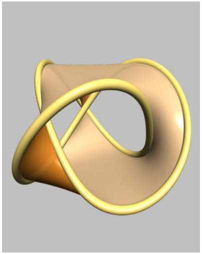
Finite type invariants of surfaces bounding links in 3-space



Michael Eisermann

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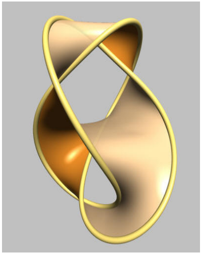
January 10, 2009



Knots in Washington XXVII, January 9–11, 2009

Overview

1. Definition and easy examples
2. The Jones polynomial of ribbon links
3. Finite type theory of surfaces in $\mathbb{R}^3$
4. Open questions

References:

The Jones polynomial of ribbon links,
Geometry & Topology 13 (2009), 623–660

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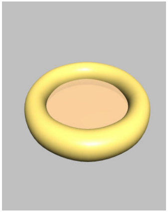
in preparation

Overview

Surfaces in 3-space

Embedded surfaces bounding knots or links:



(a) trivial knot, $\bigcirc$

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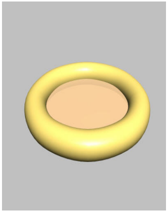

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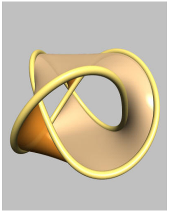

(b) trefoil knot, 3_1

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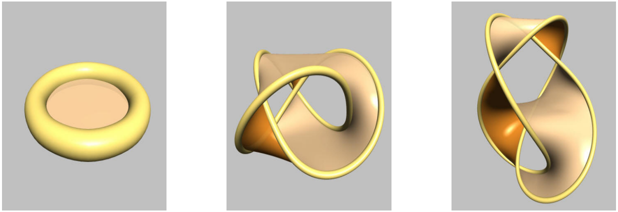

(a) trivial knot, $\bigcirc$ (b) trefoil knot, 3_1 (c) figure eight, 4_1

SeifertView, Jarke van Wijk, TUE

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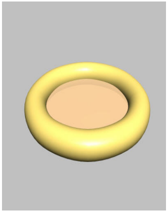

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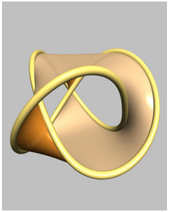

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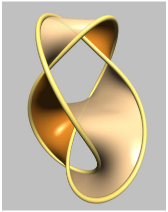

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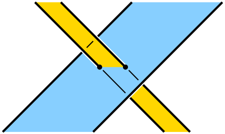

(d) ribbon singularity

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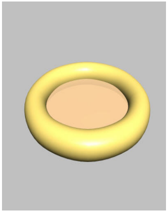
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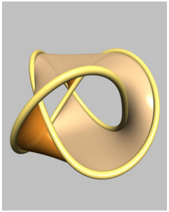
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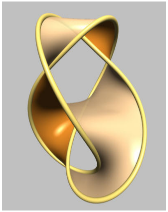
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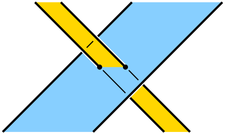
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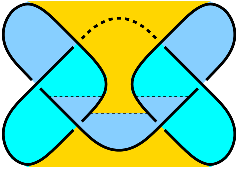
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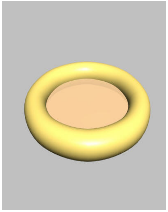

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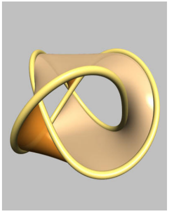

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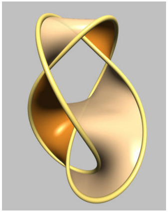

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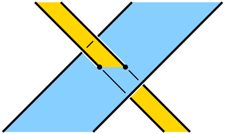

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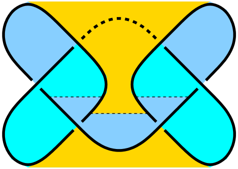

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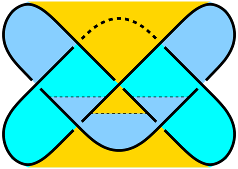

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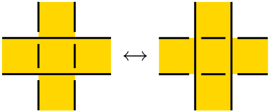

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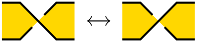
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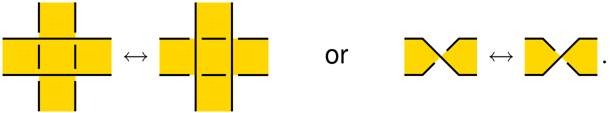

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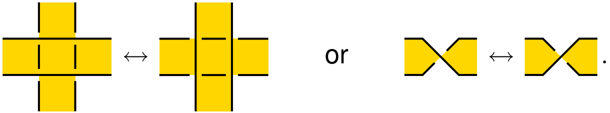

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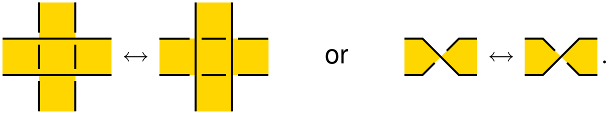

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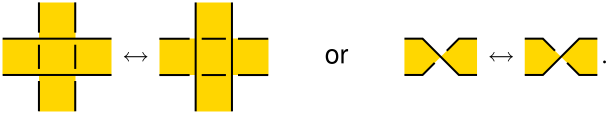

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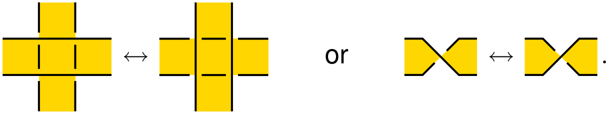

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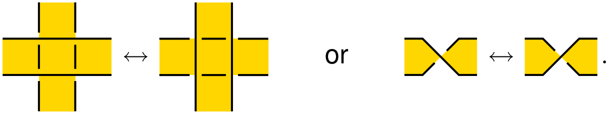

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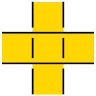
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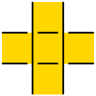
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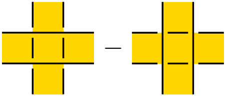

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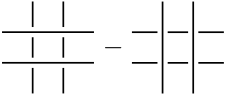

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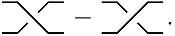

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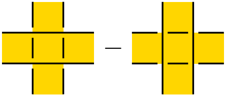

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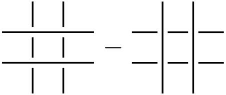

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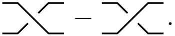

These are (telescopic sums of) crossing changes of links. □

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It is not of finite type in the sense of Vassiliev–Goussarov.

Overview

The slice-ribbon problem (Fox 1962)

A link $L \subset \mathbb{R}^3$ bounds an immersed ribbon surface $\Sigma \looparrowright \mathbb{R}^3$ iff it bounds a smoothly embedded surface $\Sigma \hookrightarrow \mathbb{R}^4_+$ without local minima.

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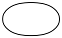

$h = 0$

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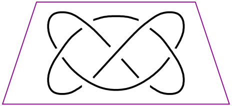

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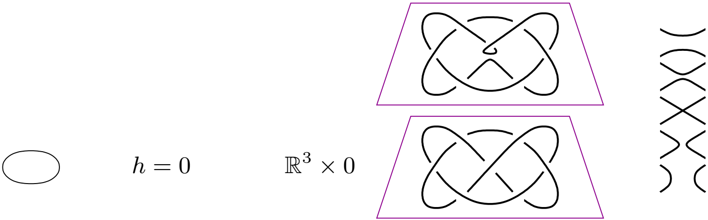

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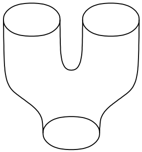
abstract surface

surface embedded in $\mathbb{R}^4_+$



saddle point

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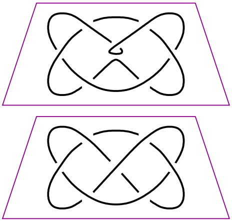

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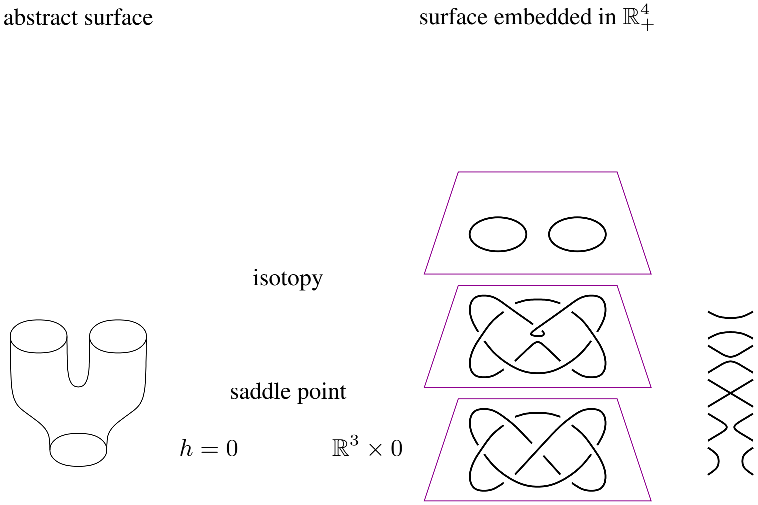

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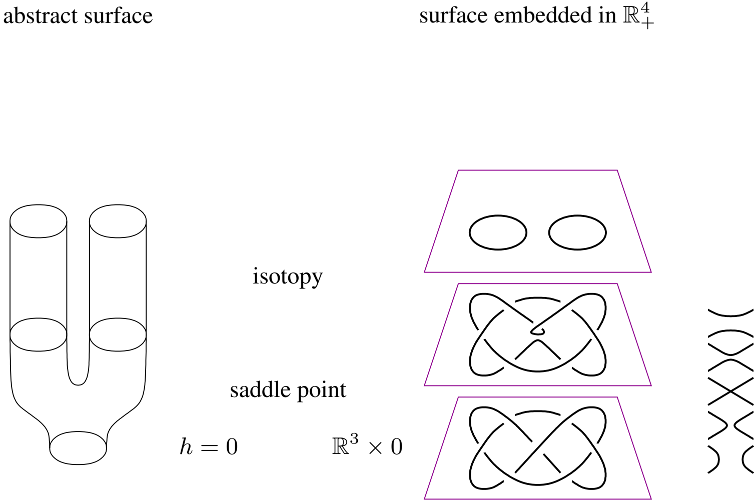

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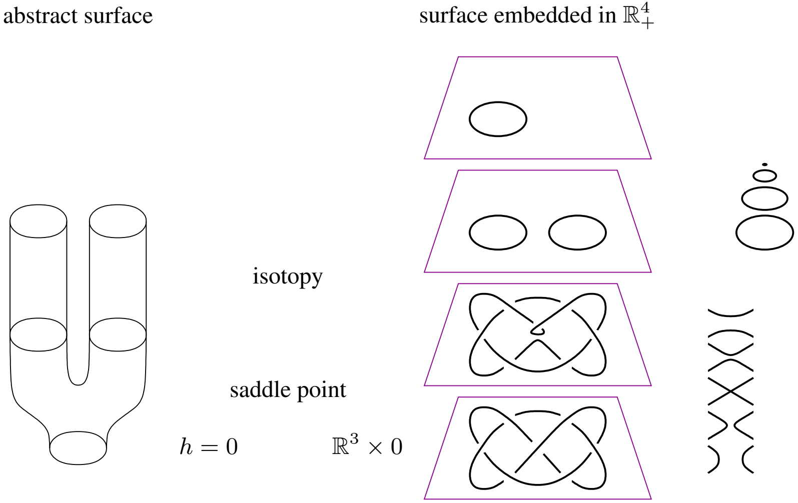

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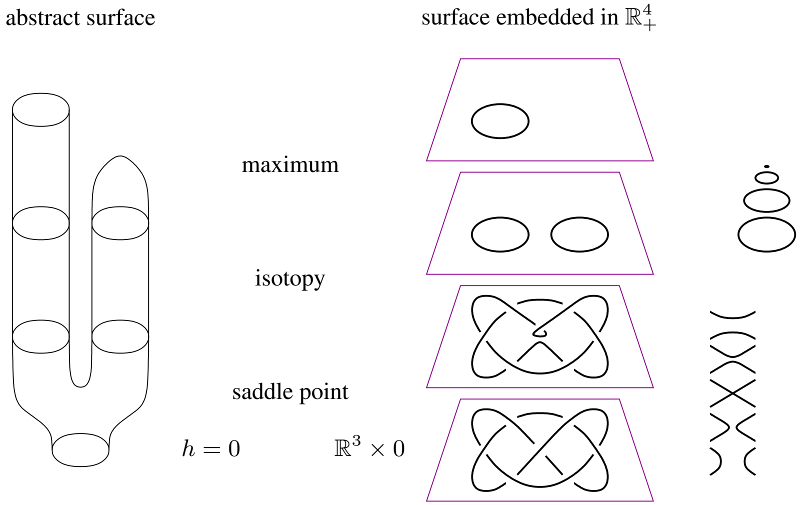

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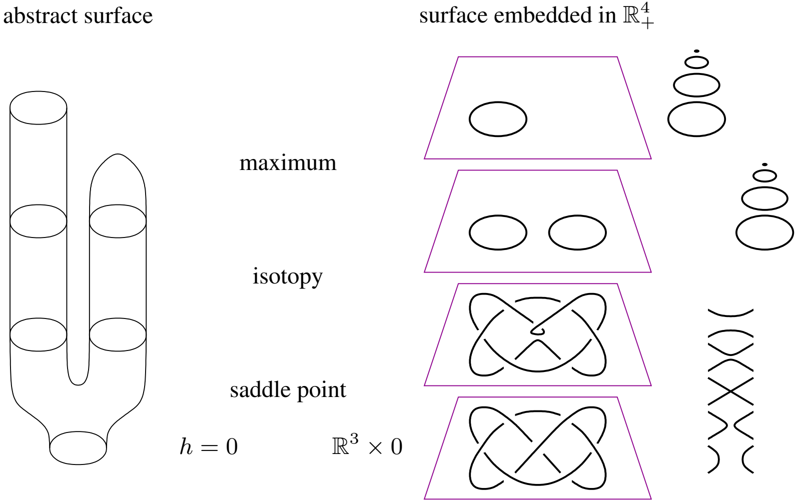

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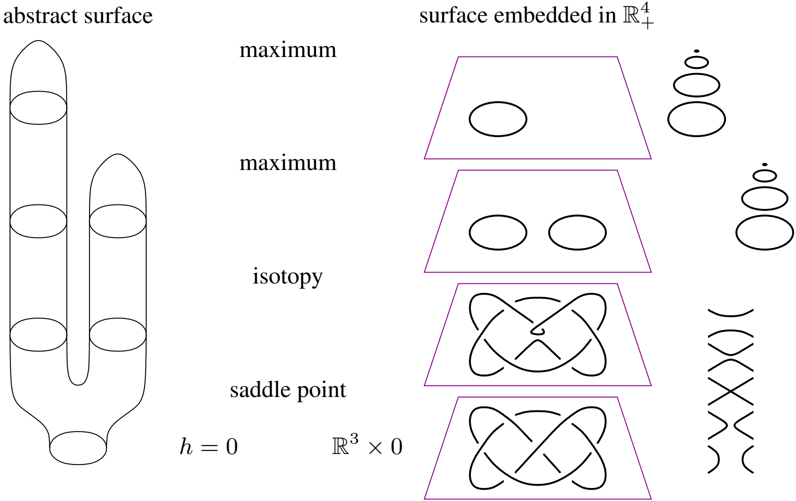

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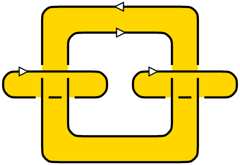
$S =$ 

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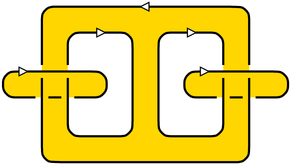
$S' =$ 

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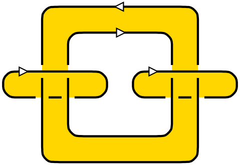
$S =$ 

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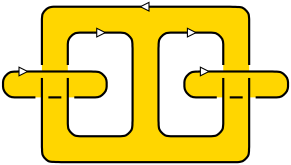
$S' =$ 

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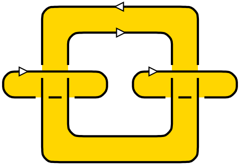
$S =$ 

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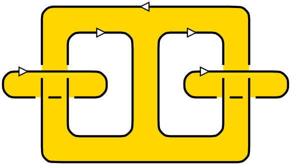
$S' =$ 

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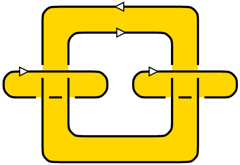
$S =$ 

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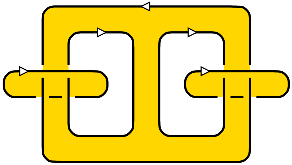
$S' =$ 

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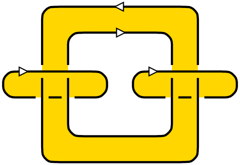
$S =$ 

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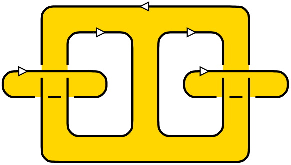
$S' =$ 

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Conjecture: This result generalizes to HOMFLYPT. (N prime)

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Example: Is Scott Morrison's 2-component link ribbon? slice?

Expansion into finite type invariants

Theorem (Birman–Lin 1993)

Expand $V(L) = \sum_{k=0}^{\infty} v_k(L) \cdot h^k$ *in* $q = \exp(h/2) = 1 + h/2 + \dots$.

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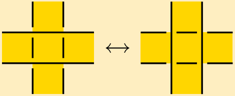

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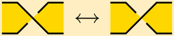.

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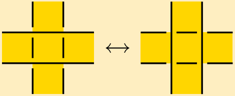

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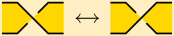

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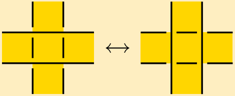

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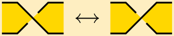.

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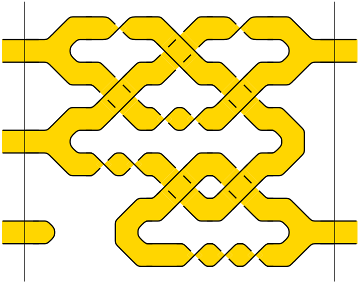
Overview

The category of entangled surfaces

Consider the category of embedded surfaces:



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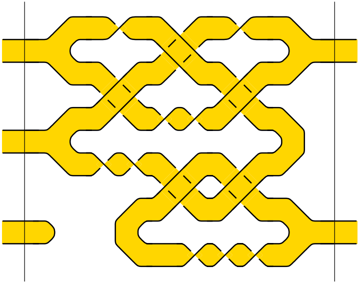

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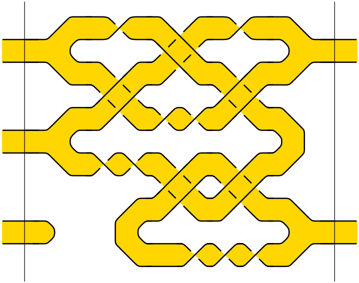

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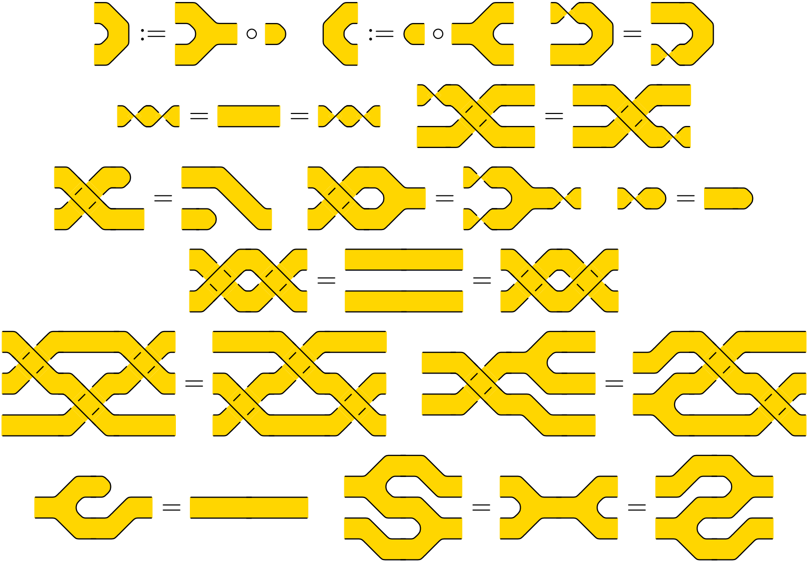
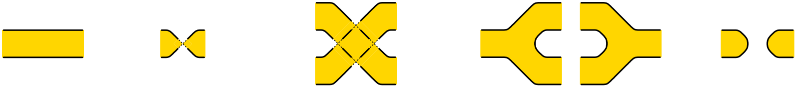
For ribbon immersions $\Sigma \looparrowright \mathbb{R}^3$ the construction is similar but longer.

Isotopy relations



Abstract surfaces

Category generated by abstract surface pieces:



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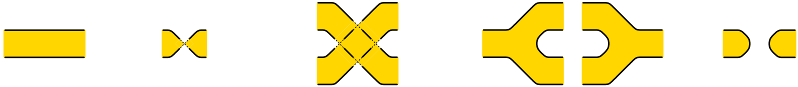

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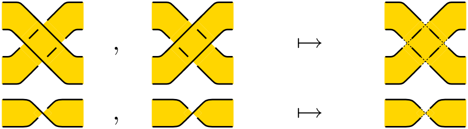
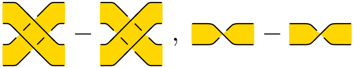


Chord diagrams on surfaces

I-adic filtration generated by band crossing changes:

$$I = \Big($$

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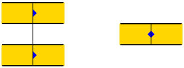

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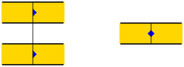

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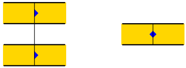

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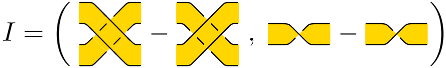

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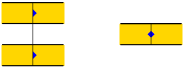

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Is the quotient finite dimensional in each degree?

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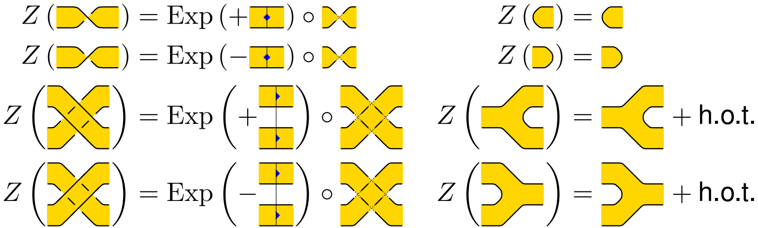

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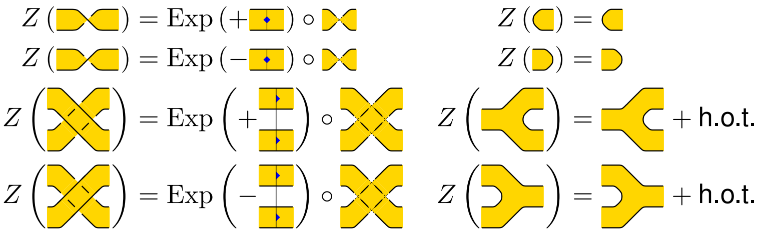

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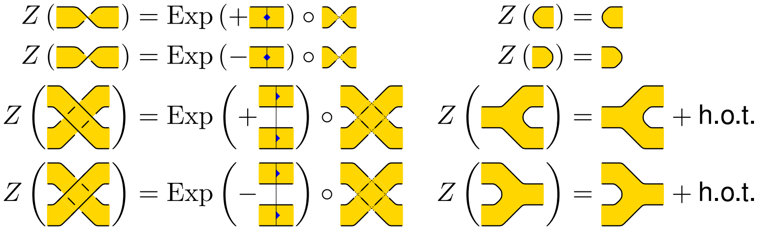

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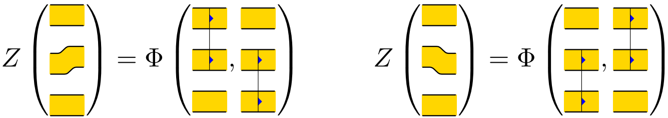

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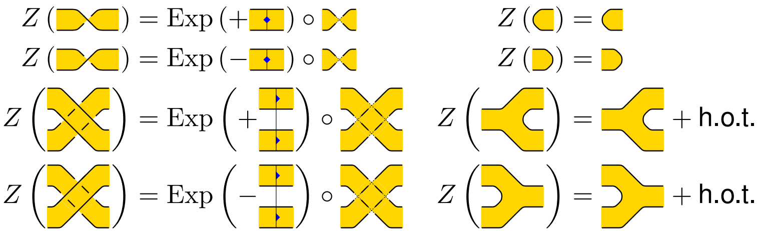

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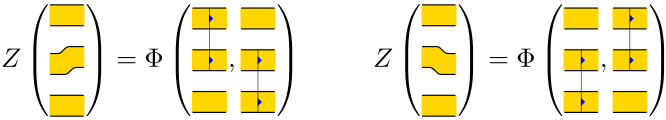

Question

Do all isotopy relations hold? If not yet, how can we arrange this?

Overview

Open questions

Jones polynomial:

- Does the Jones nullity equal the Seifert nullity?
- Generalization from Jones to HOMFLYPT? to Kauffman?
- Is this approach really 3-dimensional? or rather 4-dimensional?
- Interpretation in Khovanov homology? Spectral sequence?

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- Chord diagrams modulo relations: finite dimensional?
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Thank you for your attention!

www-fourier.ujf-grenoble.fr/˜eiserm

Overview

5 Appendix: two nice and simple proofs
- Jones nullity: lower bound
- Surface invariants of finite type

Jones nullity: lower bound

Proposition [G&T 2009]

If a link $L \subset \mathbb{R}^3$ bounds a ribbon surface $S \subset \mathbb{R}^3$ of positive Euler characteristic n, then $V(L)$ is divisible by $V(\bigcirc^n) = (q^+ + q^-)^{n-1}$.

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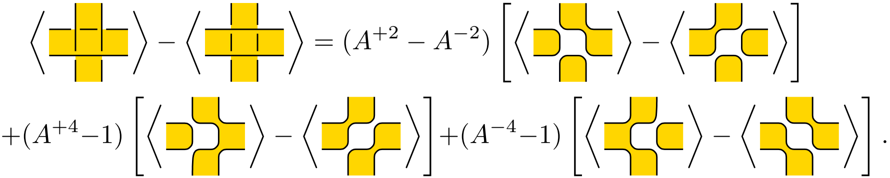

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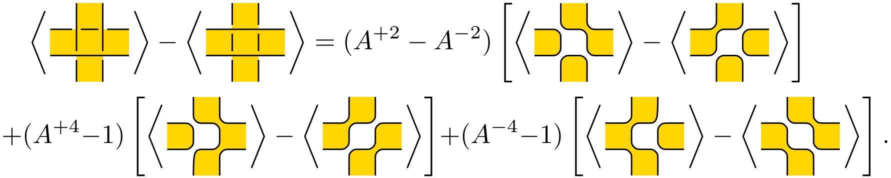

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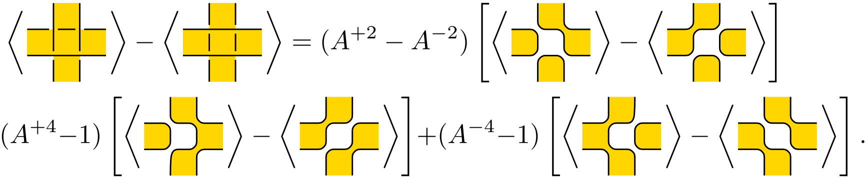

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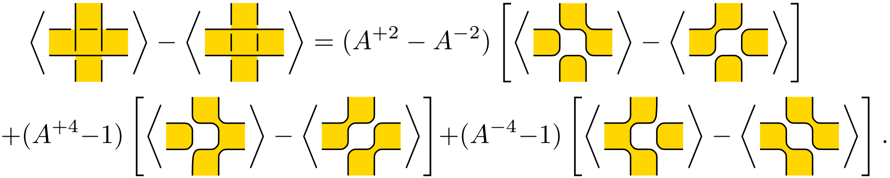

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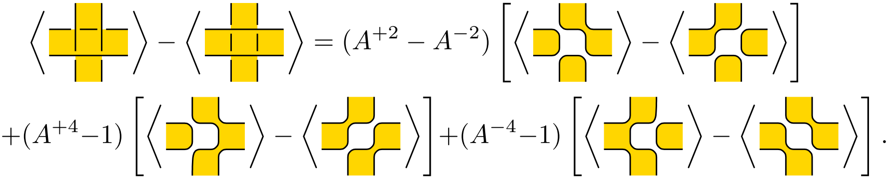

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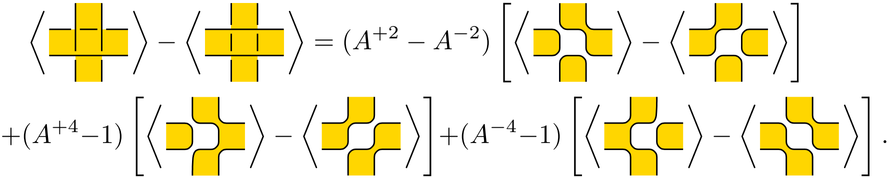

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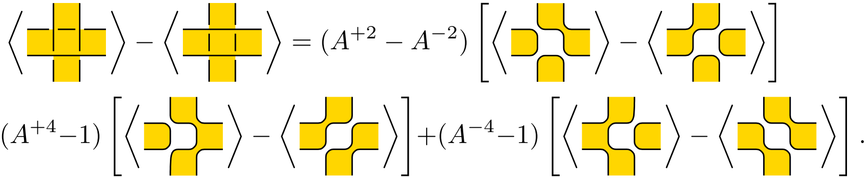

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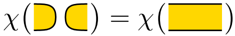
We conclude by induction using  $+ 1$. □

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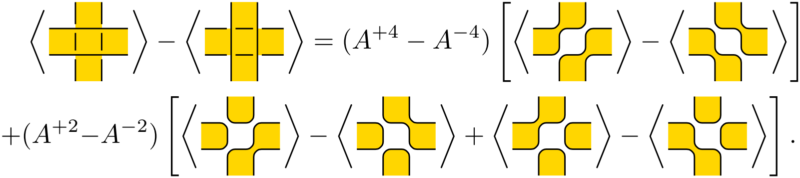

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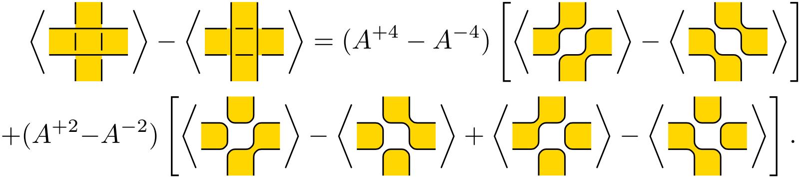

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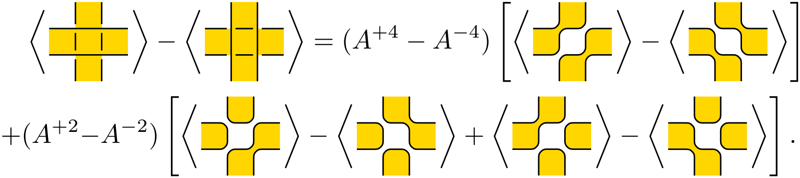

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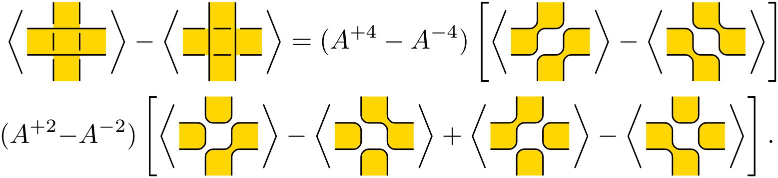

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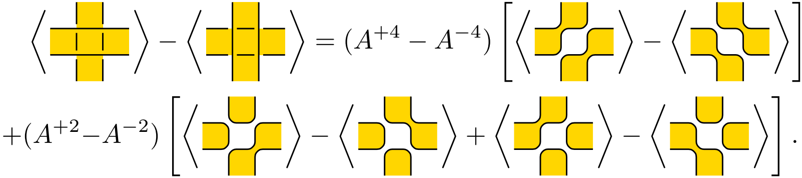

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