The Jones polynomial of ribbon knots and symmetric unions

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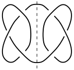
August 3, 2007

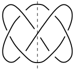
AMS-PTM International Meeting
Warsaw, July 31 – August 3, 2007

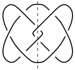
Keywords

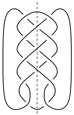
- Symmetric unions of knots
  (Kinoshita-Terasaka 1957)
- Ribbon knots & slice knots
  (Fox-Milnor 1957, 1961, 1965)
- A two-variable refinement of the Jones polynomial
  (preliminary report)

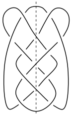
Symmetric unions (Kinoshita-Terasaka 1957)

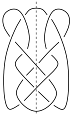

(a) symmetric sum, $3_1 \sharp 3_1^*$


(b) symmetric union, 6_1


(c) symmetric union, 8_{20}


(d) 8_8


(e) 9_{27}

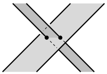

(f) 9_{27}

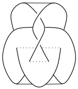
Ribbon & slice (Fox-Milnor 1957, 1961, 1965)

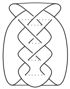
General setting

{symmetric unions} $\subset$ {ribbon knots} $\subset$ {slice knots}
¿ $\supset$? ¿ $\supset$?

Every symmetric union is a ribbon knot:


(a) ribbon singularity


(b) 8_{20}


(c) 10_{87}

Which ribbon knots are symmetric unions?

Empirical evidence

All "small" ribbon knots are symmetric unions.

- ☞ Prime knots up to 10 crossings: 21 ribbon knots.
- ✔ Can be presented as symmetric unions. (Lamm 2000)
- ☞ Two-bridge knots: three infinite families of ribbon knots. (Casson-Gordon 1975; Lisca 2007)
- ✔ Can be presented as symmetric unions. (Lamm 2005)

Question

Is every ribbon knot a symmetric union?

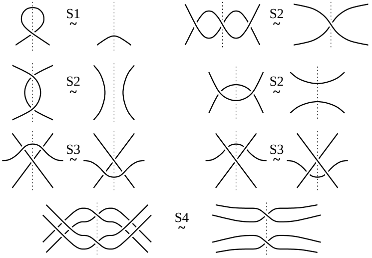
Paradigm: "knots with extra structure"

Familiar theme: $\{$knots + extra structure$\}$ $\xrightarrow[\text{structure}]{\text{forget extra}}$ $\{$knots$\}$

- $\{$diagrams$\}$ $\longrightarrow$ $\{$knots$\}$: Reidemeister's theorem.
- $\{$braids$\}$ $\longrightarrow$ $\{$knots$\}$: Theorems of Alexander and Markov.
- Reduced alternating diagrams: Menasco-Thistlethwaite.
- Legendrian knots, transverse knots, etc. . .
- Here:

{symmetric unions}	⟶	{ribbon knots + specific ribbon}	⟶	{slice knots + specific slice}
↓		↓		↓
{symmetrizable ribbon knots}	⟶	{ribbon knots}	⟶	{slice knots}

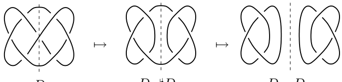
Symmetric Reidemeister moves



Remark: the associated ribbon changes only up to isotopy.

Partial knots

Resolve ⤫ ↦)(and ⤫ ↦)(. Finally split ⤫ ↦)(.



D ↦ $D_-\sharp D_+$ ↦ D_-, D_+

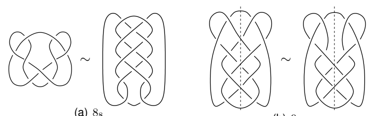
- Partial knots are invariant under S-moves.
- $\det K = \det K_- \cdot \det K_+ =$ a square
- $\Delta K = \Delta K_- \cdot \Delta K_+$ if only full twists on the axis

Question

Is there a similar *geometric* construction for ribbon knots?
Algebraically we have $\Delta K = f(t) \cdot f(t^{-1})$ for some $f \in \mathbb{Z}[t^{\pm}]$.

Symmetric union representations are not unique

The following diagrams are asymmetrically equivalent:



(a) 8_8 (b) 9_{27}

(a) Not symmetrically equivalent because partial knots differ ($4_1 \neq 5_1$).

(b) Not symmetrically equivalent although partial knots coincide (5_2).

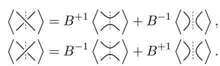
A variation of Kauffman's ansatz

Consider arbitrary diagrams (not necessarily symmetric).

- Crossings off the axis:

(A) 

- Crossings on the axis:

(B) 

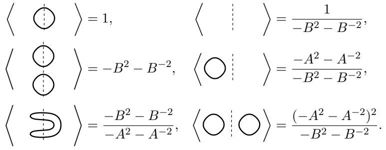
- Circle evaluation?

Circle evaluation

If C is a collection of n circles having $2m$ intersections with the axis, then

(C)
$$\langle C \rangle = (-A^2 - A^{-2})^{n-1} \left(\frac{-B^2 - B^{-2}}{-A^2 - A^{-2}} \right)^{m-1} = (-A^2 - A^{-2})^{n-m} (-B^2 - B^{-2})^{m-1}.$$

Examples:



A two-variable refinement of the Jones polynomial

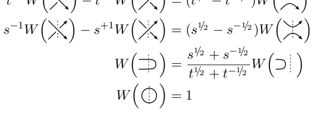
Theorem (invariance)

The normalized map $W \colon \mathscr{D} \to \mathbb{Z}(A, B)$ defined by

$$W(D) := \langle D \rangle \cdot (-A^{-3})^{\text{A-writhe}(D)} \cdot (-B^{-3})^{\text{B-writhe}(D)}$$

is invariant under all Reidemeister moves respecting the axis.

Convention: $A^2 = t^{-1/2}$, $B^2 = s^{-1/2}$.



Properties of the W-polynomial

Usual properties

- $W_{D\sharp D'} = W_D \cdot W_{D'}$.
- mirror image: $W_{D^*}(s,t) = W_D(s^{-1}, t^{-1})$.
- W_D is insensitive to orientation reversal, mutation, flypes.

Integrality

If D is a symmetric union, then $W(D) \in \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$.

$\Uparrow$

Theorem (Eisermann 2007)

For every n-component ribbon link L the Jones polynomial $V(L)$ is divisible by $V(\bigcirc^n) = (t^{1/2} + t^{-1/2})^{n-1}$.

Special values of $W(s,t)$

Specializations in t

- $W_D(s,\xi) = 1$ for $\xi \in \{1, \pm i, e^{\pm 2\pi/3}\}$
- $\frac{\partial W_D}{\partial t}(s,1) = 0$

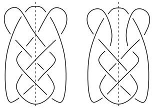
$\Rightarrow \quad W_D - 1$ is divisible by $(t-1)^2(t^2+1)(t^2+t+1)$.

Specializations in s

- $W_D(t,t) = V_K(t)$
- $W_D(-1,t) = V_{K_-}(t) \cdot V_{K_+}(t)$

$\Rightarrow \quad W_D(-1,-1) = \det(K) = \det(K_-) \cdot \det(K_+)$

Two symmetric unions for 9_{27}



(a) $D_1 \sim 9_{27}$ (b) $D_2 \sim 9_{27}$

$$W_{D_1}(s,t) = 1 + s^{-1}f(t) - s^{-2}g(t)$$
$$W_{D_2}(s,t) = 1 - s^{0}f(t) + s^{1}g(t)$$

$$f(t) = t^{-5} - 3t^{-4} + 6t^{-3} - 9t^{-2} + 11t^{-1} - 12 + 11t - 9t^2 + 6t^3 - 3t^4 + t^5$$
$$g(t) = t^{-4} - 2t^{-3} + 3t^{-2} - 4t^{-1} + 4 - 4t + 3t^2 - 2t^3 + t^4$$

$$W(t,t) = -t^{-5} + 3t^{-4} - 5t^{-3} + 7t^{-2} - 8t^{-1} + 9 - 7t + 5t^2 - 3t^3 + t^4$$
$$W(-1,t) = (t - t^2 + 2t^3 - t^4 + t^5 - t^6)(t^{-1} - t^{-2} + 2t^{-3} - t^{-4} + t^{-5} - t^{-6})$$

Summary

- *Algebraically*, a ribbon knot K looks like a connected sum of a knot K_+ with its mirror image K_-.
- *Geometrically*, this is modelled by symmetric unions.
- This allows us to define partial knots and a refined polynomial $W(s,t)$ that keep track of the symmetry.

Perspectives

- Extension to ribbon knots?

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{symmetric}\\ \text{unions}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\text{ribbon knots +}\\ \text{specific ribbon}\end{array}\right\} \\ \downarrow & & \downarrow \\ \left\{\begin{array}{c}\text{symmetrizable}\\ \text{ribbon knots}\end{array}\right\} & \longrightarrow & \{\text{ribbon knots}\} \end{array}$$

- Obstructions to being a symmetric union / ribbon knot?
- Properties of the Jones polynomial of ribbon knots & links?