# COMPLEX SURFACES OF NEGATIVE CURVATURE

by

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# ABSTRACT

We study a class of examples of negatively curved compact Kähler surfaces that are not diffeomorphic to a locally symmetric space. From the analysis of certain totally geodesic curves on these surfaces we deduce that, for infinitely many examples, the natural representation of the fundamental group into $PU(2,1)$ is nonfaithful. We also give a new construction of bounded holomorphic functions on the universal cover of our surfaces, based on lifting maps to compact Riemann surfaces.

To the memory of my father.

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# CHAPTER 1

# INTRODUCTION

Around 1980, Mostow and Siu constructed the first example of a compact four-dimensional Riemannian manifold with negative curvature not diffeomorphic to a locally symmetric space. In fact in [MS] they describe an infinite family of examples, all compact Kähler surfaces, that already at the time were considered as being of great interest.

As soon as such examples have been constructed, questions abound about their topological and complex analytic properties, and one would expect some work to be devoted to their analysis. Somehow for the past 20 years, no step has been taken in that direction. The list of examples has been expanded a little but, to the best of our knowledge, nobody has gone in any essential way beyond the construction of examples.

One goal of the present paper is to revive the interest for the subject by investigating some detailed properties of a particular class of such surfaces. This class is not quite the one described in [MS], but shares many of its interesting features.

In the original Mostow-Siu construction, the crucial ingredient is the study in [M1] of some nondiscrete groups generated by three complex reflections in the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$. It turns out that these groups, up to commensurability, can be understood as monodromy groups of certain hypergeometric functions. This is the point of view we take to construct our surfaces.

We describe the relevant class of examples in terms of the methods developed in [DM], where Deligne and Mostow study the monodromy groups $\Gamma_\mu$ of hypergeometric functions whose set of exponents $\mu$ satisfies the so-called Picard integrality condition (cf. 4.1). The relevant surfaces are then ball quotients.

We relax the condition on the exponents, so that the Picard integrality condition fails only in a controlled fashion (namely, as in 6.1), and study the corresponding monodromy groups $\Gamma_\mu$.

Just like in the Deligne-Mostow cases, hypergeometric functions define $\Gamma_\mu$-equivariant maps $\widetilde{X} \to \mathbb{B}^2$ which, because of the failure of the Picard integrality condition, exhibit some branching behavior. Most of the time the monodromy group $\Gamma_\mu$ acts in a nondiscrete fashion on the ball $\mathbb{B}^2$, but its action on $\widetilde{X}$ is always discrete.

Our surfaces are quotients $X_0 = {}_{\Gamma_0}\backslash\widetilde{X}$, where $\Gamma_0$ is a torsion free subgroup of finite index in $\Gamma_\mu$. In particular, their fundamental group comes with a natural representation $\lambda : \pi_1(X_0) \to PU(2,1)$ into the automorphism group of the ball. The starting point for the study of the fundamental group of $X_0$ is to determine whether this representation is faithful. This question was raised over 20 years ago in [M1], and we give it a partial answer in Chapter 10. More specifically, Theorem 10.4 states that for infinitely many examples the natural representation of $\pi_1(X_0)$ into $PU(2,1)$ is *not* faithful. In a sense, this result illustrates how complicated the fundamental group of our surfaces can be.

To give a rough idea of the methods involved in the proof, we mention that the representation $\lambda$ is faithful if and only if the cover $\widetilde{X}$ is simply connected. An important tool in understanding the fundamental group of the cover $\widetilde{X}$ is Theorem 9.2, which roughly states that the two-dimensional hypergeometric cover $\widetilde{X}$ contains many one-dimensional hypergeometric covers $\widetilde{X}'$ as totally geodesic divisors. The one-dimensional covers $\widetilde{X}'$ are much easier to understand, as they are closely related to triangle groups. In particular, inspired by the proof of a result of Mostow that gives a classification of certain discrete groups generated by two elliptic elements in the hyperbolic plane (see Theorem 8.1), we obtain a concrete sufficient condition for the one-dimensional analogue $\lambda'$ of $\lambda$ to be nonfaithful (see Proposition 8.3).

Once again because of the results in Chapter 9, we have many embeddings $\widetilde{X}' \subset \widetilde{X}$ as totally geodesic divisors. Since the manifold $\widetilde{X}$ has negative curvature, this inclusion induces an injection $\pi_1(\widetilde{X}') \hookrightarrow \pi_1(\widetilde{X})$ on the level of fundamental groups, which implies that whenever $\lambda'$ is nonfaithful, so is $\lambda$.

In a different direction, we make some progress towards understanding the complex analytic aspects of Mostow-Siu type surfaces. Throughout our investigations, the following questions serve as a guide. Can one construct bounded holomorphic functions on their universal cover? Are there enough such functions to separate points?

This approach was suggested by Siu around the time of the construction of the original examples, but once again, it seems that until now, nobody had done anything in that direction.

Recall that by construction, our surfaces come with maps $\widetilde{X} \to \mathbb{B}^2$ from some cover of $X_0$ to the ball. This gives an obvious way to produce many bounded holomorphic functions on the universal cover of $X_0$, since there are plenty of bounded holomorphic functions on $\mathbb{B}^2$.

One of the consequences of our work in Chapter 10 is that there are many situations where $\widetilde{X}$ is not simply connected, in which case we write $\widehat{X}$ for its universal cover. In terms of separating points in $\widehat{X}$, the functions coming from the hypergeometric maps $\widetilde{X} \to \mathbb{B}^2$ are certainly not very efficient, since they do not separate points in the same fiber of the projection $\widehat{X} \to \widetilde{X}$.

In Chapter 11, we describe a new construction of bounded holomorphic functions on the universal cover of our surfaces. It is essentially based on lifting certain maps from our surfaces to compact Riemann surfaces, as stated in Proposition 11.4. Note that it is in general a difficult task to construct nontrivial maps to lower-dimensional manifolds.

Theorem 11.2 states that for infinitely many examples, our new construction produces bounded holomorphic functions on the universal cover of $X_0$ that *do not* factor through the hypergeometric map $\widetilde{X} \to \mathbb{B}^2$. In other words, they allow one to separate some points in the same fiber of $\widehat{X} \to \widetilde{X}$.

The paper is organized as follows. In Chapters 2 through 7 we mostly recall the results in [MS], adapting the notations to our situation. Chapter 8 is for the most part devoted to some basic results on triangle groups in the hyperbolic plane. We give some details for the proof of Theorem 8.1, since our claim differs slightly from the analogous theorem in [M4]. This result, together with the ones in Chapter 9, constitute the main tools for our main two theorems, in Chapter 10 and 11 respectively. In the last chapter we mention some of the numerous open questions related to this project.

# CHAPTER 2

# BASIC RESULTS

We begin by collecting some classical results most of which date back to the late nineteenth century (Schwarz, Picard) and have been presented in various places more recently (Terada, Deligne-Mostow). Part of the point is to establish the notations.

We write $M$ for the space of $n+3$ distinct points on $\mathbb{P}^1$

$$M = \{(x_1, \dots, x_{n+3}) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 : x_i \neq x_j\}$$

and $Q$ for the space of $(n+3)$-tuples of distinct points on $\mathbb{P}^1$, modulo projective automorphisms. Since the automorphisms of $\mathbb{P}^1$ are determined uniquely by prescribing their value on three distinct points, one can think of $Q$ as

$$Q \simeq \{(x_1, \dots, x_n) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 : x_i \neq x_j, x_i \neq 0, 1, \infty\}$$

We assign weights to each of the $n+3$ points by giving an $(n+3)$-tuple of rational numbers $\mu = (\mu_0, \mu_1, \dots, \mu_{n+2})$ (we write $\mu = \frac{1}{d}(n_0, \dots, n_{n+2})$ where $d$ is the least common denominator of the $\mu_j$'s) and consider the family of curves $X_q$ parametrized by $q = (x_0, \dots, x_{n+2}) = (0, 1, x_2, \dots, x_{n+1}, \infty) \in Q$

$$v^d = \prod_{i=0}^{n+1} (u - x_i)^{n_i}$$

where we write $x_0$ and $x_1$ for 0 and 1 respectively. $X_q$ has automorphisms given by multiplying $v$ by a $d^{\text{th}}$ root of unity. Its cohomology $H^1(X_q) = H^1(X_q, \mathbb{C})$ splits as a direct sum of "eigenspaces"

$$H^1(X_q) \simeq \oplus H^1_{\zeta^k}(X_q) \tag{2.1}$$

where $H^1_{\zeta^k}(X_q)$ is the subspace of $H^1(X_q)$ invariant under the action of $\zeta^k$.

We can also split $H^1(X_q) \simeq H^{1,0}(X_q) \oplus H^{0,1}(X_q)$ and cup product gives a Hermitian inner product

$$\langle \alpha, \beta \rangle = \frac{1}{i} \int \alpha \wedge \overline{\beta} \tag{2.2}$$

such that $(\alpha, \alpha) < 0$ (resp. $(\alpha, \alpha) > 0$) for $\alpha$ a holomorphic (resp. antiholomorphic) form. In other words, $\langle \cdot, \cdot \rangle$ has signature $(g, g)$ on $H^1(X_q)$, where $g$ is the genus of $X_q$.

This inner product, when restricted to the different eigenspaces of the decomposition (2.1), might have different signatures. One can check that it restricts to an inner product of signature $(n, 1)$ on the eigenspaces $H^1_{\zeta^{-1}}(X_q)$. The point is that this summand contains a unique (up to scalar) holomorphic form $\frac{du}{v}$ that we denote by $\omega_q$.

We want to analyze this situation as $q$ varies in $Q$. The first step is the observation that the $H^1(X_q)$ form a local system on $Q$, so that we can transport cohomology classes horizontally along paths in $Q$. In particular this gives a monodromy representation

$$\overline{\rho} : \pi_1(Q, q_0) \to \mathrm{Aut}(H^1(X_{q_0}))$$

Note that horizontal transport commutes with the action of roots of unity, so that we can think of the monodromy as acting on each individual subspace of the decomposition (2.1). We will always consider its action on $H^1_{\zeta^{-1}}(X_{q_0})$. It is clear that horizontal transport also preserves the inner product (2.2), hence $\pi_1(Q, q_0)$ actually acts by isometries on $H^1_{\zeta^{-1}}(X_{q_0})$. In other words, we can think of the monodromy representation as

$$\overline{\rho} : \pi_1(Q, q_0) \to U(n, 1)$$

The importance of the monodromy representation is that it measures the multivaluedness of the hypergeometric map, that we now define. The idea is that $\omega_q$, which is just $\frac{du}{v}$ on each $X_q$, defines a map $Q \to H^1_{\zeta^{-1}}(X_q)$ and by horizontal transport, $H^1_{\zeta^{-1}}(X_q) \simeq H^1_{\zeta^{-1}}(X_{q_0})$. We would like to think of it as a map from $Q$ to the *fixed* vector space $H^1_{\zeta^{-1}}(X_{q_0})$. This is not quite well defined, since horizontal transport depends on the path chosen, but we get a multivalued map $Q \to H^1_{\zeta^{-1}}(X_{q_0})$, with image contained in the negative cone (negative with respect to the inner product $\langle \cdot, \cdot \rangle$).

We are more interested in the corresponding projectivized map to $\mathbb{B}^n \subset \mathbb{P}^n$ where we write $\mathbb{P}^n$ for $\mathbb{P}(H^1_{\zeta^{-1}}(X_{q_0}))$. Accordingly, we consider the *projective* monodromy representation, that we denote by $\rho$ (as opposed to $\overline{\rho}$ for the corresponding linear representation)

$$\rho : \pi_1(Q, q_0) \to PU(2,1)$$

and write $K$ for its kernel, $\Gamma$ for its image. $\Gamma$ is called the monodromy group. When we need to specify the dependence of this group on the set of weights $\mu$, we write $\Gamma_\mu$ instead of just $\Gamma$. The multivalued holomorphic map $w : Q \to \mathbb{B}^n$ lifts to a single valued holomorphic map $\widetilde{w} : \widetilde{Q} \to \mathbb{B}^n$, where $\widetilde{Q}$ is the cover of $Q$ corresponding to $K$.

It should be noted that our map $w$ can be described in terms of hypergeometric functions, which can be defined as

$$\int_g^h \frac{du}{v} \tag{2.3}$$

with $g, h \in \{0, 1, x_2, \dots, x_{n+1}, \infty\}$. Up to multiplicative constants independent of $q = (0, 1, x_2, \dots, x_{n+1}, \infty)$, these are integrals of $\omega_q$ along cycles in $X_q$. The expression (2.3) also makes it easy to check that $w$ is holomorphic.

The map $w$ has been studied extensively since the late nineteenth century (starting with Schwarz, Picard and others). We start by presenting the case $n = 1$ in some detail, since it motivates much of the construction in higher dimensions.

When $n = 1$, we have $Q \simeq \mathbb{P}^1 - \{0, 1, \infty\}$ and a map $\widetilde{w} : \widetilde{Q} \to \mathbb{B}^1 \simeq \mathbb{H}^2$ to the real hyperbolic plane. It is a well known fact that the multivalued map $w$ sends a hemisphere bijectively onto a triangle $T$ in $\mathbb{H}^2$ with angles given by $(1 - \mu_i - \mu_j)\pi$ (here we choose $i, j$ so that $\mu_i + \mu_j < 1$).

There are three ways to switch hemispheres, along $(0, 1)$, $(1, \infty)$ or $(-\infty, 0)$. The other hemisphere gets mapped to a reflection of $T$ in one of its three sides, accordingly. This is illustrated in Figure 2.1. Of course to be more precise, one should state this description in terms of the single valued lift $\widetilde{w}$.

Note that $\pi_1(Q)$ is generated by three loops around 0, 1 and $\infty$, and that the corresponding monodromy transformations (which generate the monodromy group) are rotations in $\mathbb{H}^2$, centered at the vertices of $T$, and with angles $2(1-\mu_i-\mu_j)\pi$.

Quite naturally, one can complete the cover $\widetilde{Q}$ to a branched cover of $\mathbb{P}^1$, branched at $0, 1$ and $\infty$, as in diagram (2.4).

$$\begin{array}{ccc} \widetilde{Q} & \hookrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Q & \hookrightarrow & \mathbb{P}^1 \end{array} \tag{2.4}$$

$\widetilde{X}$ can be described as the Fox completion of $\widetilde{Q}$ over $X$ (see Chapter 3), or as the metric completion of $\widetilde{Q}$ with respect to the pull back of the hyperbolic metric via $\widetilde{w}: \widetilde{Q} \to \mathbb{H}^2$. It carries a natural metric of negative curvature, but we do not really need this now. The map $\widetilde{w}$ then extends to $\widetilde{X}$ in such a way that the ramification points map to the vertices of $T_0$, or one of its images under the monodromy group.

The cases of interest to Schwarz et al. are the ones where $(1-\mu_i-\mu_j)^{-1}$ are integers, in which case the monodromy group is just a classical triangle group, and the map $\widetilde{w}$ is actually an isomorphism. We want to analyze the situations where the above "integrality" condition fails. The monodromy group $\Gamma \subset Aut(\mathbb{H}^2)$ is generated by two rotations, and it should be noted that most of the time it is not discrete. The relevance of $\Gamma \simeq \pi_1(Q)/K$ is that it is the deck group of the cover $\widetilde{Q} \to Q$, and even when it acts in a nondiscrete fashion on $\mathbb{H}^2$, it certainly acts discretely on $\widetilde{Q}$ as well as on the completion $\widetilde{X}$. Topologically, the quotient $\Gamma\backslash\widetilde{X}$ is

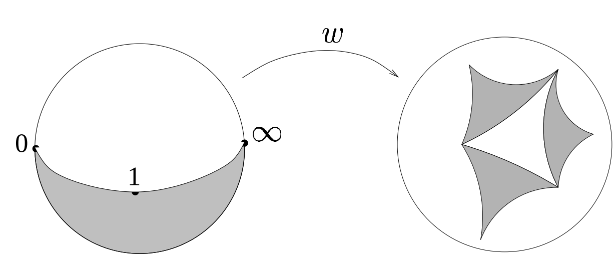


**Figure 2.1**. Hypergeometric map in dimension one.

$\mathbb{P}^1$, but since the group $\Gamma$ has torsion elements, one should think of this quotient $\mathbb{P}^1$ as an orbifold.

The constructions above carry over to higher dimensions. We describe the case $n = 2$ in detail (higher-dimensional cases are of no interest for this particular paper). Recall that $Q \simeq \{(x_1, x_2) \in \mathbb{P}^1 \times \mathbb{P}^1 : x_1 \neq x_2, x_i \neq 0, 1, \infty\}$. The fundamental group $\pi_1(Q)$ is generated by small loops around the lines $x_j = 0, 1, \infty$ and $x_1 = x_2$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Thinking of $Q$ as the set $M$ of distinct 5-tuples on $\mathbb{P}^1$ modulo projective automorphisms, we get a more symmetric set of generators by taking $\gamma_{ij}$ to be a small loop corresponding to $x_i$ going once around $x_j$, for any $i, j = 1, \ldots, 5$. These can be represented in our model for $Q$ as well, as depicted in Figure 2.2.

It is shown in [DM] (section 9) that such loops $\gamma_{ij}$ map under the monodromy representation to so-called "complex reflections". More precisely, the linear monodromy transformation $\overline{\rho}(\gamma_{ij})$ fixes a hyperplane in $\mathbb{C}^{n+1} \simeq H^1_{\zeta^{-1}}(X_{q_0})$, which we call the **mirror** of the corresponding complex reflection. In the orthogonal complement of the mirror, the action of the complex reflection is to multiply by the root of unity $\zeta_{ij} = e^{2\pi i(1-\mu_i-\mu_j)}$. If $v_{ij}$ spans this orthogonal complement, we can write

$$\overline{\rho}(\gamma_{ij})(x) = x + (\zeta_{ij} - 1)\frac{\langle w, v_{ij}\rangle}{\langle v_{ij}, v_{ij}\rangle}v_{ij} \tag{2.5}$$

In practice, this kind of formula is only useful if we can manage to get some understanding of the angle between the mirrors of two different complex reflections $\overline{\rho}(\gamma_{ij})$ and $\overline{\rho}(\gamma_{kl})$.

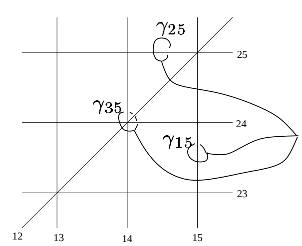


**Figure 2.2**. Some generators for $\pi_1(Q)$.

To this end, we start by observing that the notation $v_{ij}$ is quite ambiguous. For one thing, we choose a vector in the orthogonal complement of a hyperplane, which we can always multiply by any scalar. Note that this is irrelevant in terms of computing angles between the mirrors. The second and more important ambiguity comes from the fact that the linear span of $v_{ij}$ depends not just on the pair $\{i, j\}$ but also on the choice of a path $\gamma_{ij}$ around $D_{ij}$.

Whenever we need to specify a precise set of loops, we choose an embedded interval $T_{ij}$ between $x_i$ and $x_j$, and take $\gamma_{ij}$ to move $x_i$ close to $x_j$ along $T_{ij}$, go around $x_j$ once in the positive direction, and then come back along $T_{ij}$ to its original position. The relations between two different loops $\gamma_{ij}$ and $\gamma_{kl}$ then depend on the choice of embedded intervals $T_{ij}$ and $T_{kl}$ in the sphere. For instance, if $T_{ij}$ and $T_{kl}$ are disjoint, then the corresponding loops commute.

In section 12 of their paper [DM], Deligne and Mostow describe a basis for the cohomology $\mathbb{H}^1_{\zeta^{-1}}(X_{q_0})$ and write explicit matrices for generating complex reflections. A quick way to summarize their construction is the following. Take three embedded intervals $T_{ij}$, $T_{jk}$ and $T_{kl}$ that intersect only at $x_j$ and $x_k$. The corresponding loops map to complex reflections so that the corresponding vectors $v_{ij}$, $v_{jk}$ and $v_{kl}$ (see 2.5) form a basis for $\mathbb{H}^1_{\zeta^{-1}}(X_{q_0})$.

Of course, as we pointed out earlier, the loops $\gamma_{ij}$ only determine the vectors $v_{ij}$ up to multiplication by a scalar. Indeed all we know is that $v_{ij}$ is orthogonal to the mirror of the complex reflection $\overline{\rho}(\gamma_{ij})$. Deligne and Mostow explain how to make a precise choice of vectors $v_{ij}$, and compute actual matrices for some monodromy transformations in that basis. From these matrices, we can deduce a concrete expression for the matrix of the Hermitian inner product $\langle\cdot,\cdot\rangle$ in that basis, which is determined up to a real positive factor by the fact that it is invariant under the monodromy transformations.

In turn one can write a formula for the norm of the vector $v_{ij}$ and check that (up to a real positive multiple)

$$\langle v_{ij}, v_{ij}\rangle = -\frac{\sin\pi(\mu_i+\mu_j)}{\sin\pi\mu_i\sin\pi\mu_j} \tag{2.6}$$

The only piece of information we want to extract from this formula is that

$$\begin{aligned} \langle v_{ij}, v_{ij}\rangle &< 0 \text{ if and only if } \mu_i + \mu_j < 1 \\ \langle v_{ij}, v_{ij}\rangle &= 0 \text{ if and only if } \mu_i + \mu_j = 1 \\ \langle v_{ij}, v_{ij}\rangle &> 0 \text{ if and only if } \mu_i + \mu_j > 1 \end{aligned}$$

In what follows, we will always assume that

$$\mu_i + \mu_j < 1 \text{ for any } i, j \in \{1, \dots, 5\} \tag{2.7}$$

This very strong assumption is not needed until later in the paper, but it simplifies the current discussion. The fact that $1 - \mu_i - \mu_j$ is positive implies that the monodromy transformation $\rho(\gamma_{ij})$ is a complex reflection *inside* the ball $\mathbb{B}^2$. The fixed two-dimensional subspace for the linear transformation $\overline{\rho}(\gamma_{ij})$ determines a fixed one-dimensional subball in $\mathbb{B}^2$ for $\rho(\gamma_{ij})$. In the direction orthogonal to this fixed subball, $\rho(\gamma_{ij})$ rotates by a $2\pi(1 - \mu_i - \mu_j)$ angle.

We can also deduce a formula for the angle between the mirrors of $\overline{\rho}(\gamma_{ij})$ and $\overline{\rho}(\gamma_{kl})$. As long as we choose the embedded intervals $T_{ij}$ and $T_{kl}$ to meet only at possible common endpoints, we have

$$\frac{|\langle v_{ij}, v_{jk}\rangle|}{\sqrt{\langle v_{ij}, v_{ij}\rangle\langle v_{jk}, v_{jk}\rangle}} = \sqrt{\frac{\sin \pi\mu_i \sin \pi\mu_k}{\sin \pi(\mu_i + \mu_j) \sin \pi(\mu_j + \mu_k)}} \tag{2.8}$$

and $\langle v_{ij}, v_{kl}\rangle = 0$ whenever $\{i, j\} \cap \{k, l\} = \emptyset$, this last equality reflecting the fact that $\gamma_{ij}$ and $\gamma_{kl}$ commute.

Now we want to extend the multivalued map $w : Q \to \mathbb{B}^2$ to a compactification $X$ of $Q$. It turns out that $w$ does not extend to $\mathbb{P}^1 \times \mathbb{P}^1$. The appropriate compactification is the blow up $X$ of $\mathbb{P}^1 \times \mathbb{P}^1$ at the three points $(0, 0)$, $(1, 1)$ and $(\infty, \infty)$. Another way to get the same space $X$ is to blow up $\mathbb{P}^2$ at the four triple intersection points of the complete quadrilateral.

We draw a picture (Figure 2.3) that illustrates the combinatorics of the compactification divisors in $X - Q$. There are $\binom{5}{2} = 10$ compactification divisors, that we denote $D_{ij}$ for $\{i, j\} \subset \{1, \dots, 5\}$. Each of them corresponds to two of the five

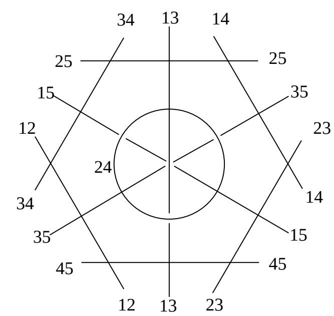


**Figure 2.3**. The stable compactification of $Q$.

points on $\mathbb{P}^1$ ($x_i$ and $x_j$) coming together. Note that we do not allow more than two points to collide. The divisors $D_{ij}$ and $D_{kl}$ intersect if and only if $\{i, j\} \cap \{k, l\} = \emptyset$.

**Remark 2.1** The description of the compactification $X$ would have to be modified without assumption (2.7). One would then get a compactification component of codimension $k - 1$ corresponding to each $S \subset \{1, \ldots, 5\}$ with $\sum_{i \in S} \mu_i < 1$, $|S| = k$. This would possibly allow for some triple intersections, and we would have to blow down some divisors in $X$. One way to think of condition (2.7) is that it guarantees that the compactification divisors have normal crossings.

We now go back to the multivalued hypergeometric map $w : Q \to \mathbb{B}^2$, which we think of as being single valued on $\widetilde{Q}$. As in the one-dimensional case, we want to complete the cover $\widetilde{Q} \to Q$ to a branched cover $\widetilde{X} \to X$, and extend the hypergeometric map to be defined on $\widetilde{X}$.

$$\begin{array}{ccccc} \widetilde{Q} & \hookrightarrow & \widetilde{X} & \overset{\widetilde{w}}{\to} & \mathbb{B}^2 \\ \downarrow & & \downarrow & & \\ Q & \hookrightarrow & X & & \end{array}$$

A little work needs to be done here to define the completion process carefully and to justify that $\widetilde{X}$ is actually a manifold. This follows from condition (2.7) and is shown by using the local description of the map $\widetilde{X} \to X$, which shall be discussed in the next chapter.

Assuming that $\widetilde{X}$ is a (complex) manifold, the Riemann removable singularities theorem implies that the holomorphic map $\widetilde{w}$ extends to $\widetilde{X}$, since it is bounded on

$\widetilde{Q}$ and $\widetilde{X} - \widetilde{Q}$ has codimension one. The local structure of this extension $\widetilde{X} \to \mathbb{B}^2$ will be described in Chapter 4.

Note that, just like in the one-dimensional situation, the deck group of the cover $\widetilde{Q} \to Q$ is $\pi_1(Q)/K \simeq \Gamma$, where $\Gamma$ is the monodromy group. The action of $\Gamma$ on $\widetilde{Q}$ extends to an action on $\widetilde{X}$, such that ${}_{\Gamma}\backslash\widetilde{X} \simeq X$ as topological spaces. The action of $\Gamma$ on $\widetilde{X}$ has entire divisors of fixed points, hence a better way to think of this quotient is in terms of orbifolds.

We also mention that, by construction, the hypergeometric map $\widetilde{w}$ is equivariant with respect the action of $\Gamma$ on both $\widetilde{X}$ and $\mathbb{B}^2$. This fact will be used several times throughout the paper.

# CHAPTER 3

# THE FOX COMPLETION

We recall some facts from [F], also presented in section 8 of [DM]. The general philosophy is that we want a way to construct branched covers, but we start with some general considerations about spreads. Let $A$ and $B$ be locally connected $T_1$ topological spaces.

**Definition 3.1** *A continuous map $f : A \to B$ is a* ***spread*** *if the connected components of inverse images of open sets in $B$ give a basis for the topology of $A$. The spread is called* ***complete*** *if for every $x \in B$,*

$$f^{-1}(x) = \varprojlim_{x \in U} \pi_0(f^{-1}(U))$$

In concrete terms, the condition for completeness expressed above in terms of inverse limits can be reformulated as follows. Fix any $x \in B$, and for each open neighborhood $U_j$ of $x$, choose a component $V_j$ of $f^{-1}(U_j)$ in such a way that $V_j \subset V_k$ whenever $U_j \subset U_k$. The condition above then requires that the intersection $\cap V_j$ be nonempty (or equivalently, that this intersection be a point).

Observe that this condition is of course interesting only for points $x$ not in the image of $f$. The basic fact proved in [F] is that any spread $f : A \to B$ can be extended uniquely to a complete spread $\overline{f} : \overline{A} \to B$. We refer to the space $\overline{A}$ as the completion of $A$ over $B$.

The Fox completion of a spread $f$ satisfies the universal property that any map to a complete spread factors through the completion $\overline{f}$. From this it is easy to deduce the following important result.

**Lemma 3.2** *Let $f : A \to B$ be a spread, with completion $\overline{f} : \overline{A} \to B$. Given an open subset $U \subset B$, any connected component of $\overline{f}^{-1}(U)$ is the completion of the corresponding component of $f^{-1}(U)$ over $U$.*

The relation to branched covers is made clear by the following observation. Suppose $Y \to Z$ is branched cover, with branch locus $Z - Z_0$, so that we have an unbranched cover $Y_0 \to Z_0$. Then the map $Y_0 \to Z$ is a spread, whose completion is precisely the branched cover $Y \to Z$.

Of course in general, given an unbranched cover $Y_0 \to Z_0$ where $Z_0$ is a subset of a certain space $Z$, the completion need not be a branched cover, even if we assume that $Z - Z_0$ is "nice." Here we are interested in the hypergeometric situation, where we have an unbranched cover $\widetilde{Q} \to Q$ and a compactification $Q \subset X$. We write $\widetilde{X}$ for the completion of $\widetilde{Q}$ over $X$. It is not clear that the space $\widetilde{X}$ is a manifold. A priori it might not even be locally compact, but in fact it turns out that this is the only obstruction for getting a (complex) manifold.

In order to check that the completion $\widetilde{X}$ is locally compact, we need to check that the map $\widetilde{X} \to X$ is locally finite-to-one. This follows from the lemma below. For any $i, j \in \{1, \ldots, 5\}$, we write $1 - \mu_i - \mu_j = \frac{n_{ij}}{d_{ij}}$ as a reduced fraction.

**Lemma 3.3** *Near a point $\widetilde{x}$ above $x \in D_{ij} - \bigcup D_{kl}$, the map $p : \widetilde{X} \to X$ looks like $(z, w) \mapsto (z^{d_{ij}}, w)$. If $x \in D_{ij} \cap D_{kl}$, then $p$ looks locally like $(z, w) \mapsto (z^{d_{ij}}, w^{d_{kl}})$*

**Proof:** We consider only the second case $x \in D_{ij} \cap D_{kl}$. Let $V$ be a small neighborhood of $x$ and write $U$ for $V \cap Q$. By "small" we mean that $\pi_1(U)$ should be abelian, generated by two loops $\gamma_{ij}$ and $\gamma_{kl}$ around $D_{ij}$ and $D_{kl}$ respectively. If $\widetilde{U}$ is a component of $p^{-1}(U)$, we have that $\pi_1(\widetilde{U}) \simeq K \cup \pi_1(U)$. But we know that $\rho(\gamma_{ij})$ has order $d_{ij}$ so that $\gamma_{ij}^{d_{ij}} \in K$ (and similarly $\gamma_{kl}^{d_{kl}} \in K$). This gives the structure of the cover $\widetilde{U} \to U$, hence the structure of $\widetilde{V} \to V$.

# CHAPTER 4

# LOCAL STRUCTURE OF THE HYPERGEOMETRIC MAPS

The first observation is that $\widetilde{w}$ is a local biholomorphism near any point of $\widetilde{Q}$. Recall that we have explicit formulas for homogeneous coordinates of $\widetilde{w}$ in terms of integrals on cycles of a certain form $\frac{du}{v}$. One can then compute the derivative explicitly. Using local triviality, one can express the coordinate functions of $\widetilde{w}$ to integrals on a constant path and bring the derivative inside the integral. This is explained in detail in [DM], section 3.

Near a point $x \notin \widetilde{Q}$, $\widetilde{w}$ looks like a branched cover. For instance, if $x \in D_{ij}$ but is in no other $D_{kl}$, the computation of the monodromy transformation described in Chapter 2 (see also [DM] section 9) implies that, near $x$, the multivalued map $X \to \mathbb{B}^2$ looks like

$$(z_1, z_2) \mapsto (z_1, z_2^{1-\mu_i-\mu_j})$$

Putting this together with the results in the previous chapter, one sees that, if $\widetilde{x}$ lies over $x$ on the cover $\widetilde{X}$, the single valued lift $\widetilde{w}$ near $\widetilde{x}$ looks like

$$(z_1, z_2) \mapsto (z_1, z_2^{n_{ij}})$$

where, as before, we write $1 - \mu_i - \mu_j = \frac{n_{ij}}{d_{ij}}$ as a reduced fraction.

Deligne and Mostow (following Picard) consider the cases where $n_{ij} = 1$ for all $i$, $j$, i.e.

$$(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z} \text{ for all } i, j \tag{4.1}$$

which we refer to as the Picard integrality condition. Then the map $\widetilde{w}$ is a local biholomorphism and one can argue, using the compactness of $X$, that it is actually an isomorphism. The monodromy group, that we can think of as the deck group of the cover $\widetilde{Q} \to Q$, is then a lattice in $Aut(\mathbb{B}^2)$, and $X = {}_{\Gamma}\backslash\widetilde{X} \simeq {}_{\Gamma}\backslash\mathbb{B}^2$ is an (orbifold) ball quotient.

From now on, we will only consider situations where the integrality condition (4.1) fails, in which case the map $\widetilde{X} \to \mathbb{B}^2$ exhibits some branching. We will see in Chapter 10 that in general $\widetilde{X}$ need not be simply connected, hence it certainly cannot always be isomorphic to the ball. Actually, even if $\widetilde{X}$ happened to be simply connected, one can still show that its universal cover cannot be biholomorphic to the ball, provided that the map $\widetilde{w}$ has branching (see Chapter 6).

It should also be noted that in general, the monodromy group $\Gamma$ is a nondiscrete subgroup of $Aut(\mathbb{B}^2)$. The list of possible choices of exponents in the hypergeometric functions that will give a discrete monodromy group is finite, and can be found in [M4]. The only nonobvious examples that satisfy our assumptions (2.7) and (6.1) are the following:

$$\left(\frac{4}{21},\frac{8}{21},\frac{10}{21},\frac{10}{21},\frac{10}{21}\right)$$

$$\left(\frac{5}{24},\frac{10}{24},\frac{11}{24},\frac{11}{24},\frac{11}{24}\right)$$

By "nonobvious," we mean that these examples do not satisfy the Picard integrality condition, or the Mostow half-integrality condition.

The important point is that even when the monodromy group is nondiscrete, it still acts discretely on $\widetilde{X}$, with (orbifold) quotient ${}_{\Gamma}\backslash\widetilde{X} = X$.

# CHAPTER 5

# THE MOSTOW-SIU SURFACES

Recall that the monodromy group $\Gamma$ is generated by torsion elements, so the quotient $X = {}_{\Gamma}\backslash\widetilde{X}$ is only an orbifold. In order to get a manifold quotient, we need to find a torsion free subgroup of finite index $\Gamma_0 \subset \Gamma$. We describe briefly how this can be done.

We go back for a while to the linear monodromy representation $\overline{\rho} : \pi_1(Q) \to U(2,1)$, where $U(2,1)$ is the group of automorphisms of $H^1_{\zeta^{-1}}(X_q, \mathbb{C})$ preserving the inner product $\langle \cdot, \cdot \rangle$ of signature (2,1). Now we consider the integral cohomology $H^1(X_q, \mathbb{Z}) \subset H^1(X_q, \mathbb{C})$. These groups also form a local system, so we can consider the monodromy transformations as acting on it.

Since our inner product, which is essentially the cup product on the cohomology of $X_q$, is invariant under parallel transport, the orthogonal projection onto the eigenspace $\mathbb{H}^1_{\zeta^{-1}}(X_q, \mathbb{C})$ commutes with the monodromy transformations. Hence the projection of $H^1(X_q, \mathbb{Z})$ onto $\mathbb{H}^1_{\zeta^{-1}}(X_q, \mathbb{C})$ is preserved by the monodromy group. It is also preserved under the action of the automorphisms of $X_q$ given by the action of $d$-th roots of unity, so the projection is actually a $\mathbb{Z}[\zeta]$-module (of rank 3).

In other words, we can think of the monodromy group as being defined over the ring $\mathcal{O}$ of integers in the cyclotomic field $\mathbb{K} = \mathbb{Q}(\zeta)$ (see [DM], section 12). This important fact can also be seen by computing the monodromy transformations explicitly. We now think of the monodromy group as $\overline{\Gamma} \subset U(2,1,\mathcal{O}) \subset GL(3,\mathcal{O})$. Torsion free subgroups of finite index in $G = GL(3,\mathcal{O})$ can be gotten by taking congruence subgroups

$$1 \to G_{\mathfrak{a}} \to G \to G/G_{\mathfrak{a}} \to 1$$

$$G_{\mathfrak{a}} = \{M \in G : M \equiv I \pmod{\mathfrak{a}}\}$$

$$G/G_{\mathfrak{a}} \subset GL(3, \mathcal{O}/\mathfrak{a})$$

For an appropriate choice of the ideal $\mathfrak{a}$, $G_{\mathfrak{a}}$ is torsion free. This produces a (normal) torsion free subgroup $\Gamma_0 \subset \Gamma$ of finite index, hence a complex manifold quotient $X_0 = {}_{\Gamma_0}\backslash\widetilde{X}$. Recall from Chapter 4 that we will always assume that the hypergeometric map $\widetilde{w}$ does have some branching.

**Definition 5.1** *We call the quotient $X_0$ a **Mostow-Siu surface**.*

It might be more appropriate to call $X_0$ a Mostow-Siu "type" surface. Strictly speaking, our construction is not the same as the one in [MS], although it exhibits many common features.

The basis for the original Mostow-Siu construction is the understanding of groups that Mostow denotes $\Gamma_{p,t}$ (see [M1]). They are generated by three complex reflections of order $p$ ($p = 3, 4$ or $5$) whose mirrors are given by three vectors $v_1$, $v_2$ and $v_3$, the inner product $\langle v_i, v_j\rangle$ being a certain known function of the rational parameter $t$. In our notations, $\Gamma_{p,t}$ is commensurable with the group $\Gamma_\mu$, for

$$\mu = \left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{4} + \frac{3}{2p} - \frac{t}{2}, \frac{1}{4} + \frac{3}{2p} + \frac{t}{2}\right) \tag{5.1}$$

A justification for this commensurability statement can be found for instance in [M3].

The careful reader will note that the 5-tuple (5.1) does not satisfy our assumption (2.7) since $\mu_4 + \mu_5$ is always greater than 1 for $p < 6$. On the other hand, it has the special property that the first three weights are equal, and the corresponding $(1 - \mu_i - \mu_j)^{-1}$ $(i, j \in \{1, 2, 3\})$ are half integers. For a finite number of values of $t$, the other $(1 - \mu_i - \mu_j)^{-1}$ are integers and the groups $\Gamma_{p,t}$ and $\Gamma_\mu$ are then discrete. This is proved in [M2] by showing that the hypergeometric map descends to a homeomorphism $\widetilde{X}/S_3 \to \mathbb{B}^2$, where $S_3$ denotes the symmetric group on three letters. Here the action of $S_3$ on $\widetilde{X}$ is induced from its obvious action on the three points with equal weights.

In general, the map has branching of order given by the numerators of $1-\mu_i-\mu_j$ ($i=1,2,3$ and $j=4,5$), just like in Chapter 4. The situation is then essentially the same as the one presented here, with $\widetilde{X}$ replaced by $\widetilde{X}/S_3$. The analysis is then much more subtle, though. For instance, the space $\widetilde{X}$ has singularities, whereas the quotient $\widetilde{X}/S_3$ is a manifold.

The original motivation behind the construction of such a surface is that it produces examples of non locally symmetric compact Kähler manifolds. This is explained in detail in [MS], making heavy use of the detailed analysis of the groups $\Gamma_{p,t}$ given in [M1]. Unfortunately, their description of the surfaces is quite different from ours. We will devote the next chapter to giving an argument using our notations.

In the last few chapters of this paper, we will focus on studying some properties of the Mostow-Siu surfaces, in essentially two directions – the analysis of their fundamental group, and the construction of bounded holomorphic functions on their universal cover.

We now go back to the fact that the each (linear) hypergeometric monodromy groups is defined over some number field $\mathbb{K}=\mathbb{Q}(\zeta)$. We discuss another important implication, that will be used repeatedly later on in this paper. Applying Galois automorphisms $\sigma\in\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ we get different embeddings of $\mathbb{K}$ in $\mathbb{C}$, which induce different embeddings of the (linear) monodromy group in $GL(n,\mathbb{C})$.

The Galois conjugate groups are still monodromy groups of hypergeometric functions, whose exponents can be computed explicitly from the original exponents (see [DM] section 12).

Recall that the weights $\mu=(\mu_1,\dots,\mu_{n+3})$ encode the monodromy group by means of the corresponding roots of unity $\mathrm{e}^{2\pi i\mu_j}$. These are powers of $\zeta$, whose behavior under the Galois automorphisms is understood. The automorphism $\sigma\in\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ given by $\zeta\mapsto\zeta^k$ (for a given $k$ prime to the order of $\zeta$) translates into $\mathrm{e}^{2\pi i\mu_j}\mapsto\mathrm{e}^{2\pi ik\mu_j}$. In other words, the conjugate tuple of weights is obtained by multiplying $\mu$ by the integer $k$, and reducing it in $\mathbb{Q}/\mathbb{Z}$. We denote by $\mu^\sigma$ the corresponding tuple of weights. We will sometimes call $\mu^\sigma$ a Galois conjugate of $\mu$, since the corresponding groups are Galois conjugates.

It is important to realize that the corresponding Galois conjugate group does not necessarily lie inside $PU(n,1)$, but rather in some $PU(n+2-r, r-1)$ subgroup of $PGL(n+1,\mathbb{C})$ (where $r = \sum_j \mu_j^\sigma \in \mathbb{Z}$). We write $\Gamma^\sigma$ for the corresponding Galois conjugate monodromy group.

For more details on Galois conjugates, we refer the reader to section 12 of [DM]. We summarize the previous discussion in the following.

**Proposition 5.2** *Let $\Gamma_\mu$ be the monodromy group corresponding to a given set of weights $\mu = (\mu_1, \ldots, \mu_{n+3})$. We write $d$ for the least common denominator of the $\mu_j$'s and $\zeta = \zeta_d = e^{2\pi i/d}$. For any $k$ prime to $d$, we denote by $\sigma$ the Galois automorphism of $\mathbb{Q}(\zeta)$ given by $\zeta \mapsto \zeta^k$. The Galois conjugate $\Gamma_\mu^\sigma$ is the monodromy group corresponding to $\mu^\sigma = (\mu_1^\sigma, \ldots, \mu_{n+3}^\sigma)$, where $0 < \mu_j^\sigma < 1$ and $\mu_j^\sigma = k\mu_j$ in $\mathbb{Q}/\mathbb{Z}$.*

**Remark 5.3** Note that for $k = d-1$, $\zeta^{d-1} = \overline{\zeta}$ so that $\Gamma_\mu^\sigma$ is obtained from $\Gamma_\mu$ by complex conjugation. More generally, if $\sigma_1 : \zeta \mapsto \zeta^k$ and $\sigma_2 : \zeta \mapsto \zeta^{d-k}$, then $\Gamma_\mu^{\sigma_2}$ is obtained from $\Gamma_\mu^{\sigma_1}$ by complex conjugation. In terms of the weights, $\mu_j^{\sigma_1} = 1 - \mu_j^{\sigma_2}$.

To make the reader more comfortable with the perhaps intimidating result of Proposition 5.2, we work out one example in detail. Consider the Deligne-Mostow 5-tuple $\mu = \frac{1}{15}(4,6,6,6,8)$. Its Galois conjugates are obtained by multiplication by $k = 1$, 2, 4, 7, 8, 11, 13 and 14. We display the results only for $k = 2$, 4, and 7, since the others can easily be deduced using Remark 5.3.

$$
\begin{aligned}
2\ \mu &= \frac{1}{15}(8,12,12,12,16) \equiv \frac{1}{15}(8,12,12,12,1)\\
4\ \mu &= \frac{1}{15}(16,24,24,24,32) \equiv \frac{1}{15}(1,9,9,9,2)\\
7\ \mu &= \frac{1}{15}(28,42,42,42,56) \equiv \frac{1}{15}(13,12,12,12,11)
\end{aligned}
$$

These give signatures $(1, 2)$, $(2, 1)$ and $(0, 3)$ respectively. One can deduce from the fact that $PU(2, 1)$ is noncompact that the discrete group $\Gamma_\mu$ is not arithmetic (see Proposition 12.7 in [DM]).

# CHAPTER 6

# THE RATIO OF CHERN CLASSES

We now argue that the universal cover of a Mostow-Siu surfaces cannot be biholomorphic to the ball. Following [MS], we compute the ratio of Chern classes $c_1^2/c_2$ and show that it is not equal to 3. The idea is that thanks to our understanding of the branching behavior of $\widetilde{X} \xrightarrow{\widetilde{w}} \mathbb{B}^2$, we can compute the Chern classes (or rather the Chern forms) of $\widetilde{X}$ in terms of those of $\mathbb{B} = \mathbb{B}^2$ and a correction factor involving the ramification divisors.

We assume from now on that the ramification divisors of the hypergeometric map $\widetilde{w} : \widetilde{X} \to \mathbb{B}^2$ are disjoint. In terms of the combinatorics of the 5-tuple $\mu$, this means that we want the integrality condition to fail ($(1-\mu_i-\mu_j)^{-1} \notin \mathbb{Z}$) only for pairs of indices that overlap.

$$\left.\begin{array}{l}(1-\mu_i-\mu_j)^{-1} \notin \mathbb{Z}\\ (1-\mu_k-\mu_l)^{-1} \notin \mathbb{Z}\end{array}\right\} \Rightarrow \{i,j\} \cap \{k,l\} \neq \emptyset \tag{6.1}$$

The first observation is that $d\widetilde{w}$ is a section of $\mathrm{T}^*\widetilde{X} \otimes \widetilde{w}^*\mathrm{T}\mathbb{B}$. By taking second exterior powers, we get the Jacobian $J(\widetilde{w})$ to be a section of $\wedge^2\mathrm{T}^*\widetilde{X} \otimes \wedge^2\mathrm{T}\mathbb{B}$. Now the zero divisor of the Jacobian gives us the ramification divisor, which we denote by $\widetilde{R}$.

$$\widetilde{R} = K_{\widetilde{X}} - \widetilde{w}^* K_{\mathbb{B}}$$

or in terms of Chern forms

$$c_1(\widetilde{X}) = \widetilde{w}^* c_1(\mathbb{B}) - \widetilde{R}$$

$$c_1^2(\widetilde{X}) = \widetilde{w}^* c_1^2(\mathbb{B}) - 2\,\widetilde{w}^* c_1(\mathbb{B}) \cdot \widetilde{R} + \widetilde{R}^2$$

We write $\widetilde{R} = \pi^*(R)$, $R = \sum b_i R_i$, where $\pi$ is the projection $\widetilde{X} \to {}_{\Gamma_0}\backslash\widetilde{X} = X_0$. The second term in the expression for $c_1^2(\widetilde{X})$ can be interpreted in terms of the $R_i$'s as follows. On a subball $\mathbb{B}'$, $c_1(\mathbb{B})$ restricts to 3/2 times the area form on $\mathbb{B}'$, hence

$$2\ \widetilde{w}^* c_1(\mathbb{B}) \cdot \pi^*(R_i) = 3\ \chi(R_i)$$

$$c_1^2(X_0) = \pi_* \widetilde{w}^* c_1^2(\mathbb{B}) - 3\ \sum b_i \chi(R_i) + R^2$$

On the other hand Riemann-Hurwitz suggests

$$c_2(X_0) = \pi_* \widetilde{w}^* c_2(\mathbb{B}) - \sum b_i \chi(R_i) \tag{6.2}$$

which gives

$$\frac{c_1^2(X_0)}{c_2(X_0)} = 3 + \frac{R^2}{\pi_* \widetilde{w} c_2(\mathbb{B}) - \sum b_i \chi(R_i)} \tag{6.3}$$

We say a few words about the justification of formula (6.2). Note that if the monodromy group $\Gamma$ acts discretely on $\mathbb{B}^2$ (which occurs only for two examples satisfying our assumptions), we get a branched cover ${}_{\Gamma_0}\backslash\widetilde{X} \to {}_{\Gamma_0}\backslash\mathbb{B}^2$ and (6.2) is just the usual Riemann-Hurwitz formula. In most cases though, we do not even have such a quotient ${}_{\Gamma_0}\backslash\mathbb{B}^2$.

One way to justify the general case of formula (6.2) is to analyze the characteristic current of the singular connection on $T\widetilde{X}$ gotten from pulling back the connection on the ball. This basis for this approach can be found in [HL].

The ratio of Chern classes of $X_0$ is equal to 3 if and only if $R^2 = 0$. We can compute the self-intersection $R^2$ more or less explicitly, but it is clear that it is actually negative (unless $\widetilde{w}$ is not branched at all, which we rule out in the definition of Mostow-Siu surfaces). Recall that the $R_i$'s are disjoint, and that $R_i^2$ is the Euler class of the normal bundle $N_{X_0} R_i$, which turns out to be equal to $\chi(R_i)/2(b_i+1)$.

Note that although this formula shows that $X_0$ is not a ball quotient, it is difficult to use in practice since the Euler characteristics $\chi(R_i)$ are hard to get our hands on. For one thing, they depend on the subgroup $\Gamma_0$, although in the end, the ratio $c_1^2/c_2$ does not.

To obtain concrete numbers in terms of the 5-tuple $\mu$, it is more convenient to compute Chern classes by thinking of $X_0$ as a branched cover of $X$, which is $\mathbb{P}^2$ blown up at four points. One then gets the following formula.

$$\frac{c_1^2(X_0)}{c_2(X_0)} = 2 + \frac{1 - \sum \frac{1}{d_{ij}^2}}{2 - \sum \frac{1}{d_{ij}} + \sum_{\{i,j\}\cap\{k,l\}=\emptyset} \frac{1}{d_{ij}d_{kl}}} \tag{6.4}$$

where, as before, $1 - \mu_i - \mu_j = \frac{n_{ij}}{d_{ij}}$. This is the formula given in Proposition 12 of [Z]. Several similar formulas can also be found in [BHH]. Note that this gives a practical way to compute the ratio of Chern classes in terms of the 5-tuple $\mu$ only (we compute the ratio $c_1^2/c_2$ for each example given in the appendix).

# CHAPTER 7

# CONSTRUCTION OF THE METRIC

Here we assume once again that condition (6.1) holds, i.e., that the branching divisors of the hypergeometric map $\widetilde{w} : \widetilde{X} \to \mathbb{B}^2$ do not intersect. Notice that this implies that the restriction of the map $\widetilde{w}$ to a component of a branching divisor maps isomorphically onto a subball in $\mathbb{B}^2$. This will be justified in detail in the proof of Theorem 9.2 (see Remark 9.4). Actually we know the local structure of the map $\widetilde{w}$ near a branching divisor $\widetilde{D}_{ij}$, namely it looks like $(z_1, z_2) \mapsto (z_1, z_2^{n_{ij}})$.

Note that this is exactly the situation envisioned in [MS]. We briefly recall their construction of a metric. The general idea is to pull back the Bergman metric of the ball using our hypergeometric map $\widetilde{w}$. Of course this does not quite define a metric on $\widetilde{X}$ because of the branching behavior of $\widetilde{w}$ (the pull back is singular along the ramification locus). One then wants to add a correction factor near the ramification divisors, but two major difficulties stand on our way.

The first one is that we want to get a *Kähler* metric, hence we need to be careful with a "partition of unity" type of argument (the way to solve this is to work with the Kähler potentials rather than with the metric themselves). The second difficulty comes from the fact that we want our metric on $\widetilde{X}$ to be invariant under the action of the monodromy group, since we eventually want a metric on the quotient $X_0$. Invariance will be guaranteed by selecting the correction metric carefully, in accordance with the local description of the map $\widetilde{w}$.

Consider the bounded domain $\mathbb{D} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2k} < 1\}$, which comes with a natural map onto the ball $\mathbb{B}^2$ given by $(z_1, z_2) \mapsto (z_1, z_2^k)$. Mostow and Siu calculate its Bergman metric explicitly, showing that its kernel is given by (7.1).

$$\Phi(z_1, z_2) = \frac{1}{k\pi^2} \frac{(k+1)(1-|z_1|^2)^{1/k} - (k-1)|z_2|^2}{(1-|z_1|^2)^{2-1/k}((1-|z_1|^2)^{1/k} - |z_2|^2)^3} \tag{7.1}$$

We write $\Psi$ for the pull-back of the Bergman kernel of the ball. The detailed computations in [MS] show that the potential $\Phi + \lambda\Psi$ gives a Kähler metric of negative curvature in a neighborhood of $w = 0$, for any $\lambda \geq 0$. The metric is by construction invariant under the action of the stabilizer of $w = 0$ in $Aut(\mathbb{B}^2)$.

This shows how we can find a metric on $\widetilde{X}$, invariant under the action of $\Gamma$. For each branching divisor $D_{ij}$ we choose a component $\widetilde{D}_{ij}^{(\nu)}$, and repeat the construction from the previous paragraph in a neighborhood $\widetilde{U}_{ij}^{\nu}$ of $\widetilde{D}_{ij}^{\nu}$. We get a potential $\Phi_{ij}^{\nu}$ invariant under the stabilizer of $\widetilde{D}_{ij}^{\nu}$ in $\Gamma$. To define the metric $\Phi_{ij}^{\nu'}$ near a different component $\widetilde{D}_{ij}^{\nu'}$, we just push forward the metric $\Phi_{ij}^{\nu}$ by some element of $\Gamma$ that maps $\widetilde{D}_{ij}^{\nu}$ to $\widetilde{D}_{ij}^{\nu'}$. This push-forward is independent of the element of $\Gamma$ we choose, since the potential $\Phi_{ij}^{\nu}$ is invariant under the stabilizer of $\widetilde{D}_{ij}^{\nu}$. Doing this for different $i$, $j$, we get a potential $\Phi$ in a neighborhood of the ramification locus, invariant under $\Gamma$.

We choose an invariant $C^\infty$ function $0 \leq \rho \leq 1$ which is 1 in a neighborhood of the ramification locus, 0 outside a larger neighborhood. Then $\Theta = \rho\Phi + (1-\rho)\Psi$ defines a Kähler metric on $\widetilde{X}$, agrees with $\Phi$ in a neighborhood of the ramification locus, and is of course still invariant under $\Gamma$.

Once again, thanks to the computation in [MS], we know that in some neighborhood of the ramification locus, the metric $\partial\overline{\partial}\log\Theta + \lambda\widetilde{w}^*(\partial\overline{\partial}\log\Psi)$ has negative sectional curvature for all $\lambda \geq 0$. Now because of the compactness of the quotient $X_0$, we can choose $\lambda$ large enough for this metric to be negatively curved everywhere on $\widetilde{X}$.

**Remark 7.1**

1. The curvature actually satisfies a stronger condition than negative curvature, which implies a strong rigidity property (see [MS]).
2. Note that it is clear from the construction that the divisors $\widetilde{D}_{ij}$ (branching or not) are totally geodesic, since they are the fixed point sets of isometries $\rho(\gamma_{ij})$ (where $\gamma_{ij}$ is an appropriate loop around $D_{ij}$).

# CHAPTER 8

# TRIANGLE GROUPS

We now describe in some detail the one-dimensional situation. Not only does it give some intuition on the behavior of the more complicated two-dimensional examples, but the results from this chapter will be used extensively later on.

We start with a 4-tuple $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ of rational numbers, satisfying as before $0 < \mu_j < 1$ and $\sum \mu_j = 2$. The relevant constructions discussed in chapter 2 can be summarized in the following diagram

$$\begin{array}{ccccccc} & & \widetilde{Q} & \hookrightarrow & \widetilde{X} & \xrightarrow{\widetilde{w}} & \mathbb{B}^1 \simeq \mathbb{H}^2 \\ & & \downarrow & & \downarrow & & \\ \mathbb{P}^1 - \{0, 1, \infty\} & \simeq & Q & \hookrightarrow & \mathbb{P}^1 & & \end{array}$$

Here $\widetilde{w}$ maps the lift of a hemisphere in $\mathbb{P}^1$ biholomorphically onto a triangle in $\mathbb{H}^2$, with angles given by $(1 - \mu_i - \mu_j)\pi$. The monodromy group $\Gamma = \Gamma_\mu \subset Aut(\mathbb{H}^2)$ is generated by (any two of) the three rotations centered at the vertices of such a triangle, with angles $2(1 - \mu_i - \mu_j)\pi$. We shall write the three numbers $1 - \mu_i - \mu_j$ as reduced fractions $\frac{k}{p}$, $\frac{m}{q}$ and $\frac{l}{r}$, and $a$, $b$ and $c$ for the corresponding rotations in $\mathbb{H}^2$ (Figure 8.1). Of course $\pi_1(Q) \simeq F_2 = \langle x, y \rangle$ is just a free group on two generators, where we think of $x$ and $y$ as small loops around two appropriate points out of

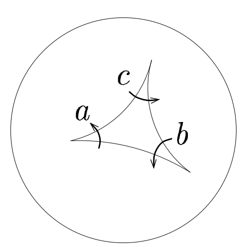


**Figure 8.1**. Generating rotations for $\Gamma$.

$\{0,1,\infty\}$. The kernel $K$ of the monodromy representation gives a presentation for the monodromy group in terms of two generators, say $a$ and $b$.

$$1 \to K \to \pi_1(Q) \simeq F_2 \to \Gamma \to 1$$

In $\Gamma$, we know that $a^p = b^q = (ab)^r = 1$. In other words, we know that $K$ contains the normal subgroup $N \subset F_2$ generated by $x^p$, $y^q$ and $(xy)^r$.

$$N = \langle\!\langle x^p, y^q, (xy)^r \rangle\!\rangle \subset K \subset F_2$$

If the angles of our hyperbolic triangle are integral parts of $\pi$ (i.e., if $k = m = l = 1$), then $\Gamma$ is just a classical triangle group, with presentation

$$\langle x, y | x^p, y^q, (xy)^r \rangle \tag{8.1}$$

In other words, if the Picard integrality condition is satisfied, then the two subgroups $N$ and $K$ coincide. In general, we could have $N \subsetneq K$, as we will see below.

The subgroups $K$ and $N$ can of course be thought of in terms of covering spaces. The inclusion $\widetilde{Q} \subset \widetilde{X}$ induces a surjection $\pi_1(\widetilde{Q}) \twoheadrightarrow \pi_1(\widetilde{X})$ (note that $\widetilde{X} - \widetilde{Q}$ has real codimension two in $\widetilde{X}$). The kernel of this surjection is the normal subgroup generated by "small" loops around the points in $\widetilde{X} - \widetilde{Q}$. Because of the local structure of the branched cover $\widetilde{X} \to X \simeq \mathbb{P}^1$, these small loops are just lifts of powers of small loops around 0, 1 and $\infty$. The exponent is given precisely by the order of the corresponding elements of the monodromy group, or in other words by the denominators $p$, $q$ and $r$ of the $1 - \mu_i - \mu_j$.

To summarize, we get a short exact sequence

$$1 \to N \to K \simeq \pi_1(\widetilde{Q}) \to \pi_1(\widetilde{X}) \to 1 \tag{8.2}$$

or $\pi_1(\widetilde{X}) \simeq K/N$. In particular, if the Picard integrality condition

$$(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z} \text{ for all } i, j \tag{8.3}$$

is satisfied, then the fact that $N = K$ means that $\widetilde{X}$ is simply connected, which is no surprise, since we know that $\widetilde{w} : \widetilde{X} \to \mathbb{B}^1$ is an isomorphism.

Here we are interested in situations where the Picard integrality (8.3) condition fails. In general, just like in the two-dimensional situation, we then get a nondiscrete monodromy group.

**Theorem 8.1** *Let $T$ be a triangle in $\mathbb{H}^2$ with angles $\alpha$, $\beta$ and $\gamma$ that are not all integral parts of $\pi$. Let $\Gamma$ be the group generated by rotations centered at the vertices of $T$ and with angles $2\alpha$, $2\beta$ and $2\gamma$ respectively. $\Gamma$ is discrete if and only if the angles of $T$ are given by one of the following*

$$\begin{array}{rl} \text{(i)} & \frac{2\pi}{s}, \frac{\pi}{t}, \frac{\pi}{t} \\ \text{(ii)} & \frac{2\pi}{t}, \frac{2\pi}{t}, \frac{2\pi}{t} \\ \text{(iii)} & \frac{2\pi}{t}, \frac{\pi}{2}, \frac{\pi}{t} \\ \text{(iv)} & \frac{3\pi}{t}, \frac{\pi}{3}, \frac{\pi}{t} \\ \text{(v)} & \frac{4\pi}{t}, \frac{\pi}{t}, \frac{\pi}{t} \\ \text{(vi)} & \frac{2\pi}{7}, \frac{\pi}{3}, \frac{\pi}{7} \end{array}$$

**Proof:** A proof of this is given in [M4] (actually Mostow omits cases (ii) and (vi), but the idea of the proof given there is entirely correct). We recall the main ideas involved in the argument.

The difficult part of the theorem is to show that our condition is necessary. The fact that it is sufficient essentially follows from the pictures in Figure 8.2.

Let us examine case (i), for instance. $T$ is a union of two copies of a triangle $T'$ with angles $\pi/2$, $\pi/s$ and $\pi/t$ (for the sake of brevity, we call $T'$ a $(2,s,t)$-triangle). This certainly implies that $\Gamma$ is a subgroup of a $(2,s,t)$-triangle group (see Figure 8.3).

It is easy to check that, as long as $s$ is odd, $\Gamma$ is not just a subgroup of a $(2,s,t)$-triangle but is actually equal to it. Indeed using the notations of Figure 8.3, we have

$$\left\{ \begin{array}{l} a = a'^2 \\ b = b' \\ c = c'b'c'^{-1} \end{array} \right. \qquad \left\{ \begin{array}{l} a' = a^{\frac{s+1}{2}} \\ b' = b \\ c' = b^{-1}a^{-\frac{s+1}{2}} = a^{\frac{s+1}{2}}b \end{array} \right. \tag{8.4}$$

In general Figure 8.2 illustrates the fact that in each case of the theorem, $\Gamma$ is a subgroup of a classical triangle group, hence it is obviously discrete. Once again,

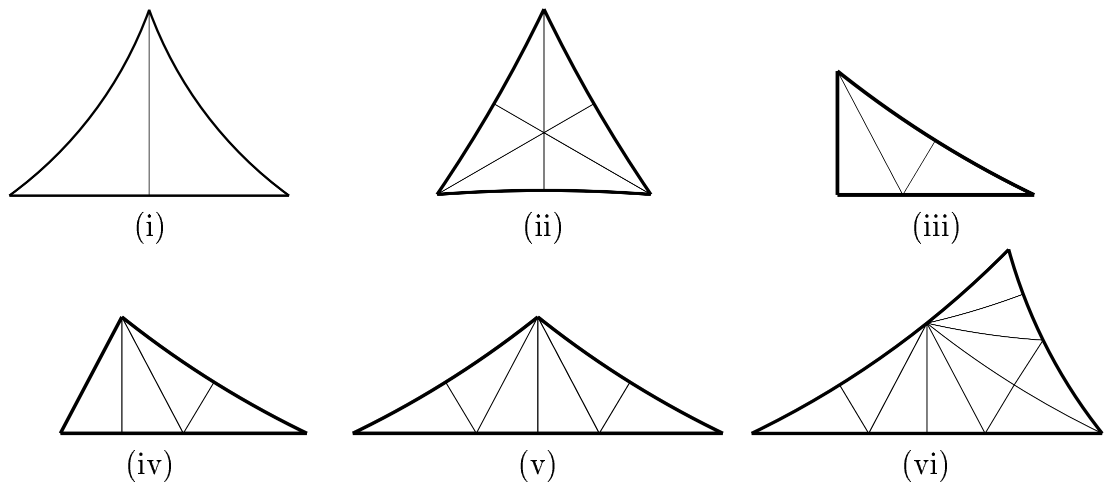


**Figure 8.2**. In each case of the theorem, $T$ is a union of copies of some smaller triangle $T'$. $T'$ is a $(2,s,t)$-triangle in case (i) and a $(2,3,t)$-triangle in all the other cases. The construction works for any $t \geq 7$ in cases (ii)-(vi) but only for $t = 7$ in case (vi).

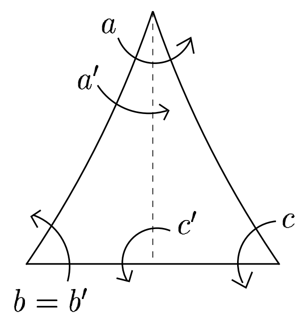


**Figure 8.3**. $T$ has angles $2\pi/s$, $\pi/t$, $\pi/t$, and $T'$ has angles $\pi/s$, $\pi/t$, $\pi/2$. $a$, $b$ and $c$ are the corresponding generators for $\Gamma$, and $a'$, $b'$, $c'$ are natural generators for the $(2,s,t)$-triangle group.

one can check that in all the cases (i)-(vi), the group $\Gamma$ is actually equal to a triangle group.

Now we give a quick sketch of the proof that the six cases (i)-(vi) are the *only* triangles for which $\Gamma$ is discrete.

Given that $\Gamma$ is discrete, we want to show that $T$ must be the union of finitely many congruent copies of a given triangle $T'$, whose angles are all integral parts of $\pi$. We shall refer to $T'$ as an elementary tile for $T$. A key ingredient in Mostow's argument is what he calls the triangulation algorithm (see [M4]).

We write $\Gamma^*$ for the group generated by the Schwarz reflections in the sides of $T$. Recall that $\Gamma \subset \Gamma^*$ is the index two subgroup of orientation preserving elements in $\Gamma^*$. Of course $\Gamma$ is discrete if and only if $\Gamma^*$ is.

Roughly, the idea behind the triangulation algorithm is to subdivide $T$ by using the components of the complement of the set of all mirrors of Schwarz reflections in $\Gamma^*$. Mostow argues that, if at least one of the angles of $T$ is not an integral part of $\pi$, then the triangulation algorithm yields an elementary tile $T'$ with angles $\pi/2$, $\pi/s$ and $\pi/t$.

Comparing the areas of $T$ and $T'$, we get

$$\pi - (\alpha + \beta + \gamma) = n\pi \left\{ 1 - (\frac{1}{2} + \frac{1}{s} + \frac{1}{t}) \right\} \tag{8.5}$$

for some integers $n$, $s$ and $t$. We may assume $3 \leq s \leq t$. Mostow considers the cases $s = 3$ and $s > 3$ separately. We present only $s = 3$ here, since it covers the two cases that are missing in [M4]. The analysis of the case $s > 3$ is similar.

Since we want $T$ to be a union of copies of $T'$, each of the rational numbers $\alpha/\pi$, $\beta/\pi$ and $\gamma/\pi$ must be chosen to be an integer multiple of one of $\frac{1}{2}$, $\frac{1}{s} = \frac{1}{3}$ or $\frac{1}{t}$. We break all the possibilities into six cases

$$(a) \quad \frac{1}{2}, \frac{1}{3}, \frac{k}{t} \qquad (b) \quad \frac{1}{3}, \frac{1}{3}, \frac{k}{t} \qquad (c) \quad \frac{2}{3}, \frac{k}{t}, \frac{l}{t}$$

$$(d) \quad \frac{1}{2}, \frac{k}{t}, \frac{l}{t} \qquad (e) \quad \frac{1}{3}, \frac{k}{t}, \frac{l}{t} \qquad (f) \quad \frac{k}{t}, \frac{l}{t}, \frac{m}{t}$$

The first three cases are easily taken care of. For instance, in case $(c)$, equation (8.5) yields

$$n = \frac{\frac{1}{3} - \frac{k+l}{t}}{\frac{1}{6} - \frac{1}{t}} = 2\,\frac{\frac{1}{3} - \frac{k+l}{t}}{\frac{1}{3} - \frac{2}{t}} \tag{8.6}$$

This expression is strictly less than 2 as long as $k + l > 2$, but $n = 1$ is impossible ($T$ would have to be equal to $T'$). Hence we must have $k = l = 1$, and the corresponding angles of $T$ fall in case (i) of our theorem.

Similarly one can rule out case $(a)$, and show that the only way $T$ can generate a discrete group in case $(b)$ is if $k = 2$, which once again falls into case (i) of our theorem.

We analyze case $(e)$ in detail. Once again we use (8.5) to get

$$n = \frac{\frac{2}{3} - \frac{k+l}{t}}{\frac{1}{6} - \frac{1}{t}} = 4\,\frac{\frac{2}{3} - \frac{k+l}{t}}{\frac{2}{3} - \frac{4}{t}} \tag{8.7}$$

Note that this implies $n \leq 4$ whenever $k + l \geq 4$. We assume without loss of generality that $k \leq l$. If $k \geq 2$, then $l \geq 2$ and the triangulation algorithm yields more than four triangles which contradicts the estimate $n \leq 4$ (Figure 8.4). The point is that if the mirrors of two Schwarz reflections in $\Gamma^*$ make an angle of $2\pi/t$, then their bisector is also a mirror in $\Gamma^*$.

Now we must have $k = 1$. Here Mostow assumes $l = 3$, in which case $n = 4$ and one gets case (iv) of our theorem, but he omits the possibility $l = 2$. A little analysis of (8.7) shows that the only odd values of $t$ for which $n$ can be an integer are $t = 7$ and $t = 9$, yielding respectively $n = 10$ and $n = 6$. Figure 8.2 shows that $t = 7$ does give a discrete group, corresponding to case (vi) of our theorem.

One can check for instance using the triangulation algorithm that the case $t = 9$ does not give a discrete group.

This concludes the analysis of case $(e)$. A similar argument would show that $(d)$ yields case (iii) our theorem, and that $(f)$ yields cases (i), (ii) and (v).

Recall that we have assumed that the elementary tile $T$ had angles $\pi/2$, $\pi/3$ and $\pi/t$. The case where $T'$ has angles $\pi/2$, $\pi/s$ and $\pi/t$ for $s, t > 3$ is treated

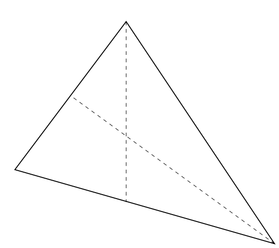

**Figure 8.4**. $T$ has angles $\pi/3$, $2\pi/t$ and $2\pi/t$. The triangulation algorithm yields at least 5 triangles, but our basic estimates show $n \leq 4$.

similarly and yields triangles of type (i). □

**Remark 8.2** 1. As mentioned in the proof, whenever $\Gamma$ is discrete, it is actually is actually some $(p, q, r)$-triangle group. If we assume that not all the angles of $T$ are integral parts of $\pi$ (in terms of the list of cases in the theorem, this amounts to having no simplification in the fractions), then $\Gamma$ is a $(2, s, t)$-triangle group in case (i), $(2, 3, t)$ in case (ii)-(v) and $(2, 3, 7)$ in case (vi).

2. It is easy to figure out which 4-tuples produce a triangle with angles as in Theorem 8.1, from the fact that the angles are given by $(1 - \mu_i - \mu_j)\pi$ for the appropriate choice of indices $i$, $j$. In fact, up to permutation of the weights, there are two 4-tuples giving angles $\alpha$, $\beta$ and $\gamma$. One of them is

$$\mu = \frac{1}{2}(1 - \alpha + \beta + \gamma, 1 + \alpha - \beta + \gamma, 1 + \alpha + \beta - \gamma, 1 - \alpha - \beta - \gamma) \quad (8.8)$$

and the other one is gotten from $\mu$ by replacing $\mu_j$ by $1 - \mu_j$.

We apply the previous considerations to $\Gamma = \Gamma_\mu$, the monodromy group of some hypergeometric map.

**Proposition 8.3** *If $\Gamma_\mu$ is discrete but $\mu$ does not satisfy the Picard integrality condition, then $N \subsetneq K$, or in other words the cover $\widetilde{X}$ is not simply connected.*

**Proof:** We write $k\pi/p$, $l\pi/q$ and $m\pi/r$ for the angles of the relevant triangle $T$, which is the image of a hemisphere under the hypergeometric map. Recall that we write $\pi_1(Q) \simeq \langle x, y\rangle$ and $N = \langle\langle x^p, y^q, (xy)^r\rangle\rangle$. The normal subgroup $N$ is a subgroup of the kernel of the monodromy $K$ and $\Gamma_\mu \simeq \pi_1(Q)/K$. In other words, $N \subset K$ with equality if and only if $\Gamma_\mu$ has a presentation

$$\langle x, y | x^p, y^q, (xy)^r\rangle \quad (8.9)$$

Now if $\Gamma_\mu$ is discrete, it must appear somewhere in the list of Theorem 8.1 and in particular it is a $(p', q', r')$-triangle group for $\{p', q', r'\} \neq \{p, q, r\}$, where this last inequality follows from Remark 8.2. Here we use the fact that the Picard integrality condition is not satisfied, so that the angles of $T$ are **not** all integral part of $\pi$.

The proposition follows at once from the fact that a $(p,q,r)$-triangle group is uniquely determined up to isomorphism by $p$, $q$ and $r$, but we can also check it directly by exhibiting elements in $K-N$. We do this in detail when the angles of $T$ are $2\pi/s$, $\pi/t$, $\pi/t$, which is case (i) of Theorem 8.1.

We refer to Figure 8.3, and write $a$, $b$ for generating rotations in $\Gamma_\mu$, with angles $4\pi/s$, $2\pi/t$ respectively. Of course we have

$$a^s = b^t = (ab)^t = 1 \tag{8.10}$$

but the claim is that there are more relations between our generators.

It is readily checked that the element $a^{\frac{s+1}{2}}b$ is a rotation centered at the midpoint of the base of the triangle $T$, with angle $\pi$. In particular it has order two. Hence we have a relation

$$(a^{\frac{s+1}{2}}b)^2 = 1 \tag{8.11}$$

between our generators $a$ and $b$, which is not a consequence of the relations (8.10).

In other words, in terms of the fundamental group $\pi_1(Q)$, the loop $(y^{\frac{s+1}{2}}x)^2$ is in $K$ but not in $N$, hence $K/N \simeq \pi_1(\widetilde{X})$ is nontrivial. $\square$

The reader should not be misled by proposition 8.3. One does not *need* $\Gamma_\mu$ to be discrete in order to get $\widetilde{X}$ not to be simply connected. Just like in the discussion at the end of Chapter 5, we can think of the linear monodromy group as being defined over the field $\mathbb{K} = \mathbb{Q}(\sqrt[d]{1})$, and consider Galois conjugates $\Gamma_\mu^\sigma$ for different Galois automorphisms $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$. These are in general nondiscrete, and have the same group theoretic properties as $\Gamma_\mu$. We summarize this in the following

**Corollary 8.4** *If some Galois conjugate $\Gamma_\mu^\sigma$ is discrete but $\mu^\sigma$ does not satisfy the Picard integrality condition, then $N \subsetneq K$ and $\widetilde{X}$ is not simply connected.*

In general it seems difficult to find *necessary* and sufficient conditions on $\mu$ in order to have $\pi_1(\widetilde{X}) \neq 1$. If some Galois conjugate does satisfy the Picard integrality condition (the corresponding monodromy group is then discrete), then we certainly know that $\widetilde{X}$ is simply connected. When $\Gamma_\mu$ is not a Galois conjugate of any discrete group, it is not clear how large the sugroup $K$ is. It is not even clear that $\Gamma_\mu$ would be finitely presented.

# CHAPTER 9

# THE COMPLETION DIVISORS

We now go back to the two-dimensional situation corresponding to some 5-tuple $\mu = (\mu_1, \ldots, \mu_5)$. Recall that we have assumed for any $i$,$j$ that $\mu_i + \mu_j < 1$, so that it makes sense to consider the 4-tuple

$$\mu' = (\mu_i + \mu_j, \mu_1, \ldots, \widehat{\mu_i}, \ldots, \widehat{\mu_j}, \ldots, \mu_5)$$

in the context the previous chapters.

**Definition 9.1** *In the situation above, we say that $\mu$ **contracts** to $\mu'$.*

We write $Q'$, $X'$, $\widetilde{w}', \ldots$ for the analogue of $Q$, $X$, $\widetilde{w}, \ldots$ corresponding to $\mu'$ instead of $\mu$. Of course, $X'$ is contained in $X$ as a topological space, but we want a stronger statement that takes into account the orbifold structure of these two spaces.

**Theorem 9.2** *$\widetilde{X}'$ embeds in $\widetilde{X}$ as a connected component of the preimage of the divisor $D_{ij}$ (corresponding to $x_i$ and $x_j$ coming together).*

**Proof:** This fact is stated in much more generality in [DM], section 8. We give a direct proof in the simpler particular case needed here. For simplicity of the notations, we will assume $\{i, j\} = \{1, 2\}$, so that

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$$
$$\mu' = (\mu_1 + \mu_2, \mu_3, \mu_4, \mu_5)$$

We write $C$ for the open subset of $D_{12}$ corresponding to having the first two points coming together, but with no other collapsing allowed ($C{=}D_{12} - \cup D_{kl}$). We want to understand the structure of a component $\widetilde{C}$ of the preimage of $C$ in $\widetilde{X}$. $C$ is isomorphic to $\mathbb{P}^1 - \{3 \text{ pts}\}$ and $\widetilde{C}$ will be a certain cover of $C$. The fact that we do get a cover follows from Lemma 3.2. The interesting part is to identify the cover $\widetilde{C} \to C$. We want to show that it is isomorphic to $\widetilde{Q}' \to Q'$. The local structure of both covers is given by the denominators of $1 - \mu_i - \mu_j$ $(i, j \in \{3, 4, 5\})$, hence they have the same local structure, but a priori they could be different globally. Part of the difficulty here comes from the fact that $\Gamma'$ is not quite a subgroup of $\Gamma$. For more on the relationship between these groups, see remark 9.4.

We select loops $\gamma_{ij}$ around $D_{ij}$ as in Figure 9.1. The loop $\gamma_{ij}$ corresponds to having $x_i$ come close to $x_j$ along $T_{ij}$, making one positive turn, then going back along $T_{ij}$ to its original position. Recall that $Q$ is the quotient of $M \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ by $Aut(\mathbb{P}^1)$. The loops $\gamma_{ij}$ were just described as loops in $M$, but we may also view them as loops in $Q$.

We claim that $\gamma_{35}\gamma_{45}\gamma_{34} = \gamma_{12}$ in $\pi_1(Q)$. This is a slightly subtle point since in $\pi_1(M)$, which is the spherical braid group on five strands, the loop $\gamma_{12}^{-1}\gamma_{35}\gamma_{45}\gamma_{34}$ is *not* trivial. It corresponds to the central element depicted in Figure 9.2.

Note that $Q$ fibers over $\mathbb{P}^1 - \{3 \text{ pts}\}$, with fibers $\mathbb{P}^1 - \{4 \text{ pts}\}$. This can easily

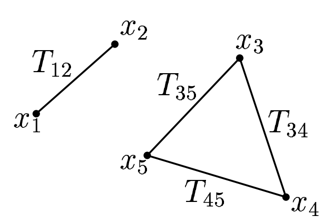


**Figure 9.1**. Each loop $\gamma_{ij}$ amounts to $x_i$ going around $x_j$, along the path $T_{ij}$.

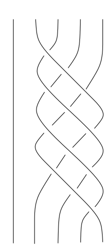

**Figure 9.2**. The loop $\gamma_{12}^{-1}\gamma_{35}\gamma_{45}\gamma_{34}$ is central in $\pi_1(M)$, hence it projects to a trivial loop in $\pi_1(Q)$.

be seen from Figure 2.2, for instance. The base and the fiber both have free fundamental group, hence $\pi_1(Q)$ has trivial center. The loop $\gamma_{12}^{-1}\gamma_{35}\gamma_{45}\gamma_{34}$, which is central in $\pi_1(M)$, must project to a trivial loop in $\pi_1(Q)$.

Recall that $\gamma_{ij}$ maps to a complex reflection $R_{ij}$, whose mirror will be denoted by $M_{ij}$. Since $\gamma_{12}$ commutes with $\gamma_{34}$, $\gamma_{35}$ and $\gamma_{45}$, we know that $M_{12}$ is orthogonal to $M_{34}$, $M_{35}$ and $M_{45}$. Using formula (2.8), one can compute the angles of the triangle in $M_{12}$ formed by the intersection points with the three other mirrors $M_{ij}$. They are just the angles between the one-dimensional orthogonal complements of the mirrors $M_{34}$, $M_{35}$ and $M_{45}$, which are given by formula (2.8). But the formula would be exactly the same if we were computing the angles between the mirrors in the case of the contracted 4-tuple (note that it involves only the weights that are not affected by the contraction). Hence the reflections $R_{ij}$ ($\{i,j\} = \{3,4\}, \{3,5\}, \{4,5\}$) act on $M_{12}$ as a copy of the monodromy group $\Gamma'$.

We know that the cover $\widetilde{Q}'$ is the smallest cover on which the corresponding hypergeometric map is single valued, but $\widetilde{C}$ could conceivably be larger cover than $\widetilde{Q}'$. To show that this is not the case, we need to analyze the completion process carefully.

We pick a tubular neighborhood $V$ of $C$, and write $U = V \cap Q$. If $\widetilde{U}$ is a component of the inverse image of $U$ in $\widetilde{Q}$, the corresponding component $\widetilde{V}$ of $\pi^{-1}(V)$ is the completion of $\widetilde{U}$ over $V$ (using Lemma 3.2 again). We choose the component $\widetilde{U}$ so that $\widetilde{C}$ is contained in $\widetilde{V}$.

Since $U$ fibers over $C \simeq \mathbb{P}^1 - \{3 \text{ pts}\}$, we have an exact sequence

$$1 \to <\gamma_{12}> \to \pi_1(U) \to \pi_1(\mathbb{P}^1 - \{3\text{pts}\}) \to 1$$

and $\pi_1(\widetilde{U}) \simeq K \cap \pi_1(U)$. Note that in what follows we choose a basepoint for $Q$ inside $U$, and take representatives for the loops $\gamma_{ij}$ that are contained in $U$.

The important point is that when we complete $\widetilde{U}$ to $\widetilde{V}$, it retracts to $\widetilde{C}$, so that

$$\pi_1(\widetilde{C}) \simeq \pi_1(\widetilde{V}) \simeq \pi_1(\widetilde{U}) \;/ <\gamma_{12}^{d_{12}}>$$

We claim that this group is the same as $\pi_1(\widetilde{Q}') = K'$. To check this, we consider the restriction to $K \cap \pi_1(U)$ of the map $\pi_1(Q) \to \pi_1(Q')$ induced by any one of the forgetful maps corresponding to forgetting $x_1$ or $x_2$. It just maps $\gamma_{12}$ to 1, and the other three $\gamma_{ij}$ to natural generators $\gamma'_{ij}$ of $\pi_1(Q')$. Note that $K \cap \pi_1(U)$ maps into $K'$. Indeed, if some product $\gamma = \Pi\gamma_{ij}$ is in $K$, then $\rho(\gamma) = 1$ acts on $M_{12}$ as $\Pi\gamma'_{ij}$, hence $\Pi\gamma'_{ij}$ is in $K'$.

We claim that the kernel of the map

$$K \cap \pi_1(U) \to K' \tag{9.1}$$

is just the cyclic subgroup generated by $\gamma_{12}^{d_{12}}$. Indeed, if $\Pi\gamma_{ij} \in K$ is such that $\Pi\gamma'_{ij} = 1$, then $\Pi\gamma_{ij}$ must be a product of conjugates of $\gamma_{35}\gamma_{45}\gamma_{34}$, which is equal to $\gamma_{12}$.

One still needs to check that the map (9.1) is onto. Given any $\prod \gamma'_{ij} \in K'$, the image of $\prod \gamma_{ij}$ in the monodromy group $\Gamma$ fixes the mirror $M_{12}$, which means that it is in $K$ possibly after changing it by power of $\gamma_{12}$, as follows from the lemma below. □

**Lemma 9.3** *Suppose $g \in \Gamma$ fixes $M_{12}$. Then $g$ is a power of $\rho(\gamma_{12})$.*

**Proof:** From equivariance, we know that $g$ preserves some component $\widetilde{D}_{12}$ of $\pi^{-1}(D_{12})$ (where $\pi : \widetilde{X} \to X$ denotes the natural projection map). It also preserves the components of $\pi^{-1}(D_{34})$, $\pi^{-1}(D_{35})$ and $\pi^{-1}(D_{45})$ that meet $D_{12}$, hence it must fix their intersection points. This shows that $g$ actually fixes $D_{12}$. Recall here that $\Gamma$ acts *discretely* on $\widetilde{X}$, hence $g$ must be of finite order. From the local description of the map $\widetilde{X} \to X$, we know that it must be of order dividing $d_{12}$ (the denominator of the reduced fraction $1 - \mu_1 - \mu_2$). □

**Remark 9.4**

1. One should be aware that the stabilizer $\mathrm{Stab}_\Gamma(M_{12})$ of $M_{12}$ in $\Gamma$ is slightly larger than $\Gamma'$. More precisely, it surjects onto $\Gamma'$, with kernel the finite cyclic subgroup generated by $R_{12} = \rho(\gamma_{12})$.

$$1 \to \langle R_{12} \rangle \to \mathrm{Stab}_\Gamma(M_{12}) \to \Gamma' \to 1$$

2. The proof shows a little more, namely that the restriction of the hypergeometric map $\widetilde{w}|_{\widetilde{D}_{ij}}$ is just the hypergeometric map in dimension one corresponding

to the contracted 4-tuple $\mu'$. In particular, if $\mu'$ satisfies the Picard integrality condition, the divisor $\widetilde{D}_{ij}$ gets mapped isomorphically onto a one-dimensional subball in $\mathbb{B}^2$.

Note that $\widetilde{X}'$ actually sits in $\widetilde{X}$ as a *totally geodesic* divisor (it is the fixed point set of an isometry), hence we know that its fundamental group injects:

$$\pi_1(\widetilde{X}') \simeq K'/N' \hookrightarrow K/N \simeq \pi(\widetilde{X})$$

In the previous chapter we have discussed how to construct examples of one-dimensional situations where $K'/N'$ is nontrivial. Hence we can get two-dimensional situations where $K/N$ is nontrivial. This will be discussed in detail in the next chapter.

# CHAPTER 10

# FUNDAMENTAL GROUP

We recall the general hypergeometric picture corresponding to an $(n+3)$-tuple $\mu$, satisfying our standing hypothesis $\sum_j \mu_j = 2$, $0 < \mu_j < 1$. We summarize the situation in the following diagram

$$\begin{array}{ccccc} \widetilde{Q} & \hookrightarrow & \widetilde{X} & \stackrel{\widetilde{w}}{\to} & \mathbb{B}^2 \\ \downarrow & & \downarrow & & \\ Q & \hookrightarrow & X & & \end{array}$$

The cover $\widetilde{Q} \to Q$ is unbranched and has deck group $\pi_1(Q)/K \simeq \Gamma$. The map $\widetilde{X} \to X$ is given by the Fox completion of $\widetilde{Q}$ over the appropriate compactification $X$ of $Q$. The action of $\Gamma$ on $\widetilde{Q}$ extends to an action on $\widetilde{X}$, and the quotient ${}_{\Gamma}\backslash\widetilde{X}$ is topologically just $X$.

The completion $\widetilde{X}$ turns out to be a manifold only for $n \leq 3$, under some quite restrictive assumptions on the weights $\mu_j$. In this paper we consider only the cases $n = 1$ and $n = 2$.

The action of $\Gamma$ has fixed points on $\widetilde{X}$. Instead of considering $X = {}_{\Gamma}\backslash\widetilde{X}$ as an orbifold, we choose a torsion free subgroup of finite index $\Gamma_0 \subset \Gamma$ and look at the manifold quotient $X_0 = {}_{\Gamma_0}\backslash\widetilde{X}$.

The long exact sequence of homotopy groups for the quotient map $\widetilde{X} \to {}_{\Gamma_0}\backslash\widetilde{X} = X_0$ gives

$$1 \to \pi_1(\widetilde{X}) \to \pi_1(X_0) \to \Gamma_0 \to 1 \tag{10.1}$$

Since $\Gamma_0 \subset \Gamma \subset PU(n,1)$, we get a representation

$$\pi_1(X_0) \to PU(n,1) \tag{10.2}$$

which is faithful if and only if $\pi_1(\widetilde{X})$ is trivial.

Note that in all Picard/Deligne-Mostow situations (i.e. $(1-\mu_i-\mu_j)^{-1} \in \mathbb{Z}\,\forall\, i,j$), we have an isomorphism $\widetilde{X} \xrightarrow{\simeq} \mathbb{B}^2$ hence we clearly get that $\pi_1(\widetilde{X}) = 1$ and $\pi_1(X_0) \xrightarrow{\simeq} \Gamma_0$. In general it is difficult to determine whether or not the representation (10.2) is faithful.

In this section, we show that in general it need not be, and give an explicit sufficient condition on $\mu$ for (10.2) to be nonfaithful (see Theorem 10.4). The preceding discussion motivates the following definition.

**Definition 10.1** *Let $\mu = (\mu_1, \ldots, \mu_r)$, $\mu_j \in \mathbb{Q}$ satisfy $0 < \mu_j < 1$ and $\sum \mu_j = 2$. We call the $r$-tuple $\mu$ **nonfaithful** if the corresponding cover $\widetilde{X}$ is not simply connected. We call it **discrete** if the corresponding monodromy group $\Gamma_\mu$ is discrete.*

We will make use of these notions only for $r$=4 or 5. In terms of Definition 10.1, Corollary 8.4 reads as follows.

**Theorem 10.2** *Suppose the 4-tuple $\mu'$ is Galois conjugate to some discrete 4-tuple that does not satisfy the Picard integrality condition. Then $\mu'$ is nonfaithful.*

**Remark 10.3**

1. Observe that to check whether the Galois conjugates of $\mu'$ are discrete or not, we need only check the ones giving signature $(1,1)$. The other ones give signature $(2,0)$ or $(0,2)$ and cannot be discrete since they are infinite subgroups of a compact group.

2. If $\mu'$ is Galois conjugate to a discrete 4-tuple that *does* satisfy the Picard integrality condition, we know that it is faithful. If none of the Galois conjugates $\mu'^{\sigma}$ is discrete, then we do not know how to determine whether $\mu'$ is faithful or not.

In principle, one can list all 4-tuples satisfying the hypotheses of Theorem 10.2, since all the discrete monodromy groups are listed in Theorem 8.1. It is not entirely obvious though how to write general formulas for the 4-tuples corresponding to their Galois conjugates (going from $\mu'$ to its Galois conjugates involves reducing fractions and taking fractional parts).

The following gives a *sufficient* condition for a 5-tuple to be nonfaithful.

**Theorem 10.4** *Suppose the 5-tuple $\mu$ contracts to a nonfaithful 4-tuple. Then $\mu$ is nonfaithful.*

**Proof:** This was already stated in the end Chapter 8. The point is that we have proved in Theorem 9.2 that the hypergeometric cover $\widetilde{X}'$ corresponding to the contracted 4-tuple $\mu'$ embeds in $\widetilde{X}$ as a totally geodesic divisor. This implies that its fundamental group injects.

$$\pi_1(\widetilde{X}') \hookrightarrow \pi_1(\widetilde{X}) \tag{10.3}$$

The hypothesis that $\mu'$ is nonfaithful implies that $\pi_1(\widetilde{X}') \neq 1$, hence $\pi_1(\widetilde{X}) \neq 1$ as well. □

We give in the appendix the list of all 5-tuples with denominator up to 200 that contract to some 4-tuple that we know to be nonfaithful. In other words we give a list of 5-tuples $\mu$ for which Theorems 10.2 and 10.4 apply to show that $\mu$ is nonfaithful. The most direct way to get such examples is to start with the list in Theorem 8.1, reconstruct the corresponding 4-tuples, and then try to split one of the weights to recover a 5-tuple that contracts to it. Such an approach only leads to a small number of examples. For instance, the $\frac{3\pi}{t}, \frac{\pi}{t}, \frac{\pi}{3}$ triangle comes from the 4-tuple

$$\mu' = \left(\frac{2}{3}+\frac{2}{t}, \frac{1}{3}+\frac{1}{t}, \frac{1}{3}-\frac{1}{t}, \frac{2}{3}-\frac{2}{t}\right) \tag{10.4}$$

which is a contraction of

$$\mu = \left(\frac{1}{3}+\frac{1}{t}, \frac{1}{3}+\frac{1}{t}, \frac{1}{3}+\frac{1}{t}, \frac{1}{3}-\frac{1}{t}, \frac{2}{3}-\frac{2}{t}\right) \tag{10.5}$$

Note that this 5-tuple satisfies our assumption of normal crossings ($\mu_i + \mu_j < 1$ for any $i, j$), but in general the branching divisors intersect. One can check that they do not intersect only for $t = 7$ or 8 (yielding respectively the third and fifth examples given in the appendix).

One way to get more examples would be to start with a nontrivial Galois conjugate of a discrete 4-tuple, rather than a discrete 4-tuple itself. The direct approach of splitting one of the weights then becomes quite cumbersome.

On the other hand, staring at the list of examples described in the appendix, it is relatively easy to find infinite families of nonfaithful examples. For instance consider the 5-tuples

$$\mu = \frac{1}{6+18k}\,(2+3k, 4+6k, 2+9k, 2+9k, 2+9k) \tag{10.6}$$

We claim that contracting two equal weights gives a nonfaithful 4-tuple $\mu'$. Recall that this means that one of the Galois conjugates $\mu'^{\sigma}$ is discrete, which we will show by using Proposition 5.2 and Theorem 8.1. The behavior of $\mu'$ depends on the parity of the parameter $k$, and we shall go into the details of the argument only for $k$ even. The analysis is similar for $k$ odd.

We write $k = 2n$, so that $\mu'$ becomes

$$\frac{1}{3+18n}(1+3n, 2+6n, 1+9n, 2+18n)$$

It corresponds to a triangle with angles $\frac{3n\pi}{1+6n}$, $\frac{n\pi}{1+6n}$ and $\frac{\pi}{3}$. One might suspect that it be Galois conjugate to the discrete group corresponding to angles $\frac{3\pi}{1+6n}$, $\frac{\pi}{1+6n}$ and $\frac{\pi}{3}$, which we now proceed to show.

Looking back at Remark 5.3, we know that $\mu'$ is Galois conjugate (in fact complex conjugate) to

$$\overline{\mu}' = \frac{1}{3+18n}(2+15n, 1+12n, 2+9n, 1)$$

Multiplying $\overline{\mu}'$ by $(-5+6n)$ and reducing in $\mathbb{Q}/\mathbb{Z}$, we get

$$\frac{1}{3+18n}(5+12n, 7+6n, -1+12n, -5+6n) \tag{10.7}$$

We check for instance

$$\begin{aligned}(2+15n)(-5+6n) &= -10-63n+5n\cdot 18n\\ &\equiv -10-63n+5n\cdot(-3) = -10+78n \equiv 5+12n \pmod{3+18n}\end{aligned}$$

Now the 4-tuple (10.7) corresponds to angles $\frac{3\pi}{1+6n}$, $\frac{\pi}{1+6n}$ and $\frac{\pi}{3}$ which fits into case (iv) of Theorem 8.1.

When $k$ is odd, $\mu'$ can be shown to be nonfaithful by showing that the two groups corresponding to angles $\frac{3k\pi}{2+6k}$, $\frac{k\pi}{2+6k}$, $\frac{\pi}{3}$ and $\frac{3\pi}{2+6k}$, $\frac{\pi}{2+6k}$, $\frac{\pi}{3}$ respectively are Galois conjugates. We will describe two more families explicitly in the next chapter.

**Remark 10.5** 1. The theorems of this chapter produce examples of *nonfaithful* 5-tuples. It is natural to ask whether there are non obvious *faithful* 5-tuples. The obvious ones are the Picard examples (it turns out there are only finitely many such). Interestingly enough, we do not know of any other faithful example. A natural approach would be to consider Galois conjugates of the Picard examples. Unfortunately their Galois conjugates never satisfy our assumptions that the compactification divisors have normal crossings.

2. As the preceding discussion illustrates, the fundamental group $\pi_1(X_0)$ is quite a complicated object. It is an extension

$$1 \to K/N \simeq \pi_1(\widetilde{X}) \to \pi_1(X_0) \to \Gamma_0 \to 1$$

and it already takes some work to determine just if $K/N$ is trivial or not. In the cases where we know the representation to be nonfaithful (namely when Theorem 10.4 applies), $K/N$ is in fact not finitely generated. A natural question then comes to mind – is $\pi_1(X_0)$ is residually finite?

# CHAPTER 11

# BOUNDED HOLOMORPHIC FUNCTIONS AND MAPS TO RIEMANN SURFACES

We recall some notations. $X - Q$ consists of 10 divisors $D_{ij}$ corresponding to $x_i$ and $x_j$ coming together. We pick loops $\gamma_{ij}$ around $D_{ij}$, and write $1 - \mu_i - \mu_j = \frac{n_{ij}}{d_{ij}}$ as a reduced fraction. Then the loops $\gamma_{ij}^{d_{ij}}$ are in the kernel $K$ of the monodromy, and we write

$$N = \langle\langle \gamma_{ij}^{d_{ij}}, i, j \in \{0, \ldots, 5\} \rangle\rangle \subset K$$

for the normal subgroup of $\pi_1(Q)$ generated by the loops $\gamma_{ij}^{d_{ij}}$. In what follows we assume that $N \subsetneq K$ (we presented sufficient conditions for this to happen in Chapter 10). We then get nontrivial covers

$$\begin{array}{ccc} \widehat{Q} & \hookrightarrow & \widehat{X} \\ \downarrow & & \downarrow \\ \widetilde{Q} & \hookrightarrow & \widetilde{X} \end{array}$$

where $\widehat{Q}$ is the cover of $Q$ so that $\pi_1(\widehat{Q}) = N$, and $\widehat{X}$ is the Fox completion of $\widehat{Q}$ over $X$ (or equivalently over $\widetilde{X}$). One checks at once that the space $\widehat{X}$ is simply connected. Indeed, since the completion divisors have complex codimension one in $\widehat{X}$, $\pi_1(\widehat{Q})$ surjects onto $\pi_1(\widehat{X})$, the kernel being given by the normal subgroup generated by small loops around the components of $\widehat{X} - \widehat{Q}$, which is precisely $N$.

The map $\widehat{X} \to \widetilde{X}$ is an unbranched cover. One way to see this is to observe that $K/N = \pi_1(\widetilde{X})$, which is the deck group of the cover $\widehat{Q} \to \widetilde{Q}$, must be torsion free since $\widetilde{X}$ has negative curvature. Another approach is to look at the local structure of the cover $\widehat{X} \to X$, which is the same as that of the cover $\widetilde{X} \to X$.

Now $\widehat{X}$ is the universal cover of $X_0$, and we want to construct bounded holomorphic functions on $\widehat{X}$. There is an obvious way of getting such functions, using our hypergeometric map $\widetilde{X} \to \mathbb{B}^2$, pre-composing it with the projection $\widehat{X} \to \widetilde{X}$ and post-composing with any bounded holomorphic function on $\mathbb{B}^2$. These were of course already known when the Mostow-Siu surfaces were first constructed.

We give another construction that produces bounded holomorphic functions on $\widehat{X}$, and show that in certain cases the functions we get do not factor through the corresponding hypergeometric map. The basis for our construction lies in the use of forgetful maps $Q \to Q'$, where $Q$ (resp. $Q'$) is the configuration space of five points (resp. four points) on $\mathbb{P}^1$. In terms of the appropriate description of $Q$ as a subset of $\mathbb{P}^1 \times \mathbb{P}^1$ (see Chapter 2), these forgetful maps are just the projections onto one of the factors.

These maps extend to maps $X \to X'$ between the compactifications, but in general these extensions are not maps of orbifolds. In terms of covers, $Q \to Q'$ does not lift to a map $\widehat{Q} \to \widehat{Q}'$ in general. In other words, the induced homomorphism $\pi_1(Q) \to \pi_1(Q')$ does not in general map $N$ into $N'$. We point out that when the map *is* a map of orbifolds, we can lift it to a holomorphic map from some complex surface to a compact Riemann surface. Lifting it further to the universal cover of the complex surface yields a bounded holomorphic function. We shall come back to maps to Riemann surfaces later on in the chapter (see Proposition 11.4). For now we concentrate on the description of a simple necessary and sufficient condition for $N$ to map into $N'$.

We write $\gamma'_{ij}$ for the image of $\gamma_{ij}$. Note that some $\gamma'_{ij}$ are trivial. To fix the ideas, we assume that we forget the first point, and project onto the second factor, as in Figure 11.1. Then $\gamma_{1j} = 1$ for all $j$, and $\gamma'_{34} = \gamma'_{25}$, $\gamma'_{35} = \gamma'_{24}$ and $\gamma'_{45} = \gamma'_{23}$ are standard generators for $\pi_1(Q')$.

$N$ is the normal subgroup generated by $\gamma_{ij}^{d_{ij}}$. In order for these loops to map into $N'$, we need

$$\begin{array}{c} d_{34} \mid d_{25} \\ d_{35} \mid d_{24} \\ d_{45} \mid d_{23} \end{array} \tag{11.1}$$

and this divisibility condition is clearly also sufficient to get $N \to N'$.

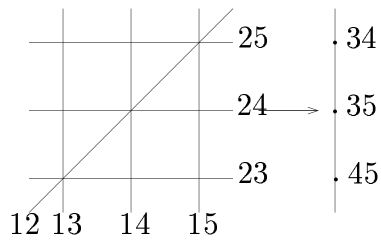


**Figure 11.1**. A forgetful map $Q \to Q'$.

In general, for other forgetful maps, we also get a necessary and sufficient condition for $N$ to map into $N'$ in terms of three divisibility conditions (just apply the appropriate permutation of indices in (11.1)).

We assume some choice of a forgetful map $Q \to Q'$ does lift to $\widehat{Q} \to \widehat{Q}'$, or in other words that $N$ maps into $N'$, or in yet other words that condition (11.1) is satisfied for some permutation of the indices. Then we can extend this lift to a map $\widehat{\phi} : \widehat{X} \to \widehat{X}'$. Note that $\widehat{X}'$, the universal cover of $\widetilde{X}'$, is just $\mathbb{B}^1 \simeq \mathbb{H}^2$. Hence $\widehat{\phi}$ is a bounded holomorphic function on the universal cover of our Mostow-Siu surface. Summarizing our discussion, we get

**Theorem 11.1** *The natural map to $D_{ij}$ forgetting $x_i$ lifts to a bounded holomorphic function $\widehat{\phi} : \widehat{X} \to \widehat{X}'$ if and only if $N$ maps into $N'$, which is equivalent to the three divisibility conditions*

$$\begin{array}{r|l} d_{kl} & d_{jm} \\ d_{km} & d_{jl} \\ d_{lm} & d_{jk} \end{array} \tag{11.2}$$

*where* $\{1, \ldots, 5\} = \{i, j\} \cup \{k, l, m\}$.

The divisibility condition (11.2) is easy to test on any given 5-tuple of weights and should give an efficient way to construct bounded holomorphic functions on the universal cover $\widehat{X}$ of $X_0$, but it is not at all clear that there should be any examples where it is satisfied. Recall that we require that the compactification divisors have normal crossings and that the ramification divisors be disjoint. These requirements can be translated into concrete conditions on the weights $\mu_j$, namely (2.7) and (6.1). In order to apply Theorem 11.1 we need $\mu$ to admit a contraction to some 4-

tuple satisfying (11.2). All these numerical conditions taken together seem very restrictive, but it turns out that one can produce examples where Theorem 11.1 applies. We will exhibit two infinite families of such examples at the end of this chapter.

Another possible objection to the relevance of Theorem 11.1 is that the bounded holomorphic functions it produces might a priori be of "hypergeometric origin." In other words, they could conceivably come from $\widetilde{w} : \widetilde{X} \to \mathbb{B}^2$. If this were the case, then $\widehat{\phi}$ would be constant on the fibers of the projection $\widehat{X} \to \widetilde{X}$.

**Theorem 11.2** *Assume that the 5-tuple $\mu$ is such that some forgetful map to $D_{ij}$ lifts to a bounded holomorphic function and that the corresponding contracted 4-tuple $\mu'$, obtained from contracting $\mu_i$ and $\mu_j$, is nonfaithful. Then $\widehat{\phi}$ does not factor through the corresponding hypergeometric map.*

**Proof:** All we need to show is that $K/N$ does not map trivially under $\pi_1(Q)/N \to \pi_1(Q')/N'$ ($K/N$ is the deck group of $\widehat{Q} \to \widetilde{Q}$). Our assumption that $\mu'$ be nonfaithful ensures that $K'/N'$ is nontrivial.

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\phi}} & \widehat{X}' \\ \downarrow & & \downarrow \\ \widetilde{X} & & \widetilde{X}' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & X' \end{array}$$

We distinguish two cases.

CASE 1: $K \not\twoheadrightarrow K'$

Then clearly $K/N$ does not map trivially, and $\widehat{\phi}$ is not constant on the fibers of $\widehat{X} \to \widetilde{X}$.

CASE 2: $K \twoheadrightarrow K'$

Then $K/N$ maps into $K'/N'$, and we want this homomorphism to be nontrivial. Actually the map is onto, and we have assumed that $K'/N' \neq 1$ ($\mu'$ is nonfaithful). The fact that it is onto follows from the arguments given in the proof of Theorem 9.2. There we showed that there is a subgroup of $K$ that maps onto $K'$. □

**Remark 11.3** 1. One difficulty here is that in general, we do not know how to check whether $K$ maps into $K'$. An obvious necessary condition is that $N$ map into $N'$, but it is not clear whether that condition is also sufficient. In terms of covers, we do not know when the forgetful maps lift to maps $\widetilde{\phi} : \widetilde{X} \to \widetilde{X}'$ between the hypergeometric covers.

2. Our assumption that $K'/N'$ be nonfaithful is sufficient, but most likely not necessary to produce holomorphic functions that do not factor through the hypergeometric map. There is no a priori reason, given that $\widehat{\phi}$ be constant on the fibers of $\widehat{X} \to \widetilde{X}$, for it to factor through $\mathbb{B}^2$. Note also that, as we shall discuss in more detail below, our bounded holomorphic functions have the very special property that they descend to maps to Riemann surfaces.

3. A natural question is of course whether our bounded holomorphic functions allow one to separate points in $\widehat{X}$. It is easy to see that, in general, these functions (together with the ones coming from the hypergeometric map) cannot separate points. One way to describe this is to look at the number of compactification divisors $D_{ij}$ for which the forgetful map is a map of orbifolds. In many examples, our constructions produce only one map of orbifolds, which clearly is not enough to separate points of $\widehat{X}$.

4. One way to generalize the construction of our holomorphic functions would be to map $X$ to any orbifold $\mathbb{P}^1$ with weights at 0, 1 and $\infty$ satisfying some divisibility condition. We would then get a map $\widehat{X} \to \mathbb{B}^1$ but since the weights are unrelated to the original hypergeometric situation, it is not clear how to check which points of $\widehat{X}$ it separates.

Recall that each of our new bounded holomorphic function is obtained as the lift of map of orbifolds $\phi : X \to X'$. In terms of manifolds, we may think of it as the lift of a map

$$\phi_0 : {}_{G_0}\backslash\widehat{X} \to {}_{G'_0}\backslash\widehat{X}' \tag{11.3}$$

One way to get such quotients on which $\phi$ lifts is the following. We write $G$ for the quotient $\pi_1(Q)/N$, and similarly $G' = \pi_1(Q')/N'$. Recall that $\phi : X \to X'$ is a map of orbifolds exactly when $N$ maps into $N'$, or in other words when $\pi_1(Q) \to \pi_1(Q')$ descends to a map $G \to G'$.

We choose a torsion free subgroup $G'_0$ of finite index in $G'$, and pull it back to a subgroup of finite index $\overline{G}_0 \subset G$. This group $\overline{G}_0$ might not be torsion free, but we can take a torsion free subgroup $G_0 \subset \overline{G}_0$ of finite index. By construction $G_0$ maps into $G'_0$, which means that $\widehat{\phi}$ descends to a map to a Riemann surface as in (11.3).

Note that in general we cannot choose the manifold ${}_{G_0}\backslash\widehat{X}$ to be a quotient of the monodromy cover $\widetilde{X}$. This would be the case if we could guarantee that $G_0$ be saturated with respect to the map $G \to \Gamma$, or in other words that $G_0$ contain $K/N$. This can certainly be arranged when $K$ maps into $K'$ (but this condition is difficult to check, as we stated in Remark 11.3).

On the other hand, the two surfaces ${}_{G_0}\backslash\widehat{X}$ and ${}_{\Gamma_0}\backslash\widetilde{X}$ are commensurable, hence we consider them both as Mostow-Siu type surfaces. We can then think of our bounded holomorphic functions as lifts of maps from Mostow-Siu type surfaces to compact Riemann surfaces. Observe that in general it is difficult to produce nontrivial maps to lower-dimensional manifolds.

We summarize the preceding discussion in the following.

**Proposition 11.4** *If one of the forgetful maps $\phi : X \to X'$ is a map of orbifolds, then it lifts to a holomorphic map $\phi_0 : {}_{G_0}\backslash\widehat{X} \to {}_{G'_0}\backslash\widehat{X}'$ to a compact Riemann surface, whose lift to the universal covers is the bounded holomorphic function $\widehat{\phi} : \widehat{X} \to \widehat{X}'$.*

We now go back to the quite restrictive hypotheses we need to make on our 5-tuples $\mu$ in order to get new bounded holomorphic functions. Once again, it is not at all clear that there should be situations where Theorem 11.2 applies. A quick look at the list of examples described in the appendix shows that there are examples, namely every time $N+$, $N-$ or $N\pm$ appears in some column $D_{ij}$, the hypotheses of the theorem are satisfied. $N$ means that the corresponding contracted

4-tuple $\mu'$ is nonfaithful, $+$ or $-$ indicates that one of the two natural forgetful maps to $D_{ij}$ lifts to a bounded holomorphic function on the universal cover $\widehat{X}$.

In fact, one can construct two infinite families of examples where Theorem 11.2 applies, parameterized by an integer $k \geq 0$:

$$\mu = \frac{1}{20+16k}(6+4k, 6+4k, 9+8k, 9+8k, 10+8k) \tag{11.4}$$

$$\mu = \frac{1}{18+12k}(5+4k, 7+4k, 7+4k, 7+4k, 10+8k) \tag{11.5}$$

It is readily checked that these 5-tuples always satisfy our standing assumptions (the compactification divisors have normal crossings and the branching divisors do not intersect). We analyze the family (11.5) in some detail.

The first observation is that we always gets an orbifold map to $D_{45}$ by forgetting the point $x_4$ (of course since $\mu_3 = \mu_4$ we could map to $D_{35}$ as well). Once again, one needs to check the three divisibility conditions (11.2). The contracted 4-tuple is given by (11.6).

$$\mu' = \frac{1}{10+8k}(3+2k, 3+2k, 5+4k, 9+8k) \tag{11.6}$$

By multiplying $\mu'$ by $1+4k$ (which is prime to the denominator $10+8k$), we see that it is Galois conjugate (see Proposition 5.2) to

$$\mu'^{\sigma} = \frac{1}{10+8k}(3+4k, 3+4k, 5+4k, 9+4k) \tag{11.7}$$

This last 4-tuple gives a triangle with angles $\frac{2\pi}{5+4k}$, $\frac{\pi}{4}$, $\frac{\pi}{4}$ which is in the list of Theorem 8.1.

The computations for family (11.5) work essentially the same way (although the behavior depends on the divisibility by 3 of the index $k$), to give an orbifold map to the divisor $D_{15}$ by forgetting the first point.

# CHAPTER 12

# FURTHER DEVELOPMENTS

## 12.1 Maps between surfaces

Recall that an interesting by-product of our construction of bounded holomorphic functions in Chapter 11 is that it gives nontrivial maps from Mostow-Siu type surfaces to compact Riemann surfaces, as in Proposition 11.4. In a similar fashion, one can construct maps between different Mostow-Siu surfaces, or from a Mostow-Siu surface to a Picard ball quotient. These seem to exhibit interesting behavior, and are certainly worth investing.

## 12.2 The three-dimensional example

Our two dimensional construction depends heavily on the fact that the Fox completion of the cover of $Q$ corresponding to the kernel of the monodromy is in fact a manifold. One way to state our starting assumption is that we did not allow for triple intersections between the completion divisors. Strictly speaking, this condition is sufficient but not necessary in order to get the Fox completion to be a manifold.

Considering necessary and sufficient conditions, we can make the construction work in some more cases in dimension two (at least we still get $\widetilde{X}$ to be a manifold). In higher dimensions, aside from the Deligne-Mostow ball quotients, only one hypergeometric cover gives a manifold, and it occurs in dimension three, for the choice of weights $\mu = (\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$. Coincidentally, this example is mentioned in [M4]. It happens to be, in dimension higher than two, the only "nonobvious" choice of weights that produces a discrete monodromy group. By "nonobvious," we mean that it does not satisfy the Picard integrality condition, nor the Mostow half-integrality condition (see [M2]).

Still we get a threefold $\widetilde{X}$ that branches over $\mathbb{B}^3$, and it is conceivable that this would produce the first example in dimension higher than two of a negatively curved compact Kähler manifold which is not locally symmetric.

# APPENDIX

# LIST OF EXAMPLES

Table A.1 lists all possible 5-tuples $\mu = \frac{1}{d}(n_1, n_2, n_3, n_4, n_5)$ with denominators up to 200, satisfying our assumptions

1. $0 < \mu_j < 1$ and $\sum \mu_j = 2$ (essentially this says that the monodromy group lies in $Aut(\mathbb{B}^2) \simeq PU(2,1)$).

2. The compactification divisors have normal crossings ($\mu_i + \mu_j < 1$ for all $i, j$).

3. The branching divisors are disjoint.

and satisfying the hypotheses of Theorems 10.2 and 10.4. This last condition means that some contraction of $\mu$ should be Galois conjugate to some discrete non Picard 4-tuple (in terms of the table below, each row contains at least one $N$).

We compute the ratio of Chern classes and, for each divisor $D_{ij}$, we give the following information on the corresponding contracted 4-tuple $\mu'$.

- $F$ means $\mu'$ is faithful.

- $N$ means $\mu'$ is not faithful.

- Neither $F$ nor $N$ means that no Galois conjugate of $\mu'$ is discrete, in which case we do not know whether $\mu'$ is faithful or not.

- $+$ (resp. $-$) indicates that the natural forgetful map to $D_{ij}$ forgetting $i$ (resp. $j$) is an orbifold map, or in other words that it lifts to give a bounded holomorphic function on the universal cover of the corresponding surface.

**Table A.1**. List of all examples with denominators up to 200 where our theorems in Chapters 10 and 11 apply. For each example we compute the ratio of Chern classes and give some information on the structure of the completion divisors $D_{ij}$.

| $d$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $n_5$ | $c_1^2/c_2$ | $D_{12}$ | $D_{13}$ | $D_{14}$ | $D_{15}$ | $D_{23}$ | $D_{24}$ | $D_{25}$ | $D_{34}$ | $D_{35}$ | $D_{45}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 18 | 5 | 7 | 7 | 7 | 10 | 2.9412 | $N$ | $N$ | $N$ | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 20 | 6 | 6 | 9 | 9 | 10 | 2.9389 | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $N$ | $N-$ | $N-$ |
| 21 | 4 | 8 | 10 | 10 | 10 | 2.8763 | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 24 | 5 | 10 | 10 | 11 | 12 | 2.8065 | $F$ | $F$ | $F$ | $F$ | | | $N\pm$ | | $N\pm$ | $F$ |
| 24 | 5 | 10 | 11 | 11 | 11 | 2.9032 | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 30 | 6 | 13 | 13 | 14 | 14 | 2.7622 | $F+$ | $F+$ | $N+$ | $N+$ | $F$ | | | | | |
| 30 | 9 | 11 | 11 | 11 | 18 | 2.8421 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 30 | 10 | 11 | 11 | 14 | 14 | 2.8667 | $F+$ | $F+$ | $N+$ | $N+$ | $F$ | | | | | $N$ |
| 30 | 11 | 11 | 11 | 13 | 14 | 2.8657 | $F$ | $F$ | $N$ | | $F$ | $N$ | | $N$ | | $N$ |
| 36 | 10 | 10 | 17 | 17 | 18 | 2.8391 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 39 | 7 | 14 | 19 | 19 | 19 | 2.7946 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 42 | 13 | 15 | 15 | 15 | 26 | 2.9072 | $N$ | $N$ | $N$ | $N+$ | $F$ | $F$ | $N+$ | $F$ | $N+$ | $N+$ |
| 52 | 14 | 14 | 25 | 25 | 26 | 2.7890 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 54 | 17 | 19 | 19 | 19 | 34 | 2.7656 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 57 | 10 | 20 | 28 | 28 | 28 | 2.7567 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 60 | 11 | 22 | 29 | 29 | 29 | 2.7747 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 66 | 21 | 23 | 23 | 23 | 42 | 2.7477 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 68 | 18 | 18 | 33 | 33 | 34 | 2.7610 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 75 | 13 | 26 | 37 | 37 | 37 | 2.7359 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 78 | 25 | 27 | 27 | 27 | 50 | 2.8020 | | | | $N+$ | $F$ | $F$ | $N+$ | $F$ | $N+$ | $N+$ |
| 84 | 22 | 22 | 41 | 41 | 42 | 2.7434 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 90 | 29 | 31 | 31 | 31 | 58 | 2.7261 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 93 | 16 | 32 | 46 | 46 | 46 | 2.7228 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 96 | 17 | 34 | 47 | 47 | 47 | 2.7352 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 100 | 26 | 26 | 49 | 49 | 50 | 2.7313 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 102 | 33 | 35 | 35 | 35 | 66 | 2.7191 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 111 | 19 | 38 | 55 | 55 | 55 | 2.7139 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 114 | 37 | 39 | 39 | 39 | 74 | 2.7599 | | | | $N+$ | $F$ | $F$ | $N+$ | $F$ | $N+$ | $N+$ |
| 116 | 30 | 30 | 57 | 57 | 58 | 2.7225 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 126 | 41 | 43 | 43 | 43 | 82 | 2.7091 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 129 | 22 | 44 | 64 | 64 | 64 | 2.7074 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 132 | 23 | 46 | 65 | 65 | 65 | 2.7167 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 132 | 34 | 34 | 65 | 65 | 66 | 2.7158 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 138 | 45 | 47 | 47 | 47 | 90 | 2.7054 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 147 | 25 | 50 | 73 | 73 | 73 | 2.7025 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 148 | 38 | 38 | 73 | 73 | 74 | 2.7105 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 150 | 49 | 51 | 51 | 51 | 98 | 2.7377 | | | | $N+$ | $F$ | $F$ | $N+$ | $F$ | $N+$ | $N+$ |
| 162 | 53 | 55 | 55 | 55 | 106 | 2.6997 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 164 | 42 | 42 | 81 | 81 | 82 | 2.7062 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 165 | 28 | 56 | 82 | 82 | 82 | 2.6986 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 168 | 29 | 58 | 83 | 83 | 83 | 2.7061 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 174 | 57 | 59 | 59 | 59 | 114 | 2.6974 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |
| 180 | 46 | 46 | 89 | 89 | 90 | 2.7028 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 183 | 31 | 62 | 91 | 91 | 91 | 2.6955 | $F$ | | | | $F$ | $F$ | $F$ | $N$ | $N$ | $N$ |
| 186 | 61 | 63 | 63 | 63 | 122 | 2.7240 | | | | $N+$ | $F$ | $F$ | $N+$ | $F$ | $N+$ | $N+$ |
| 196 | 50 | 50 | 97 | 97 | 98 | 2.6998 | $F$ | | | $F$ | | | $F$ | $N$ | $N-$ | $N-$ |
| 198 | 65 | 67 | 67 | 67 | 130 | 2.6937 | | | | $N+$ | $F$ | $F$ | $N$ | $F$ | $N$ | $N$ |

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