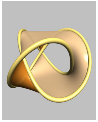
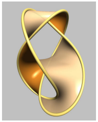
Finite type invariants of surfaces bounding links in 3-space



Michael Eisermann

Institut Fourier, Université Grenoble
www-fourier.ujf-grenoble.fr/~eiserm

January 10, 2009



Knots in Washington XXVII, January 9–11, 2009

Overview

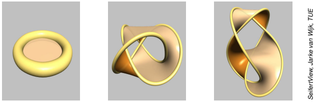
References:

The Jones polynomial of ribbon links,
Geometry & Topology 13 (2009), 623–660

Finite type invariants of surfaces bounding links in 3-space,
in preparation

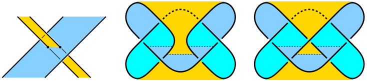
Surfaces in 3-space

Embedded surfaces bounding knots or links:



(a) trivial knot, ◯ (b) trefoil knot, 3_1 (c) figure eight, 4_1

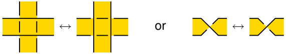
Immersed surfaces having only ribbon singularities:



(d) ribbon singularity (e) $3_1 \sharp 3_1^*$ (f) 6_1

Surface invariants of finite type

Set $\mathscr{S} = \{F\colon \Sigma \looparrowright \mathbb{R}^3\}/\text{isotopy}$ and consider *band crossing changes:*



Let $F \leftrightarrow F_X$ be obtained by changing a family X of crossings.

Definition

A surface invariant $v\colon \mathscr{S} \to A$ is of degree $\leq m$ if

$$\sum_{Y \subset X} (-1)^{|Y|}\, v(F_Y) = 0 \quad \text{for all } X \text{ with } |X| > m.$$

v is of degree < 0 $\iff$ $v = 0$,
v is of degree ≤ 0 $\iff$ v is "constant",
v is of degree ≤ 1 $\iff$ v is "at most linear",
v is of degree ≤ 2 $\iff$ v is "at most quadratic", etc.

Link invariants of finite type

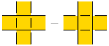
Example

The map $\chi\colon \mathscr{S} \to \mathbb{Z}$, $S \mapsto \chi(\Sigma)$, is a surface invariant of degree 0.

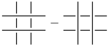
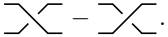
Proposition

If $\mathscr{L} \xrightarrow{v} A$, $L \mapsto v(L)$, *is a link invariant of degree* $\le m$, *then* $\mathscr{S} \xrightarrow{\partial} \mathscr{L} \xrightarrow{v} A$, $S \mapsto v(\partial S)$, *is a surface invariant of degree* $\le m$.

Proof. Consider band crossing changes:

 and 

Forgetting the surfaces, we simply obtain

 and 

These are (telescopic sums of) crossing changes of links. □

Seifert matrix and determinant

Assume the surface Σ to be compact, oriented and connected:
$H_0(\Sigma) = \mathbb{Z}$ and $H_2(\Sigma) = 0$ and $H_1(\Sigma)$ is free of rank $m = 1 - \chi(\Sigma)$.
To each embedding $F\colon \Sigma \hookrightarrow \mathbb{R}^3$ we associate its Seifert form

$$\theta_F\colon H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}, \qquad \theta_F(a,b) = \operatorname{lk}(F^{\uparrow}(a), F^{\downarrow}(b)).$$

Observation

The coefficients of θ_F are of degree ≤ 1.

The determinant of F is defined by $\det(F) := \det\big[-i(\theta_F + \theta_F^*)\big]$.
It is a homogeneous polynomial of degree m in the coefficients of θ_F.

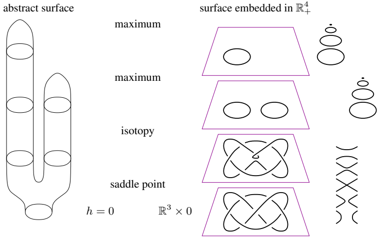
Conclusion

The surface invariant $F \mapsto \det(F)$ is of degree $\le m = 1 - \chi(\Sigma)$.

⚠ The invariant $\det(F)$ depends only on the link $L = F(\partial\Sigma)$.
Its degree $m = 1 - \chi(\Sigma)$ depends on the abstract surface Σ.
It is not of finite type in the sense of Vassiliev–Goussarov.

The slice-ribbon problem (Fox 1962)

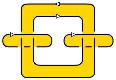
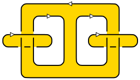
A link $L \subset \mathbb{R}^3$ bounds an immersed ribbon surface $\Sigma \looparrowright \mathbb{R}^3$ iff it bounds a smoothly embedded surface $\Sigma \hookrightarrow \mathbb{R}^4_+$ without local minima.



An integrality property of the Jones polynomial

Proposition [G&T 2009]

If a link $L \subset \mathbb{R}^3$ *bounds a ribbon surface* $S \subset \mathbb{R}^3$ *of positive Euler characteristic* n, *then* $V(L)$ *is divisible by* $V(\bigcirc^n) = (q^+ + q^-)^{n-1}$.

$S =$  $S' =$ 

$\chi(S) = 1 + 1 + 0 = 2$ $\qquad$ $\chi(S') = 1 + 1 - 1 = 1$

We find $\quad V(L) = (q^+ + q^-) \cdot \big(q^6 - q^4 + 2q^2 + 2q^{-2} - q^{-4} + q^{-6}\big)$.
Hence L bounds ribbon surfaces with $\chi \le 2$ but not with $\chi \ge 3$.

We find $\quad V(L') = (q^{+1} + q^{+5})^2 \cdot (q^{-1} + q^{-5})^2$.
Hence L' bounds ribbon surfaces with $\chi \le 1$ but not with $\chi \ge 2$.

The nullity of the Jones polynomial

Definition

The Jones nullity $\operatorname{null} V(L)$ is the order of the zero at $q = i$.

We have $V(L) = (q^+ + q^-)^{\operatorname{null}} \tilde{V}$ with $\tilde{V}_{(q \mapsto i)} \neq 0$.

Lemma

Every n-component link L satisfies $0 \leq \operatorname{null} V(L) \leq n - 1$.

This corresponds to the Seifert nullity $\operatorname{null}(L) = \operatorname{null}(\theta + \theta^*)$.

Question

Do we have $\operatorname{null} V(L) = \operatorname{null}(L)$ for all links L?

Partial answer:

Theorem [G&T 2009]

For every n-component ribbon link we have $\operatorname{null} V(L) = n - 1$.

Conjecture: This result generalizes to HOMFLYPT. (N prime)

The Jones polynomial of ribbon links

Theorem [G&T 2009]

Let $L = K_1 \cup \cdots \cup K_n$ be a ribbon link.
Then $V(L)$ is divisible by $V(\bigcirc^n) = (q^+ + q^-)^{n-1}$. The value

$$\det V(L) := [V(L)/V(\bigcirc^n)]_{(q \mapsto i)}$$

satisfies the congruence

$$\det V(L) \equiv \det(K_1) \cdots \det(K_n) \mod 32.$$

In particular we obtain $\det V(L) \equiv 1 \mod 8$.

$\Rightarrow$ Analogy with Seifert nullity: here $\operatorname{null} V(L) = \operatorname{null}(L)$.

$\Rightarrow$ Analogy with the Arf invariant: here $\det(K) = 1$ modulo 8.

$\Rightarrow$ Obstruction for links to be ribbon or slice?

Example: Is Scott Morrison's 2-component link ribbon? slice?

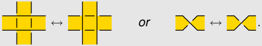
Expansion into finite type invariants

Theorem (Birman–Lin 1993)

Expand $V(L) = \sum_{k=0}^{\infty} v_k(L) \cdot h^k$ *in* $q = \exp(h/2) = 1 + h/2 + \ldots$.
Then $L \mapsto v_k(L)$ is of degree $\leq k$ w.r.t. crossing changes:

Theorem [G&T 2009]

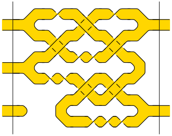
Expand $V(L) = \sum_{k=0}^{\infty} d_k(L) \cdot h^k$ *in* $q = i\exp(h/2) = i + ih/2 + \ldots$.
Then $S \mapsto d_k(\partial S)$ is of degree $\leq k + 1 - \chi(S)$ w.r.t. band crossing changes:



⚠ $d_k(L)$ is not of finite type in the sense of Vassiliev–Goussarov.
In particular not $d_0(L) = V(L)_{q \mapsto i} = \det(L) = \det\big[-i(\theta + \theta^*)\big]$.

The category of entangled surfaces

Consider the category of embedded surfaces:

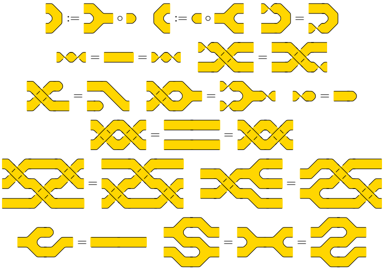


Standard generators:

(a) id (b) twists (c) crossings (d) junctions (e) ends

For ribbon immersions $\Sigma \looparrowright \mathbb{R}^3$ the construction is similar but longer.

Isotopy relations



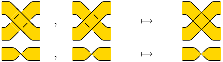
Abstract surfaces

Category generated by abstract surface pieces:

(a) id (b) twist (c) crossing (d) junctions (e) ends

Relations as before (but abstract = non-embedded)

Forgetful functor:



Chord diagrams on surfaces

I-adic filtration generated by band crossing changes:

Encoded by abstract surfaces with chords:

Tensor functor resolving chord diagrams:

This maps abstract surfaces with m chords to I^m/I^{m+1}.

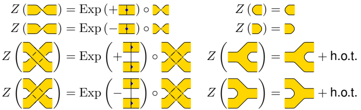
Quotient chord diagrams by the obvious relations induced by isotopy.

Question

Is the quotient finite dimensional in each degree?

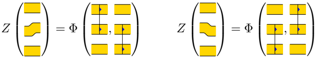
Towards a universal invariant

We wish to define a universal invariant Z as follows:



The naïve construction does not work (same problem as for tangles).

Use non-associative tangles and introduce an associator Φ:



Question

Do all isotopy relations hold? If not yet, how can we arrange this?

Open questions

Jones polynomial:

- Does the Jones nullity equal the Seifert nullity?
- Generalization from Jones to HOMFLYPT? to Kauffman?
- Is this approach really 3-dimensional? or rather 4-dimensional?
- Interpretation in Khovanov homology? Spectral sequence?

Surface invariants of finite type:

- Chord diagrams modulo relations: finite dimensional?
- Examine further examples: HOMFLYPT, Kauffman, . . .
- Understand invariants of low degree.
- Construct a universal invariant.

Thank you for your attention!

www-fourier.ujf-grenoble.fr/˜eiserm

Ribbon surfaces and band diagrams

Let Σ be a compact oriented surface *without closed components*.

Definition

A *ribbon immersion* $F\colon \Sigma \looparrowright \mathbb{R}^3$ has only ribbon singularities.

A *ribbon surface* $S = F(\Sigma)$ is the image of a ribbon immersion F.

A *band diagram* is a planar diagram formed by the following pieces:

Proposition

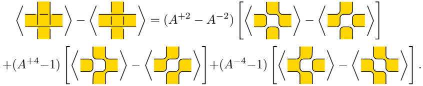
Every ribbon surface S in $\mathbb{R}^3$ can be presented by a band diagram.

Jones nullity: lower bound

Proposition [G&T 2009]

If a link $L \subset \mathbb{R}^3$ bounds a ribbon surface $S \subset \mathbb{R}^3$ of positive Euler characteristic n, then $V(L)$ is divisible by $V(\bigcirc^n) = (q^+ + q^-)^{n-1}$.

Key observation for the induction:



Proof. By hypothesis each component S_i has a boundary.

Thus $\chi(S_i) > 0 \;\Leftrightarrow\; S_i = \bigcirc \;\Leftrightarrow\; \chi(S_i) = 1$.

Induction on the number $r(S)$ of ribbon singularities.

If $r(S) = 0$ then S is embedded and $L = L_0 \sqcup \bigcirc^n$.

If $r(S) \geq 1$ then we employ the above equation.

We conclude by induction using $\chi($$) = \chi($$) + 1$. □

Surface invariants of finite type

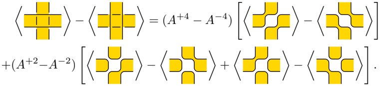
Expand $V(L) = \sum_{k=0}^{\infty} d_k(L) \cdot h^k$ in $q = i\exp(h/2)$.

Proposition [G&T 2009]

The surface invariant $S \mapsto d_k(\partial S)$ is of degree $\leq m := k + 1 - \chi(S)$.

The case $d_0 = \det$ has already been derived from the Seifert matrix.

Key observation for the induction:



Proof. We know that $V(\partial S)$ is divisible by $(q^+ + q^-)^{\chi(S)-1}$.

This means that $d_k(\partial S) = 0$ for $k < \chi(S) - 1$, in which case $m < 0$.

Then $\sum_{Y \subset X} (-1)^{|Y|}\, V(\partial D_Y)$ is divisible by $(q^+ + q^-)^{|X|+\chi(S)-1}$:

Employ the above equation and conclude by induction on $|X|$. □