

THE PROBLEM OF GAUGE THEORY

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ABSTRACT. I sketch what it is supposed to mean to quantize gauge theory, and how this can be made more concrete in perturbation theory and also by starting with a finite-dimensional lattice approximation. Based on real experiments and computer simulations, quantum gauge theory in four dimensions is believed to have a mass gap. This is one of the most fundamental facts that makes the Universe the way it is. This article is the written form of a lecture presented at the conference “Geometric Analysis: Past and Future” (Harvard University, August 27-September 1, 2008), in honor of the 60th birthday of S.-T. Yau.

1. YANG-MILLS EQUATIONS

The strong, weak, and electromagnetic interactions – in other words, more or less everything we know about in nature except gravity – are all described by gauge theory. Mathematically, in a gauge theory with gauge group G , formulated on a spacetime M , a gauge field is a connection on a G -bundle $E \rightarrow M$. For our purposes, M is a four-manifold with a metric of Lorentz signature $-+++$. In fact, for our purposes there is no essential loss to take M to be Minkowski spacetime $\mathbb{R}^{3,1}$ (that is, \mathbb{R}^4 with a flat pseudo-Riemannian metric of Lorentz signature).

In addition to gauge fields, in nature there also are matter fields. The matter fields describe things such as electrons, neutrinos, quarks, and possibly Higgs particles. Gauge fields mediate “forces” between particles described by matter fields (and between additional particles described by the gauge fields themselves). For simplicity, in this talk, I will omit the matter fields and just describe the gauge fields.

One thing I should say before getting too far is that, for a large variety of reasons, it is unrealistic, in a talk or a short article, to expect to fully describe the problem of Yang-Mills theory. To really appreciate the problem, it is necessary to delve into quantum field theory in some depth. There is a large physics literature on quantum field theory, and there is also a large math literature (for example, see [1]).

I felt in preparing the lecture that to make it comprehensible, I could not simply skip the preliminaries. One can skip the preliminaries and make some formal statements, but such statements are rather opaque. On the other hand, it is also not possible to fully explain the preliminaries in a short space or time, so I have had to seek a middle path. This inevitably involves some debateable choices of what to explain. The main idiosyncrasy in my presentation is that I have decided to rely on a Hamiltonian approach (including the well-established [2] but relatively unfamiliar notion of Hamiltonian lattice gauge theory) rather than converting everything to

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path integrals. The path integral approach is very powerful but involves an extra layer of abstraction.

Classically, there is no problem to explain what is meant by Yang-Mills gauge theory. A gauge field, that is a connection A on a G -bundle $E \rightarrow M$, has a curvature $F = dA + A \wedge A$. The curvature is a two-form on M valued in the adjoint bundle $\text{ad}(E)$ derived from E . The classical Yang-Mills equations are

$$(1.1) \quad D \star F = 0,$$

where \star is the Hodge star (mapping two-forms to $n - 2$ -forms, where $n = \dim M$), and $D = d + A$ is the gauge-covariant extension of the exterior derivative.

The Yang-Mills equations may be most familiar in the abelian case, that is in the case that $G = U(1)$. Maxwell's equations of electromagnetism (in vacuum) can be described in terms of a two-form F that obeys

$$(1.2) \quad 0 = d \star F = dF.$$

The first of these equations is the Yang-Mills equation as written in eqn. (1.1). (Recall that D reduces to d when G is abelian.) We can omit the second equation if we define $F = dA$ as the curvature of a connection A , for it is then an identity – the Bianchi identity.

For $G = U(1)$, Maxwell's equations describe propagation of light waves in vacuum. These are linear equations, so, for example, in the approximation that Maxwell's equations are valid, two beams of light pass through each other without scattering. (In a more precise description of nature, there are all sorts of corrections to Maxwell's equations, involving things such as quantum mechanics, electrons, and gravity, so it is not expected that the propagation of light is precisely linear even in vacuum. The nonlinearities are expected to be very small and have not yet been observed, though it is believed that this may be possible in the near future.)

In the nonabelian case, the Yang-Mills equations $D \star F = 0$ are *nonlinear* wave equations. So classical Yang-Mills waves do scatter each other, although in the case of weak waves, the nonlinearities and the scattering effects are small. In this respect, the Yang-Mills equations are analogous to the vacuum Einstein equations $R_{\mu\nu} = 0$ (where R is the Ricci tensor). They are nonlinear hyperbolic wave equations.

Having a hyperbolic wave equation means that, optimistically speaking, solutions of the equation are in one-to-one correspondence with initial data, given on a global Cauchy hypersurface (fig. 1). However, there is a fundamental difference between Yang-Mills theory and General Relativity. In General Relativity there is a phenomenon of gravitational collapse – the formation of a black hole. As a result, a solution on a Cauchy hypersurface cannot be extended for all times, in general. In Yang-Mills theory, instead, there is a global-in-time existence result for classical solutions (see [3], [4]). This is probably one of the reasons that quantum Yang-Mills theory is simpler than quantum gravity, though it is not one of the reasons that is easiest to interpret.

For gauge group $G = U(1)$, we observe classical solutions of Maxwell's equations all the time – light waves. For nonabelian G , even though there are beautiful nonlinear classical wave equations, we do not observe these nonlinear classical waves in practice. That is actually because of a phenomenon known as the mass gap. The mass gap means that the description in terms of nonlinear classical waves is only a good approximation above a certain minimum energy and frequency. At lower

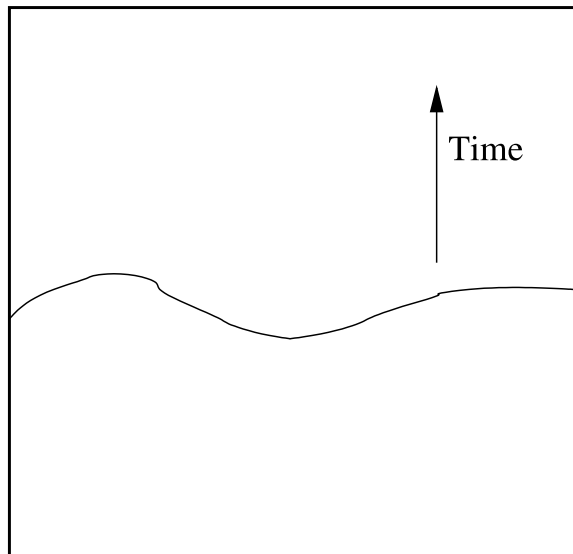


FIGURE 1. A spacetime with a global Cauchy hypersurface.

energies and frequencies, one must use quantum field theory rather than classical field theory.

According to theory, physical conditions that are well-described by classical non-linear wave equations can exist, but because of the minimum frequency involved, our technology does not enable use to generate the appropriate initial conditions. In practice, all manifestations of Yang-Mills theory that we observe, except in the abelian case, involve quantum behavior – that is, they involve phenomena that cannot be described by the classical field equations. That is why the role of Yang-Mills theory in physics cannot be described without talking about the quantum theory.

2. CLASSICAL PHASE SPACE

Formally speaking, the starting point in going to the quantum theory is to observe that what I will call W , the space of all solutions of the classical Yang-Mills equations modulo gauge transformations, is an (infinite-dimensional) symplectic manifold. The real reason for this is that the Yang-Mills equations are not just equations. They are the Euler-Lagrange equations associated with an action function

$$(2.1) \quad I = \frac{1}{4g^2} \int_M \text{Tr} F \wedge \star F.$$

Here Tr is an invariant quadratic form on the Lie algebra of G , and g is a constant, known as the gauge coupling constant.

In general, starting with any action, the space of classical solutions of the Euler-Lagrange equations, modulo the relevant gauge equivalence, is always a symplectic manifold. Quantization has to do with quantizing this symplectic manifold.

Actually, W is a cotangent bundle. This can be established as follows. Pick an initial value surface $S \subset M$, and let \mathcal{Y} be the space of all gauge fields on S (that is all connections on a G -bundle $E \rightarrow S$) modulo gauge equivalence. Then W is the

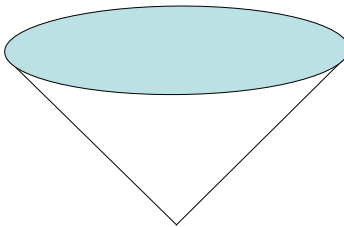


FIGURE 2. The future light cone.

cotangent bundle of \mathcal{Y} , for any choice of S . In effect, to specify a classical solution of Yang-Mills theory corresponding to a point in W , we must pick an initial value of the gauge field along S – that is a point in \mathcal{Y} – and also, as the Yang-Mills equations are of second order, we must specify the normal derivative to the gauge field along S . By forgetting the normal derivative, we get a map $W \rightarrow \mathcal{Y}$, and W is the cotangent bundle to \mathcal{Y} .

In finite dimensions, there is no problem in quantizing a cotangent bundle. But W is infinite-dimensional, and in infinite dimensions, we have to be careful. For an elementary illustration of the problem, recall that by a well-known theorem of Stone and von Neumann, the quantization of \mathbb{R}^{2n} , with a symplectic structure that comes from a nondegenerate skew form, is unique (provided that one requires that this quantization should admit an action of the Heisenberg group, the central extension of the group of translations of \mathbb{R}^{2n}). The analog of this for $n = \infty$ is more subtle.

We need more information about what sort of answer we want to get. The additional information is that the energy should be bounded below. With this information, quantization becomes unique again, at least in the abelian case, as we will explain.

Instead of just describing the energy in an ad hoc way, let us provide a framework for this discussion. Our Yang-Mills action (2.1) is invariant under the symmetries of the pseudo-Riemannian manifold M . Classically, we need only endow M with a conformal structure (rather than a metric), since the action is defined using the Hodge star operator \star , and in four-dimensions, this operator is conformally-invariant when acting on two-forms. We take M to be Minkowski spacetime $\mathbb{R}^{3,1}$, with its standard conformal structure (induced from a flat pseudo-Riemannian metric). The group of conformal motions of M (or more precisely of its conformal compactification) is $SO(2, 4)$, and this is a group of symmetries of the classical theory.

For $G = U(1)$, $SO(2, 4)$ is realized as a group of symmetries of the quantum theory, but this is actually not true for nonabelian G . The quantum theory is obtained via a kind of limiting procedure, concerning which I will try to give a few hints. This limiting procedure does not preserve the full $SO(2, 4)$ symmetry, but only a subgroup. The details depend on exactly how one proceeds with quantization.

A maximal subgroup of $SO(2, 4)$ that can be preserved in the quantization, for nonabelian G , is the subgroup P – known as the Poincaré group – that preserves a flat metric on $\mathbb{R}^{3,1}$. This group acts on linear coordinates x on $\mathbb{R}^{3,1}$ by $x \rightarrow ax + b$, where b is a constant “translation,” and a is a linear transformation that belongs to the Lorentz group $SO(1, 3)$ (the group of symmetries of a quadratic form of



FIGURE 3. Sketched here is an example of what the spectrum of m looks like when there is a “mass gap.” Apart from the isolated eigenvalue at $m = 0$ corresponding to the vacuum, there is in this example a single discrete eigenvalue of m at a positive value. There is also a continuous spectrum (corresponding to multiparticle states) which begins at twice the value of the smallest positive eigenvalue.

Lorentz signature). We are supposed to quantize in such a way that we get a unitary representation of P .

Representation theory of P is simple because P is an extension:

$$(2.2) \quad 0 \rightarrow \mathbb{R}^{3,1} \rightarrow P \rightarrow SO(3,1) \rightarrow 1.$$

Here $\mathbb{R}^{3,1}$ is the abelian group of translations. Its spectrum defines a point in the dual space to $\mathbb{R}^{3,1}$. This dual space, which I will denote $\tilde{\mathbb{R}}^{3,1}$, is usually called momentum space. Of course, $\tilde{\mathbb{R}}^{3,1}$ is also endowed with a flat metric. (We could use the metrics to identify them, but this would be confusing.)

We write p for a point in $\tilde{\mathbb{R}}^{3,1}$, usually called the energy-momentum. The condition $(p,p) = 0$ defines a cone, called the light cone. p is said to be lightlike if it lies on this cone. As is usual in Lorentz signature, the light cone is the union of two components, the “past” and “future” parts of the light cone. The future light cone is sketched in fig. 2.

We want a quantization such that the spectrum of energy-momentum lies inside (and on) the future light cone. This is usually described by saying that the energy is bounded below by zero. Here, energy is a suitable linear function on $\tilde{\mathbb{R}}^{3,1}$.

Boundary points of the future light cone are allowed, but they play a very special role. The Hilbert space \mathcal{H} that we get by quantization is supposed to have a special state, the vacuum state $|\Omega\rangle$, whose energy-momentum is supported at the apex of the cone, in other words at the point $p = 0$. This state transforms in a one-dimensional trivial representation of the Poincaré group.

Apart from this one trivial representation, the other representations that are relevant are constructed as follows. Let \mathcal{O} be a non-trivial $SO(3,1)$ orbit that is inside, or on, the future light cone. Let $V \rightarrow \mathcal{O}$ be a homogeneous vector bundle over \mathcal{O} . Then the space of L^2 sections of V is a positive energy representation of the Poincaré group, and, apart from the trivial representation that corresponds to the vacuum state, these are the representations that we allow.

An orbit is characterized by the invariant $m^2 = (p,p)$. This invariant is non-negative (since we consider only orbits inside or on the light cone), and m , defined by the positive square root, is called the mass. m can have either a discrete or a continuous spectrum. More exactly, in quantum field theory, m always has a continuous spectrum (from what are known as multiparticle states) above some minimum value m_* . There may also be a discrete spectrum from single particle states with $m < m_*$. Positive discrete eigenvalues of m are called masses of particles.

The vacuum state always has $m = 0$. If this is a discrete eigenvalue of m , in other words if there is $\epsilon > 0$ such that every state orthogonal to the vacuum has $m \geq \epsilon$, then we say that the theory has a mass gap. In fig. 3, we sketch a typical spectrum of a quantum field theory with a mass gap. The discrete eigenvalues of m are 0 and one positive value.

Now, let us examine the problem of quantizing gauge theory, armed with the information that the energy should be non-negative. Classically, if we write the curvature in non-relativistic terms as $F = dt \wedge d\vec{x} \cdot \vec{E} + \frac{1}{2} d\vec{x} \cdot d\vec{x} \times \vec{B}$, where \vec{E} and \vec{B} are the electric and magnetic fields, then the energy is the conserved quantity

$$(2.3) \quad H = \frac{1}{2g^2} \int d^3x \operatorname{Tr} (\vec{E}^2 + \vec{B}^2).$$

It is non-negative, and vanishes precisely for the trivial solution with $F = 0$.

3. QUANTIZATION

We next discuss the quantization, beginning with the abelian case, that is $G = U(1)$. In the abelian case, the curvature F is linear in the connection A , that is $F = dA$, and Maxwell's equations $d \star F = 0$ are also linear. The space W of solutions modulo gauge transformations is therefore also a linear space – an infinite-dimensional one.

Now in the case of a *finite*-dimensional affine space, \mathbb{R}^{2n} for some n , quantization is unique (once one requires that it should respect the affine structure, in a suitable sense) according to a theorem of Stone and von Neumann.

This is far from being true in infinite dimensions. Quantization of an infinite-dimensional affine or linear space W is far from unique. But we do get the uniqueness again if we are given a positive-definite quadratic function Q on W , and we ask for a quantization in which Q is represented by a hermitian operator that is bounded below.

In abelian gauge theory, because the curvature is a linear function, the Hamiltonian or energy function H is a quadratic function, which moreover is positive-definite. This puts us in the situation just described.

The result of quantization can be described as follows. As the space of solutions of Maxwell's equations, W has a natural symplectic structure. This symplectic structure is translation-invariant – that is, it comes from a constant two-form (on the infinite-dimensional linear space W). The positive-definite quadratic function Q on W is equivalent to a translation-invariant Riemannian metric on W . Combining these, W is endowed with a translation-invariant complex structure and therefore can be regarded as a complex vector space with a hermitian metric, that is, a Hilbert space. Let us write \mathcal{W} for W regarded in this way as a Hilbert space.

Quantization of W is supposed to give us a Hilbert space \mathcal{H} . This turns out to be the Hilbert space completion of the “Fock space” constructed from \mathcal{W} , which by definition is

$$(3.1) \quad \mathcal{H}_0 = \bigoplus_{n=0}^{\infty} \operatorname{Sym}^n \mathcal{W}.$$

Here $\operatorname{Sym}^n \mathcal{W}$ is the n -fold symmetric product of \mathcal{W} , with $\operatorname{Sym}^0 \mathcal{W} = \mathbb{C}$.

In (3.1), $\mathbb{C} = \operatorname{Sym}^0 \mathcal{W}$ is the one-dimensional space of “vacuum” states (Poincaré-invariant states of zero energy, as discussed earlier). $\mathcal{W} = \operatorname{Sym}^1 \mathcal{W}$ is the space of “single-particle states.” Concretely, as a representation of the conformal group, \mathcal{W} is the space of sections of a certain homogeneous vector bundle over the future

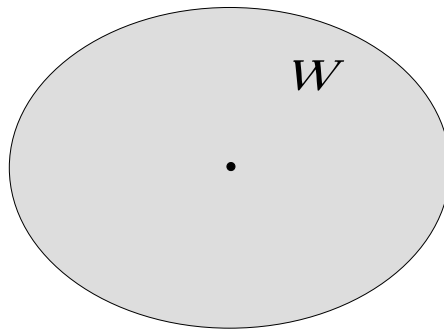


FIGURE 4. A space W endowed with a function with a unique isolated minimum, indicated by the black dot.

light cone. (In four dimensions, this bundle is of rank 2, the number of polarization states of an electromagnetic wave.) Support on the cone means that \mathcal{W} is a space of massless states, that is states of $m^2 = (p, p) = 0$. These are called the one-photon states. Similarly, $\text{Sym}^n W$ is the space of n -photon states.

This is our answer for quantization of abelian gauge theory, though we need to say more about the physical interpretation in terms of photons, and about how various classical expressions are realized as operators acting on this Hilbert space.

Now let us discuss the nonabelian case. The curvature F is no longer linear, so the space W of classical solutions is no longer a linear or affine space. Similarly, the energy function H is not quadratic. However, it is still true, as suggested in fig. 4, that H is positive semidefinite with a unique zero corresponding to the trivial solution $F = 0$.

Like any function with an isolated minimum, H looks quadratic near its minimum. One may ask whether this simple fact can be a starting point for understanding the quantization.

The constant $1/g^2$ in H is very important. In general, suppose that we have a not-quadratic function H of variables x_i with a minimum at the origin:

$$(3.2) \quad H = \frac{1}{g^2} \left(\sum_{i,j} a_{ij} x_i x_j + \sum_{i,j,k} b_{ijk} x_i x_j x_k + \dots \right).$$

We write $y_i = x_i/g$, so that

$$(3.3) \quad H = \left(\sum_{i,j} a_{ij} y_i y_j + g \sum_{i,j,k} b_{ijk} y_i y_j y_k + \dots \right).$$

We do not know how to diagonalize H as an operator in a Hilbert space, but we can diagonalize its quadratic part H_0 :

$$(3.4) \quad H_0 = \sum_{i,j} a_{ij} y_i y_j.$$

This means that if g is small, we can approximately diagonalize H . The first step is to diagonalize H_0 , and then one makes successive corrections, treating the higher order terms in H as perturbations, so as to construct the eigenfunctions of H in an asymptotic expansion in powers of g .

Classically it does not make sense to say that g is large or small; g is just an uninteresting constant multiplying the action. But quantum mechanically there is a dimensionless number that in the usual units is $g^2/\hbar c$. This is really the small parameter in the asymptotic expansion that was just suggested. In this asymptotic expansion, one diagonalizes H – and computes all quantities of physical interest – in an asymptotic expansion in powers of $g^2/\hbar c$.

There is really a lot to explain here. There are many important details and techniques in constructing the formal expansion, and there is actually much more to explain about what are the interesting and important things to calculate. The techniques include Feynman diagrams, renormalization, path integrals, gauge fixing, and BRST symmetry. What one wants to calculate are masses, other static quantities such as magnetic moments and other matrix elements of local operators, and especially scattering amplitudes.

After a lot of work, one does end up with a systematic asymptotic expansion. Moreover, there are many physics problems for which the asymptotic expansion is enough, in practice. That is the case for the electromagnetic and weak interactions, because $g^2/\hbar c$ is small (roughly $1/137$ for electromagnetism).

Apart from the fact that the asymptotic expansion – known as perturbation theory – is satisfactory for many questions, it is important in another way: it is unrealistic to expect to develop an exact theory without having a thorough understanding of perturbation theory. Trying to do this would be somewhat analogous to trying to study Riemannian geometry without learning linear algebra.

4. NONPERTURBATIVE APPROACH

However, to understand the strong interactions, or nuclear force, the asymptotic expansion is not enough. We need to understand something about the exact theory. I will therefore conclude by trying to say something about this.

Any known approach to understanding the exact theory requires, one way or another, modifying it by introducing a “cut-off” so as to make the number of variables effectively finite, and then taking a limit in which the cutoff is removed. The usual way to do this is via Euclidean lattice gauge theory, but we will not follow that route because in this lecture we have avoided introducing path integrals. (For an introduction to that subject, see for instance [1].) Instead, I will describe the Hamiltonian version of lattice gauge theory [2]. This approach is not widely used in practice, though possibly it could be useful. At any rate, whether or not this approach is useful in practice, it is easily described and gives a good orientation about what it means to introduce a cut-off and then remove it.

A fact that was mentioned in section 2 is helpful here. This concerns the space W of solutions of the Yang-Mills equations modulo gauge transformations. Pick an initial value hypersurface S ; we may as well simply take S to be a “time zero” subspace $\mathbb{R}^3 \subset \mathbb{R}^4$. Let \mathcal{A} be the space of gauge connections on $S = \mathbb{R}^3$ and \mathcal{G} the group of gauge transformations on \mathbb{R}^3 . Thus the quotient $\mathcal{Y} = \mathcal{A}/\mathcal{G}$ is the space of connections modulo gauge transformations on \mathbb{R}^3 . And W can be identified as a cotangent bundle, $W = T^*(\mathcal{A}/\mathcal{G})$. The idea behind this identification is that to determine a classical solution, we must give the initial value of the connection (a point in \mathcal{A}/\mathcal{G}) and its time derivative (a cotangent vector). So to get a finite-dimensional approximation to W , it suffices to get finite-dimensional approximations to \mathcal{A} and \mathcal{G} .

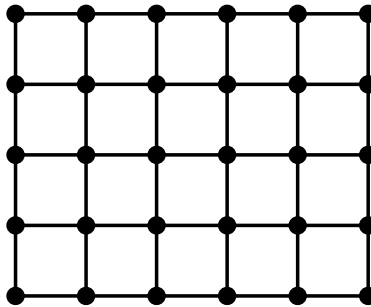


FIGURE 5. We introduce a cutoff by formulating gauge theory on a lattice of finite spatial extent (for clarity, what is drawn here is a two-dimensional rather than three-dimensional lattice). To describe quantum field theory, we must take the spatial extent of the lattice to infinity while also shrinking the lattice spacing to zero.

To do this, we approximate \mathbb{R}^3 by a finite set Γ of points – later we will take the number of points to infinity. In fact, as in fig. 5, we arrange the finite set of points in a regular array, as part of a lattice.¹ Of course, a finite set of points is not a very good approximation to \mathbb{R}^3 . To recover \mathbb{R}^3 , we increase the number of points. In the process, we must improve the approximation in two directions: we take the spacing between the lattice points to be smaller and smaller, so as to recover the continuum, while also taking the spatial extent of our chosen array of points to be bigger and bigger, so that in the limit the array of points covers all of \mathbb{R}^3 .

So now, for each such finite set Γ , we must give an approximation to \mathcal{A} , the space of all connections, and \mathcal{G} , the group of gauge transformations. Taking Γ to be part of a rectangular lattice, we connect the nearest neighbor pairs, as is shown in the figure. Then we approximate gauge theory by only allowing parallel transport along lattice paths. For each oriented link ℓ between nearest neighbors in Γ , we introduce a group element $U_\ell \in G$ that describes parallel transport along ℓ (we take $U_{-\ell} = U_\ell^{-1}$). By a connection on the finite lattice corresponding to Γ , we mean a collection of group elements U_ℓ for all ℓ . So our finite-dimensional approximation to \mathcal{A} is $\mathcal{A}_\Gamma = G^{n_1}$, where n_1 is the number of nearest neighbor pairs in Γ .

Similarly, we permit ourselves to make gauge transformations at all points in Γ . A gauge transformation is thus specified by giving an element $g_p \in G$ for all $p \in \Gamma$. If ℓ connects points p and q , the gauge transformation acts on U_ℓ by $U_\ell \rightarrow g_p U_\ell g_q^{-1}$. So our finite-dimensional approximation to the group \mathcal{G} of gauge transformations is $\mathcal{G}_\Gamma = G^{n_0}$, where n_0 is the number of points in Γ .

The corresponding approximation to the space W of classical solutions is $W_\Gamma = T^*(\mathcal{Y}_\Gamma)$, where $\mathcal{Y}_\Gamma = \mathcal{A}_\Gamma/\mathcal{G}_\Gamma$ is the space of gauge fields modulo gauge transformations. Since we are in finite dimensions and W_Γ is a cotangent bundle, quantization

¹ Actually, it is better to consider a discrete approximation to a three-torus, rather than a discrete approximation to a parallelepiped in \mathbb{R}^3 as sketched in the figure. This avoids edge effects in the spectrum of the Hamiltonian. A discrete approximation to a three-torus is made by taking an array of points that is finite and is periodic in three directions. The limit is still taken by letting the number of points in the array go to infinity (so that it approximates a larger and larger three-torus) while the distance between nearest neighbor points goes to zero.

is straightforward: the Hilbert space associated to Γ is just $\mathcal{H}_\Gamma = L^2(\mathcal{Y}_\Gamma)$. Equivalently, this Hilbert space is the \mathcal{G}_Γ -invariant subspace of $\mathcal{H}_\Gamma^0 = L^2(\mathcal{A}_\Gamma)$. Any operator on \mathcal{H}_Γ^0 that commutes with \mathcal{G}_Γ descends to an operator on \mathcal{H}_Γ .

To complete the description of the finite-dimensional approximation, we must give a suitable approximation H_Γ to the Hamiltonian or energy operator H . It is not hard to do this. It turns out that the first term in H , namely $H' = (1/2g^2) \int d^3x \text{Tr } \vec{E}^2$, can be approximated by $(g^2/2)\Delta$, where Δ is the Laplace operator on the Riemannian manifold \mathcal{A}_Γ . (Δ commutes with \mathcal{G}_Γ , and so descends to an operator on \mathcal{H}_Γ by virtue of the comment at the end of the last paragraph.)

The second term in H , namely $H'' = (1/2g^2) \int d^3x \text{Tr } \vec{B}^2$, is the L^2 norm of the curvature of a connection on the initial value surface. To approximate this for the lattice Γ , the main point is to know what we mean by curvature in the context of such a finite-dimensional approximation. This can be defined in terms of parallel transport around a small loop; in the lattice Γ , the smallest possible nontrivial loops are the squares of minimal area, which in lattice gauge theory are usually called plaquettes. For any plaquette $s \in \Gamma$, consisting of four nearest neighbor links ℓ_1, \dots, ℓ_4 , we let V_s be the function on \mathcal{A}_Γ that associates to a connection $\{U_\ell\}$ the trace of the holonomy of that connection around s (thus, $V_s = \text{Tr } U_{\ell_1} U_{\ell_2} U_{\ell_3} U_{\ell_4}$; the trace is taken in the same representation used to define H''). Then H'' can be approximated by a suitable linear function of $\hat{V} = \sum_s V_s$. (One subtracts a constant from \hat{V} so that it vanishes when $U_\ell = 1$ for all ℓ ; and one then multiplies by a constant to get an approximation to H'' .)

We also want to define lattice approximations to other expressions of classical and quantum gauge theory. But I will not go farther; the examples that have been given hopefully suffice to illustrate the idea.

The problem of defining quantum gauge theory is, in this formulation, to show that when we “remove the cutoff” by refining and enlarging the finite set Γ , the lattice Hamiltonian H_Γ (and other operators) converge to a limit.² While removing the cutoff, one must also adjust the coupling constant g in a suitable fashion.

There is a precise theory of how g must be adjusted; if a is the lattice spacing (the distance between points in Γ), then one requires $g \sim f/|\ln a|$ for $a \rightarrow 0$. Here f is a constant that depends on G ; it was computed by Gross, Wilczek, and Politzer in 1973. This computation led to the 2004 Nobel Prize for “asymptotic freedom,” which is the statement that g must go to zero as a does.

5. BREAKING OF CONFORMAL INVARIANCE AND THE MASS GAP

For $a, g \rightarrow 0$, it is believed that there is a limiting theory that obeys all of the axioms of quantum field theory, including invariance under the Poincaré group. However, it is believed that in the limit one does *not* recover the $SO(2, 4)$ conformal symmetry of the classical theory.

A very basic aspect of the violation of the conformal symmetry is that it is believed that the spectrum has a mass gap, and thus is qualitatively as depicted in fig. 3. We recall that the mass gap means simply that the mass m of any state (orthogonal to the vacuum) is bounded strictly above zero.

²Here to avoid edge effects, it is best to take Γ to be an array of points that is periodic in three directions, as mentioned in footnote 1. Otherwise, there are edge effects at the boundary of Γ and the statement that H_Γ converges to a limit needs to be formulated carefully.

There is no mass gap in electromagnetism; the photon is massless, so electromagnetic waves can have any positive frequency. That is why we can experience light waves in everyday life. By contrast, the mass gap in strong interactions means that the minimum frequency needed to probe the world of $SU(3)$ gauge theory (which describes the strong interactions) is mc^2/\hbar , where m is the smallest mass. Taking from experiment the value of the smallest mass, this frequency is of order 10^{24} sec^{-1} , which is high enough (with room to spare) that this world is way outside of our ordinary experience.

While it is very large compared to our ordinary experience, the mass gap is in one sense very small: it is zero in the asymptotic expansion described in Section 3. As a result, we do not have a really good way to calculate it, though we know it is there from real experiments and computer simulations.

So in short, this mass gap is one of the most basic things that makes the Universe the way it is, with electromagnetism obvious in everyday life and other forces³ only accessible to study with modern technology.

The mass gap is the reason, if you will, that we do not see classical nonlinear Yang-Mills waves. They are a good approximation only under inaccessible conditions.

I have spent most of my career wishing that we had a really good way to quantitatively understand the mass gap in four-dimensional gauge theory. I hope that this problem will be solved one day.

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³The weak interactions are also affected by a mass gap, but for very different reasons from the strong interactions, which have been our subject here. See [5] for an introduction.