CHARACTERIZATIONS OF PROJECTIVE SPACES AND HYPERQUADRICS

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Abstract. In this paper we prove that if the r-th tensor power of the tangent bundle of a smooth projective variety X contains the determinant of an ample vector bundle of rank at least r, then X is isomorphic either to a projective space or to a smooth quadric hypersurface. Our result generalizes Mori's, Wahl's, Andreatta-Wiśniewski's and Araujo-Druel-Kovács's characterizations of projective spaces and hyperquadrics.

1. Introduction

Projective spaces and hyperquadrics are the simplest projective algebraic varieties, and they can be characterized in many ways. The aim of this paper is to provide a new characterization of them in terms of positivity properties of the tangent bundle. We refer the reader to the article [ADK08] which reviews these matters. Notice that our results generalize Mori's (see [Mor79]), Wahl's (see [Wah83] and [Dru04]), Andreatta-Wiśniewski's (see [AW01] and [Ara06]), Araujo-Druel-Kovács's (see [ADK08]) and Paris's (see [Par10]) characterizations of projective spaces and hyperquadrics. K. Ross recently posted a somewhat related result (see [ROS10]).

In this paper, we prove the following theorems. Here Q_n denotes a smooth quadric hypersurface in $\mathbf{P}^{n+1}$, and $\mathscr{O}_{Q_n}(1)$ denotes the restriction of $\mathscr{O}_{\mathbf{P}^{n+1}}(1)$ to Q_n. When $n=1$, $(Q_1, \mathscr{O}_{Q_1}(1))$ is just $(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(2))$.

Theorem A. *Let X be a smooth complex projective n-dimensional variety and $\mathscr{E}$ an ample vector bundle on X of rank $r+k$ with $r \geqslant 1$ and $k \geqslant 1$. If $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$, then $(X, \det(\mathscr{E})) \simeq (\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(l))$ with $r+k \leqslant l \leqslant \frac{r(n+1)}{n}$.*

Theorem B. *Let X be a smooth complex projective n-dimensional variety and $\mathscr{E}$ an ample vector bundle on X of rank $r \geqslant 1$. If $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$, then either $(X, \det(\mathscr{E})) \simeq (\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(l))$ with $r \leqslant l \leqslant \frac{r(n+1)}{n}$, or $(X, \mathscr{E}) \simeq (Q_n, \mathscr{O}_{Q_n}(1)^{\oplus r})$ and $r = 2i + nj$ with $i \geqslant 0$ and $j \geqslant 0$.*

The line of argumentation follows [AW01] (see also [ADK08] and [Par10]). We first prove Theorem A and Theorem B for Fano manifolds with Picard number $\rho(X) = 1$ (see Proposition 14). Then the argument for the proof of the main Theorem goes as follows. We argue by induction on $\dim(X)$. We may assume $\rho(X) \geqslant 2$. Hence the H-rationally connected quotient of X with respect to an unsplit covering family H of rational curves on X is non-trivial. It can be extended in codimension one so that we can produce a normal variety X_B equipped with a surjective morphism π_B with integral fibers onto a smooth curve B such that either $B \simeq \mathbf{P}^1$, $X_B \to B$ is a $\mathbf{P}^d$-bundle for some $d \geqslant 1$ and $h^0(X_B, T_{X_B/\mathbf{P}^1}^{\otimes i} \otimes \pi^*\mathscr{G}^{\otimes r-i} \otimes \det(\mathscr{E})_{|X_B}^{\otimes -1}) \neq 0$ for some integer $1 \leqslant i \leqslant r$ where $\mathscr{G}$ be a vector bundle on $\mathbf{P}^1$ such that $\mathscr{G}^*(2)$ is nef, or $X_B \to B$ is a

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$\mathbf{P}^d$-bundle for some $d \geqslant 1$ and $h^0(X_B, T_{X_B/B}^{\otimes r} \otimes \det(\mathscr{E})_{|X_B}^{\otimes -1} \otimes \pi_B^* \mathscr{G}^*) \neq 0$ where $\mathscr{G}$ is a nef vector bundle on C, or the geometric generic fiber of π_B is isomorphic to a smooth hyperquadric and $h^0(X_B, T_{X_B/B}^{[\otimes r]} \otimes \det(\mathscr{E})_{|X_B}^{\otimes -1} \otimes \pi_B^* \mathscr{G}^*) \neq 0$ where $\mathscr{G}$ is a nef vector bundle on C. But this is impossible unless $X \simeq \mathbf{P}^1 \times \mathbf{P}^1$ (see Lemma 3, Lemma 4 and Proposition 6).

Throughout this paper we work over the field of complex numbers.

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2. Proofs

2.1. **Projective spaces and hyperquadrics.** In this section, we gather some properties of the tangent bundle to projective spaces and smooth hyperquadrics.

Lemma 1. *Let n, r and k be integers with $n \geqslant 1$ and $r \geqslant 1$. Then $h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes r}(-k)) \neq 0$ if and only if $k \leqslant \frac{r(n+1)}{n}$.*

Proof. It is well-known that $T_{\mathbf{P}^n}$ is stable in the sense of Mumford-Takemoto with slope $\mu(T_{\mathbf{P}^n}) = \frac{n+1}{n}$ with respect to $\mathscr{O}_{\mathbf{P}^n}(1)$. By [HL97, Theorem 3.1.4], $T_{\mathbf{P}^n}^{\otimes r}(-r)$ is semistable with slope $\mu(T_{\mathbf{P}^n}^{\otimes r}(-k)) = \frac{r(n+1)}{n} - k$. It follows that if $h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes r}(-k)) \neq 0$ then $k \leqslant \frac{r(n+1)}{n}$. Conversely, let us assume that $k \leqslant \frac{r(n+1)}{n}$. Write $r = an + b$ where a and b are integers with $a \geqslant 0$ and $0 \leqslant b < n$. Then $k - a(n+1) = \llcorner k - a(n+1) \lrcorner \leqslant \llcorner \frac{b(n+1)}{n} \lrcorner = \llcorner b + \frac{b}{n} \lrcorner = b$ and

$$
\begin{aligned}
h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes r}(-k) &= h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes an}(-a(n+1)) \otimes T_{\mathbf{P}^n}^{\otimes b}(-k + a(n+1))) \\
&\geqslant h^0(\mathbf{P}^n, [T_{\mathbf{P}^n}^{\otimes n}(-(n+1))]^{\otimes a} \otimes T_{\mathbf{P}^n}^{\otimes b}(-b)) \\
&\geqslant h^0(\mathbf{P}^n, [\det(T_{\mathbf{P}^n})(-(n+1))]^{\otimes a} \otimes T_{\mathbf{P}^n}^{\otimes b}(-b)) \\
&= h^0(\mathbf{P}^n, T_{\mathbf{P}^n}^{\otimes b}(-b)) \geqslant 1,
\end{aligned}
$$

as claimed. □

Let d be a positive integer. Let $Q \subset \mathbf{P}^{d+1} = \mathbf{P}(W)$ be a smooth hyperquadric defined by a non degenerate quadratic form q on $W := \mathbf{C}^{d+2}$ and let $\mathscr{O}_Q(1)$ denote the restriction of $\mathscr{O}_{\mathbf{P}^{d+1}}(1)$ to Q. Let x be a point of Q and $w \in W \setminus \{0\}$ representing x; then $T_Q(-1)_x$ identifies with $x^\perp / < x >$ and q induces an isomorphism $T_Q(-1) \simeq \Omega_Q^1(1)$ or equivalently a nonzero section in $H^0(Q, (T_Q(-1))^{\otimes 2})$ still denoted by q. Let $V := x^\perp / < x >$. Let $G := SO(W)$ and let $P \subset SO(W)$ be the parabolic subgroup such that $G/P \simeq Q$ corresponding to $x \in Q$. Let $\det \in H^0(Q, \det(T_Q(-1))$ be a nonzero section.

Lemma 2. *Let the notations be as above.*

(1) *The vector bundle T_Q is stable in the sense of Mumford-Takemoto; in particular, one has $h^0(Q, T_Q^{\otimes r}(-k)) = 0$ for $k > r \geqslant 1$.*
(2) *The space of sections $H^0(Q, (T_Q(-1))^{\otimes r})$ is generated as a $\mathbf{C}$-vector space by the $\sigma \cdot q^{\otimes i} \otimes \det^{\otimes j}$'s where i and j are nonnegative integers such that $r = 2i + dj$ and $\sigma \in \mathfrak{S}_r$ the symmetric group on r letters acting as usual on the vector bundle $(T_Q(-1))^{\otimes r}$.*

Proof. Observe that $T_Q(-1)$ is homogeneous or equivalently that

$$T_Q(-1) \simeq (G \times V)/P$$

over $Q \simeq G/P$ where $g \in P$ acts on $G \times V$ by the formula

$$g \cdot (g', v) = (g'g, \rho(g^{-1}) \cdot v)$$

and

$$\rho : P \to GL(T_Q(-1)_x) = GL(V)$$

is the stabilizer representation. It vanishes on the unipotent radical U of P and can be viewed as the representation of the Levi subgoup $L \simeq \mathbf{C}^* \times SO(V) \subset P$ on V given by the standard representation of $SO(V)$ on V. It is irreducible and therefore $T_Q(-1)$ is indecomposable hence stable by [Ram66] and [Ume78] with slope $\mu(T_Q(-1)) = 0$ with respect to $\mathscr{O}_Q(1)$. By [HL97, Theorem 3.1.4], $(T_Q(-1))^{\otimes k}$ is semistable with slope $\mu((T_Q(-1))^{\otimes r}) = 0$. This ends the proof of the first part of the Lemma.

Observe that $(T_Q(-1))^{\otimes r}$ is homogeneous and that the stabilizer representation

$$P \to GL((T_Q(-1))_x^{\otimes r})$$

is $\rho^{\otimes r}$. In particular, $(T_Q(-1))^{\otimes r}$ decomposes as the direct sum of indecomposable vector bundles hence as the direct sum of stable vector bundles with slope 0. It follows that there is a one-to-one correspondence between the set of nonzero section in $H^0(Q, (T_Q(-1))^{\otimes r})$ and the set of rank one direct summands of $T_Q(-1))^{\otimes r}$. Finally, we obtain an isomorphism

$$H^0(Q, (T_Q(-1))^{\otimes r}) \simeq (V^{\otimes r})^{SO(V)}$$

since $SO(V)$ has no nontrivial character. The result now follows from [Wey39, Theorem 2.9 A]. □

2.2. **Fibrations over curves.** In this section, we prove our main Theorems for fibrations over curves.

Lemma 3. *Let $\mathscr{F}$ be a vector bundle on $\mathbf{P}^1$ of rank $m \geqslant 2$, $X := \mathbf{P}_{\mathbf{P}^1}(\mathscr{F})$ and $\pi : X \to \mathbf{P}^1$ the natural morphism. Let $\mathscr{E}$ be an ample vector bundle on X of rank $r+k$ with $r \geqslant 2$ and $k \geqslant 0$. Let $\mathscr{G}$ be a vector bundle on $\mathbf{P}^1$ such that $\mathscr{G}^*(2)$ is nef. If $h^0(X, T_{X/\mathbf{P}^1}^{\otimes i} \otimes \pi^*\mathscr{G}^{\otimes r-i} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$ for some integer $0 \leqslant i \leqslant r$ then $X \simeq \mathbf{P}^1 \times \mathbf{P}^1$, $\mathscr{F} = \mathscr{O}_{\mathbf{P}^1}(a)^{\oplus 2}$ for some integer a, $k = 0$, $2i = r$ and $\det(\mathscr{E}) \simeq \mathscr{O}_{\mathbf{P}^1}(2) \boxtimes \mathscr{O}_{\mathbf{P}^1}(2)$.*

Proof. Write $\mathscr{F} \simeq \mathscr{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1}(a_m)$ with $a_1 \leqslant \cdots \leqslant a_m$. Let $b := a_m - a_1 \geqslant 0$. Let σ be a section of π corresponding to a surjective morphism $\mathscr{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1}(a_m) \twoheadrightarrow \mathscr{O}_{\mathbf{P}^1}(a_m)$ and let σ_1 the section of π corresponding to the projection map $\mathscr{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1}(a_m) \twoheadrightarrow \mathscr{O}_{\mathbf{P}^1}(a_1)$. Then $\sigma \equiv \sigma_1 + b\ell$ where ℓ is vertical line and

$$\det(\mathscr{E}) \cdot \sigma \geqslant r + k + b(r+k) = (r+k)(b+1).$$

We may assume that $h^0(\sigma, (T_{X/\mathbf{P}^1}^{\otimes i} \otimes \pi^*\mathscr{G}^{\otimes r-i} \det(\mathscr{E})^{\otimes -1})_{|\sigma}) \neq 0$ since σ is a free rational curve. But

$$T_{X/\mathbf{P}^1}{}_{|\sigma} \simeq N_{\sigma/X} \simeq \mathscr{O}_{\mathbf{P}^1}(a_m - a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1}(a_m - a_{m-1})$$

and we obtain

$$(r+k)(b+1) \leqslant \det(\mathscr{E}) \cdot \sigma \leqslant ib + 2(r-i). \tag{1}$$

By Lemma 1, we must have

$$r + k \leqslant i\frac{m}{m-1}. \tag{2}$$

We obtain

$$(3)\qquad (r+k)(b+1)+2k \leqslant ib+2(r+k)-2i \leqslant ib+2i\frac{m}{m-1}-2i = i(b+\frac{2}{m-1}).$$

It follows that $m=2$, $b=k=0$ and $r=2i$. □

Lemma 4. *Let X be a smooth complex projective variety, $\mathscr{E}$ an ample vector bundle on X of rank $r+k$ with $r \geqslant 1$ and $k \geqslant 0$. Let $\pi : X \to B$ be a surjective morphism onto a smooth connected curve with integral fibers. Let $\mathscr{G}$ be a numerically effective vector bundle on B of rank > 0. Assume that the geometric generic fiber is isomorphic to a projective space. Then $h^0(X, T_{X/B}^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*) = 0$.*

Proof. Let η be the generic point of B. Tsen's Theorem implies that $X_\eta \simeq \mathbf{P}^d_{k(\eta)}$. Thus there exists a divisor H on X such that $\mathscr{O}_X(H)_{|X_\eta} \simeq \mathscr{O}_{\mathbf{P}^d_{k(\eta)}}(1)$. Let $\mathscr{L} := \mathscr{O}_X(H)$. Let $r' \geqslant r+k$ be defined by the formula $\det(\mathscr{E})_{|X_\eta} \simeq \mathscr{O}_{\mathbf{P}^d_{k(\eta)}}(r')$. It follows from the semicontinuity Theorem that $h^0(X_b, (\det(\mathscr{E}) \otimes \mathscr{L}^{\otimes -r'})_{|X_b}) \geqslant 1$ and $h^0(X_b, (\mathscr{L}^{\otimes r'} \otimes \det(\mathscr{E})^{\otimes -1})_{|X_b}) \geqslant 1$ for any point b in B. Thus $h^0(X_b, (\det(\mathscr{E}) \otimes \mathscr{L}^{\otimes -r'})_{|X_b}) = 1$ since X_b is integral. By the base change Theorem, $\det(\mathscr{E}) \simeq \mathscr{L}^{\otimes r'} \otimes \pi^*\mathscr{M}$ for some line bundle $\mathscr{M}$ on B. Thus $\mathscr{L}$ is ample$/B$ and by [Fuj75, Corollary 5.4], π is a $\mathbf{P}^d$-bundle. By replacing B with a finite cover $\bar{B} \to B$ and X with $X \times_B \bar{B}$ we may assume that $g(B) \geqslant 1$. Let $\mathscr{M}'$ be a line bundle on B such that $\mathscr{M} \simeq \mathscr{M}'^{\otimes r'}$. Set $\mathscr{L}' := \mathscr{L} \otimes \pi^*\mathscr{M}'^{\otimes -1}$. Then $\mathscr{L}'^{\otimes r'} \simeq \det(\mathscr{E})$ hence $\mathscr{L}'$ is ample. Let $\mathscr{F} := \pi_*(\mathscr{L}')$. Then $\mathscr{F}$ is an ample vector bundle on B and $X := \mathbf{P}_B(\mathscr{F})$. By [CF90], By replacing B with a finite cover $\bar{B} \to B$ and X with $X \times_B \bar{B}$, we may assume that there exist an ample line bundle $\mathscr{M}$ on B, a positive integer m and a surjective map of $\mathscr{O}_B$-modules $\mathscr{M}^{\oplus m} \twoheadrightarrow \mathscr{F}$. Observe that the line bundle $\mathscr{L}' \otimes \pi^*\mathscr{M}^{\otimes -1}$ is generated by its global sections. Let $C = D_1 \cap \cdots \cap D_{\dim(X)-1}$ be general complete intersection curve with $D_i \in |\mathscr{L}' \otimes \pi^*\mathscr{M}^{\otimes -1}|$ (C is a section of π). Then $(T_{X/B})_{|C} \simeq N_{C/X} \simeq (\mathscr{L}' \otimes \pi^*\mathscr{M}^{\otimes -1})^{\oplus \dim(X)-1}_{|C}$. But

$$h^0(C, (\mathscr{L}' \otimes \pi^*\mathscr{M}^{\otimes -1})^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}_{|C}) = h^0(C, \mathscr{L}'^{\otimes r - r'} \otimes \pi^*\mathscr{M}^{\otimes -r}_{|C}) = 0$$

and the claim follows. □

When dealing with sheaves that are not necessarily locally free, we use square brackets to indicate taking the reflexive hull.

Notation 5 (Reflexive tensor operations)**.** Let X be a normal variety and $\mathscr{Q}$ a coherent sheaf of $\mathscr{O}_X$-modules. For $n \in \mathbf{N}$, set $\mathscr{Q}^{[\otimes n]} := (\mathscr{Q}^{\otimes n})^{**}$, $S^{[n]}\mathscr{Q} := (S^n\mathscr{Q})^{**}$ and $\det(\mathscr{Q}) := (\wedge^{\mathrm{rank}(\mathscr{Q})}(\mathscr{Q}))^{**}$.

Proposition 6. *Let X be a normal complex projective variety, $\mathscr{E}$ an ample vector bundle on X of rank $r+k$ with $r \geqslant 1$ and $k \geqslant 0$. Let $\pi : X \to B$ be a surjective morphism onto a smooth connected curve with integral fibers. Let $\mathscr{G}$ be a numerically effective vector bundle on B of rank > 0. Assume that the geometric generic fiber is isomorphic to a smooth hyperquadric. Then $h^0(X, T_{X/B}^{[\otimes r]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*) = 0$.*

Proof. Let η be the generic point of B and $k(\bar{\eta})$ be an algebraic closure of $k(\eta)$. Let $q_{\bar{\eta}}$ be a non degenerate quadratic form defining $X_{\bar{\eta}} \subset \mathbf{P}^{d+1}_{k(\bar{\eta})}$ where $d := \dim(X)-1$. By Lemma 2, $k=0$ and $\det(\mathscr{E})_{|X_{\bar{\eta}}} \simeq \mathscr{O}_{X_{\bar{\eta}}}(r)$.

Let us assume to the contrary that $h^0(X, T_{X/B}^{[\otimes r]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*) \neq 0$ and let $s \in H^0(X, T_{X/B}^{[\otimes r]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*)$ be a nonzero section. Notice that, for any $\sigma \in \mathfrak{S}_r$ and any non negative integers i and j such that $r = 2i + dj$,

$$\sigma \cdot [(S^{[2]}T_{X/B})^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]}] \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*$$

is a direct summand of

$$T_{X/B}^{[\otimes r]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*.$$

By Lemma 2, we may assume that

$$s \in H^0(X, (S^{[2]}T_{X/B})^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*)$$

and

$$s_{|X_{\bar\eta}} = q_{\bar\eta}^{\otimes i} \otimes {\det}_{\bar\eta}^{\otimes j} \otimes g_{\bar\eta}$$

for some non negative integers i and j with $r = 2i+dj$ and some non zero section $g_{\bar\eta} \in \pi^*H^0(\bar\eta, \mathscr{G}_{|\bar\eta})$. It follows that the induced map

$$\mathscr{G} \to \pi_*((S^{[2]}T_{X/B})^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]} \otimes \det(\mathscr{E})^{\otimes -1})$$

has rank one and therefore, we may assume that $\mathscr{G}$ is a line bundle (with $\deg(\mathscr{G}) \geqslant 0$). We obtain a map

$$\varphi_s : {\Omega^1_{X/B}}^{[\otimes i]} \to {T_{X/B}}^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*$$

whose restriction to $X_{\bar\eta}$ is an isomorphism. Finally, we obtain a nonzero section

$$\begin{aligned} s' := \det(\varphi_s) \in H^0(X, \det({T_{X/B}}^{[\otimes i]}) \otimes \det({T_{X/B}}^{[\otimes i]} \otimes \det(T_{X/B})^{[\otimes j]} \otimes \det(\mathscr{E})^{\otimes -1} \otimes \pi^*\mathscr{G}^*)) \\ \simeq H^0(X, \det(T_{X/B})^{[\otimes(2id^{i-1}+d^ij)]} \otimes \det(\mathscr{E})^{\otimes -d^i} \otimes \pi^*\mathscr{G}^{\otimes -d^i}). \end{aligned}$$

Observe that s' does not vanish anywhere on a general fiber of π and that any fiber of π is integral. Thus

$$-K_{X/B} \equiv \frac{d^i}{2id^{i-1} + d^ij} c_1(\det(\mathscr{E})) + \pi^*\Delta$$

for some (integral) effective divisor $\Delta \geqslant \frac{d^i}{2id^{i-1}+d^ij} c_1(\mathscr{G})$ and $-K_{X/B}$ is ample. But that contradicts Lemma 7. □

Lemma 7 ([ADK08, Theorem 3.1])**.** *Let X be a normal projective variety, $f : X \to C$ a surjective morphism onto a smooth curve, and $\Delta \subseteq X$ a Weil divisor such that (X, Δ) is log canonical over the generic point of C. Then $-(K_{X/C} + \Delta)$ is not ample.*

Lemma 8. *Let S be a smooth projective surface equipped with a surjective morphism $\pi : S \to B$ with connected fibers onto a smooth connected curve. Let $\mathscr{M}$ be a nef and big line bundle on S. Assume that, for a general point b in B, $\mathscr{M} \cdot S_b = 2r$ for some $r \geqslant 1$ and either $g(B) \geqslant 1$ or $B = \mathbf{P}^1$ and S is a ruled surface over B. Then $h^0(S, T_S^{\otimes r} \otimes \mathscr{M}^{\otimes -1}) = 0$.*

Proof. Let $c : S \to \bar S$ be a minimal model. Write $\mathscr{M} = c^*\bar{\mathscr{M}}(-E)$ for some divisor E on S supported on the exceptional locus of c. Observe that E is effective and $\bar{\mathscr{M}}$ is nef since $\mathscr{M}$ is nef. Therefore, the natural map $T_S \to c^*T_{\bar S}$ induces an inclusion $H^0(S, T_S^{\otimes r} \otimes \mathscr{M}^{\otimes -1}) \subset H^0(\bar S, T_{\bar S}^{\otimes r} \otimes \bar{\mathscr{M}}^{\otimes -1})$. If

$g(B) \geqslant 1$ then π induces a morphism $\bar{S} \to B$ and $\bar{S}$ is a ruled surface over B. We may thus assume that $S \to B$ is smooth. Since $\mathscr{M} \cdot S_b = 2r$ and $T_{S/C} \cdot S_b = 2$ for $b \in B$, we must have

$$H^0(S, T_{S/B}^{\otimes r} \otimes \mathscr{M}^{\otimes -1}) = H^0(S, T_S^{\otimes r} \otimes \mathscr{M}^{\otimes -1}).$$

Let us assume to the contrary that $h^0(S, T_S^{\otimes r} \otimes \mathscr{M}^{\otimes -1}) \neq 0$. Then $r(-K_{S/B}) \sim c_1(\mathscr{M}) + \pi^*\Delta$ where Δ is an effective divisor on C and $K_{S/B}$ is nef and big. But $K_{S/B}^2 = 0$ for any (geometrically) ruled surface, a contradiction. □

2.3. **Tools.** The proof of the main Theorem will apply rational curves on X. Our notation is consistent with that of [Kol96].

Let X be a smooth complex projective uniruled variety and H an irreducible component of $\mathrm{RatCurves}(X)$. Recall that only general points in H are in 1:1-correpondence with the associated curves in X. Let ℓ be a rational curve corresponding to a general point in H, with normalization morphism $f : \mathbf{P}^1 \to \ell \subset X$. We denote by $[\ell]$ or $[f]$ the point in H corresponding to ℓ.

We say that H is a *dominating family of rational curves on* X if the corresponding universal family dominates X. A dominating family H of rational curves on X is called *unsplit* if it is proper. It is called *minimal* if, for a general point $x \in X$, the subfamily of H parametrizing curves through x is proper.

Let $H_1, \ldots, H_k$ be minimal dominating families of rational curves on X. For each i, let $\overline{H}_i$ denote the closure of H_i in $\mathrm{Chow}(X)$. We define the following equivalence relation on X, which we call $(H_1, \ldots, H_k)$-equivalence. Two points $x, y \in X$ are $(H_1, \ldots, H_k)$-equivalent if they can be connected by a chain of 1-cycles from $\overline{H}_1 \cup \cdots \cup \overline{H}_k$. By [Cam92] (see also [Kol96, IV.4.16]), there exists a proper surjective morphism $\pi_0 : X_0 \to Y_0$ from a dense open subset of X onto a normal variety whose fibers are $(H_1, \ldots, H_k)$-equivalence classes. We call this map the $(H_1, \ldots, H_k)$-*rationally connected quotient of* X. For more details see [Kol96].

Lemma 9. *Let X be a smooth complex projective variety and $H_1, \ldots, H_k$ unsplit dominating families of rational curves on X. Let $\pi_0 : X_0 \to Y_0$ be the $(H_1, \ldots, H_k)$-rationally connected quotient of X. If the geometric generic fiber is isomorphic to a projective space, then π_0 is a $\mathbf{P}^d$-bundle in codimension one in Y_0 with $d := \dim(X_0) - \dim(Y_0)$.*

Proof. By [ADK08, Lemma 2.2], we may assume that π_0 is a proper surjective equidimensional morphism with integral fibers. Let $C_0 \subset Y_0$ be a general complete intersection curve. Set $X_{C_0} := \pi_0^{-1}(C_0)$. Then X_{C_0} is a smooth variety. Let η be the generic point of C_0 and let $\mathscr{L}_{C_0}$ be a line bundle on X_{C_0} that restricts to $\mathscr{O}_{\mathbf{P}^d_{k(\eta)}}(1)$ on $X_{C_0\,\eta} \simeq \mathbf{P}^d_{k(\eta)}$ $(d \geqslant 1)$ (see the proof of Lemma 4). Let $\mathscr{M}$ be an ample line bundle on X and r a positive integer such that $\mathscr{M}_{|X_{C_0\,\eta}} \simeq \mathscr{O}_{\mathbf{P}^d_{k(\eta)}}(r)$.

For each i, denote by H_i^j, $1 \leq j \leq n_i$, the unsplit covering families of rational curves on X_{C_0} whose general members correspond to rational curves on X from the family H_i. Then $\pi_{C_0} := \pi_{0|X_{C_0}} : X_{C_0} \to C_0$ is the $(H_1^1, \ldots, H_1^{n_1}, \ldots, H_k^1, \ldots, H_k^{n_k})$-rationally connected quotient of X_{C_0}. Let F be a fiber of π_{C_0}. Let $[H_i^j]$ denote the class of a member of H_i^j in $N_1(F)$ and $\mathcal{H} := \{[H_i^j] \mid i = 1, \ldots, k, j = 1, \ldots, n_i\}$. Then by [Kol96, Proposition IV 3.13.3], $N_1(F)$ is generated by $\mathcal{H}$. Therefore any curve contained in any fiber of π_{C_0} is numerically proportional in $N_1(X_{C_0}/C_0)$ to a linear combination of the $[H_i^j]$'s. Hence $N_1(X_{C_0}/C_0)$ is generated by $\mathcal{H}$ and $c_1(\mathscr{M}_{X_{C_0}}) = rc_1(\mathscr{L}_{C_0}) \in N_1(X_{C_0}/C_0)$. Thus $\mathscr{L}_{X_{C_0}}$ is ample/C_0 and the claim follows from [Fuj75, Corollary 5.4]. □

Notation 10. Let X be a normal variety and $\mathscr{Q}$ be a coherent torsion free sheaf of $\mathscr{O}_X$-modules. Say that a curve $C \subset X$ is a general complete intersection curve for $\mathscr{Q}$ in the sense of Mehta-Ramanathan if $C = H_1 \cap \cdots \cap H_{\dim(X)-1}$, where $H_i \in |m_i H|$ are general, H is an ample line bundle on X and the $m_i \in \mathbf{N}$ are large enough so that the Harder-Narasimhan filtration of $\mathscr{Q}$ commutes with restriction to C.

The following result was established in [Par10, Proposition 4.1].

Lemma 11. *Let X and Y be a smooth complex projective varieties with $\dim(Y) \geqslant 1$, X_0 an open subset of X with $\operatorname{codim}_X(X \setminus X_0) \geq 2$, Y_0 a dense open subset of Y and $\pi_0 : X_0 \to Y_0$ a proper surjective equidimensional morphism. Let $C \subset X_0$ be a general complete intersection curve for $\pi_0^*\Omega^1_{Y_0}$ in the sense of Mehta-Ramanathan. If $(\pi_0^*\Omega^1_{Y_0})_{|C}$ is not nef then Y is uniruled.*

Proof. Let us sketch the proof for the reader's convenience. Fix an ample line bundle H on X, and consider general elements $H_i \in |m_i H|$, for $i \in \{1, \dots, \dim(X) - 1\}$, where the $m_i \in \mathbf{N}$ are large enough so that the Harder-Narasimhan filtration of $\pi_0^*\Omega^1_{Y_0}$ commutes with restriction to $C := H_1 \cap \cdots \cap H_{\dim(X)-1}$. Setting $Z := H_1 \cap \cdots \cap H_{\dim(X)-\dim(Y)}$ and $Z_0 := Z \cap X_0$, we may assume that Z is a smooth variety of dimension $\dim(Y)$, and that the restriction $\varphi_0 := \pi_0|_{Z_0}$ is a finite morphism.

By the hypothesis $(\varphi_0^*\Omega^1_{Y_0})|_C$ is not nef, therefore $(\varphi_0^* T_{Y_0})|_C$ contains a subsheaf with positive slope. Thus if we denote by $i : Z_0 \hookrightarrow Z$ the inclusion and by $\mathscr{F}$ the reflexive sheaf $i_*(\varphi_0^* T_{Y_0})$, then the maximally destabilizing subsheaf $\mathscr{E}$ of $\mathscr{F}$ has positive slope (with respect to $H_{|Z}$).

Let K be a splitting field of the function field $K(Z_0)$ over $K(Y_0)$, and let $\psi : T \to Z$ be the normalization of Z in K. Consider $T_0 := \psi^{-1}(Z_0)$, and let $j : T_0 \hookrightarrow T$ be the inclusion. If we denote by ψ_0 the restriction of ψ to T_0, then the reflexive sheaf $\mathscr{F}' := (\psi^*\mathscr{F})^{**} = j_*(\psi_0^*\varphi_0^* T_{Y_0})$ contains the sheaf $(\psi^*\mathscr{E})^{**}$. Notice that $(\psi^*\mathscr{E})^{**}$ has positive slope. Consequently the maximally destabilizing subsheaf $\mathscr{E}'$ of $\mathscr{F}'$ has positive slope. Hence by replacing Z_0 with T_0, φ_0 with $\varphi_0 \circ \psi_0$, and $(\mathscr{F}, \mathscr{E})$ with $(\mathscr{F}', \mathscr{E}')$ if necessary, we may assume that $K(Z_0) \supset K(Y_0)$ is a Galois extension with Galois group G.

Because of its uniqueness, the maximally destabilizing subsheaf $\mathscr{E}$ of $\mathscr{F}$ is invariant under the action of G. Thus by replacing Z_0 with another open subset of Z if necessary, we may assume that there exists a saturated subsheaf $\mathscr{G}$ of T_{Y_0} such that $\mathscr{E} = i_*(\varphi_0^*\mathscr{G})$.

As $\mathscr{E}$ has positive slope, it follows from [KSCT07, Proposition 29 and Proposition 30] that the vector bundles $\mathscr{E}_{|C}$ and $(\mathscr{E} \otimes \mathscr{E} \otimes (\mathscr{F}/\mathscr{E})^*)_{|C}$ are ample. The morphism φ_0 being finite, this implies that $\mathscr{G}_{|\varphi_0(C)}$ and $(\mathscr{G} \otimes \mathscr{G} \otimes (T_{Y_0}/\mathscr{G})^*)_{|\varphi_0(C)}$ are ample vector bundles too. In particular we deduce from this that $\operatorname{Hom}(\mathscr{G} \otimes \mathscr{G}, T_{Y_0}/\mathscr{G}) = 0$, because the deformations of the curve $\varphi_0(C)$ dominate the variety Y_0. As a consequence $\mathscr{G}$ is a foliation on Y_0.

Finally, by extending $\mathscr{G}$ to a foliation $\widetilde{\mathscr{G}}$ on the whole variety Y, we can conclude by using [KSCT07, Theorem 1]. Indeed it follows from the fact that $\widetilde{\mathscr{G}}_{|\varphi_0(C)}$ is ample that the leaf of the foliation $\widetilde{\mathscr{G}}$ passing through a general point of $\varphi_0(C)$ is rationally connected; in particular Y is uniruled. □

The proof of our main result is based on the following result wich appears essentially in [Par10].

Corollary 12. *Let X be a smooth complex projective variety, X_0 an open subset of X with $\operatorname{codim}_X(X \setminus X_0) \geq 2$, Y_0 a smooth variety with $\dim(Y_0) \geqslant 1$ and $\pi_0 : X_0 \to Y_0$ a proper surjective equidimensional morphism. Assume that the generic fiber of π_0 is isomorphic to a projective* *space. Let C be a general complete intersection curve for $\pi_0^*\Omega^1_{Y_0}$ in the sense of Mehta-Ramanathan. If $(\pi_0^*\Omega^1_{Y_0})_{|C}$ is not nef then there exists a minimal free morphism $f : \mathbf{P}^1 \to Y_0$.*

Proof. Let Y be a smooth projective variety containing Y_0 as a dense open subset. By Lemma 11, Y is uniruled. Let H_Y be a minimal dominating family of rational curves on Y. Since the generic fiber of π_0 is isomorphic to a projective space, there exists a dominating family H_X of rational curves on X such that for a general member $[f] \in H_X$, $[\pi_0 \circ f]$ is a general member of H_Y. By [Kol96, Proposition II 3.7], if $[f] \in H_X$ is a general member then $f(\mathbf{P}^1) \subset X_0$. The claim follows from [Kol96, Corollary IV 2.9]. □

The following Lemma is certainly well known to experts. We include a proof for lack of an adequate reference.

Lemma 13. *Let X be a smooth complex variety and H a minimal dominating family of rational curves on X. Let x be a general point in X and $[\ell] \in H$ with $x \in \ell$. If $T_{\ell,x}$ does not depend on $\ell \ni x$ then there exists a non empty open subset X_0 in X and a proper surjective morphism $\pi_0 : X_0 \to Y_0$ onto a variety Y_0 such that any fiber of π_0 is a rational curve from the family H.*

Proof. Let $[f] \in H$ be a general member. By [Kol96, Corollary IV 2.9], $f^*T_X \simeq \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbf{P}^1}^{\oplus(n-d-1)}$ with $d := -K_X \cdot f_*\mathbf{P}^1 - 2$. Let x be a general point in X with $x \in \ell := f(\mathbf{P}^1)$. By [Hwa01, Proposition 2.3], $d = 0$ using the fact that $T_{\ell,x}$ does not depend on $\ell \ni x$.

Let $\bar{H}$ be the normalization of the closure of H in $\mathrm{Chow}(X)$ and $\bar{U}$ the normalization of the universal family. Let us denote by $\bar{\pi} : \bar{U} \to \bar{H}$ and $\bar{e} : \bar{U} \to X$ the universal morphisms. By shrinking H if necessary, we may assume that H parametrizes free morphisms. Then H is smooth (see [Kol96, Theorem I 2.16]) and $e := \bar{e}_{|U} : U \to X$ is étale where $U := \bar{\pi}^{-1}(H)$ (see [Kol96, Proposition II 3.4]).

It remains to show that there exists a dense open subset H_0 of H such that the restriction of $\bar{e}$ to $\bar{\pi}^{-1}(H_0)$ induces an isomorphism onto the open set $\bar{e}(\bar{\pi}^{-1}(H_0))$. By Zariski's main Theorem, it is enough to prove that $\bar{e}$ is birational. We argue by contradiction. Then there exists a curve $C \subset \bar{U}$ such that $\dim(\bar{\pi}(C)) = 1$ and $\bar{e}(C) = \ell$. Let c be a general point in C. Then $d_c\bar{e}(T_{C,c}) = d_c\bar{e}(T_{\bar{\pi}^{-1}(\bar{\pi}(c)),c}) = T_{\ell,\bar{e}(c)}$. But that contradicts the fact that $\bar{e}$ is étale at c. The claim follows. □

2.4. **Characterizations of projective spaces and hyperquadrics.** The proof of the main Theorem stated in the introduction is based on the following result whose proof is similar to that of [ADK08, Theorem 6.3].

Proposition 14. *Let X be a smooth complex projective n-dimensional variety with $\rho(X) = 1$ and $\mathscr{E}$ an ample vector bundle on X of rank $r+k$ with $r \geqslant 1$ and $k \geqslant 0$. If $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$, then either $X \simeq \mathbf{P}^n$, or $k = 0$ and $X \simeq Q_n$ ($n \neq 2$).*

Proof. Let us give the proof following [ADK08]. First notice that X is uniruled by [Miy87], and hence a Fano manifold with $\rho(X) = 1$. The result is clear if $\dim X = 1$, so we assume that $n \geq 2$. Fix a minimal dominating family H of rational curves on X. Let $\mathscr{L}$ be an ample line bundle on X such that $\mathrm{Pic}(X) = \mathbf{Z}[\mathscr{L}]$.

Let $\mathscr{E}' \subset T_X$ be the maximally destabilizing subsheaf of T_X; $\mathscr{E}'$ is a reflexive sheaf of rank $r' \geqslant 1$. By [ADK08, Lemma 6.2], $\mu_{\mathscr{L}}(\mathscr{E}') \geq \frac{\mu_{\mathscr{L}}(\det(\mathscr{E}))}{r}$. Notice that $\mu_{\mathscr{L}}(\det(\mathscr{E})) \geqslant r+k$ since $\mathscr{E}$ is ample. This implies that $\frac{\deg(f^*\mathscr{E}')}{r'} \geq \frac{\deg(f^*\det(\mathscr{E}))}{r} \geqslant \frac{r+k}{r} \geqslant 1$ for a general member $[f] \in H$. If $r' = 1$, then $\mathscr{E}'$ is ample and we are done by Wahl's Theorem. If $f^*\mathscr{E}'$ is ample, then $X \simeq \mathbf{P}^n$ by [ADK08, Proposition 2.7], using the fact that $\rho(X) = 1$.

Otherwise, as $f^*\mathscr{E}'$ is a subsheaf of $f^*T_X \simeq \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbf{P}^1}^{\oplus(n-d-1)}$ (see [Kol96, Corollary IV 2.9]), we must have $\deg(f^*\det(\mathscr{E}')) = r'$, $\deg(f^*\det(\mathscr{E})) = r$, $k = 0$ and $f^*\mathscr{E}' \simeq \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus r'-2} \oplus \mathscr{O}_{\mathbf{P}^1}$ for a general $[f] \in H$. Then $\mathscr{O}_{\mathbf{P}^1}(2) \subset f^*\mathscr{E}'$ for general $[f] \in H$. Thus by [Hwa01, Proposition 2.3], $(f^*T_X^+)_p \subset (f^*\mathscr{E}')_p$ for a general $p \in \mathbf{P}^1$ and a general $[f] \in H$. Since $f^*\mathscr{E}'$ is a subbundle of f^*T_X, we have an inclusion of sheaves $f^*T_X^+ \hookrightarrow f^*\mathscr{E}'$, and thus $f^*\det(\mathscr{E}') = f^*\omega_X^{-1}$. Since $\rho(X) = 1$, this implies that $\det\mathscr{E}' = \omega_X^{-1}$, and thus $0 \neq h^0(X, \wedge^{r'}T_X \otimes \omega_X) = h^{n-r'}(X, \mathscr{O}_X)$. The latter is zero unless $r' = n$ since X is a Fano manifold. Notice that $\deg(f^*\det(\mathscr{E})) = r$. It follows that, for any $[f] \in H$, $f^*\mathscr{E} \simeq \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus r}$. By [AW01, Proposition 1.2] (see also [ROS10, Theorem 4.3]), $\mathscr{E} \simeq \mathscr{L}^{\oplus r}$ and $\deg(f^*\mathscr{L}) = 1$. If $n = r'$, then we must have $\omega_X^{-1} \simeq \det(\mathscr{E}') \simeq \mathscr{L}^{\otimes n}$. Hence $X \simeq Q_n$ by [KO73]. □

We will need the following auxiliary result.

Lemma 15. *Let X be a smooth complex projective variety and $\mathscr{E}$ an ample vector bundle on X of rank $r + k$ with $r \geqslant 2$ and $k \geqslant 0$. Assume that X is uniruled and fix a minimal dominating family H of rational curves on X. If $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$, then H is unsplit.*

Proof. The proof is similar to that of [Par10, Proposition 4.2]. Let $[f] \in H$ be a general member. Let us assume to the contrary that $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$ and $f_*(\mathbf{P}^1) \equiv C_1 + C_2$ with C_1 and C_2 nonzero integral effective rational 1-cycles. Notice first that $\det(\mathscr{E}) \cdot C \geqslant r + k$ for all rational curve $C \subset X$. By [Kol96, Corollary IV 2.9], $f^*T_X \simeq \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbf{P}^1}^{\oplus(n-d-1)}$ and we must have $\deg(f^*\det(\mathscr{E})) \leqslant 2r$. Finally, $2(r + k) \leqslant \deg(f^*\det(\mathscr{E})) \leqslant 2r$ and we must have $k = 0$, $\deg(f^*\det(\mathscr{E})) = 2r$ and $f^*\det(\mathscr{E}) \simeq \mathscr{O}_{\mathbf{P}^1}(2r) \subset f^* \wedge^r (T_X) \simeq \wedge^r(\mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbf{P}^1}^{\oplus(n-d-1)})$. Hence $T_{\ell,x}^{\otimes r} = \det(\mathscr{E})_x \subset T_{X,x}^{\otimes r}$ for a general point x in ℓ and therefore, $T_{\ell,x}$ does not depend on $\ell \ni x$. Thus, by Lemma 13, there exists a non empty open subset X_0 in X and a proper surjective morphism $\pi_0 : X_0 \to Y_0$ onto a variety Y_0 such that any fiber of π_0 is a rational curve from the family H and $\det(\mathscr{E})_{|X_0} \simeq T_{X_0/Y_0}^{\otimes r}$. Let $\mathscr{L} \subset T_X$ be the saturated line bundle such that $T_{X_0/Y_0} \simeq \mathscr{L}_{|X_0}$. Notice that $\det(\mathscr{E}) \subset \mathscr{L}^{\otimes r}$ with equality on X_0. Let $C \subset X$ be a general complete intersection curve and let S be the normalization of the closure in X of $\pi_0^{-1}(\pi_0(C \cap X_0))$. By [Dru04, Lemme 1.2] (or [ADK08, Proposition 4.5]), the map $\Omega_X^1 \to \mathscr{L}^{\otimes -1}$ induces a map $\Omega_S^1 \to \mathscr{L}_S{}^{\otimes -1}$ where $\mathscr{L}_S$ denotes the pull-back of $\mathscr{L}$ to S. Notice that π_0 induces a surjective morphism $\pi_S : S \to B$ onto a smooth curve. By Lemma 8, $\dim(X_0) \neq 2$. Thus, we may assume $g(B) \geqslant 1$. Let $\tilde{S} \to S$ be a minimal desingularization of S. By [BW74, Proposition 1.2], $\Omega_S^1 \to \mathscr{L}_S{}^{\otimes -1}$ extends to $\Omega_{\tilde{S}}^1 \to \mathscr{L}_{\tilde{S}}{}^{\otimes -1}$. Let $\pi_{\tilde{S}} : \tilde{S} \to B$ be the induced morphism. By replacing $\mathscr{L}_{\tilde{S}}$ with its saturation in $T_{\tilde{S}}$, we may assume $\det(\mathscr{E})_{\tilde{S}} \subset \mathscr{L}_{\tilde{S}}^{\otimes r} \subset T_{\tilde{S}}^{\otimes r}$. Observe also that, for a general point b in B, $\det(\mathscr{E})_{\tilde{S}} \cdot \tilde{S}_b = 2r$. But that contradicts Lemma 8. □

Now we can prove our main theorems.

Theorem 16. *Let X be a smooth complex projective variety and $\mathscr{E}$ an ample vector bundle on X of rank $r + k$ with $r \geqslant 1$ and $k \geqslant 0$ and such that $h^0(X, T_X^{\otimes r} \otimes \det(\mathscr{E})^{\otimes -1}) \neq 0$.*

1. *If $k \geqslant 1$ then $X \simeq \mathbf{P}^n$.*
2. *If $k = 0$ then either $X \simeq \mathbf{P}^n$, or $X \simeq Q_n$.*

Proof. We shall proceed by induction on $n := \dim(X)$. The result is clear if $n = 1$, so we assume that $n \geq 2$. If $r + k = 1$ then we are done by Wahl's Theorem so we assume that $r + k \geqslant 2$. Notice that X is uniruled by [Miy87]. Fix a minimal dominating family H of rational curves on X. By Lemma 15, H is unsplit. Let $\pi_0 : X_0 \to Y_0$ be the H-rationally connected quotient of X. By [ADK08, Lemma 2.2], we may assume $\mathrm{codim}_X(X \setminus X_0) \geqslant 2$ and π_0 is an equidimensional surjective morphism with integral fibers. By shrinking Y_0 if necessary, we may also assume that Y_0 is smooth.

By Proposition 14, we may assume $\rho(X) \geqslant 2$. By [Kol96, Proposition IV 3.13.3], we must have $\dim(Y_0) \geqslant 1$.

Let F be a general fiber of π_0. There exist (see [ADK08, Lemma 5.1] or [Par10, Lemme 2.1]) non negative integers i and j with $i + j = r$ such that $h^0(X, T_{X_0/Y_0}^{[\otimes i]} \otimes \det(\mathscr{E})_{|X_0}^{\otimes -1} \otimes \pi_0^* T_{Y_0}^{\otimes j}) \neq 0$ and $h^0(F, T_F^{\otimes i} \otimes \det(\mathscr{E})_{|F}^{\otimes -1}) \neq 0$. Notice that $i \geqslant 1$ since $\det(\mathscr{E})_{|F}$ is an ample line bundle and $d := \dim(F) \geqslant 1$. The induction hypothesis implies that $F \simeq \mathbf{P}^d$ if $i < r$ or $k \geqslant 1$ and either $F \simeq \mathbf{P}^d$ or $F \simeq Q_d$ if $i = r$ and $k = 0$.

Let $C \subset X_0$ be a general complete intersection curve (with respect to some very ample line bundle on X). Let X_C be the normalization of $\pi_0^{-1}(\pi_0(C))$. Let $\pi_C : X_C \to C$ be the induced map. Notice that X_C is the normalization of $C \times_{Y_0} X_0$ and that $C \times_{Y_0} X_0$ is regular in codimension one. Hence, we must have $h^0(X_C, T_{X_C/C}^{[\otimes i]} \otimes \det(\mathscr{E})_{|X_C}^{\otimes -1} \otimes \pi_C^*(\Omega^1_{Y_0|C}{}^{\otimes -j})) \neq 0$. Let us assume that either $(\pi_0^*\Omega^1_{Y_0})_{|C}$ is a nef vector bundle or $i = r$. If the geometric generic fiber of π_0 is isomorphic to a projective space then π_0 is a $\mathbf{P}^d$-bundle by Lemma 9. But that contradicts Lemma 4. Thus the geometric generic fiber of π_0 is isomorphic to a (smooth) hyperquadric. But that contradicts Proposition 6.

Thus $i < r$, $F \simeq \mathbf{P}^d$ and by Lemma 12, there exists a minimal free morphism $f : \mathbf{P}^1 \to Y_0$. By generic smoothness, we may assume that $X_f := \mathbf{P}^1 \times_{Y_0} X_0$ is smooth. We may also assume that $h^0(X_f, T_{X_f/\mathbf{P}^1}^{[\otimes i]} \otimes \det(\mathscr{E})_{|X_f}^{\otimes -1} \otimes \pi_f^*(T_{Y_0|\mathbf{P}^1}^{\otimes j})) \neq 0$. Let $\mathscr{L}_f$ be a line bundle on X_f that restricts to $\mathscr{O}_{\mathbf{P}^d}(1)$ on $F \simeq \mathbf{P}^d$ (see the proof of Lemma 4). By [Fuj75, Corollary 5.4], $\pi_f : X_f \to \mathbf{P}^1$ is a $\mathbf{P}^d$ bundle. It follows from Lemma 3 that $k = 0$, $d = 1$, $(X_f/\mathbf{P}^1) \simeq (\mathbf{P}^1 \times \mathbf{P}^1/\mathbf{P}^1)$ and $\det(\mathscr{E})_{|X_f} \simeq \mathscr{O}_{\mathbf{P}^1}(2) \boxtimes \mathscr{O}_{\mathbf{P}^1}(2)$. Since $\mathscr{E}$ is ample, X admits an unsplit dominating covering family H' of rational curves whose general member corresponds to a ruling of X_f that is not contracted by π. Let $\pi_1 : X_1 \to Y_1$ be the (H, H')-rationally connected quotient of X. By [ADK08, Lemma 2.2], we may assume $\mathrm{codim}_X(X \setminus X_1) \geqslant 2$ and π_1 is an equidimensional surjective morphism with integral fibers. By shrinking Y_1 if necessary, we may also assume that Y_1 is smooth. Replacing $\pi_0 : X_0 \to Y_0$ with $\pi_1 : X_1 \to Y_1$ above, we obtain a contradiction unless $X \simeq \mathbf{P}^1 \times \mathbf{P}^1$. □

Proof of Theorem A. By Theorem 16, $X \simeq \mathbf{P}^n$ and by Lemma 1, $\det(\mathscr{E}) \simeq \mathscr{O}_{\mathbf{P}^n}(l))$ with $r + k \leqslant l \leqslant \frac{r(n+1)}{n}$. □

Proof of Theorem B. By Theorem 16, either $X \simeq \mathbf{P}^n$ or $X \simeq Q_n$. If $X \simeq \mathbf{P}^n$, then the claim follows from Lemma 1. Let us assume $X \simeq Q_n$. By Lemma 2, $\det(\mathscr{E}) \simeq \mathscr{O}_{Q_n}(r)$. Thus, for any line $\mathbf{P}^1 \subset Q_n \subset \mathbf{P}^{n+1}$, $\mathscr{E}_{|\mathbf{P}^1} \simeq \mathscr{O}_{\mathbf{P}^1}(1)^{\oplus r}$, and the claim follows from [AW01, Proposition 1.2] (see also [ROS10, Theorem 4.3]). □

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