Surface invariants of finite type

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Summary

Motivation

Finite-type theory of knots and links:
- Common framework
- Beautiful structure

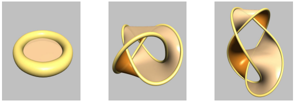
But difficult to interpret in terms of classical topology.

Naïve but natural idea: bring surfaces into play.
- Study surface invariants of finite type
- Analyze their interplay with links

First modest results indicate that this is successful.

Embedded and immersed surfaces in $\mathbb{R}^3$

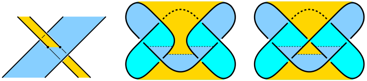
Embedded surfaces bounding knots or links:



(a) trivial knot, ◯ (b) trefoil knot, 3_1 (c) figure eight, 4_1

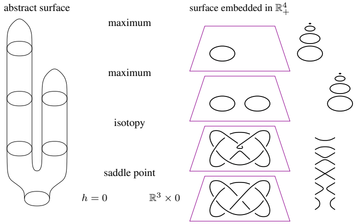
SeifertView, Jarke van Wijk, TUE

Immersed surfaces having only ribbon singularities:



(d) ribbon singularity (e) $3_1 \sharp 3_1^*$ (f) 6_1

Relationship with surfaces in $\mathbb{R}^4_+$



Proposition (Fox 1962)

A link $L \subset \mathbb{R}^3$ bounds an immersed ribbon surface $\Sigma \looparrowright \mathbb{R}^3$ iff it bounds a smoothly embedded surface $\Sigma \hookrightarrow \mathbb{R}^4_+$ without local minima.

Band diagrams

Let Σ be a compact oriented surface without closed components.

Definition

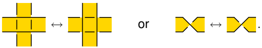
A *ribbon immersion* $F\colon \Sigma \looparrowright \mathbb{R}^3$ has only ribbon singularities.
A *ribbon surface* $S = F(\Sigma)$ is the image of a ribbon immersion F.
A *band diagram* is a planar diagram formed by the following pieces:

Proposition

Every ribbon surface S in $\mathbb{R}^3$ can be presented by a band diagram.

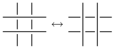
Band crossing changes

We consider *band crossing changes:*



⚠ It is important to respect the surface:
we are dealing with *links with extra structure*!

Forgetting the surface would lead to a coarser theory:

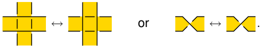


Kauffman's band pass moves

ordinary crossing changes

Surface invariants of finite type

Let D be a band diagram and let X be a set of band crossings.
Given $Y \subset X$ we obtain D_Y by changing the crossings in Y:



Definition

An invariant $v\colon \mathscr{S} \to A$ is of degree $\le m$ if

$$\sum_{Y \subset X} (-1)^{|Y|} v(D_Y) = 0 \quad \text{for all } X \text{ with } |X| > m.$$

We say that v is of *finite type* if v is of degree $\le m$ for some $m \in \mathbb{N}$.

v is of degree < 0 $\iff$ $v = 0$,
v is of degree ≤ 0 $\iff$ v is constant,
v is of degree ≤ 1 $\iff$ v is "at most linear",
v is of degree ≤ 2 $\iff$ v is "at most quadratic", etc.

Invariants of finite type

Example

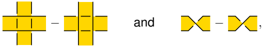
The Euler characteristic $S \mapsto \chi(\Sigma)$ is a surface invariant of degree 0.

(The universal invariant of degree 0 is the abstract surface Σ.)

Proposition

If $\mathscr{L} \xrightarrow{v} A,\quad L \mapsto v(L),$ *is a link invariant of degree* $\leq m$, *then*
$\mathscr{S} \xrightarrow{\partial} \mathscr{L} \xrightarrow{v} A,\quad S \mapsto v(\partial S),$ *is a surface invariant of degree* $\leq m$.

Proof. If we forget the surfaces in the band crossings



then we obtain (telescopic sums of) crossing changes of links. □

Seifert matrix and determinant

Assume the surface Σ to be compact, oriented and connected.

We have $\chi(\Sigma) = 1 - \operatorname{rk} H_1(\Sigma)$ because $H_0(\Sigma) = 1$ and $H_2(\Sigma) = 0$.

The module $H_1(\Sigma) \cong \mathbb{Z}^m$ is free of rank $m = 1 - \chi(\Sigma)$.

To each embedding $F\colon \Sigma \hookrightarrow \mathbb{R}^3$ we associate its Seifert form

$$\theta_F\colon H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}, \qquad \theta_F(a,b) = \operatorname{lk}(F^{\uparrow}(a), F^{\downarrow}(b)).$$

Observation

The coefficients of θ_F are of degree ≤ 1.

The determinant of F is defined by $\det(F) := \det\big[-i(\theta_F + \theta_F^*)\big]$.

It is a homogeneous polynomial of degree m in the coefficients of θ_F.

Conclusion

The surface invariant $F \mapsto \det(F)$ is of degree $\leq m = 1 - \chi(\Sigma)$.

⚠ The invariant $\det(F)$ depends only on the link $L = F(\partial\Sigma)$, but $L \mapsto \det(L)$ is not of finite type in the sense of Vassiliev–Goussarov.

Alexander polynomial

The same arguments hold for $\Delta(F) = \det(q^- \theta_F^* - q^+ \theta_F)$.

(We recover the determinant $\det(F) = \Delta(F)_{q \mapsto i}$ as a specialization.)

Proposition

$\Delta(F)$ *is a surface invariant of degree* $\leq m = 1 - \chi(\Sigma)$.

Theorem (Seifert 1934)

The invariant $\Delta(F)$ *depends only on the link* $L = F(\partial\Sigma)$*:*

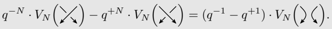
Question (cf. Murakami–Ohtsuki 2001)

Which polynomials in θ_F are invariants of $L = F(\partial\Sigma)$?

Skein relations

Theorem (HOMFLY-PT)

For each $N \in \mathbb{N}$ *there exists a unique invariant* $V_N\colon \mathscr{L} \to \mathbb{Z}[q^{\pm}]$ *satisfying* $V_N(\bigcirc) = 1$ *and the skein relation*



$N = 0$: Alexander 1928, Conway 1969
$N = 1$: trivial invariant, $V_1 = 1$
$N = 2$: Jones 1984, $V := V_2$
$N > 2$: HOMFLY-PT 1985-1987

Remark

$V(\bigcirc^n) = (q^{-1} + q^{+1})^{n-1}$ and $q^{-1} + q^{+1}$ is the minimal polynomial of i.

The situation is similar for V_N if N is prime.

Kauffman's bracket

Definition

There exists a unique map $\mathscr{D} \to \mathbb{Z}[A^{\pm}]$, denoted $D \mapsto \langle D \rangle$, such that

$$\langle \bigcirc \rangle = 1,$$

$$\langle D \sqcup \bigcirc \rangle = \langle D \rangle \cdot (-A^{+2} - A^{-2}),$$

Theorem (Kauffman 1987)

We have $V(L)|_{(q \mapsto -A^{-2})} = \langle D \rangle \cdot (-A^{-3})^{\mathrm{writhe}(D)}$.

Similar construction for V_N by Murakami–Ohtsuki–Yamada 1998

Jones nullity

Definition

The Jones nullity $\mathrm{null}\, V(L)$ is the order of the zero at $q = i$.

Lemma

Every n-component link L satisfies $0 \le \mathrm{null}\, V(L) \le n - 1$.

This corresponds to the Seifert nullity $\mathrm{null}(L) = \mathrm{null}(\theta + \theta^*)$.

Question

Do we have $\mathrm{null}\, V(L) = \mathrm{null}(L)$ for all links L?

Partial answer:

Theorem (E 2007)

For every n-component ribbon link we have $\mathrm{null}\, V(L) = n - 1$.

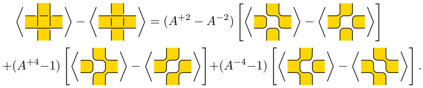
Jones nullity (lower bound)

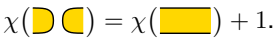
Proposition (E 2007)

If a link $L \subset \mathbb{R}^3$ bounds a ribbon surface $S \subset \mathbb{R}^3$ of positive Euler characteristic n, then $V(L)$ is divisible by $V(\bigcirc^n) = (q^+ + q^-)^{n-1}$.

More succinctly: $V(\partial S)$ is divisible by $(q^+ + q^-)^{\chi(S)-1}$.

Proof. By hypothesis each component S_i has a boundary.
Thus $\chi(S_i) > 0 \Leftrightarrow S_i = \bigcirc \Leftrightarrow \chi(S_i) = 1$.
Induction on the number $r(S)$ of ribbon singularities.
If $r(S) = 0$ then S is embedded and $L = L_0 \sqcup \bigcirc^n$.
If $r(S) \ge 1$ then we consider the Kauffman bracket:



We conclude by induction using  □

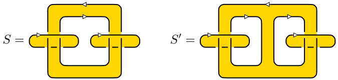
Jones nullity (examples)

Example

We have $\chi(S) = 1 + 1 + 0 = 2$ and

$$V(L) = (q^+ + q^-) \cdot (q^6 - q^4 + 2q^2 + 2q^{-2} - q^{-4} + q^{-6}).$$

Hence L bounds surfaces with $\chi \le 2$ but not with $\chi \ge 3$.



Example

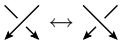
We have $\chi(S') = 1 + 1 - 1 = 1$. Notice that $L' = \partial S'$ is the connected sum $H_+ \sharp H_- \sharp H_+ \sharp H_-$ of Hopf links, whence

$$V(L) = (q^{+1} + q^{+5})^2 \cdot (q^{-1} + q^{-5})^2.$$

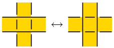
Thus L' bounds surfaces with $\chi \le 1$ but not with $\chi \ge 2$.

Expansion into finite type invariants

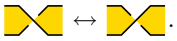
Expand $V(L) = \sum_{k=0}^{\infty} v_k(L) \cdot h^k$ in $q = \exp(h/2)$.
Then $L \mapsto v_k(L)$ is of degree $\leq k$ w.r.t. crossing changes:



Expand $V(L) = \sum_{k=0}^{\infty} d_k(L) \cdot h^k$ in $q = i\exp(h/2)$.
Then $S \mapsto d_k(\partial S)$ is of finite type w.r.t. band crossing changes:



or



⚠ $d_k(L)$ is not of finite type in the sense of Vassiliev–Goussarov.
In particular $d_0(L) = V(L)_{q\mapsto i} = \Delta(L)_{q\mapsto i} = \det(L) = \det\big[-i(\theta+\theta^*)\big]$.

Inductive proof

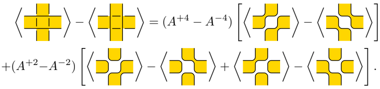
Proposition (E 2007)

The surface invariant $S \mapsto d_k(\partial S)$ is of degree $\leq m := k + 1 - \chi(S)$.

The case $d_0 = \det$ has already been derived from the Seifert matrix.
Here we consider $V(L) = \sum_{k=0}^{\infty} d_k(L) \cdot h^k$ in $q = i\exp(h/2)$.

Proof. We know that $V(\partial S)$ is divisible by $(q^+ + q^-)^{\chi(S)-1}$.
This means that $d_k(\partial S) = 0$ for $k < \chi(S) - 1$, that is, $m < 0$.

Moreover, $\sum_{Y\subset X}(-1)^{|Y|}V(\partial D_Y)$ is divisible by $(q^+ + q^-)^{|X|+\chi(S)-1}$:

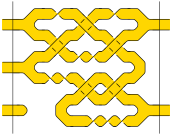


We conclude by induction on $|X|$. □

Tangled surfaces

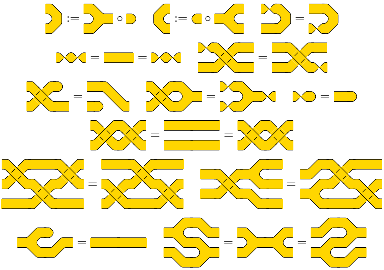
Consider the category generated by embedded surface pieces:

(a) id (b) twists (c) crossings (d) junctions (e) ends



For ribbon immersions $\Sigma \looparrowright \mathbb{R}^3$ the construction is similar but longer.

Isotopy relations



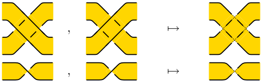
Abstract surfaces

Category generated by abstract surface pieces:

(a) id (b) twist (c) crossing (d) junctions (e) ends

Relations as before (but abstract = non-embedded)

Forgetful functor:



Chord diagrams on surfaces

I-adic filtration generated by band crossing changes:

Encoded by abstract surfaces with chords:

Tensor functor resolving chord diagrams:

This maps abstract surfaces with m chords to I^m/I^{m+1}.

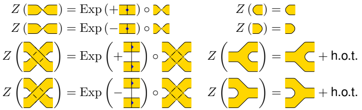
Quotient chord diagrams by the obvious relations induced by isotopy.

Question

Is the quotient finite dimensional in each degree?

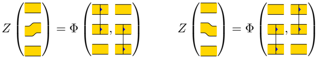
Towards a universal invariant

We wish to define a universal invariant Z as follows:



The naïve construction does not work (same problem as for tangles).

Use non-associative tangles and introduce an associator Φ:



Question

Do all isotopy relations hold? Can we arrange this?

Open questions

Alexander polynomial:

- Which polynomials in θ_F are invariants of $L = F(\partial\Sigma)$?

Jones polynomial:

- Does the Jones nullity equal the Seifert nullity?
- Generalization from Jones to HOMFLYPT? to Kauffman?
- Is this approach really 3-dimensional? or rather 4-dimensional?
- Interpretation in terms of Khovanov homology?

Surface invariants of finite type:

- Chord diagrams modulo relations $\Longrightarrow$ finite dimensional?
- Examine further examples: HOMFLYPT, Kauffman, ...
- Understand invariants of low degree.
- Construct a universal invariant.

Thank you for your attention.