A negatively curved Kähler threefold not covered by the ball

Martin Deraux

Abstract

We construct examples of three-dimensional compact Kähler manifolds with negative curvature, not covered by the ball. Our manifolds are obtained as a natural generalization of the two-dimensional examples discovered by Mostow and Siu, using their description in terms of monodromy covers of hypergeometric functions.

Each example is obtained from a hypergeometric monodromy group in $PU(3,1)$ that is not discrete but has finite local monodromy. We describe a manifold on which the group acts discretely, and check that it has a compact quotient with the above features. The examples are locally described as branched covers of the ball, with totally geodesic branch locus.

1 Introduction

Compact Riemann surfaces (complex curves) are covered by the disk, the complex plane or the Riemann sphere, and these cases can be distinguished by the existence of a metric of negative, zero or positive curvature. In higher dimension, the situation is much more complicated, as was first illustrated by Mostow and Siu, who constructed an infinite family of compact negatively curved Kähler surfaces whose universal cover is not biholomorphic to the ball; it is unknown whether their universal cover is even a bounded domain. Another striking aspect of these examples is that they are not diffeomorphic to any locally symmetric space.

In this paper, we generalize the Mostow-Siu contruction to produce analogous examples in dimension three. A natural strategy to construct such non locally symmetric examples is the following : start with a smooth totally geodesic complex hypersurface in a given compact ball quotient, and use it as the branch locus of a cyclic cover. The topological obstruction for such a cover to exist, namely that the integer homology class of the hypersurface be divisible, seems to be difficult to check. In fact, neither the original

Mostow-Siu surfaces nor our threefolds are exactly of this nature, though they do draw on this idea since they are *locally* described as branched covers of the ball.

More specifically, each example X has an unramified covering $\tilde{X}$ and an equivariant holomorphic map $\tilde{w} : \tilde{X} \to \mathbb{B}^n$ ($n = 2$ or 3) that is locally a simple branched cover. However, the representation of the deck group Γ into $PU(n, 1)$ is not discrete, and the branch locus in the ball is a nondiscrete collection of totally geodesic subballs. We do not know whether our examples admit any nonzero degree maps to compact ball quotients.

We describe our examples in terms of monodromy covers of hypergeometric functions, using the techniques developed by Deligne and Mostow (see [DM] and [M2]). Since this differs quite a bit from the original description of the Mostow-Siu surfaces, we devote the first section to a proof of their main result using hypergeometric terminology. After writing down a quick argument based on the results in [M2], we go through a more detailed proof that gives much more information about the surfaces, and guides us toward the three-dimensional generalization.

One of the main ideas is that we want the relevant subgroups of $PU(n, 1)$ to be "mildly" nondiscrete, in the sense that the obvious point stabilizers in the group should be finite; we refer to this condition as finite local monodromy. In turn, this condition translates quite naturally into the existence of a locally compact (but possibly singular) covering of the ball on which the group acts discretely. Further work is required to check when this cover is actually a manifold.

The finite local monodromy condition can be translated into a purely numerical condition, and we can list all the choices of exponents for hypergeometric functions that yield only finite local monodromy. The list contains all the examples in [DM] and [M2] since those groups are discrete. It also contains an infinite number of two-dimensional non discrete examples, including the ones from [MS] and [D]. In dimension 3 or higher, the list of examples is finite: apart from the examples from [M2], there are nine groups with finite local monodromy, all in dimension three. The exponents for these nine examples are given in Table 1.

In particular, our hypergeometric technique for producing non locally symmetric negatively curved Kähler manifolds does not work in dimension four or higher, where the corresponding monodromy groups do not have finite local monodromy, unless the group is itself discrete.

It turns out that only three of the nine groups yield smooth, compact, negatively curved manifolds. Our main result is stated in Theorem 3.3, where we summarize the construction as well as the main features of our examples. We present a short proof that relies on some technical lemmas in [DM] and [M2], but a more direct proof, similar to the second proof of the two-dimensional case, can easily be deduced from our paper.

Section 4 is devoted to checking that our examples have the announced features. The construction of the Kähler metric is a natural 3-dimensional extension of the one given in [MS], and we simply refer the reader to [Z] for the details of the curvature computations.

In order to verify that our examples are not covered by the ball, recall that any three-dimensional ball quotient must have the same ratios of Chern numbers as $\mathbb{C}P^3$, and in particular it must satisfy $c_1^3 = 16c_3$. It is an easy consequence of the fact that our examples look locally like a branch cover of the ball that they do not satisfy the above equality.

Another way to see that our 3-folds are ball quotients is to analyze their totally geodesic divisors: Proposition 4.1 states that they contain some of the original Mostow-Siu surfaces as totally geodesic divisors.

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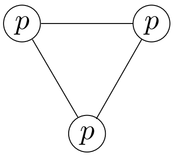
2 The Mostow-Siu surfaces

In this section we review the description of the original Mostow-Siu surfaces in terms of monodromy covers of hypergeometric functions, using the notations from [DM], [M3]. The Mostow-Siu surfaces were originally described in terms of certain reflection groups $\Gamma(p,t)$, generated by three complex reflections of order p ($p = 3$, 4 or 5), with Coxeter diagram



and phase shift φ given by $\varphi^3 = e^{\pi i t}$. We shall not use that description here, and we simply mention that this means that the group is generated by certain braiding complex reflections. As mentioned by Mostow in a number of survey papers on the subject (see [M3], [M4], or [M2]), these reflection groups can be identified as finite index subgroups of certain monodromy groups of hypergeometric functions.

The surfaces in [MS] are indexed by a positive integer parameter s, which is related to the rational number t by $t = \frac{1}{2} - \frac{1}{p} - \frac{2}{s}$ (we replace the greek letters ρ and σ from [MS] by r and s). The corresponding $\Gamma(p, t)$ has finite index in the hypergeometric monodromy group $\Gamma_{\mu,\Sigma}$, where

$$\begin{aligned} \mu_1 = \mu_2 = \mu_3 &= \frac{1}{2} - \frac{1}{p} \\ \mu_4 &= \frac{2}{p} + \frac{1}{s} \\ \mu_5 &= \frac{1}{2} + \frac{1}{p} - \frac{1}{s} \end{aligned} \tag{1}$$

and Σ is the symmetric group on three letters, permuting the three equal weights.

We follow [MS] in choosing p and s so that

$$\begin{aligned} &p = 3,\ 4 \text{ or } 5 \\ &\frac{1}{s} < \frac{1}{2} - \frac{1}{p} \end{aligned} \tag{2}$$

In terms of the 5-tuple μ, these two conditions imply respectively

$$\begin{aligned} &\mu_4 + \mu_5 > 1 \\ &\mu_1 + \mu_4 = \mu_2 + \mu_4 = \mu_3 + \mu_4 < 1 \end{aligned} \tag{3}$$

In other words, for such a choice of p and s, the only set $\{i, j\}$ of two indices with $\mu_i + \mu_j > 1$ is $\{4, 5\}$. In the terminology of [DM], this implies that Q_{st} is $\mathbb{P}^2$ blown up at three non collinear points.

We remind the reader of some of the notations used in [DM2]. $M \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ consists of 5-tuples of distinct points and $Q = M/PGL_2$. $Q_{st} = M_{st}/PGL_2$ where, in M_{st}, points $x_j \in \mathbb{P}^1$ are allowed to coalesce if and only if the corresponding weights μ_j add up to strictly less than 1.

Definition 2.1 *When $\mu_i + \mu_j < 1$, we denote by D_{ij} the divisor in Q_{st} corresponding to $x_i = x_j$.*

In particular for the 5-tuples coming from pairs (p, s) satisfying condition (2), $Q_{st} - Q$ consists of 9 divisors, whose combinatorics are described in Figure 1, on the left.

Given the symmetry of our 5-tuples, it is useful (and in fact needed, see Remark 2.5) to consider the natural action of the symmetric group Σ on Q_{st} (on the level of M_{st} this action simply permutes the first three factors in

$\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$). One way to think of Q_{st}/Σ is as the moduli space of five points on $\mathbb{P}^1$, the first three unordered and the last two ordered. Accordingly, we use the labels A, B and C as follows: A stands for 1, 2 or 3, B stands for 4 and C stands for 5.

Definition 2.2 *The divisor D_{AB} is the image in Q_{st}/Σ of the divisors D_{14}, D_{24} and D_{34}. D_{AC} is the image of D_{15}, D_{25} and D_{35} and D_{AA} is the image of D_{12}, D_{23} and D_{13}.*

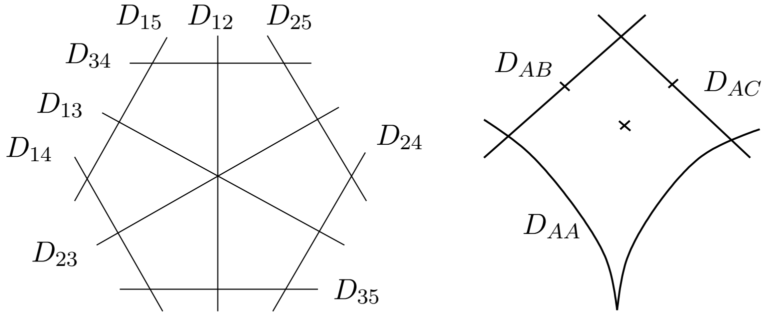


Figure 1: Combinatorics of the compactification divisors on Q_{st} and Q_{st}/Σ

For each of the divisors D_{ij} we compute the rational number $1-\mu_i-\mu_j$, where we allow in this notation to take i, j=A, B or C.

$$\begin{aligned} 1-\mu_A-\mu_A &= \frac{2}{p} \\ 1-\mu_A-\mu_B &= \frac{1}{2}-\frac{1}{p}-\frac{1}{s} \\ 1-\mu_A-\mu_C &= \frac{1}{s} \end{aligned} \tag{4}$$

In the terminology of [M2], the only divisor along which condition ΣINT could fail is D_{AB}. The 5-tuple μ satisfies ΣINT if and only if $1-\mu_A-\mu_B$ is the inverse of an integer. In particular we get

Theorem 2.3 *Let μ be given by (1) with p and s satisfying (2). Then the group Γ_μ, Σ is discrete in $PU(2,1)$ if and only if*

$$(1-\mu_A-\mu_B)^{-1} \in \mathbb{Z} \tag{5}$$

The fact that this condition is sufficient is a special case of the main theorem in [M2], and it is proved to be necessary in [M5]. There are only finitely many choices of (p,s) so that (5) holds, and these are listed for instance in [MS], page 360. When condition (5) fails, we write

$$1-\mu_A-\mu_B=\frac{1}{2}-\frac{1}{p}-\frac{1}{s}=\frac{m}{r} \tag{6}$$

as a reduced fraction (r is the same as the parameter ρ in [MS]). Condition ΣINT fails if and only if $m>1$. We restate the main results from [MS].

Theorem 2.4 *When $\Gamma_{\mu,\Sigma}$ is not a discrete subgroup of $PU(2,1)$, the action can be made discrete on some complex manifold $\widetilde{X}$ with an equivariant holomorphic map $\widetilde{w}:\widetilde{X}\to\mathbb{B}^2$, such that:*

1. *$\widetilde{w}$ is a local biholomorphism away from a smooth divisor in $\widetilde{X}$, whose components map to a non discrete collection of totally geodesic subballs in $\mathbb{B}^2$. Along this divisor, $\widetilde{w}$ has simple branching of order m.*
2. *There exists a Kähler metric of negative sectional curvature on $\widetilde{X}$ that is $\Gamma_{\mu,\Sigma}$-invariant.*
3. *For any torsion free subgroup of finite index $\Gamma_0\subset\Gamma_{\mu,\Sigma}$ the quotient $X={}_{\Gamma_0}\backslash\widetilde{X}$ is a compact negatively curved Kähler manifold.*
4. *$c_1^2(X)/c_2(X)<3$, hence X cannot be a ball quotient. In fact, X is not diffeomorphic to any locally symmetric space.*

We will focus mainly on the construction of the cover $\widetilde{X}$, since it is the only part where the arguments are significantly different from [MS]. For instance the construction of a Kähler metric is done by starting with the pull-back of the ball metric by the hypergeometric map, and carefully patching to it the Bergman metric of the bounded domain $|z_1|^{2m}+|z_2|^2<1$. This carries over word for word in our description precisely because of part 1 of the theorem. Part 4 follows from Remark 2.13 below.

The original construction was based on gluing copies of certain polyhedra in complex hyperbolic space, bounded by bisectors, a bare hands approach that turns out to be very difficult. On the other hand, it was soon noticed that the complex manifold $\widetilde{X}$ can also be constructed as a monodromy cover, as we now explain. Following Mostow (see [M2]), we define $Q'\subset Q$ to be the subset of Q on which Σ acts freely. $Q-Q'$ consists of two points (see Lemma 2.7), and in particular $\pi_1(Q')\simeq\pi_1(Q)$.

The monodromy representation $\rho_\mu : \pi_1(Q) \to \mathrm{Aut}(\mathbb{B}^2)$ from [DM] extends to a representation $\rho_{\mu,\Sigma} : \pi_1(Q'/\Sigma) \to \mathrm{Aut}(\mathbb{B}^2)$ as in [M2]. Hypergeometric functions yield a multivalued map $Q'/\Sigma \to \mathbb{B}^2$. We write K_μ (resp. $K_{\mu,\Sigma}$) for the kernel of ρ_μ (resp. $\rho_{\mu,\Sigma}$) and Γ_μ (resp. $\Gamma_{\mu,\Sigma}$) for its image.

$$\begin{tikzcd}
K_\mu \arrow[r, hook] \arrow[d, phantom, "\cap"] & \pi_1(Q) \arrow[r] \arrow[d, phantom, "\cap"] & \Gamma_\mu \arrow[d, phantom, "\cap"] \\
K_{\mu,\Sigma} \arrow[r, hook] & \pi_1(Q'/\Sigma) \arrow[r] & \Gamma_{\mu,\Sigma}
\end{tikzcd}$$

Since $\pi_1(Q'/\Sigma)/\pi_1(Q) = \Sigma$, we know that $|\Gamma_{\mu,\Sigma}/\Gamma_\mu|$ divides $|\Sigma| = 6$, but the index depends on μ.

The hypergeometric map becomes single-valued on the cover $\widetilde{Q'/\Sigma}$ of Q'/Σ with deck group $\Gamma_{\mu,\Sigma}$, i.e. the cover whose fundamental group is the kernel $K_{\mu,\Sigma}$ of $\rho_{\mu,\Sigma}$. It is readily checked that the single-valued lift $\widetilde{w}$ is a local biholomorphism on $\widetilde{Q'/\Sigma}$ (see [DM] and [M2]). $\widetilde{X}$ is the Fox completion of that cover over Q_{st}/Σ.

$$\begin{tikzcd}
\widetilde{Q'/\Sigma} \arrow[r, hook] \arrow[d] & \widetilde{X} \arrow[r, "\widetilde{w}"] \arrow[d] & \mathbb{B}^2 \\
Q'/\Sigma \arrow[r, hook] & Q_{st}/\Sigma &
\end{tikzcd} \tag{7}$$

For a definition and some properties of the Fox completion see section 8 of [DM]. The completion is a priori only a topological space, and it is not at all obvious that $\widetilde{X}$ is a manifold, or even that it is locally compact.

Remark 2.5

1. It is crucial here that we take the *smallest* cover on which the hypergeometric map becomes single valued. For instance, the Fox completion of the universal cover of Q'/Σ would not be locally compact.
2. The same construction makes sense to obtain a monodromy cover of Q_{st} rather than Q_{st}/Σ. The corresponding Fox completion is denoted by $\widetilde{Q}_{st}$ in [DM], but that completion is not a manifold unless condition INT holds on D_{AA}, i.e. with the notations above unless $p = 4$.

The action of $\Gamma_{\mu,\Sigma}$ extends to the completion $\widetilde{X}$, and it is quite clear that ${}_{\Gamma_{\mu,\Sigma}}\backslash\widetilde{X}$ is homeomorphic to Q_{st}/Σ. As we shall see from the discussion below, when $\widetilde{X}$ is smooth, the quotient map gives Q_{st}/Σ the structure of an orbifold, where D_{AA} has weight p, D_{AB} has weight r, and D_{AC} has weight s. Note that these weights are the denominators of the fractions in (4) and (6).

One of the key points in proving Theorem 2.4 is to show that the completion $\widetilde{X}$ is a manifold, in which case the map $\widetilde{w}$ extends to $\widetilde{X}$. One then analyzes the local structure of this extension. We summarize the results in the following:

Proposition 2.6 *The Fox completion $\widetilde{X}$ is a manifold, and it has a unique complex structure such that $\widetilde{w}:\widetilde{X}\to\mathbb{B}^2$ is holomorphic. $\widetilde{w}$ is a local biholomorphism away from the divisor in the preimage of D_{AB}. Each component $\widetilde{D}_{AB}$ of that preimage maps biholomorphically to a subball in $\mathbb{B}^2$, and $\widetilde{w}$ branches with order m around $\widetilde{D}_{AB}$, with m as in (6).*

We describe two ways to prove this proposition: one using the results in [M2], the other based on a detailed case by case local analysis. The first one is very short but a lot of its difficulty is hidden in [M2], whereas the second has the advantage of concreteness. In particular, the second proof yields enough local information to compute ratios of Chern classes, as mentioned in Remark 2.13.

First proof: The arguments given in Mostow essentially show that it is enough to check our claim in codimension one, where the result is obvious. More precisely, let $\widetilde{U}$ be the preimage of $U=Q_{st}/\Sigma-D_{AB}$ in $\widetilde{X}$.

Claim 1: $\widetilde{w}|_{\widetilde{U}}$ is a local homeomorphism.

This follows from the methods in [M2]. Q_{st}/Σ is stratified by the image of the natural strata in Q_{st}, deduced from the intersection pattern of the compactification divisors. This induces a stratification on U, and condition ΣINT holds along each stratum. The proof of Theorem (3.12) in [M2] applies verbatim to give claim 1. Now consider a connected component $\widetilde{D}_{AB}$ of the preimage of D_{AB} in $\widetilde{X}$.

Claim 2: In a neighborhood of $\widetilde{D}_{AB}$, the map $\widetilde{w}$ is a simple branched cover of order m.

This is clearly true in codimension one, by the monodromy calculations given in section 9 of [DM]. Indeed, at a generic point of D_{14}, the multivalued hypergeometric map looks like $(u_1,u_2)\mapsto(u_1^{m/r},u_2)$. The local monodromy cover near such a point is of order r, and looks like $(z_1,z_2)\mapsto(z_1^r,z_2)$, with the single valued lift of the hypergeometric map given by $(z_1,z_2)\mapsto(z_1^m,z_2)$.

The codimension two strata on D_{AB} are $q_1=D_{AA}\cap D_{AB}$ and $q_2=D_{AB}\cap D_{AC}$. The local structure of the monodromy cover above these two points is easily understood. We only discuss the situation near q_1. Let V_{q_1}

be a small neighborhood of q_1, and let $U_{q_1} = V_{q_1} \cap Q$. The local fundamental group at q_1 is simply $\pi_1(U_{q_1}) \simeq \mathbb{Z} \times \mathbb{Z}$ (the divisors have normal crossings at q_1). The corresponding monodromy transformations are two commuting complex reflections, with nontrivial eigenvalue given by $e^{2\pi i m/r}$ and $e^{2\pi i 2/p}$ respectively (see section 9 of [DM]). In any case, the local monodromy cover is simply a product of two cyclic covers of order r and p. The hypergeometric map on the monodromy cover looks like a product of the two maps near generic points of D_{AA} and D_{AB}, one of them being a biholomorphism, the other a branched cover of order m. In other words, it looks like $(z_1, z_2) \mapsto (z_1^m, z_2)$. $\square$

All the difficulty of the previous proof is hidden in the proof of the results in [DM] and [M2]. It is not at all obvious that $\widetilde{U}$ is smooth, since $\widetilde{U}$ is a Fox completion over U, which has singularities (see Lemma 2.7). As mentioned in [M2], the point is that most singularities come from small fixed point sets for the action of Σ on Q_{st}, and only codimension one fixed point sets matter.

We devote the second part of this section to writing a more instructive (and also much longer) argument. One the one hand, it has the advantage of being independent of the technical lemmas in [M2], and on the other it gives much more information about the constructed surfaces.

Recall that the surfaces we obtain are only locally described as branched covers of the ball, but that we do not know have any map to an actual compact ball quotient - this makes the surfaces difficult to understand and, for instance, to compute Chern numbers explicitly. A reasonable substitute for such a map is the map to the singular space Q_{st}/Σ, since we can analyze its local structure by simple topological techniques.

Another advantage of the second proof is clarifies what is needed in order to generalize the construction to higher dimension.

Second proof: In order to study the local structure of the Fox completion $\widetilde{X}$, we analyze in detail the local monodromy near the various fixed points of the action of Σ on Q_{st}, a list of which can be obtained just as in Lemma 8.3.2 from [DM2].

Lemma 2.7 *The fixed points for the action of* Σ *on* Q_{st} *are in the* Σ*-orbit of one of the following*

1. *Points on* $D_{12} \setminus D_{23}$*, fixed by* $(1,2)$.
2. $D_{12} \cap D_{23}$*, fixed by all of* Σ.
3. *The point on* D_{34} *given by* $(1,-1,0,0,\infty)$*, fixed by* $(1,2)$.
4. *The point on* D_{35} *given by* $(1,-1,0,\infty,0)$*, fixed by* $(1,2)$.
5. *The point in* Q *given by* $(1,\omega,\omega^2,0,\infty)$*,* $\omega = e^{2\pi i/3}$*, fixed by* $(1,2,3)$.

The singularities of the quotient Q_{st}/Σ are described in [DM2]. The isolated fixed points on D_{34} and D_{35} yield A_2 singularities on Q_{st}/Σ, and the isolated fixed point in Q yields an A_3 singularity. The image of $D_{12} \cap D_{23}$ is a smooth point of the quotient, but D_{AA} has a cusp there. Note that the points in M_{st} that project to singular points of Q_{st}/Σ are not fixed by the Σ-action, only their images in Q_{st} are. For instance $(1, \omega, \omega^2, 0, \infty)$ and $(\omega, \omega^2, 1, 0, \infty)$ are distinct in M_{st}, but they yield the same point in Q_{st}.

We now analyze the local monodromy at various points $q \in Q_{st}/\Sigma - Q'/\Sigma$. For every such point let $V_{q,\Sigma}$ be a small neighborhood of q and let $U_{q,\Sigma} = V_{q,\Sigma} \cap (Q'/\Sigma)$. Consider the local monodromy representation at q, i.e. the restriction of $\rho_{\mu,\Sigma}$ to $\pi_1(U_{q,\Sigma})$ (of course this is only defined up to conjugacy, because of the dependence on base points). We denote by $K_{q,\Sigma}$ the kernel of this restriction, and by $G_{q,\Sigma}$ its image. Let $\widetilde{U}_{q,\Sigma}$ be the cover of $U_{q,\Sigma}$ with $\pi_1(\widetilde{U}_{q,\Sigma}) = K_{q,\Sigma}$. Its deck group is the group $G_{q,\Sigma}$. Finally we write $\widetilde{V}_{q,\Sigma}$ for the Fox completion of $\widetilde{U}_{q,\Sigma}$ over $V_{q,\Sigma}$.

Definition 2.8 *We call $G_{q,\Sigma}$ the local monodromy group at q. We refer to both $\widetilde{U}_{q,\Sigma}$ and its completion $\widetilde{V}_{q,\Sigma}$ as local monodromy covers at q.*

We emphasize the importance of these notions in the following result, which follows from the definition of the Fox completion.

Proposition 2.9 *Viewing $\widetilde{V}_{q,\Sigma}$ as a subset of the completion $\widetilde{X}$, it gives a neighborhood of any point $\widetilde{q} \in \widetilde{X}$ that lies over q. The local monodromy group $G_{q,\Sigma}$ is the isotropy group at $\widetilde{q}$ of the action of $\Gamma_{\mu,\Sigma}$ on $\widetilde{X}$.*

To give some intuition about the notions just introduced, we list the various local monodromy groups (justification will be provided along the proof of Proposition 2.6). The order of the isotropy groups $G_{q,\Sigma}$ are also the isotropy groups for the cover $X \to Q_{st}/\Sigma$ (where $X = \Gamma_0 \backslash \widetilde{X}$, see Theorem 2.4), and Lemma 2.10 can also be used to compute the ratio of Chern classes c_1^2/c_2 for X, cf. Remark 2.13. By Q'_{st} we mean the complement of the set of fixed points for the action of Σ on Q_{st}.

Lemma 2.10 *For $q \in Q_{st}/\Sigma - Q'_{st}/\Sigma$ given by the image of the various points in Lemma 2.7, the local monodromy group $G_{q,\Sigma}$ is given by*

1. *(a) $\textcircled{p}$ for q generic on D_{AA}*
   (b) $\textcircled{p} \times \textcircled{r}$ for $q = D_{AA} \cap D_{AB}$
   (c) $\textcircled{p} \times \textcircled{s}$ for $q = D_{AA} \cap D_{AC}$
2. $\textcircled{p}\!-\!\!-\!\textcircled{p}$

3. $\textcircled{1}^{*3}$
4. $\textcircled{r}^{*2}$
5. $\textcircled{s}^{*2}$

where $\textcircled{k}$ *denotes the cyclic group generated by one complex reflection of order* k*, and* $\textcircled{k}^{*l}$ *denotes a cyclic group whose generator is not a complex reflection, but whose* l^{th} *power is a reflection of order* k*. The group* $\textcircled{k}^{*l}$ *contains* $\textcircled{k}$ *with index* l*.*

Remark 2.11

1. The Coxeter diagrams from case 2 are finite reflection groups #4, #8 and #16 from [ST], when $p = 3$, 4 and 5 respectively. They are generated by two braiding reflections R_1 and R_2 of order p, or in other words they have the presentation

   $$\langle R_1, R_2 | R_1^p = R_2^p = 1, R_1 R_2 R_1 = R_2 R_1 R_2 \rangle$$

   These groups have order 24, 96 and 600.

2. Lemma 2.10 is consistent with Chevalley's theorem, since the local monodromy groups at singular points of Q_{st}/Σ are not generated by complex reflections.

For some points $q \in Q_{st}/\Sigma$ it is useful to compare the local monodromy at q to the local monodromy at a point $\hat{q} \in Q_{st}$ that projects to q. To that end we use notations similar to the ones introduced with Definition 2.8, but we drop the subscript Σ. In other words we take U_q to be a component of the preimage in Q' of $U_{q,\Sigma} \subset Q'/\Sigma$, and V_q is the corresponding preimage of $V_{q,\Sigma}$. $K_q = \text{Ker } \rho_\mu|_{\pi_1(U_q)}$ is the kernel of the local monodromy, G_q is its image. $\widetilde{U}_q$ is the corresponding monodromy cover of U_q. The completion $\widetilde{V}_q$ of $\widetilde{U}_q$ over V_q gives the local structure of the completion $\widetilde{Q}_{st}$ of $\widetilde{Q'}$ over Q_{st}. Of course these groups fit into a diagram

$$\begin{tikzcd}
K_q \arrow[r, hook] \arrow[d, phantom, "\cap"] & \pi_1(U_q) \arrow[r] \arrow[d, phantom, "\cap"] & G_q \arrow[d, phantom, "\cap"] \\
K_{q,\Sigma} \arrow[r, hook] & \pi_1(U_{q,\Sigma}) \arrow[r] & G_{q,\Sigma}
\end{tikzcd}$$

and there is a short exact sequence

$$K_{q,\Sigma}/K_q \hookrightarrow I_q \to G_{q,\Sigma}/G_q \tag{8}$$

where we write I_q for the isotropy group of Σ at $\hat{q}$, or in other words $I_q = \pi_1(U_{q,\Sigma})/\pi_1(U_q)$.

The situation is quite simple if q is not in the image of the fixed points of Σ, i.e. if $q \in Q'_{st}/\Sigma$. In that case $V_{q,\Sigma} = V_q$, $\widetilde{V}_{q,\Sigma} = \widetilde{V}_q$ and all the groups in (8) are trivial. At such a point the arguments from [DM] or [D] to analyze the local structure of $\widetilde{Q}_{st}$ suffice to get the result.

For instance if q is a generic point on D_{AB}, $\pi_1(U_{q,\Sigma}) = \pi_1(U_q) = \mathbb{Z}$, and the local monodromy is generated by a complex reflection with non trivial eigenvalue $e^{2\pi i m/r}$ since $1 - \mu_A - \mu_B = m/r$. This implies that the monodromy cover $\widetilde{V}_{q,\Sigma} \to V_{q,\Sigma}$ is the standard monodromy cover for the multivalued map $(u_1^{m/r}, u_2)$, i.e. it is given by $(z_1, z_2) \mapsto (z_1^r, z_2)$. The hypergeometric map is the single-valued lift to that cover, namely it looks like $(z_1, z_2) \mapsto (z_1^m, z_2)$.

The situation is similar for q generic on D_{AC}, but then $1 - \mu_A - \mu_C = 1/s$ so the integrality condition holds along D_{AC}, and the hypergeometric map is a local biholomorphism on $\widetilde{V}_{q,\Sigma}$.

If $q = D_{AB} \cap D_{AC}$, the situation is a product of the two previous cases, since the divisor $D_{AB} \cup D_{AC}$ has normal crossings at that point. $\pi_1(U_q)$ is simply a product $\mathbb{Z} \times \mathbb{Z}$. The local monodromy group G_q is generated by two commuting complex reflections, with nontrivial eigenvalues $e^{2\pi i m/r}$ and $e^{2\pi i/s}$, respectively. The cover $\widetilde{V}_q$ is a product, given by $(z_1, z_2) \mapsto (z_1^r, z_2^s)$, and the hypergeometric map on $\widetilde{V}_q$ given by $(z_1, z_2) \mapsto (z_1^m, z_2)$.

When the relevant divisors do not have normal crossings, an important tool is the following general result about monodromy groups of hypergeometric functions, which is crucial in higher-dimensional situations. Let μ be an N-tuple of rational numbers such that $0 < \mu_j < 1$, $\sum \mu_j = 2$, giving the exponents of some hypergeometric integrals. Suppose $\mu_1 + \mu_2 + \mu_3 < 1$, and $\hat{q} \in Q_{st}$ is a generic point in $D_{12} \cap D_{13}$, which has codimension two in Q_{st}. We choose a basis of $N - 2$ hypergeometric integrals given by integrating along the paths in Figure 2 (this is the same choice as in Lemma 2.5 of [M5]).

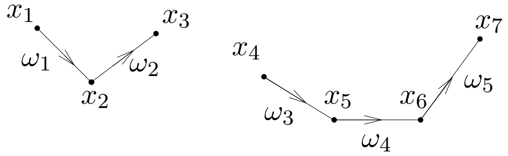


Figure 2: Basis of hypergeometric integrals

We write W for the span of ω_1 and ω_2. The local monodromy group $G_{\hat{q}}$ clearly preserves W, and is the identity on the span of $\omega_3, \ldots, \omega_{n+1}$.

Lemma 2.12 *(three points collision) The local monodromy group $G_{\hat{q}}$ acts on $\mathbb{P}(W) \simeq \mathbb{P}^1$ as a group $T_{\hat{q}}$ generated by the rotations by angles $2\pi(1-\mu_i-\mu_j)$ about the vertices of a spherical triangle with angles $\pi(1-\mu_i-\mu_j)$, where $i \neq j \in \{1,2,3\}$. $G_{\hat{q}}$ is a central extension of $T_{\hat{q}}$, whose center is generated by the matrix that multiplies W by* $\exp 2\pi i(1-\mu_1-\mu_2-\mu_3)$.

In fact one can write explicit matrices for the natural generators for $G_{\hat{q}}$ (corresponding to x_i going once around x_j), and check that they are complex reflections with nontrivial eigenvalues given by $\exp 2\pi i(1-\mu_i-\mu_j)$. We omit the details (see Lemma 2.5 in [M5] or Lemma (10.3) in [DM]).

We now go through the various cases where q is the image of a fixed point of the action of Σ on Q_{st} (see Lemma 2.7).

1. (a) Let q be a generic point in D_{AA}, i.e. the image of $(0,0,\alpha,1,\infty)$, with $\alpha \neq 0,1,\infty$. We then have $\pi_1(U_{q,\Sigma}) = \mathbb{Z}$ and $\pi_1(U_q) = 2\mathbb{Z}$. The natural generator τ of $\pi_1(U_{q,\Sigma})$ maps to a complex reflection T with exponent $e^{2\pi i/p}$.

   If p is odd, T and T^2 both have order p, and $G_q = G_{q,\Sigma}$. The monodromy kernels satisfy $K_q = K_{q,\Sigma} \cap \pi_1(U_q)$, and $K_{q,\Sigma} = 2p\mathbb{Z} \subset p\mathbb{Z} = K_q$. The map $\widetilde{V}_q \to \mathbb{B}^2$ has branching of order 2 since $1-\mu_A-\mu_A = 2/p$, but it factors through $\widetilde{V}_{q,\Sigma}$, and the map $\widetilde{V}_q \to \widetilde{V}_{q,\Sigma}$ branches with order 2 as well, so that $\widetilde{V}_q \to \mathbb{B}^2$ is a local biholomorphism.

   If $p = 4$, T has order 4 and T^2 has order 2. $G_q \subsetneq G_{q,\Sigma}$ has index two, but now the kernels K_q and $K_{q,\Sigma}$ are the same, so the covers $\widetilde{V}_q$ and $\widetilde{V}_{q,\Sigma}$ are the same. Since $p = 4$, $1-\mu_A-\mu_A = 1/2$, and condition INT holds on D_{AA}, which implies that the hypergeometric map is a local biholomorphism on $\widetilde{V}_q$.

   (b) Let q be the image of $(0,0,1,1,\infty)$. Then $q = D_{AA} \cup D_{AB}$, and this divisor has normal crossings at q. The local fundamental group is a product $\pi_1(U_{q,\Sigma}) = \mathbb{Z} \times \mathbb{Z}$, and the local monodromy group $G_{q,\Sigma}$ is generated by two commuting complex reflections, with non trivial eigenvalues $e^{2\pi i/p}$ and $e^{2\pi i m/r}$. The monodromy cover $\widetilde{V}_{q,\Sigma}$ is a product, and the hypergeometric map is understood in each factor from the situation near generic points of D_{AA} and D_{AB} respectively. It has branching of order m around the divisor above D_{AB} and is biholomorphic near the divisor above D_{AA}.

   (c) If $q = D_{AA} \cap D_{AC}$, i.e the image of $(0,0,\infty,1,\infty)$, the analysis is the same as in case (b), with B replaced by C. The map is biholomorphic on $\widetilde{V}_{q,\Sigma}$ because $1 - \mu_A - \mu_C = 1/2$.

2. Let q be the cusp point of D_{AA}. It is quite clear that $\pi_1(U_q)$ is the pure braid group P_3 on three strands, and $\pi_1(U_{q,\Sigma})$ is the corresponding full braid group B_3. The structure of the group G_q is understood from Lemma 2.12, where we take $\mu_1 = \mu_2 = \mu_3 = \frac{1}{2} - \frac{1}{p}$. We have $1 - \mu_A - \mu_A = \frac{2}{p}$ and (μ_1, μ_2, μ_3) satisfies ΣINT, but it satisfies INT only if p is even.

   By Lemma 2.12, the natural generators of P_3 (full turn of x_i around x_j) map to complex reflections with nontrivial eigenvalue $e^{4\pi i/p}$, that induce on $\mathbb{P}(W)$ rotations by an angle $4\pi/p$ about the vertices of an equilateral triangle whose angles are $2\pi/p$. Note that such a triangle is a union of six congruent triangles, with angles $\pi/2$, $\pi/3$, π/p (see Figure 3). G_q is a central extension of T_q, where T_q is contained in the triangle group $T(2,3,p)$. It is readily seen that $T_q = T(2,3,p)$ if p is odd, and $T_q \subsetneq T(2,3,p)$ has index 6 if p is even. When $p = 4$, T_q is simply the triangle group $T(2,2,2)$, which has order 4. When $p = 3$ or 5 the order of T_q is

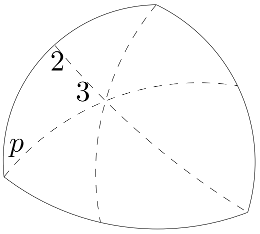
   

   Figure 3: Part of a tessellation of the sphere

   $$2(\frac{1}{2} + \frac{1}{3} + \frac{1}{p} - 1)^{-1} = \frac{12p}{6-p}$$

   The center of G_q has order given by the denominator of $3(\frac{1}{2} - \frac{1}{p})$, which is 2, 4 or 10 if $p = 3$, 4 or 5 respectively. In particular we get that the order of G_q is 24, 16 or 600 if $p = 3$, 4 or 5 respectively. This allows us to identify these finite reflection groups in the classification given in [ST]. They are groups #4, G(4,2,2) and #16 respectively.

   We now describe $G_{q,\Sigma}$. The two natural generators of the braid group B_3 (half twist between x_i and x_j) also map to complex reflections, with the same mirrors as the generators for G_q, but now their non-trivial eigenvalue is $-e^{2\pi i(1/2-1/p)} = e^{2\pi i/p}$ (see [M2] page 98). These reflections induce rotations of order p in $\mathbb{P}(W)$ centered at the same vertices as the generators of G_q (see Figure 3).

This implies that $G_{q,\Sigma}$ always induces $T(2,3,p)$ on $\mathbb{P}(W)$, regardless of the parity of p. The centers of G_q and $G_{q,\Sigma}$ are the same since the center of B_3 is contained in P_3. In particular we get that $G_{q,\Sigma} = G_q$ if p is odd. When $p = 4$, $G_{q,\Sigma}$ has order $24 \cdot 4 = 96$. In turn one identifies this group in the Shephard-Todd classification as being #8.

For the description of the groups #4, #8 and #16 in terms of Coxeter diagrams we refer the reader to [M1].

Note that since $G_{q,\Sigma}$ is a finite group generated by complex reflections, the quotient $\mathbb{C}^2/G_{q,\Sigma}$ is smooth, and in fact the branching structure is understood. The branch locus is known to be cuspidal cubic $z_1^2 = z_2^3$ in appropriate coordinates on the quotient (see [B]).

Now the local monodromy cover $\widetilde{V}_{q,\Sigma} \to V_{q,\Sigma}$ must be the same as the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/G_{q,\Sigma}$, since the deck group of the cover $\widetilde{U}_{q,\Sigma} \to U_{q,\Sigma}$ is $G_{q,\Sigma}$, and the branch loci coincide. In particular the hypergeometric map is a local biholomorphism on $\widetilde{V}_{q,\Sigma}$.

3. Let q be the image of $(1, \omega, \omega^2, 0, \infty)$. This point comes from an isolated fixed point $\hat{q}$ in Q, hence $\pi_1(U_q) = \{1\}$, and $\pi_1(U_{q,\Sigma}) = \mathbb{Z}_3$ ($\hat{q}$ is fixed by a 3-cycle). A loop that generates $\pi_1(U_{q,\Sigma})$ is described in Figure 4. One can check directly that the corresponding monodromy transformation T is non trivial, so that $K_{q,\Sigma} = \{1\}$, and $\widetilde{U}_{q,\Sigma} \simeq U_q$, $\widetilde{V}_{q,\Sigma} = V_q$. In fact the monodromy on $U_{q,\Sigma}$ could not be trivial, since we know from [DM] that the hypergeometric map is a local biholomorphism on V_q, which is contained in Q. Note that T cannot be a complex reflection since Q_{st}/Σ is singular at q. Of course T has order 3.

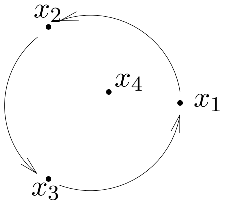


Figure 4: A generator for $\pi_1(U_{p,\Sigma})$

4. If q is the singular point on D_{AB} (which is the Σ-orbit of the isolated fixed point of a transposition in Σ), we have $\pi_1(U_{q,\Sigma}) = \mathbb{Z}$ and under this isomorphism $\pi_1(U_q) = 2\mathbb{Z}$. A generator for $\pi_1(U_{q,\Sigma})$ is given by the loop τ in Figure 5 (left), where we chose the base point to be the image of $(1, -1, 0, \epsilon, \infty)$. The corresponding monodromy transformation T can be computed explicitly as in section 12 of [DM]. One checks that in the basis $\{\omega_1, \omega_2, \omega_3\}$ described in Figure 5 we have

$$T = \begin{bmatrix} 0 & 1 & 0 \\ \overline{\alpha}_3\overline{\alpha}_4 & 0 & 0 \\ 1 - \overline{\alpha}_4 & 0 & 1 \end{bmatrix} \tag{9}$$

T is not a complex reflection, but its square is. In fact T^2 is a complex reflection with non trivial eigenvalue $\alpha_3\alpha_4 = e^{-2\pi i m/r}$ as expected, since τ^2 generates $\pi_1(U_q)$ and gives a small loop around D_{34} in Q_{st}. The conclusion of this computation is that $K_q = K_{q,\Sigma}$, so that $\widetilde{U}_{q,\Sigma} = \widetilde{U}_q$ is a cover of U_q, whose structure is understood from the analysis in [DM]. Since $1 - \mu_A - \mu_B = m/r$, it is isomorphic to the obvious monodromy cover for the multivalued map $(z_1, z_2) \mapsto (z_1^{m/r}, z_2)$, which is just a cyclic cover with order r. The hypergeometric map on $\widetilde{V}_{q,\Sigma} = \widetilde{V}_q$ is given by $(z_1, z_2) \mapsto (z_1^m, z_2)$.

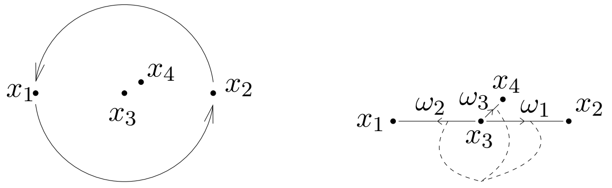


Figure 5: The loop τ (left) and the basis $\{\omega_1, \omega_2, \omega_3\}$ (right). The dotted paths indicate how to choose a branch for the multivalued integrand.

5. The description of the local behavior near the singular point of D_{AC} is of course entirely the same as the previous case, with B replaced by C in the proof. Since condition INT is satisfied for D_{AC}, the hypergeometric map is a local biholomorphism on the corresponding $\widetilde{V}_{q,\Sigma} = \widetilde{V}_q$.

□

Remark 2.13 The information from Lemma 2.10 can be used to compute the ratio of Chern classes c_1^2/c_2 for the surfaces X from Theorem 2.4. Using the description of the branching behavior of the projection to Q_{st}/Σ and the branching of the map $\widetilde{X} \to \mathbb{B}^2$, one obtains the following formula:

$$\frac{c_1^2(X)}{c_2(X)} = 3 + \frac{\frac{1}{2}\frac{(m-1)^2}{m}\frac{1}{r}\left(\frac{1}{p}+\frac{1}{s}-\frac{1}{2}\right)}{\frac{1}{3}-\frac{1}{p}+\frac{(p-6)^2}{24p^2}-\left(\frac{1}{2}-\frac{1}{p}\right)\left(\frac{1}{r}+\frac{1}{s}\right)+\frac{1}{rs}} \tag{10}$$

This shows in particular that X is a ball quotient if and only if $m = 1$. Note that this differs slightly from the formula given in [MS] (one would get the formula given there by replacing the factor $(m-1)^2/m$ by $m-1$).

3 The 3-dimensional examples

We now explain how to generalize the constructions from section 2 to obtain three-dimensional analogues of the Mostow-Siu surfaces. The examples will be described in terms of hypergeometric functions, by listing their exponents in the form of 6-tuples of rational numbers.

We first justify our choice of exponents. One of the crucial properties of the 2-dimensional groups analyzed in section 2 is that all the local monodromy groups are finite. This is to be the case in any generalization to higher dimensions since, at the very least, we want the Fox completion to be locally compact. It is easy to write down explicit numerical conditions on the exponents of our hypergeometric functions for finiteness of the local monodromy groups, based on the following ideas:

1. Lemma 2.12, together with Schwarz's classification of finite hypergeometric monodromy groups acting on $\mathbb{P}^1$ (see also [M5]), tells us when three point collisions yield finite monodromy.
2. The stabilizers of mirrors of the generating reflections (corresponding to x_i turning once around x_j) have a hypergeometric description, where in the set of exponents we replace μ_i and μ_j by one single exponent $\mu_i + \mu_j$. We call any set of exponents obtained by applying this process repeatedly a contraction of μ.

Given an N-tuples μ so that $0 < \mu_j < 1$, $\sum \mu_j = 2$ (Mostow calls such sets of exponents *ball N-tuples*), we formulate the finite local monodromy condition as follows:

Whenever $\mu_i + \mu_j + \mu_k < 1$, the triple $\{\mu_i, \mu_j, \mu_k\}$ appears in Schwarz's list (see [M5], p.557). Moreover, we require that the same property hold for any contraction of μ, i.e. any set of exponents obtained by replacing some subsets of the μ_j by their sum (as long as that sum is less than 1).

It is easy to see that this condition is actually equivalent to finite local monodromy. In particular, it contains a condition for multiple collisions where more than three points coalesce (these appear in three point collisions corresponding to some contraction of μ). A quick computer search gives the following:

Proposition 3.1 *The only ball N-tuples, $N \geq 6$ such that all the local monodromy groups are finite are the ΣINT examples and the 6-tuples in Table 1, where we write $\mu_j = n_j/d$.*

	d	n_1	n_2	n_3	n_4	n_5	n_6
1.	30	9	9	9	11	11	11
2.	30	9	9	9	9	11	13
3.	30	5	5	5	5	19	21
4.	20	3	7	7	7	7	9
5.	20	1	3	9	9	9	9
6.	12	1	3	5	5	5	5
7.	10	3	3	3	3	4	4
8.	10	2	2	3	3	3	7
9.	10	1	3	3	3	3	7

Table 1: The list of potential examples, using the finite local monodromy condition. Only the first three yield smooth compact examples with simple branching.

In particular our method does not produce any example analogous to the Mostow-Siu surfaces in dimension larger than three. Note also that the list is finite for $N \geq 6$, in contrast to the cases $N \leq 5$ where there are infinitely many examples (see also Remark 4.2).

It turns out the finiteness of the local monodromy is only necessary, but not sufficient to obtain smooth examples. Further analysis, based on Lemma 2.12, shows that only cases 1, 2 and 3 yield compact smooth examples with simple branching to the ball. We first present the main features that make it plausible that cases 1, 2 and 3 do indeed produce examples with simple branching behavior.

1. Let $\mu = (9,9,9,11,11,11)/30$ and let $\Sigma = S_3$ permute the first three weights. In this case the various $1 - \mu_i - \mu_j$ are $2/5$, $1/3$ and $4/15$ and condition ΣINT fails only on the three divisors D_{45}, D_{56} and D_{46}. Note that these are disjoint in Q_{st}, since $11/3 + 11/30 + 11/30 > 1$. Their images in Q_{st}/Σ remain disjoint. In this case Q_{st} is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ blown up at three points on the diagonal (see [KLW]).

2. Let $\mu = (9,9,9,9,11,13)/30$ and $\Sigma = S_4$, and as in Section 2 we write A for either of 1, 2, 3 or 4, and B, C for 5, 6 respectively. Condition ΣINT fails only on the compactification divisor D_{AC} of Q_{st}/Σ, since $1 - \mu_A - \mu_C = 4/15$. On the level of Q_{st}, the four divisors D_{i6}, $i = 1, \ldots, 4$ that project to D_{AC} are disjoint since $(9+9+13)/30 > 1$. For this choice of exponents, Q_{st} is obtained from $\mathbb{P}^3$ by blowing up five points in general position. The combinatorics of the compactification divisors are suggested in Figure 6, except that we would have to blow up the barycenter of the tetrahedron as well. The divisors D_{i6}, with $i = 1, \ldots, 4$ correspond to the exceptional obtained by truncating the four vertices, and D_{56} is the exceptional divisor over the barycenter of the tetrahedron.

3. Let $\mu = (5,5,5,5,19,21)/30$ and $\Sigma = S_4$. Condition ΣINT fails on D_{AC} only, where we get $1 - \mu_A - \mu_C = 2/15$, and once again the corresponding divisors D_{i6} in Q_{st} are disjoint. The combinatorics of the compaction divisors in Q_{st} is depicted in Figure 6. The divisors D_{i6} are the four exceptionals obtained by truncating the four vertices of the tetrahedron.

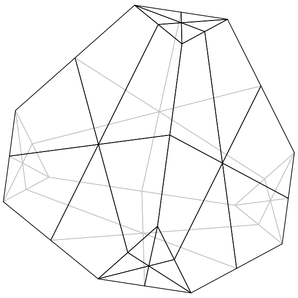


Figure 6: Q_{st} is a blow up of $\mathbb{P}^3$

Definition 3.2 *We call the divisors where condition* ΣINT *fails* **branching** *divisors. For each branching divisor* $D_{\alpha\beta}$*, the branching order is the numerator of the reduced fraction* $1 - \mu_\alpha - \mu_\beta$.

Indeed the branch locus of the hypergeometric map $\widetilde{w} : \widetilde{X} \to \mathbb{B}^3$ in the monodromy cover $\widetilde{X}$ is the preimage of the branching divisors in Q_{st}/Σ.

The main difficulty is to analyze the various (irreducible) components of that preimage, and to show that they are in fact disjoint. We check this below by working locally near every point in Q_{st}/Σ, and obtain our main result:

Theorem 3.3 *Let* μ *be given by* $(9,9,9,11,11,11)/30$*,* $(9,9,9,9,11,13)/30$ *or* $(5,5,5,5,19,21)/30$*.*

(i) The group $\Gamma_{\mu,\Sigma}$ *is not discrete in* $\mathrm{Aut}(\mathbb{B}^3)$*, but it acts discretely on a complex manifold* $\widetilde{X}$ *with an equivariant holomorphic map* $\widetilde{w}:\widetilde{X}\to\mathbb{B}^3$*, which is locally a branched cover. The singular locus of* $\widetilde{w}$ *in* $\widetilde{X}$ *is a smooth divisor in* $\widetilde{X}$*, that maps to a non discrete collection of totally geodesic subballs in* $\mathbb{B}^3$*, all with the same branching order.*

(ii) If $\Gamma_0\subset\Gamma_{\mu,\Sigma}$ *is a torsion free subgroup of finite index, the quotient* $X={}_{\Gamma_0}\backslash\widetilde{X}$ *is a compact manifold that admits a Kähler metric of strictly negative curvature, but it is not a ball quotient.*

The manifold $\widetilde{X}$ from Theorem 3.3 is of course the Fox completion of the monodromy cover $\widetilde{Q'/\Sigma}\to Q'/\Sigma$ over Q_{st}/Σ, and the map $\widetilde{w}$ is given by hypergeometric functions. Before starting the proof, we explain briefly why the other choices from Table 1 are not good enough.

The monodromy covers for cases 4, 5 and 6 are not smooth. Let us check this for $\mu=(1,3,5,5,5,5)/12$, for instance. Note that Q_{st} only has one branching divisor, namely D_{12}. Note that the weights $1/12$ and $3/12$ are *distinct* but $1-1/12-3/12=2/3$. The local monodromy group near a generic point of $D_{12}\cap D_{23}\simeq\mathbb{P}^1$ is easily deduced from Lemma 2.12, namely it is (an embedding into $U(3)$ of) group #6 in the Shephard-Todd classification of finite complex reflection groups (see [ST]). The local monodromy cover is a product $\widetilde{V}\times\mathbb{C}\to V\times\mathbb{C}$, which is the identity on the $\mathbb{C}$ factor. The cover $\widetilde{V}\to V$ branches over three lines through the origin in $\mathbb{C}^2$, with orders 2, 3 and 3 (note that $1-\mu_i-\mu_j$ is $1/2$, $1/3$ or $2/3$ for $i\neq j$ in $\{1,2,3\}$). The universal cover $\hat{V}$ of $\mathbb{C}^2$, branched with these orders along three lines is described in Lemma 10.3 of [DM], and its deck group is group #7 from [ST]. In particular $\widetilde{V}\neq\hat{V}$, which shows that $\widetilde{V}$ cannot be smooth. Indeed, otherwise we would get a nontrivial covering $\mathbb{C}^2-\{0\}\simeq\hat{V}-\{pt\}\to\widetilde{V}-\{pt\}\simeq\mathbb{C}^2-\{0\}$, but this is impossible since $\mathbb{C}^2-\{0\}$ is simply connected.

We disregard the examples obtained from cases 6, 7 and 8 in Table 1, since they cannot be compact (some of the exponents add up to one). In fact their branching is not simple, so that we do not know how to construct a Kähler metric of strictly negative curvature on the corresponding examples.

For more on why only cases 1, 2 and 3 work, see Remark 3.6. In what follows we assume μ is one of the 6-tuple from Theorem 3.3. As in the 2-dimensional case we mainly insist on proving the following:

Proposition 3.4 *The Fox completion $\widetilde{X}$ is a complex manifold, and the hypergeometric map is a local biholomorphism away from the preimage of the branching divisor D. The components $\widetilde{D}$ of that preimage are disjoint and smooth, and $\widetilde{w}$ has simple branching around each of them.*

Remark 3.5 If $\mu = (9, 9, 9, 11, 11, 11)/30$, D has three disjoint components. In the other two cases D has only one component. The order of branching is given by the numerator of the appropriate $1 - \mu_i - \mu_j$, namely it is 4 in the first two cases and 2 in the third one.

Proof: Let $U = Q_{st}/\Sigma - D$ and let $\widetilde{U}$ be its preimage in the Fox completion $\widetilde{X}$. Just as in the 2-dimensional case, it follows from the proof of Theorem (3.12) in [M2] that the hypergeometric map $\widetilde{w}$ is a local homeomorphism on $\widetilde{U}$. Note that the key is that U has a natural stratification that satisfies the conditions of Proposition (10.16.1) of [DM].

Let D be the branching divisor, and let $\widetilde{D}$ be a connected component of its preimage in $\widetilde{X}$. We look at the local monodromy covers near the various strata of D, organizing them by increasing codimension. In fact, one could replace this case by case argument by induction on the codimension, along the lines of [DM].

The behavior of $\widetilde{w}$ is clearly as claimed in the proposition near a generic point of $\widetilde{D}$. Once again, this simply follows from the monodromy calculations from [DM].

The 1-dimensional strata of D are easily handled as well, since the compactification divisor has normal crossings there. We consider for instance a generic point $q \in D_{AA} \cap D$. The local fundamental group is $\mathbb{Z} \times \mathbb{Z}$, and the monodromy corresponding to each factor is simply the monodromy of small loops around D_{AA} and D, respectively. In other words, the Fox completion near q is a product of the two monodromy covers. This shows that the hypergeometric map has simple branching around $\widetilde{D}$ above q.

There are two types of 0-dimensional strata, depending on whether or not the compactification divisors have normal crossings at the corresponding points. When they do have normal crossings, it is quite clear that the result follows from the analysis near the 1-dimensional strata: the local fundamental group, the monodromy cover and the hypergeometric map are all products and each factor is analyzed as in the 1-dimensional case.

When the divisors do not have normal crossings, the local structure is a bit more complicated, but a quick look at the structure of the compactification divisors in Q_{st} for the three choices of exponents in the theorem shows why we still get the result. In each case, the non normal character of the crossing is caused by a line that is common to three planes (or rather a quotient by a symmetric group of that situation), but the branching divisor cuts these three planes transversally. The local fundamental group is still a product but the factor transverse to the branching is a bit more complicated, namely it is a braid group on three strands, more or less pure depending on which 0-dimensional stratum we consider. The monodromy cover is a product $A \times B$, where A is a cover corresponding to a three points collision where condition ΣINT holds and B is simply a one-dimensional cyclic cover. In particular we get the smoothness of the cover and the simple branching structure. ☐

Remark 3.6 The arguments from the last paragraph would fail with the other choices of exponents from Table 1, since in these cases the points that collide on the branching divisors are involved in three points collisions as well.

It is of course possible to write an argument along the lines of the second proof of Proposition 2.6, but we avoid doing so for the sake of saving space. Indeed, the analogues of Lemma 2.7 and Lemma 2.10 would be much longer to state, and a case by case analysis would be much more tedious. Still, for the sake of concreteness, we list the triple collisions that come up in the cases from the theorem.

1. For $\mu = (9, 9, 9, 11, 11, 11)/30$ there are two types of three points collisions, given up to symmetry by $x_1 = x_2 = x_3$ and $x_2 = x_3 = x_4$.

    (a) Let us consider $q = D_{AA} \cap D_{A4} \cap D_{56}$. At this point $\pi_1(U_{q,\Sigma}) = H \times \mathbb{Z}$ where H is the subgroup of the braid group B_3 generated by a half twist between x_2 and x_3 and full twists between x_i and x_4 (i = 2, 3). The local monodromy group is a product $Q_{q,\Sigma} = G^{(1)} \times G^{(2)}$, where $G^{(2)}$ is generated by a single complex reflection with non trivial eigenvalue $e^{2\pi i(1-\mu_5-\mu_6)}$ and $G^{(1)}$ can be understood by using Lemma 2.12 : it is a central extension of $T_{2,3,5}$, with center of order 30. In particular it has order 1800, and it is group #30 in the Shephard-Todd list. It is readily checked that $G^{(1)}$ can also be described by the Coxeter diagram ③═⑤, and has the presentation

$$\langle a, b | a^3 = b^5 = 1, abab = baba \rangle$$

    As far as the monodromy covers are concerned, we obtain a similar product structure $\widetilde{V}_{q,\Sigma} = \widetilde{V}^{(1)} \times \widetilde{V}^{(2)}$ and the hypergeometric map $\widetilde{w}$ is a local biholomorphism on $\widetilde{V}^{(1)}$ because condition ΣINT holds for the triple of weights corresponding to this three points collision (namely $(9, 9, 11)/30$). The second factor $\widetilde{V}^{(2)}$ gives simple branching of order 4, since $1 - \mu_5 - \mu_6 = 4/15$.

  (b) Let q be the image in Q_{st}/Σ of $(0, 0, 0, 1, \infty, \infty)$, where we have $\pi_1(U_{q,\Sigma}) = B_3 \times \mathbb{Z}$. The B_3 factor maps to a ⑤—⑤ group, as in the corresponding two-dimensional situation near the cups point of D_{AA}. This group is group #16 in [ST], or more precisely an embedding of that group into a $U(2)$ subgroup of $U(3) \subset PU(3, 1)$. The $\mathbb{Z}$ factor maps once again to a complex reflection with eigenvalue $e^{8\pi i/15}$, that commutes with the ⑤—⑤ subgroup. Once again the local monodromy cover is a product, and the hypergeometric map is biholomorphic on the first factor (because it is so in the corresponding 2-dimensional situation) and the second factor gives branching of order 4.

2. For $\mu = (9, 9, 9, 9, 11, 13)/30$, the three points collisions are the same as in the previous case.

3. For $\mu = (5, 5, 5, 5, 19, 21)/30$, there are two types of three points collisions, given (up to symmetry) by $x_1 = x_2 = x_3$ and $x_3 = x_4 = x_5$.

   (a) For a generic point on the image of $D_{12} \cap D_{23}$, the local monodromy group is ③—③. For the image of $D_{12} \cap D_{23} \cap D_{46}$ we get a product ③—③ × ⑮.

   (b) For a generic point on the image of $D_{34} \cap D_{45}$, the local monodromy has Coxeter diagram ③═⑤, and at the image of $D_{34} \cap D_{45} \cap D_{16}$, it is a product ③═⑤ × ⑮.

In both cases (a) and (b), the group ⑮ is generated by a complex reflection with non trivial eigenvalue $e^{4\pi i/15}$, so the hypergeometric map branches with order 2.

Remark 3.7 1. Some of the points that are not on the branch locus are difficult to analyze directly (we took care of them by quoting the techniques in [M2]). It is quite instructive to study the complicated local monodromy covers explicitly. For instance, the interested reader should investigate the local description near the image of $D_{12} \cap D_{34} \cap D_{56}$. The isotropy group of the Σ-action on Q_{st} at that point has order 8, and is generated by $(1,2)$, $(3,4)$ and $(1,3)(2,4)$. The local monodromy group G_q is generated by three commuting reflections of order 5, whose non trivial eigenvalue is $e^{4\pi i/5}$, $e^{4\pi i/5}$ and $e^{2\pi i/5}$ respectively. The group $G_{q,\Sigma}$ is not generated by complex reflections, since Q_{st}/Σ is singular at that point, but one checks that $G_{q,\Sigma}$ contains G_q as a subgroup of index 2. In particular, the exact sequence (8) implies that $K_{q,\Sigma}/K_q \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (see section 2 for the definitions of these groups).

2. The 6-tuple $(5,5,5,5,19,21)/30$ has the interesting feature that it allows the first four points to collide in Q_{st}. Let q be the image of the point $x_1 = x_2 = x_3 = x_4$. Since we know by the techniques in [M2] that the hypergeometric map is a local biholomorphism above that point, $G_{q,\Sigma}$ must be some three-dimensional finite group generated by complex reflections. It is not too hard to check that it is in fact group #25 in the Shephard-Todd list.

The proof of Proposition 3.4 is complete, hence we have part (i) of Theorem 3.3. We briefly discuss part (ii) in the next section.

4 Totally geodesic divisors and construction of the metric

The construction of a Kähler metric is essentially the same as in [MS]. The branch locus of the hypergeometric map $\widetilde{X} \to \mathbb{B}^3$ is a union of disjoint divisors, each of which maps biholomorphically to a totally geodesic subball in $\mathbb{B}^3$. The pull-back of the Bergmann metric on $\mathbb{B}^3$ yields a negatively curved *singular* metric on $\widetilde{X}$, which can be made into a genuine Kähler metric by adding a correction factor near the branch locus, in an equivariant fashion. The only difficulty is to generalize the correction term from [MS] to higher dimensions, which was done in [Z] (in fact we only need the easier part of his results, since our branch locus is not only normal crossings but it is actually smooth).

Writing g_B for the Bergman metric on the ball, the metric on $\widetilde{X}$ is of the form $\tilde{w}^* g_B + \varepsilon\partial\overline{\partial} \log \Phi$, where ε is a small positive number and the correction term is constructed as follows: near each component of the branch locus, in coordinates where the map is described by $(z, u_1, u_2) \mapsto (z^m, u_1, u_2)$, Φ is given by

$$\frac{(m+1)(1-\|u\|^2)^{1/m} - (m-1)|z|^2}{(1-\|u\|^2)^{2-1/m}((1-\|w\|^2)^{2-1/m} - |z|^2)^3} \tag{11}$$

Zheng describes the metric in terms of the distance to the branch locus in the ball, showing that it is actually independent of the chosen coordinates. We refer the reader to [Z] for the computations that show that this metric does have negative curvature. Here we simply mention that, just like in [MS], the patching of the correction term can be made in such a way that the resulting metric is invariant under the action of the monodromy group on $\widetilde{X}$, so that it descends to a metric on the compact quotient ${}_{\Gamma}\backslash\tilde{X}$.

A quick way to check that $X = {}_{\Gamma_0}\backslash\widetilde{X}$ is not a ball quotient is to observe that it contains some Mostow-Siu surfaces as totally geodesic divisors. The codimension one strata in Q_{st} or Q_{st}/Σ correspond to the conjugacy classes of mirrors of complex reflection in the monodromy group (the corresponding reflection is of course the monodromy image of a small loop around the appropriate divisor). It follows from the equivariance of the hypergeometric map $\widetilde{X} \to \mathbb{B}^3$ with respect to the action of the monodormy group, that the corresponding divisors in $\widetilde{X}$ map to the mirrors of these complex reflections. Moreover, the components $\widetilde{D}_{\alpha\beta}$ of the preimage of the divisors $D_{\alpha\beta}$ are fixed by the action on $\widetilde{X}$ of the appropriate reflection $R_{\alpha\beta}$ which gives monodromy around that divisor. Since the monodromy group $\Gamma_{\mu,\Sigma}$ acts on $\widetilde{X}$ by isometries (this is simply a restatement of the fact that the metric is invariant by the group, so that we get a metric on the quotient), the components of the preimages of the compactification divisors are totally geodesic in $\widetilde{X}$.

Since the compactification divisors are obtained from letting two points collide, it is natural to expect them to be described in terms of hypergeometric functions as well, but where we replace two of the original exponents by their sum.

Proposition 4.1

1. *If* $\mu = (9, 9, 9, 11, 11, 11)/30$*, then* $\widetilde{X}$ *contains totally geodesic copies of the monodromy cover for* $(9, 11, 11, 11, 18)/30$*, which is one of the surfaces studied in [D].*

2. *If* $\mu = (9, 9, 9, 9, 11, 13)/30$*, then* $\widetilde{X}$ *contains totally geodesic copies of the monodromy cover for* $(9, 9, 9, 13, 18)/30$*, which corresponds to one of the original surfaces in [MS] (in the notations from section 2, take* $p = 5$ *and* $s = 10$*).*

3. *If* $\mu = (5,5,5,5,19,21)/30$*, then* $\widetilde{X}$ *contains totally geodesic copies of the monodromy cover for* $(5,5,5,21,24)/30$*, which corresponds to taking* $p = 3$ *and* $s = 30$ *in section 2.*

Proof: Although it is not completely obvious that the components of the preimages are exactly the same as the lower-dimensional monodromy cover, it is clear that they are indeed totally geodesic and that they cannot be (two-dimensional) ball quotients, since the map to the mirror exhibits some branching. For the precise claim from Proposition 4.1, we shall leave the details to the reader. Observe for instance that $\mu = (9,9,9,9,11,13)/30$ contracts to $\mu' = (9,9,9,13,20)/30$ when adding the weights μ_4 and μ_5. It is easy to verify that the group $\Gamma_{\mu',\Sigma'}$ is essentially a subgroup of the group $\Gamma_{\mu,\Sigma}$. More specifically, consider the stabilizer in $\Gamma_{\mu,\Sigma}$ of the mirror $M_{45} \subset \mathbb{B}^3$ of a complex reflection R_{45} corresponding to a half twist between x_4 and x_5 (the mirror depends on the choice of a path for this half twist). This stabilizer is a central extension of $\Gamma_{\mu',\Sigma'}$, with center generated by R_{45}, or in other words the stabilizer modulo the point stabilizer of M_{45} is simply $\Gamma_{\mu',\Sigma'}$. This follows from explicit monodromy calculations, taking a basis with one integral is taken between x_4 and x_5, and the others between the other points x_j. See also Proposition (8.9) from [DM], or section 2 from [M5]. $\square$

Remark 4.2 1. Each example above contains other totally geodesic surfaces, given by adding up other pairs of exponents. The only ball quotients obtained in this way are of course the components of the branch locus.

2. Because of our choice of exponents, contracting the two exponents corresponding to the branch locus should yield 5-tuple that satisfies ΣINT. This is indeed the case, and in particular this explains why there is no hope to obtain more than finitely many examples in dimension at least three with this method, since there are only finitely many ΣINT examples in dimension at least two. On the other hand, there are infinitely many ΣINT 4-tuples, so it is not surprising that there are infinitely many Mostow-Siu surfaces.

Another way to check that our threefolds are not ball quotients is to compute some ratio of Chern numbers, for instance c_1^3/c_3. This can be done explicitly by using the branching structure of the maps $\widetilde{w} : \widetilde{X} \to \mathbb{B}^3$ and $X \to Q_{st}/\Sigma$.

We sketch the computations for a branched cover $f : X \to B$ of a compact ball quotient, even though our situation is slightly more general, since the group $\Gamma_{\mu,\Sigma}$ does not act discretely on $\mathbb{B}^3$. The computations below can be interpreted in equivariant terms to yield the same ratio of Chern classes (see [D]).

We have $c_1(X) = f^*c_1(B) - R$, where R is the ramification divisor. In view of our analysis in Section 3, we assume that the various components of R are smooth and have multiplicity $m-1$, where m is the order of branching.

The obvious higher-dimensional analogue of the Riemann-Hurwitz formula yields $c_3(X) = f^*c_3(B) - \sum \frac{m-1}{m}\deg(f)\chi(R_i)$, and in turn we have

$$c_1^3(X) - 16c_3(X) = 3f^*c_1(B)R^2 - R^3 \tag{12}$$

Here we used the fact that $c_1^2(B)|_S = \dfrac{8}{3}c_2(B)|_S = \dfrac{16}{3}c_2(S)$, for S a subball quotient in B. Note that if S is a component of the ramification locus in B, then $f^*S = \sum \dfrac{m-1}{m}R_i$. We use this to compute (12) on the level of B.

$$c_1^3(X) - 16c_3(X) = \deg(f)\left(\frac{m-1}{m}\right)^2\left(3c_1(B)S^2 - S^3\right) \tag{13}$$

Using $c_1(B)|S = c_1(S) + c_1(N_{S/B})$ and expressing the intersections above in terms of $c_1(N_{S/B}) = \dfrac{1}{3}c_1(S)$, we get

$$c_1^3(X) - 16c_3(X) = \deg(f)\left(\frac{m-1}{m}\right)^2\frac{11m+1}{3m}c_2(S) \tag{14}$$

This expression makes it clear that we cannot have $c_1^3(X) = 16c_3(X)$, hence that X cannot be a ball quotient. We can also get a numerical expression for the ratio c_1^3/c_3 by computing $c_2(S)$ from the description of the components of the branch locus given in Proposition 4.1, which reduces the computation to using formula 10. A formula for $c_3(X)$ can be deduced from the local description of the monodromy cover given by writing down a three-dimensional analogue of Lemma 2.10. A tedious calculation gives $c_1^3/c_3 = 9313/616$, $2781/176$ and $3061/208$ respectively for our three choices of the exponents.

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