# Semiclassical estimates for non-selfadjoint operators with double characteristics

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Joint work with Karel Pravda-Starov

# Introduction

The study of operators with double characteristics has a long tradition in the analysis of linear PDE. Boutet de Monvel, Grigis, Helffer, Hörmander, Ivrii, Petkov, Sjöstrand... (classical works on hypoellipticity from the 1970's).

Recent works on Kramers–Fokker–Planck type operators have brought about a renewed interest in this subject. Hérau – Nier, Helffer – Nier, Hérau – Sjöstrand – Stolk (2004 – 2006).

In a recent work with K. Pravda–Starov we have investigated spectral and semigroup properties for a class of non-selfadjoint quadratic operators that are also non-elliptic.

Specifically, let

$$q : T^*\mathbf{R}^n \to \mathbf{C}$$

be a quadratic form such that $\operatorname{Re} q(x,\xi) \geq 0$, $(x,\xi) \in T^*\mathbf{R}^n$. Associated to $q$ is the Hamilton map

$$F : T^*\mathbf{C}^n \to T^*\mathbf{C}^n$$

defined by

$$q(X,Y) = \sigma(X, FY), \quad X,\, Y \in T^*\mathbf{C}^n,$$

where $\sigma$ is the canonical symplectic form on $T^*\mathbf{R}^n$.

It turns out that in order to understand the quadratic operator

$$Q = q^w(x, D_x),$$

defined as the Weyl quantization of $q$, it is both helpful and natural to introduce the **singular space** $S$ defined as follows :

$$S = \left( \bigcap_{j=0}^{\infty} \operatorname{Ker} \left( \operatorname{Re} F \ (\operatorname{Im} F)^j \right) \right) \cap \mathbf{R}^{2n}.$$

Notice that

$$\operatorname{Re} F(S) = \{0\} \quad \text{and} \quad (\operatorname{Im} F)\, S \subset S.$$

**Example.** The one-dimensional quadratic Kramers–Fokker–Planck operator is given by

$$K = q^w(x, y, D_x, D_y) - 1,$$

where

$$q(x, y, \xi, \eta) = \eta^2 + y^2 + i\,(y\xi - ax\eta)\,, \quad a \in \mathbf{R}\backslash\{0\}. \tag{1.1}$$

In this case,

$$S = \mathrm{Ker}(\mathrm{Re}\, F) \cap \mathrm{Ker}(\mathrm{Re}\, F\, \mathrm{Im}\, F) \cap \mathbf{R}^4 = \{0\}.$$

## Theorem

*(Pravda-Starov – H., 2008). Assume that the quadratic form $q$ is such that $\mathrm{Re}\, q \geq 0$ and that the restriction of $q$ to $S$ is* **elliptic**,

$$X \in S, \quad q(X) = 0 \quad \Rightarrow \quad X = 0.$$

*Then the singular space $S$ is symplectic and the spectrum of $q^w(x, D_x)$ on $L^2$ is discrete. The eigenvalues are of the form*

$$\sum_{\substack{\lambda \in \mathrm{Spec}(F), \\ -i\lambda \in \mathbf{C}_+ \cup (\Sigma(q|_S) \setminus \{0\})}} (r_\lambda + 2k_\lambda)(-i\lambda), \quad k_\lambda \in \mathbf{N},$$

*where $r_\lambda$ is the dimension of the space of generalized eigenvectors of $F$ in $T^*\mathbf{C}^n$ associated to the eigenvalue $\lambda \in \mathbf{C}$,*

$$\Sigma(q|_S) = \overline{q(S)} \ \textit{and} \ \mathbf{C}_+ = \{z \in \mathbf{C}; \mathrm{Re}\, z > 0\}.$$

*Remarks*.

- The structure of the spectrum of $q^w(x, D_x)$ in the globally elliptic case is known since the work of J. Sjöstrand (1974).
- In the quadratic Kramers–Fokker–Planck case, this result is known (Helffer – Nier, Hérau – Sjöstrand – Stolk, Risken).

**This talk :** work in progress on non-selfadjoint semiclassical operators with double characteristics, when the quadratic approximations at doubly characteristic points are merely partially elliptic along the singular space.

In addition to the classical PDE works, our main source of inspiration is the work by Hérau – Sjöstrand – Stolk (2004) and also the recent work by Hérau – Sjöstrand – H. (2007) on second order differential operators of Kramers–Fokker–Planck type.

# Statement of the results

Let $m \geq 1$ be an order function on $\mathbf{R}^{2n}$, satisfying for some $C_0 > 0$, $N_0 > 0$,

$$m(X) \leq C_0 \langle X - Y \rangle^{N_0} m(Y), \quad X, Y \in \mathbf{R}^{2n}.$$

Associated to $m$ is the symbol space $S(m)$ defined by

$$a \in S(m) \Leftrightarrow \partial^\alpha a(X) = \mathcal{O}_\alpha(1) m(x), \quad \alpha \in \mathbf{N}^{2n}.$$

Let

$$P(x, \xi; h) \sim p(x, \xi) + h p_1(x, \xi) + \dots \quad \text{in} \quad S(m)$$

be such that

$$\mathrm{Re}\, p(X) \geq 0, \quad X = (x, \xi) \in \mathbf{R}^{2n}.$$

Assume that for some $C > 0$,

$$\operatorname{Re} p(X) \geq \frac{m(X)}{C}, \quad |X| \geq C.$$

For $h > 0$ small enough, we introduce the $h$ – Weyl quantization of $P(x, \xi; h)$,

$$P = P^w(x, hD_x; h).$$

The spectrum of $P$ in a fixed neighborhood of $0 \in \mathbf{C}$ is discrete.

Assume that the set

$$\{X \in \mathbf{R}^{2n}; \operatorname{Re} p(X) = 0\}$$

is finite

$$= \{X_1, \ldots X_N\}.$$

Then necessarily

$$\operatorname{Re} p(X) = \mathcal{O}\left((X - X_j)^2\right), \quad X \to X_j, \quad 1 \leq j \leq N,$$

and assume that the same holds for $\operatorname{Im} p$,

$$\operatorname{Im} p(X) = \mathcal{O}\left((X - X_j)^2\right), \quad X \to X_j, \quad 1 \leq j \leq N.$$

Write

$$p(X_j + Y) = q_j(Y) + \mathcal{O}(Y^3), \quad Y \to 0,$$

where $q_j$ is quadratic with $\operatorname{Re} q_j \geq 0$. Let $S_j$ stand for the singular space of $q_j$, $1 \leq j \leq N$.

## Theorem

*(Pravda-Starov – H., 2008) Assume that* $q_j$ *is* **elliptic** *along* $S_j$*, for each* $1 \leq j \leq N$*,*

$$X \in S_j, \quad q_j(X) = 0 \;\Rightarrow\; X = 0.$$

*Then for each* $B > 1$ *and for every fixed neighborhood* $\Omega \subset \mathbf{C}$ *of*

$$\bigcup_{j=1}^{N} \left(p_1(X_j) + \mathrm{Spec}(q_j^w(x, D_x))\right)$$

*there exists* $h_0 > 0$ *and* $C > 0$ *such that for* $|z| \leq B$*,* $z \notin \Omega$*, and* $h \in (0, h_0]$*, we have*

$$||\,(P - hz)\,u\,||_{L^2} \geq \frac{h}{C}||\,u\,||_{L^2}, \quad u \in \mathcal{S}(\mathbf{R}^n).$$

*Remarks*.

- We get the same estimate as in the quadratic case, when $P = q^w(x, hD_x)$, with $q$ quadratic, elliptic along $S$.
- In the case when $q_j$ are globally elliptic, $1 \leq j \leq N$, this result is essentially well-known (J. Sjöstrand).
- For Kramers–Fokker–Planck type operators, this result was established by Hérau – Sjöstrand – Stolk.
- For $m = 1$, say, the result implies that for $z \in \mathbf{C}$ with $|z| \leq B$ as in the theorem,
$$(P - hz)^{-1} = \mathcal{O}\left(\frac{1}{h}\right) : L^2 \to L^2.$$
Following the methods of Hérau – Sjöstrand – Stolk, one can go further and compute $\mathrm{Spec}(P)$ for $|z| < Bh$, modulo $\mathcal{O}(h^\infty)$.

**Example.** Let $q = q(x', \xi')$ be the quadratic form defined in (1.1) and let $\widetilde{q} = \widetilde{q}(x'', \xi'')$ be a real-valued elliptic quadratic form in another group of symplectic variables $(x'', \xi'')$. Then the quadratic form

$$Q(x', x'', \xi', \xi'') = q(x', \xi') + i\widetilde{q}(x'', \xi'')$$

is elliptic along the associated singular space

$$S = \{(x', x'', \xi', \xi''); x' = \xi' = 0\}.$$

# Ideas of the proof

Take $m = 1$ and assume for simplicity that $N = 1$ and that $X_1 = (0,0) \in T^*\mathbf{R}^n$. Write

$$p(X) = q(X) + \mathcal{O}(X^3), \quad X \to 0,$$

where $q$ is elliptic when restricted to $S$. It follows then that $S$ is symplectic.

We have the $F$ – invariant decomposition

$$T^*\mathbf{R}^n = S^\sigma \oplus S,$$

with linear symplectic coordinates $(x', \xi') \in S^\sigma$, $(x'', \xi'') \in S$, so that

$$q(x,\xi) = q_1(x',\xi') + iq_2(x'',\xi''), \quad X = (x,\xi) = (x',x'',\xi',\xi'').$$

Here $q_2$ is elliptic and real-valued. The quadratic form $q_1$ enjoys the following **dynamical property** : for each $T > 0$,

$$\frac{1}{T}\int_0^T \mathrm{Re}\, q_1 \circ \exp(tH_{\mathrm{Im}\, q_1})\, dt > 0.$$

It follows that the flow average

$$\langle \mathrm{Re}\, p \rangle_{T,\mathrm{Im}\, p} = \frac{1}{T}\int_0^T \mathrm{Re}\, p \circ \exp(tH_{\mathrm{Im}\, p})\, dt$$

satisfies

$$\langle \mathrm{Re}\, p \rangle_{T,\mathrm{Im}\, p}(X) = \widetilde{q}(x', \xi') + \mathcal{O}(X^3),$$

where the quadratic form

$$\widetilde{q}(x', \xi') > 0.$$

We shall introduce a weight corresponding to the procedure of averaging along the $H_{\mathrm{Im}\, p}$ – flow in a small neighborhood of 0.

Let $g \in C^\infty([0,\infty);[0,1])$ be decreasing and such that

$$g(t) = 1, \quad t \in [0,1], \quad g(t) = t^{-1}, \quad t \geq 2.$$

Let

$$(\mathrm{Re}\, p)_\varepsilon (X) = g\left(\frac{|X|^2}{\varepsilon}\right) \mathrm{Re}\, p(X), \quad \varepsilon > 0.$$

Then

$$(\mathrm{Re}\, p)_\varepsilon (X) = \mathcal{O}(\varepsilon).$$

Set, for $T > 0$

$$G_\varepsilon = -\int J\left(-\frac{t}{T}\right)(\mathrm{Re}\, p)_\varepsilon \circ \exp(tH_{\mathrm{Im}\, p})\, dt.$$

Here $J$ is the compactly supported piecewise affine function solving

$$J'(t) = \delta(t) - 1_{[-1,0]}(t).$$

Then $G_\varepsilon = \mathcal{O}(\varepsilon)$ satisfies

$$H_{\mathrm{Im}\, p} G_\varepsilon = \langle (\mathrm{Re}\, p)_\varepsilon \rangle_{T, \mathrm{Im}\, p} - (\mathrm{Re}\, p)_\varepsilon$$

Associated to $G_\varepsilon$ we have the IR–manifold

$$\Lambda_{\delta,\varepsilon} = \{X + i\delta H_{G_\varepsilon}(X);\, X \in T^*\mathbf{R}^n\}, \quad 0 < \delta \ll 1.$$

The distorted symbol

$$p|_{\Lambda_{\delta,\varepsilon}} = p\,(X + i\delta H_{G_\varepsilon}(X))$$

satisfies

$$\mathrm{Re}\,\left(p|_{\Lambda_{\delta,\varepsilon}}\right) = \mathrm{Re}\,p + \delta H_{\mathrm{Im}\,p} G_\varepsilon + \mathcal{O}(\delta^2\,|\nabla G_\varepsilon|^2),$$

and

$$\mathrm{Im}\,\left(p|_{\Lambda_{\delta,\varepsilon}}\right) = \mathrm{Im}\,p + \mathcal{O}(\delta\,|\nabla G_\varepsilon|).$$

Using the ellipticity of

$$\mathrm{Re}\left(p|_{\Lambda_{\delta,\varepsilon}}\right) \quad \text{along} \quad S^{\sigma}$$

and the ellipticity of

$$\mathrm{Im}\ \left(p|_{\Lambda_{\delta,\varepsilon}}\right) \quad \text{along} \quad S,$$

we obtain that for $X \in T^*\mathbf{R}^n$ in a small but fixed neighborhood of 0,

$$|p\left(X + i\delta H_{G_\varepsilon}(X)\right)| \geq \frac{\delta}{C}\min\left(|X|^2, \varepsilon\right), \quad C > 1,$$

for $\delta \in (0, \delta_0]$ and $\varepsilon \in (0, \varepsilon_0]$, with $\delta_0 > 0$ and $\varepsilon_0 > 0$ sufficiently small.

More precisely, in the entire region $|X| \geq \sqrt{\varepsilon}$, we have for some $C > 1$,

$$\mathrm{Re}\left(\left(1 - \frac{\delta\varepsilon}{C\,|X|^2}\right) p\left(X + i\delta H_{G_\varepsilon}(X)\right)\right) \geq \frac{\delta\varepsilon}{C}.$$

Associated to $\Lambda_{\delta,\varepsilon}$ is the Hilbert space $H(\Lambda_{\delta,\varepsilon})$ of functions that are microlocally

$$\mathcal{O}\left(\exp\left(\frac{G_\varepsilon}{h}\right)\right)$$

in the $L^2$–sense.

We shall take $\varepsilon = Ah$, where $A$ is a constant. Then we have

$$||\cdot||_{H(\Lambda_{\delta,\varepsilon})} \sim ||\cdot||_{L^2},$$

uniformly as $h \to 0$, for each fixed $A > 1$.

When proving our theorem, we work in the weighted space $H(\Lambda_{\delta,\varepsilon})$.

- In a $\sqrt{\varepsilon}$ – neighborhood of 0 we use the quadratic approximation of $p|_{\Lambda_{\delta,\varepsilon}}$, which now becomes globally elliptic.
- Away from such a neighborhood we have a lower bound for $p|_{\Lambda_{\delta,\varepsilon}}$ and we use a version of the sharp Gårding inequality applied to a rescaled symbol. It is here that we need to choose $A$ sufficiently large.

# Some future work/work in progress

- Study the behavior of the semiclassical propagator $e^{-tP/h}$ as $t \to \infty$, $h \to 0$, in relation to the low lying eigenvalues of $P$.
- Analyze the subelliptic case under the assumption that $S = \{0\}$. (Cf with the recent work by K. Pravda-Starov).
- Study resolvent estimates that are polynomial in $\frac{1}{h}$ further away from the boundary of $\mathbf{C}_+ = \{z \in \mathbf{C}; \mathrm{Re}\, z > 0\}$.
- Investigate the behavior of the higher eigenvalues of $P$ in the analytic case.