On the universal cover of certain exotic Kähler surfaces of negative curvature

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Abstract. We study a class of examples of negatively curved compact Kähler surfaces that are not diffeomorphic to any locally symmetric space. From the analysis of certain totally geodesic curves on these surfaces we deduce that, for infinitely many examples, the natural representation of the fundamental group into $PU(2,1)$ is non-faithful. We also give a new construction of bounded holomorphic functions on the universal cover of our surfaces, based on lifting maps to compact Riemann surfaces.

1. Introduction

Around 1980, Mostow and Siu constructed the first example of a compact four-dimensional Riemannian manifold with negative curvature not diffeomorphic to any locally symmetric space. In fact in [MS] they describe an infinite family of examples, all compact Kähler surfaces, that already at the time were considered as being of great interest.

Once such examples have been constructed, questions abound about their topological and complex analytic properties, and one would expect some work to be devoted to their analysis. Somehow for the past 20 years, no step has been taken in that direction. The list of examples has been expanded a little but, to the best of our knowledge, nobody has gone in any essential way beyond the construction of examples.

One goal of the present paper is to revive the interest for the subject by investigating some detailed properties of a particular class of such surfaces. This class is not quite the one described in [MS], but shares many of its interesting features.

In the original Mostow-Siu construction, the crucial ingredient is the study in [M1] of some non-discrete groups generated by three complex reflections in the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$. It turns out that these groups, up to commensurability, can be understood as monodromy groups of certain hypergeometric functions. This is the point of view we take to construct our surfaces.

We describe the relevant class of examples in terms of the methods developed in [DM], where Deligne and Mostow study the monodromy groups Γ_μ of hypergeometric functions whose set of exponents μ satisfies the so-called Picard integrality condition (cf. 2.5). The relevant surfaces are then ball quotients. We relax the condition on the exponents, so that the Picard integrality condition fails only in a controlled fashion (namely, as in 5.1), and study the corresponding monodromy groups Γ_μ.

As in the Deligne-Mostow cases, hypergeometric functions define Γ_μ-equivariant maps $\widetilde{X} \to \mathbb{B}^2$ which, because of the failure of the Picard integrality condition, exhibit some branching behavior. Most of the time the monodromy group Γ_μ acts in a non-discrete fashion on the ball $\mathbb{B}^2$, but its action on $\widetilde{X}$ is always discrete. Our surfaces are quotients $X_0 = {}_{\Gamma_0}\backslash \widetilde{X}$, where Γ_0 is a torsion free subgroup of finite index in Γ_μ. In particular, their fundamental group comes with a natural representation $\lambda : \pi_1(X_0) \to PU(2,1)$ into the automorphism group of the ball. The starting point for the study of the fundamental group of X_0 is to determine whether this representation is faithful. This question was raised over 20 years ago in [M1], and we give it a partial answer in section 9. More specifically, Theorem 9.4 states that for infinitely many examples the natural representation of $\pi_1(X_0)$ into $PU(2,1)$ is *not* faithful. In a sense, this result illustrates how complicated the fundamental group of our surfaces can be.

To give a rough idea of the methods involved in the proof, we mention that the representation λ is faithful if and only if the cover $\widetilde{X}$ is simply connected (cf. section 9). An important tool in understanding the fundamental group of the cover $\widetilde{X}$ is Theorem 8.2, which roughly states that the two-dimensional hypergeometric cover $\widetilde{X}$ contains many one-dimensional hypergeometric covers $\widetilde{X}'$ as totally geodesic divisors. These covers $\widetilde{X}'$ are much easier to understand, since they are just Riemann surfaces, closely related to triangle groups. In particular, inspired by the proof of a result of Mostow that gives a classification of certain discrete groups generated by two elliptic elements in the hyperbolic plane (see Theorem 7.1), we obtain a concrete sufficient condition for the one-dimensional analogue λ' of λ to be non-faithful (see Proposition 7.3).

Once again because of the results in section 8, we have many embeddings $\widetilde{X}' \subset \widetilde{X}$ as totally geodesic divisors. Since the manifold $\widetilde{X}$ has negative curvature, this inclusion induces an injection $\pi_1(\widetilde{X}') \hookrightarrow \pi_1(\widetilde{X})$ on the level of fundamental groups, which implies that whenever λ' is non-faithful, so is λ.

In a different direction, we make some progress towards understanding the complex analytic aspects of Mostow-Siu type surfaces. Throughout our investigations, the following questions serve as a guide. Can one construct bounded holomorphic functions on their universal cover? Are there enough such functions to separate points? This approach was suggested by Siu around the time of the construction of the original examples, but once again, it seems that until now, nobody had done anything in that direction.

Recall that by construction, our surfaces come with maps $\widetilde{X} \to \mathbb{B}^2$ from some cover of X_0 to the ball. This gives an obvious way to produce many bounded holomorphic functions on the universal cover of X_0, since there are plenty of bounded holomorphic functions on $\mathbb{B}^2$. A natural question is whether every bounded holomorphic function must be of that type. In section 9, we give this question a negative answer. We describe a new construction of bounded holomorphic functions on the universal cover of our surfaces, essentially based on lifting certain maps from our surfaces to compact Riemann surfaces, as stated in Proposition 10.4. Note that it is in general a difficult task to construct non-trivial maps to lower-dimensional manifolds. Theorem 10.2 states that for infinitely many examples, our new construction produces bounded holomorphic functions on the universal cover of X_0 that *do not* factor through the hypergeometric map $\widetilde{X} \to \mathbb{B}^2$.

The paper is organized as follows. In sections 2 through 6 we mostly recall the main results from [MS], adapting the notations when necessary. Section 7 is for the most part devoted to some basic results on triangle groups in the hyperbolic plane. We give some details for the proof of Theorem 7.1, since our claim differs slightly from the analogous theorem in [M4]. This result, together with the ones in section 8, constitute the main tools for our main two theorems, in section 9 and 10 respectively.

2. Basic Results

We start by collecting some classical results on hypergeometric functions. The reader will find proofs and much more on this beautiful construction in [DM] and the references given there.

Consider the integrals

$$\int_{x_i}^{x_j} \frac{dz}{\prod_k (z - x_k)^{\mu_k}} \tag{2.1}$$

where $x \in \mathbb{P}^1$, and the rational exponents μ_k satisfy $0 < \mu_k < 1$ and $\sum \mu_k = 2$. The natural domain is

$$M = \{(x_1, \dots, x_{n+3}) \in (\mathbb{P}^1)^{n+3} : x_i \neq x_j \ \forall\ i \neq j\}$$

but the map clearly factors to

$$Q = M/\mathrm{Aut}(\mathbb{P}^1) \simeq \{(x_1, \dots, x_n) \in (\mathbb{P}^1)^n : x_i \neq x_j, x_k \neq 0, 1, \infty\}$$

It turns out that there are only $n+1$ linearly independent integrals of the type (2.1), and these yield projective coordinates for a multivalued holomorphic map $Q \to \mathbb{P}^n$. In fact the image lies in a copy of the unit ball in $\mathbb{C}^n$, which we always write $\mathbb{B}^n \subset \mathbb{P}^n$. The multivaluedness is measured by the monodromy representation

$$\rho : \pi_1(Q) \to \mathrm{Aut}(\mathbb{B}^n) \tag{2.2}$$

The fundamental group $\pi_1(M)$ is the spherical braid group on $n+3$ strands and $\pi_1(Q) = \pi_1(M/\mathrm{Aut}(\mathbb{P}^1))$ is isomorphic to this braid group modulo its center. Both groups are generated by loops γ_{ij} corresponding to letting x_i go once around x_j. Here, with a slight abuse of notation, we view γ_{ij} as a loop either in M or in Q. We give a schematic picture of some of these loops in Q in the case $n = 2$ in Figure 2.1. The notation γ_{ij} is of course a little ambiguous, since the indices i, j only determine the loop up to conjugacy.

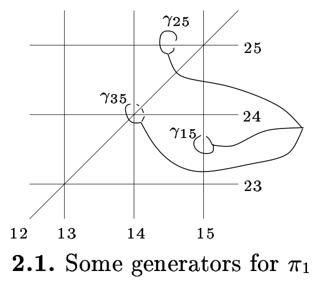

Fig. 2.1. Some generators for $\pi_1(Q)$.

Whenever we need to specify a precise set of loops, we choose an embedded interval T_{ij} between x_i and x_j, and take γ_{ij} to move x_i close to x_j along T_{ij}, go around x_j once in the positive direction, and then come back along T_{ij} to its original position. The relations between two different loops γ_{ij} and γ_{kl} then depend on the choice of embedded intervals T_{ij} and T_{kl} in the sphere. For instance, if T_{ij} and T_{kl} are disjoint, then the corresponding loops commute.

Theorem 2.1. *The monodromy transformation $\rho(\gamma_{ij})$ is a complex reflection in $\mathbb{P}^n$. Its mirror intersects $\mathbb{B}^n$ if and only if $\mu_i + \mu_j < 1$, in which case it is a totally geodesic sub-ball in $\mathbb{B}^n$. $\rho(\gamma_{ij})$ rotates by an angle of $2\pi(1 - \mu_i - \mu_j)$ in the directions orthogonal to the mirror.*

The multivalued map $Q \to \mathbb{B}^n$ extends to a certain compactification Q_{sst} of Q, whose description depends on the combinatorics of the exponents μ_k. For a general description, see [DM]. Here we only need the cases $n = 1$ or 2. When $n = 1$, $Q \simeq \mathbb{P}^1 - \{0, 1, \infty\}$ and we compactify it to $\mathbb{P}^1$, regardless of the exponents. Note that in dimension one the complex reflections of Theorem 2.1 are just rotations in the real hyperbolic plane. We shall come back to this in section 7. When $n = 2$, we describe the appropriate compactification only under the assumption that

$$\mu_i + \mu_j < 1 \quad \forall\, i \neq j \in \{1, \ldots, 5\} \tag{2.3}$$

The relevance of this condition is stated in Theorem 2.1, and we sketch in Remark 2.2 how the compactification needs to be modified in more general situations . In fact condition (2.3) is not needed in any essential way, and we shall explain how our theorems carry over without (2.3) in a subsequent paper.

Since $n = 2$, Q is the complement in $\mathbb{P}^1 \times \mathbb{P}^1$ of the seven lines depicted in Figure 2.1. We write X for the appropriate compactification, which is obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at the three triple intersections of that configuration of seven lines. The complement of Q in X consists of 10 divisors D_{ij}, each of them corresponding to letting x_i and x_j coalesce. Recall that, since $n = 2$, Q is the moduli space of five points on $\mathbb{P}^1$, and $\binom{5}{2} = 10$. Figure 2.2 gives an idea of the combinatorics of the compactification divisors. We do not allow more than two points to collide, or equivalently D_{ij} and D_{kl} intersect if and only if $\{i, j\} \cap \{k, l\} = \emptyset$.

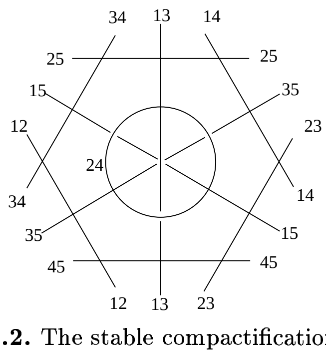

Fig. 2.2. The stable compactification of Q.

Remark 2.2. The combinatorics of the divisors D_{ij} is designed to match the intersection pattern of the mirrors of the corresponding complex reflections $\rho(\gamma_{ij})$. If $\mu_i + \mu_j > 1$ for some indices i, j, then $\rho(\gamma_{ij})$ has an isolated fixed point in $\mathbb{B}^2$, which is the intersection point of three mirrors of the $\rho(\gamma_{kl})$, $\{i,j\}\cap\{k,l\}=\emptyset$. In that case the appropriate compactification would allow for the three divisors D_{kl} to have a common intersection point. In fact the way to get the corresponding compactification is to blow down the divisors D_{ij} in X for which $\mu_i+\mu_j>1$. We think of condition (2.3) as a way to guarantee that the compactification divisors have normal crossings.

The multivalued map $Q\to\mathbb{B}^n$ lifts to a single valued holomorphic map $\widetilde{w}:\widetilde{Q}\to\mathbb{B}^n$, where $\widetilde{Q}$ is the cover of Q such that $\pi_1(\widetilde{Q})$ is the kernel of the monodromy representation (2.2). Note once again that in this paper we only consider the cases $n=1$ and $n=2$. We write K for the kernel of ρ and Γ for its image. Of course K and Γ depend on the set of exponents $\mu=(\mu_0,\dots,\mu_{n+2})$, and we shall sometimes write K_μ and Γ_μ to insist on this dependence.

The monodromy group $\Gamma\simeq\pi_1(Q)/K$ is the deck group of the cover $\widetilde{Q}\to Q$, and the hypergeometric map $\widetilde{w}$ is equivariant with respect to the action of Γ on $\widetilde{Q}$ and $\mathbb{B}^n$. The cover $\widetilde{Q}\to Q$ can be extended in a natural way to a branched cover $\widetilde{X}\to X$, branched over the 10 compactification divisors D_{ij}. The action of Γ extends to $\widetilde{X}$ in such a way that ${}_{\Gamma}\backslash\widetilde{X}\simeq X$, at least as topological spaces.

Theorem 2.3. *The Fox completion $\widetilde{X}$ of $\widetilde{Q}$ over X is a complex manifold, and hypergeometric functions yield a map $\widetilde{w}:\widetilde{X}\to\mathbb{B}^n$, which is a local biholomorphism near every point of $\widetilde{Q}$, and branches along certain completion divisors $\widetilde{D}_{ij}$, where $\widetilde{D}_{ij}$ is a component of the pre-image of D_{ij} in $\widetilde{X}$. The branching order of $\widetilde{w}$ around $\widetilde{D}_{ij}$ is given by the numerator n_{ij} of the reduced fraction $1-\mu_i-\mu_j=\frac{n_{ij}}{d_{ij}}$.*

For a definition of Fox completions we refer the reader to section 3, where we also sketch the proof of Theorem 2.3. For now we just give the following diagram

$$\begin{array}{ccccc}\widetilde{Q} & \hookrightarrow & \widetilde{X} & \stackrel{\widetilde{w}}{\to} & \mathbb{B}^n\\ \downarrow & & \downarrow & & \\ Q & \hookrightarrow & X & & \end{array}\tag{2.4}$$

In [DM], Deligne and Mostow assume that $n_{ij}=1$ for all i, j, in which case $\widetilde{w}$ does not have any branching. In fact, $\widetilde{w}$ is then a biholomorphism and $X={}_{\Gamma}\backslash\widetilde{X}$ is an orbifold ball quotient. The condition

$$1-\mu_i-\mu_j=\frac{1}{d_{ij}},\ d_{ij}\in\mathbb{Z}\quad\forall\, i\neq j\tag{2.5}$$

is known as the Picard integrality condition, and it turns out there are only finitely many examples where it is satisfied. In this paper we analyze the situations where the Picard integrality condition fails, or in other words where the hypergeometric map $\widetilde{w} : \widetilde{X} \to \mathbb{B}^n$ does have some branching. Most of the time, the monodromy group is then a non-discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, but since it is the deck group of the cover of $\widetilde{Q} \to Q$, it always acts discretely on $\widetilde{Q}$, hence on $\widetilde{X}$ as well. Of course the action of Γ has fixed points in the completion divisors $\widetilde{D}_{ij}$. In fact each complex reflection $\rho(\gamma_{ij})$ fixes some $\widetilde{D}_{ij}$, and its order is given by the denominator d_{ij} of the reduced fraction $1 - \mu_i - \mu_j$. Accordingly, one can think of the quotient $X = {}_{\Gamma}\backslash\widetilde{X}$ as an orbifold, where each divisor D_{ij} gets assigned the weight d_{ij}. In terms of branched covers, the projection $\widetilde{X} \to X$ branches with order d_{ij} around D_{ij}.

Remark 2.4. 1. We will see in section 9 that in general $\widetilde{X}$ need not be simply connected, hence it certainly cannot always be isomorphic to the ball. Actually even if $\widetilde{X}$ happened to be simply connected one can still show that, due to the branching nature of $\widetilde{w}$, its universal cover cannot be biholomorphic to the ball (see section 5).
2. The list of possible choices of exponents in the hypergeometric functions that give a discrete monodromy group is finite, and can be found in [M4]. The only non-obvious examples that satisfy our assumptions (2.3) and (5.1) are given by $\mu = \frac{1}{21}(4, 8, 10, 10, 10)$ or $\frac{1}{24}(5, 10, 11, 11, 11)$. By "non-obvious," we mean that these examples do not satisfy the Picard integrality condition, nor the Mostow half-integrality condition.
3. One can think of the integrals (2.1) as periods of the one-form dz/v on the family of curves $v^d = \prod_k (z - x_k)^{n_k}$, where $\mu = \frac{1}{d}(n_0, \ldots, n_{n+2})$. We do not need this here, but the monodromy transformations can then be interpreted in terms of parallel transport corresponding to the local system of cohomology groups of these curves.

3. Local structure of the hypergeometric maps

As announced in section 2, we now describe the Fox completion process in some detail, and explain how it applies to our hypergeometric monodromy covers. We start with some general considerations about spreads (a good reference is [F], or section 8 of [DM]). Let A and B be locally connected T_1 topological spaces.

Definition 3.1. *A continuous map $f : A \to B$ is a **spread** if the connected components of inverse images of open sets in B give a basis for the topology of A. The spread is called **complete** if for every $x \in B$,*

$$f^{-1}(x) = \varprojlim_{x \in U} \pi_0(f^{-1}(U))$$

In concrete terms, the condition for completeness expressed above in terms of inverse limits can be reformulated as follows. Fix any $x \in B$, and for each open neighborhood U_j of x, choose a component V_j of $f^{-1}(U_j)$ in such a way that $V_j \subset V_k$ whenever $U_j \subset U_k$. The condition above then requires that the intersection $\cap V_j$ be non-empty (or equivalently, that this intersection be a point).

Observe that this condition is of course interesting only for points x not in the image of f. The basic fact proved in [F] is that any spread $f : A \to B$ can be extended uniquely to a complete spread $\overline{f} : \overline{A} \to B$. We refer to the space $\overline{A}$ as the completion of A over B. The Fox completion of a spread f satisfies the universal property that any map to a complete spread factors through the completion $\overline{f}$. From this it is easy to deduce the following important result.

Lemma 3.2. *Let $f : A \to B$ be a spread, with completion $\overline{f} : \overline{A} \to B$. Given an open subset $U \subset B$, any connected component of $\overline{f}^{-1}(U)$ is the completion of the corresponding component of $f^{-1}(U)$ over U.*

The relation to branched covers is made clear by the following observation. Suppose $Y \to Z$ is a branched cover, with branch locus $Z - Z_0$, so that we have an unbranched cover $Y_0 \to Z_0$. Then the map $Y_0 \to Z$ is a spread, whose completion is precisely the branched cover $Y \to Z$. Of course in general, given an arbitrary unbranched cover $Y_0 \to Z_0$ where Z_0 is a subset of a certain space Z, the corresponding completion need not be a branched cover. Here we are interested in the hypergeometric situation, where we have an unbranched cover $\widetilde{Q} \to Q$ and a compactification $Q \subset X$. We write $\widetilde{X}$ for the Fox completion of $\widetilde{Q}$ over X. It is not clear that the space $\widetilde{X}$ is a manifold. A priori it might not even be locally compact, but in fact it turns out that this is the only obstruction to getting a complex manifold.

In order to check that the completion $\widetilde{X}$ is locally compact, we need to check that the map $\widetilde{X} \to X$ is locally finite-to-one. This follows from the lemma below. As in section 2, we write $1 - \mu_i - \mu_j = \frac{n_{ij}}{d_{ij}}$ as a reduced fraction.

Lemma 3.3. *Near a point $\widetilde{x}$ above $x \in D_{ij} - \bigcup D_{kl}$, the map $\pi : \widetilde{X} \to X$ looks like $(z, w) \mapsto (z^{d_{ij}}, w)$. If $x \in D_{ij} \cap D_{kl}$, then π looks locally like $(z, w) \mapsto (z^{d_{ij}}, w^{d_{kl}})$*

Proof. We consider only the case $x \in D_{ij} \cap D_{kl}$. Let V be a small neighborhood of x and write U for $V \cap Q$. By "small" we mean that $\pi_1(U)$ should be abelian, generated by two loops γ_{ij} and γ_{kl} around D_{ij} and D_{kl} respectively. If $\widetilde{U}$ is a component of $\pi^{-1}(U)$, we have that $\pi_1(\widetilde{U}) \simeq K \cap \pi_1(U)$. But we know that $\rho(\gamma_{ij})$ has order d_{ij} so that $\gamma_{ij}^{d_{ij}} \in K$ (and similarly $\gamma_{kl}^{d_{kl}} \in K$). This gives the structure of the cover $\widetilde{U} \to U$, hence the structure of $\widetilde{V} \to V$. □

Note that Lemma 3.3 also shows how to get complex coordinates on $\widetilde{X}$. Now that it is established that the completion $\widetilde{X}$ of the monodromy cover is a manifold, we analyze the local structure of the hypergeometric map. The first observation is that $\widetilde{w}$ is a local biholomorphism near any point of $\widetilde{Q}$. This is explained in detail in [DM], section 3.

Near a point $x \notin \widetilde{Q}$, $\widetilde{w}$ looks like a branched cover. For instance, if $x \in D_{ij}$ but is in no other D_{kl}, the computation of the monodromy transformation described in section 2 (see also [DM] section 9) implies that, near x, the multivalued map $X \to \mathbb{B}^2$ looks like

$$(z_1, z_2) \mapsto (z_1, z_2^{1-\mu_i-\mu_j})$$

In fact this local description is quite natural in terms of the hypergeometric integrals (2.1).

Putting this together with the result of Lemma 3.3, one sees that, if $\widetilde{x}$ lies over x on the cover $\widetilde{X}$, the single valued lift $\widetilde{w}$ near $\widetilde{x}$ looks like

$$(z_1, z_2) \mapsto (z_1, z_2^{n_{ij}})$$

which proves Theorem 2.3.

4. The Mostow-Siu surfaces

Recall that the monodromy group Γ is generated by torsion elements, hence the quotient $X = {}_{\Gamma}\backslash\widetilde{X}$ is only an orbifold. In order to get a manifold quotient, we need to find a torsion free subgroup of finite index $\Gamma_0 \subset \Gamma$. We briefly describe how this can be done.

The key observation is that the monodromy group is defined over the ring $\mathcal{O}$ of integers in the cyclotomic field $\mathbb{K} = \mathbb{Q}(\zeta)$ (see [DM], section 12). This important fact is quite natural in terms of the interpretation of part 3 of Remark 2.4. One can also check it by computing explicit matrices for the monodromy transformations. Instead of working with $\Gamma \subset PU(2,1,\mathcal{O})$, we can take a torsion free subgroup in the corresponding linear group $\overline{\Gamma} \subset U(2,1,\mathcal{O}) \subset GL(3,\mathcal{O})$. Torsion free subgroups of finite index in $G = GL(3,\mathcal{O})$ can be obtained by taking congruence subgroups. For an appropriate choice of the ideal $\mathfrak{a}$, the reduction of G modulo $\mathfrak{a}$ is torsion free. This produces a normal torsion free subgroup $\Gamma_0 \subset \Gamma$ of finite index, hence a complex manifold quotient $X_0 = {}_{\Gamma_0}\backslash\widetilde{X}$. Once again, we mention that if μ satisfies the Picard integrality condition, the hypergeometric map $\widetilde{w} : \widetilde{X} \to \mathbb{B}^2$ is a biholomorphism, and X_0 is just a ball quotient. In what follows, we will always assume that μ does not satisfy the Picard integrality condition, or in other words that $\widetilde{w}$ does have some branching.

Definition 4.1. *In the cases where the hypergeometric map* $\widetilde{w}$ *has some branching, we call the quotient* X_0 *a* ***Mostow-Siu surface****.*

It might be more appropriate to call X_0 a Mostow-Siu "type" surface. Strictly speaking, our construction is not the same as the one in [MS], although it exhibits many common features. The basis for the original Mostow-Siu construction is the understanding of groups that Mostow denotes $\Gamma_{p,t}$ (see [M1]). They are generated by three complex reflections of order p ($p = 3, 4$ or 5) whose mirrors are given by three vectors v_1, v_2 and v_3, the inner product $\langle v_i, v_j \rangle$ being a certain known function of the rational parameter t. In terms of our notations, $\Gamma_{p,t}$ is commensurable with the group Γ_μ, for $\mu = (1/2 - 1/p, 1/2 - 1/p, 1/2 - 1/p, 1/4 + 3/2p - t/2, 1/4 + 3/2p + t/2)$. A justification for this commensurability statement can be found for instance in [M3].

The careful reader will note that the above 5-tuple does not satisfy our assumption (2.3) since $\mu_4 + \mu_5$ is always greater than 1 for $p < 6$. In fact most of the considerations in this paper carry over to that situation, but the discussion is significantly more complicated. We give a rough idea of the corresponding construction, and sketch the relation of our surfaces to the actual Mostow-Siu construction.

The 5-tuple μ has the special property that the first three weights are equal, and the corresponding $(1 - \mu_i - \mu_j)^{-1}$ ($i, j \in \{1, 2, 3\}$) are either integers (if $p = 4$) or half integers (if $p = 3$ or 5). This is closely related to the situation envisioned in [M2]. In what follows we assume that $p = 3$ or 5. For a finite number of values of t, the other $(1 - \mu_i - \mu_j)^{-1}$ are integers and the groups $\Gamma_{p,t}$, Γ_μ are then discrete. This is proved in [M2] by showing that the hypergeometric map descends to a homeomorphism $\widetilde{X}/S_3 \to \mathbb{B}^2$, where S_3 denotes the symmetric group on three letters. Here the action of S_3 on $\widetilde{X}$ is induced from its obvious action on the three points with equal weights.

In general, the map has branching of order given by the numerators of $1-\mu_i-\mu_j$ ($i = 1, 2, 3$ and $j = 4, 5$), exactly like in section 3. The situation is then essentially the same as the one presented here, with $\widetilde{X}$ replaced by $\widetilde{X}/S_3$. The analysis is much more subtle, though. For instance, the space $\widetilde{X}$ has singularities, whereas the quotient $\widetilde{X}/S_3$ is a manifold.

The original motivation behind the construction of such a surface is that it produces examples of non-locally symmetric compact Kähler manifolds. This is explained in detail in [MS], making heavy use of the detailed analysis of the groups $\Gamma_{p,t}$ given in [M1]. Since their description of the surfaces is quite different from ours, we will sketch the argument in the next two sections. In the last sections of this paper, we then focus on studying some properties of the Mostow-Siu surfaces, in essentially two directions – the analysis of their fundamental group, and the construction of bounded holomorphic functions on their universal cover.

We now go back to the observation that each linear hypergeometric monodromy group is defined over some number field $\mathbb{K} = \mathbb{Q}(\zeta)$. We discuss another important implication, that will be used repeatedly later on in this paper. Applying Galois automorphisms $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ we get different embeddings of $\mathbb{K}$ in $\mathbb{C}$, which induce different embeddings of the monodromy group in $PGL(n+1,\mathbb{C})$.

The Galois conjugate groups are still monodromy groups of hypergeometric functions, whose exponents can be computed explicitly from the original exponents (see [DM] section 12). Recall that the weights $\mu = (\mu_1, \dots, \mu_{n+3})$ encode the monodromy group by means of the corresponding roots of unity $e^{2\pi i \mu_j}$. These are powers of ζ, whose behavior under the Galois automorphisms is understood. The automorphism $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ given by $\zeta \mapsto \zeta^k$ (for a given k prime to the order of ζ) translates into $e^{2\pi i \mu_j} \mapsto e^{2\pi i k \mu_j}$. In other words, the conjugate tuple of weights is obtained by multiplying μ by the integer k, and reducing it in $\mathbb{Q}/\mathbb{Z}$. We denote by μ^σ the corresponding tuple of weights. We will sometimes call μ^σ a Galois conjugate of μ, since the corresponding monodromy groups are Galois conjugates.

It is important to realize that the corresponding Galois conjugate group does not necessarily lie inside $PU(n,1)$, but rather in some $PU(n+2-r, r-1)$ subgroup of $PGL(n+1,\mathbb{C})$ (where $r = \sum_j \mu_j^\sigma \in \mathbb{Z}$). We write Γ^σ for the corresponding Galois conjugate monodromy group. For more details on Galois conjugates, we refer the reader to section 12 of [DM]. We summarize the previous discussion in the following.

Proposition 4.2. *Let Γ_μ be the monodromy group corresponding to a given set of weights $\mu = (\mu_1, \dots, \mu_{n+3})$. We write d for the least common denominator of the μ_j's and $\zeta = \zeta_d = e^{2\pi i/d}$. For any k prime to d, we denote by σ the Galois automorphism of $\mathbb{Q}(\zeta)$ given by $\zeta \mapsto \zeta^k$. The Galois conjugate Γ_μ^σ is the monodromy group corresponding to $\mu^\sigma = (\mu_1^\sigma, \dots, \mu_{n+3}^\sigma)$, where $0 < \mu_j^\sigma < 1$ and $\mu_j^\sigma = k\mu_j$ in $\mathbb{Q}/\mathbb{Z}$.*

Remark 4.3. Note that for $k = d-1$, $\zeta^{d-1} = \overline{\zeta}$ so that Γ_μ^σ is obtained from Γ_μ by complex conjugation. More generally, if $\sigma_1 : \zeta \mapsto \zeta^k$ and $\sigma_2 : \zeta \mapsto \zeta^{d-k}$, then $\Gamma_\mu^{\sigma_2}$ is obtained from $\Gamma_\mu^{\sigma_1}$ by complex conjugation. In terms of the weights, $\mu_j^{\sigma_1} = 1 - \mu_j^{\sigma_2}$.

5. The ratio of Chern classes

We briefly recall how to argue that the universal cover of a Mostow-Siu surface cannot be biholomorphic to the ball. Following [MS], we compute the ratio of Chern classes c_1^2/c_2 and show that it is not equal to 3. The idea is that thanks to our understanding of the branching behavior of $\widetilde{X} \xrightarrow{\widetilde{w}} \mathbb{B}^2$, we can compute the Chern classes (or rather the Chern forms) of $\widetilde{X}$ in terms of those of $\mathbb{B} = \mathbb{B}^2$ and a correction term involving the ramification divisors.

We assume from now on that the ramification divisors of the hypergeometric map $\widetilde{w} : \widetilde{X} \to \mathbb{B}^2$ are disjoint. In terms of the combinatorics of the 5-tuple μ, this means that we want the integrality condition to fail only for pairs of indices that overlap.

$$\left.\begin{array}{l}(1-\mu_i-\mu_j)^{-1} \notin \mathbb{Z}\\ (1-\mu_k-\mu_l)^{-1} \notin \mathbb{Z}\end{array}\right\} \Rightarrow \{i,j\} \cap \{k,l\} \neq \emptyset \tag{5.1}$$

One views $d\widetilde{w}$ as a section of $\mathrm{T}^*\widetilde{X} \otimes \widetilde{w}^*\mathrm{T}\mathbb{B}$. Taking second exterior powers, we view the Jacobian $J(\widetilde{w})$ as a section of $\wedge^2\mathrm{T}^*\widetilde{X} \otimes \wedge^2\mathrm{T}\mathbb{B}$. Now the zero divisor of the Jacobian gives us the ramification divisor $\widetilde{R}$, and we have $\widetilde{R} = K_{\widetilde{X}} - \widetilde{w}^*K_{\mathbb{B}}$. In terms of the square of the first Chern class, this becomes

$$c_1^2(\widetilde{X}) = \widetilde{w}^*c_1^2(\mathbb{B}) - 2\ \widetilde{w}^*c_1(\mathbb{B}) \cdot \widetilde{R} + \widetilde{R}^2$$

We write $\widetilde{R} = \pi^*(R)$, $R = \sum b_i R_i$, where π is the projection $\widetilde{X} \to \Gamma_0\backslash\widetilde{X} = X_0$. The second term in the expression for $c_1^2(\widetilde{X})$ can be interpreted in terms of the R_i's as follows. On a subball $\mathbb{B}'$, $c_1(\mathbb{B})$ restricts to $3/2$ times the area form on $\mathbb{B}'$, hence

$$2\ \widetilde{w}^*c_1(\mathbb{B}) \cdot \pi^*(R_i) = 3\ \chi(R_i)$$

$$c_1^2(X_0) = \pi_*\widetilde{w}^*c_1^2(\mathbb{B}) - 3\ \sum b_i\chi(R_i) + R^2$$

On the other hand, using the appropriate version of the Riemann-Hurwitz formula we have

$$c_2(X_0) = \pi_*\widetilde{w}^*c_2(\mathbb{B}) - \sum b_i\chi(R_i) \tag{5.2}$$

which gives

$$c_1^2(X_0) - 3c_2(X_0) = R^2 \tag{5.3}$$

Equation (5.3) shows that the ratio of Chern classes of X_0 is equal to 3 if and only if $R^2 = 0$. We can compute the self-intersection R^2 more or less explicitly but it is clear that it is actually negative, unless $\widetilde{w}$ is not branched at all, which we rule out in the definition of Mostow-Siu surfaces. Recall that the R_i's are disjoint, and that R_i^2 is the Euler class of the normal bundle $N_{X_0}R_i$, which turns out to be equal to $\chi(R_i)/2(b_i+1)$.

Note that the formula above is difficult to use in practice to compute the ratio of Chern classes for X_0, since the Euler characteristics

$\chi(R_i)$ are hard to get our hands on. To obtain concrete numbers in terms of the 5-tuple μ, it is more convenient to compute Chern classes by thinking of X_0 as a branched cover of X. One then gets

$$\frac{c_1^2(X_0)}{c_2(X_0)} = 2 + \frac{1 - \sum \frac{1}{d_{ij}^2}}{2 - \sum \frac{1}{d_{ij}} + \sum_{\{i,j\}\cap\{k,l\}=\emptyset} \frac{1}{d_{ij}d_{kl}}} \tag{5.4}$$

where, as before, $1 - \mu_i - \mu_j = \frac{n_{ij}}{d_{ij}}$. This is the formula given in Proposition 12 of [Z]. It can also be thought of as giving the ratio of orbifold Chern classes of the orbifold $X = {}_\Gamma\backslash\widetilde{X}$. Note that it gives a practical way to compute the ratio of Chern classes in terms of the 5-tuple μ only. We use it to compute the ratio c_1^2/c_2 for each example given in section 11.

6. Construction of the metric

Here we assume once again that condition (5.1) holds, i.e., that the branching divisors of the hypergeometric map $\widetilde{w} : \widetilde{X} \to \mathbb{B}^2$ do not intersect. Note that this implies that the restriction of the map $\widetilde{w}$ to a component of a branching divisor maps isomorphically onto a subball in $\mathbb{B}^2$. This will be justified in detail in the proof of Theorem 8.2 (see Remark 8.4). We also understand the local structure of the map $\widetilde{w}$ near a branching divisor $\widetilde{D}_{ij}$, namely it looks like $(z_1, z_2) \mapsto (z_1, z_2^{n_{ij}})$.

Note that this is exactly the situation envisioned in [MS]. We briefly recall their construction of a metric. The general idea is to pull back the Bergman metric of the ball using our hypergeometric map $\widetilde{w}$. Of course this does not quite define a metric on $\widetilde{X}$ because of the branching behavior of $\widetilde{w}$ (the pull back is singular along the ramification locus). One then wants to add a correction term near the ramification divisors, but two major difficulties stand on our way.

The first one is that we want to get a *Kähler* metric, hence we need to be careful with a "partition of unity" type of argument – the way to solve this is to work with the Kähler potentials rather than with the metric themselves. The second difficulty comes from the fact that we want our metric on $\widetilde{X}$ to be invariant under the action of the monodromy group, since we eventually want a metric on the quotient X_0. Invariance will be guaranteed by selecting the correction metric carefully, in accordance with the local description of the map $\widetilde{w}$.

The key to getting an invariant metric is to use the bounded domain

$$\mathbb{D} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2k} < 1\}$$

and its natural map to the ball $\mathbb{B}^2$ given by $(z_1, z_2) \mapsto (z_1, z_2^k)$. Mostow and Siu calculate its Bergman kernel Φ explicitly, and construct a potential on $\mathbb{D}$ by adding to Φ a multiple of the pull-back of the Bergman kernel of the ball. They show that this gives a metric of negative curvature near the ramification locus.

This describes the local construction of the metric on $\widetilde{X}$, since, near a component $\widetilde{D}_{ij}$ of the ramification locus, the map $\widetilde{w} : \widetilde{X} \to \mathbb{B}^2$ looks like $(z_1, z_2) \mapsto (z_1, z_2^{n_{ij}})$. Once again, we mention that this construction can be made to work globally by a standard partition of unity argument (on the Kähler potentials rather than on the metrics). The fact that the metric can be made invariant under the action of Γ follows from the invariance property of Bergman metrics.

Remark 6.1. 1. The curvature actually satisfies a stronger condition than negative curvature, which implies a strong rigidity property (see [MS]).
2. Note that it follows from the construction that the divisors $\widetilde{D}_{ij}$ (branching or not) are totally geodesic, since they are the fixed point sets of isometries $\rho(\gamma_{ij})$ (where γ_{ij} is an appropriate loop around D_{ij}).
3. It is not clear how to modify the construction of the metric if we relax the condition (5.1) to allow the ramification locus to have normal crossings. The Bergman metric of the domain $|z_1|^{2l} + |z_2|^{2k} < 1$ is considerably more complicated when $l \neq 1$.

7. Triangle groups

We now describe in some detail the one-dimensional situation. Not only does it give some intuition on the behavior of the more complicated two-dimensional examples, but the results from this section will be used extensively later on.

We start with a 4-tuple $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ of rational numbers, satisfying as before $0 < \mu_j < 1$ and $\sum \mu_j = 2$. The one-dimensional analogue of the constructions discussed in section 2 can be summarized in the following diagram

$$\begin{array}{ccccc} & & \widetilde{Q} \hookrightarrow \widetilde{X} \xrightarrow{\widetilde{w}} \mathbb{B}^1 \simeq \mathbb{H}^2 \\ & & \downarrow \qquad \downarrow \\ \mathbb{P}^1 - \{0,1,\infty\} \simeq Q \hookrightarrow \mathbb{P}^1 \end{array}$$

Here $\widetilde{w}$ maps the lift of a hemisphere in $\mathbb{P}^1$ biholomorphically onto a triangle in $\mathbb{H}^2$, with angles given by $(1 - \mu_i - \mu_j)\pi$. The monodromy group $\Gamma = \Gamma_\mu \subset Aut(\mathbb{H}^2)$ is generated by any two of the three rotations centered at the vertices of such a triangle, with angles $2(1 - \mu_i - \mu_j)\pi$. We shall write the three numbers $1 - \mu_i - \mu_j$ as reduced fractions $\frac{k}{p}$, $\frac{m}{q}$ and $\frac{l}{r}$, and a, b and c for the corresponding

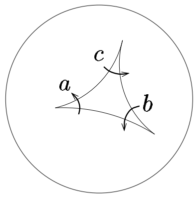

Fig. 7.1. Generating rotations for Γ.

rotations in $\mathbb{H}^2$ (Figure 7.1). Of course $\pi_1(Q) \simeq F_2 = \langle x, y \rangle$ is just a free group on two generators, where we think of x and y as small loops around two appropriate points out of $\{0, 1, \infty\}$. The kernel K of the monodromy representation gives a presentation for the monodromy group in terms of two generators, say a and b.

$$1 \to K \to \pi_1(Q) \simeq F_2 \to \Gamma \to 1$$

In Γ, we know that $a^p = b^q = (ab)^r = 1$. In other words, we know that K contains the normal subgroup $N \subset F_2$ generated by x^p, y^q and $(xy)^r$.

$$N = \langle\!\langle x^p, y^q, (xy)^r \rangle\!\rangle \subset K \subset F_2$$

If the angles of our hyperbolic triangle are integral parts of π (i.e., if $k = m = l = 1$), then Γ is just a classical triangle group, with presentation

$$\langle x, y | x^p, y^q, (xy)^r \rangle \tag{7.1}$$

In other words, if the Picard integrality condition is satisfied, then the two subgroups N and K coincide. In general, we can have $N \subsetneq K$, as we will see below.

The subgroups K and N can of course be thought of in terms of covering spaces. The inclusion $\widetilde{Q} \subset \widetilde{X}$ induces a surjection $\pi_1(\widetilde{Q}) \twoheadrightarrow \pi_1(\widetilde{X})$, since $\widetilde{X} - \widetilde{Q}$ has real codimension two in $\widetilde{X}$. The kernel of this surjection is the normal subgroup generated by small loops around the points in $\widetilde{X} - \widetilde{Q}$. Because of the local structure of the branched cover $\widetilde{X} \to X \simeq \mathbb{P}^1$, these small loops are just lifts of powers of small loops around 0, 1 and ∞. The exponent is given precisely by the order of the corresponding elements of the monodromy group, or in other words by the denominators p, q and r of the $1 - \mu_i - \mu_j$.

To summarize, we get a short exact sequence

$$1 \to N \to K \simeq \pi_1(\widetilde{Q}) \to \pi_1(\widetilde{X}) \to 1 \tag{7.2}$$

or $\pi_1(\widetilde{X}) \simeq K/N$. In particular, if the Picard integrality condition

$$(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z} \text{ for all } i, j \tag{7.3}$$

is satisfied, then the fact that $N = K$ means that $\widetilde{X}$ is simply connected. In fact in that case the map $\widetilde{w} : \widetilde{X} \to \mathbb{B}^1$ is an biholomorphism. Here we are interested in situations where the Picard integrality (7.3) condition fails. In general, just like in the two-dimensional situation, we then get a non-discrete monodromy group.

Theorem 7.1. *Let T be a triangle in $\mathbb{H}^2$ with angles α, β and γ that are not all integral parts of π. Let Γ be the group generated by rotations centered at the vertices of T and with angles 2α, 2β and 2γ respectively. Γ is discrete if and only if the angles of T are given by one of the following*

(i) $\frac{2\pi}{s}, \frac{\pi}{t}, \frac{\pi}{t}$	(ii) $\frac{2\pi}{t}, \frac{2\pi}{t}, \frac{2\pi}{t}$	(iii) $\frac{2\pi}{t}, \frac{\pi}{2}, \frac{\pi}{t}$
(iv) $\frac{3\pi}{t}, \frac{\pi}{3}, \frac{\pi}{t}$	(v) $\frac{4\pi}{t}, \frac{\pi}{t}, \frac{\pi}{t}$	(vi) $\frac{2\pi}{7}, \frac{\pi}{3}, \frac{\pi}{7}$

Proof. A proof of this is given in [M4] (actually Mostow omits cases (ii) and (vi), but the idea of the proof given there is entirely correct). We recall the main ideas involved in the argument.

The difficult part of the theorem is to show that our condition is necessary. The fact that it is sufficient essentially follows from the pictures in Figure 7.2.

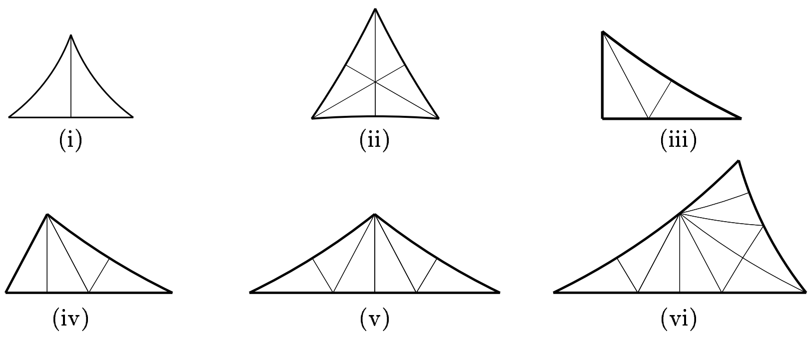

Fig. 7.2. In each case of the theorem, T is a union of copies of some smaller triangle T'. T' is a $(2, s, t)$-triangle in case (i) and a $(2, 3, t)$-triangle in all the other cases. The construction works for any $t \geq 7$ in cases (ii)-(vi) but only for $t = 7$ in case (vi).

Let us examine case (i), for instance. T is a union of two copies of a triangle T' with angles $\pi/2$, π/s and π/t (for the sake of brevity, we call T' a $(2, s, t)$-triangle). This certainly implies that Γ is a subgroup of a $(2, s, t)$-triangle group (see Figure 7.3).

It is easy to check that, as long as s is odd, Γ is not just a subgroup of the $(2, s, t)$-triangle but is actually equal to it. Indeed using the

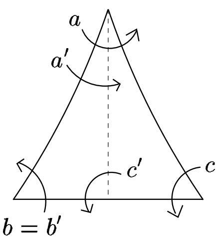

Fig. 7.3. T has angles $2\pi/s$, π/t, π/t, and T' has angles π/s, π/t, $\pi/2$. a, b and c are the corresponding generators for Γ, and a', b', c' are natural generators for the $(2, s, t)$-triangle group.

notations of Figure 7.3, we have

$$\begin{cases} a = a'^2 \\ b = b' \\ c = c'b'c'^{-1} \end{cases} \qquad \begin{cases} a' = a^{\frac{s+1}{2}} \\ b' = b \\ c' = b^{-1}a^{-\frac{s+1}{2}} = a^{\frac{s+1}{2}}b \end{cases} \tag{7.4}$$

In general Figure 7.2 illustrates the fact that in each case of the theorem, Γ is a subgroup of a classical triangle group, hence it is obviously discrete. Once again, one can check that in all the cases (i)-(vi), the group Γ is actually equal to a triangle group.

Now we give a quick sketch of the proof that the six cases (i)-(vi) are the only triangles for which Γ is discrete. Given that Γ is discrete, we want to show that T must be the union of finitely many congruent copies of a given triangle T', whose angles are all integral parts of π. We shall refer to T' as an elementary tile for T. A key ingredient in Mostow's argument is what he calls the triangulation algorithm (see [M4]).

We write Γ^* for the group generated by the Schwarz reflections in the sides of T. Recall that $\Gamma \subset \Gamma^*$ is the index two subgroup of orientation preserving elements in Γ^*. Of course Γ is discrete if and only if Γ^* is discrete. Roughly, the idea behind the triangulation algorithm is to subdivide T by using the components of the complement of the set of all mirrors of Schwarz reflections in Γ^*. Mostow argues that, if at least one of the angles of T is not an integral part of π, then the triangulation algorithm yields an elementary tile T' with angles $\pi/2$, π/s and π/t.

Comparing the areas of T and T', we get

$$\pi - (\alpha + \beta + \gamma) = n\pi \left\{ 1 - (\frac{1}{2} + \frac{1}{s} + \frac{1}{t}) \right\} \tag{7.5}$$

for some integers n, s and t. We may assume $3 \leq s \leq t$. Mostow considers the cases $s = 3$ and $s > 3$ separately. We present only $s = 3$ here, since it covers the two cases that are missing in [M4]. The analysis of the case $s > 3$ is similar.

Since we want T to be a union of copies of T', each of the rational numbers α/π, β/π and γ/π must be chosen to be an integer multiple of one of $\frac{1}{2}$, $\frac{1}{s}=\frac{1}{3}$ or $\frac{1}{t}$. We break all the possibilities into six cases

$$\begin{array}{llllll} (a) & \frac{1}{2},\frac{1}{3},\frac{k}{t} & (b) & \frac{1}{3},\frac{1}{3},\frac{k}{t} & (c) & \frac{2}{3},\frac{k}{t},\frac{l}{t} \\ (d) & \frac{1}{2},\frac{k}{t},\frac{l}{t} & (e) & \frac{1}{3},\frac{k}{t},\frac{l}{t} & (f) & \frac{k}{t},\frac{l}{t},\frac{m}{t} \end{array}$$

The first three cases are easily taken care of. For instance, in case (c), equation (7.5) yields

$$n = \frac{\frac{1}{3}-\frac{k+l}{t}}{\frac{1}{6}-\frac{1}{t}} = 2\,\frac{\frac{1}{3}-\frac{k+l}{t}}{\frac{1}{3}-\frac{2}{t}} \tag{7.6}$$

This expression is strictly less than 2 as long as $k+l>2$, but $n=1$ is impossible (T would have to be equal to T'). Hence we must have $k=l=1$, and the corresponding angles of T fall in case (i) of our theorem.

Similarly one can rule out case (a), and show that the only way T can generate a discrete group in case (b) is if $k=2$, which once again falls into case (i) of our theorem. We analyze case (e) in detail. Once again we use (7.5) to get

$$n = \frac{\frac{2}{3}-\frac{k+l}{t}}{\frac{1}{6}-\frac{1}{t}} = 4\,\frac{\frac{2}{3}-\frac{k+l}{t}}{\frac{2}{3}-\frac{4}{t}} \tag{7.7}$$

This implies $n\le 4$ whenever $k+l\ge 4$. We assume without loss of generality that $k\le l$. If $k\ge 2$, then $l\ge 2$ and the triangulation algorithm yields more than four triangles which contradicts the estimate $n\le 4$. The point is that if the mirrors of two Schwarz reflections in Γ^* make an angle of $2\pi/t$, then their bisector is also a mirror in Γ^*.

Now we must have $k=1$. Here Mostow assumes $l=3$, in which case $n=4$ and one gets case (iv) of our theorem, but he omits the possibility $l=2$. A little analysis of (7.7) shows that the only odd values of t for which n can be an integer are $t=7$ and $t=9$, yielding respectively $n=10$ and $n=6$. Figure 7.2 shows that $t=7$ does give a discrete group, corresponding to case (vi) of our theorem. One can check for instance using the triangulation algorithm that the case $t=9$ does not give a discrete group.

This concludes the analysis of case (e). A similar argument would show that (d) yields case (iii) our theorem, and that (f) yields cases (i), (ii) and (v). Recall that we have assumed that the elementary tile T had angles $\pi/2$, $\pi/3$ and π/t. The case where T' has angles $\pi/2$, π/s and π/t for $s,t>3$ is treated similarly and yields triangles of type (i). □

Remark 7.2. 1. As mentioned in the proof, whenever Γ is discrete, it is actually a (p,q,r)-triangle group, for some p, q and r. If we assume that not all the angles of T are integral parts of π, we get one of the cases in the Theorem 7.1, and the group Γ is a $(2,s,t)$-triangle group in case (i), $(2,3,t)$ in case (ii)-(v) and $(2,3,7)$ in case (vi).

2. It is easy to figure out which 4-tuples produce a triangle with angles as in Theorem 7.1, from the fact that the angles are given by $(1-\mu_i-\mu_j)\pi$ for the appropriate choice of indices i, j. In fact, up to permutation of the weights, there are two 4-tuples giving angles α, β and γ. One of them is

$$\mu = \frac{1}{2}(1-\alpha+\beta+\gamma, 1+\alpha-\beta+\gamma, 1+\alpha+\beta-\gamma, 1-\alpha-\beta-\gamma)$$

and the other one is gotten from μ by replacing μ_j by $1-\mu_j$.

We now state the consequences of 7.1 for $\Gamma=\Gamma_\mu$, the monodromy group of some hypergeometric map.

Proposition 7.3. *If Γ_μ is discrete but μ does not satisfy the Picard integrality condition, then $N \subsetneq K$, or in other words the cover $\widetilde{X}$ is not simply connected.*

Proof. We write $k\pi/p$, $l\pi/q$ and $m\pi/r$ for the angles of the relevant triangle T, which is the image of a hemisphere under the hypergeometric map. Recall that we write $\pi_1(Q) \simeq \langle x,y\rangle$ and $N = \langle\langle x^p, y^q, (xy)^r\rangle\rangle$. The normal subgroup N is a subgroup of the kernel of the monodromy K and $\Gamma_\mu \simeq \pi_1(Q)/K$. In other words, $N \subset K$ with equality if and only if Γ_μ has a presentation

$$\langle x,y | x^p, y^q, (xy)^r\rangle \tag{7.8}$$

Now if Γ_μ is discrete, it must appear somewhere in the list of Theorem 7.1 and in particular it is a (p',q',r')-triangle group for $\{p',q',r'\} \neq \{p,q,r\}$, where this last inequality follows from Remark 7.2. Here we use the fact that the Picard integrality condition is not satisfied, so that the angles of T are not all integral part of π.

The proposition follows at once from the fact that a (p,q,r)-triangle group is uniquely determined up to isomorphism by p, q and r, but we can also check it directly by exhibiting elements in $K-N$. We do this in detail when the angles of T are $2\pi/s$, π/t, π/t, which is case (i) of Theorem 7.1 (we assume s is odd).

We refer to Figure 7.3, and write a, b for generating rotations in Γ_μ, with angles $4\pi/s$, $2\pi/t$ respectively. Of course we have

$$a^s = b^t = (ab)^t = 1 \tag{7.9}$$

but the claim is that there are more relations between our generators. It is readily checked that the element $a^{\frac{s+1}{2}}b$ is a rotation centered at the midpoint of the base of the triangle T, with angle π. In particular it has order two. Hence we have a relation

$$(a^{\frac{s+1}{2}}b)^2 = 1 \tag{7.10}$$

between our generators a and b, which is not a consequence of the relations (7.9). In other words, in terms of the fundamental group $\pi_1(Q)$, the loop $(y^{\frac{s+1}{2}}x)^2$ is in K but not in N, hence $K/N \simeq \pi_1(\widetilde{X})$ is non-trivial. □

The reader should not be misled by proposition 7.3. One does not need Γ_μ to be discrete in order to get $\widetilde{X}$ not to be simply connected. Just like in the discussion at the end of section 4, we can think of the monodromy group as being defined over the field $\mathbb{K} = \mathbb{Q}(\sqrt[d]{1})$, and consider Galois conjugates Γ_μ^σ for different Galois automorphisms $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$. These are in general non-discrete but they have the same group theoretic properties as Γ_μ, for instance in terms of writing group presentations. We summarize this observation in the following

Corollary 7.4. *If some Galois conjugate Γ_μ^σ is discrete but μ^σ does not satisfy the Picard integrality condition, then $N \subsetneq K$ and $\widetilde{X}$ is not simply connected.*

In general it seems difficult to find *necessary* and sufficient conditions on μ in order to have $\pi_1(\widetilde{X}) \neq 1$. If some Galois conjugate is discrete and satisfies the Picard integrality condition, then we know that $\widetilde{X}$ is simply connected. When Γ_μ is not a Galois conjugate of any discrete group, it is not clear how large the subgroup K is. It is not even clear that Γ_μ would be finitely presented.

8. The completion divisors

We now go back to the two-dimensional situation corresponding to some 5-tuple $\mu = (\mu_1, \dots, \mu_5)$. Recall that we have assumed for any i,j that $\mu_i + \mu_j < 1$, so that it makes sense to consider the 4-tuple

$$\mu' = (\mu_i + \mu_j, \mu_1, \dots, \widehat{\mu_i}, \dots, \widehat{\mu_j}, \dots, \mu_5)$$

in the context of the previous sections.

Definition 8.1. *In the situation above, we say that μ* ***contracts*** *to μ'.*

We write Q', X', $\widetilde{w}', \ldots$ for the analogue of Q, X, $\widetilde{w}, \ldots$ corresponding to μ' instead of μ. Of course, X' is contained in X as divisor, but one can get a stronger statement that takes into account the orbifold structure of these two spaces. We express this in terms of the monodromy covers.

Theorem 8.2. *$\widetilde{X}'$ embeds in $\widetilde{X}$ as a connected component of the preimage of the divisor D_{ij} (corresponding to x_i and x_j coming together).*

Proof. This fact is stated in much more generality in [DM], section 8. We give a direct proof in the simpler particular case needed here. For simplicity of the notations, we will assume $\{i, j\} = \{1, 2\}$, so that

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$$
$$\mu' = (\mu_1 + \mu_2, \mu_3, \mu_4, \mu_5)$$

We write $\widetilde{D}_{12}$ for a connected component of the preimage of D_{12} in $\widetilde{X}$. Note that the branched covers $\widetilde{D}_{12} \to D_{12}$ and $\widetilde{X}' \to X'$ have the same local structure, since their branching orders are given by the denominators of $1 - \mu_i - \mu_j$, $i, j \in \{3, 4, 5\}$. A priori they could be different globally.

We think of these two branched covers as Fox completions of the corresponding unbranched covers. More precisely, we write C for the open subset of D_{12} corresponding to letting the first two points coming together, but with no other collapsing allowed ($C{=}D_{12} - \cup D_{kl}$). The cover $\widetilde{D}_{12} \to D_{12}$ is the Fox completion of $\widetilde{C} \to C$, where $\widetilde{C}$ is the component of the preimage of C contained in $\widetilde{D}_{12}$.

To prove the theorem, we need to identify $\widetilde{C} \to C$ as the monodromy cover $\widetilde{Q}' \to Q$ corresponding to the 4-tuple μ'. We select

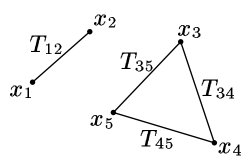

Fig. 8.1. Each loop γ_{ij} amounts to x_i going around x_j, along the path T_{ij}.

loops γ_{ij} around D_{ij} as in Figure 8.1. The loop γ_{ij} corresponds to having x_i come close to x_j along T_{ij}, making one positive turn, then going back along T_{ij} to its original position. Recall that Q is the quotient of $M \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ by $Aut(\mathbb{P}^1)$. The loops γ_{ij} were just described as loops in M, but we may also view them as loops in Q.

We claim that $\gamma_{35}\gamma_{45}\gamma_{34} = \gamma_{12}$ in $\pi_1(Q)$. This is a subtle point since in $\pi_1(M)$, which is the spherical braid group on five strands, the loop $\gamma_{12}^{-1}\gamma_{35}\gamma_{45}\gamma_{34}$ is *not* trivial. It corresponds to the central

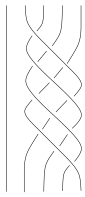

Fig. 8.2. The loop $\gamma_{12}^{-1}\gamma_{35}\gamma_{45}\gamma_{34}$ is central in $\pi_1(M)$, hence it projects to a trivial loop in $\pi_1(Q)$.

element depicted in Figure 8.2. Note that Q fibers over $\mathbb{P}^1 - \{3 \text{ pts}\}$, with fibers $\mathbb{P}^1 - \{4 \text{ pts}\}$, as can easily be seen from Figure 2.1. The base and the fiber both have free fundamental group, hence $\pi_1(Q)$ has trivial center. The loop $\gamma_{12}^{-1}\gamma_{35}\gamma_{45}\gamma_{34}$, which is central in $\pi_1(M)$, must project to a trivial loop in $\pi_1(Q)$.

Recall that γ_{ij} maps to a complex reflection R_{ij}, whose mirror will be denoted by M_{ij}. Since γ_{12} commutes with γ_{34}, γ_{35} and γ_{45}, we know that M_{12} is orthogonal to M_{34}, M_{35} and M_{45}. Accordingly, the three reflections R_{34}, R_{35} and R_{45} stabilize M_{12}, and one can check that they act on this sub-ball like the standard generators of the monodromy group Γ'. The point is that we can compute angles between the mirrors M_{34}, M_{35} and M_{45}, and check that they cut out in M_{12} a triangle with angles given by $\pi(1-\mu_i-\mu_j)$, $i,j = 3,4,5$. Note that each R_{ij} acts on M_{12} as rotations by an angle $2\pi(1-\mu_i-\mu_j)$.

To justify that the two covers above are isomorphic, we give an explicit isomorphism between $\pi_1(\widetilde{C})$ and $\pi_1(\widetilde{Q}')$. Pick a tubular neighborhood V of C, and write $U = V \cap Q$. If $\widetilde{U}$ is a component of the inverse image of U in $\widetilde{Q}$, the corresponding component $\widetilde{V}$ of $\pi^{-1}(V)$ is the completion of $\widetilde{U}$ over V (using Lemma 3.2 again). We choose the component $\widetilde{U}$ so that $\widetilde{C}$ is contained in $\widetilde{V}$.

Since U fibers over $C \simeq \mathbb{P}^1 - \{3 \text{ pts}\}$, we have an exact sequence

$$1 \to <\gamma_{12}> \to \pi_1(U) \to \pi_1(\mathbb{P}^1 - \{3\text{pts}\}) \to 1$$

and $\pi_1(\widetilde{U}) \simeq K \cap \pi_1(U)$. Note that in what follows we choose a basepoint for Q inside U, and take representatives for the loops γ_{ij} that are contained in U.

The important point is that when we complete $\widetilde{U}$ to $\widetilde{V}$, it retracts to $\widetilde{C}$, so that

$$\pi_1(\widetilde{C}) \simeq \pi_1(\widetilde{V}) \simeq \pi_1(\widetilde{U}) \; / <\gamma_{12}^{d_{12}}>$$

We claim that this group is the same as $\pi_1(\widetilde{Q}') = K'$. To check this, we consider the restriction to $K \cap \pi_1(U)$ of the map $\pi_1(Q) \to \pi_1(Q')$ induced by any one of the forgetful maps corresponding to forgetting x_1 or x_2. It just maps γ_{12} to 1, and the other three γ_{ij} to natural generators γ'_{ij} of $\pi_1(Q')$. Note that $K \cap \pi_1(U)$ maps into K'. Indeed, if some product $\gamma = \prod \gamma_{ij}$ is in K, then $\rho(\gamma) = 1$ acts on M_{12} as $\prod \gamma'_{ij}$, hence $\prod \gamma'_{ij}$ is in K'.

We claim that the kernel of the map

$$K \cap \pi_1(U) \to K' \tag{8.1}$$

is just the cyclic subgroup generated by $\gamma_{12}^{d_{12}}$. Indeed, if $\prod \gamma_{ij} \in K$ is such that $\prod \gamma'_{ij} = 1$, then $\prod \gamma_{ij}$ must be a product of conjugates of $\gamma_{35}\gamma_{45}\gamma_{34}$, which is equal to γ_{12}.

One still needs to check that the map (8.1) is onto. Given any $\prod \gamma'_{ij} \in K'$, the image of $\prod \gamma_{ij}$ in the monodromy group Γ fixes the mirror M_{12}, which means that it is in K possibly after changing it by a power of γ_{12}, as follows from the lemma below. □

Lemma 8.3. *Suppose $g \in \Gamma$ fixes M_{12}. Then g is a power of $\rho(\gamma_{12})$.*

Proof. From equivariance, we know that g preserves some component $\widetilde{D}_{12}$ of $\pi^{-1}(D_{12})$ (where $\pi : \widetilde{X} \to X$ denotes the natural projection map). It also preserves the components of $\pi^{-1}(D_{34})$, $\pi^{-1}(D_{35})$ and $\pi^{-1}(D_{45})$ that meet D_{12}, hence it must fix their intersection points. This shows that g actually fixes D_{12}. Recall here that Γ acts *discretely* on $\widetilde{X}$, hence g must be of finite order. From the local description of the map $\widetilde{X} \to X$, we know that it must be of order dividing d_{12} (the denominator of the reduced fraction $1 - \mu_1 - \mu_2$). □

Remark 8.4. 1. One should be aware that the stabilizer $\mathrm{Stab}_\Gamma(M_{12})$ of M_{12} in Γ is slightly larger than Γ'. More precisely, it surjects onto Γ', with kernel the finite cyclic subgroup generated by $R_{12} = \rho(\gamma_{12})$.

$$1 \to \langle R_{12} \rangle \to \mathrm{Stab}_\Gamma(M_{12}) \to \Gamma' \to 1$$

2. The proof shows a little more, namely that the restriction of the hypergeometric map $\widetilde{w}|_{\widetilde{D}_{ij}}$ is just the hypergeometric map in dimension one corresponding to the contracted 4-tuple μ'. In particular, if μ' satisfies the Picard integrality condition, the divisor $\widetilde{D}_{ij}$ gets mapped isomorphically onto a one-dimensional subball in $\mathbb{B}^2$.

Note that $\widetilde{X}'$ actually sits in the negatively curved manifold $\widetilde{X}$ as a *totally geodesic* divisor (it is the fixed point set of an isometry), hence we know that its fundamental group injects

$$\pi_1(\widetilde{X}') \simeq K'/N' \hookrightarrow K/N \simeq \pi(\widetilde{X})$$

as can easily be seen using the uniqueness of closed geodesics in a given homotopy class. In the previous section we have discussed how to construct examples of one-dimensional situations where K'/N' is non-trivial. Hence we can get two-dimensional situations where K/N is non-trivial. This will be discussed in detail in the next section.

9. Fundamental group

We recall the general hypergeometric picture corresponding to an $(n+3)$-tuple μ, satisfying our standing hypothesis $\sum_j \mu_j = 2$, $0 < \mu_j < 1$. We get diagram (2.4) again. The cover $\widetilde{Q} \to Q$ is unbranched and has deck group $\pi_1(Q)/K \simeq \Gamma$. The map $\widetilde{X} \to X$ is given by the Fox completion of $\widetilde{Q}$ over the appropriate compactification X of Q. The action of Γ on $\widetilde{Q}$ extends to an action on $\widetilde{X}$, and the quotient ${}_{\Gamma}\backslash\widetilde{X}$ is topologically just X.

The completion $\widetilde{X}$ turns out to be a manifold only for $n \leq 3$, under some quite restrictive assumptions on the weights μ_j. In this paper we consider only the cases $n = 1$ and $n = 2$.

The action of Γ has fixed points on $\widetilde{X}$. Instead of considering $X = {}_{\Gamma}\backslash\widetilde{X}$ as an orbifold, we choose a torsion free subgroup of finite index $\Gamma_0 \subset \Gamma$ and look at the manifold quotient $X_0 = {}_{\Gamma_0}\backslash\widetilde{X}$. The long exact sequence of homotopy groups of the cover $\widetilde{X} \to X_0$ gives

$$1 \to \pi_1(\widetilde{X}) \to \pi_1(X_0) \to \Gamma_0 \to 1 \tag{9.1}$$

Since $\Gamma_0 \subset \Gamma \subset PU(n,1)$, we get a representation

$$\pi_1(X_0) \to PU(n,1) \tag{9.2}$$

which is faithful if and only if $\pi_1(\widetilde{X})$ is trivial.

Note that in all Picard situations (i.e. $(1-\mu_i-\mu_j)^{-1} \in \mathbb{Z}\ \forall\, i,j$), we have an isomorphism $\widetilde{X} \xrightarrow{\simeq} \mathbb{B}^2$ hence we clearly get that $\pi_1(\widetilde{X}) = 1$ and $\pi_1(X_0) \xrightarrow{\simeq} \Gamma_0$. If the integrality condition fails, it is difficult to determine whether or not the representation (9.2) is faithful. In this section we show that in general it need not be, and give an explicit sufficient condition on μ for (9.2) to be non-faithful (see Theorem 9.4). The preceding discussion motivates the following definition.

Definition 9.1. *Let $\mu = (\mu_1, \dots, \mu_r)$, $\mu_j \in \mathbb{Q}$ satisfy $0 < \mu_j < 1$ and $\sum \mu_j = 2$. We call the r-tuple μ* ***non-faithful*** *if the corresponding cover $\widetilde{X}$ is not simply connected. We call it* ***discrete*** *if the corresponding monodromy group Γ_μ is discrete.*

We will make use of these notions only for r=4 or 5. In terms of Definition 9.1, Corollary 7.4 reads as follows.

Theorem 9.2. *Suppose the 4-tuple μ' is Galois conjugate to some discrete 4-tuple that does not satisfy the Picard integrality condition. Then μ' is non-faithful.*

Remark 9.3. If μ' is Galois conjugate to a discrete 4-tuple that *does* satisfy the Picard integrality condition, we know that it is faithful. If none of the Galois conjugates μ'^{σ} is discrete, then we do not know how to determine whether μ' is faithful or not.

In principle, one can list all 4-tuples satisfying the hypotheses of Theorem 9.2, since all the discrete monodromy groups are listed in Theorem 7.1. It is not entirely obvious though how to write general formulas for the 4-tuples corresponding to their Galois conjugates (going from μ' to its Galois conjugates involves reducing fractions and taking fractional parts). The following gives a *sufficient* condition for a 5-tuple to be non-faithful.

Theorem 9.4. *Suppose the 5-tuple μ contracts to a non-faithful 4-tuple. Then μ is non-faithful.*

Proof. This was already stated in the end section 8. The point is that we have proved in Theorem 8.2 that the hypergeometric cover $\widetilde{X}'$ corresponding to the contracted 4-tuple μ' embeds in $\widetilde{X}$ as a totally geodesic divisor. This yields an injection $\pi_1(\widetilde{X}') \hookrightarrow \pi_1(\widetilde{X})$ on the level of fundamental groups since $\widetilde{X}$ has negative curvature. The hypothesis that μ' is non-faithful implies that $\pi_1(\widetilde{X}') \neq 1$, hence $\pi_1(\widetilde{X}) \neq 1$ as well. □

We give in section 11 the list of all 5-tuples with denominator up to 200 that contract to some 4-tuple that we know to be non-faithful. In other words we give a list of 5-tuples μ for which Theorems 9.2 and 9.4 apply to show that μ is non-faithful.

Staring at the list of examples given in Table 11.1, it is relatively easy to find infinite families of non-faithful examples. For instance consider the 5-tuples

$$\mu = \frac{1}{6+18k}(2+3k, 4+6k, 2+9k, 2+9k, 2+9k) \qquad (9.3)$$

One can check that contracting two equal weights gives a non-faithful 4-tuple μ'. Recall that this means that one of the Galois conjugates μ'^{σ} is discrete, which can be verified by using Proposition 4.2 and Theorem 7.1. The behavior of μ' depends on the parity of the parameter k, and we shall go into the details of the argument only for k even. The analysis is similar for k odd.

We write $k = 2n$, so that μ' becomes

$$\frac{1}{3+18n}(1+3n, 2+6n, 1+9n, 2+18n)$$

It corresponds to a triangle with angles $\frac{3n\pi}{1+6n}$, $\frac{n\pi}{1+6n}$ and $\frac{\pi}{3}$, hence μ is not discrete. After multiplying μ' by $(8+12n)$ and reducing in $\mathbb{Q}/\mathbb{Z}$, we get a Galois conjugate 4-tuple that corresponds to angles $\frac{3\pi}{1+6n}$, $\frac{\pi}{1+6n}$ and $\frac{\pi}{3}$, which fits into case (iv) of Theorem 7.1.

When k is odd, μ' can be shown to be non-faithful by showing that the two groups corresponding to angles $\frac{3k\pi}{2+6k}$, $\frac{k\pi}{2+6k}$, $\frac{\pi}{3}$ and $\frac{3\pi}{2+6k}$, $\frac{\pi}{2+6k}$, $\frac{\pi}{3}$ are Galois conjugates. We will describe two more families explicitly in the next section.

Remark 9.5. 1. The theorems of this section produce examples of non-faithful 5-tuples. It is natural to ask whether there are non obvious faithful 5-tuples. The obvious ones are the Picard examples (it turns out there are only finitely many such). Interestingly enough, we do not know of any other faithful example. A natural approach would be to consider Galois conjugates of the Picard examples. Unfortunately their Galois conjugates never satisfy our assumptions that the compactification divisors have normal crossings.

2. As the preceding discussion illustrates, the fundamental group $\pi_1(X_0)$ is quite a complicated object. It is an extension

$$1 \to K/N \simeq \pi_1(\widetilde{X}) \to \pi_1(X_0) \to \Gamma_0 \to 1$$

and it already takes some work to determine just if K/N is trivial or not. In the cases where we know the representation to be non-faithful (namely when Theorem 9.4 applies), K/N seems not to be finitely generated. A natural question then comes to mind – is $\pi_1(X_0)$ residually finite?

10. Bounded holomorphic functions and maps to Riemann surfaces

We recall some notations. $X - Q$ consists of 10 divisors D_{ij} corresponding to x_i and x_j coming together. We pick loops γ_{ij} around D_{ij}, and write $1-\mu_i-\mu_j = \frac{n_{ij}}{d_{ij}}$ as a reduced fraction. Then the loops $\gamma_{ij}^{d_{ij}}$ are in the kernel K of the monodromy, and we write

$$N = \langle\langle \gamma_{ij}^{d_{ij}}, i,j \in \{1,\dots,5\} \rangle\rangle \subset K$$

for the normal subgroup of $\pi_1(Q)$ generated by the loops $\gamma_{ij}^{d_{ij}}$. In what follows we assume that $N \subsetneq K$ (we presented sufficient conditions for this to happen in section 9). We then get a non-trivial cover $\widehat{Q} \to \widetilde{Q}$, where $\widehat{Q}$ is the cover of Q such that $\pi_1(\widehat{Q}) = N$. We consider the corresponding Fox completion $\widehat{X} \to \widetilde{X}$. $\widehat{X}$ is the completion of $\widehat{Q}$ over

$\widetilde{X}$, or equivalently over X. One checks at once that the space $\widehat{X}$ is simply connected. Indeed, since the completion divisors have complex codimension one in $\widehat{X}$, $\pi_1(\widehat{Q})$ surjects onto $\pi_1(\widehat{X})$, the kernel being given by the normal subgroup generated by small loops around the components of $\widehat{X} - \widehat{Q}$, which is precisely N.

The map $\widehat{X} \to \widetilde{X}$ is an unbranched cover. One way to see this is to observe that $K/N = \pi_1(\widetilde{X})$, which is the deck group of the cover $\widehat{Q} \to \widetilde{Q}$, must be torsion free since $\widetilde{X}$ has negative curvature. Another approach is to look at the local structure of the cover $\widehat{X} \to X$, which is the same as that of the cover $\widetilde{X} \to X$.

Now $\widehat{X}$ is the universal cover of X_0, and we want to construct bounded holomorphic functions on $\widehat{X}$. There is an obvious way of getting such functions, using our hypergeometric map $\widetilde{X} \to \mathbb{B}^2$, precomposing it with the projection $\widehat{X} \to \widetilde{X}$ and post-composing with any bounded holomorphic function on $\mathbb{B}^2$. These were of course already known when the Mostow-Siu surfaces were first constructed.

We give another construction that produces bounded holomorphic functions on $\widehat{X}$, and show that in certain cases the functions we get do not factor through the corresponding hypergeometric map. The basis for our construction lies in the use of forgetful maps $Q \to Q'$, where Q (resp. Q') is the configuration space of five points (resp. four points) on $\mathbb{P}^1$. In terms of the appropriate description of Q as a subset of $\mathbb{P}^1 \times \mathbb{P}^1$ (see section 2), these forgetful maps are just the projections onto one of the factors.

These maps extend to maps $X \to X'$ between the compactifications, but in general these extensions are not maps of orbifolds. In terms of covers, $Q \to Q'$ does not lift to a map $\widehat{Q} \to \widehat{Q}'$ in general. In other words, the induced homomorphism $\pi_1(Q) \to \pi_1(Q')$ does not in general map N into N'. We point out that when the map *is* a map of orbifolds, we can lift it to a holomorphic map from some complex surface to a compact Riemann surface. Lifting it further to the universal cover of the complex surface yields a bounded holomorphic function. We shall come back to maps to Riemann surfaces later on in this section (see Proposition 10.4). For now we concentrate on the description of a simple necessary and sufficient condition for N to map into N'.

We write γ'_{ij} for the image of γ_{ij}. Note that some γ'_{ij} are trivial. To fix the ideas, we assume that we forget the first point, and project onto the second factor, as in Figure 10.1. Then $\gamma'_{1j} = 1$ for all j, and $\gamma'_{34} = \gamma'_{25}$, $\gamma'_{35} = \gamma'_{24}$ and $\gamma'_{45} = \gamma'_{23}$ are standard generators for $\pi_1(Q')$. N is the normal subgroup generated by $\gamma_{ij}^{d_{ij}}$. In order for these loops to map into N', we need

$$d_{34} \mid d_{25}, \quad d_{35} \mid d_{24}, \quad d_{45} \mid d_{23} \tag{10.1}$$

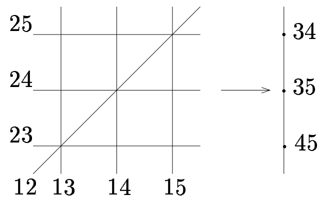

Fig. 10.1. A forgetful map $Q \to Q'$.

and these conditions are clearly also sufficient to get $N \to N'$.

In general, for other forgetful maps, we also get a necessary and sufficient condition for N to map into N' in terms of three divisibility conditions, applying the appropriate permutation of indices in (10.1).

We assume some choice of a forgetful map $Q \to Q'$ does lift to $\widehat{Q} \to \widehat{Q}'$, or in other words that N maps into N', or yet in other words that condition (10.1) is satisfied for some permutation of the indices. This lift has a holomorphic extension $\widehat{\phi} : \widehat{X} \to \widehat{X}' \simeq \mathbb{B}^1 \simeq \mathbb{R}H^2$, since $\widehat{X} - \widehat{Q}$ has codimension one in $\widehat{X}$. Hence $\widehat{\phi}$ is a bounded holomorphic function on the universal cover of our Mostow-Siu surface. Summarizing our discussion, we get

Theorem 10.1. *The natural map to D_{ij} obtained by forgetting x_i lifts to a bounded holomorphic function $\widehat{\phi} : \widehat{X} \to \widehat{X}'$ if and only if N maps into N', which is equivalent to the three divisibility conditions*

$$d_{kl} \mid d_{jm}, \quad d_{km} \mid d_{jl}, \quad d_{lm} \mid d_{jk} \tag{10.2}$$

where $\{1, \dots, 5\} = \{i, j\} \cup \{k, l, m\}$.

The divisibility condition (10.2) is easy to test on any given 5-tuple of weights and should give an efficient way to construct bounded holomorphic functions on the universal cover $\widehat{X}$ of X_0, but it is not at all clear that there should be any examples where it is satisfied. Recall that we require that the compactification divisors have normal crossings and that the ramification divisors are disjoint. These requirements can be translated into concrete conditions on the weights μ_j, namely (2.3) and (5.1). In order to apply Theorem 10.1 we need μ to admit a contraction to some 4-tuple satisfying (10.2). All these numerical conditions taken together seem very restrictive, but it turns out that one can produce examples where Theorem 10.1 applies. We will exhibit two infinite families of such examples at the end of this section.

Another possible objection to the relevance of Theorem 10.1 is that the bounded holomorphic functions it produces might a priori be of "hypergeometric origin." In other words, they could conceivably come from $\widetilde{w} : \widetilde{X} \to \mathbb{B}^2$. If this were the case, then $\widehat{\phi}$ would be constant on the fibers of the projection $\widehat{X} \to \widetilde{X}$.

Theorem 10.2. *Assume that the 5-tuple μ is such that some forgetful map to D_{ij} lifts to a bounded holomorphic function and that the corresponding contracted 4-tuple μ', obtained from contracting μ_i and μ_j, is non-faithful. Then $\widehat{\phi}$ does not factor through the corresponding hypergeometric map.*

Proof. All we need to show is that K/N does not map trivially under $\pi_1(Q)/N \to \pi_1(Q')/N'$. Here we recall that K/N is the deck group of the unbranched cover $\widehat{X} \to \widetilde{X}$. Our assumption that μ' be non-faithful ensures that K'/N' is non-trivial. We distinguish two cases.
CASE 1: $K \not\to K'$

Then clearly K/N does not map trivially, and $\widehat{\phi}$ is not constant on the fibers of $\widehat{X} \to \widetilde{X}$.
CASE 2: $K \to K'$

Then K/N maps into K'/N', and we want this homomorphism to be non-trivial. Actually the map is onto, and we have assumed that $K'/N' \neq 1$ (μ' is non-faithful). The fact that it is onto follows from the arguments given in the proof of Theorem 8.2. There we showed that there is a subgroup of K that maps onto K'. □

Remark 10.3. 1. One difficulty here is that in general, we do not know how to check whether K maps into K'. An obvious necessary condition is that N map into N', but it is not clear whether that condition is also sufficient. In terms of covers, we do not know when the forgetful maps lift to maps $\widetilde{\phi} : \widetilde{X} \to \widetilde{X}'$ between the hypergeometric covers.

2. Our assumption that K'/N' be non-faithful is sufficient, but most likely not necessary to produce holomorphic functions that do not factor through the hypergeometric map. If $\widehat{\phi}$ happens to descend to $\widetilde{X}$, it might still not factor through $\mathbb{B}^2$. Note that, as we shall discuss in more detail below, our bounded holomorphic functions have the very special property that they descend to maps to Riemann surfaces.
3. A natural question is of course whether our bounded holomorphic functions allow one to separate points in $\widehat{X}$. In fact it is not difficult to see that, in general, these functions (together with the ones coming from the hypergeometric map) cannot separate points.
4. One way to generalize our construction and obtain more bounded holomorphic functions would be to map X to any orbifold $\mathbb{P}^1$ with three orbifold points, with weights satisfying some divisibility condition. We would then get a map $\widehat{X} \to \mathbb{B}^1$ but since the weights are unrelated to the original hypergeometric situation, it is not clear how to check which points of $\widehat{X}$ it separates.

Recall that each of our new bounded holomorphic function is obtained as the lift of map of orbifolds $\phi : X \to X'$. In terms of manifolds, we may think of it as the lift of a map

$$\phi_0 : {}_{G_0}\backslash\widehat{X} \to {}_{G'_0}\backslash\widehat{X}' \tag{10.3}$$

Here we write G for $\pi_1(Q)/N$, and similarly $G' = \pi_1(Q')/N'$. G_0 (resp. G'_0) stands for a torsion free subgroup of finite index of G (resp. G'). Recall that $\phi : X \to X'$ is a map of orbifolds exactly when N maps into N', or in other words when $\pi_1(Q) \to \pi_1(Q')$ descends to a map $G \to G'$.

We can think of our bounded holomorphic functions as lifts of maps from Mostow-Siu type surfaces to compact Riemann surfaces. Observe that in general it is difficult to produce non-trivial maps to lower-dimensional manifolds.

Proposition 10.4. *If one of the forgetful maps $\phi : X \to X'$ is a map of orbifolds, then it lifts to a holomorphic map $\phi_0 : {}_{G_0}\backslash\widehat{X} \to {}_{G'_0}\backslash\widehat{X}'$ to a compact Riemann surface, whose lift to the universal covers is the bounded holomorphic function $\widehat{\phi} : \widehat{X} \to \widehat{X}'$.*

Remark 10.5. In general we cannot choose the manifold ${}_{G_0}\backslash\widehat{X}$ to be a quotient of the monodromy cover $\widetilde{X}$. This would be the case if we could guarantee that G_0 is saturated with respect to the map $G \to \Gamma$, or in other words that G_0 contains K/N. This can certainly be arranged when K maps into K' (but this condition is difficult to check, as we stated in Remark 10.3). On the other hand, the two surfaces ${}_{G_0}\backslash\widehat{X}$ and ${}_{\Gamma_0}\backslash\widetilde{X}$ are commensurable, hence we consider them both as Mostow-Siu type surfaces.

We now go back to the quite restrictive hypotheses we need to make on our 5-tuples μ in order to get new bounded holomorphic functions. Once again, it is not at all clear that there should be situations where Theorem 10.2 applies. A quick look at the list of examples given in section 11 shows that there are examples, namely every time $N+$, $N-$ or $N\pm$ appears in some column D_{ij}, the hypotheses of the theorem are satisfied. N means that the corresponding contracted 4-tuple μ' is non-faithful, $+$ or $-$ indicates that one of the two natural forgetful maps to D_{ij} lifts to a bounded holomorphic function on the universal cover $\widehat{X}$.

In fact, one can construct two infinite families of examples where Theorem 10.2 applies, parameterized by an integer $k \geq 0$:

$$\mu = \frac{1}{20+16k}(6+4k, 6+4k, 9+8k, 9+8k, 10+8k) \tag{10.4}$$

$$\mu = \frac{1}{18+12k}(5+4k, 7+4k, 7+4k, 7+4k, 10+8k) \tag{10.5}$$

It is readily checked that these 5-tuples always satisfy our standing assumptions (the compactification divisors have normal crossings and the branching divisors do not intersect). We analyze the family (10.4) in some detail.

The first observation is that we always get an orbifold map to D_{45} by forgetting the point x_4 (of course since $\mu_3 = \mu_4$ we could map to D_{35} as well). This amounts to checking that the three divisibility conditions (10.2) hold. The contracted 4-tuple is given by (10.6).

$$\mu' = \frac{1}{10+8k}(3+2k, 3+2k, 5+4k, 9+8k) \tag{10.6}$$

After multiplication by $1+4k$ (which is prime to the denominator $10+8k$), we get a Galois conjugate that corresponds to a triangle with angles $\frac{2\pi}{5+4k}$, $\frac{\pi}{4}$, $\frac{\pi}{4}$ which is in the list of Theorem 7.1. Once again in order to compute Galois conjugates, we use Proposition 4.2.

The computations for the family (10.5) work essentially the same way, although the behavior depends on the divisibility by 3 of the index k. One gets an orbifold map to the divisor D_{15} by forgetting the first point.

11. List of examples

Table 11.1 lists all possible 5-tuples $\mu = \frac{1}{d}(n_1, n_2, n_3, n_4, n_5)$ with denominators up to 200, such that

1. $0 < \mu_j < 1$ and $\sum \mu_j = 2$ (essentially this says that the monodromy group lies in $Aut(\mathbb{B}^2) \simeq PU(2,1)$).
2. The compactification divisors have normal crossings ($\mu_i + \mu_j < 1$ for all i, j).
3. The branching divisors are disjoint.

and satisfying the hypotheses of Theorems 9.2 and 9.4. This last condition means that some contraction of μ should be Galois conjugate to some discrete non Picard 4-tuple (in terms of the table below, each row contains at least one N). We compute the ratio of Chern classes and, for each divisor D_{ij}, we give the following information on the corresponding contracted 4-tuple μ'.

1. F means μ' is faithful.
2. N means μ' is not faithful.
3. Neither F nor N means that no Galois conjugate of μ' is discrete, in which case we do not know whether μ' is faithful or not.
4. $+$ (resp. $-$) indicates that the natural forgetful map to D_{ij} forgetting i (resp. j) is an orbifold map, or in other words that it lifts to give a bounded holomorphic function on the universal cover of the corresponding surface.

Table 11.1. List of all examples with denominators up to 200 where our theorems in sections 9 and 10 apply. For each example we compute the ratio of Chern classes and give some information on the structure of the completion divisors D_{ij}.

d	n_1	n_2	n_3	n_4	n_5	c_1^2/c_2	D_{12}	D_{13}	D_{14}	D_{15}	D_{23}	D_{24}	D_{25}	D_{34}	D_{35}	D_{45}
18	5	7	7	7	10	2.9412	N	N	N	$N+$	F	F	N	F	N	N
20	6	6	9	9	10	2.9389	F	F	F	F	F	F	F	N	$N-$	$N-$
21	4	8	10	10	10	2.8763	F	F	F	F	F	F	F	N	N	N
24	5	10	10	11	12	2.8065	F	F	F	F			$N\pm$		$N\pm$	F
24	5	10	11	11	11	2.9032	F	F	F	F	F	F	F	N	N	N
30	6	13	13	14	14	2.7622	$F+$	$F+$	$N+$	$N+$	F					
30	9	11	11	11	18	2.8421				$N+$	F	F	N	F	N	N
30	10	11	11	14	14	2.8667	$F+$	$F+$	$N+$	$N+$	F					N
30	11	11	11	13	14	2.8657	F	F	N		F	N		N		N
36	10	10	17	17	18	2.8391	F			F			F	N	$N-$	$N-$
39	7	14	19	19	19	2.7946	F				F	F	F	N	N	N
42	13	15	15	15	26	2.9072	N	N	N	$N+$	F	F	$N+$	F	$N+$	$N+$
52	14	14	25	25	26	2.7890	F			F			F	N	$N-$	$N-$
54	17	19	19	19	34	2.7656				$N+$	F	F	N	F	N	N
57	10	20	28	28	28	2.7567	F				F	F	F	N	N	N
60	11	22	29	29	29	2.7747	F				F	F	F	N	N	N
66	21	23	23	23	42	2.7477				$N+$	F	F	N	F	N	N
68	18	18	33	33	34	2.7610	F			F			F	N	$N-$	$N-$
75	13	26	37	37	37	2.7359	F				F	F	F	N	N	N
78	25	27	27	27	50	2.8020				$N+$	F	F	$N+$	F	$N+$	$N+$
84	22	22	41	41	42	2.7434	F			F			F	N	$N-$	$N-$
90	29	31	31	31	58	2.7261				$N+$	F	F	N	F	N	N
93	16	32	46	46	46	2.7228	F				F	F	F	N	N	N
96	17	34	47	47	47	2.7352	F				F	F	F	N	N	N
100	26	26	49	49	50	2.7313	F			F			F	N	$N-$	$N-$
102	33	35	35	35	66	2.7191				$N+$	F	F	N	F	N	N
111	19	38	55	55	55	2.7139	F				F	F	F	N	N	N
114	37	39	39	39	74	2.7599				$N+$	F	F	$N+$	F	$N+$	$N+$
116	30	30	57	57	58	2.7225	F			F			F	N	$N-$	$N-$
126	41	43	43	43	82	2.7091				$N+$	F	F	N	F	N	N
129	22	44	64	64	64	2.7074	F				F	F	F	N	N	N
132	23	46	65	65	65	2.7167	F				F	F	F	N	N	N
132	34	34	65	65	66	2.7158	F			F			F	N	$N-$	$N-$
138	45	47	47	47	90	2.7054				$N+$	F	F	N	F	N	N
147	25	50	73	73	73	2.7025	F				F	F	F	N	N	N
148	38	38	73	73	74	2.7105	F			F			F	N	$N-$	$N-$
150	49	51	51	51	98	2.7377				$N+$	F	F	$N+$	F	$N+$	$N+$
162	53	55	55	55	106	2.6997				$N+$	F	F	N	F	N	N
164	42	42	81	81	82	2.7062	F			F			F	N	$N-$	$N-$
165	28	56	82	82	82	2.6986	F				F	F	F	N	N	N
168	29	58	83	83	83	2.7061	F				F	F	F	N	N	N
174	57	59	59	59	114	2.6974				$N+$	F	F	N	F	N	N
180	46	46	89	89	90	2.7028	F			F			F	N	$N-$	$N-$
183	31	62	91	91	91	2.6955	F				F	F	F	N	N	N
186	61	63	63	63	122	2.7240				$N+$	F	F	$N+$	F	$N+$	$N+$
196	50	50	97	97	98	2.6998	F			F			F	N	$N-$	$N-$
198	65	67	67	67	130	2.6937				$N+$	F	F	N	F	N	N

References

[DM] Deligne P., Mostow G. D., Monodromy of hypergeometric functions and non-lattice integral monodromy. *Inst. Hautes Études Sci. Publ. Math.*, 63 (1986), 5–89.

[D] Deraux M., *Complex surfaces of negative curvature*, Ph.D. thesis, University of Utah, 2001.

[F] Fox R. H., Covering spaces with singularities, in *A symposium in honor of S. Lefschetz*, pp. 243–257. Princeton University Press, Princeton, N.J., 1957.

[M1] Mostow G. D., On a remarkable class of polyhedra in complex hyperbolic space. *Pacific J. Math.*, 86 (1980), 171–276.

[M2] Mostow G. D., Generalized Picard lattices arising from half-integral conditions. *Inst. Hautes Études Sci. Publ. Math.*, 63 (1986), 91–106.

[M3] Mostow G. D., Braids, hypergeometric functions, and lattices. *Bull. Amer. Math. Soc.*, 16 (1987), 225–246.

[M4] Mostow G. D., On discontinuous action of monodromy groups on the complex n-ball. *J. Amer. Math. Soc.*, 1 (1988), 555–586.

[MS] Mostow, G. D., Siu, Y. T., A compact Kähler surface of negative curvature not covered by the ball. *Ann. of Math.*, 112 (1980), 321–360.

[Z] Zheng F., Hirzebruch-Kato surfaces, Deligne-Mostow's construction, and new examples of negatively curved compact Kähler surfaces. *Comm. Anal. Geom.*, 7 (1999), 755–786.