Weyl asymptotics for non-self-adjoint operators with small random perturbations

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1. Introduction

Non-self-adjoint spectral problems appear naturally e.g.:

- Resonances, (scattering poles) for self-adjoint operators, like the Schrödinger operator,
- The Kramers–Fokker–Planck operator

$$y \cdot h\partial_x - V'(x) \cdot h\partial_y + \frac{\gamma}{2}(y - h\partial_y) \cdot (y + h\partial_y).$$

A major difficulty is that the resolvent may be very large even when the spectral parameter is far from the spectrum:

$$\|(z-P)^{-1}\| \gg \frac{1}{\mathrm{dist}\,(z, \sigma(P))},$$

$\sigma(P)$ = spectrum of P. This implies that $\sigma(P)$ is unstable under small perturbations of the operator. (Here $P : \mathcal{H} \to \mathcal{H}$ is a closed operator and $\mathcal{H}$ a complex Hilbert space.)

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In the case of (pseudo)differential operators, this follows from the Hörmander (1960) – Davies – Zworski quasimode construction: Let

$$P = P(x, hD_x) = \sum_{|\alpha|\le m} a_\alpha(x)(hD_x)^\alpha,\ \ D_x = \frac{1}{i}\frac{\partial}{\partial x},$$

be a differential operator with smooth coefficients on some open set in $\mathbf{R}^n$, with leading symbol

$$p(x, \xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha,$$

using standard multiindex notation. If $z = p(x, \xi)$, $i^{-1}\{p, \overline{p}\}(x, \xi) > 0$, then $\exists u = u_h \in C_0^\infty(\mathrm{neigh}\,(x, \mathbf{R}^n))$ such that $\|u\|_{L^2} = 1$, $\|(P - z)u\| = \mathcal{O}(h^\infty)$, $h \to 0$.

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Related problems:

- Numerical instability,
- No spectral resolution theorem in general,
- Difficult to study the distribution of eigenvalues.

In this talk we shall discuss the latter problem in the case of (pseudo)differential operators, in the semi-classical limit ($h \to 0$) and in the high frequency limit ($h = 1$).

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In the selfadjoint case, p will be real-valued (up to terms that are $\mathcal{O}(h)$ and we neglect for brevity) and under suitable additional assumptions, P will have discrete spectrum near some given interval I and we have the Weyl asymptotic distribution of the eigenvalues in the semiclassical limit:

$$\#(\sigma(P)\cap I)=\frac{1}{(2\pi h)^n}(\mathrm{vol}\,(p^{-1}(I))+o(1)),\ \ h\to 0.$$

In the high frequency limit, we take $h=1$ and look for the distribution of large eigenvalues:

$$\#(\sigma(P)\cap]-\infty,\lambda])=\frac{1}{(2\pi)^n}(\mathrm{vol}\,(p_m^{-1}(]-\infty,\lambda]))+o(\lambda^{\frac{n}{m}})),\ \ \lambda\to+\infty,$$

where p_m is the classical principal symbol obtained by restricting the summation in the formula for p to $\alpha\in\mathbf{N}^n$ with $|\alpha|=m$.

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In the non-self-adjoint case, we do not always have Weyl asymptotics and actually almost never when we are able to compute the eigenvalues "by hand". (Weyl asymptotics in the semi-classical case would be to replace intervals in the formula above by more general sets in $\mathbf{C}$, say with smooth boundary.)

Example 1. $P = hD_x + g(x)$ on S^1. The range of $p(x,\xi) = \xi + g(x)$ is the band $\{z \in \mathbf{C}; \min \Im g \le \Im z \le \max \Im g\}$ while $\sigma(P) \subset \{z;\, \Im z = (2\pi)^{-1} \int_0^{2\pi} \Im g(x) dx\}$.

Example 2. $P = (hD_x)^2 + ix^2$ with $p(x,\xi) = \xi^2 + ix^2$. $\sigma(P) \subset e^{i\pi/4}[0,\infty[$ while the range of p is the closed first quadrant.

Example 3. $P = f(x)D_x$ gives a counter-example similar to the one in Ex 1, now for the high frequency limit.

For operators with analytic coefficients in two dimensions we (Hitrik-Melin-Sj-VuNgoc) have several results where the asymptotic distribution is determined by the extension of the symbol to the complex domain, leading to other counter-examples.

In her thesis in 2005, M. Hager showed for a class of non-self-adjoint semi-classical operators on **R** that if we add a small random perturbation, then with probability tending to 1 very fast, when $h \to 0$, we do have Weyl asymptotics.

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2. Some results in higher dimensions

The original 1D result of Hager was generalized in many ways by Hager–Sj (Math Ann 2008), we were able to count eigenvalues also near the boundary of the range of p. One weakness of this generalization was however that the random perturbations were no more multiplicative so the perturbed operator could not be a differential one but rather a pseudodifferential operator.

To get further it seemed necessary to have a more general approach to the random perturbations and get rid of the restriction to Gaussian random variables. I got the following result, still a little technical to state.

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Let X be a compact n-dimensional manifold,

$$P = \sum_{|\alpha|\le m} a_\alpha(x;h)(hD)^\alpha, \tag{1}$$

Assume

$$\begin{aligned} &a_\alpha(x;h) = a_\alpha^0(x) + \mathcal{O}(h) \text{ in } C^\infty, \\ &a_\alpha(x;h) = a_\alpha(x) \text{ is independent of } h \text{ for } |\alpha| = m. \end{aligned} \tag{2}$$

Let

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \tag{3}$$

Assume that P is elliptic,

$$|p_m(x,\xi)| \ge \frac{1}{C}|\xi|^m, \tag{4}$$

and that $p_m(T^*X) \neq \mathbf{C}$.

Let $p = \sum_{|\alpha|\le m} a_\alpha^0(x)\xi^\alpha$ be the semi-classical principal symbol. We make the symmetry assumption

$$P^* = \Gamma P \Gamma, \tag{5}$$

where P^* denotes the complex adjoint with respect to some fixed smooth positive density of integration and Γ is the antilinear operator of complex conjugation; $\Gamma u = \overline{u}$. Notice that this assumption implies that

$$p(x,-\xi) = p(x,\xi). \tag{6}$$

Let $V_z(t) := \mathrm{vol}\left(\{\rho \in T^*X;\ |p(\rho)-z|^2 \le t\}\right)$. For $\kappa \in]0,1]$, $z \in \mathbf{C}$, we consider the non-flatness property that

$$V_z(t) = \mathcal{O}(t^\kappa),\ 0 \le t \ll 1. \tag{7}$$

We see that (7) holds with $\kappa = 1/(2m)$.

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Random potential:

$$q_\omega(x) = \sum_{0<h\mu_k\le L} \alpha_k(\omega)\epsilon_k(x),\ |\alpha|_{\mathbf{C}^D} \le R, \tag{8}$$

where ϵ_k is the orthonormal basis of eigenfunctions of $\widetilde{R}$, where $\widetilde{R}$ is an h-independent positive elliptic 2nd order operator on X with smooth coefficients. Moreover, $\widetilde{R}\epsilon_k = \mu_k^2\epsilon_k$, $\mu_k > 0$.

We choose $L = L(h)$, $R = R(h)$ in the interval

$$h^{\frac{\kappa-3n}{s-\frac{n}{2}-\epsilon}} \ll L \le h^{-M}, \qquad M \ge \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}, \tag{9}$$

$$\frac{1}{C}h^{-(\frac{n}{2}+\epsilon)M+\kappa-\frac{3n}{2}} \le R \le Ch^{-\widetilde{M}}, \qquad \widetilde{M} \ge \frac{3n}{2} - \kappa + (\frac{n}{2}+\epsilon)M,$$

for some $\epsilon \in]0, s-\frac{n}{2}[$, $s > \frac{n}{2}$. Put $\delta = \tau_0 h^{N_1+n}$, $0 < \tau_0 \le \sqrt{h}$, where

$$N_1 := \widetilde{M} + sM + \frac{n}{2}. \tag{10}$$

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The randomly perturbed operator is

$$P_\delta = P + \delta h^{N_1} q_\omega =: P + \delta Q_\omega. \tag{11}$$

The random variables $\alpha_j(\omega)$ will have a joint probability distribution

$$\mathbb{P}(d\alpha) = C(h) e^{\Phi(\alpha;h)} L(d\alpha), \tag{12}$$

where for some $N_4 > 0$,

$$|\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_4}), \tag{13}$$

and $L(d\alpha)$ is the Lebesgue measure. ($C(h)$ is the normalizing constant, assuring that the probability of $B_{\mathbf{C}^D}(0, R)$ is equal to 1.) We also need the parameter

$$\epsilon_0(h) = (h^\kappa + h^n \ln \frac{1}{h})(\ln \frac{1}{\tau_0} + (\ln \frac{1}{h})^2) \tag{14}$$

and assume that $\tau_0 = \tau_0(h)$ is not too small, so that $\epsilon_0(h)$ is small.

Theorem (Sj 2008)

Let $\Gamma \Subset \mathbf{C}$ *have smooth boundary, let* $\kappa \in]0,1]$ *be the parameter in (8), (9), (14) and assume that (7) holds uniformly for* z *in a neighborhood of* $\partial\Gamma$*. Then, for* $C^{-1} \ge r > 0$, $\widetilde{\epsilon} \ge C\epsilon_0(h)$ *we have with probability*

$$\ge 1 - \frac{C\epsilon_0(h)}{rh^{n+\max(n(M+1),N_5+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} \tag{15}$$

that:

$$|\#(\sigma(P_\delta)\cap\Gamma) - \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma))| \le \frac{C}{h^n}\left(\frac{\widetilde{\epsilon}}{r} + C\left(r + \ln(\frac{1}{r})\mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0,r)))\right)\right). \tag{16}$$

Here $\#(\sigma(P_\delta)\cap\Gamma)$ *denotes the number of eigenvalues of* P_δ *in* Γ*, counted with their algebraic multiplicity.*

Explain the choice of parameters!

Almost sure Weyl distribution of large eigenvalues.

$h = 1$. Let P^0 be an elliptic differential operator on X of order $m \geq 2$ with smooth coefficients and with principal symbol $p_m(x,\xi) = p(x,\xi)$. We assume that

$$p(T^*X) \neq \mathbf{C}. \tag{17}$$

We keep the symmetry assumption

$$(P^0)^* = \Gamma P^0 \Gamma. \tag{18}$$

Our randomly perturbed operator is

$$P^0_\omega = P^0 + q^0_\omega(x), \tag{19}$$

where ω is the random parameter and

$$q^0_\omega(x) = \sum_0^\infty \alpha^0_j(\omega)\epsilon_j(x). \tag{20}$$

Here ϵ_j, μ_j, $\widetilde{R}$ are as before and we assume that $\alpha_j^0(\omega)$ are independent complex Gaussian random variables of variance σ_j^2 and mean value 0:

$$\sigma_j^0 \sim \mathcal{N}(0, \sigma_j^2), \tag{21}$$

where

$$\sigma_j \asymp (\mu_j)^{-\rho}, \quad \rho > n \tag{22}$$

Then almost surely: $q_\omega^0 \in L^\infty$, so P_ω^0 has purely discrete spectrum.
Consider the function $F(\omega) = \arg p(\omega)$ on S^*X. For a given $\theta_0 \in S^1 \simeq \mathbf{R}/(2\pi\mathbf{Z})$, $N_0 \in \dot{\mathbf{N}} := \mathbf{N} \setminus \{0\}$, we introduce the property:

$$P(\theta_0, N_0): \quad \sum_1^{N_0} |\nabla^k F(\omega)| \neq 0 \text{ on } \{\omega \in S^*X;\, F(\omega) = \theta_0\}. \tag{23}$$

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Theorem (Bordeaux M, Sj 2008)

Assume that $m \geq 2$. *Let* $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$ *and assume that* $P(\theta_1, N_0)$ *and* $P(\theta_2, N_0)$ *hold for some* $N_0 \in \dot{\mathbf{N}}$. *Let* $g \in C^\infty([\theta_1, \theta_2];]0, \infty[)$ *and put*

$$\Gamma(0, \lambda g) = \{re^{i\theta}; \theta_1 \leq \theta \leq \theta_2,\ 0 \leq r \leq \lambda g(\theta)\}.$$

Then for every $\delta \in]0, \frac{1}{2}[$ *there exists* $C > 0$ *such that almost surely:* $\exists C(\omega) < \infty$ *such that for all* $\lambda \in [1, \infty[$:

$$\begin{aligned} &|\#(\sigma(P_\omega^0) \cap \Gamma(0, \lambda g)) - \frac{1}{(2\pi)^n} \mathrm{vol}\, p^{-1}(\Gamma(0, \lambda g))| \\ &\leq C(\omega) + C\lambda^{\frac{n}{m} - (\frac{1}{2} - \delta)\frac{1}{N_0+1}}. \end{aligned} \tag{24}$$

3. Some ideas in the proofs

In the proofs of the semi-classical theorems, a common feature is to identify the eigenvalues of the operator with the zeros of a holomorphic function with exponential growth and to show that with probability close to 1 this function really is exponentially large at finitely many points distributed nicely along the boundary of Γ, then apply a proposition about the the number of zeros of such functions.

In the one dimensional results by Hager (and Bordeaux-Montrieux for matrix-valued operators) this is done via a Grushin (Feschbach) problem that makes use of the Davies-Hörmander quasimodes for P and P^* and we get quite a concrete holomorphic function.

In the higher dimensional results we have a more general approach that we shall outline:

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In the higher dimensional results we have a more general approach that we shall outline:

First we construct a symbol $\widetilde{p}$, equal to p outside a compact set such that $\widetilde{p} - z \neq 0$ for $z \in \mathrm{neigh}(\Gamma)$, and put on the operator level: $\widetilde{P} = P + (\widetilde{p} - p)$. Then $\widetilde{P} - z$ has a bounded (pseudodifferential) inverse for every z in some simply connected neighborhood of Γ. The eigenvalues of P coincide with the zeros of the holomorphic function,

$$z \mapsto \det(\widetilde{P} - z)^{-1}(P - z) = \det(1 - (\widetilde{P} - z)^{-1}(\widetilde{P} - P)).$$

If $P_\delta = P + \delta Q_\omega$, put $\widetilde{P}_\delta := \widetilde{P} + \delta Q_\omega$ which has no spectrum in near Γ. The eigenvalues of P_δ in that region are the zeros of

$$z \mapsto \det(\widetilde{P}_{\delta,z}),$$

where

$$\widetilde{P}_{\delta,z} = (\widetilde{P}_\delta - z)^{-1}(P_\delta - z) = 1 - (\widetilde{P}_\delta - z)^{-1}(\widetilde{P} - P).$$

The general strategy is the following:

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The general strategy is the following:

- Step 1. Show that with probability close to 1, we have for all z in a neighborhood of $\partial\Gamma$ with $p_z = (\widetilde{p} - z)^{-1}(p - z)$:

$$\ln|\det P_{\delta,z}| \leq \frac{1}{(2\pi h)^n}\left(\int \ln|p_z(\rho)|d\rho + o(1)\right). \tag{25}$$

- Step 2. Show that for each z in a neighborhood of $\partial\Gamma$ we have with probability close to one that

$$\ln|\det P_{\delta,z}| \geq \frac{1}{(2\pi h)^n}\left(\int \ln|p_z(\rho)|d\rho + o(1)\right). \tag{26}$$

- Step 3. Apply results ([Ha, HaSj]) about counting zeros of holomorphic functions: Roughly, if $u(z) = u(z;\widetilde{h})$ is holomorphic with respect to z in a neighborhood of $\overline{\Gamma}$, $|u(z)| \leq \exp(\phi(z)/\widetilde{h})$ near $\partial\Gamma$ and we have a reverse estimate $|u(z_j)| \geq \exp((\phi(z_j) - \text{"small"})/\widetilde{h})$ for a finite set of points, distributed "densely" along the boundary, then the number of zeros of u in Γ is equal to $(2\pi\widetilde{h})^{-1}(\int\int_\Gamma \Delta\phi(z)d\Re z d\Im z + \text{"small"})$. This is applied with $\widetilde{h} = (2\pi h)^n$, $\phi(z) = \int \ln|p_z(\rho)|d\rho$.

- Step 1. Show that with probability close to 1, we have for all z in a neighborhood of $\partial\Gamma$ with $p_z = (\widetilde{p} - z)^{-1}(p - z)$:

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Step 1 can be carried out using microlocal analysis for the unperturbed operator (cf Melin–Sj) and the fact that the perturbation is small in a suitable sense.

Step 2 is the delicate one. In the other results, (Hager, Bordeaux M., Hager–Sj) with the Gaussianity assumption we are lead to the problem of finding lower bounds on the determinant of a random matrix which is close to a Gaussian one, but in the case of multiplicative perturbations in higher dimension, this does not seem to work. Instead, we forget about Gaussianity, and make a complex analysis argument in the α-variables[1]. Then we come down to the task of constructing (for each fixed z near $\partial\Gamma$) at least one perturbation of the requested form for which we have a nice lower bound on the determinant. This is again a delicate problem.

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Theorem

Let $\Gamma \Subset \mathbf{C}$ *open,* $\gamma := \partial\Gamma$, $r : \gamma \to]0,1[$ *Lipschitz:* $|r(x) - r(y)| \le \frac{1}{2}|x - y|$ *and assume that for each* $z \in \gamma$, $D(z, r(z)) \cap \gamma$ *is the graph of a Lipschitz function after a translation and rotation, uniformly with respect to* x*. Let* $z_1, z_2, ..., z_N$ *run through* γ *with cyclic convention; "*$N + 1 = 1$*" such that* $C^{-1}r(z_k) \le |z_{k+1} - z_k| \le \frac{1}{2}r(z_k)$*. Let* ϕ *be continuous and subharmonic in a neighborhood of the closure of* $\gamma_r := \cup_{x\in\gamma} D(x, r(x))$*. Then* $\exists$ $\widetilde{z}_j \in D(z_j, \frac{1}{C}r(z_j))$ *such that:*

If $u = u_{\widetilde{h}}$, $0 < \widetilde{h} \le 1$ *is holomorphic in* $\Gamma \cup \gamma_r$ *such that* $\widetilde{h}\ln|u| \le \phi$ *on* γ_r, $\widetilde{h}\ln|u(\widetilde{z}_j)| \ge \phi(\widetilde{z}_j) - \epsilon_j$, $j = 1, ..., N$,

then with $\mu := \Delta\phi$ *(where* ϕ *denotes any extension from* γ_r *to* $\Gamma \cup \gamma_r$*):*

$$|\#(u^{-1}(0) \cup \Gamma) - \frac{1}{2\pi\widetilde{h}}\mu(\Gamma)| \le \frac{\widetilde{C}}{\widetilde{h}}(\mu(\gamma_r) + \sum \epsilon_j)$$

Large eigenvalues

Put (with fixed values of θ_1, θ_2): $\Gamma_{\lambda_1,\lambda_2} = \Gamma(0,\lambda_2 g) \setminus \Gamma(0,\lambda_1 g)$ and make the dyadic decomposition:

$$\Gamma(0,\lambda g) = \Gamma(0,g) \cup \Gamma_{1,2} \cup \Gamma_{2,2^2} \ldots \cup \Gamma_{2^{k-1},2^k} \cup \Gamma_{2^k,\lambda},$$

where k is the largest integer such that $2^k \leq \lambda$. Counting the eigenvalues of P_ω in $\Gamma_{2^j,2^{j+1}}$ amounts to counting the eigenvalues of $2^{-j}P_\omega$ in $\Gamma_{1,2}$ which is a semi-classical problem when j is large. Similarly for $\Gamma_{2^k,\lambda}$. In the 1D case, it then suffices to apply Hager (or Bordeaux M. in the case of systems), together with the Borel-Cantelli lemma, and in the higher dimensional case we use the corresponding semi-classical result (Sj) instead of Hager – Bordeaux M.