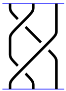
Set-theoretic Yang-Baxter operators and their deformations



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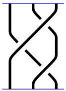

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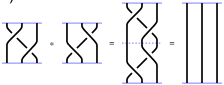
AMS–MAA Joint Mathematics Meetings in Washington DC
Special Session on Algebraic Structures in Knot Theory

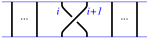
Overview

Artin's braid group (1925)

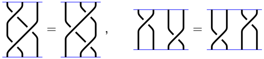
Braids form a group:



Standard generators: $s_i =$



Obvious relations:



Theorem (Artin 1925)

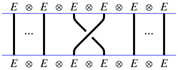
The group B_n of braids on n strands is presented by

$$B_n = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_j s_i = s_j s_i s_j & \text{if } |i-j| = 1 \\ s_i s_j = s_j s_i & \text{if } |i-j| \geq 2 \end{array} \right\rangle.$$

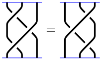
Yang-Baxter representations

Each automorphism $c \in \mathrm{Aut}(E \otimes E)$ acts on $E^{\otimes n}$ as

$c_i =$



The braid relation now becomes the Yang-Baxter equation:



$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(\mathrm{id} \otimes c) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$

Corollary (of Artin's theorem)

Every Yang-Baxter operator c induces a braid group representation $\rho_n^c \colon B_n \to \mathrm{Aut}(E^{\otimes n})$ given by $s_i \mapsto c_i$.

The Jones polynomial (1984)

Let $E = \mathbb{A}u \oplus \mathbb{A}v$ and thus $E \otimes E = \mathbb{A}uu \oplus \mathbb{A}uv \oplus \mathbb{A}vu \oplus \mathbb{A}vv$.

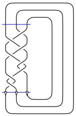
Theorem (Jones 1984)

For each $q \in \mathbb{A}$ *we have a Yang-Baxter operator*

$$c(q) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & q^2 & q - q^3 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \xrightarrow{q \mapsto 1} \tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Over $\mathbb{A} = \mathbb{Z}[q^{\pm}]$ we obtain the Jones polynomial:

$$\begin{array}{ccc} B_n & \xrightarrow{\text{closure}} & \mathscr{L} \\ \rho_n^c \downarrow & & \downarrow V \\ \operatorname{Aut}(E^{\otimes n}) & \xrightarrow[\operatorname{tr}_q]{} & \mathbb{Z}[q^{\pm}] \end{array}$$



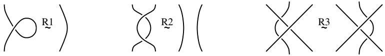
This approach generalizes to quantum invariants.

Braids act on groups

Consider conjugation $a * b := b^{-1}ab$ and $a \mathbin{\bar{*}} b := bab^{-1}$ in a group:

- (Q1) $a * a = a$ (idempotent)
- (Q2) $(a * b) \mathbin{\bar{*}} b = (a \mathbin{\bar{*}} b) * b = a$ (left-invertible)
- (Q3) $(a * b) * c = (a * c) * (b * c)$ (self-distributive)

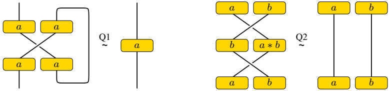
This corresponds to the three Reidemeister moves:

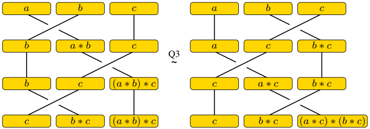


Definition

Consider a set Q with two operations $*, \bar{*} \colon Q \times Q \to Q$.
We call $(Q, *, \bar{*})$ a *quandle* if it satisfies (Q1–Q3),
and a *rack* if it satisfies (Q2–Q3).

Braids act on racks and quandles





Set-theoretic Yang-Baxter operators

Consider a set Q and a bijective map $s \colon Q \times Q \to Q \times Q$.

Definition (Drinfel'd 1990)

The *set-theoretic Yang-Baxter equation* is

$$(\mathrm{id} \times s)(s \times \mathrm{id})(\mathrm{id} \times s) = (s \times \mathrm{id})(\mathrm{id} \times s)(s \times \mathrm{id}).$$

Here $s(x, y) = (x \rhd y, x \lhd y)$ defines two operations $\rhd, \lhd \colon Q \times Q \to Q$.
For racks/quandles the left operation is trivial: $x \rhd y = y$.
In the quantum case (transposition) both operations are trivial.

Remark

Every set-theoretic Yang-Baxter operator $s \colon Q \times Q \to Q \times Q$ induces a Yang-Baxter operator $c \colon \mathbb{A}Q \otimes \mathbb{A}Q \to \mathbb{A}Q \otimes \mathbb{A}Q$ over the ring $\mathbb{A}$.

It is then natural to study deformations over $\mathbb{A}$:

Question

What are the deformations of c over $\mathbb{A}$?

Yang-Baxter deformations

We fix a ring $\mathbb{A}$ and an ideal $\mathfrak{m} \subset \mathbb{A}$. Typical examples:

- The power series ring $\mathbb{K}[\![h]\!]$ over a field $\mathbb{K}$ with maximal ideal (h).
- The p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/_{p^n}$ with maximal ideal (p).

Consider a Yang-Baxter operator $c \colon E \otimes E \to E \otimes E$ over $\mathbb{A}$.

Definition

A map $\tilde{c} \colon E \otimes E \to E \otimes E$ is a *Yang-Baxter deformation* of c if $\tilde{c}$ is itself a Yang-Baxter operator and satisfies $\tilde{c} \equiv c$ modulo $\mathfrak{m}$.

Definition

A *gauge equivalence* is an automorphism $\alpha \colon E \to E$ satisfying $\alpha \equiv \mathrm{id}_E$ modulo $\mathfrak{m}$.

Definition

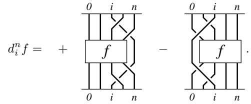
Two Yang-Baxter operators c and $\tilde{c}$ are *gauge equivalent* if there is a gauge transformation $\alpha \colon E \to E$ such that $\tilde{c} = (\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha)$.

Yang-Baxter cohomology

First assume $\mathfrak{m}^2 = 0$ and study infinitesimal deformations.

For each $n \in \mathbb{N}$ we set $C^n = \mathrm{Hom}(E^{\otimes n}, \mathfrak{m}E^{\otimes n})$.

We define the ith partial coboundary $d_i^n \colon C^n \to C^{n+1}$ by



Definition & proposition

The *Yang-Baxter cochain complex* $C^*_{\mathrm{YB}}(c; \mathfrak{m})$ consists of the modules $C^n = \mathrm{Hom}(E^{\otimes n}, \mathfrak{m}E^{\otimes n})$ with coboundary $d^n = \sum_{i=0}^{n} (-1)^i d_i^n$.

Infinitesimal deformations

Proposition

The second cohomology $H^2_{\mathrm{YB}}(c; \mathfrak{m})$ classifies infinitesimal Yang-Baxter deformations of c over $(\mathbb{A}, \mathfrak{m})$.

The deformation $\tilde{c} = c \circ (\mathrm{id}_E^{\otimes 2} + f)$ satisfies

$$\begin{aligned}(\mathrm{id} \otimes \tilde{c})^{-1}(\tilde{c} \otimes \mathrm{id})^{-1}(\mathrm{id} \otimes \tilde{c})^{-1}(\tilde{c} \otimes \mathrm{id})(\mathrm{id} \otimes \tilde{c})(\tilde{c} \otimes \mathrm{id}) \\ = (\mathrm{id} \otimes c)^{-1}(c \otimes \mathrm{id})^{-1}(\mathrm{id} \otimes c)^{-1}(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) + d^2 f.\end{aligned}$$

This means that $\tilde{c}$ is a Yang-Baxter operator if and only if $d^2 f = 0$.

Likewise, for $\alpha = (\mathrm{id}_E + g)$ we find

$$(\alpha \otimes \alpha)^{-1} \circ c \circ (\alpha \otimes \alpha) = c \circ (\mathrm{id}_E^{\otimes 2} + d^1 g).$$

Therefore, c and $\tilde{c}$ are gauge equivalent if and only if $f = d^1 g$.

Special case: Yang-Baxter cohomology of racks

Consider a rack Q and $c \colon \mathbb{A}Q^2 \to \mathbb{A}Q^2$ given by $x \otimes y \mapsto y \otimes (x * y)$.

Use the canonical basis Q to identify $f \colon \mathbb{A}Q^n \to \mathbb{A}Q^n$ with its matrix

$$f \colon (x_1 \otimes \cdots \otimes x_n) \mapsto \sum_{y_1, \ldots, y_n} f\begin{bmatrix} x_1, \ldots, x_n \\ y_1, \ldots, y_n \end{bmatrix} \cdot (y_1 \otimes \cdots \otimes y_n).$$

In this notation the coboundary can be rewritten more explicitly as

$$\begin{aligned}(d_i^n f)\begin{bmatrix} x_0, \ldots, x_n \\ y_0, \ldots, y_n \end{bmatrix} &= +f\begin{bmatrix} x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \\ y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_i^{x_{i+1} \cdots x_n} \\ y_i^{y_{i+1} \cdots y_n} \end{bmatrix} \\ &\quad - f\begin{bmatrix} x_0^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1}, \ldots, x_n \\ y_0^{y_i}, \ldots, y_{i-1}^{y_i}, y_{i+1}, \ldots, y_n \end{bmatrix} \cdot \mathrm{id}\begin{bmatrix} x_i \\ y_i \end{bmatrix}.\end{aligned}$$

This is cumbersome and costly to calculate, even in small degrees.

Diagonal cochains form a subcomplex

The matrix f is *diagonal* if $f[\,{}^{x_1,\dots,x_n}_{y_1,\dots,y_n}\,]$ vanishes whenever $x_i \neq y_i$.

Proposition

The diagonal cochains form a subcomplex $C^*_{\mathrm{Diag}} \subset C^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$.

The diagonal subcomplex C^*_{Diag} is known from rack cohomology:

$$(\delta^n f)(x_0,\dots,x_n) = \sum_{i=1}^{n} (-1)^i \big[\, f(x_0\ ,\dots,x_{i-1},x_{i+1},\dots,x_n) \\ - f(x_0^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n)\,\big].$$

Theorem

The Yang-Baxter complex $C^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ retracts to C^*_{Diag}.

As a consequence, H^*_{Diag} is a direct summand of H^*_{YB}.
In general $H^*_{\mathrm{Diag}} \subsetneq H^*_{\mathrm{YB}}$ and so we do *not* have a homotopy retract.

Quasi-diagonal cochains

Definition

Two rack elements $y, z \in Q$ are *behaviourally equivalent*, denoted $y \equiv z$, if they satisfy $x * y = x * z$ for all $x \in Q$.

This means that y, z are identified under $\rho\colon Q \to \mathrm{Inn}(Q)$.

Definition

The matrix f is *diagonal* if $f[\,{}^{x_1,\dots,x_n}_{y_1,\dots,y_n}\,]$ vanishes whenever $x_i \neq y_i$.
It is *quasi-diagonal* if $f[\,{}^{x_1,\dots,x_n}_{y_1,\dots,y_n}\,]$ vanishes whenever $x_i \not\equiv y_i$.

Stated informally, only behaviourally equivalent elements mix.

Proposition

The quasi-diagonal cochains form a subcomplex $C^*_{\Delta} \subset C^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$.

$$C^*_{\mathrm{YB}}(c_Q;\mathfrak{m}) \quad\supset\quad C^*_{\Delta}(c_Q;\mathfrak{m}) \quad\supset\quad C^*_{\mathrm{Diag}}(c_Q;\mathfrak{m})$$

Homotopy retraction to quasi-diagonal deformations

$$C^*_{\mathrm{YB}}(c_Q;\mathfrak{m}) \quad\supset\quad C^*_{\Delta}(c_Q;\mathfrak{m}) \quad\supset\quad C^*_{\mathrm{Diag}}(c_Q;\mathfrak{m})$$

Theorem — arxiv 2008

The quasi-diagonal subcomplex $C^*_{\Delta} \subset C^*_{\mathrm{YB}}$ is a homotopy retract, whence $H^*_{\Delta}(c_Q;\mathfrak{m}) \to H^*_{\mathrm{YB}}(c_Q;\mathfrak{m})$ is an isomorphism.

There are two extreme cases:

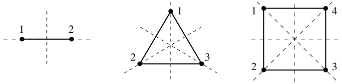
- If the rack Q is trivial, then $C^*_{\Delta} = C^*_{\mathrm{YB}}$ with $d^*_{\mathrm{YB}} = 0$. This is the setting of quantum invariants.
- If $Q \to \mathrm{Inn}(Q)$ is injective, then $C^*_{\Delta} = C^*_{\mathrm{Diag}}$. This is the setting of rack cohomology.

Assume that the ring $\mathbb{A}$ is complete with respect to the ideal $\mathfrak{m}$.

Theorem — arxiv 2008

Every Yang-Baxter deformation c of c_Q over $(\mathbb{A},\mathfrak{m})$ is gauge-equivalent to a quasi-diagonal deformation of c_Q.

Examples and applications



trivial but deformable — non−trivial but rigid — neither trivial nor rigid

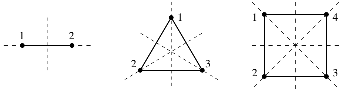
Example

For the dihedral quandle $Q = \{(12),(13),(23)\} \subset S_3$ the associated Yang-Baxter operator c_Q is rigid over every complete ring.

Sketch of proof:

- Since $Q \to \mathrm{Inn}(Q)$ is injective, we only consider rack cohomology.
- If $|\,\mathrm{Inn}(Q)| = 6$ is invertible in $\mathbb{A}$, then c_Q is rigid (symmetrization).
- In characteristic 2 and 3 we obtain $H^2 = 0$ by direct calculation.

Examples and applications



trivial but deformable | non–trivial but rigid | neither trivial nor rigid

Example

The dihedral quandle $Q = \{\ (13)\,,\ (24)\,,\ (12)(34)\,,\ (14)(23)\ \} \subset S_4$ admits a 16-parameter deformation (to all orders) over any ring.

Over $\mathbb{Z}/_2$ it admits an additional 4-parameter infinitesimal deformation. This is quasi-diagonal but not diagonal.

Conclusion and open questions

We analyze the Yang-Baxter cohomology of a rack Q:

$$C^*_{\mathrm{YB}}(c_Q;\mathfrak{m}) \quad\supset\quad C^*_{\Delta}(c_Q;\mathfrak{m}) \quad\supset\quad C^*_{\mathrm{Diag}}(c_Q;\mathfrak{m})$$

The diagonal part $H^*_{\mathrm{Diag}} = H^*_{\mathrm{R}}$ is interesting and quite well understood.
In general $H^*_{\mathrm{YB}} = H^*_{\Delta}$ is more complicated... and more interesting.

Some open questions:

- Passage from infinitesimal to complete deformations?
  $\rightarrow$ Study higher order obstructions in $H^3_{\mathrm{YB}}(c;\mathfrak{m}^2/\mathfrak{m}^3)$.
- What sort of topological information can we expect?
  $\rightarrow$ Completely solved for rack/diagonal cohomology.
- Do our results generalize to general set-theoretic YB operators?
  $\rightarrow$ Their Yang-Baxter cohomology is more difficult to analyze...

Thank you for your attention!

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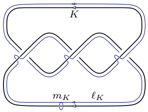
Appendix: colouring numbers

Consider the knot group $\pi_K := \pi_1(\mathbb{R}^3 \smallsetminus K)$ equipped with meridian m_k and longitude ℓ_k.



Let G be a group and let $Q = x^G$ be a conjugacy class.
Let $E = \mathbb{Z}Q$ and $c_Q \colon E \otimes E \to E \otimes E$, $x \otimes y \mapsto y \otimes (x * y)$.

Theorem (Freyd-Yetter 1989)

The induced invariant $F^x_G \colon \mathscr{K} \to \mathbb{Z}$ is the colouring number

$$F^x_G(K) = \sharp\,\mathrm{Hom}(\pi_K, m_K; G, x).$$

Natural question: Can we deform c_Q? What invariants do we get?

Appendix: colouring polynomials

Let G be a group and let $x \in G$ be an element. We set $\Lambda = C(x) \cap G'$.

Definition

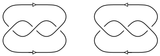
The colouring polynomial $P^x_G \colon \mathscr{K} \to \mathbb{Z}\Lambda$ is the sum

$$P^x_G(K) := \sum_\rho \rho(\ell_K) \quad \text{over} \quad \rho\colon \pi_K \to G \ \text{ with } \ \rho(m_k) = x.$$

We have $F^x_G = \varepsilon P^x_G$ via the augmentation $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$, $g \mapsto 1$.

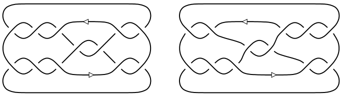
Example

The two trefoil knots are not isotopic. (Dehn 1914)



Simple proof: choose $G = A_5$ with basepoint $x = (12345)$.
We find $P^x_G(3_1) = 1 + 5x$ and $P^x_G(3_1^*) = 1 + 5x^{-1}$.

Appendix: colouring polynomial examples



Let M_{11} be the Mathieu group of order $7920 = 2^4{\cdot}3^2{\cdot}5{\cdot}11$,

$$M_{11} = \langle\, x = (\texttt{abcdefghijk}),\ y = (\texttt{abcejikdghf})\,\rangle \subset A_{11}.$$

Example

For the Kinoshita-Terasaka knot K and the Conway knot C we find

$$P(K) = 1 + 11x^3 + 11x^7 \qquad P(C) = 1 + 11x^3 + 11x^7$$
$$P(K^*) = 1 + 11x^4 + 11x^8 \qquad P(C^*) = 1 + 11x^4 + 11x^8$$
$$P(K^\times) = 1 + 11x^4 + 22x^8 \qquad P(C^\times) = 1 + 11x^4 + 11x^6 + 11x^8$$
$$P(K^!) = 1 + 22x^3 + 11x^7 \qquad P(C^!) = 1 + 11x^3 + 11x^5 + 11x^7$$

Colouring polynomials are Yang-Baxter invariants

$$\left\{\begin{matrix}\text{Yang-Baxter}\\ \text{invariants}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{colouring}\\ \text{polynomials}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{quandle 2-cocycle}\\ \text{state-sum invariants}\end{matrix}\right\}$$

Theorem PJM 2007

Every colouring polynomial $P_G^x \colon \mathscr{K} \to \mathbb{Z}\Lambda$ is a Yang-Baxter invariant.

There exists a Yang-Baxter operator $\tilde{c}$ deforming c_Q over the ring $\mathbb{Z}\Lambda$, such that the associated knot invariant is a constant multiple of P_G^x.

Corollary

Yang-Baxter invariants can detect non-invertible knots.

For quantum invariants this question remains open.

Appendix: quandle 2-cocycle invariants

$$\left\{\begin{matrix}\text{Yang-Baxter}\\ \text{invariants}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{colouring}\\ \text{polynomials}\end{matrix}\right\} \supset \left\{\begin{matrix}\text{quandle 2-cocycle}\\ \text{state-sum invariants}\end{matrix}\right\}$$

Theorem PJM 2007

Every quandle 2-cocycle state-sum invariant of knots
is the specialization of some knot colouring polynomial.

Main tool: Galois correspondence for quandles.

Theorem (Fenn-Rourke, Carter *et al*, Eisermann)

There is a correspondence between

- *diagonal deformations of c_Q over $\mathbb{A} = \mathbb{Z} \oplus \Lambda$,*
- *quandle cohomology $H^2(Q, \Lambda)$ with coefficients in Λ,*
- *central extensions $\mathscr{E}(Q, \mathbb{K})$ of Q by Λ.*

References

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