Spectral analysis of a completely positive map and thermal relaxation of a QED cavity

Joint work with Laurent Bruneau (Cergy)

Overview

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- Cavity QED and the Jaynes-Cummings Hamiltonian

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- Open questions

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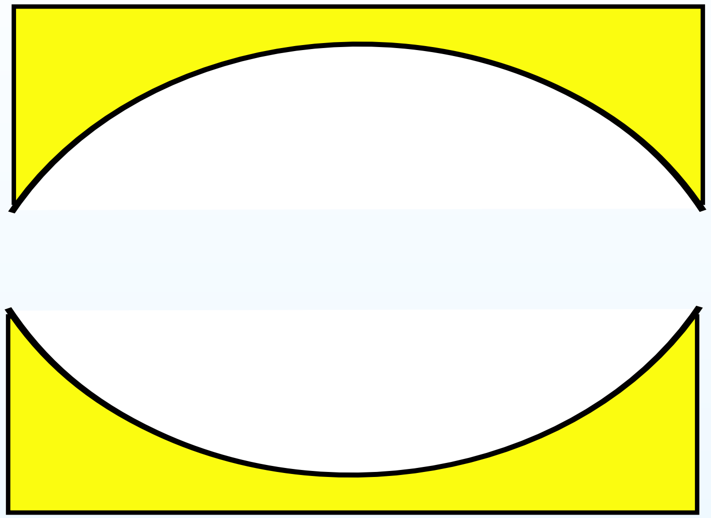

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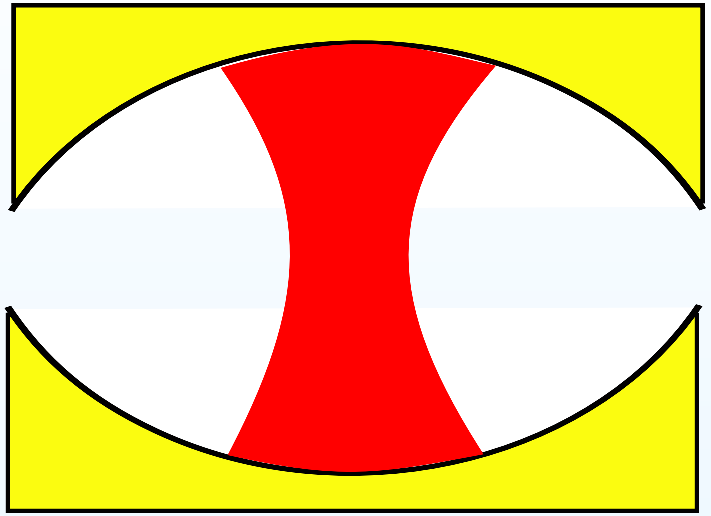

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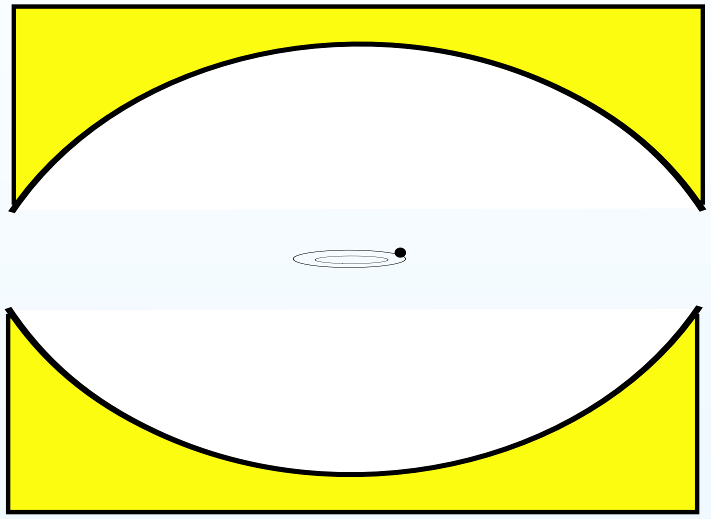

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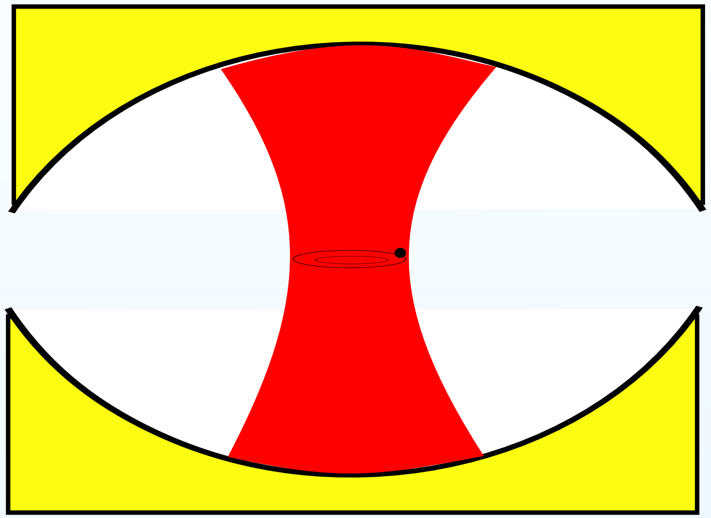

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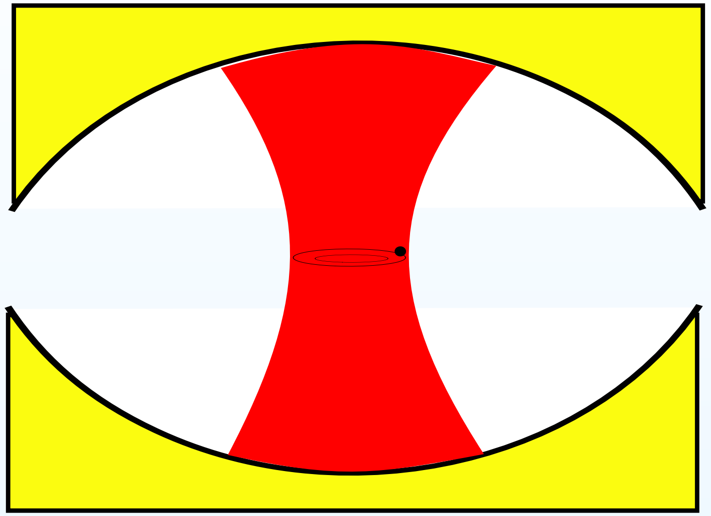

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The Jaynes-Cummings Hamiltonian

$$H_{\mathrm{JC}} = \omega a^* a + \omega_0 b^* b + \frac{\lambda}{2}(b^* a + b a^*)$$

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Detuning parameter $\Delta = \omega - \omega_0$

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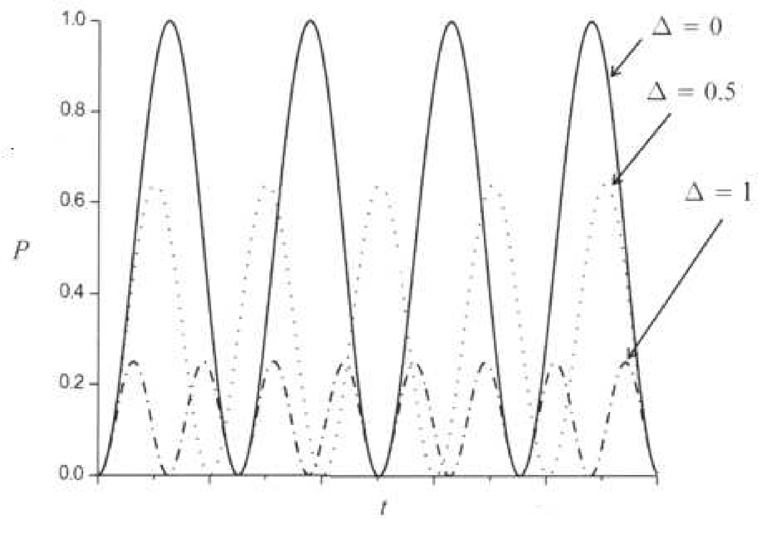

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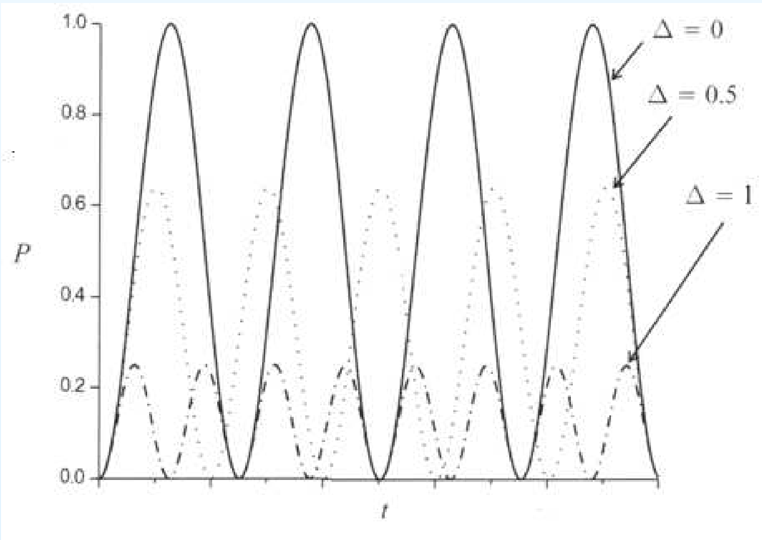

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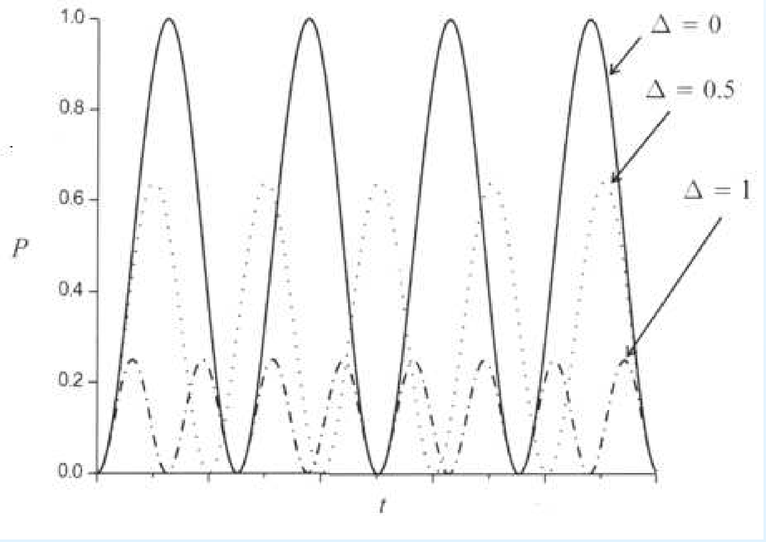
$$\Omega_{\mathrm{Rabi}}(n) = \sqrt{\lambda^2 n + \Delta^2}$$



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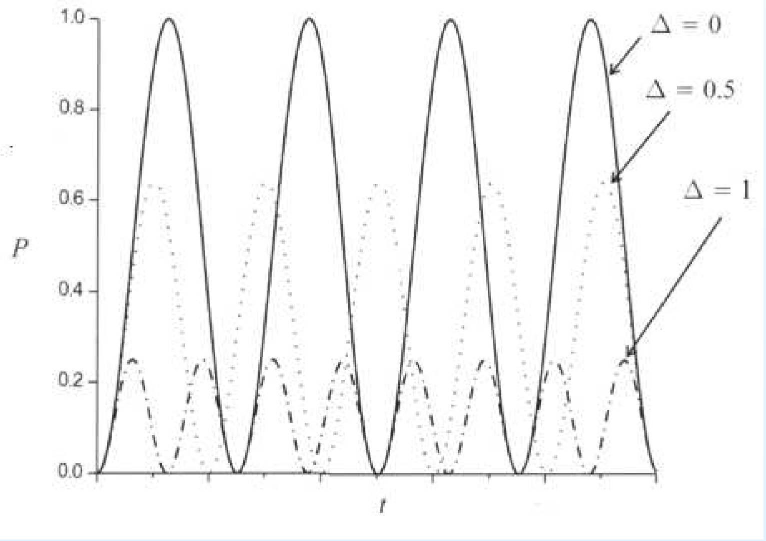
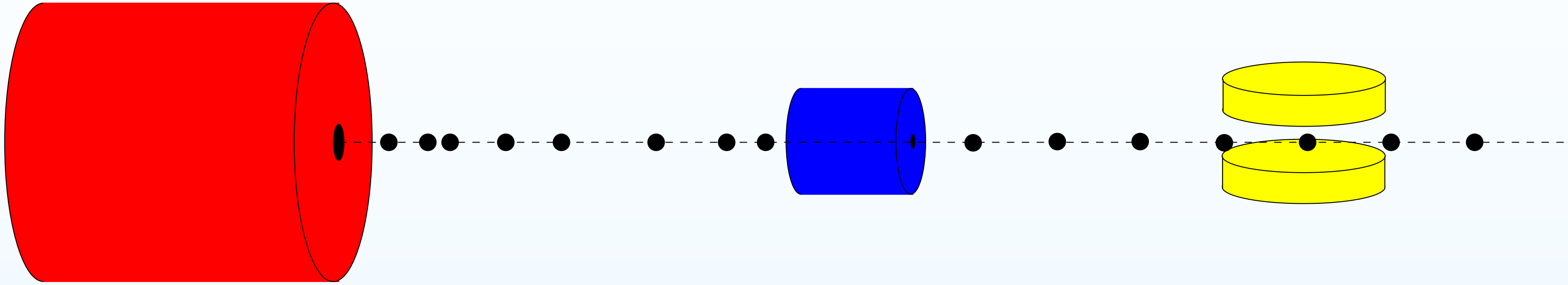
$$P(t) = \left[1 - \left(\frac{\Delta}{\Omega_{\mathrm{Rabi}}(n)} \right)^2 \right] \sin^2 \Omega_{\mathrm{Rabi}}(n) t/2$$



3. The one-atom maser



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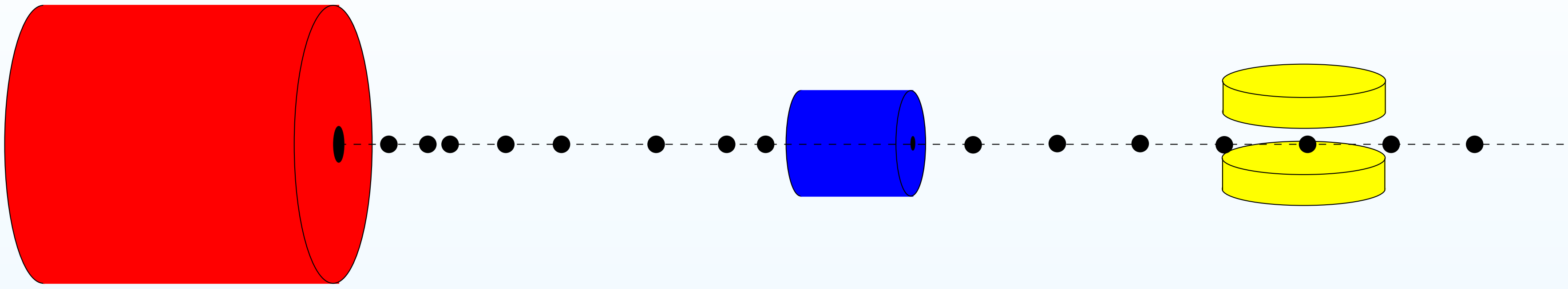

Repeated interaction scheme

$$\mathcal{H} = \mathcal{H}_{\text{cavity}} \otimes \mathcal{H}_{\text{beam}}, \qquad \mathcal{H}_{\text{beam}} = \bigotimes_{n\geq 1} \mathcal{H}_{\text{atom } n}$$

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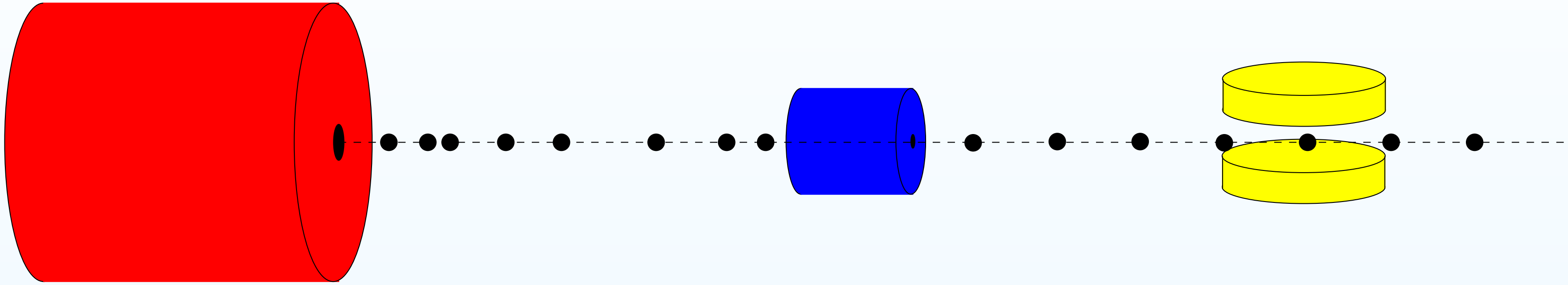

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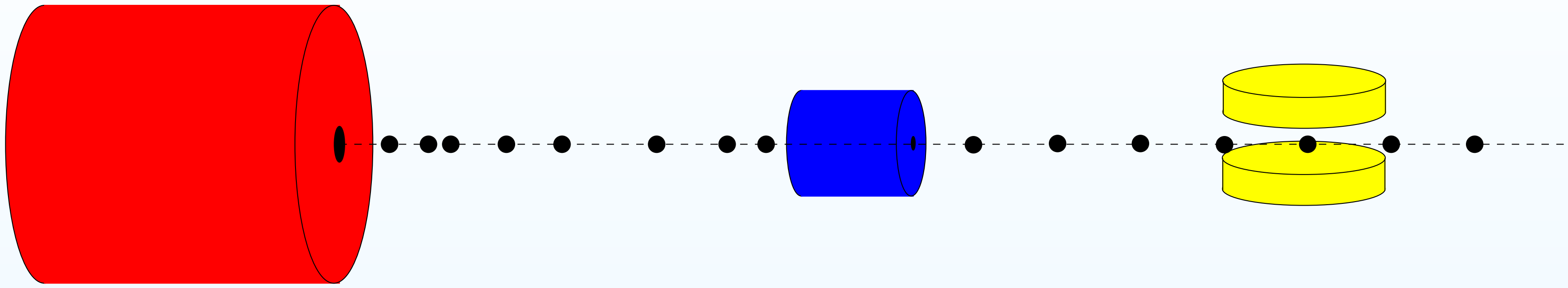

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Cavity state after n interactions

$$\rho_n = \text{Tr}_{\mathcal{H}_{\text{beam}}} \left[\mathrm{e}^{-\mathrm{i}\tau H_n} \cdots \mathrm{e}^{-\mathrm{i}\tau H_1} \left(\rho_0 \otimes \bigotimes_{k=1}^{n} \rho_{\text{atom } k} \right) \mathrm{e}^{\mathrm{i}\tau H_1} \cdots \mathrm{e}^{\mathrm{i}\tau H_n} \right]$$

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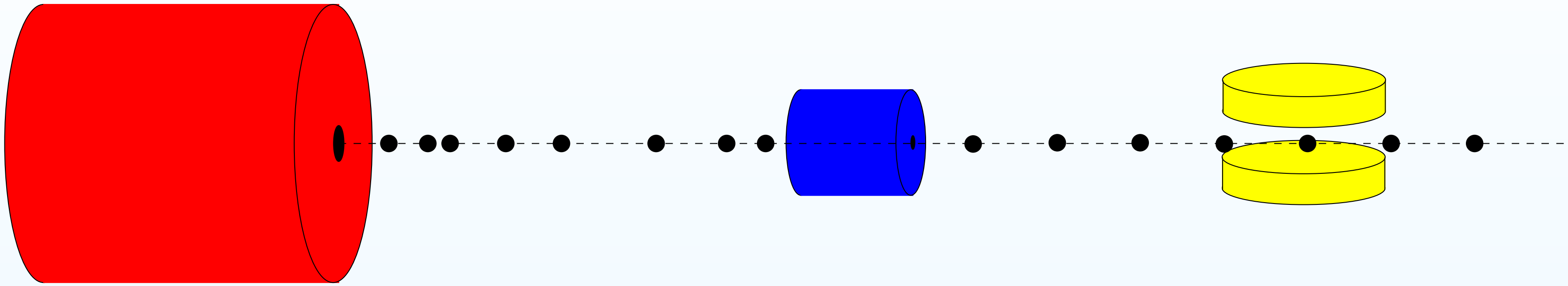

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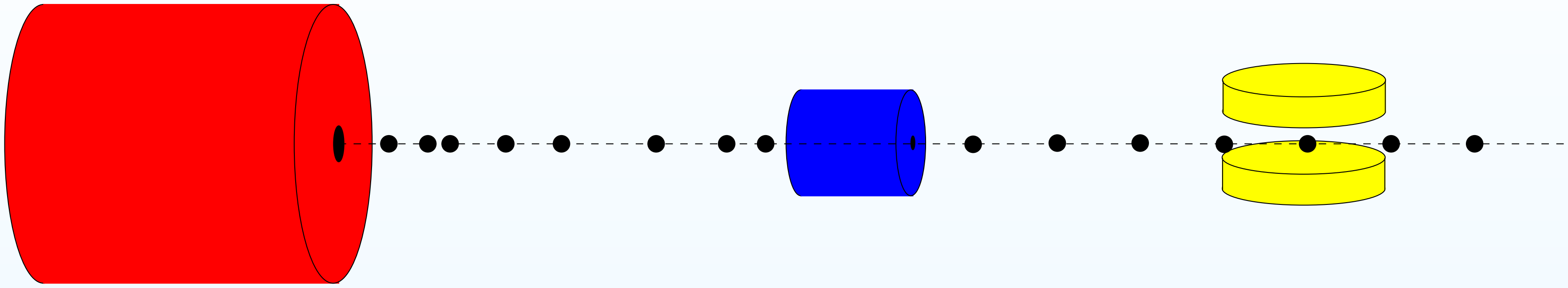

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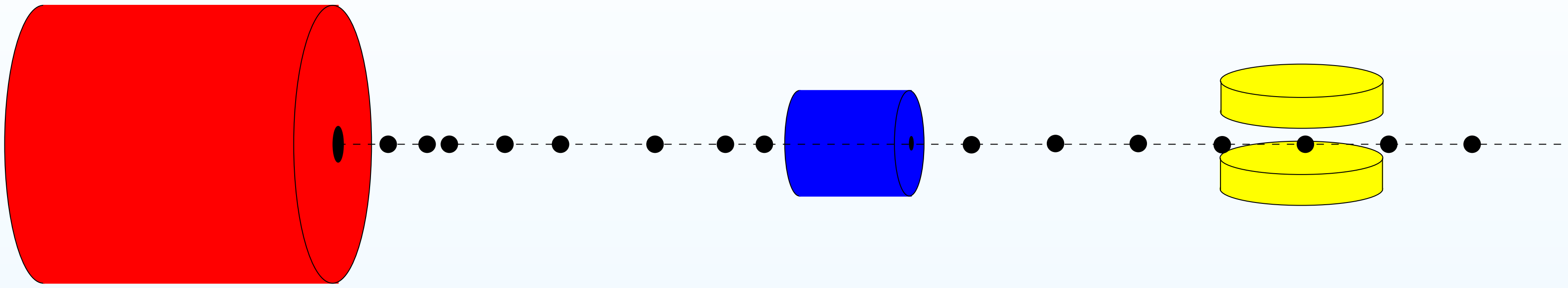

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Reduced dynamics

$$\boxed{\mathcal{L}_\beta(\rho) = \mathrm{Tr}_{\mathcal{H}_{\text{atom}}}\left[\mathrm{e}^{-\mathrm{i}\tau H_{\text{JC}}}\left(\rho \otimes \rho^{\beta}\right)\mathrm{e}^{\mathrm{i}\tau H_{\text{JC}}}\right]}$$

Completely positive, trace preserving map on the trace ideal $\mathcal{J}^1(\mathcal{H}_{\text{cavity}})$

4. Rabi resonances

A resonance occurs when the interaction time τ is a multiple of the Rabi period

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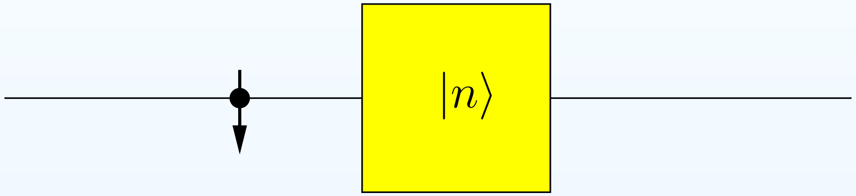

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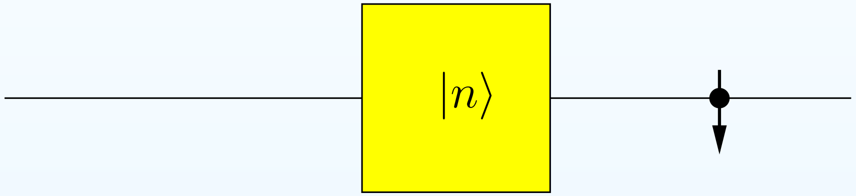

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Definition. The system is

- Non resonant: $R(\eta,\xi)$ is empty.
- Simply resonant: $R(\eta,\xi) = \{n_1\}$.
- Fully resonant: $R(\eta,\xi) = \{n_1, n_2, \ldots\}$ i.e. has ∞-many resonances.
- Degenerate: fully resonant and there exist $n \in R(\eta,\xi) \cup \{0\}$ and $m \in R(\eta,\xi)$ such that $n+1, m+1 \in R(\eta,\xi)$.

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- both rational: $\eta = a/b$, $\xi = c/d$ (irreducible) and $m = \mathrm{LCM}(b, d)$

$$\mathfrak{X} = \{x \in \{0, \ldots, \xi m - 1\} \,|\, x^2 m \equiv \eta m \pmod{\xi m}\}$$

then non-resonant if $\mathfrak{X}$ is empty or fully resonant

$$R(\eta, \xi) = \{(k^2 - \eta)/\xi \,|\, k = jm\xi + x, j \in \mathbb{N}^*, x \in \mathfrak{X}\} \cap \mathbb{N}^*$$

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Remark. This lemma is elementary but characterizing integers η, ξ for which the system is degenerate is a very hard (open) problem in Diophantine analysis.

5. Rabi sectors

Decomposition into Rabi sectors

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Partial Gibbs state in $\mathcal{H}^{(k)}$:

$$\rho_{\text{cavity}}^{(k)\beta^*} = \frac{\mathrm{e}^{-\beta^* H_{\text{cavity}} P_k}}{\operatorname{Tr} \mathrm{e}^{-\beta^* H_{\text{cavity}} P_k}}, \qquad \beta^* = \beta \frac{\omega_0}{\omega}$$

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$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}^n(\mu)\right)(A) = \rho(A)$$

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and exponentially mixing iff

$$\left|\left(\mathcal{L}^n(\mu)\right)(A) - \rho(A)\right| \leq C_{A,\mu}\,\mathrm{e}^{-\alpha n},$$

for some constants $C_{A,\mu}$ and $\alpha > 0$.

7. Ergodic properties of the one-atom maser

Main Theorem. 1. If the system is non-resonant then $\mathcal{L}_\beta$ has no invariant state for $\beta \le 0$ and a unique ergodic state

$$\rho_{\mathrm{cavity}}^{\beta^*} = \frac{\mathrm{e}^{-\beta^* H_{\mathrm{cavity}}}}{\mathrm{Tr}\,\mathrm{e}^{-\beta^* H_{\mathrm{cavity}}}}, \qquad \beta^* = \beta \frac{\omega_0}{\omega}$$

for $\beta > 0$. In the latter case any initial state relaxes in the mean to this thermal equilibrium state

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}_\beta^n(\mu) \right)(A) = \rho_{\mathrm{cavity}}^{\beta^*}(A)$$

for any $A \in \mathcal{B}(\mathcal{H}_{\mathrm{cavity}})$.

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Main Theorem. 2. If the system is simply resonant then $\mathcal{L}_\beta$ has the unique ergodic state $\rho_{\mathrm{cavity}}^{(1)\,\beta^*}$ if $\beta \leq 0$ and two ergodic states $\rho_{\mathrm{cavity}}^{(1)\,\beta^*}$, $\rho_{\mathrm{cavity}}^{(2)\,\beta^*}$ if $\beta > 0$. In the latter case, for any state μ, one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \mu(P_1)\,\rho_{\mathrm{cavity}}^{(1)\,\beta^*}(A) + \mu(P_2)\,\rho_{\mathrm{cavity}}^{(2)\,\beta^*}(A),$$

for any $A \in \mathcal{B}(\mathcal{H}_{\mathrm{cavity}})$.

7. Ergodic properties of the one-atom maser

Main Theorem. 3. If the system is fully resonant then for any $\beta \in \mathbb{R}$, $\mathcal{L}_\beta$ has infinitely many ergodic states $\rho_{\text{cavity}}^{(k)\,\beta*}$, $k = 1, 2, \ldots$ Moreover, if the system is non-degenerate,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \sum_{k=1}^{\infty} \mu(P_k)\, \rho_{\text{cavity}}^{(k)\,\beta*}(A),$$

holds for any state μ and all $A \in \mathcal{B}(\mathcal{H}_{\text{cavity}})$.

7. Ergodic properties of the one-atom maser

Main Theorem. 4. If the system is fully resonant and degenerate there exists a finite set $\mathcal{D}(\eta,\xi) \subset \mathbb{Z}$ such that the conclusions of 3. still hold provided the non-resonance condition

$$(\mathrm{NR}) \qquad \mathrm{e}^{\mathrm{i}(\tau\omega+\xi\pi)d} \neq 1$$

is satisfied for all $d \in \mathcal{D}(\eta,\xi)$.

5. In all the previous cases any invariant state is diagonal and can be represented as a convex linear combination of ergodic states, *i.e.,* the set of invariant states is a simplex whose extremal points are ergodic states.
In the remaining case, *i.e.,* if condition (NR) fails, there are non-diagonal invariant states.

6. Whenever the state $\rho_{\mathrm{cavity}}^{(k)\,\beta^*}$ is ergodic it is also exponentially mixing if the Rabi sector $\mathcal{H}_{\mathrm{cavity}}^{(k)}$ is finite dimensional.

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- Degenerate fully resonant systems exist. If $\eta = 241$ and $\xi = 720$ then

$$720 + 241 = 29^2, \quad 2 \cdot 720 + 241 = 41^2, \quad 3 \cdot 720 + 241 = 49^2$$

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- Another example is $\eta = 1$ and $\xi = 840$ for which 1, 2, 52 and 53 are Rabi resonances

$$840 + 1 = 29^2, \quad 2 \cdot 840 + 1 = 41^2, \quad 52 \cdot 840 + 1 = 209^2, \quad 53 \cdot 840 + 1 = 211^2$$

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- We do not know of any example where $\mathcal{D}(\eta,\xi)$ contains more than one element.

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so that

$$\mathrm{sp}(\mathcal{L}_\beta) = \mathrm{sp}_{\mathrm{pp}}(\mathcal{L}_\beta) = \{\mathrm{e}^{\mathrm{i}\tau\omega d} \,|\, d \in \mathbb{Z}\}$$

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- Use the block structure induced by Rabi sectors.
- Use Schrader's version of Perron-Frobenius theory for trace preserving CP maps on trace ideals [Fields Inst. Commun. 30 (2001)].

9. Metastable states of the one-atom maser

By gauge symmetry, the subspace of diagonal states is invariant. The action of $\mathcal{L}_\beta$ on this subspace is conjugated to that of

$$L = I - \nabla^* D(N) \mathrm{e}^{-\beta\omega_0 N} \nabla \mathrm{e}^{\beta\omega_0 N}$$

on $\ell^1(\mathbb{N})$ where

$$(Nx)_n = n x_n, \qquad (\nabla x)_n = \begin{cases} x_0 & \text{for } n = 0; \\ x_n - x_{n-1} & \text{for } n \geq 1; \end{cases} \qquad (\nabla^* x)_n = x_n - x_{n+1}$$

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Rabi resonances are integers n such that $D(n) = 0$. They decouple L.

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Setting $L_0 = I - \nabla^* D_0(N) \mathrm{e}^{-\beta\omega_0 N} \nabla \mathrm{e}^{\beta\omega_0 N}$ with

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L_0 has infinitely degenerate eigenvalue 1: eigenvectors are metastable states of L

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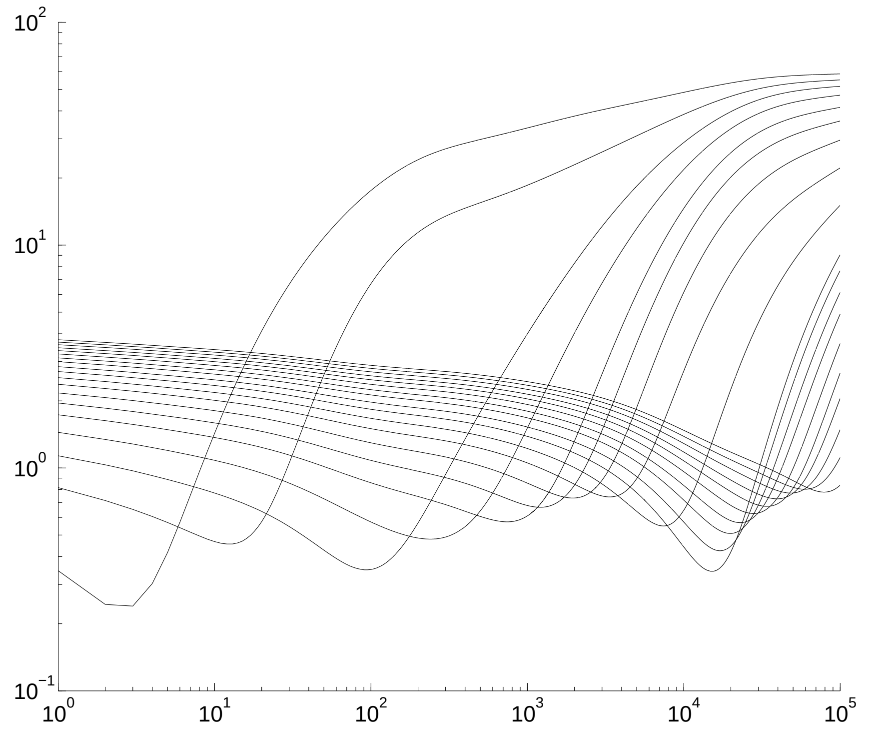
The metastable cascade



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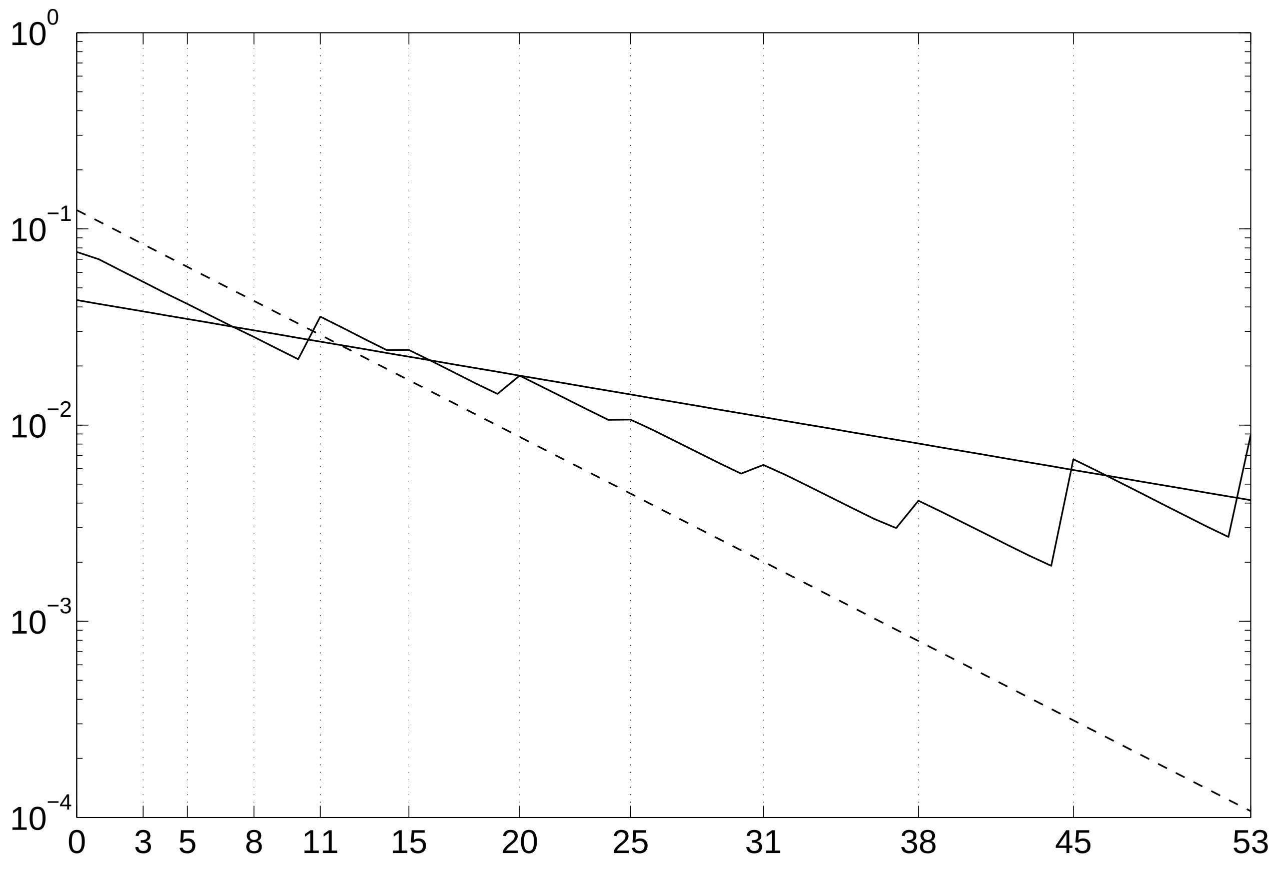
Local equilibrium after 5000 interactions



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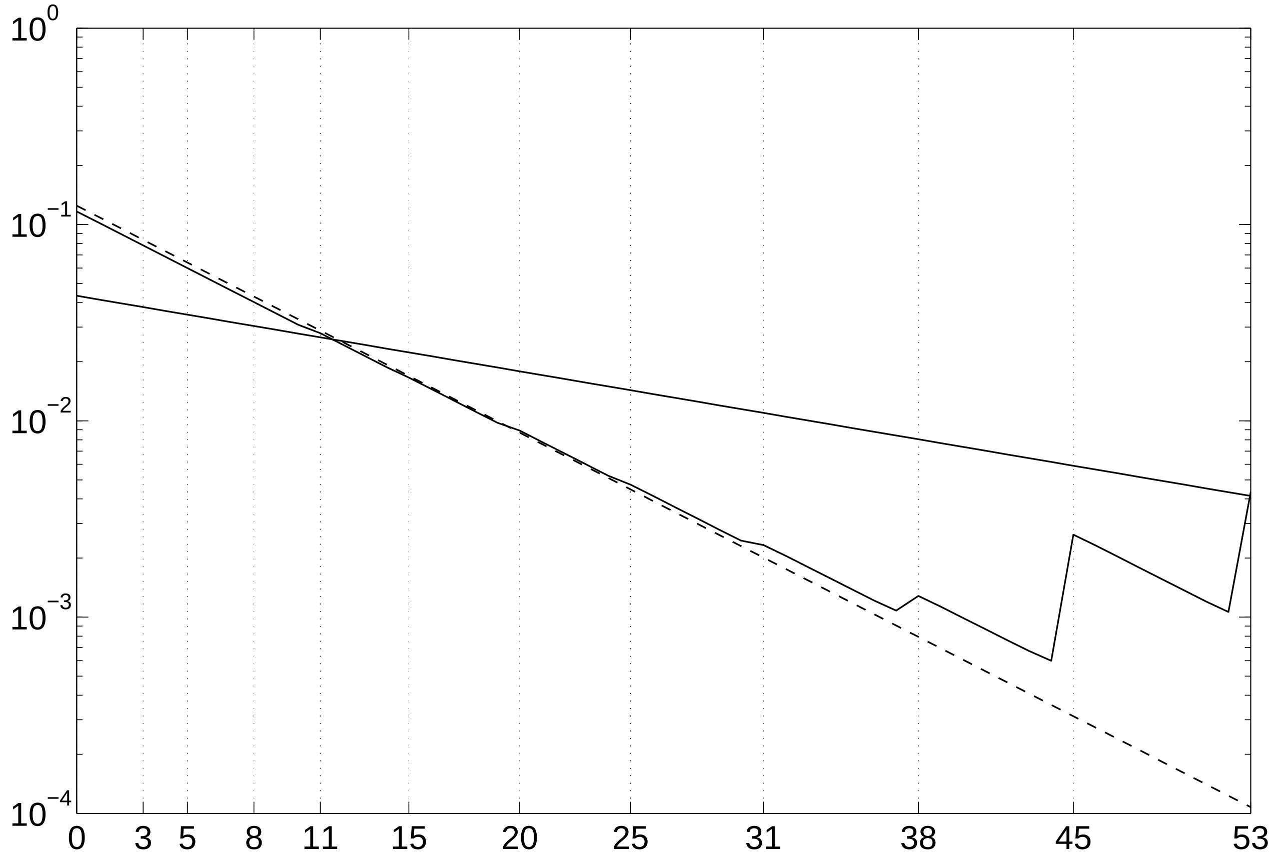

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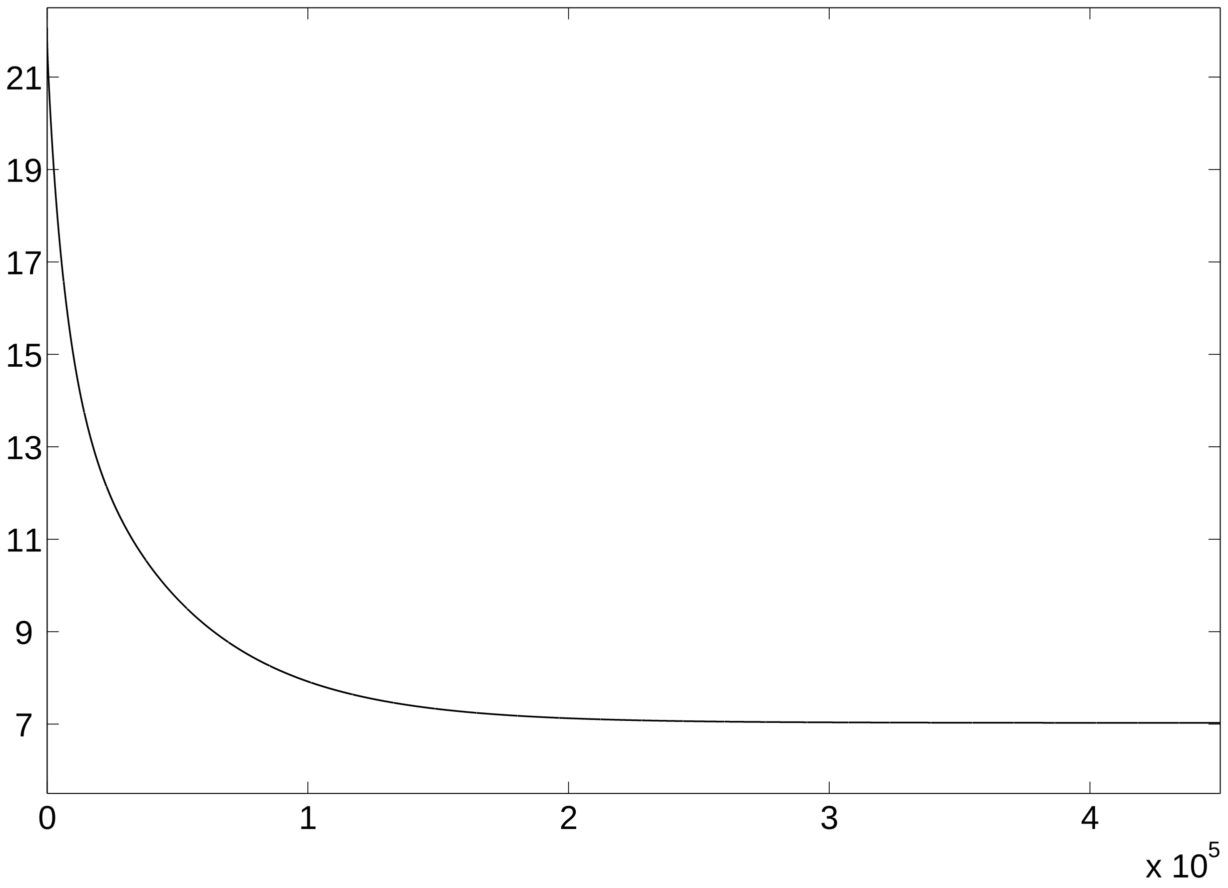
Mean photon number



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