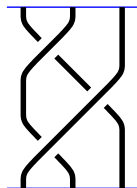
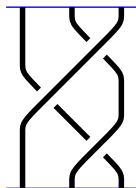


Set-theoretic Yang-Baxter operators and their deformations



Michael Eisermann

Institut Fourier, Université Grenoble
www-fourier.ujf-grenoble.fr/~eiserm



January 5, 2009

AMS–MAA Joint Mathematics Meetings in Washington DC
Special Session on Algebraic Structures in Knot Theory

Overview

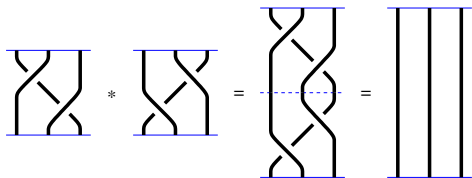
- 1 Braid groups and Yang-Baxter representations
- 2 Yang-Baxter deformations
- 3 Yang-Baxter cohomology of racks
- 4 Conclusion and open questions

Overview

- 1** Braid groups and Yang-Baxter representations
 - Artin's braid group
 - Yang-Baxter representations
 - Set-theoretic operators
- 2 Yang-Baxter deformations
- 3 Yang-Baxter cohomology of racks
- 4 Conclusion and open questions

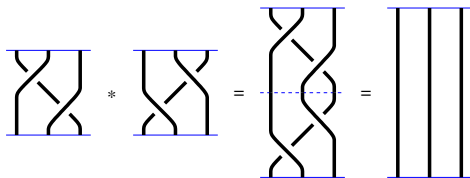
Artin's braid group (1925)

Braids form a group:

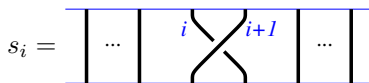


Artin's braid group (1925)

Braids form a group:

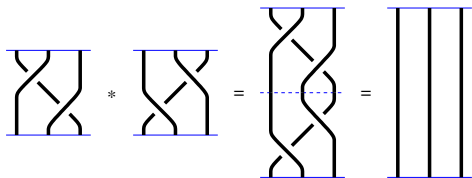


Standard generators:

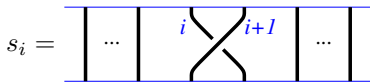


Artin's braid group (1925)

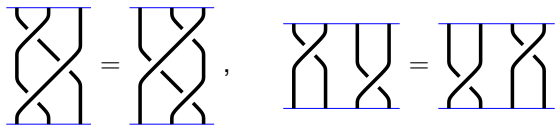
Braids form a group:



Standard generators:

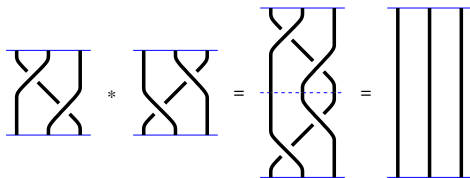


Obvious relations:

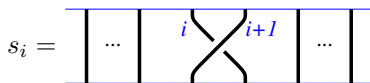


Artin's braid group (1925)

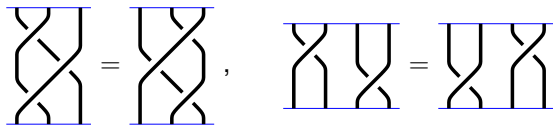
Braids form a group:



Standard generators:



Obvious relations:



Theorem (Artin 1925)

The group B_n of braids on n strands is presented by

$$B_n = \left\langle s_1, \dots, s_{n-1} \mid \begin{array}{ll} s_i s_j s_i = s_j s_i s_j & \text{if } |i - j| = 1 \\ s_i s_j = s_i s_j & \text{if } |i - j| \geq 2 \end{array} \right\rangle.$$

Yang-Baxter representations

Each automorphism $c \in \text{Aut}(E \otimes E)$ acts on $E^{\otimes n}$ as

$$c_i = \begin{array}{cccccccc} & E \otimes E & \otimes & E \otimes E & \otimes & E \otimes E & \otimes & E \otimes E \\ & \text{---} & & \text{---} & & \text{---} & & \text{---} \\ & | & & | & & | & & | \\ c_i = & \dots & & & & & & \dots \\ & | & & | & & | & & | \\ & E \otimes E & \otimes & E \otimes E & \otimes & E \otimes E & \otimes & E \otimes E \end{array}$$

Yang-Baxter representations

Each automorphism $c \in \text{Aut}(E \otimes E)$ acts on $E^{\otimes n}$ as

$$c_i = \begin{array}{c} E \otimes E \otimes E \otimes E \\ \hline \begin{array}{c} \vdots \\ \vdots \\ \text{---} \\ \vdots \\ \vdots \end{array} \\ \hline E \otimes E \otimes E \otimes E \end{array}$$

The braid relation now becomes the Yang-Baxter equation:

$$\begin{array}{c} \begin{array}{c} \vdots \\ \vdots \\ \text{---} \\ \vdots \\ \vdots \end{array} \\ \hline \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \\ \hline \begin{array}{c} \vdots \\ \vdots \\ \text{---} \\ \vdots \\ \vdots \end{array} \end{array} = \begin{array}{c} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \\ \hline \begin{array}{c} \vdots \\ \vdots \\ \text{---} \\ \vdots \\ \vdots \end{array} \\ \hline \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \end{array}$$
$$(c \otimes \text{id})(\text{id} \otimes c)(\text{id} \otimes c) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)$$

Yang-Baxter representations

Each automorphism $c \in \text{Aut}(E \otimes E)$ acts on $E^{\otimes n}$ as

$$c_i = \begin{array}{c} E \otimes E \otimes E \otimes E \\ \hline \begin{array}{c} \vdots \\ \vdots \\ \text{---} \\ \vdots \\ \vdots \end{array} \\ \hline E \otimes E \otimes E \otimes E \end{array}$$

The braid relation now becomes the Yang-Baxter equation:

$$(c \otimes \text{id})(\text{id} \otimes c)(\text{id} \otimes c) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)$$

Corollary (of Artin's theorem)

Every Yang-Baxter operator c induces a braid group representation $\rho_n^c: B_n \rightarrow \text{Aut}(E^{\otimes n})$ given by $s_i \mapsto c_i$.

The Jones polynomial (1984)

Let $E = \mathbb{A}u \oplus \mathbb{A}v$ and thus $E \otimes E = \mathbb{A}uu \oplus \mathbb{A}uv \oplus \mathbb{A}vu \oplus \mathbb{A}vv$.

The Jones polynomial (1984)

Let $E = \mathbb{A}u \oplus \mathbb{A}v$ and thus $E \otimes E = \mathbb{A}uu \oplus \mathbb{A}uv \oplus \mathbb{A}vu \oplus \mathbb{A}vv$.

Theorem (Jones 1984)

For each $q \in \mathbb{A}$ we have a Yang-Baxter operator

$$c(q) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & q^2 & q - q^3 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \xrightarrow{q \mapsto 1} \tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Jones polynomial (1984)

Let $E = \mathbb{A}u \oplus \mathbb{A}v$ and thus $E \otimes E = \mathbb{A}uu \oplus \mathbb{A}uv \oplus \mathbb{A}vu \oplus \mathbb{A}vv$.

Theorem (Jones 1984)

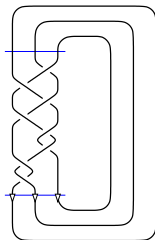
For each $q \in \mathbb{A}$ we have a Yang-Baxter operator

$$c(q) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & q^2 & q - q^3 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \xrightarrow{q \mapsto 1} \tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Over $\mathbb{A} = \mathbb{Z}[q^{\pm}]$ we obtain the Jones polynomial:



$$\begin{array}{ccc} B_n & \xrightarrow{\text{closure}} & \mathcal{L} \\ \rho_n^c \downarrow & & \downarrow V \\ \text{Aut}(E^{\otimes n}) & \xrightarrow{\text{tr}_q} & \mathbb{Z}[q^{\pm}] \end{array}$$



This approach generalizes to quantum invariants.

Braids act on groups

Consider conjugation $a * b := b^{-1}ab$ and $a \bar{*} b := bab^{-1}$ in a group:

Braids act on groups

Consider conjugation $a * b := b^{-1}ab$ and $a \bar{*} b := bab^{-1}$ in a group:

$$(Q1) \quad a * a = a$$

(idempotent)

Braids act on groups

Consider conjugation $a * b := b^{-1}ab$ and $a \bar{*} b := bab^{-1}$ in a group:

$$(Q1) \quad a * a = a \quad \text{(idempotent)}$$

$$(Q2) \quad (a * b) \bar{*} b = (a \bar{*} b) * b = a \quad \text{(left-invertible)}$$

Braids act on groups

Consider conjugation $a * b := b^{-1}ab$ and $a \bar{*} b := bab^{-1}$ in a group:

$$(Q1) \quad a * a = a \quad \text{(idempotent)}$$

$$(Q2) \quad (a * b) \bar{*} b = (a \bar{*} b) * b = a \quad \text{(left-invertible)}$$

$$(Q3) \quad (a * b) * c = (a * c) * (b * c) \quad \text{(self-distributive)}$$

Braids act on groups

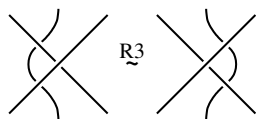
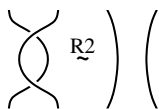
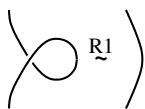
Consider conjugation $a * b := b^{-1}ab$ and $a \bar{*} b := bab^{-1}$ in a group:

(Q1) $a * a = a$ (idempotent)

(Q2) $(a * b) \bar{*} b = (a \bar{*} b) * b = a$ (left-invertible)

(Q3) $(a * b) * c = (a * c) * (b * c)$ (self-distributive)

This corresponds to the three Reidemeister moves:



Braids act on groups

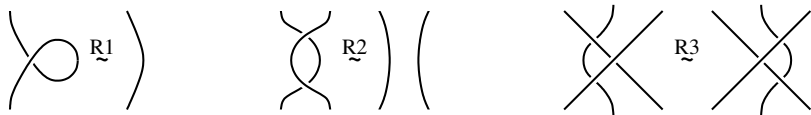
Consider conjugation $a * b := b^{-1}ab$ and $a \bar{*} b := bab^{-1}$ in a group:

$$(Q1) \quad a * a = a \quad \text{(idempotent)}$$

$$(Q2) \quad (a * b) \bar{*} b = (a \bar{*} b) * b = a \quad \text{(left-invertible)}$$

$$(Q3) \quad (a * b) * c = (a * c) * (b * c) \quad \text{(self-distributive)}$$

This corresponds to the three Reidemeister moves:



Definition

Consider a set Q with two operations $*, \bar{*}: Q \times Q \rightarrow Q$.

Braids act on groups

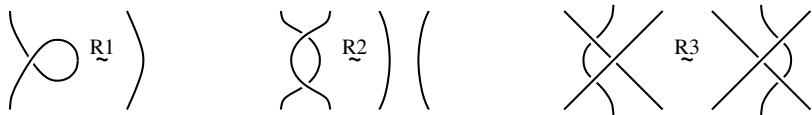
Consider conjugation $a * b := b^{-1}ab$ and $a \bar{*} b := bab^{-1}$ in a group:

$$(Q1) \quad a * a = a \quad (\text{idempotent})$$

$$(Q2) \quad (a * b) \bar{*} b = (a \bar{*} b) * b = a \quad (\text{left-invertible})$$

$$(Q3) \quad (a * b) * c = (a * c) * (b * c) \quad (\text{self-distributive})$$

This corresponds to the three Reidemeister moves:



Definition

Consider a set Q with two operations $*, \bar{*}: Q \times Q \rightarrow Q$.

We call $(Q, *, \bar{*})$ a *quandle* if it satisfies (Q1–Q3),

Braids act on groups

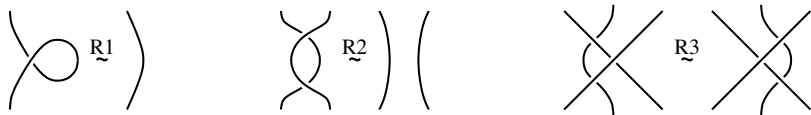
Consider conjugation $a * b := b^{-1}ab$ and $a \bar{*} b := bab^{-1}$ in a group:

(Q1) $a * a = a$ (idempotent)

(Q2) $(a * b) \bar{*} b = (a \bar{*} b) * b = a$ (left-invertible)

(Q3) $(a * b) * c = (a * c) * (b * c)$ (self-distributive)

This corresponds to the three Reidemeister moves:



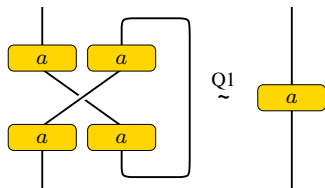
Definition

Consider a set Q with two operations $*, \bar{*}: Q \times Q \rightarrow Q$.

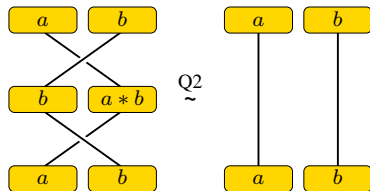
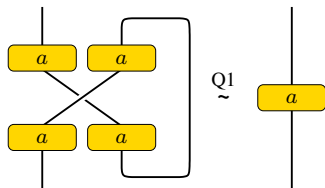
We call $(Q, *, \bar{*})$ a *quandle* if it satisfies (Q1–Q3),

and a *rack* if it satisfies (Q2–Q3).

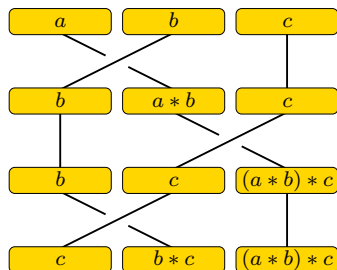
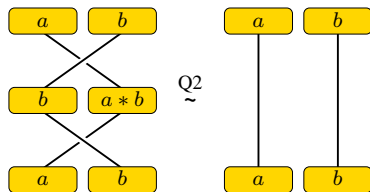
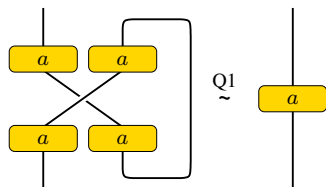
Braids act on racks and quandles



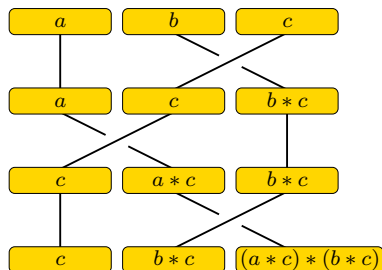
Braids act on racks and quandles



Braids act on racks and quandles



\sim



Set-theoretic Yang-Baxter operators

Consider a set Q and a bijective map $s: Q \times Q \rightarrow Q \times Q$.

Set-theoretic Yang-Baxter operators

Consider a set Q and a bijective map $s: Q \times Q \rightarrow Q \times Q$.

Definition (Drinfel'd 1990)

The *set-theoretic Yang-Baxter equation* is

$$(\text{id} \times s)(s \times \text{id})(\text{id} \times s) = (s \times \text{id})(\text{id} \times s)(s \times \text{id}).$$

Set-theoretic Yang-Baxter operators

Consider a set Q and a bijective map $s: Q \times Q \rightarrow Q \times Q$.

Definition (Drinfel'd 1990)

The *set-theoretic Yang-Baxter equation* is

$$(\text{id} \times s)(s \times \text{id})(\text{id} \times s) = (s \times \text{id})(\text{id} \times s)(s \times \text{id}).$$

Here $s(x, y) = (x \triangleright y, x \triangleleft y)$ defines two operations $\triangleright, \triangleleft: Q \times Q \rightarrow Q$.

Set-theoretic Yang-Baxter operators

Consider a set Q and a bijective map $s: Q \times Q \rightarrow Q \times Q$.

Definition (Drinfel'd 1990)

The *set-theoretic Yang-Baxter equation* is

$$(\text{id} \times s)(s \times \text{id})(\text{id} \times s) = (s \times \text{id})(\text{id} \times s)(s \times \text{id}).$$

Here $s(x, y) = (x \triangleright y, x \triangleleft y)$ defines two operations $\triangleright, \triangleleft: Q \times Q \rightarrow Q$.
For racks/quandles the left operation is trivial: $x \triangleright y = y$.

Set-theoretic Yang-Baxter operators

Consider a set Q and a bijective map $s: Q \times Q \rightarrow Q \times Q$.

Definition (Drinfel'd 1990)

The *set-theoretic Yang-Baxter equation* is

$$(\text{id} \times s)(s \times \text{id})(\text{id} \times s) = (s \times \text{id})(\text{id} \times s)(s \times \text{id}).$$

Here $s(x, y) = (x \triangleright y, x \triangleleft y)$ defines two operations $\triangleright, \triangleleft: Q \times Q \rightarrow Q$.

For racks/quandles the left operation is trivial: $x \triangleright y = y$.

In the quantum case (transposition) both operations are trivial.

Set-theoretic Yang-Baxter operators

Consider a set Q and a bijective map $s: Q \times Q \rightarrow Q \times Q$.

Definition (Drinfel'd 1990)

The *set-theoretic Yang-Baxter equation* is

$$(\text{id} \times s)(s \times \text{id})(\text{id} \times s) = (s \times \text{id})(\text{id} \times s)(s \times \text{id}).$$

Here $s(x, y) = (x \triangleright y, x \triangleleft y)$ defines two operations $\triangleright, \triangleleft: Q \times Q \rightarrow Q$.

For racks/quandles the left operation is trivial: $x \triangleright y = y$.

In the quantum case (transposition) both operations are trivial.

Remark

Every set-theoretic Yang-Baxter operator $s: Q \times Q \rightarrow Q \times Q$ induces a Yang-Baxter operator $c: \mathbb{A}Q \otimes \mathbb{A}Q \rightarrow \mathbb{A}Q \otimes \mathbb{A}Q$ over the ring \mathbb{A} .

Set-theoretic Yang-Baxter operators

Consider a set Q and a bijective map $s: Q \times Q \rightarrow Q \times Q$.

Definition (Drinfel'd 1990)

The *set-theoretic Yang-Baxter equation* is

$$(\text{id} \times s)(s \times \text{id})(\text{id} \times s) = (s \times \text{id})(\text{id} \times s)(s \times \text{id}).$$

Here $s(x, y) = (x \triangleright y, x \triangleleft y)$ defines two operations $\triangleright, \triangleleft: Q \times Q \rightarrow Q$.

For racks/quandles the left operation is trivial: $x \triangleright y = y$.

In the quantum case (transposition) both operations are trivial.

Remark

Every set-theoretic Yang-Baxter operator $s: Q \times Q \rightarrow Q \times Q$ induces a Yang-Baxter operator $c: \mathbb{A}Q \otimes \mathbb{A}Q \rightarrow \mathbb{A}Q \otimes \mathbb{A}Q$ over the ring \mathbb{A} .

It is then natural to study deformations over \mathbb{A} :

Set-theoretic Yang-Baxter operators

Consider a set Q and a bijective map $s: Q \times Q \rightarrow Q \times Q$.

Definition (Drinfel'd 1990)

The *set-theoretic Yang-Baxter equation* is

$$(\text{id} \times s)(s \times \text{id})(\text{id} \times s) = (s \times \text{id})(\text{id} \times s)(s \times \text{id}).$$

Here $s(x, y) = (x \triangleright y, x \triangleleft y)$ defines two operations $\triangleright, \triangleleft: Q \times Q \rightarrow Q$.

For racks/quandles the left operation is trivial: $x \triangleright y = y$.

In the quantum case (transposition) both operations are trivial.

Remark

Every set-theoretic Yang-Baxter operator $s: Q \times Q \rightarrow Q \times Q$ induces a Yang-Baxter operator $c: \mathbb{A}Q \otimes \mathbb{A}Q \rightarrow \mathbb{A}Q \otimes \mathbb{A}Q$ over the ring \mathbb{A} .

It is then natural to study deformations over \mathbb{A} :

Question

What are the deformations of c over \mathbb{A} ?

Overview

- 1 Braid groups and Yang-Baxter representations
- 2 Yang-Baxter deformations
 - Yang-Baxter deformations
 - Yang-Baxter cohomology
 - Infinitesimal deformations
- 3 Yang-Baxter cohomology of racks
- 4 Conclusion and open questions

Yang-Baxter deformations

We fix a ring \mathbb{A} and an ideal $\mathfrak{m} \subset \mathbb{A}$.

Yang-Baxter deformations

We fix a ring \mathbb{A} and an ideal $\mathfrak{m} \subset \mathbb{A}$. Typical examples:

Yang-Baxter deformations

We fix a ring \mathbb{A} and an ideal $\mathfrak{m} \subset \mathbb{A}$. Typical examples:

- The power series ring $\mathbb{K}[[h]]$ over a field \mathbb{K} with maximal ideal (h) .

Yang-Baxter deformations

We fix a ring \mathbb{A} and an ideal $\mathfrak{m} \subset \mathbb{A}$. Typical examples:

- The power series ring $\mathbb{K}[[h]]$ over a field \mathbb{K} with maximal ideal (h) .
- The p -adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$ with maximal ideal (p) .

Yang-Baxter deformations

We fix a ring \mathbb{A} and an ideal $\mathfrak{m} \subset \mathbb{A}$. Typical examples:

- The power series ring $\mathbb{K}[[h]]$ over a field \mathbb{K} with maximal ideal (h) .
- The p -adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$ with maximal ideal (p) .

Consider a Yang-Baxter operator $c: E \otimes E \rightarrow E \otimes E$ over \mathbb{A} .

Yang-Baxter deformations

We fix a ring \mathbb{A} and an ideal $\mathfrak{m} \subset \mathbb{A}$. Typical examples:

- The power series ring $\mathbb{K}[[h]]$ over a field \mathbb{K} with maximal ideal (h) .
- The p -adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$ with maximal ideal (p) .

Consider a Yang-Baxter operator $c: E \otimes E \rightarrow E \otimes E$ over \mathbb{A} .

Definition

A map $\tilde{c}: E \otimes E \rightarrow E \otimes E$ is a *Yang-Baxter deformation* of c if \tilde{c} is itself a Yang-Baxter operator and satisfies $\tilde{c} \equiv c$ modulo \mathfrak{m} .

Yang-Baxter deformations

We fix a ring \mathbb{A} and an ideal $\mathfrak{m} \subset \mathbb{A}$. Typical examples:

- The power series ring $\mathbb{K}[[h]]$ over a field \mathbb{K} with maximal ideal (h) .
- The p -adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$ with maximal ideal (p) .

Consider a Yang-Baxter operator $c: E \otimes E \rightarrow E \otimes E$ over \mathbb{A} .

Definition

A map $\tilde{c}: E \otimes E \rightarrow E \otimes E$ is a *Yang-Baxter deformation* of c if \tilde{c} is itself a Yang-Baxter operator and satisfies $\tilde{c} \equiv c$ modulo \mathfrak{m} .

Definition

A *gauge equivalence* is an automorphism $\alpha: E \rightarrow E$ satisfying $\alpha \equiv \text{id}_E$ modulo \mathfrak{m} .

Yang-Baxter deformations

We fix a ring \mathbb{A} and an ideal $\mathfrak{m} \subset \mathbb{A}$. Typical examples:

- The power series ring $\mathbb{K}[[h]]$ over a field \mathbb{K} with maximal ideal (h) .
- The p -adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$ with maximal ideal (p) .

Consider a Yang-Baxter operator $c: E \otimes E \rightarrow E \otimes E$ over \mathbb{A} .

Definition

A map $\tilde{c}: E \otimes E \rightarrow E \otimes E$ is a *Yang-Baxter deformation* of c if \tilde{c} is itself a Yang-Baxter operator and satisfies $\tilde{c} \equiv c$ modulo \mathfrak{m} .

Definition

A *gauge equivalence* is an automorphism $\alpha: E \rightarrow E$ satisfying $\alpha \equiv \text{id}_E$ modulo \mathfrak{m} .

Definition

Two Yang-Baxter operators c and \tilde{c} are *gauge equivalent* if there is a gauge transformation $\alpha: E \rightarrow E$ such that $\tilde{c} = (\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha)$.

Yang-Baxter cohomology

First assume $m^2 = 0$ and study infinitesimal deformations.

Yang-Baxter cohomology

First assume $\mathfrak{m}^2 = 0$ and study infinitesimal deformations.

For each $n \in \mathbb{N}$ we set $C^n = \text{Hom}(E^{\otimes n}, \mathfrak{m}E^{\otimes n})$.

Yang-Baxter cohomology

First assume $\mathfrak{m}^2 = 0$ and study infinitesimal deformations.

For each $n \in \mathbb{N}$ we set $C^n = \text{Hom}(E^{\otimes n}, \mathfrak{m}E^{\otimes n})$.

We define the i th partial coboundary $d_i^n: C^n \rightarrow C^{n+1}$ by

$$d_i^n f = + \begin{array}{c} 0 \quad i \quad n \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 0 \quad i \quad n \end{array} - \begin{array}{c} 0 \quad i \quad n \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 0 \quad i \quad n \end{array} .$$

Definition & proposition

The *Yang-Baxter cochain complex* $C_{\text{YB}}^*(c; \mathfrak{m})$ consists of the modules $C^n = \text{Hom}(E^{\otimes n}, \mathfrak{m}E^{\otimes n})$ with coboundary $d^n = \sum_{i=0}^n (-1)^i d_i^n$.

Proposition

The second cohomology $H_{\text{YB}}^2(c; \mathfrak{m})$ classifies infinitesimal Yang-Baxter deformations of c over $(\mathbb{A}, \mathfrak{m})$.

Infinitesimal deformations

Proposition

The second cohomology $H_{\text{YB}}^2(c; \mathfrak{m})$ classifies infinitesimal Yang-Baxter deformations of c over $(\mathbb{A}, \mathfrak{m})$.

The deformation $\tilde{c} = c \circ (\text{id}_E^{\otimes 2} + f)$ satisfies

$$\begin{aligned} & (\text{id} \otimes \tilde{c})^{-1} (\tilde{c} \otimes \text{id})^{-1} (\text{id} \otimes \tilde{c})^{-1} (\tilde{c} \otimes \text{id}) (\text{id} \otimes \tilde{c}) (\tilde{c} \otimes \text{id}) \\ &= (\text{id} \otimes c)^{-1} (c \otimes \text{id})^{-1} (\text{id} \otimes c)^{-1} (c \otimes \text{id}) (\text{id} \otimes c) (c \otimes \text{id}) + d^2 f. \end{aligned}$$

Infinitesimal deformations

Proposition

The second cohomology $H_{\text{YB}}^2(c; \mathfrak{m})$ classifies infinitesimal Yang-Baxter deformations of c over $(\mathbb{A}, \mathfrak{m})$.

The deformation $\tilde{c} = c \circ (\text{id}_E^{\otimes 2} + f)$ satisfies

$$\begin{aligned} & (\text{id} \otimes \tilde{c})^{-1} (\tilde{c} \otimes \text{id})^{-1} (\text{id} \otimes \tilde{c})^{-1} (\tilde{c} \otimes \text{id}) (\text{id} \otimes \tilde{c}) (\tilde{c} \otimes \text{id}) \\ &= (\text{id} \otimes c)^{-1} (c \otimes \text{id})^{-1} (\text{id} \otimes c)^{-1} (c \otimes \text{id}) (\text{id} \otimes c) (c \otimes \text{id}) + d^2 f. \end{aligned}$$

This means that \tilde{c} is a Yang-Baxter operator if and only if $d^2 f = 0$.

Infinitesimal deformations

Proposition

The second cohomology $H_{\text{YB}}^2(c; \mathfrak{m})$ classifies infinitesimal Yang-Baxter deformations of c over $(\mathbb{A}, \mathfrak{m})$.

The deformation $\tilde{c} = c \circ (\text{id}_E^{\otimes 2} + f)$ satisfies

$$\begin{aligned} & (\text{id} \otimes \tilde{c})^{-1} (\tilde{c} \otimes \text{id})^{-1} (\text{id} \otimes \tilde{c})^{-1} (\tilde{c} \otimes \text{id}) (\text{id} \otimes \tilde{c}) (\tilde{c} \otimes \text{id}) \\ &= (\text{id} \otimes c)^{-1} (c \otimes \text{id})^{-1} (\text{id} \otimes c)^{-1} (c \otimes \text{id}) (\text{id} \otimes c) (c \otimes \text{id}) + d^2 f. \end{aligned}$$

This means that \tilde{c} is a Yang-Baxter operator if and only if $d^2 f = 0$.

Likewise, for $\alpha = (\text{id}_E + g)$ we find

$$(\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha) = c \circ (\text{id}_E^{\otimes 2} + d^1 g).$$

Infinitesimal deformations

Proposition

The second cohomology $H_{\text{YB}}^2(c; \mathfrak{m})$ classifies infinitesimal Yang-Baxter deformations of c over $(\mathbb{A}, \mathfrak{m})$.

The deformation $\tilde{c} = c \circ (\text{id}_E^{\otimes 2} + f)$ satisfies

$$\begin{aligned} & (\text{id} \otimes \tilde{c})^{-1} (\tilde{c} \otimes \text{id})^{-1} (\text{id} \otimes \tilde{c})^{-1} (\tilde{c} \otimes \text{id}) (\text{id} \otimes \tilde{c}) (\tilde{c} \otimes \text{id}) \\ &= (\text{id} \otimes c)^{-1} (c \otimes \text{id})^{-1} (\text{id} \otimes c)^{-1} (c \otimes \text{id}) (\text{id} \otimes c) (c \otimes \text{id}) + d^2 f. \end{aligned}$$

This means that \tilde{c} is a Yang-Baxter operator if and only if $d^2 f = 0$.

Likewise, for $\alpha = (\text{id}_E + g)$ we find

$$(\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha) = c \circ (\text{id}_E^{\otimes 2} + d^1 g).$$

Therefore, c and \tilde{c} are gauge equivalent if and only if $f = d^1 g$.

Overview

- 1 Braid groups and Yang-Baxter representations
- 2 Yang-Baxter deformations
- 3 Yang-Baxter cohomology of racks**
 - Yang-Baxter cohomology of racks
 - The diagonal cochain complex
 - The quasi-diagonal cochain complex
- 4 Conclusion and open questions

Special case: Yang-Baxter cohomology of racks

Consider a rack Q and $c: \mathbb{A}Q^2 \rightarrow \mathbb{A}Q^2$ given by $x \otimes y \mapsto y \otimes (x * y)$.

Special case: Yang-Baxter cohomology of racks

Consider a rack Q and $c: \mathbb{A}Q^2 \rightarrow \mathbb{A}Q^2$ given by $x \otimes y \mapsto y \otimes (x * y)$.

Use the canonical basis Q to identify $f: \mathbb{A}Q^n \rightarrow \mathbb{A}Q^n$ with its matrix

$$f: (x_1 \otimes \cdots \otimes x_n) \mapsto \sum_{y_1, \dots, y_n} f \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} \cdot (y_1 \otimes \cdots \otimes y_n).$$

Special case: Yang-Baxter cohomology of racks

Consider a rack Q and $c: \mathbb{A}Q^2 \rightarrow \mathbb{A}Q^2$ given by $x \otimes y \mapsto y \otimes (x * y)$.

Use the canonical basis Q to identify $f: \mathbb{A}Q^n \rightarrow \mathbb{A}Q^n$ with its matrix

$$f: (x_1 \otimes \cdots \otimes x_n) \mapsto \sum_{y_1, \dots, y_n} f \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} \cdot (y_1 \otimes \cdots \otimes y_n).$$

In this notation the coboundary can be rewritten more explicitly as

$$\begin{aligned} (d_i^n f) \begin{bmatrix} x_0, \dots, x_n \\ y_0, \dots, y_n \end{bmatrix} &= +f \begin{bmatrix} x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n \\ y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_n \end{bmatrix} \cdot \text{id} \begin{bmatrix} x_i^{x_{i+1} \cdots x_n} \\ y_i^{y_{i+1} \cdots y_n} \end{bmatrix} \\ &\quad - f \begin{bmatrix} x_0^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n \\ y_0^{y_i}, \dots, y_{i-1}^{y_i}, y_{i+1}, \dots, y_n \end{bmatrix} \cdot \text{id} \begin{bmatrix} x_i \\ y_i \end{bmatrix}. \end{aligned}$$

Special case: Yang-Baxter cohomology of racks

Consider a rack Q and $c: \mathbb{A}Q^2 \rightarrow \mathbb{A}Q^2$ given by $x \otimes y \mapsto y \otimes (x * y)$.

Use the canonical basis Q to identify $f: \mathbb{A}Q^n \rightarrow \mathbb{A}Q^n$ with its matrix

$$f: (x_1 \otimes \cdots \otimes x_n) \mapsto \sum_{y_1, \dots, y_n} f \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix} \cdot (y_1 \otimes \cdots \otimes y_n).$$

In this notation the coboundary can be rewritten more explicitly as

$$\begin{aligned} (d_i^n f) \begin{bmatrix} x_0, \dots, x_n \\ y_0, \dots, y_n \end{bmatrix} &= +f \begin{bmatrix} x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n \\ y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_n \end{bmatrix} \cdot \text{id} \begin{bmatrix} x_i^{x_{i+1} \cdots x_n} \\ y_i^{y_{i+1} \cdots y_n} \end{bmatrix} \\ &\quad - f \begin{bmatrix} x_0^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n \\ y_0^{y_i}, \dots, y_{i-1}^{y_i}, y_{i+1}, \dots, y_n \end{bmatrix} \cdot \text{id} \begin{bmatrix} x_i \\ y_i \end{bmatrix}. \end{aligned}$$

This is cumbersome and costly to calculate, even in small degrees.

Diagonal cochains form a subcomplex

The matrix f is *diagonal* if $f\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \neq y_i$.

Diagonal cochains form a subcomplex

The matrix f is *diagonal* if $f\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \neq y_i$.

Proposition

The diagonal cochains form a subcomplex $C_{\text{Diag}}^ \subset C_{\text{YB}}^*(c_Q; \mathfrak{m})$.*

Diagonal cochains form a subcomplex

The matrix f is *diagonal* if $f\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \neq y_i$.

Proposition

The diagonal cochains form a subcomplex $C_{\text{Diag}}^ \subset C_{\text{YB}}^*(c_Q; \mathfrak{m})$.*

The diagonal subcomplex C_{Diag}^* is known from rack cohomology:

$$(\delta^n f)(x_0, \dots, x_n) = \sum_{i=1}^n (-1)^i \left[f(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \right. \\ \left. - f(x_0^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n) \right].$$

Diagonal cochains form a subcomplex

The matrix f is *diagonal* if $f[x_1, \dots, x_n]$ vanishes whenever $x_i \neq y_i$.

Proposition

The diagonal cochains form a subcomplex $C_{\text{Diag}}^* \subset C_{\text{YB}}^*(c_Q; \mathfrak{m})$.

The diagonal subcomplex C_{Diag}^* is known from rack cohomology:

$$(\delta^n f)(x_0, \dots, x_n) = \sum_{i=1}^n (-1)^i \left[f(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \right. \\ \left. - f(x_0^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n) \right].$$

Theorem

The Yang-Baxter complex $C_{\text{YB}}^*(c_Q; \mathfrak{m})$ retracts to C_{Diag}^* .

Diagonal cochains form a subcomplex

The matrix f is *diagonal* if $f[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}]$ vanishes whenever $x_i \neq y_i$.

Proposition

The diagonal cochains form a subcomplex $C_{\text{Diag}}^* \subset C_{\text{YB}}^*(c_Q; \mathfrak{m})$.

The diagonal subcomplex C_{Diag}^* is known from rack cohomology:

$$(\delta^n f)(x_0, \dots, x_n) = \sum_{i=1}^n (-1)^i \left[f(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) - f(x_0^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n) \right].$$

Theorem

The Yang-Baxter complex $C_{\text{YB}}^*(c_Q; \mathfrak{m})$ retracts to C_{Diag}^* .

As a consequence, H_{Diag}^* is a direct summand of H_{YB}^* .

Diagonal cochains form a subcomplex

The matrix f is *diagonal* if $f[x_1, \dots, x_n]$ vanishes whenever $x_i \neq y_i$.

Proposition

The diagonal cochains form a subcomplex $C_{\text{Diag}}^* \subset C_{\text{YB}}^*(c_Q; \mathfrak{m})$.

The diagonal subcomplex C_{Diag}^* is known from rack cohomology:

$$(\delta^n f)(x_0, \dots, x_n) = \sum_{i=1}^n (-1)^i \left[f(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \right. \\ \left. - f(x_0^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n) \right].$$

Theorem

The Yang-Baxter complex $C_{\text{YB}}^*(c_Q; \mathfrak{m})$ retracts to C_{Diag}^* .

As a consequence, H_{Diag}^* is a direct summand of H_{YB}^* .

In general $H_{\text{Diag}}^* \subsetneq H_{\text{YB}}^*$ and so we do *not* have a homotopy retract.

Quasi-diagonal cochains

Definition

Two rack elements $y, z \in Q$ are *behaviourally equivalent*, denoted $y \equiv z$, if they satisfy $x * y = x * z$ for all $x \in Q$.

Quasi-diagonal cochains

Definition

Two rack elements $y, z \in Q$ are *behaviourally equivalent*, denoted $y \equiv z$, if they satisfy $x * y = x * z$ for all $x \in Q$.

This means that y, z are identified under $\rho: Q \rightarrow \text{Inn}(Q)$.

Quasi-diagonal cochains

Definition

Two rack elements $y, z \in Q$ are *behaviourally equivalent*, denoted $y \equiv z$, if they satisfy $x * y = x * z$ for all $x \in Q$.

This means that y, z are identified under $\rho: Q \rightarrow \text{Inn}(Q)$.

Definition

The matrix f is *diagonal* if $f \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix}$ vanishes whenever $x_i \neq y_i$.

Quasi-diagonal cochains

Definition

Two rack elements $y, z \in Q$ are *behaviourally equivalent*, denoted $y \equiv z$, if they satisfy $x * y = x * z$ for all $x \in Q$.

This means that y, z are identified under $\rho: Q \rightarrow \text{Inn}(Q)$.

Definition

The matrix f is *diagonal* if $f \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix}$ vanishes whenever $x_i \neq y_i$.
It is *quasi-diagonal* if $f \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix}$ vanishes whenever $x_i \not\equiv y_i$.

Quasi-diagonal cochains

Definition

Two rack elements $y, z \in Q$ are *behaviourally equivalent*, denoted $y \equiv z$, if they satisfy $x * y = x * z$ for all $x \in Q$.

This means that y, z are identified under $\rho: Q \rightarrow \text{Inn}(Q)$.

Definition

The matrix f is *diagonal* if $f \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix}$ vanishes whenever $x_i \neq y_i$.
It is *quasi-diagonal* if $f \begin{bmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{bmatrix}$ vanishes whenever $x_i \not\equiv y_i$.

Stated informally, only behaviourally equivalent elements mix.

Quasi-diagonal cochains

Definition

Two rack elements $y, z \in Q$ are *behaviourally equivalent*, denoted $y \equiv z$, if they satisfy $x * y = x * z$ for all $x \in Q$.

This means that y, z are identified under $\rho: Q \rightarrow \text{Inn}(Q)$.

Definition

The matrix f is *diagonal* if $f\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \neq y_i$.
It is *quasi-diagonal* if $f\left[\begin{smallmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{smallmatrix}\right]$ vanishes whenever $x_i \not\equiv y_i$.

Stated informally, only behaviourally equivalent elements mix.

Proposition

The quasi-diagonal cochains form a subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*(c_Q; \mathfrak{m})$.

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Theorem

arxiv 2008

The quasi-diagonal subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*$ is a homotopy retract,

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Theorem

arxiv 2008

The quasi-diagonal subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*$ is a homotopy retract, whence $H_{\Delta}^*(c_Q; \mathfrak{m}) \rightarrow H_{\text{YB}}^*(c_Q; \mathfrak{m})$ is an isomorphism.

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Theorem

arxiv 2008

The quasi-diagonal subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*$ is a homotopy retract, whence $H_{\Delta}^*(c_Q; \mathfrak{m}) \rightarrow H_{\text{YB}}^*(c_Q; \mathfrak{m})$ is an isomorphism.

There are two extreme cases:

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Theorem

arxiv 2008

The quasi-diagonal subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*$ is a homotopy retract, whence $H_{\Delta}^*(c_Q; \mathfrak{m}) \rightarrow H_{\text{YB}}^*(c_Q; \mathfrak{m})$ is an isomorphism.

There are two extreme cases:

- If the rack Q is trivial, then $C_{\Delta}^* = C_{\text{YB}}^*$ with $d_{\text{YB}}^* = 0$.

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Theorem

arxiv 2008

The quasi-diagonal subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*$ is a homotopy retract, whence $H_{\Delta}^*(c_Q; \mathfrak{m}) \rightarrow H_{\text{YB}}^*(c_Q; \mathfrak{m})$ is an isomorphism.

There are two extreme cases:

- If the rack Q is trivial, then $C_{\Delta}^* = C_{\text{YB}}^*$ with $d_{\text{YB}}^* = 0$.
This is the setting of quantum invariants.

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Theorem

arxiv 2008

The quasi-diagonal subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*$ is a homotopy retract, whence $H_{\Delta}^*(c_Q; \mathfrak{m}) \rightarrow H_{\text{YB}}^*(c_Q; \mathfrak{m})$ is an isomorphism.

There are two extreme cases:

- If the rack Q is trivial, then $C_{\Delta}^* = C_{\text{YB}}^*$ with $d_{\text{YB}}^* = 0$.
This is the setting of quantum invariants.
- If $Q \rightarrow \text{Inn}(Q)$ is injective, then $C_{\Delta}^* = C_{\text{Diag}}^*$.

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Theorem

arxiv 2008

The quasi-diagonal subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*$ is a homotopy retract, whence $H_{\Delta}^*(c_Q; \mathfrak{m}) \rightarrow H_{\text{YB}}^*(c_Q; \mathfrak{m})$ is an isomorphism.

There are two extreme cases:

- If the rack Q is trivial, then $C_{\Delta}^* = C_{\text{YB}}^*$ with $d_{\text{YB}}^* = 0$.
This is the setting of quantum invariants.
- If $Q \rightarrow \text{Inn}(Q)$ is injective, then $C_{\Delta}^* = C_{\text{Diag}}^*$.
This is the setting of rack cohomology.

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Theorem

arxiv 2008

The quasi-diagonal subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*$ is a homotopy retract, whence $H_{\Delta}^*(c_Q; \mathfrak{m}) \rightarrow H_{\text{YB}}^*(c_Q; \mathfrak{m})$ is an isomorphism.

There are two extreme cases:

- If the rack Q is trivial, then $C_{\Delta}^* = C_{\text{YB}}^*$ with $d_{\text{YB}}^* = 0$.
This is the setting of quantum invariants.
- If $Q \rightarrow \text{Inn}(Q)$ is injective, then $C_{\Delta}^* = C_{\text{Diag}}^*$.
This is the setting of rack cohomology.

Assume that the ring \mathbb{A} is complete with respect to the ideal \mathfrak{m} .

Homotopy retraction to quasi-diagonal deformations

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Theorem

arxiv 2008

The quasi-diagonal subcomplex $C_{\Delta}^* \subset C_{\text{YB}}^*$ is a homotopy retract, whence $H_{\Delta}^*(c_Q; \mathfrak{m}) \rightarrow H_{\text{YB}}^*(c_Q; \mathfrak{m})$ is an isomorphism.

There are two extreme cases:

- If the rack Q is trivial, then $C_{\Delta}^* = C_{\text{YB}}^*$ with $d_{\text{YB}}^* = 0$.
This is the setting of quantum invariants.
- If $Q \rightarrow \text{Inn}(Q)$ is injective, then $C_{\Delta}^* = C_{\text{Diag}}^*$.
This is the setting of rack cohomology.

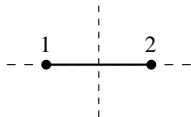
Assume that the ring \mathbb{A} is complete with respect to the ideal \mathfrak{m} .

Theorem

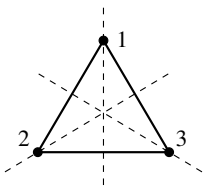
arxiv 2008

Every Yang-Baxter deformation c of c_Q over $(\mathbb{A}, \mathfrak{m})$ is gauge-equivalent to a quasi-diagonal deformation of c_Q .

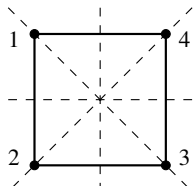
Examples and applications



trivial but
deformable

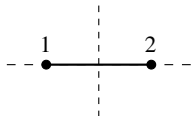


non-trivial
but rigid

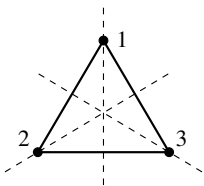


neither trivial
nor rigid

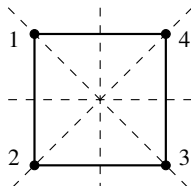
Examples and applications



trivial but
deformable



non-trivial
but rigid

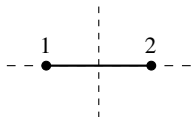


neither trivial
nor rigid

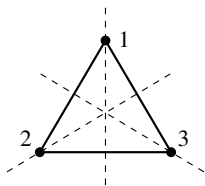
Example

For the dihedral quandle $Q = \{(12), (13), (23)\} \subset S_3$ the associated Yang-Baxter operator c_Q is rigid over every complete ring.

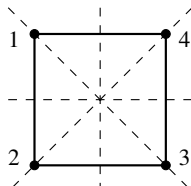
Examples and applications



trivial but
deformable



non-trivial
but rigid



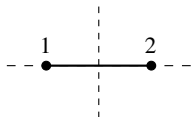
neither trivial
nor rigid

Example

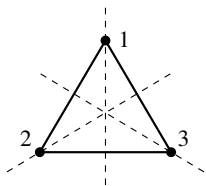
For the dihedral quandle $Q = \{(12), (13), (23)\} \subset S_3$ the associated Yang-Baxter operator c_Q is rigid over every complete ring.

Sketch of proof:

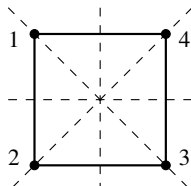
Examples and applications



trivial but
deformable



non-trivial
but rigid



neither trivial
nor rigid

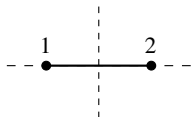
Example

For the dihedral quandle $Q = \{(12), (13), (23)\} \subset S_3$ the associated Yang-Baxter operator c_Q is rigid over every complete ring.

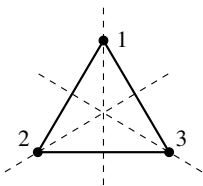
Sketch of proof:

- Since $Q \rightarrow \text{Inn}(Q)$ is injective, we only consider rack cohomology.

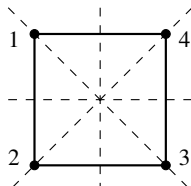
Examples and applications



trivial but
deformable



non-trivial
but rigid



neither trivial
nor rigid

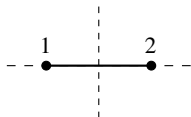
Example

For the dihedral quandle $Q = \{(12), (13), (23)\} \subset S_3$ the associated Yang-Baxter operator c_Q is rigid over every complete ring.

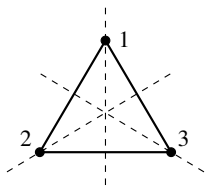
Sketch of proof:

- Since $Q \rightarrow \text{Inn}(Q)$ is injective, we only consider rack cohomology.
- If $|\text{Inn}(Q)| = 6$ is invertible in \mathbb{A} , then c_Q is rigid (symmetrization).

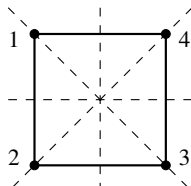
Examples and applications



trivial but
deformable



non-trivial
but rigid



neither trivial
nor rigid

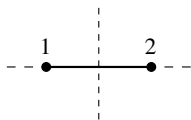
Example

For the dihedral quandle $Q = \{(12), (13), (23)\} \subset S_3$ the associated Yang-Baxter operator c_Q is rigid over every complete ring.

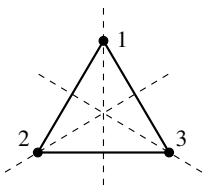
Sketch of proof:

- Since $Q \rightarrow \text{Inn}(Q)$ is injective, we only consider rack cohomology.
- If $|\text{Inn}(Q)| = 6$ is invertible in \mathbb{A} , then c_Q is rigid (symmetrization).
- In characteristic 2 and 3 we obtain $H^2 = 0$ by direct calculation.

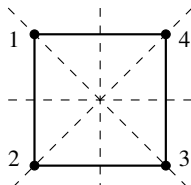
Examples and applications



trivial but
deformable

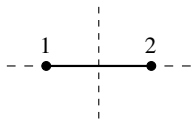


non-trivial
but rigid

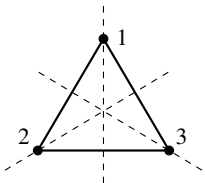


neither trivial
nor rigid

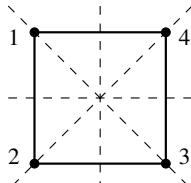
Examples and applications



trivial but
deformable



non-trivial
but rigid

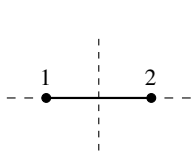


neither trivial
nor rigid

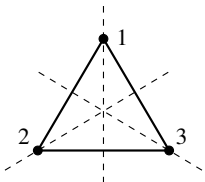
Example

The dihedral quandle $Q = \{ (13), (24), (12)(34), (14)(23) \} \subset S_4$ admits a 16-parameter deformation (to all orders) over any ring.

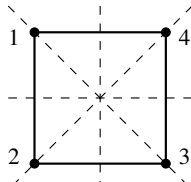
Examples and applications



trivial but
deformable



non-trivial
but rigid



neither trivial
nor rigid

Example

The dihedral quandle $Q = \{ (13), (24), (12)(34), (14)(23) \} \subset S_4$ admits a 16-parameter deformation (to all orders) over any ring.

Over $\mathbb{Z}/2$ it admits an additional 4-parameter infinitesimal deformation. This is quasi-diagonal but not diagonal.

Overview

- 1 Braid groups and Yang-Baxter representations
- 2 Yang-Baxter deformations
- 3 Yang-Baxter cohomology of racks
- 4 Conclusion and open questions**

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood.

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood.
In general $H_{\text{YB}}^* = H_{\Delta}^*$ is more complicated... and more interesting.

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood. In general $H_{\text{YB}}^* = H_{\Delta}^*$ is more complicated... and more interesting.

Some open questions:

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood. In general $H_{\text{YB}}^* = H_{\Delta}^*$ is more complicated... and more interesting.

Some open questions:

- Passage from infinitesimal to complete deformations?

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood. In general $H_{\text{YB}}^* = H_{\Delta}^*$ is more complicated... and more interesting.

Some open questions:

- Passage from infinitesimal to complete deformations?
→ Study higher order obstructions in $H_{\text{YB}}^3(c; \mathfrak{m}^2/\mathfrak{m}^3)$.

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood. In general $H_{\text{YB}}^* = H_{\Delta}^*$ is more complicated... and more interesting.

Some open questions:

- Passage from infinitesimal to complete deformations?
→ Study higher order obstructions in $H_{\text{YB}}^3(c; \mathfrak{m}^2/\mathfrak{m}^3)$.
- What sort of topological information can we expect?

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood. In general $H_{\text{YB}}^* = H_{\Delta}^*$ is more complicated... and more interesting.

Some open questions:

- Passage from infinitesimal to complete deformations?
→ Study higher order obstructions in $H_{\text{YB}}^3(c; \mathfrak{m}^2/\mathfrak{m}^3)$.
- What sort of topological information can we expect?
→ Completely solved for rack/diagonal cohomology.

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood. In general $H_{\text{YB}}^* = H_{\Delta}^*$ is more complicated... and more interesting.

Some open questions:

- Passage from infinitesimal to complete deformations?
→ Study higher order obstructions in $H_{\text{YB}}^3(c; \mathfrak{m}^2/\mathfrak{m}^3)$.
- What sort of topological information can we expect?
→ Completely solved for rack/diagonal cohomology.
- Do our results generalize to general set-theoretic YB operators?

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood. In general $H_{\text{YB}}^* = H_{\Delta}^*$ is more complicated... and more interesting.

Some open questions:

- Passage from infinitesimal to complete deformations?
→ Study higher order obstructions in $H_{\text{YB}}^3(c; \mathfrak{m}^2/\mathfrak{m}^3)$.
- What sort of topological information can we expect?
→ Completely solved for rack/diagonal cohomology.
- Do our results generalize to general set-theoretic YB operators?
→ Their Yang-Baxter cohomology is more difficult to analyze...

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q :

$$C_{\text{YB}}^*(c_Q; \mathfrak{m}) \supset C_{\Delta}^*(c_Q; \mathfrak{m}) \supset C_{\text{Diag}}^*(c_Q; \mathfrak{m})$$

The diagonal part $H_{\text{Diag}}^* = H_{\text{R}}^*$ is interesting and quite well understood. In general $H_{\text{YB}}^* = H_{\Delta}^*$ is more complicated... and more interesting.

Some open questions:

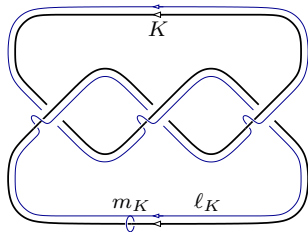
- Passage from infinitesimal to complete deformations?
→ Study higher order obstructions in $H_{\text{YB}}^3(c; \mathfrak{m}^2/\mathfrak{m}^3)$.
- What sort of topological information can we expect?
→ Completely solved for rack/diagonal cohomology.
- Do our results generalize to general set-theoretic YB operators?
→ Their Yang-Baxter cohomology is more difficult to analyze...

Thank you for your attention!

- 5 Appendix: colouring polynomials
 - colouring numbers
 - Colouring polynomials
 - Colouring polynomials are Yang-Baxter invariants

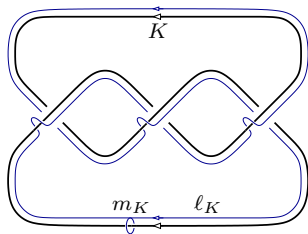
Appendix: colouring numbers

Consider the knot group $\pi_K := \pi_1(\mathbb{R}^3 \setminus K)$
equipped with meridian m_k and longitude ℓ_k .



Appendix: colouring numbers

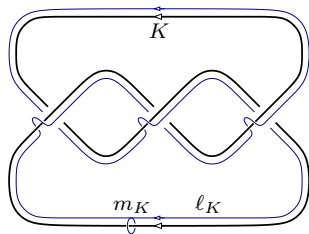
Consider the knot group $\pi_K := \pi_1(\mathbb{R}^3 \setminus K)$
equipped with meridian m_k and longitude ℓ_k .



Let G be a group and let $Q = x^G$ be a conjugacy class.

Appendix: colouring numbers

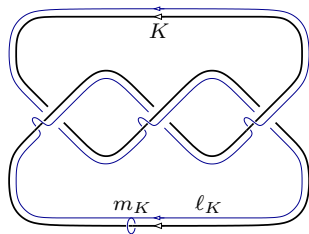
Consider the knot group $\pi_K := \pi_1(\mathbb{R}^3 \setminus K)$
equipped with meridian m_k and longitude ℓ_k .



Let G be a group and let $Q = x^G$ be a conjugacy class.
Let $E = \mathbb{Z}Q$ and $c_Q: E \otimes E \rightarrow E \otimes E, x \otimes y \mapsto y \otimes (x * y)$.

Appendix: colouring numbers

Consider the knot group $\pi_K := \pi_1(\mathbb{R}^3 \setminus K)$
equipped with meridian m_k and longitude ℓ_k .



Let G be a group and let $Q = x^G$ be a conjugacy class.
Let $E = \mathbb{Z}Q$ and $c_Q: E \otimes E \rightarrow E \otimes E, x \otimes y \mapsto y \otimes (x * y)$.

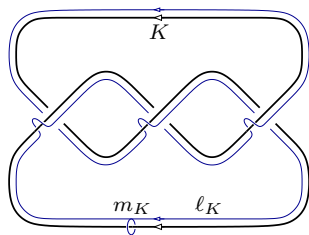
Theorem (Freyd-Yetter 1989)

The induced invariant $F_G^x: \mathcal{K} \rightarrow \mathbb{Z}$ is the colouring number

$$F_G^x(K) = \# \text{Hom}(\pi_K, m_K; G, x).$$

Appendix: colouring numbers

Consider the knot group $\pi_K := \pi_1(\mathbb{R}^3 \setminus K)$
equipped with meridian m_k and longitude ℓ_k .



Let G be a group and let $Q = x^G$ be a conjugacy class.
Let $E = \mathbb{Z}Q$ and $c_Q: E \otimes E \rightarrow E \otimes E$, $x \otimes y \mapsto y \otimes (x * y)$.

Theorem (Freyd-Yetter 1989)

The induced invariant $F_G^x: \mathcal{K} \rightarrow \mathbb{Z}$ is the colouring number

$$F_G^x(K) = \# \text{Hom}(\pi_K, m_K; G, x).$$

Natural question: Can we deform c_Q ? What invariants do we get?

Appendix: colouring polynomials

Let G be a group and let $x \in G$ be an element. We set $\Lambda = C(x) \cap G'$.

Appendix: colouring polynomials

Let G be a group and let $x \in G$ be an element. We set $\Lambda = C(x) \cap G'$.

Definition

The colouring polynomial $P_G^x: \mathcal{K} \rightarrow \mathbb{Z}\Lambda$ is the sum

$$P_G^x(K) := \sum_{\rho} \rho(\ell_K) \quad \text{over } \rho: \pi_K \rightarrow G \text{ with } \rho(m_k) = x.$$

Appendix: colouring polynomials

Let G be a group and let $x \in G$ be an element. We set $\Lambda = C(x) \cap G'$.

Definition

The colouring polynomial $P_G^x: \mathcal{K} \rightarrow \mathbb{Z}\Lambda$ is the sum

$$P_G^x(K) := \sum_{\rho} \rho(\ell_K) \quad \text{over } \rho: \pi_K \rightarrow G \text{ with } \rho(m_k) = x.$$

We have $F_G^x = \varepsilon P_G^x$ via the augmentation $\varepsilon: \mathbb{Z}G \rightarrow \mathbb{Z}$, $g \mapsto 1$.

Appendix: colouring polynomials

Let G be a group and let $x \in G$ be an element. We set $\Lambda = C(x) \cap G'$.

Definition

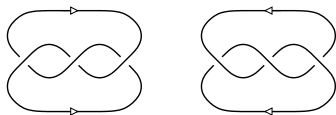
The colouring polynomial $P_G^x: \mathcal{K} \rightarrow \mathbb{Z}\Lambda$ is the sum

$$P_G^x(K) := \sum_{\rho} \rho(\ell_K) \quad \text{over} \quad \rho: \pi_K \rightarrow G \quad \text{with} \quad \rho(m_k) = x.$$

We have $F_G^x = \varepsilon P_G^x$ via the augmentation $\varepsilon: \mathbb{Z}G \rightarrow \mathbb{Z}$, $g \mapsto 1$.

Example

The two trefoil knots are not isotopic. (Dehn 1914)



Appendix: colouring polynomials

Let G be a group and let $x \in G$ be an element. We set $\Lambda = C(x) \cap G'$.

Definition

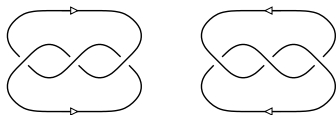
The colouring polynomial $P_G^x: \mathcal{K} \rightarrow \mathbb{Z}\Lambda$ is the sum

$$P_G^x(K) := \sum_{\rho} \rho(\ell_K) \quad \text{over } \rho: \pi_K \rightarrow G \text{ with } \rho(m_k) = x.$$

We have $F_G^x = \varepsilon P_G^x$ via the augmentation $\varepsilon: \mathbb{Z}G \rightarrow \mathbb{Z}$, $g \mapsto 1$.

Example

The two trefoil knots are not isotopic. (Dehn 1914)



Simple proof: choose $G = A_5$ with basepoint $x = (12345)$.

Appendix: colouring polynomials

Let G be a group and let $x \in G$ be an element. We set $\Lambda = C(x) \cap G'$.

Definition

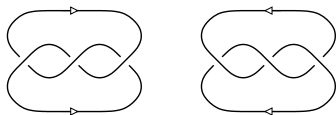
The colouring polynomial $P_G^x: \mathcal{K} \rightarrow \mathbb{Z}\Lambda$ is the sum

$$P_G^x(K) := \sum_{\rho} \rho(\ell_K) \quad \text{over } \rho: \pi_K \rightarrow G \text{ with } \rho(m_k) = x.$$

We have $F_G^x = \varepsilon P_G^x$ via the augmentation $\varepsilon: \mathbb{Z}G \rightarrow \mathbb{Z}$, $g \mapsto 1$.

Example

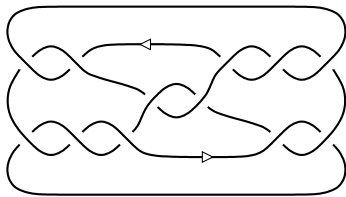
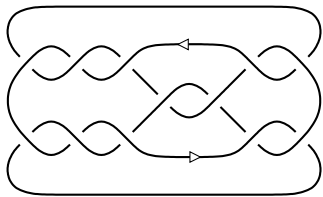
The two trefoil knots are not isotopic. (Dehn 1914)



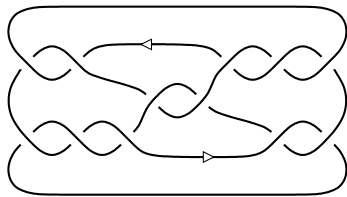
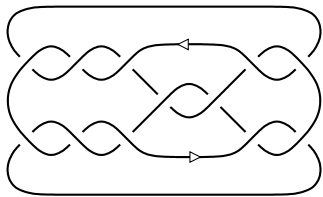
Simple proof: choose $G = A_5$ with basepoint $x = (12345)$.

We find $P_G^x(3_1) = 1 + 5x$ and $P_G^x(3_1^*) = 1 + 5x^{-1}$.

Appendix: colouring polynomial examples



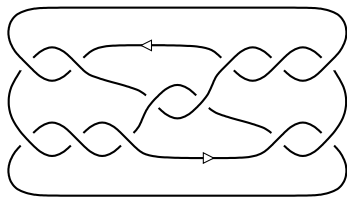
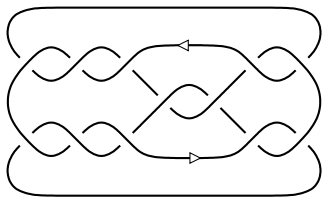
Appendix: colouring polynomial examples



Let M_{11} be the Mathieu group of order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$,

$$M_{11} = \langle x = (abcdefghijk), y = (abcejikdghf) \rangle \subset A_{11}.$$

Appendix: colouring polynomial examples



Let M_{11} be the Mathieu group of order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$,

$$M_{11} = \langle x = (abcdefghijk), y = (abcejikdghf) \rangle \subset A_{11}.$$

Example

For the Kinoshita-Terasaka knot K and the Conway knot C we find

$$P(K) = 1 + 11x^3 + 11x^7$$

$$P(C) = 1 + 11x^3 + 11x^7$$

$$P(K^*) = 1 + 11x^4 + 11x^8$$

$$P(C^*) = 1 + 11x^4 + 11x^8$$

$$P(K^\times) = 1 + 11x^4 + 22x^8$$

$$P(C^\times) = 1 + 11x^4 + 11x^6 + 11x^8$$

$$P(K^!) = 1 + 22x^3 + 11x^7$$

$$P(C^!) = 1 + 11x^3 + 11x^5 + 11x^7$$

Colouring polynomials are Yang-Baxter invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Colouring polynomials are Yang-Baxter invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every colouring polynomial $P_G^x: \mathcal{K} \rightarrow \mathbb{Z}\Lambda$ is a Yang-Baxter invariant.

Colouring polynomials are Yang-Baxter invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every colouring polynomial $P_G^x: \mathcal{K} \rightarrow \mathbb{Z}\Lambda$ is a Yang-Baxter invariant.

There exists a Yang-Baxter operator \tilde{c} deforming c_Q over the ring $\mathbb{Z}\Lambda$, such that the associated knot invariant is a constant multiple of P_G^x .

Colouring polynomials are Yang-Baxter invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every colouring polynomial $P_G^x: \mathcal{K} \rightarrow \mathbb{Z}\Lambda$ is a Yang-Baxter invariant.

There exists a Yang-Baxter operator \tilde{c} deforming c_Q over the ring $\mathbb{Z}\Lambda$, such that the associated knot invariant is a constant multiple of P_G^x .

Corollary

Yang-Baxter invariants can detect non-invertible knots.

Colouring polynomials are Yang-Baxter invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every colouring polynomial $P_G^x: \mathcal{K} \rightarrow \mathbb{Z}\Lambda$ is a Yang-Baxter invariant.

There exists a Yang-Baxter operator \tilde{c} deforming c_Q over the ring $\mathbb{Z}\Lambda$, such that the associated knot invariant is a constant multiple of P_G^x .

Corollary

Yang-Baxter invariants can detect non-invertible knots.

For quantum invariants this question remains open.

Appendix: quandle 2-cocycle invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Appendix: quandle 2-cocycle invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every quandle 2-cocycle state-sum invariant of knots is the specialization of some knot colouring polynomial.

Appendix: quandle 2-cocycle invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every quandle 2-cocycle state-sum invariant of knots is the specialization of some knot colouring polynomial.

Main tool: Galois correspondence for quandles.

Appendix: quandle 2-cocycle invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every quandle 2-cocycle state-sum invariant of knots is the specialization of some knot colouring polynomial.

Main tool: Galois correspondence for quandles.

Theorem (Fenn-Rourke, Carter *et al*, Eisermann)

There is a correspondence between

Appendix: quandle 2-cocycle invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every quandle 2-cocycle state-sum invariant of knots is the specialization of some knot colouring polynomial.

Main tool: Galois correspondence for quandles.

Theorem (Fenn-Rourke, Carter *et al*, Eisermann)

There is a correspondence between

- *diagonal deformations of c_Q over $\mathbb{A} = \mathbb{Z} \oplus \Lambda$,*

Appendix: quandle 2-cocycle invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every quandle 2-cocycle state-sum invariant of knots is the specialization of some knot colouring polynomial.

Main tool: Galois correspondence for quandles.

Theorem (Fenn-Rourke, Carter *et al*, Eisermann)

There is a correspondence between

- *diagonal deformations of c_Q over $\mathbb{A} = \mathbb{Z} \oplus \Lambda$,*
- *quandle cohomology $H^2(Q, \Lambda)$ with coefficients in Λ ,*

Appendix: quandle 2-cocycle invariants

$$\left\{ \begin{array}{l} \text{Yang-Baxter} \\ \text{invariants} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{colouring} \\ \text{polynomials} \end{array} \right\} \supset \left\{ \begin{array}{l} \text{quandle 2-cocycle} \\ \text{state-sum invariants} \end{array} \right\}$$

Theorem

PJM 2007

Every quandle 2-cocycle state-sum invariant of knots is the specialization of some knot colouring polynomial.

Main tool: Galois correspondence for quandles.

Theorem (Fenn-Rourke, Carter *et al*, Eisermann)

There is a correspondence between

- diagonal deformations of c_Q over $\mathbb{A} = \mathbb{Z} \oplus \Lambda$,
- quandle cohomology $H^2(Q, \Lambda)$ with coefficients in Λ ,
- central extensions $\mathcal{E}(Q, \mathbb{K})$ of Q by Λ .

References

- Yang-Baxter deformations and rack cohomology.
preprint, arXiv:0808.0108
- Yang-Baxter deformations of quandles and racks.
Algebraic and Geometric Topology 5 (2005), 537-562.
- Knot colouring polynomials.
Pacific Journal of Mathematics 231 (2007), 305-336.
- Homological characterization of the unknot.
Journal of Pure and Applied Algebra 177 (2003), 131-157.