Reductions of irreducible symplectic varieties defined over number fields

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Hasse-Witt invariant of X = stable rank of F acting on $\mathrm{H}^{\dim(X)}(X, \mathcal{O}_X)$

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Question

Does there exist a finite field extension L/K such that

$$\lim_{x \to \infty} \frac{\sharp\{v \in \Lambda(L, X) \mid q_v \leqslant x\}}{\sharp\{v \text{ finite place of } K \mid q_v \leqslant x\}} = 1?$$

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$$A \times \prod_{i \in I} Y_i \times \prod_{j \in J} Z_j$$

where

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Cartier operator (σ^{-1}-linear) $C : Z\Omega_X^i \to Z\Omega_X^i/B\Omega_X^i = \mathcal{H}^i(\Omega_X^\bullet) \overset{\gamma^{-1}}{\to} \Omega_X^i$

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For almost all v, ρ is unramified at v; for such a v, let $F_{v,\rho} \in \mathrm{GL}(\mathrm{H}^2_{\text{ét}}(X \otimes \bar{K}, \mathbf{Q}_\ell))$ be a geometric Frobenius element at v

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2. $\rightsquigarrow$ α_i are algebraic integers,

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Then, there exists a finite field extension L/K such that $\Phi(L, X)$ has density 0 where $\Phi(L, X) = \{v$ finite place of K s.t. $p_v \mid \mathrm{Tr}(F_{v,\rho})\}$.

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Let $\Phi^1(K,X) = \{v$ finite place of K s.t. k_v is not a prime field$\}$ and $\Phi^2(K,X) = \{v$ finite place of K s.t. $F_{v,\rho\otimes\chi_\ell} = \mathrm{Id}_{\mathrm{H}^2_{\text{ét}}(X\otimes\bar{K},\mathbf{Q}_\ell)(1)}\}$

$\Phi^1(K,X)$ has density 0 (Čebotarev density theorem)

$\overset{2+4}{\rightsquigarrow}$ the ℓ-adic Lie group $\rho \otimes \chi_\ell(\mathrm{Gal}(\bar{K}/K)) \subset \mathrm{GL}(\mathrm{H}^2_{\text{ét}}(X \otimes \bar{K}, \mathbf{Q}_\ell)(1))$ has dimension $\geqslant 1 \overset{\text{Serre}}{\rightsquigarrow} \Phi^2(K,X)$ has density 0

Sketch of proof of Proposition 2

Assumptions:

1. $\ell > 2b_2(X \otimes \mathbf{C})$
2. $\mathrm{NS}(X \otimes \bar{K})) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \twoheadrightarrow \bigcup_U \mathrm{H}^2_{\text{ét}}(X \otimes \bar{K}, \mathbf{Q}_\ell)(1)^U$ where $U \subset \mathrm{Gal}(\bar{K}/K)$ open subgroup,
3. $F_{v,\rho}$ is semisimple for almost all places of K,
4. $\mathrm{rang}(\mathrm{NS}(X \otimes \bar{K})) < \dim_{\mathbf{Q}_\ell}(\mathrm{H}^2_{\text{ét}}(X \otimes \bar{K}, \mathbf{Q}_\ell))$,
5. $\mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(\mathrm{H}^2_{\text{ét}}(X \otimes \bar{K}, \mathbf{Z}_\ell) \otimes (\mathbf{Z}_\ell/\ell\mathbf{Z}_\ell))$ is trivial,
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$\rightsquigarrow \Phi^1(K,X) \cup \Phi^2(K,X)$ has density 0

$\Phi(K,X) \subset \Phi^1(K,X) \cup \Phi^2(K,X)$
$(\Phi(K,X) = \{v \text{ finite place of } K \text{ s.t. } p_v \mid \mathrm{Tr}(F_{v,\rho})\})$

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