) = e^{\left(\int^t V\right)(x)} \delta\left(y - \phi_{-t}(x)\right)$ (this is the "graph of the flow"). For $t > 0$, the **"flat trace"** is:

$$\mathrm{Tr}^\flat\left(\hat{F}_t\right) := \int_M K_t(x,x)\, dx = \int_M e^{\int^t V}.\delta\left(x - \phi_{-t}(x)\right) dx$$

See figure 4.1.

As in Proposition 3.26 we obtain[21] the **"Atiyah-Bott trace formula"** as a sum over periodic orbits of the flow $\phi_t$:

$$\mathrm{Tr}^\flat\left(\hat{F}_t\right) = \sum_{\gamma:o.p.} |\gamma| \sum_{n\geq 1} \frac{e^{\int^t V}.\delta\left(t - n|\gamma|\right)}{\left|\det\left(1 - D_{(u,s)}\phi_{-t}(\gamma)\right)\right|} \tag{4.5}$$

with $|\gamma| > 0$ : period of $\gamma$ and $n$: number of repetitions. This is a distribution in $\mathcal{D}'(\mathbb{R}_t)$.

**Question:** relation between the periodic orbits $\gamma$ and the Ruelle spectrum of $A = -X + V$, generator of $\hat{F}_t = e^{tA}$?

[21] For this we use that if $f : \mathbb{R}^n \to \mathbb{R}^n$ with fixed point $f(0) = 0$, with the change of variable $y = f(x)$, we write $\int \delta(f(x))\, dx = \frac{1}{|\det Df(0)|} \int \delta(y)\, dy = \frac{1}{|\det Df(0)|}$.

### 4.2.1 Zeta function

- Observation: in linear algebra, the eigenvalues of a matrix $\mathbf{A}$ are zeroes of the holomorphic function[22]

$$\mathbf{d}(z) := \det(z-\mathbf{A}) = \mathbf{d}(z_0).\exp\left(\lim_{\varepsilon\to 0}\left[-\int_\varepsilon^\infty \frac{1}{t}e^{-zt}\mathrm{Tr}\left(e^{t\mathbf{A}}\right)dt\right]_{z_0}^{z}\right), \quad z_0 \notin \mathrm{Spec}(\mathbf{A}).$$

For $\mathrm{Re}(z) \gg 1$ we define the "**spectral determinant**" or **zeta function**:

$$\begin{aligned} d(z) :&= \exp\left(-\int_{|\gamma|_{min}}^\infty \frac{1}{t}e^{-zt}\mathrm{Tr}^\flat\left(e^{tA}\right)dt\right) \\ &\underset{(4.5)}{=} \exp\left(-\sum_\gamma\sum_{n\geq 1}\frac{e^{\int^t V}.e^{-zn|\gamma|}}{n\left|\det\left(1-D_{(u,s)}\phi_{n|\gamma|}(\gamma)\right)\right|}\right)\end{aligned}$$

**Theorem 4.6.** *[GLP13]For an Anosov vector field $X$, $d(z)$ has an analytic extension on $\mathbb{C}$. Its zeroes are Ruelle resonances with multiplicities.*

*Remark* 4.7. in 2008, Baladi-Tsujii [BT08] have a similar result for Anosov diffeomorphisms.

### 4.2.2 Application: counting periodic orbits

The objective is to express in term of Ruelle spectrum the **counting function**:

$$\pi(T) := \sharp\{\gamma : periodic-orbit, \quad |\gamma| \leq T\} = \sum_{\gamma, |\gamma|\leq T} 1$$

Observe that $\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1} \underset{t\infty}{\simeq} \det\left(D\phi_{t/E_u}\right)^{-1}$. The choice of potential $V = \mathrm{div}X_{/E_u}$ gives $e^{\int^t V} = \det\left(D\phi_{t/E_u}\right)$ and $e^{\int^t V}\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1} \simeq 1$.

**Theorem 4.8.** *[GLP13](with pinching hypothesis) there exists $\delta > 0$ s.t.*

$$\pi(T) = \mathrm{Ei}(h_{top}T) + O\left(e^{(h_{top}-\delta)T}\right) \underset{T\to\infty}{\sim} \frac{e^{h_{top}T}}{h_{top}T}$$

*with $\mathrm{Ei}(x) := \int_{x_0}^x \frac{e^y}{y}dy$ and $h_{top}$ dominant eigenvalue of $A = -X + \mathrm{div}X_{/E_u(x)}$ called* ***topological entropy****.*

[22]Write $(z-A)^{-1} = \int_0^\infty e^{-(z-A)t}dt$, and $d(z) = \det(z-A) = \exp(\mathrm{Tr}(\log(z-A)))$ hence $\frac{d}{dz}\log d(z) = \mathrm{Tr}(z-A)^{-1} = \int_0^\infty e^{-zt}\mathrm{Tr}\left(e^{tA}\right)dt$.

### 4.2.3 Semiclassical zeta function

Observe that we have $\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1} \underset{t\infty}{\simeq} \det\left(D\phi_{t/E_u}\right)^{-1/2}\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1/2}$ and $\det\left(D\phi_{t/E_u}\right)^{-1/2} = e^{-\frac{1}{2}\int^t \mathrm{div}X_{/E_u}} e^{-\frac{1}{2}\int^t V_0}$ so in (4.5) we have

$$e^{\int^t V}\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1} \underset{t\infty}{\simeq} e^{\int^t D}\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1/2}.$$

We define the "**Gutzwiller-Voros zeta function**" or "**semi-classical zeta function**" by

$$d_{G-V}(z) := \exp\left(-\sum_{\gamma}\sum_{n\geq 1}\frac{e^{-zn|\gamma|}e^{\int^t D}}{n\left|\det\left(1-D_{(u,s)}\phi_{n|\gamma|}(\gamma)\right)\right|^{1/2}}\right) \tag{4.6}$$

**Theorem 4.9.** *[FT16]The semiclassical zeta function $d_{G-V}(z)$ has an meromorphic extension on $\mathbb{C}$. On $\mathrm{Re}(z) > \gamma_1^+$, $d_{G-V}(z)$ has finite number of poles and its zeroes coincide (up to finite number) with the Ruelle eigenvalues of $A$.*

See figure 3.15. The motivation for studying $d_{G-V}(z)$ comes from the Gutzwiller semiclassical trace formula in quantum chaos. Also in the case of surface with constant curvature, and $V = V_0 = \frac{1}{2}$, we have $D_{(u,s)}\phi_{n|\gamma|}(\gamma) = \begin{pmatrix} e^{|\gamma|n} & 0 \\ 0 & e^{-|\gamma|n}\end{pmatrix}$. This gives

$$\begin{aligned} d_{G-V}(z) &\underset{(4.6)}{=} \exp\left(-\sum_{\gamma}\sum_{n\geq 1}\sum_{m\geq 0}\frac{1}{n}e^{-n|\gamma|\left(z+\frac{1}{2}+m\right)}\right) \\ &= \prod_{\gamma}\prod_{m\geq 0}\left(1-e^{-\left(z+\frac{1}{2}+m\right)|\gamma|}\right) =: \zeta_{Selberg}\left(z+\frac{1}{2}\right)\end{aligned}$$

*Proof.* Put $x = e^{-|\gamma|n}$ and use that $\left|\det\left(1-\begin{pmatrix} 1-x^{-1} & 0 \\ 0 & 1-x\end{pmatrix}\right)\right|^{-1/2} = x^{1/2}(1-x)^{-1} = x^{1/2}\sum_{m\geq 0}x^m$. □

Therefore $d_{G-V}(z)$ **"generalizes" the Selberg zeta function** $\zeta_{Selberg}$ for case of variable curvature (or contact Anosov flows). Compare figure 4.2 with figure 3.15.

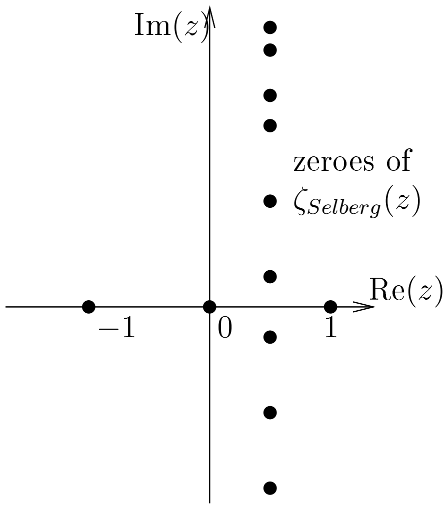


Figure 4.2: Zeroes of $\zeta_{Selberg}$.

# A Some definitions and theorems of semiclassical analysis

## A.1 Class of symbols

**Notations:** For $x \in \mathbb{R}^n$, $\langle x \rangle := \sqrt{1+|x|^2}$ and we use the standard multi-indices notation $\partial_x^\alpha f := \frac{\partial^{\alpha_i} f}{\partial x_i^{\alpha_i}} \dots \frac{\partial^{\alpha_n} f}{\partial x_n^{\alpha_n}}$.

### A.1.1 Symbols with constant order

The folloing classes of symbols have been introduced by Hörmander [Hör83]. Let $M$ be a smooth compact manifold.

**Definition A.1.** Let $\mu \in \mathbb{R}$ called the **order**. Let $0 \leq \delta < \frac{1}{2} < \rho \leq 1$. The **class of symbols** $S^\mu_{\rho,\delta}$ contains smooth functions $p \in C^\infty(T^*M)$ such that on any charts of $U \subset M$ with coordinates $x = (x_1, \dots x_n)$ and associated dual coordinates $\xi = (\xi_1, \dots \xi_n)$ on $T_x^*U$, any multi-index $\alpha, \beta \in \mathbb{N}^n$, there is a constant $C_{\alpha,\beta}$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leq C_{\alpha,\beta} \langle \xi \rangle^{\mu - \rho|\alpha| + \delta|\beta|} \tag{A.1}$$

The case $\rho = 1, \delta = 0$ is very common. We denote $S^\mu := S^\mu_{1,0}$.

For example on a chart, $p(x,\xi) = \langle \xi \rangle^\mu$ is a symbol $p \in S^\mu$.
If $\mu \leq \mu'$ then $S^\mu \subset S^{\mu'}$. We have $S^{-\infty} := \bigcap_{\mu \in \mathbb{R}} S^\mu = \mathcal{S}(T^*M)$.

### A.1.2 Symbols with variable order in $T^*M$

We refer to [FRS08, Section A.2.2] for a precise description of theorems related to symbols with variable orders. This class of symbols is useful for Anosov diffeomorphisms and Anosov flows on a manifold. Let $M$ be a smooth compact manifold.

**Definition A.2.** Let $m(x,\xi) \in S^0_{1,0}$ be a real-valued called **variable order** and let $0 \le \delta < \frac{1}{2} < \rho \le 1$. The **class of symbols** $S^{m(x,\xi)}_{\rho,\delta}$ contains smooth functions $p \in C^\infty(T^*M)$ such that on any charts of $U \subset M$ with coordinates $x = (x_1, \dots x_n)$ and associated dual coordinates $\xi = (\xi_1, \dots \xi_n)$ on $T^*_x U$, any multi-index $\alpha, \beta \in \mathbb{N}^n$, there is a constant $C_{\alpha,\beta}$ such that

$$|\partial^\alpha_\xi \partial^\beta_x p(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m(x,\xi) - \rho|\alpha| + \delta|\beta|} \tag{A.2}$$

**Example A.3.** For example $A(x,\xi) = \langle \xi \rangle^{m(x,\xi)}$ in (3.31) belongs to $S^{m(x,\xi)}_{\rho,\delta}$ with any $0 < \delta < \frac{1}{2} < \rho < 1$.

### A.1.3 Symbols with variable order in $\mathbb{R}^{2d}$

Here we introduce a class of symbol specifically for application to Section 3.1 on $\mathbb{R}^{2d}$. We denote $z = (x,\xi) \in \mathbb{R}^{2d}$.

**Definition A.4.** Let $\mu \in \mathbb{R}$ and $0 < \rho \le 1$. A symbol $p(z) \in S^\mu_\rho$ is a function $p \in C^\infty\left(\mathbb{R}^{2d}\right)$ such that $\forall \alpha \in \mathbb{N}^{2d}$,$\exists C_\alpha > 0$,

$$|\partial^\alpha_z p(z)| \le C_\alpha \langle z \rangle^{\mu - \rho|\alpha|} \tag{A.3}$$

Example: $m(z)$ after eq.(3.14) belongs to $S^0 := S^0_1$.

**Definition A.5.** Let $m(z) \in S^0$. The **class of symbols** $S^{m(z)}_\rho$ **with variable order** $m(z)$ contains smooth functions $p \in C^\infty\left(\mathbb{R}^{2d}\right)$ such that $\forall \alpha \in \mathbb{N}^{2d}$,$\exists C_\alpha > 0$,

$$|\partial^\alpha_z p(z)| \le C_\alpha \langle z \rangle^{m(z) - \rho|\alpha|} \tag{A.4}$$

Example: $A_C(z)$ in eq.(3.14) belongs to $S^{m(z)}_\rho$ with any $0 < \rho < 1$.

*Proof.* Let us observe: we have $\partial_x A = (\partial_x m) \log \langle \xi \rangle .A$ but $(\partial_x m) \in S^0$ and $\log \langle \xi \rangle \in S^\varepsilon$ for every $\varepsilon > 0$ so $\partial_x A \in S^\varepsilon$. We have

$$\partial_\xi A = \left( (\partial_\xi m) \log \langle \xi \rangle + m. \frac{\partial_\xi \langle \xi \rangle}{\langle \xi \rangle} \right) .A$$

but $(\partial_\xi m) \in S^{-1}$, $\log \langle \xi \rangle \in S^\varepsilon$ for any $\varepsilon > 0$, $m \in S^0$, $\partial_\xi \langle \xi \rangle \in S^0$, $\langle \xi \rangle^{-1} \in S^{-1}$ so $\partial_\xi A \in S^{m-\rho}$ with $\rho = 1 - \varepsilon$. ☐

## A.2 Pseudo-differential operators (PDO)

### A.2.1 Quantization

"Quantization" is a map Op which maps a symbol $p$ to an operator $\mathrm{Op}(p)$ with specific properties. For example, its inverse maps the algebra of operators (for the composition) to an algebra on the symbols which coincide with the ordinary product of functions at first order.

**Definition A.6.** If $p \in S^m_{\rho,\delta}(T^*M)$ is a symbol with order $m$, its **standard quantization** is the operator $\mathrm{Op}(p) : \mathcal{D}'(M) \to \mathcal{D}'(M)$, $C^\infty(M) \to C^\infty(M)$ whose distribution kernel is smooth outside the diagonal and such that on a local coordinate chart $U \subset \mathbb{R}^n$,it is given up to a smoothing operator by

$$(\mathrm{Op}(p)\, u)(x) := \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi} p(x,\xi) u(y) dy d\xi \tag{A.5}$$

We say that $\mathrm{Op}(p)$ is a **pseudo-differential operator** or **PDO** with ordinary symbol $p$.

- For example if $X$ is a vector field on $M$, the operator $\hat{p} = \mathrm{Op}(p) = -iX$ is a PDO with ordinary symbol
$$p(x,\xi) = X(\xi) \tag{A.6}$$
- For example on $M = \mathbb{R}^d$, if $p(x,\xi) = \sum_{\alpha \in \mathbb{N}^d} p_\alpha(x) \xi^\alpha$ (with a finite number of terms) then $\mathrm{Op}(p)$ is the differential operator:
$$\mathrm{Op}(p)\, u = \sum_{\alpha \in \mathbb{N}^d} p_\alpha(x) (-i\partial_x)^\alpha u$$

**Definition A.7.** For **Weyl quantization**, (A.5) is replaced by [Tay96b, (14.5) p.60]:

$$\left(\mathrm{Op}_W\left(p\right)u\right)\left(x\right) := \frac{1}{\left(2\pi\right)^n}\iint e^{i(x-y)\cdot\xi}p\left(\frac{x+y}{2},\xi\right)u\left(y\right)dyd\xi \tag{A.7}$$

We say that $\mathrm{Op}_{\mathrm{W}}\left(p\right)$ is a **pseudo-differential operator** or **PDO** with Weyl symbol $p$.

*Remark* A.8. Weyl quantization is often prefered other standard quantization because it has specific interesting properties. First a real symbol $p \in S^m\left(M\right)$,$m \in \mathbb{R}$, is quantized in a formally self-adjoint operator $\hat{P} = \mathrm{Op}\left(p\right)$. Secondly, a change of coordinate systems preserving the volume form changes the symbol at a subleading order $S^{\mu-2}$ only. In other words, on a manifold with a fixed smooth density $dx$, the Weyl symbol $p$ of a given pseudodifferential operator $\hat{P}$ is well defined modulo terms in $S^{\mu-2}$.

For example if $X$ is a vector field on $M$, the operator $\hat{p} = -iX$ is a PDO with Weyl symbol

$$p_W\left(x,\xi\right) = X\left(\xi\right) + \frac{i}{2}\mathrm{div}\left(X\right) \tag{A.8}$$

Indeed from [Tay96b, (14.7) p.60], in a given chart where $X = \sum X_j\left(x\right)\frac{\partial}{\partial x^j} \equiv X\left(x\right)\partial_x$,

$$p_W\left(x,\xi\right) = \exp\left(\frac{i}{2}\partial_x\partial_\xi\right)\left(X\left(x\right).\xi\right) = X\left(x\right).\xi + \frac{i}{2}\partial_x X = X\left(\xi\right) + \frac{i}{2}\mathrm{div}\left(X\right)$$

and $\mathrm{div}\left(X\right)$ depends only on the choice of the volume form, see [Tay96a, p.125]. Notice that this symbol does not depend on the choice of coordinates systems provided the volume form is expressed by $dx = dx_1\dots dx_n$. The first term $p_0\left(x,\xi\right) = X\left(\xi\right)$ in (A.8) belongs to $S^1$ is called the **principal symbol** of $\hat{p}$. The second term $\frac{i}{2}\mathrm{div}\left(X\right)$ in (A.8) belongs to $S^0$ and is called the **subprincipal symbol** of $\hat{p}$.

### A.2.2 Composition

**Theorem A.9.** *[Tay96b, Prop.(3.3) p.11]**"Composition of PDO"**. If $A \in S^{m_1}_{\rho,\delta}$ and $B \in S^{m_2}_{\rho,\delta}$ then*

$$\mathrm{Op}\left(A\right)\mathrm{Op}\left(B\right) = \mathrm{Op}\left(AB\right) + \mathcal{O}\left(\mathrm{Op}(S^{m_1+m_2-(\rho-\delta)}_{\rho,\delta})\right)$$

*i.e. the symbol of* $\mathrm{Op}\left(A\right)\mathrm{Op}\left(B\right)$ *is the product $AB$ and belongs to $S^{m_1+m_2}_{\rho,\delta}$ modulo terms in $S^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}$.*

We also have:

**Theorem A.10.** *[Tay96b, Eq.(3.24)(3.25) p.13]The symbol of the commutator* $[\mathrm{Op}(A), \mathrm{Op}(B)]$ *is the Poisson bracket* $-i\{A,B\}$ *modulo* $S^{m_1+m_2-2(\rho-\delta)}_{\rho,\delta}$*. The symbol* $-i\{A,B\}$ *belongs to* $S^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}$ *. We also recall [Tay96a, (10.8) p.43] that* $\{A,B\} = -\mathbf{X}_B(A)$ *where* $\mathbf{X}_B$ *is the Hamiltonian vector field generated by* $B$*.*

### A.2.3 Bounded and compact PDO

For PDO with order zero we have:

**Theorem A.11.** *"$L^2$* ***continuity theorem****". Let* $p \in S^0_\rho$*. Then* $\mathrm{Op}(p)$ *is a* ***bounded operator*** *and for any* $\varepsilon > 0$ *there is a decomposition*

$$\mathrm{Op}(p) = \hat{p}_\varepsilon + \hat{K}_\varepsilon$$

*with* $\hat{K}_\varepsilon \in \mathrm{Op}(S^{-\infty})$ *smoothing operator,* $\|\hat{p}_\varepsilon\| \leq L + \varepsilon$ *and*

$$L = \limsup_{(x,\xi)\in T^*M} |p(x,\xi)|.$$

For PDO with negative order we have:

**Theorem A.12.** *Let* $p \in S^\mu_\rho$ *with* $\mu < 0$ *then* $\mathrm{Op}(p)$ *is a* ***compact operator****. If* $\mu < -d$ *so that* $\int_{T^*M} |p(x,\xi)|\, dx d\xi < \infty$*, then* $\mathrm{Op}(p)$ *is a* ***trace class operator*** *and*

$$\mathrm{Tr}(\mathrm{Op}(p)) = \frac{1}{(2\pi)^d} \int p(x,\xi)\, dx d\xi$$

## A.3 Wavefront

The wavefront set of a distribution has been introduced by Hörmander. The wavefront set corresponds to the directions in $T^*X$ where the distribution is not $C^\infty$ (i.e. the local Fourier transform is not rapidly decreasing). The wavefront set of a PDO is the directions in $T^*X$ where the symbol is not rapidly decreasing:

**Definition A.13.** ([GS94, p.77][Tay96b, p.27]) If $(x_0,\xi_0)\in T^*M\backslash 0$, we say that $A\in S^m$ is **non characteristic (or elliptic)** at $(x_0,\xi_0)$ if $\left|A(x,\xi)^{-1}\right|\leq C\left|\xi\right|^{-m}$ for $(x,\xi)$ in a small conic neighborhood of $(x_0,\xi_0)$ and $|\xi|$ large. If $u\in\mathcal{D}'(M)$ is a distribution, we say that $u$ **is** $C^\infty$ at $(x_0,\xi_0)\in T^*X\backslash 0$ if there exists $A\in S^m$ non characteristic (or elliptic) at $(x_0,\xi_0)$ such that $(\mathrm{Op}(A)\,u)\in C^\infty(M)$. The **wavefront set of the distribution** $u$ is

$$\mathrm{WF}(u):=\{(x_0,\xi_0)\in T^*M\backslash 0,\quad u\text{ is not }C^\infty\text{ at }\ (x_0,\xi_0)\}$$

The **wavefront set of the operator** $\mathrm{Op}(A)$ is the smallest closed cone $\Gamma\subset T^*M\backslash 0$ such that $A_{/\complement\Gamma}\in S^{-\infty}\left(\complement\Gamma\right)$.

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