

The Jones polynomial of ribbon knots and symmetric unions

Michael Eisermann (Institut Fourier, Grenoble)
Christoph Lamm (Dresdner Bank, Frankfurt)

August 3, 2007



AMS-PTM International Meeting
Warsaw, July 31 – August 3, 2007

Keywords

- Symmetric unions of knots
(Kinoshita-Terasaka 1957)

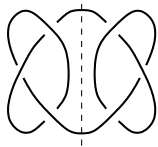
Keywords

- Symmetric unions of knots
(Kinoshita-Terasaka 1957)
- Ribbon knots & slice knots
(Fox-Milnor 1957, 1961, 1965)

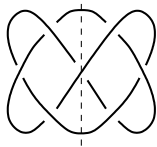
Keywords

- Symmetric unions of knots
(Kinoshita-Terasaka 1957)
- Ribbon knots & slice knots
(Fox-Milnor 1957, 1961, 1965)
- A two-variable refinement of the Jones polynomial
(preliminary report)

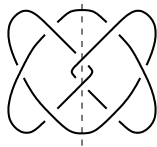
Symmetric unions (Kinoshita-Terasaka 1957)



(a) symmetric
sum, $3_1 \sharp 3_1^*$

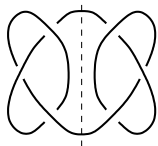


(b) symmetric
union, 6_1

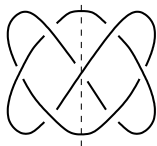


(c) symmetric
union, 8_{20}

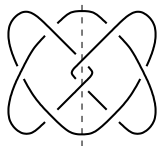
Symmetric unions (Kinoshita-Terasaka 1957)



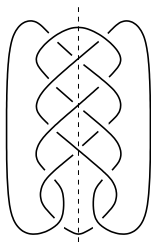
(a) symmetric
sum, $3_1 \sharp 3_1^*$



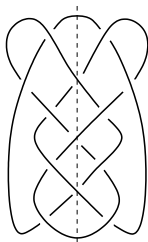
(b) symmetric
union, 6_1



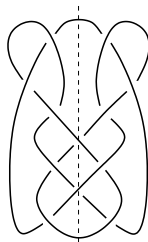
(c) symmetric
union, 8_{20}



(d) 8_8



(e) 9_{27}



(f) 9_{27}

Ribbon & slice (Fox-Milnor 1957, 1961, 1965)

General setting

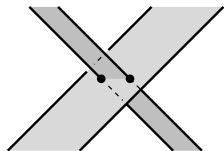
$$\{\text{symmetric unions}\} \subset \{\text{ribbon knots}\} \subset \{\text{slice knots}\}$$
$$\downarrow \supset ? \qquad \qquad \downarrow \supset ?$$

Ribbon & slice (Fox-Milnor 1957, 1961, 1965)

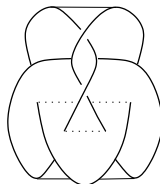
General setting

$\{\text{symmetric unions}\} \subset \{\text{ribbon knots}\} \subset \{\text{slice knots}\}$
 $\downarrow \supset ? \qquad \downarrow \supset ?$

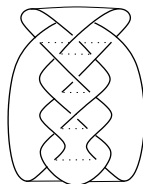
Every symmetric union is a ribbon knot:



(a) ribbon singularity



(b) 8_{20}



(c) 10_{87}

Which ribbon knots are symmetric unions?

Empirical evidence

All “small” ribbon knots are symmetric unions.

Which ribbon knots are symmetric unions?

Empirical evidence

All “small” ribbon knots are symmetric unions.

- ☞ Prime knots up to 10 crossings: 21 ribbon knots.
- ✓ Can be presented as symmetric unions. (Lamm 2000)

Which ribbon knots are symmetric unions?

Empirical evidence

All “small” ribbon knots are symmetric unions.

- ☞ Prime knots up to 10 crossings: 21 ribbon knots.
- ✓ Can be presented as symmetric unions. (Lamm 2000)
- ☞ Two-bridge knots: three infinite families of ribbon knots. (Casson-Gordon 1975; Lisca 2007)
- ✓ Can be presented as symmetric unions. (Lamm 2005)

Which ribbon knots are symmetric unions?

Empirical evidence

All “small” ribbon knots are symmetric unions.

- ☞ Prime knots up to 10 crossings: 21 ribbon knots.
 - ✓ Can be presented as symmetric unions. (Lamm 2000)
- ☞ Two-bridge knots: three infinite families of ribbon knots. (Casson-Gordon 1975; Lisca 2007)
 - ✓ Can be presented as symmetric unions. (Lamm 2005)

Question

Is every ribbon knot a symmetric union?

Paradigm: “knots with extra structure”

Familiar theme: {knots + extra structure} $\xrightarrow[\text{structure}]{\text{forget extra}}$ {knots}

Paradigm: “knots with extra structure”

Familiar theme: $\{\text{knots} + \text{extra structure}\} \xrightarrow[\text{structure}]{\text{forget extra}} \{\text{knots}\}$

- $\{\text{diagrams}\} \longrightarrow \{\text{knots}\}$: Reidemeister's theorem.

Paradigm: “knots with extra structure”

Familiar theme: $\{\text{knots} + \text{extra structure}\} \xrightarrow[\text{structure}]{\text{forget extra}} \{\text{knots}\}$

- $\{\text{diagrams}\} \longrightarrow \{\text{knots}\}$: Reidemeister's theorem.
- $\{\text{braids}\} \longrightarrow \{\text{knots}\}$: Theorems of Alexander and Markov.

Paradigm: “knots with extra structure”

Familiar theme: {knots + extra structure} $\xrightarrow[\text{structure}]{\text{forget extra}}$ {knots}

- {diagrams} \longrightarrow {knots}: Reidemeister's theorem.
- {braids} \longrightarrow {knots}: Theorems of Alexander and Markov.
- Reduced alternating diagrams: Menasco-Thistlethwaite.

Paradigm: “knots with extra structure”

Familiar theme: $\{\text{knots} + \text{extra structure}\} \xrightarrow[\text{structure}]{\text{forget extra}} \{\text{knots}\}$

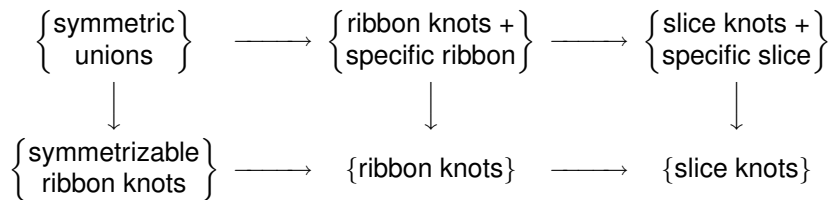
- $\{\text{diagrams}\} \longrightarrow \{\text{knots}\}$: Reidemeister's theorem.
- $\{\text{braids}\} \longrightarrow \{\text{knots}\}$: Theorems of Alexander and Markov.
- Reduced alternating diagrams: Menasco-Thistlethwaite.
- Legendrian knots, transverse knots, etc. . .

Paradigm: “knots with extra structure”

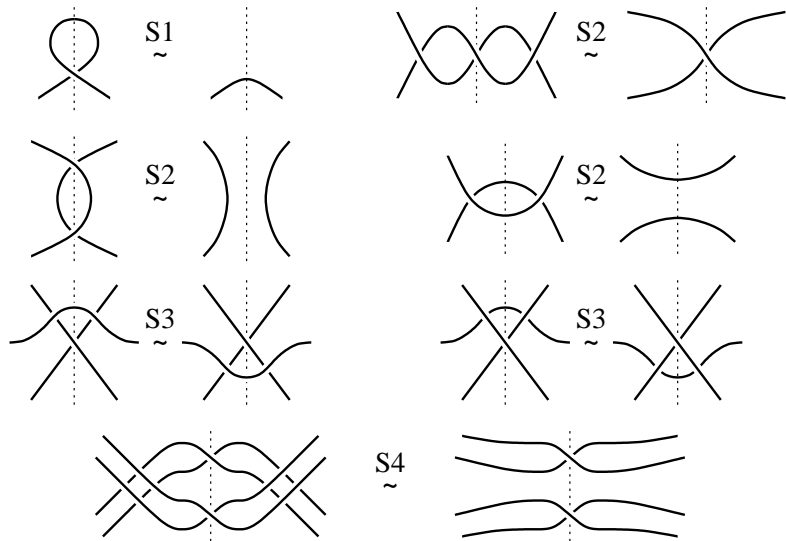
Familiar theme: $\{\text{knots} + \text{extra structure}\} \xrightarrow[\text{structure}]{\text{forget extra}} \{\text{knots}\}$

- $\{\text{diagrams}\} \longrightarrow \{\text{knots}\}$: Reidemeister’s theorem.
- $\{\text{braids}\} \longrightarrow \{\text{knots}\}$: Theorems of Alexander and Markov.
- Reduced alternating diagrams: Menasco-Thistlethwaite.
- Legendrian knots, transverse knots, etc. . .

■ Here:



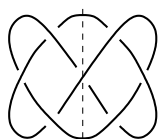
Symmetric Reidemeister moves



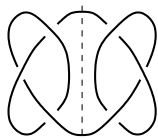
Remark: the associated ribbon changes only up to isotopy.

Partial knots

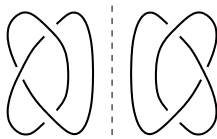
Resolve $\times \mapsto \rangle \langle$ and $\times \mapsto \langle \langle$.



D



$D_- \# D_+$

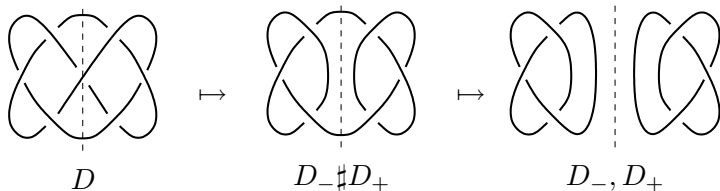


D_-, D_+

Finally split $\times \mapsto \rangle \langle$.

Partial knots

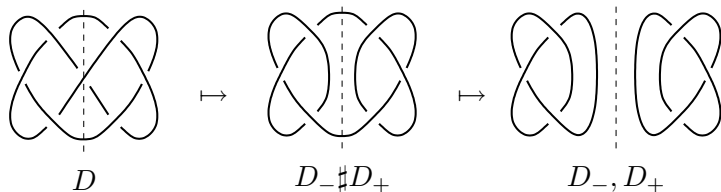
Resolve $\times \mapsto \rangle\langle$ and $\times \mapsto \langle\langle$. Finally split $\times \mapsto \rangle\langle$.



- Partial knots are invariant under S-moves.
- $\det K = \det K_- \cdot \det K_+ = \text{a square}$
- $\Delta K = \Delta K_- \cdot \Delta K_+$ if only full twists on the axis

Partial knots

Resolve $\times \mapsto \rangle \langle$ and $\times \mapsto \rangle \langle$. Finally split $\times \mapsto \rangle \langle$.



- Partial knots are invariant under S-moves.
- $\det K = \det K_- \cdot \det K_+ = \text{a square}$
- $\Delta K = \Delta K_- \cdot \Delta K_+$ if only full twists on the axis

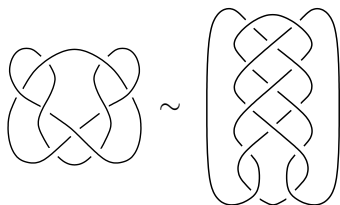
Question

Is there a similar *geometric* construction for ribbon knots?

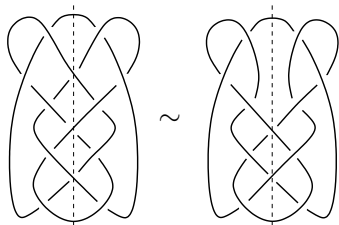
Algebraically we have $\Delta K = f(t) \cdot f(t^{-1})$ for some $f \in \mathbb{Z}[t^{\pm}]$.

Symmetric union representations are not unique

The following diagrams are asymmetrically equivalent:



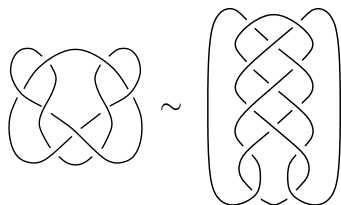
(a) 8_8



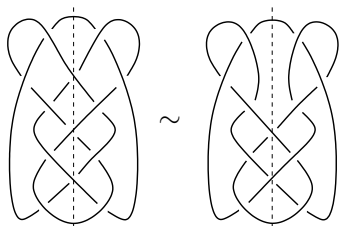
(b) 9_{27}

Symmetric union representations are not unique

The following diagrams are asymmetrically equivalent:



(a) 8_8

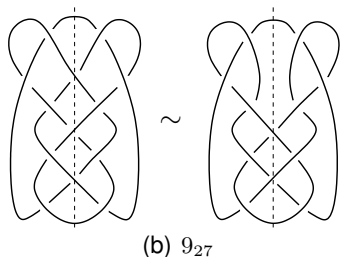
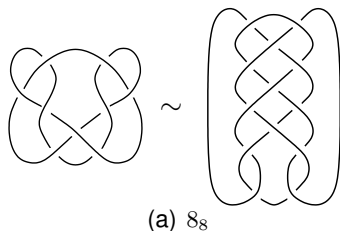


(b) 9_{27}

(a) Not symmetrically equivalent
because partial knots differ ($4_1 \neq 5_1$).

Symmetric union representations are not unique

The following diagrams are asymmetrically equivalent:



- (a) Not symmetrically equivalent because partial knots differ ($4_1 \neq 5_1$).
- (b) Not symmetrically equivalent although partial knots coincide (5_2).

A variation of Kauffman's ansatz

Consider arbitrary diagrams (not necessarily symmetric).

- Crossings off the axis:

$$(A) \quad \langle \text{crossing} \rangle = A^{+1} \langle \text{cup} \rangle + A^{-1} \langle \rangle \langle \rangle.$$

A variation of Kauffman's ansatz

Consider arbitrary diagrams (not necessarily symmetric).

■ Crossings off the axis:

$$(A) \quad \langle \text{crossing} \rangle = A^{+1} \langle \text{cup} \rangle + A^{-1} \langle \rangle \langle \rangle .$$

■ Crossings on the axis:

$$(B) \quad \begin{aligned} \langle \text{crossing} \rangle &= B^{+1} \langle \text{cup} \rangle + B^{-1} \langle \rangle \langle \rangle , \\ \langle \text{crossing} \rangle &= B^{-1} \langle \text{cup} \rangle + B^{+1} \langle \rangle \langle \rangle . \end{aligned}$$

A variation of Kauffman's ansatz

Consider arbitrary diagrams (not necessarily symmetric).

- Crossings off the axis:

$$(A) \quad \langle \text{crossing} \rangle = A^{+1} \langle \text{cup} \rangle + A^{-1} \langle \text{cap} \rangle.$$

- Crossings on the axis:

$$(B) \quad \begin{aligned} \langle \text{crossing} \rangle &= B^{+1} \langle \text{cup} \rangle + B^{-1} \langle \text{cap} \rangle, \\ \langle \text{crossing} \rangle &= B^{-1} \langle \text{cup} \rangle + B^{+1} \langle \text{cap} \rangle. \end{aligned}$$

- Circle evaluation?

Circle evaluation

If C is a collection of n circles
having $2m$ intersections with the axis, then

$$\begin{aligned} \text{(C)} \quad \langle C \rangle &= (-A^2 - A^{-2})^{n-1} \left(\frac{-B^2 - B^{-2}}{-A^2 - A^{-2}} \right)^{m-1} \\ &= (-A^2 - A^{-2})^{n-m} (-B^2 - B^{-2})^{m-1}. \end{aligned}$$

Circle evaluation

If C is a collection of n circles having $2m$ intersections with the axis, then

$$\begin{aligned} \text{(C)} \quad \langle C \rangle &= (-A^2 - A^{-2})^{n-1} \left(\frac{-B^2 - B^{-2}}{-A^2 - A^{-2}} \right)^{m-1} \\ &= (-A^2 - A^{-2})^{n-m} (-B^2 - B^{-2})^{m-1}. \end{aligned}$$

Examples:

$$\begin{aligned} \langle \text{○} \rangle &= 1, & \langle \text{---} \rangle &= \frac{1}{-B^2 - B^{-2}}, \\ \langle \text{○} \rangle &= -B^2 - B^{-2}, & \langle \text{○} \text{---} \rangle &= \frac{-A^2 - A^{-2}}{-B^2 - B^{-2}}, \\ \langle \text{∩} \rangle &= \frac{-B^2 - B^{-2}}{-A^2 - A^{-2}}, & \langle \text{○} \text{---} \text{○} \rangle &= \frac{(-A^2 - A^{-2})^2}{-B^2 - B^{-2}}. \end{aligned}$$

A two-variable refinement of the Jones polynomial

Theorem (invariance)

The normalized map $W : \mathcal{D} \rightarrow \mathbb{Z}(A, B)$ defined by

$$W(D) := \langle D \rangle \cdot (-A^{-3})^{\text{A-writhe}(D)} \cdot (-B^{-3})^{\text{B-writhe}(D)}$$

is invariant under all Reidemeister moves respecting the axis.

A two-variable refinement of the Jones polynomial

Theorem (invariance)

The normalized map $W : \mathcal{D} \rightarrow \mathbb{Z}(A, B)$ defined by

$$W(D) := \langle D \rangle \cdot (-A^{-3})^{A\text{-writhe}(D)} \cdot (-B^{-3})^{B\text{-writhe}(D)}$$

is invariant under all Reidemeister moves respecting the axis.

Convention: $A^2 = t^{-1/2}$, $B^2 = s^{-1/2}$.

$$t^{-1}W\left(\begin{array}{c} \nearrow \\ \searrow \end{array}\right) - t^{+1}W\left(\begin{array}{c} \nwarrow \\ \searrow \end{array}\right) = (t^{1/2} - t^{-1/2})W\left(\begin{array}{c} \smile \\ \smile \end{array}\right)$$

$$s^{-1}W\left(\begin{array}{c} \nearrow \\ \swarrow \end{array}\right) - s^{+1}W\left(\begin{array}{c} \nwarrow \\ \swarrow \end{array}\right) = (s^{1/2} - s^{-1/2})W\left(\begin{array}{c} \smile \\ \searrow \end{array}\right)$$

$$W\left(\begin{array}{c} \supset \\ \vdots \end{array}\right) = \frac{s^{1/2} + s^{-1/2}}{t^{1/2} + t^{-1/2}}W\left(\begin{array}{c} \supset \\ \vdots \end{array}\right)$$

$$W\left(\begin{array}{c} \bigcirc \\ \vdots \end{array}\right) = 1$$

Properties of the W -polynomial

Usual properties

- $W_{D\sharp D'} = W_D \cdot W_{D'}$.
- mirror image: $W_{D^*}(s, t) = W_D(s^{-1}, t^{-1})$.
- W_D is insensitive to orientation reversal, mutation, flypes.

Properties of the W -polynomial

Usual properties

- $W_{D\sharp D'} = W_D \cdot W_{D'}$.
- mirror image: $W_{D^*}(s, t) = W_D(s^{-1}, t^{-1})$.
- W_D is insensitive to orientation reversal, mutation, flypes.

Integrality

If D is a symmetric union, then $W(D) \in \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$.

Properties of the W -polynomial

Usual properties

- $W_{D\sharp D'} = W_D \cdot W_{D'}$.
- mirror image: $W_{D^*}(s, t) = W_D(s^{-1}, t^{-1})$.
- W_D is insensitive to orientation reversal, mutation, flypes.

Integrality

If D is a symmetric union, then $W(D) \in \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$.

↑

Theorem (Eisermann 2007)

For every n -component ribbon link L the Jones polynomial $V(L)$ is divisible by $V(\bigcirc^n) = (t^{1/2} + t^{-1/2})^{n-1}$.

Special values of $W(s, t)$

Specializations in t

- $W_D(s, \xi) = 1$ for $\xi \in \{1, \pm i, e^{\pm 2\pi/3}\}$
- $\frac{\partial W_D}{\partial t}(s, 1) = 0$

$\Rightarrow W_D - 1$ is divisible by $(t - 1)^2(t^2 + 1)(t^2 + t + 1)$.

Special values of $W(s, t)$

Specializations in t

- $W_D(s, \xi) = 1$ for $\xi \in \{1, \pm i, e^{\pm 2\pi/3}\}$
- $\frac{\partial W_D}{\partial t}(s, 1) = 0$

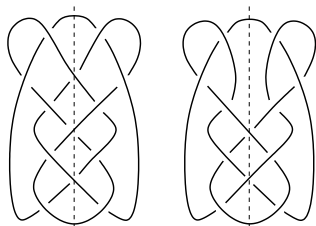
$\Rightarrow W_D - 1$ is divisible by $(t - 1)^2(t^2 + 1)(t^2 + t + 1)$.

Specializations in s

- $W_D(t, t) = V_K(t)$
- $W_D(-1, t) = V_{K_-}(t) \cdot V_{K_+}(t)$

$\Rightarrow W_D(-1, -1) = \det(K) = \det(K_-) \cdot \det(K_+)$

Two symmetric unions for 9_{27}



(a) $D_1 \sim 9_{27}$

(b) $D_2 \sim 9_{27}$

$$W_{D_1}(s, t) = 1 + s^{-1}f(t) - s^{-2}g(t)$$

$$W_{D_2}(s, t) = 1 - s^0f(t) + s^1g(t)$$

$$f(t) = t^{-5} - 3t^{-4} + 6t^{-3} - 9t^{-2} + 11t^{-1} - 12 + 11t - 9t^2 + 6t^3 - 3t^4 + t^5$$

$$g(t) = t^{-4} - 2t^{-3} + 3t^{-2} - 4t^{-1} + 4 - 4t + 3t^2 - 2t^3 + t^4$$

$$W(t, t) = -t^{-5} + 3t^{-4} - 5t^{-3} + 7t^{-2} - 8t^{-1} + 9 - 7t + 5t^2 - 3t^3 + t^4$$

$$W(-1, t) = (t - t^2 + 2t^3 - t^4 + t^5 - t^6)(t^{-1} - t^{-2} + 2t^{-3} - t^{-4} + t^{-5} - t^{-6})$$

Summary

- *Algebraically*, a ribbon knot K looks like a connected sum of a knot K_+ with its mirror image K_- .

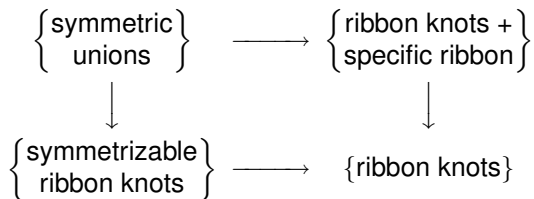
Summary

- *Algebraically*, a ribbon knot K looks like a connected sum of a knot K_+ with its mirror image K_- .
- *Geometrically*, this is modelled by symmetric unions.

Summary

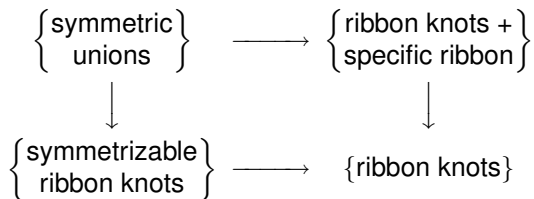
- *Algebraically*, a ribbon knot K looks like a connected sum of a knot K_+ with its mirror image K_- .
- *Geometrically*, this is modelled by symmetric unions.
- This allows us to define partial knots and a refined polynomial $W(s, t)$ that keep track of the symmetry.

■ Extension to ribbon knots?



Perspectives

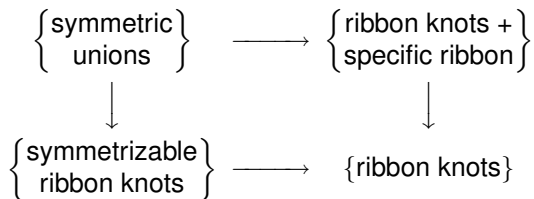
■ Extension to ribbon knots?



■ Obstructions to being a symmetric union / ribbon knot?

Perspectives

- Extension to ribbon knots?



- Obstructions to being a symmetric union / ribbon knot?

- Properties of the Jones polynomial of ribbon knots & links?