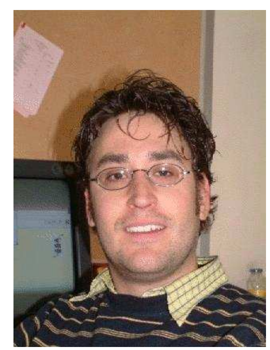
Leaky Repeated Interaction Quantum Systems *

Alain JOYE

INSTITUT FOURIER

* Joint work with



Laurent BRUNEAU (Université de Cergy) & Marco MERKLI (Memorial University)

The Formal Model

Quantum system $\mathcal{S}$:

- Finite dimensional system, driven by Hamiltonian $H_{\mathcal{S}}$ on $\mathfrak{H}_{\mathcal{S}}$, s.t. $\sigma(H_{\mathcal{S}}) = \{e_1, \cdots, e_d\}$.

The Formal Model

Quantum system $\mathcal{S}$:

- Finite dimensional system, driven by Hamiltonian $H_{\mathcal{S}}$ on $\mathfrak{H}_{\mathcal{S}}$, s.t. $\sigma(H_{\mathcal{S}}) = \{e_1, \cdots, e_d\}$.

Chain $\mathcal{C}$ of identical quantum sub-systems $\mathcal{E}_k \equiv \mathcal{E}$, $k = 1, 2, \cdots$:

$$\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \cdots$$

The Formal Model

Quantum system $\mathcal{S}$:

- Finite dimensional system, driven by Hamiltonian $H_{\mathcal{S}}$ on $\mathfrak{H}_{\mathcal{S}}$, s.t. $\sigma(H_{\mathcal{S}}) = \{e_1, \cdots, e_d\}$.

Chain $\mathcal{C}$ of identical quantum sub-systems $\mathcal{E}_k \equiv \mathcal{E}$, $k = 1, 2, \cdots$:

$$\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \cdots$$

- Each $\mathcal{E}_k$ is driven by the Hamiltonian $H_{\mathcal{E}\,k} \equiv H_{\mathcal{E}}$ on $\mathfrak{H}_{\mathcal{E}_k} \equiv \mathfrak{H}_{\mathcal{E}}$, $\dim \mathfrak{H}_{\mathcal{E}} \leq \infty$
- The chain $\mathcal{C}$ is driven by $H_{\mathcal{C}} \equiv H_{\mathcal{E}\,1} + H_{\mathcal{E}\,2} + \cdots$ on $\mathfrak{H}_{\mathcal{C}} \equiv \mathfrak{H}_{\mathcal{E}_1} \otimes \mathfrak{H}_{\mathcal{E}_2} \otimes \cdots$, with $[H_{\mathcal{E}\,j}, H_{\mathcal{E}\,k}] = 0, \;\; \forall j, k.$

The Formal Model

Quantum system $\mathcal{S}$:

- Finite dimensional system, driven by Hamiltonian $H_{\mathcal{S}}$ on $\mathfrak{H}_{\mathcal{S}}$, s.t. $\sigma(H_{\mathcal{S}}) = \{e_1, \cdots, e_d\}$.

Chain $\mathcal{C}$ of identical quantum sub-systems $\mathcal{E}_k \equiv \mathcal{E}$, $k = 1, 2, \cdots$:

$$\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \cdots$$

- Each $\mathcal{E}_k$ is driven by the Hamiltonian $H_{\mathcal{E}\,k} \equiv H_{\mathcal{E}}$ on $\mathfrak{H}_{\mathcal{E}_k} \equiv \mathfrak{H}_{\mathcal{E}}$, $\dim \mathfrak{H}_{\mathcal{E}} \leq \infty$
- The chain $\mathcal{C}$ is driven by $H_{\mathcal{C}} \equiv H_{\mathcal{E}\,1} + H_{\mathcal{E}\,2} + \cdots$ on $\mathfrak{H}_{\mathcal{C}} \equiv \mathfrak{H}_{\mathcal{E}_1} \otimes \mathfrak{H}_{\mathcal{E}_2} \otimes \cdots$, with $[H_{\mathcal{E}\,j}, H_{\mathcal{E}\,k}] = 0, \ \ \forall j, k.$

Fermionic reservoir $\mathcal{R}$:

- ∞ -ly extended gas of indep. fermions at temperature β , driven by "$H_{\mathcal{R}}$" on "$\mathfrak{H}_{\mathcal{R}}$" .

The Formal Model

Complete system $\mathcal{S} + \mathcal{R} + \mathcal{C}$

- Formal Hilbert space $\mathfrak{H}_{\mathcal{S}} \otimes \text{''}\mathfrak{H}_{\mathcal{R}}\text{''} \otimes \mathfrak{H}_{\mathcal{E}}$

The Formal Model

Complete system $\mathcal{S}+\mathcal{R}+\mathcal{C}$

- Formal Hilbert space $\mathfrak{H}_{\mathcal{S}} \otimes {}''\mathfrak{H}_{\mathcal{R}}'' \otimes \mathfrak{H}_{\mathcal{E}}$

Interaction $\mathcal{S}-\mathcal{C}$

- $W_{\mathcal{S}\mathcal{E}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k = 1, 2, \cdots$.

The Formal Model

Complete system $\mathcal{S} + \mathcal{R} + \mathcal{C}$

- Formal Hilbert space $\mathfrak{H}_{\mathcal{S}} \otimes$ "$\mathfrak{H}_{\mathcal{R}}$" $\otimes \mathfrak{H}_{\mathcal{E}}$

Interaction $\mathcal{S} - \mathcal{C}$

- $W_{\mathcal{S}\mathcal{E}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k = 1, 2, \cdots$.

Interaction $\mathcal{S} - \mathcal{R}$

- $W_{\mathcal{S}\mathcal{R}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes$ "$\mathfrak{H}_{\mathcal{R}}$".

The Formal Model

Complete system $\mathcal{S} + \mathcal{R} + \mathcal{C}$

- Formal Hilbert space $\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{E}}$

Interaction $\mathcal{S} - \mathcal{C}$

- $W_{\mathcal{SE}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k = 1, 2, \cdots$.

Interaction $\mathcal{S} - \mathcal{R}$

- $W_{\mathcal{SR}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}"$.

Evolution Let $\tau > 0$ be a duration, $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}}) \in \mathbb{R}^2$ be couplings

For $t = (m-1)\tau + s, \ \ 0 \leq s < \tau$,

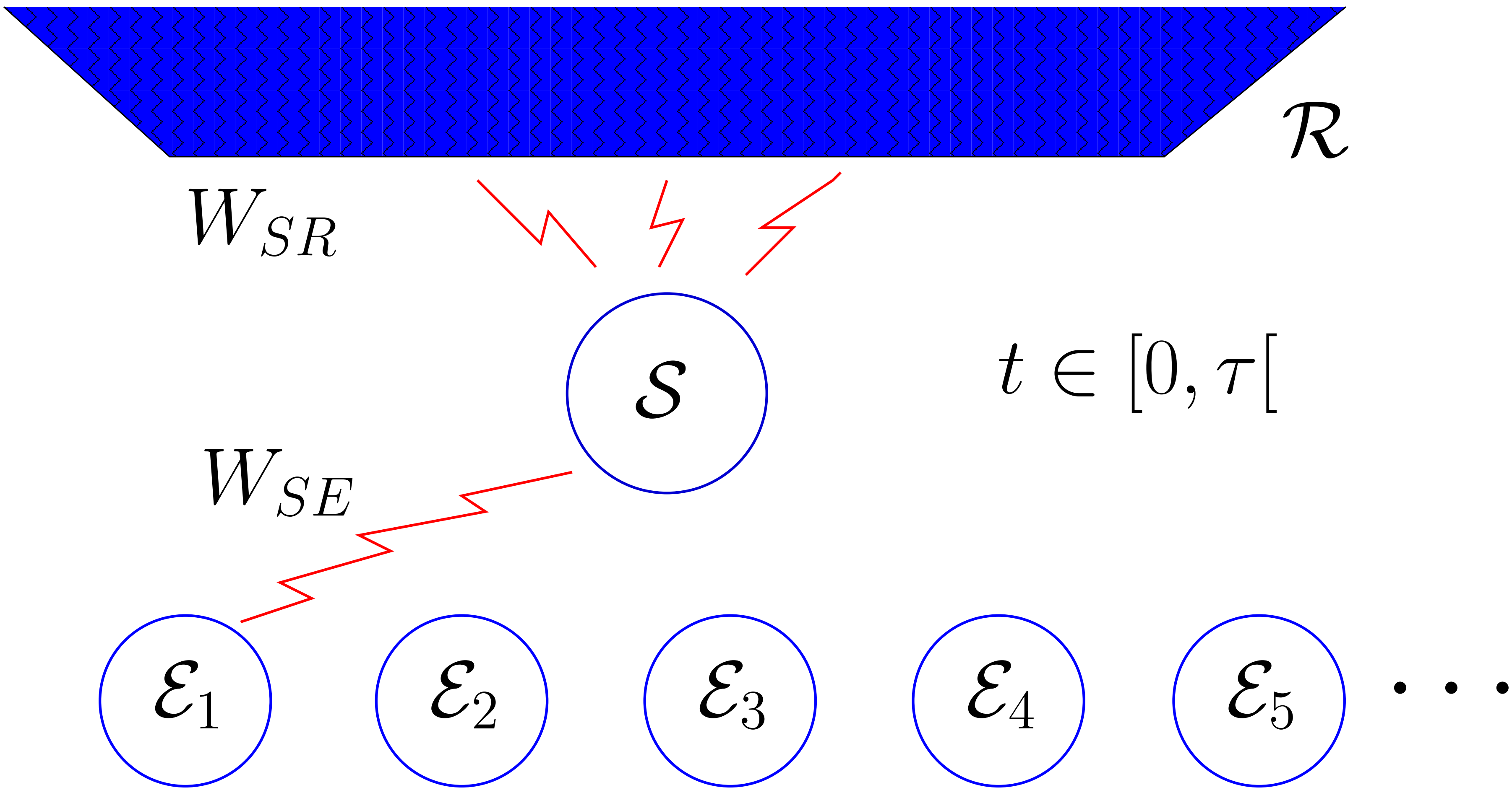
- $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{E}_m$ are driven by $H_{\mathcal{S}} + "H_{\mathcal{R}}" + H_{\mathcal{E}} + \lambda_{\mathcal{R}} W_{\mathcal{SR}} + \lambda_{\mathcal{E}} W_{\mathcal{SE}}$
- $\mathcal{E}_k$ evolve freely with $H_{\mathcal{E}}$, $\forall \ k \neq m$

Leaky Repeated Interactions Quantum Systems

Pictorially

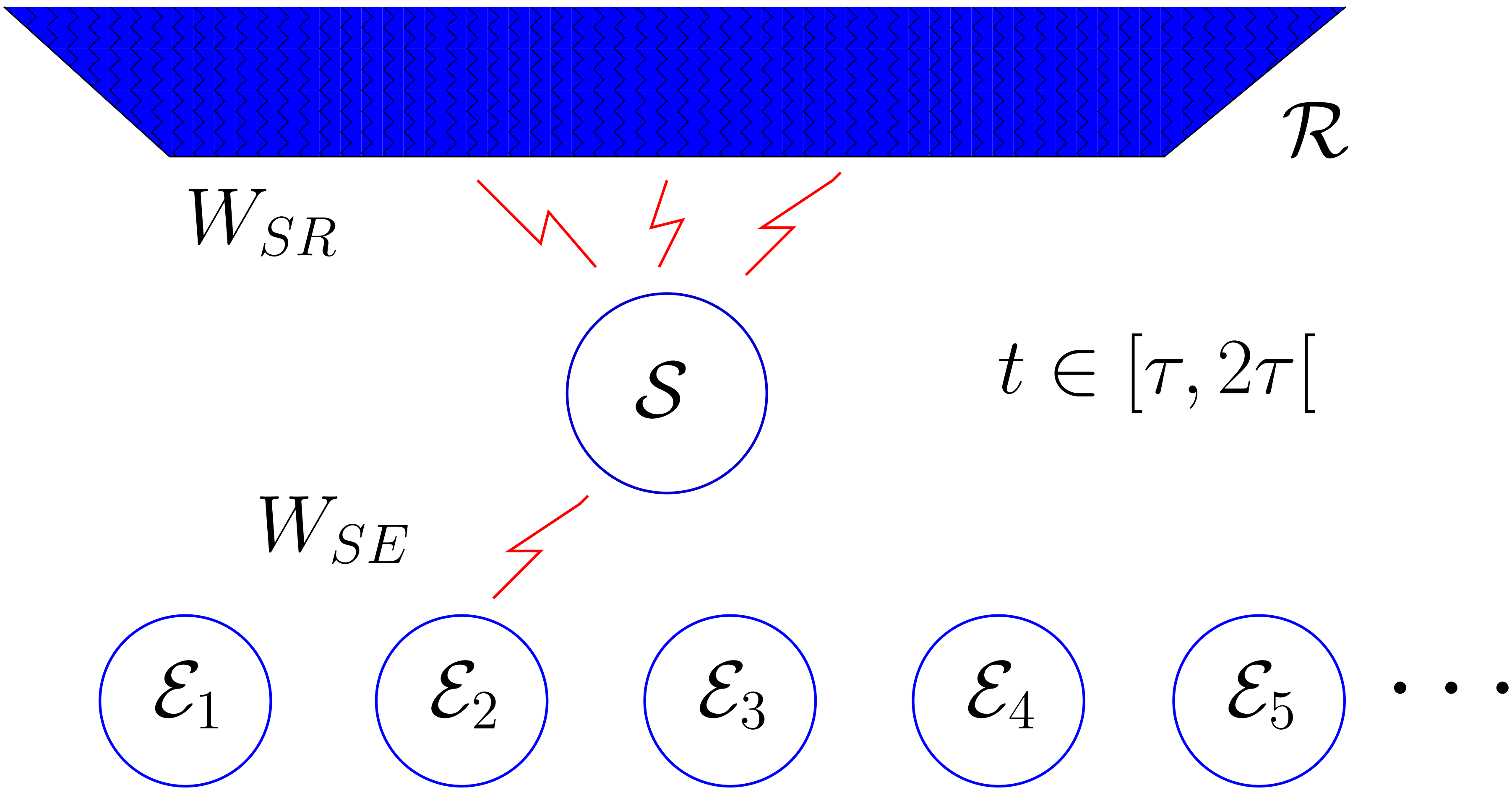
Leaky Repeated Interactions Quantum Systems

Pictorially



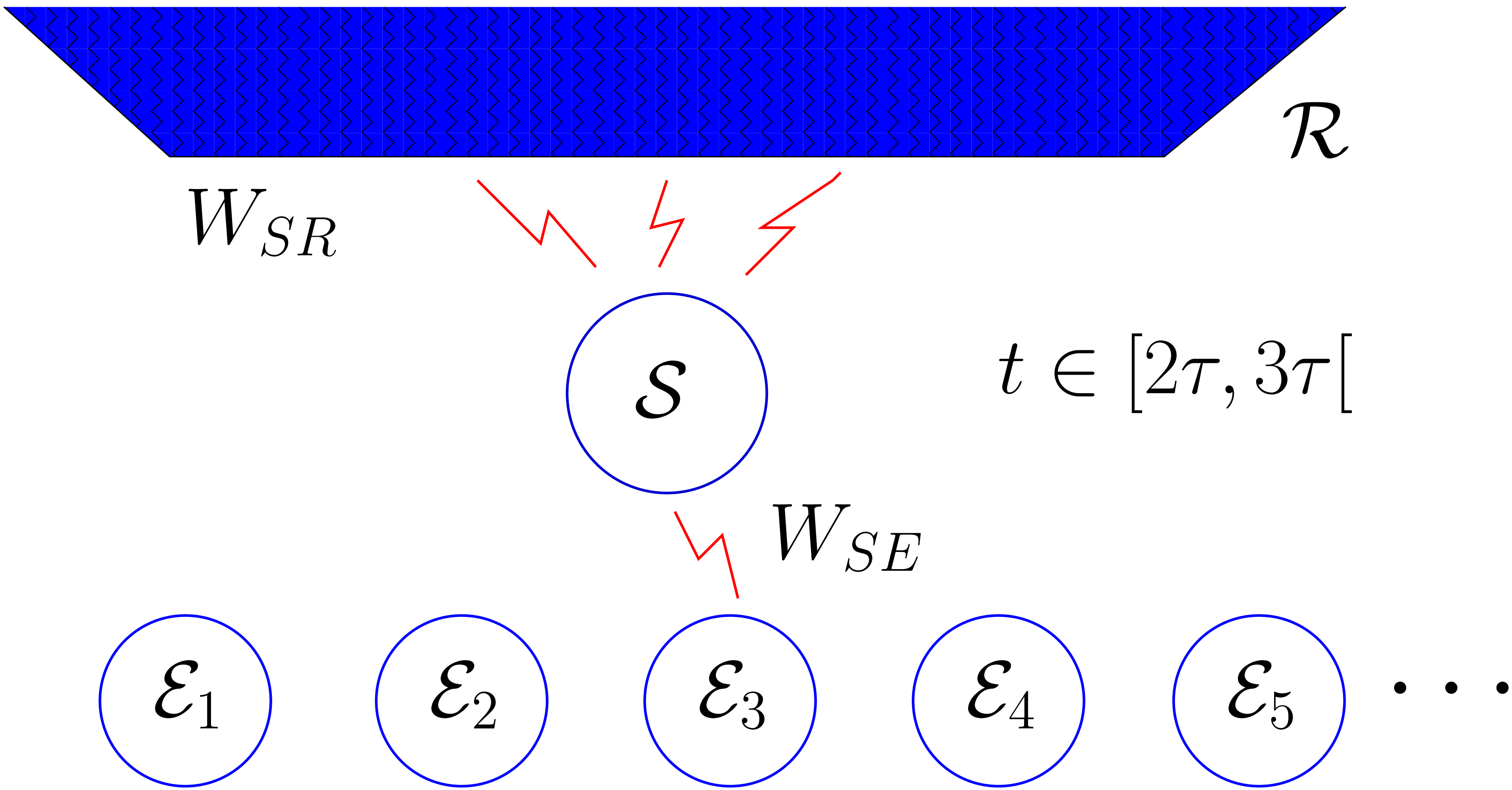
Leaky Repeated Interactions Quantum Systems

Pictorially



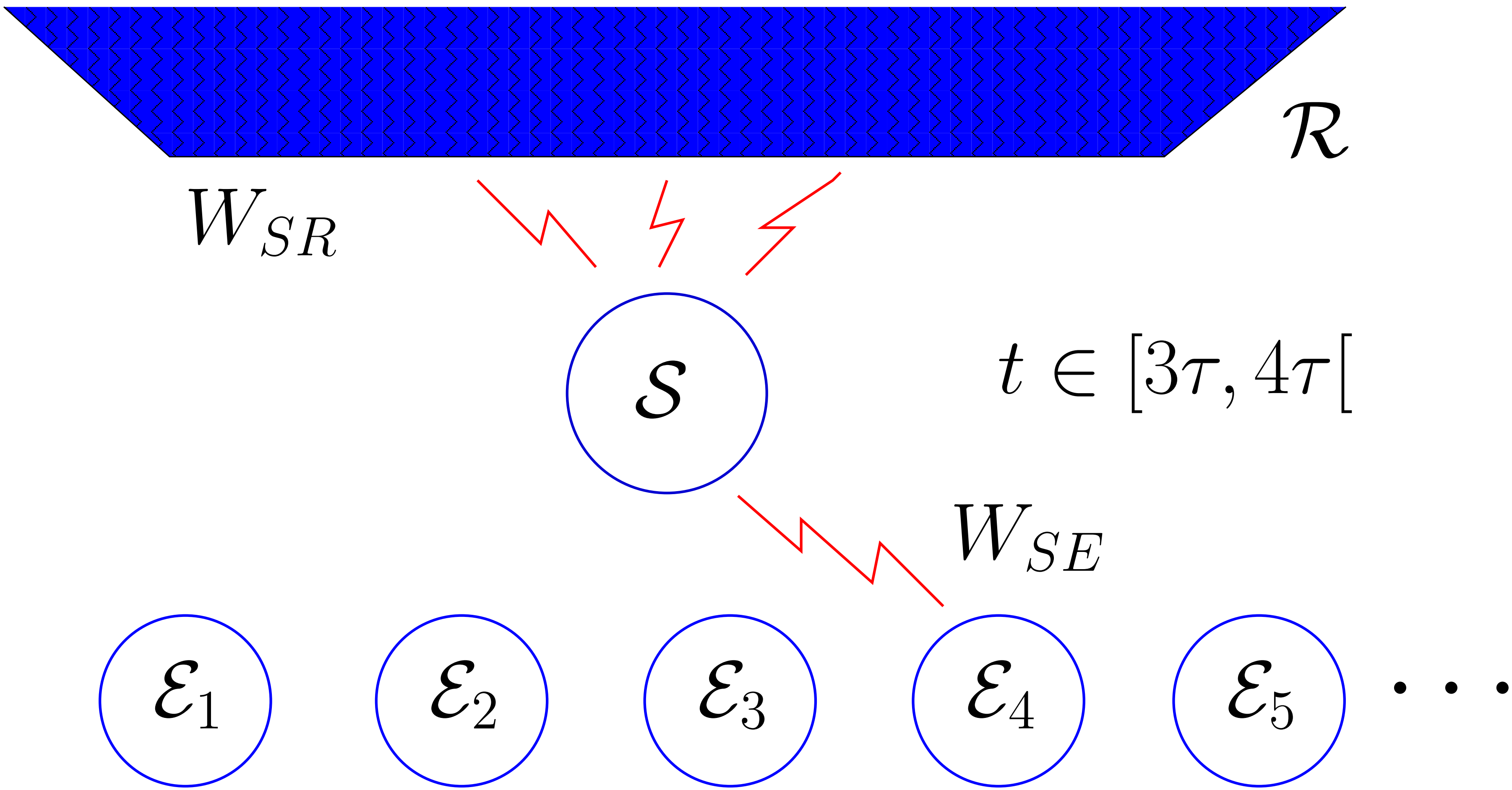
Leaky Repeated Interactions Quantum Systems

Pictorially



Leaky Repeated Interactions Quantum Systems

Pictorially



Questions about Open Quantum Systems

Large times asymptotics

Let $A = A_{SR} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}})$ be an observable on acting on $\mathcal{S} + \mathcal{R}$

Let $\alpha^t(A)$ be its Heisenberg evolution (yet to be defined), at time $t = m\tau$

Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state ("density matrix") on observables

Questions about Open Quantum Systems

Large times asymptotics

Let $A = A_{SR} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}})$ be an observable on acting on $\mathcal{S} + \mathcal{R}$

Let $\alpha^t(A)$ be its Heisenberg evolution (yet to be defined), at time $t = m\tau$

Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state (“density matrix”) on observables

- Existence of asymptotic behavior of $\lim_{m\to\infty} \rho \circ \alpha^{m\tau}(A)$?
  Dependence of an asympt. state on the initial state ρ ?
  Dependence of an asympt. state on the coupling constants $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})$?

Questions about Open Quantum Systems

Large times asymptotics

Let $A = A_{SR} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}})$ be an observable on acting on $\mathcal{S} + \mathcal{R}$

Let $\alpha^t(A)$ be its Heisenberg evolution (yet to be defined), at time $t = m\tau$

Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state (“density matrix”) on observables

- Existence of asymptotic behavior of $\lim_{m\to\infty} \rho \circ \alpha^{m\tau}(A)$?
  Dependence of an asympt. state on the initial state ρ ?
  Dependence of an asympt. state on the coupling constants $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})$?
- More general observables, like the energy variation ?
  Exchanges between $\mathcal{R}$ and $\mathcal{C}$ through $\mathcal{S}$?

Questions about Open Quantum Systems

Large times asymptotics

Let $A = A_{SR} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \text{"}\mathfrak{H}_{\mathcal{R}}\text{"} \otimes \mathfrak{H}_{\mathcal{C}})$ be an observable on acting on $\mathcal{S} + \mathcal{R}$

Let $\alpha^t(A)$ be its Heisenberg evolution (yet to be defined), at time $t = m\tau$

Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \text{"}\mathfrak{H}_{\mathcal{R}}\text{"} \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state ("density matrix") on observables

- Existence of asymptotic behavior of $\lim_{m\to\infty} \rho \circ \alpha^{m\tau}(A)$?
  Dependence of an asympt. state on the initial state ρ ?
  Dependence of an asympt. state on the coupling constants $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})$?
- More general observables, like the energy variation ?
  Exchanges between $\mathcal{R}$ and $\mathcal{C}$ through $\mathcal{S}$?
- Non-trivial examples ?

Questions about Open Quantum Systems

Large times asymptotics

Let $A = A_{SR} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \text{"}\mathfrak{H}_{\mathcal{R}}\text{"} \otimes \mathfrak{H}_{\mathcal{C}})$ be an observable on acting on $\mathcal{S} + \mathcal{R}$

Let $\alpha^t(A)$ be its Heisenberg evolution (yet to be defined), at time $t = m\tau$

Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \text{"}\mathfrak{H}_{\mathcal{R}}\text{"} \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state ("density matrix") on observables

- Existence of asymptotic behavior of $\lim_{m\to\infty} \rho \circ \alpha^{m\tau}(A)$?
  Dependence of an asympt. state on the initial state ρ ?
  Dependence of an asympt. state on the coupling constants $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})$?
- More general observables, like the energy variation ?
  Exchanges between $\mathcal{R}$ and $\mathcal{C}$ through $\mathcal{S}$?
- Non-trivial examples ?

Remark :

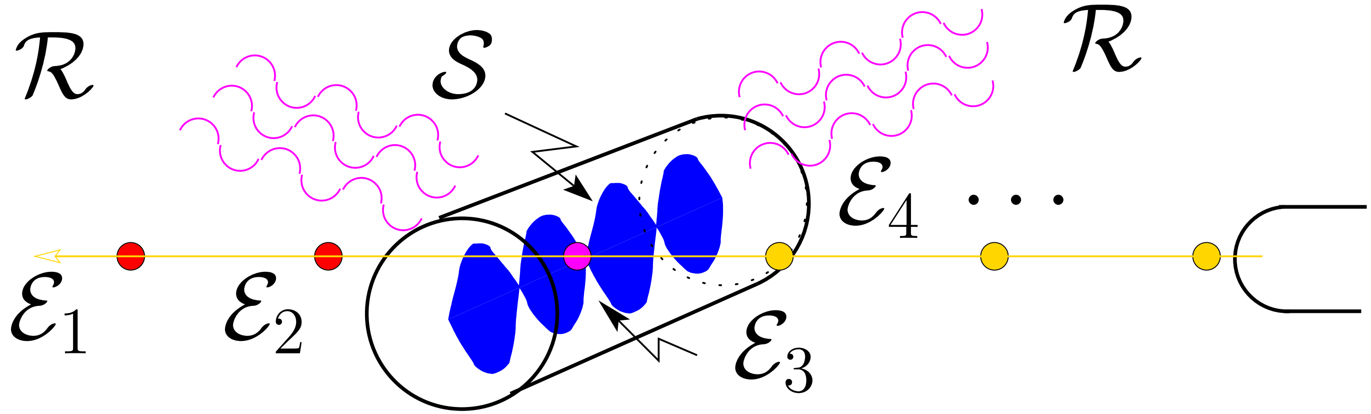
If $\lambda_{\mathcal{R}} = 0$, then $\mathcal{S} + \mathcal{C}$ $\Rightarrow$ convergence to a NESS Bruneau-J.-Merkli 06

If $\lambda_{\mathcal{E}} = 0$, then $\mathcal{S} + \mathcal{R}$ $\Rightarrow$ return to equilibrium Jaksic-Pillet 96

Motivations

One-atom maser

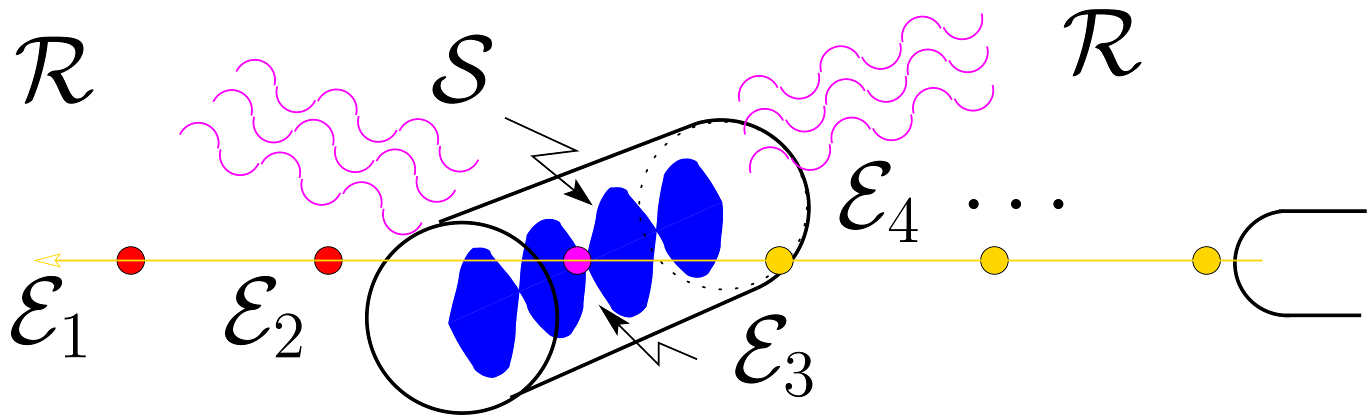
Walther et al ’85, Haroche et al ’92



Motivations

One-atom maser

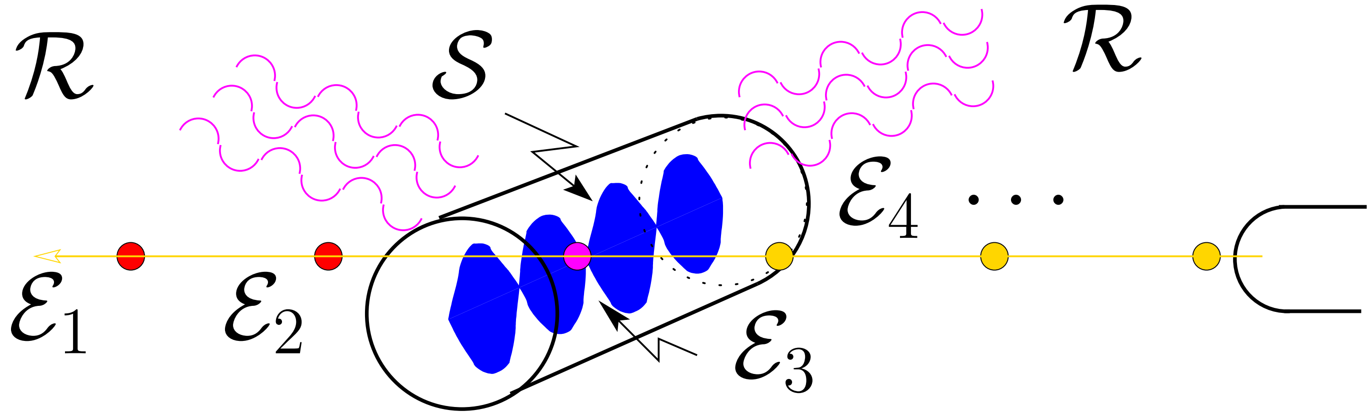
Walther et al '85, Haroche et al '92



- $\mathcal{S}$: one mode of E-M field in a cavity
- $\mathcal{E}_k$: atom $\#k$ interacting with the mode
- $\mathcal{C}$: sequence of atoms passing through the cavity
- $\mathcal{R}$: environment responsible for losses

Motivations

One-atom maser — Walther et al '85, Haroche et al '92



- $\mathcal{S}$: one mode of E-M field in a cavity
- $\mathcal{E}_k$: atom $\#k$ interacting with the mode
- $\mathcal{C}$: sequence of atoms passing through the cavity
- $\mathcal{R}$: environment responsible for losses

Ideal RIQS used as simple models — Vogel et al 93, Wellens et al 00, BJM 06

Random RIQS to model fluctuations — BJM 08

Leaky RIQS to account for losses

Mathematical Framework

GNS representation

Let $\rho \in \mathcal{B}_1(\mathfrak{H})$ be a density matrix on $\mathfrak{H}$

$$0 < \rho = \sum \lambda_j |\varphi_j\rangle\langle\varphi_j| \quad \text{and} \quad \text{Tr}\rho = 1$$

Mathematical Framework

GNS representation

Let $\rho \in \mathcal{B}_1(\mathfrak{H})$ be a density matrix on $\mathfrak{H}$

$$0 < \rho = \sum \lambda_j |\varphi_j\rangle\langle\varphi_j| \text{ and } \mathrm{Tr}\rho = 1$$

GNS:

$\rho \longrightarrow$ pure state $|\Psi_\rho\rangle\langle\Psi_\rho|$ on enlarged Hilbert space

Mathematical Framework

GNS representation

Let $\rho \in \mathcal{B}_1(\mathfrak{H})$ be a density matrix on $\mathfrak{H}$

$$0 < \rho = \sum \lambda_j |\varphi_j\rangle\langle\varphi_j| \text{ and } \mathrm{Tr}\rho = 1$$

GNS:

$$\rho \to \text{pure state } |\Psi_\rho\rangle\langle\Psi_\rho| \text{ on enlarged Hilbert space}$$

- $\mathfrak{H} \to \mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$
- $\rho \in \mathcal{B}_1(\mathfrak{H}) \to \Psi_\rho = \sum_j \sqrt{\lambda_j}\varphi_j \otimes \varphi_j \in \mathcal{H}$
- $A \in \mathcal{B}(\mathfrak{H}) \to \Pi(A) = A \otimes \mathbb{I}_\mathfrak{H} \in \mathcal{B}(\mathcal{H})$

$$\Rightarrow \mathrm{Tr}_\mathfrak{H}(\rho A) = \langle\Psi_\rho|A \otimes \mathbb{I}_\mathfrak{H}\Psi_\rho\rangle_\mathcal{H} = \mathrm{Tr}_\mathcal{H}(|\Psi_\rho\rangle\langle\Psi_\rho|\Pi(A))$$

- $\mathbb{I}_\mathfrak{H} \otimes B \in \mathcal{B}(\mathcal{H})$ don't play any role
- For gas of ∞-ly many particles, GNS is required and non-trivial, see below

Liouvillean

Evolution of observables

$$\alpha^t(A) = e^{itH} A e^{-itH} \in \mathcal{B}(\mathfrak{H})$$

Evolution of states

$$\rho \in \mathcal{B}_1(\mathfrak{H}) \mapsto e^{-itH} \rho e^{itH} \in \mathcal{B}_1(\mathfrak{H})$$

Liouvillean

Evolution of observables

$$\alpha^t(A) = e^{itH} A e^{-itH} \in \mathcal{B}(\mathfrak{H})$$

Evolution of states

$$\rho \in \mathcal{B}_1(\mathfrak{H}) \mapsto e^{-itH} \rho e^{itH} \in \mathcal{B}_1(\mathfrak{H})$$

Invariant states

$$\mathsf{Tr}(e^{-itH} \rho e^{itH} A) = \mathsf{Tr}(\rho A), \;\; \forall A \in \mathcal{B}(\mathfrak{H})$$

Liouvillean

Evolution of observables

$$\alpha^t(A) = e^{itH} A e^{-itH} \in \mathcal{B}(\mathfrak{H})$$

Evolution of states

$$\rho \in \mathcal{B}_1(\mathfrak{H}) \mapsto e^{-itH} \rho e^{itH} \in \mathcal{B}_1(\mathfrak{H})$$

Invariant states

$$\mathrm{Tr}(e^{-itH} \rho e^{itH} A) = \mathrm{Tr}(\rho A), \ \ \forall A \in \mathcal{B}(\mathfrak{H})$$

Liouville operator

Given ρ invariant, $\exists$ a unique self-adjoint L on $\mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$ s.t.

$$\begin{cases} \Pi(\alpha^t(A)) & = & e^{itL} \Pi(A) e^{-itL} \in \mathcal{H} \\ L\Psi_\rho & = & 0 \end{cases}$$

Liouvillean

Evolution of observables

$$\alpha^t(A) = e^{itH} A e^{-itH} \in \mathcal{B}(\mathfrak{H})$$

Evolution of states

$$\rho \in \mathcal{B}_1(\mathfrak{H}) \mapsto e^{-itH} \rho e^{itH} \in \mathcal{B}_1(\mathfrak{H})$$

Invariant states

$$\mathrm{Tr}(e^{-itH} \rho e^{itH} A) = \mathrm{Tr}(\rho A), \;\; \forall A \in \mathcal{B}(\mathfrak{H})$$

Liouville operator

Given ρ invariant, $\exists$ a unique self-adjoint L on $\mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$ s.t.

$$\begin{cases} \Pi(\alpha^t(A)) & = & e^{itL} \Pi(A) e^{-itL} \in \mathcal{H} \\ L\Psi_\rho & = & 0 \end{cases}$$

Simple setup

$$L = H \otimes \mathbb{I}_{\mathfrak{H}} - \mathbb{I}_{\mathfrak{H}} \otimes H$$

Temperature β^{-1}

Originally Hamiltonian $d\Gamma_-(\tilde{h})$ on $\Gamma_-(\tilde{\mathfrak{h}}) = \oplus_{n=0}^{\infty}\Gamma_-^n(\tilde{\mathfrak{h}})$ where

$\tilde{\mathfrak{h}} = L^2(\mathbb{R}^+, \mathfrak{G})$ one part. Hilbert sp., $\mathfrak{G}$ auxil. Hilbert sp. and

one part. Hamiltonian $\tilde{h}$ s.t.

$(\tilde{h}\tilde{f})(s) = s\tilde{f}(s)$, $s \in \mathbb{R}^+$, $\forall \tilde{f} \in \tilde{\mathfrak{h}} = L^2(\mathbb{R}^+, \mathfrak{G})$

$a(\tilde{g})$, $a^*(\tilde{g})$ annih. and creat. op's on $\Gamma_-(\tilde{\mathfrak{h}})$, $\tilde{g} \in \tilde{\mathfrak{h}}$

Equilibrium State ω_β characterized by

$\omega_\beta(a^*(\tilde{g})a(\tilde{f})) = \langle \tilde{f}|(1 + e^{\beta\tilde{h}})^{-1}\tilde{g}\rangle$ and

$\omega_\beta(a^*(\tilde{g}_n)\cdots a^*(\tilde{g}_1)a(\tilde{f}_1)\cdots a(\tilde{f}_n)) = \det(\omega_\beta(a^*(\tilde{g}_i)a(\tilde{f}_j)))$

GNS for Fermi Bath

Araki-Wyss 64 + Jaksic-Pillet Gluing 02:

Enlarged Hilbert space $\mathcal{H}_{\mathcal{R}} = \Gamma_-(\mathfrak{h})$, $\mathfrak{h} = L^2(\mathbb{R}, \mathfrak{G})$

Liouvillean $L_{\mathcal{R}} = d\Gamma(h)$, with h s.t.

$(hf)(s) = sf(s)$, $s \in \mathbb{R}$, $\forall f \in \mathfrak{h} = L^2(\mathbb{R}, \mathfrak{G})$

Creat., annih. op's $a^*(g_\beta)$, $a(g_\beta)$, where $g_\beta \leftrightarrow \tilde{g}$ via

$$g_\beta(s) = (e^{-\beta s} + 1)^{-1/2} g(s)\,, \; g(s) = \begin{cases} \tilde{g}(s) & \text{if } s \geq 0 \\ \bar{\tilde{g}}(-s) & \text{if } s < 0. \end{cases}$$

Equilibrium State $|\Psi_R\rangle\langle\Psi_{\mathcal{R}}|$, $\Psi_{\mathcal{R}}$ vacuum of $\Gamma_-(\mathfrak{h})$

Note

$$\text{“}L^2(\mathbb{R}^+, \mathfrak{G}) + L^2(\mathbb{R}^+, \mathfrak{G}) = L^2(\mathbb{R}, \mathfrak{G})\text{”}$$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{R}}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{R}}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$
- Algebras of observables $\mathfrak{M}_{\#} \subset \mathcal{B}(\mathcal{H}_{\#})$, $\# = \mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{R}}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$
- Algebras of observables $\mathfrak{M}_{\#} \subset \mathcal{B}(\mathcal{H}_{\#})$, $\# = \mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$
- States on $\mathcal{S}$, $\mathcal{R}$, $\mathcal{E}$, $\mathcal{C}$ are density matrices on $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{R}}$, $\mathcal{H}_{\mathcal{E}}$, $\mathcal{H}_{\mathcal{C}}$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{R}}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$
- Algebras of observables $\mathfrak{M}_{\#} \subset \mathcal{B}(\mathcal{H}_{\#})$, $\# = \mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$
- States on $\mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$ are density matrices on $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{R}}$, $\mathcal{H}_{\mathcal{E}}$, $\mathcal{H}_{\mathcal{C}}$
- Evolution of observables $A_{\mathcal{S}} \mapsto \alpha_{\mathcal{S}}^t(A_{\mathcal{S}})$, $A_{\mathcal{R}} \mapsto \alpha_{\mathcal{R}}^t(A_{\mathcal{R}})$, $A_{\mathcal{E}} \mapsto \alpha_{\mathcal{E}}^t(A_{\mathcal{E}})$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_\mathcal{R}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_\mathcal{C} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$
- Algebras of observables $\mathfrak{M}_\# \subset \mathcal{B}(\mathcal{H}_\#)$, $\# = \mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$
- States on $\mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$ are density matrices on $\mathcal{H}_\mathcal{S}, \mathcal{H}_\mathcal{R}, \mathcal{H}_\mathcal{E}, \mathcal{H}_\mathcal{C}$
- Evolution of observables $A_\mathcal{S} \mapsto \alpha^t_\mathcal{S}(A_\mathcal{S})$, $A_\mathcal{R} \mapsto \alpha^t_\mathcal{R}(A_\mathcal{R})$, $A_\mathcal{E} \mapsto \alpha^t_\mathcal{E}(A_\mathcal{E})$

- Assumption: $\exists$ invariant states (cyclic and separating)

$\Psi_\mathcal{S} \in \mathcal{H}_\mathcal{S}$, $\Psi_\mathcal{R} \in \mathcal{H}_\mathcal{R}$ and $\Psi_\mathcal{E} \in \mathcal{H}_\mathcal{E}$ s.t.

$$\alpha^t_\#(A_\#) = e^{itL_\#} A_\# e^{-itL_\#}, \text{ and } L_\# \Psi_\# = 0, \quad \text{where } \# = \mathcal{S}, \mathcal{R} \text{ and } \mathcal{E}$$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_\mathcal{R}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_\mathcal{C} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$
- Algebras of observables $\mathfrak{M}_\# \subset \mathcal{B}(\mathcal{H}_\#)$, $\# = \mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$
- States on $\mathcal{S}$, $\mathcal{R}$, $\mathcal{E}$, $\mathcal{C}$ are density matrices on $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_\mathcal{R}$, $\mathcal{H}_\mathcal{E}$, $\mathcal{H}_\mathcal{C}$
- Evolution of observables $A_\mathcal{S} \mapsto \alpha^t_\mathcal{S}(A_\mathcal{S})$, $A_\mathcal{R} \mapsto \alpha^t_\mathcal{R}(A_\mathcal{R})$, $A_\mathcal{E} \mapsto \alpha^t_\mathcal{E}(A_\mathcal{E})$

- Assumption: $\exists$ invariant states (cyclic and separating)

$\Psi_\mathcal{S} \in \mathcal{H}_\mathcal{S}$, $\Psi_\mathcal{R} \in \mathcal{H}_\mathcal{R}$ and $\Psi_\mathcal{E} \in \mathcal{H}_\mathcal{E}$ s.t.

$\alpha^t_\#(A_\#) = e^{itL_\#} A_\# e^{-itL_\#}$, and $L_\# \Psi_\# = 0$, where $\# = \mathcal{S}, \mathcal{R}$ and $\mathcal{E}$

System $\mathcal{S} + \mathcal{R} + \mathcal{C}$

On $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{C}$, driven by $L_{\text{free}} = L_\mathcal{S} + L_\mathcal{R} + \sum_k L_{\mathcal{E}_k}$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{R}}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$
- Algebras of observables $\mathfrak{M}_{\#} \subset \mathcal{B}(\mathcal{H}_{\#})$, $\# = \mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$
- States on $\mathcal{S}$, $\mathcal{R}$, $\mathcal{E}$, $\mathcal{C}$ are density matrices on $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{R}}$, $\mathcal{H}_{\mathcal{E}}$, $\mathcal{H}_{\mathcal{C}}$
- Evolution of observables $A_{\mathcal{S}} \mapsto \alpha^t_{\mathcal{S}}(A_{\mathcal{S}})$, $A_{\mathcal{R}} \mapsto \alpha^t_{\mathcal{R}}(A_{\mathcal{R}})$, $A_{\mathcal{E}} \mapsto \alpha^t_{\mathcal{E}}(A_{\mathcal{E}})$

- Assumption: $\exists$ invariant states (cyclic and separating)

$$\Psi_{\mathcal{S}} \in \mathcal{H}_{\mathcal{S}}, \ \Psi_{\mathcal{R}} \in \mathcal{H}_{\mathcal{R}} \text{ and } \Psi_{\mathcal{E}} \in \mathcal{H}_{\mathcal{E}} \text{ s.t.}$$

$$\alpha^t_{\#}(A_{\#}) = e^{itL_{\#}} A_{\#} e^{-itL_{\#}}, \quad \text{and} \quad L_{\#}\Psi_{\#} = 0, \qquad \text{where} \quad \# = \mathcal{S}, \mathcal{R} \text{ and } \mathcal{E}$$

System $\mathcal{S} + \mathcal{R} + \mathcal{C}$

On $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{C}}$, driven by $L_{\text{free}} = L_{\mathcal{S}} + L_{\mathcal{R}} + \sum_k L_{\mathcal{E}_k}$

Interactions

$V_{\mathcal{S}\#} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\#}$, the GNS repres. of $W_{\mathcal{S}\#}$, $\# = \mathcal{R}, \mathcal{E}$ + tech. hyp.

Dynamics

Repeated interaction Schrödinger dynamics

For any $m \in \mathbb{N}$, if $t = m\tau$ and $\psi \in \mathcal{H}$,

$$U(m)\psi := e^{-i\tilde{L}_m} e^{-i\tilde{L}_{m-1}} \cdots e^{-i\tilde{L}_1} \psi$$

where the generator for the duration τ is

$$\tilde{L}_m = \tau L_m + \tau \sum_{k \neq m} L_{\mathcal{E},k}$$

with

$$\begin{cases} L_m &= L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}} + V_m & \text{on } \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{E}_m} & \text{coupled} \\ V_m &= \lambda_{\mathcal{R}} V_{\mathcal{SR}} + \lambda_{\mathcal{E}} V_{\mathcal{SE}} & & \\ L_{\mathcal{E},k} &= L_{\mathcal{E}} & \text{on } \mathcal{H}_{\mathcal{E}_k} & \text{free} \end{cases}$$

Dynamics

Repeated interaction Schrödinger dynamics

For any $m \in \mathbb{N}$, if $t = m\tau$ and $\psi \in \mathcal{H}$,

$$U(m)\psi := e^{-i\tilde{L}_m} e^{-i\tilde{L}_{m-1}} \cdots e^{-i\tilde{L}_1} \psi$$

where the generator for the duration τ is

$$\tilde{L}_m = \tau L_m + \tau \sum_{k \neq m} L_{\mathcal{E},k}$$

with

$$\begin{cases} L_m = L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}} + V_m & \text{on } \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{E}_m} \quad \text{coupled} \\ V_m = \lambda_{\mathcal{R}} V_{\mathcal{S}\mathcal{R}} + \lambda_{\mathcal{E}} V_{\mathcal{S}\mathcal{E}} & \\ L_{\mathcal{E},k} = L_{\mathcal{E}} & \text{on } \mathcal{H}_{\mathcal{E}_k} \quad \text{free} \end{cases}$$

To be studied

Let $\varrho \in \mathcal{B}_1(\mathcal{H})$ be a state on $\mathcal{H}$ and $A_{\mathcal{S}\mathcal{R}} \in \mathfrak{M}$ an observable on $\mathcal{S} + \mathcal{R}$

$$m \mapsto \varrho(U^*(m) A_{\mathcal{S}\mathcal{R}} U(m)) \equiv \varrho(\alpha^{m\tau}(A_{\mathcal{S}\mathcal{R}})), \quad \text{as} \quad m \to \infty$$

Reduction to a Product of Operators

Special state

$$\varrho_0 = \langle \Psi_0 | \cdot \Psi_0 \rangle$$ where $$\Psi_0 = \Psi_{\mathcal{SR}} \otimes \Psi_{\mathcal{C}}$$ with

$$\Psi_{\mathcal{SR}} = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \equiv \mathcal{H}_{\mathcal{SR}}$$ and

$$\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}_1} \otimes \Psi_{\mathcal{E}_2} \otimes \cdots \in \mathcal{H}_{\mathcal{C}}$$

$P = \mathbb{I}_{\mathcal{H}_{\mathcal{SR}}} \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$ is the projector on $\mathcal{H}_{\mathcal{SR}} \otimes \mathbb{C}\Psi_{\mathcal{C}} \simeq \mathcal{H}_{\mathcal{SR}}$

Reduction to a Product of Operators

Special state

$\varrho_0 = \langle \Psi_0 | \cdot \Psi_0 \rangle$ where $\Psi_0 = \Psi_{\mathcal{SR}} \otimes \Psi_{\mathcal{C}}$ with

$\Psi_{\mathcal{SR}} = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \equiv \mathcal{H}_{\mathcal{SR}}$ and

$\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}_1} \otimes \Psi_{\mathcal{E}_2} \otimes \cdots \in \mathcal{H}_{\mathcal{C}}$

$P = \mathbb{I}_{\mathcal{H}_{\mathcal{SR}}} \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$ is the projector on $\mathcal{H}_{\mathcal{SR}} \otimes \mathbb{C}\Psi_{\mathcal{C}} \simeq \mathcal{H}_{\mathcal{SR}}$

$C-$ Liouvillean Given $L_{\mathcal{S}} + L_{\mathcal{R}}$, $L_{\mathcal{E}}$ and $V_m \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{E}_m}$,

$$\exists \quad K_m \text{ s.t. } \begin{cases} e^{i\tilde{L}_m} A e^{-i\tilde{L}_m} = e^{iK_m} A e^{-iK_m} \quad \forall A \in \mathfrak{M}_{\mathcal{SR}} \otimes \mathfrak{M}_{\mathcal{C}} \\ K_m \Psi_{\mathcal{SR}} \otimes \Psi_{\mathcal{C}} = 0. \end{cases}$$

K_m is not self-adjoint, not even normal ! Jaksic, Pillet '02

Reduction to a Product of Operators

Special state

$$\varrho_0 = \langle\Psi_0|\cdot\Psi_0\rangle \quad \text{where} \quad \Psi_0 = \Psi_{\mathcal{SR}}\otimes\Psi_{\mathcal{C}} \quad \text{with}$$
$$\Psi_{\mathcal{SR}} = \Psi_{\mathcal{S}}\otimes\Psi_{\mathcal{R}} \in \mathcal{H}_{\mathcal{S}}\otimes\mathcal{H}_{\mathcal{R}} \equiv \mathcal{H}_{\mathcal{SR}} \quad \text{and}$$
$$\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}_1}\otimes\Psi_{\mathcal{E}_2}\otimes\cdots\in\mathcal{H}_{\mathcal{C}}$$

$P = \mathbb{I}_{\mathcal{H}_{\mathcal{SR}}}\otimes|\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$ is the projector on $\mathcal{H}_{\mathcal{SR}}\otimes\mathbb{C}\Psi_{\mathcal{C}}\simeq\mathcal{H}_{\mathcal{SR}}$

$C-$ Liouvillean Given $L_{\mathcal{S}}+L_{\mathcal{R}}$, $L_{\mathcal{E}}$ and $V_m\in\mathfrak{M}_{\mathcal{S}}\otimes\mathfrak{M}_{\mathcal{R}}\otimes\mathfrak{M}_{\mathcal{E}_m}$,

$$\exists \quad K_m \text{ s.t.} \begin{cases} e^{i\tilde{L}_m}Ae^{-i\tilde{L}_m} = e^{iK_m}Ae^{-iK_m} \quad \forall A\in\mathfrak{M}_{\mathcal{SR}}\otimes\mathfrak{M}_{\mathcal{C}} \\ K_m\Psi_{\mathcal{SR}}\otimes\Psi_{\mathcal{C}} = 0. \end{cases}$$

K_m is not self-adjoint, not even normal ! Jaksic, Pillet '02

$$K_m = \tau(L_{\text{free}}+V_m-V'_m), \quad V'_m = J_m\Delta_m^{\frac{1}{2}}V_m\Delta_m^{-\frac{1}{2}}J_m$$
$$:= \tau(L_{\text{free}}+\tilde{V}_m)$$

Tomita-Takesaki '57

Reduction to a Product of Matrices

Evolution of ϱ_0

$$\begin{aligned}
\varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle \Psi_0 | e^{i\tilde{L}_1} \cdots e^{i\tilde{L}_m} A_{SR} e^{-i\tilde{L}_m} \cdots e^{-i\tilde{L}_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | e^{iK_1} \cdots e^{iK_m} A_{SR} e^{-iK_m} \cdots e^{-iK_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | P e^{iK_1} \cdots e^{iK_m} A_{SR} P \Psi_0 \rangle \\
&= \langle \Psi_0 | (P e^{iK_1} P)(P e^{iK_2} P) \cdots (P e^{iK_m} P) A_{SR} \Psi_0 \rangle \\
&\equiv \langle \Psi_{SR} | M_1 M_2 \cdots M_m A_{SR} \Psi_{SR} \rangle \\
&= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle
\end{aligned}$$

where $M_j \simeq P e^{iK_j} P$ on $\mathcal{H}_{SR}$ are all identical.

Reduction to a Product of Matrices

Evolution of ϱ_0

$$\begin{aligned}
\varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle \Psi_0 | e^{i\tilde{L}_1} \cdots e^{i\tilde{L}_m} A_{SR} e^{-i\tilde{L}_m} \cdots e^{-i\tilde{L}_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | e^{iK_1} \cdots e^{iK_m} A_{SR} e^{-iK_m} \cdots e^{-iK_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | P e^{iK_1} \cdots e^{iK_m} A_{SR} P \Psi_0 \rangle \\
&= \langle \Psi_0 | (P e^{iK_1} P)(P e^{iK_2} P) \cdots (P e^{iK_m} P) A_{SR} \Psi_0 \rangle \\
&\equiv \langle \Psi_{SR} | M_1 M_2 \cdots M_m A_{SR} \Psi_{SR} \rangle \\
&= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle
\end{aligned}$$

where $M_j \simeq P e^{iK_j} P$ on $\mathcal{H}_{SR}$ are all identical.

Reduced Dynamical Operators

$M \in \mathcal{B}(\mathcal{H}_{\mathcal{SR}})$ s.t.

$$\begin{cases} M \Psi_{\mathcal{SR}} = \Psi_{\mathcal{SR}} \\ \|M^n \varphi\| \leq C(\varphi), \quad \forall n \in \mathbb{N}, \quad \forall \varphi \text{ in a dense set} \end{cases}$$

Note: Ψ_{SR} cyclic and evolution is unitary.

Spectral Properties of RDO's

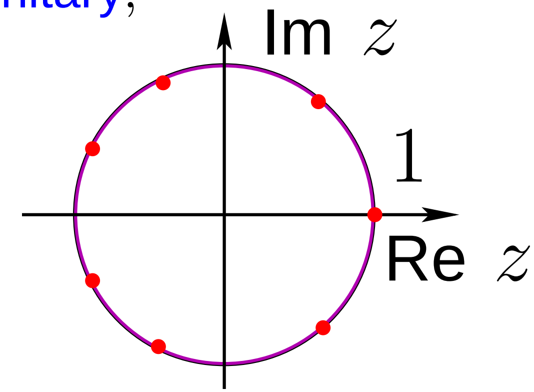
RDO

$$M = M_{(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})}$$

Uncoupled case

$$M_{(0,0)} = e^{i\tau(L_{\mathcal{S}} + L_{\mathcal{R}})} \text{ unitary,}$$

- eigenvalues of $M_{(0,0)}$: $\{e^{i\tau(e_k - e_l)}\}_{k,l}$
- 1 is $\dim \mathfrak{h}_{\mathcal{S}}$-fold degenerate
- ess spec $M_{(0,0)} = \mathbb{S}^1$



Spectral Properties of RDO's

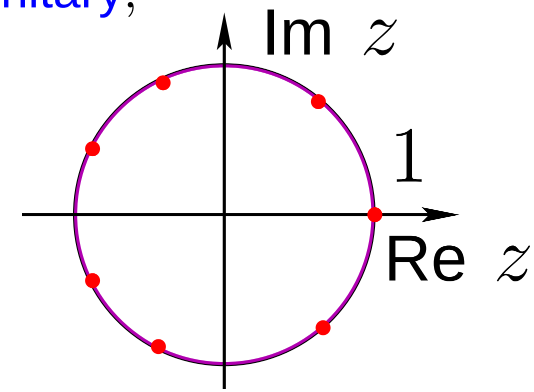
RDO

$$M = M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}$$

Uncoupled case

$$M_{(0,0)} = e^{i\tau(L_{\mathcal{S}}+L_{\mathcal{R}})}$$ unitary,

- eigenvalues of $M_{(0,0)}$: $\{e^{i\tau(e_k - e_l)}\}_{k,l}$
- 1 is $\dim \mathfrak{h}_{\mathcal{S}}$-fold degenerate
- ess spec $M_{(0,0)} = \mathbb{S}^1$



$(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}}) \neq (0,0)$ $\Rightarrow$ Perturbation of embedded eigenvalues

Spectral Properties of RDO's

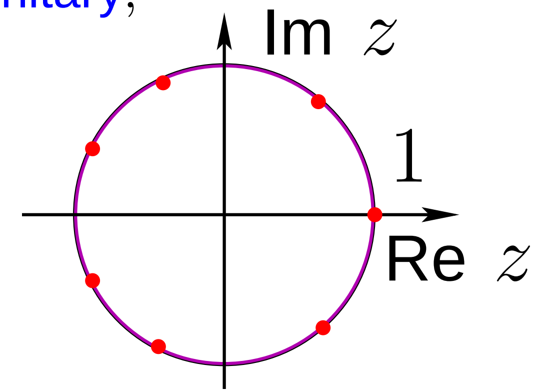
RDO

$$M = M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}$$

Uncoupled case

$$M_{(0,0)} = e^{i\tau(L_{\mathcal{S}}+L_{\mathcal{R}})}$$ unitary,

- eigenvalues of $M_{(0,0)}$: $\{e^{i\tau(e_k - e_l)}\}_{k,l}$
- 1 is $\dim \mathfrak{h}_{\mathcal{S}}$-fold degenerate
- ess spec $M_{(0,0)} = \mathbb{S}^1$



$(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}}) \neq (0,0)$ $\Rightarrow$ Perturbation of embedded eigenvalues

$L_{\mathcal{R}} = d\Gamma(h)$ with h mult. by s on $L^2(\mathbb{R}, \mathcal{G})$ is suitable for translation analyticity

Avron-Herbst 77

Translation Analyticity

Translation Group

$$\mathbb{R} \ni \theta \mapsto T(\theta) = \Gamma(e^{-\theta\partial_s}) \text{ on } \Gamma_-(L^2(\mathbb{R},\mathcal{G}))$$

$$\text{s.t.} \quad (e^{-\theta\partial_s} f)(s) = f(s-\theta), \quad \forall f \in L^2(\mathbb{R},\mathcal{G})$$

Assumption (A)

$\mathbb{R} \ni \theta \mapsto \tilde{V}_{\mathcal{SR}}(\theta) := T(\theta)^{-1}\tilde{V}_{\mathcal{SR}}T(\theta)$ admits an analytic extension to $\kappa_{\theta_0} = \{z \in \mathbb{C} \mid 0 < \text{Im } z < \theta_0\}$

Translation Analyticity

Translation Group

$$\mathbb{R} \ni \theta \mapsto T(\theta) = \Gamma(e^{-\theta\partial_s}) \text{ on } \Gamma_-(L^2(\mathbb{R},\mathcal{G}))$$

$$\text{s.t.} \quad (e^{-\theta\partial_s} f)(s) = f(s-\theta), \ \ \forall f \in L^2(\mathbb{R},\mathcal{G})$$

Assumption (A)

$\mathbb{R} \ni \theta \mapsto \tilde{V}_{\mathcal{SR}}(\theta) := T(\theta)^{-1}\tilde{V}_{\mathcal{SR}}T(\theta)$ admits an analytic extension to $\kappa_{\theta_0} = \{z \in \mathbb{C} \mid 0 < \text{Im } z < \theta_0\}$

Recall

$$M = P\exp(iK)P\,, \quad \text{where}$$

$$K = \tau(L_0 + \lambda_{\mathcal{R}}\tilde{V}_{\mathcal{SR}} + \lambda_{\mathcal{E}}\tilde{V}_{\mathcal{SE}})\ ,\ \ L_0 = L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}}$$

Theorem The following op's are analytic $\forall\theta \in \kappa_{\theta_0}$

$$K(\theta) = \tau(L_0 + \theta N + \lambda_{\mathcal{R}}\tilde{V}_{\mathcal{SR}}(\theta) + \lambda_{\mathcal{E}}\tilde{V}_{\mathcal{SE}}) \text{ on } D(L_0)\cap D(N),$$

$$M(\theta) = P\exp(iK(\theta))P \in \mathcal{B}(\mathcal{H}_{\mathcal{SR}})$$

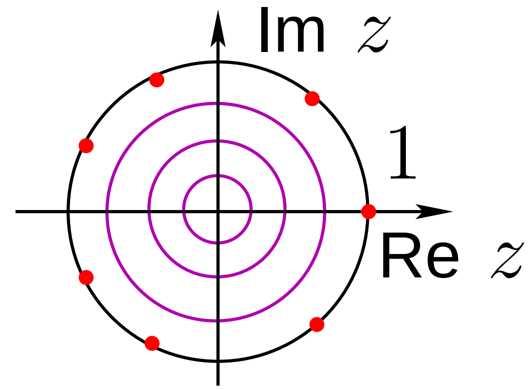
Translation Analyticity

Consequences

Discrete e.v. of $M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta)$ are θ-independent

Spectrum of $M_{(0,0)}(\theta) = \exp(i\tau(L_{\mathcal{S}} + L_{\mathcal{R}} + \theta N))$

$$\left\{ \begin{array}{l} \text{eigenvalues of } M_{(0,0)}(\theta) : \quad \{e^{i\tau(e_k - e_l)}\}_{k,l} \\ 1 \text{ is } \dim \mathfrak{h}_{\mathcal{S}}\text{-fold degenerate} \\ \text{ess spec } M_{(0,0)}(\theta) = \cup_{n=1}^{\infty}\{|z| = e^{-n\tau \mathrm{Im}\theta}\} \end{array} \right.$$



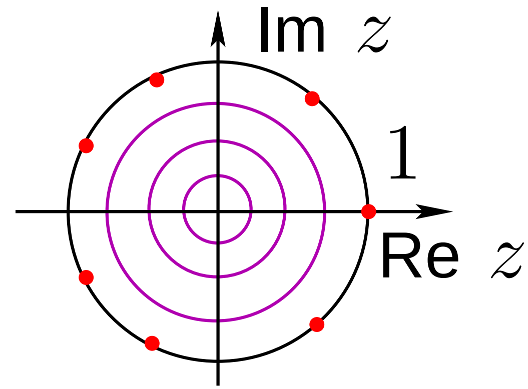
Translation Analyticity

Consequences

Discrete e.v. of $M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta)$ are θ-independent

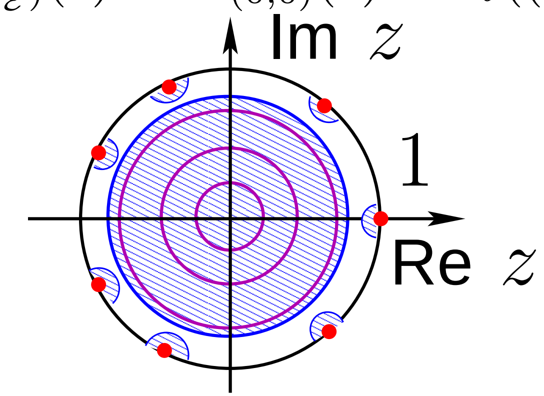
Spectrum of $M_{(0,0)}(\theta) = \exp(i\tau(L_{\mathcal{S}} + L_{\mathcal{R}} + \theta N))$

$$\left\{ \begin{array}{l} \text{eigenvalues of } M_{(0,0)}(\theta): \quad \{e^{i\tau(e_k - e_l)}\}_{k,l} \\ 1 \text{ is } \dim \mathfrak{h}_{\mathcal{S}}\text{-fold degenerate} \\ \text{ess spec } M_{(0,0)}(\theta) = \cup_{n=1}^{\infty}\{|z| = e^{-n\tau \mathrm{Im}\theta}\} \end{array} \right.$$



Perturbative approach

$$M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta) = M_{(0,0)}(\theta) + O_\theta((\lambda_{\mathcal{R}},\lambda_{\mathcal{E}}))$$



Lemma

$\|(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})\| < \lambda_0(\theta) \Rightarrow \sigma(M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta))$:

Asymptotic State

Analytic observables

$$A_{\mathcal{SR}} \text{ s.t. } A_{\mathcal{SR}}(\theta) = T(\theta)^{-1} A_{\mathcal{SR}} T(\theta) \text{ analytic in } \kappa_{\theta_0}$$

Note: For $A_{\mathcal{SR}}$ analytic,

$$\begin{aligned} \varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle \\ &= \langle \Psi_{SR} | M(\theta)^m A_{SR}(\theta) \Psi_{SR} \rangle \end{aligned}$$

Asymptotic State

Analytic observables

$$A_{\mathcal{SR}} \text{ s.t. } A_{\mathcal{SR}}(\theta) = T(\theta)^{-1} A_{\mathcal{SR}} T(\theta) \text{ analytic in } \kappa_{\theta_0}$$

Note: For $A_{\mathcal{SR}}$ analytic,

$$\begin{aligned} \varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle \\ &= \langle \Psi_{SR} | M(\theta)^m A_{SR}(\theta) \Psi_{SR} \rangle \end{aligned}$$

Assumption (FGR)

$\exists \theta_1 \in \kappa_{\theta_0}, \lambda_0(\theta_1) > 0$ s.t. $\|(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})\| < \lambda_0(\theta_1)$ implies

$\sigma(M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)) \cap \mathbb{S} = \{1\}$ and 1 is simple

Consequences

$$\lim_{n\to\infty} M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)^n = P_{1,M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)} = |\Psi_{\mathcal{SR}}\rangle\langle\psi^*_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)|$$

exponentially fast, and

$$\varrho_0(\alpha^{m\tau}(A_{SR})) \overset{m\to\infty}{\longrightarrow} \langle \psi^*_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1) | A_{SR}(\theta_1) \Psi_{SR} \rangle$$

Main Result

Theorem

Assume (A) and (FRG). For any state (density matrix) ϱ on $\mathcal{H}_{\mathcal{SR}} \otimes \mathcal{H}_{\mathcal{C}}$, any analytic observable A_{SR} , we have

$$\lim_{n\to\infty} \varrho(\alpha^{\tau n}_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(A_{SR})) = \langle \psi^{*}_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1) | A_{SR}(\theta_1)\Psi_{SR}\rangle$$

Main Result

Theorem

Assume (A) and (FRG). For any state (density matrix) ϱ on $\mathcal{H}_{\mathcal{SR}} \otimes \mathcal{H}_{\mathcal{C}}$, any analytic observable A_{SR} , we have

$$\lim_{n\to\infty} \varrho(\alpha^{\tau n}_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(A_{SR})) = \langle \psi^{*}_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1) | A_{SR}(\theta_1)\Psi_{SR}\rangle$$

Application

- $\mathcal{S}$ and $\mathcal{E}$ spins with e.v. $\{0, E_{\mathcal{S}}\}$, resp. $\{0, E_{\mathcal{E}}\}$
- $\mathcal{R}$ Fermi gas at $\beta_{\mathcal{R}}$, eq. state $\omega_{\beta_{\mathcal{R}}}$
- $W_{\mathcal{SE}} = a_S \otimes a_E^* + a_S^* \otimes a_E$
- $\Psi_{\mathcal{S}}$ tracial, $\Psi_{\mathcal{E}} \simeq \omega_{\beta,\mathcal{E}} = e^{-\beta_{\mathcal{E}} H_{\mathcal{E}}}/Z_{\beta_{\mathcal{E}}}$
- $W_{\mathcal{SR}} = \sigma_x \otimes (a_R^*(\tilde{f}) + a_R(\tilde{f}))$, $f \in L^2(\mathbb{R}^+, \mathcal{G})$ "regular".

Application

Perturbation theory

1) If $\|f(E_{\mathcal{S}})\| > 0$ and $\text{sinc}(\tau(E_{\mathcal{S}} - E_{\mathcal{E}})/2) \neq 0$, then (FGR) holds

2) The asymptotic state ω_+ is given by

$$\omega_+ = (\gamma_1 \omega_{\beta_{\mathcal{R}},\mathcal{S}} + \gamma_2 \omega_{\tilde{\beta}_{\mathcal{E}},\mathcal{S}}) \otimes \omega_{\beta_{\mathcal{R}}} + \mathcal{O}_{\theta_1,\beta_{\mathcal{R}},\cdots}(\|(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})\|)$$

with

$$\gamma_1 = \frac{\lambda_{\mathcal{R}}^2 2\pi\tau\|f(E_{\mathcal{S}})\|^2}{\lambda_{\mathcal{R}}^2 2\pi\tau\|f(E_{\mathcal{S}})\|^2 + \lambda_{\mathcal{E}}^2 \tau^2 \text{sinc}(\tau(E_{\mathcal{S}} - E_{\mathcal{E}})/2)^2},$$

$$\gamma_2 = \frac{\lambda_{\mathcal{E}}^2 \tau^2 \text{sinc}(\tau(E_{\mathcal{S}} - E_{\mathcal{E}})/2)^2}{\lambda_{\mathcal{R}}^2 2\pi\tau\|f(E_{\mathcal{S}})\|^2 + \lambda_{\mathcal{E}}^2 \tau^2 \text{sinc}(\tau(E_{\mathcal{S}} - E_{\mathcal{E}})/2)^2}$$

$$\tilde{\beta}_{\mathcal{E}} = \beta_{\mathcal{E}} \frac{E_{\mathcal{E}}}{E_{\mathcal{S}}} \text{ and } \gamma_1 + \gamma_2 = 1\,.$$