One-cusped complex hyperbolic 2-manifolds

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Abstract

This paper builds one-cusped complex hyperbolic 2-manifolds by an explicit geometric construction. Specifically, for each odd $d \geq 1$ there is a smooth projective surface Z_d with $c_1^2(Z_d) = c_2(Z_d) = 6d$ and a smooth irreducible curve E_d on Z_d of genus one so that $Z_d \smallsetminus E_d$ admits a finite volume uniformization by the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$. This produces one-cusped complex hyperbolic 2-manifolds of arbitrarily large volume. As a consequence, the 3-dimensional nilmanifold of Euler number $12d$ bounds geometrically for all odd $d \geq 1$.

1 Introduction

This introduction describes the main result along with some history and context in §1.1, a more precise result in §1.2, and an application to geometric bounding in §1.3.

1.1 The main result, history, and context

The purpose of this paper is to provide the first geometric construction of complete one-cusped complex hyperbolic 2-manifolds of finite volume. The basic main result is as follows.

Theorem 1.1. *For each odd $d \geq 1$, there is a one-cusped complex hyperbolic 2-manifold of volume $16\pi^2 d$.*

The existence of one-cusped manifolds was observed by the first author using Magma experimentation [4, Thm. 1.4], and a number of geometric properties were also recorded there, including the structure of the cusp cross-section. This paper gives an explicit, completely disjoint, computer-independent construction[1]. There is strong computational evidence suggesting that the examples in [4] are precisely those constructed here, but this paper does not address that question.

There is significant interest in existence of special metrics (e.g., Kähler–Einstein metrics) on a smooth projective variety V, or more generally metrics on $V \smallsetminus D$ for a given divisor D; see for example [29, 28]. Within this context, Theorem 1.1 produces the first concrete example of a pair (V, D) where $\dim(V) > 1$, D is smooth and irreducible, and $V \smallsetminus D$ admits a complete Kähler metric of constant biholomorphic sectional curvature -1 and finite volume. However, the primary interest in one-cusped complex hyperbolic manifolds stems from the fact that one-cusped locally symmetric manifolds of negative curvature are exceptional, rare, and elusive in (real) dimension greater than three.

With deep geometrization results in hand it becomes easy to construct 2- and 3-dimensional manifolds that admit complete, finite volume metrics of constant curvature -1 with exactly one cusp. Indeed, uniformization implies that any once-punctured Riemann surface of genus $g \geq 1$ admits such a metric, and Thurston's hyperbolization theorem [27, Thm. 2.3] leads to a plethora of examples (e.g., many knot complements in the 3-sphere). However, shockingly little is known in higher dimensions, especially in the more generally setting of locally symmetric manifolds with negative curvature (i.e., rank one). Hyperbolic 4-manifolds with exactly one cusp were first constructed by Kolpakov and Martelli [13, Thm. 1.1], and see [25, 14, 15, 22] for more examples. This paper and [4, Thm. 1.4] provide a fourth known symmetric space of negative curvature admitting a one-cusped manifold quotient of finite volume; still no example of any kind is known in dimension greater than four.

For some perspective on why one-cusped manifolds are rare and special, the second author proved that the theory of arithmetic subgroups of algebraic groups is not as useful for producing one-cusped examples as one might think or hope. More specifically, for each $k \geq 1$ the rank one arithmetic locally symmetric spaces with k cusps fall into finitely many commensurability classes over all possible universal covers and dimensions [26, Thm. 1.1]. In particular, there is a dimension $d(n)$ so that n-cusped arithmetic locally symmetric manifolds of negative curvature cannot exist above dimension $d(n)$. For example, 1-cusped arithmetic hyperbolic manifolds (in fact, orb-

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[1] While some computations are more easily done with the aid of computer algebra software, they can in principle all be done by hand.

ifolds) cannot exist above dimension 30 [26, Thm. 1.3]. Thus it could well be the case that there are no one-cusped rank one locally symmetric spaces in sufficiently high dimensions.

Remark 1.2. The minimal possible volume of a complex hyperbolic 2-manifold is $8\pi^2/3$. Work of Kamishima [10] asserts that if M is a one-cusped finite volume complex hyperbolic 2-manifold, then its cusp cross section must have trivial holonomy, which implies that M admits a smooth toroidal compactification; see §1.3 for further discussion. L. Di Cerbo and the second author proved that there is no minimal volume smooth toroidal compactification with one cusp [6, Thm. 1.1], so one-cusped manifolds must have volume at least $16\pi^2/3$. On the other hand, Di Cerbo and the second author also proved that there is a two-cusped complex hyperbolic manifold realizing every possible volume [5, Thm. 1.4].

1.2 The main technical result

Theorem 1.1 is a direct consequence of the following more precise result.

Theorem 1.3. *For each odd $d \geq 1$ there is a minimal smooth projective surface Z_d of general type with $c_1^2(Z) = c_2(Z) = 6d$ and a smooth irreducible curve E_d on Z_d of genus one with self-intersection $-12d$ so that $Z_d \smallsetminus E_d$ is uniformized by the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$.*

The pair (Z_1, E_1) is constructed in §3 and the proof of Theorem 1.3, which immediately implies Theorem 1.1 by Chern–Gauss–Bonnet, is the content of §4. The initial example Z_1 is one of the desingularized product-quotient surfaces with $p_g = q = 1$ studied by Polizzi [21]. Briefly, there is a product $X = C_1 \times C_2$ of hyperbolic Riemann surfaces and a finite group F acting on X so that Z_1 is the minimal desingularization of X/F. The other examples are built using a covering construction.

Remark 1.4. As in the case of Riemann surfaces, the proof that (Z_1, E_1) is a ball quotient applies a uniformization theorem, here due to Kobayashi [12, Thm. 2] (also see [28, Thm. 3.1]). A key difference makes the higher-dimensional case much more subtle and special. A Riemann surface can be uniformized by a constant curvature metric if and only if a characteristic class (namely the Euler characteristic) has the appropriate sign. In higher dimensions, uniformization requires equality between Chern numbers, not merely an inequality, which is a significantly more stringent requirement.

1.3 Geometric bounding

One application of the construction is to *geometric bounding.* A famous theorem of Rohlin states that all closed, connected, orientable 3-manifolds are diffeomorphic to the boundary of a compact 4-manifold [23]. For a 3-manifold admitting one of the eight geometries [24, §4], it is then of interest to know whether it can be realized as the boundary of a geometric 4-manifold, and now there are obstructions, say from index theory. For example, Long and Reid studied when flat or hyperbolic manifolds can geometrically bound a hyperbolic manifold [17]. For flat manifolds, this means realizing the manifold as the cusp cross-section of a one-cusped hyperbolic manifold of one dimension higher, whereas in the hyperbolic case it means realizing the manifold as a totally geodesic boundary.

For infranil 3-manifolds (i.e., those with Nil geometry), the relevant question is whether or not it can be realized as the cusp-cross section of a complex hyperbolic manifold. Restrictions on which manifolds can bound were given by unpublished work of Walter Neumann and Alan Reid, and work of Kamishima [10] indicates that a manifold that bounds must have trivial holonomy. In other words, the cross-section should be a nilmanifold. In the language of toroidal compactifications, this is equivalent to saying that the complex hyperbolic manifold admits a *smooth* toroidal compactification by an elliptic curve of self-intersection $-d$, where $d \geq 1$ and d is the Euler number of the nilmanifold (as defined on [24, p. 435]). In the smooth toroidal case, Corollary 2.2 below shows that the Euler number is moreover divisible by four. The examples in this paper cover many, but not all, remaining possibilities.

Theorem 1.5. *For every odd $d \geq 1$, the 3-dimensional nilmanifold with Euler number $12d$ geometrically bounds a complex hyperbolic manifold.*

As in the flat case [18], every infranil 3-manifold is the cusp cross-section of some complex hyperbolic 2-manifold, possibly with many cusps [19, 20]. It would be interesting if every other nilmanifold with Euler number divisible by four can be realized as the cusp cross-section of a one-cusped manifold. It may even be the case that examples can be found that are commensurable with the examples in this paper.

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2 Background

This paper assumes basic familiarity with complex hyperbolic manifolds and the topology of smooth projective varieties. See for example [9, 1] for basic references on what follows.

Let $\mathbb{B}^2$ be complex hyperbolic 2-space, that is the unit ball in $\mathbb{C}^2$ with its metric of constant holomorphic sectional curvature -1. A complex hyperbolic manifold is a quotient $M = \mathbb{B}^2/\Gamma$ of $\mathbb{B}^2$ by a torsion-free discrete group of holomorphic isometries. If M has finite volume but is not compact, then it has a finite number of cusps, each diffeomorphic to the product of an infranil 3-manifold N with $[0, \infty)$, and N is called a *cusp cross-section* of M. See [3, §2.2] for the definition of an infranil manifold.

When a cusp cross-section is a nilmanifold (i.e., has *trivial holonomy*), the cusp is naturally diffeomorphic to a torus bundle over a punctured disk. Smoothly filling in the puncture gives a *smooth toroidal compactification* of the cusp. Since it is the case relevant to this paper, suppose that M has one cusp with smooth toroidal compactification X obtained by adding the elliptic curve E. Then (X, E) is called a *ball quotient pair.* In that case, X is in fact a smooth projective variety and E a smooth curve of genus one and self-intersection $-d$, where d is the Euler number of the nil 3-manifold in the cusp cross-section (see [9, §4.2]).

Moreover, Kobayashi [12, Thm. 2] proved a uniformization theorem that determines precisely when a pair (X, E) consisting of a smooth projective variety and a smooth curve on X determine a ball quotient pair. Suppose that X is a smooth projective variety with canonical divisor K_X and E is a smooth curve of genus one on X. Then $X \smallsetminus E$ admits a complete complex hyperbolic metric of finite volume if and only if the divisor $K_X + E$ is nef and big, there are no (-2) curves on X disjoint from E, and

$$(K_X + E)^2 = 3c_2(X), \tag{1}$$

where $c_2(X)$ is the topological Euler characteristic. Here $(K_X + E)^2$ and $c_2(X)$ are the relative Chern numbers $c_1^2(X, E)$ and $c_2(X, E)$ of the pair (X, E), respectively.

Lemma 2.1. *Let (X, E) be a ball quotient pair with E an elliptic curve of self-intersection $-d$. Then d is divisible by four.*

Proof. Suppose (X, E) is a ball quotient pair with E an elliptic curve of self-intersection $-d$. The adjunction formula implies that

$$\begin{aligned}(K_X + E)^2 &= K_X^2 - E^2 \\ &= c_1(X)^2 + d \\ &= 3c_2(X)\end{aligned}$$

and so $d = 3c_2(X) - c_1^2(X)$. However $c_1^2(X) = 12\chi(\mathcal{O}_X) - c_2(X)$ by Noether's formula, so

$$d = 4c_2(X) - 12\chi(\mathcal{O}_X),$$

which is divisible by four. □

Reinterpreting Lemma 2.1 in terms of cusp cross-sections gives the following corollary.

Corollary 2.2. *If M is a one-cusped complex hyperbolic manifold with cusp cross-section a nilmanifold N with Euler number d, then d is divisible by four.*

3 Construction of the first example

This section follows work of Polizzi [21] to construct the crucial first example (Z, E) needed to prove Theorem 1.1, and assumes that the reader is very familiar with the language of Fuchsian groups, for example as in [11]. The notation $\Delta(g; \mathfrak{n})$ will denote the Fuchsian group of signature $(g; \mathfrak{n})$. In other words, the quotient $\mathbb{H}^2/\Delta(g; \mathfrak{n})$ has genus g and a_j cone points of order n_j, where $\mathfrak{n} = n_1^{a_1}, \dots, n_k^{a_k}$ and any n_j^1 is simply given as n_j.

Briefly, the surface Z is the minimal desingularization of the quotient of a product of curves. The curves C_1 and C_2 with action of the alternating group $\mathfrak{A}_4$ are constructed in §3.1. The quotient of $C_1 \times C_2$ by the diagonal action of $\mathfrak{A}_4$ and its resolution Z are studied in §3.2. The elliptic curve E on Z is constructed in §3.3. Finally, the fact that (Z, E) is a ball quotient pair is proved in §3.4.

Remark 3.1. The example constructed in this section is of the kind covered by [21, Prop. 7.3]. However, the precise example presented here is not the one contained in Polizzi's proof. Polizzi is only concerned with showing a given triple consisting of two Fuchsian groups and a finite quotient produces at least one surface, and does not analyze all the possibilities for a given triple. Computer experiment indicates that each triple can in fact produce distinct surfaces that are not even homotopy equivalent.

3.1 The curves of genus four

The construction begins with the $(2,3,12)$ triangle group

$$\Gamma_0 = \left\langle p, q, r \mid p^2,\ q^3,\ r^{12},\ pqr \right\rangle$$

and the elements:

$$\begin{aligned} a &= r^6 & g &= (prq)^{-1} \\ b &= q^2 & h &= (qpr)^{-1} \\ c &= (pr)q^2(pr)^{-1} \\ d &= (pr)^{-1}q^2(pr) \end{aligned}$$

One can directly check that

$$\begin{aligned} \Gamma_1 &= \left\langle a, b, c, d \right\rangle \\ &\cong \left\langle a, b, c, d \mid a^2,\ b^3,\ c^3,\ d^3,\ abcd \right\rangle \\ \Gamma_2 &= \left\langle g, h \right\rangle \\ &\cong \left\langle g, h \mid [g,h]^2 \right\rangle \end{aligned}$$

are both index six subgroups of Γ_0. Fundamental domains for the action of Γ_1 and Γ_2 are shown in Figure 1 as unions of copies of the obvious fundamental domain for the action of the triangle group Γ_0 (i.e., the union of two adjacent triangles, one white and one gray). In terms of the orbifold quotients, there are maps of orbifolds depicted in the top half of Figure 2. In particular, $\Gamma_1 \cong \Delta(0; 2, 3^3)$ and $\Gamma_2 \cong \Delta(1; 2)$.

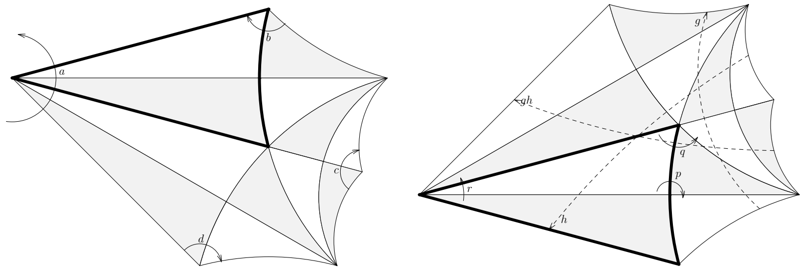

Figure 1: Fundamental domains for Γ_1 (left) and Γ_2 (right), as copies of a fundamental domain for Γ_0 (shown in bold on both pictures)

Now consider the homomorphisms $\rho_j : \Gamma_j \to \mathfrak{A}_4$ induced by

$$\rho_1(a) = (1\,3)(2\,4) \qquad\qquad \rho_2(g) = (1\,2\,3)$$

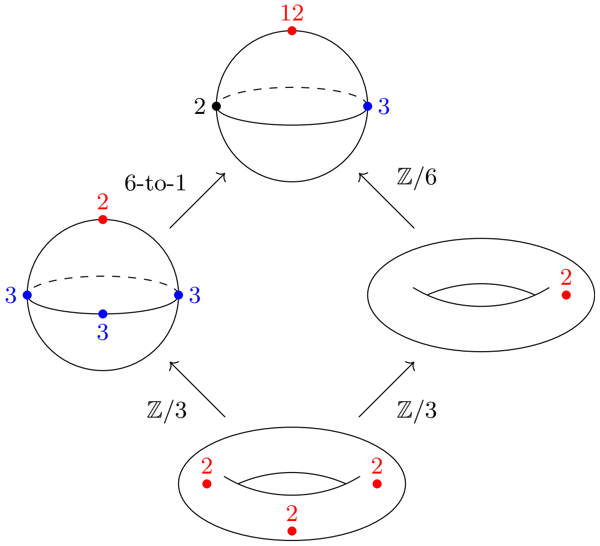

Figure 2: Coverings of hyperbolic orbifolds

$$\begin{aligned} \rho_1(b) &= (1\,3\,2) & \qquad \rho_2(h) &= (1\,4\,2) \\ \rho_1(c) &= (1\,2\,4) \\ \rho_1(d) &= (1\,2\,4) \end{aligned}$$

with the convention that the alternating group $\mathfrak{A}_4$ acts on $\{1, 2, 3, 4\}$ on the right[2]. Let Λ_j be the kernel of ρ_j. Since ρ_j is injective on the finite order elements of Γ_j, it follows that Λ_j is torsion-free.

Up to a choice of orientation on the upper half-plane $\mathbb{H}^2$, there is a unique uniformization of the $(2, 3, 12)$ triangle group, which by restriction to Λ_j determines a unique Riemann surface $C_j = \mathbb{H}^2/\Lambda_j$ with an action of $\mathfrak{A}_4$ induced by ρ_j. An orbifold Euler characteristic calculation implies that C_1 and C_2 both have genus four. The following lemma is direct from the definitions of the homomorphisms and combinatorics of the associated coverings of orbifolds.

Lemma 3.2. *Each order two element of $\mathfrak{A}_4$ has exactly two fixed points on C_j. These fixed points are precisely the six lifts to C_j of the unique cone point of order two on $\mathbb{H}^2/\Gamma_j$, hence the $\mathfrak{A}_4$ action on these six points is transitive. No order three element of $\mathfrak{A}_4$ has fixed point on C_2.*

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[2] While not the authors' preference, this choice is for consistency with standard computer algebra software.

3.2 The product action

Retaining the notation of §3.1, define $X = C_1 \times C_2$ and consider the diagonal action of $\mathfrak{A}_4$ on X. Lemma 3.2 implies that each 3-cycle acts freely on X, since it acts freely on the second coordinate. Each order two element has exactly four fixed points, namely the points of the form (z, w) with z one of its two fixed points on C_1 and w one of its two fixed points on C_2.

Therefore the quotient

$$Y = X/\mathfrak{A}_4$$

has exactly two singular points, which are quotient singularities of type A_1. Indeed, the action of an order two element $\sigma \in \mathfrak{A}_4$ on the tangent space to a fixed point (z, w) is by $-\,\mathrm{Id}$. Let $\varphi : Z \to Y$ be the minimal resolution of singularities of Y, which gives a diagram

$$\begin{tikzcd} X \arrow[d, "\pi"] & \\ Y & Z \arrow[l, "\varphi"'] \end{tikzcd}$$

where π is projection for the $\mathfrak{A}_4$ action. The minimal resolution has the property that if y_1, y_2 are the singular points of Y, then φ is an isomorphism on $Z \smallsetminus \varphi^{-1}(\{y_1, y_2\})$ and each $\varphi^{-1}(y_j)$ is a smooth rational curve F_j of self-intersection -2. See [1, §III.1-7].

Note that X has Euler characteristic 36, the action of $\mathfrak{A}_4$ has twelve fixed points, and each singular point of Y is replaced in Z by $\mathbb{P}^1$ (i.e., topologically a 2-sphere), hence the Euler characteristic of Z is

$$c_2(Z) = \frac{1}{12}(36 - 12) + 2 \times 2 = 6.$$

Moreover, $K_Z = \varphi^* K_Y$ since the resolution is of two A_1 singularities. Indeed, K_Z can be written as $\varphi^* K_Y + \alpha_1 F_1 + \alpha_2 F_2$ for some $\alpha_1, \alpha_2 \in \mathbb{Q}$ by basic properties of the canonical bundle, but adjunction implies that each α_j must be zero. Therefore

$$c_1(Z)^2 = \varphi^*(K_Y)^2 = \frac{1}{12} K_X^2 = 6,$$

since $c_1^2(X) = 2c_2(X)$ for smooth compact quotients of $\mathbb{H}^2 \times \mathbb{H}^2$ by Hirzebruch proportionality [7, Satz 2]. Thus Z is a smooth surface such that $c_1^2(Z) = c_2(Z) = 6$.

Finally, observe that Z is minimal of general type; see [21, Prop. 5.6]. Moreover, as discussed in [21, §5], $p_g(Z) = q(Z) = 1$ where the Albanese map of Z is φ followed by the projection of Y onto $\mathbb{H}^2/\Gamma_2$ induced by projection of $C_1 \times C_2$ onto the second factor. The contents of this subsection are collected in the following result.

Proposition 3.3. *Let C_1 and C_2 be the genus four curves with $\mathfrak{A}_4$ action described in §3.1. Set $X = C_1 \times C_2$ and consider the diagonal action of $\mathfrak{A}_4$ with quotient Y. Then Y has two singularities of type A_1 and its minimal resolution Z is a minimal surface of general type with $p_g = q = 1$ and $c_1^2 = c_2 = 6$. The Albanese map of Z is induced by the natural projection of Y onto the smooth curve $C_2/\mathfrak{A}_4$ of genus one.*

3.3 The elliptic curve E

The surface X in §3.2 is uniformized by $\Lambda_1 \times \Lambda_2$, where the action on $\mathbb{H}^2 \times \mathbb{H}^2$ is induced by the product action of $\Gamma_0 \times \Gamma_0$. Consider the intermediate group $\Gamma_1 \times \Gamma_2$, which produces a sequence of orbifold covers

$$\begin{tikzcd}
\mathbb{H}^2/\Gamma_1 \times \mathbb{H}^2/\Gamma_2 \\
Y \arrow[u] \\
C_1 \times C_2 \arrow[u, "\mathfrak{A}_4"] \arrow[uu, bend right=60, "\mathfrak{A}_4 \times \mathfrak{A}_4"']
\end{tikzcd}$$

where the bottom covering is the diagonal action of $\mathfrak{A}_4$ and the composition of the two covers is the product action of $\mathfrak{A}_4 \times \mathfrak{A}_4$. The goal of this subsection is to use this diagram to prove that the image of the diagonal $\widetilde{D}$ of $\mathbb{H}^2 \times \mathbb{H}^2$ in Y is a singular curve whose normalization has genus one.

The stabilizer in $\Gamma_1 \times \Gamma_2$ of $\widetilde{D}$ is naturally isomorphic to the subgroup $\Gamma_1 \cap \Gamma_2$ of Γ_0. The relations

$$\begin{aligned}
t_1 &= g^3 & \qquad s_1 &= [g,h] \\
&= bdc & &= a \\
t_2 &= g^{-1}h & s_2 &= (hgh^{-1})[g,h](hgh^{-1})^{-1} \\
&= c^{-1}b & &= (d^2cb^2)a(d^2cb^2)^{-1} \\
& & s_3 &= (hg^{-1}h^{-1})[g,h](hg^{-1}h^{-1})^{-1} \\
& & &= dad^{-1}
\end{aligned}$$

in Γ_0 imply that $\Gamma_1 \cap \Gamma_2$ contains the group Γ_3 generated by these elements. In fact, $\Gamma_3 \cong \Gamma(1; 2^3)$ is generated by these five elements with the relation

$$[t_1, t_2] = s_1 s_2 s_3$$

along with the relations that each s_j has order two.

Moreover, Γ_3 is the kernel of the homomorphism $\Gamma_2 \to \mathbb{Z}/3 = \langle \sigma \rangle$ defined by sending both g and h to σ. It is also the kernel of the homomorphism $\Gamma_1 \to \mathbb{Z}/3$ defined by sending a to the identity and b, c, d all to σ. It follows that $\Gamma_3 = \Gamma_1 \cap \Gamma_2$ is normal of index 3 in each, and $\mathbb{H}^2/\Gamma_3$ is the orbifold of genus one with three cone points of order two completing the diagram in Figure 2.

If $\Omega < \Gamma_1 \times \Gamma_2$ is the subgroup associated with Y, then Ω is the preimage in $\Gamma_1 \times \Gamma_2$ of the diagonal subgroup of $\mathfrak{A}_4 \times \mathfrak{A}_4$ under $\rho_1 \times \rho_2$. Thus the stabilizer in Ω of $\widetilde{D}$ is

$$\Gamma_4 = \big\{ \gamma \in \Gamma_3 \ : \ \rho_1(\gamma) = \rho_2(\gamma) \big\}.$$

Direct checks show that

$$\begin{aligned} \rho_1(t_1) &= (1\,3)(2\,4) & \qquad \rho_2(t_1) &= \mathrm{Id} \\ \rho_1(t_2) &= (1\,4)(2\,3) & \qquad \rho_2(t_2) &= (1\,3)(2\,4) \end{aligned}$$

and that ρ_1 and ρ_2 agree on each s_j. Then

$$\begin{aligned} \rho_1(t_1^2) &= \rho_2(t_1^2) \\ \rho_1(t_2^2) &= \rho_2(t_2^2) \end{aligned}$$

and so Γ_4 contains the index four subgroup of Γ_3 isomorphic to $\Delta(1; 2^{12})$ induced by the unique $(\mathbb{Z}/2)^2$ unramified cover of the torus. From the fact that ρ_1 and ρ_2 differ on $t_j^{\ell_1} t_k^{\ell_2}$ with $\ell_1, \ell_2 \in \{0, 1\}$ not both zero, it follows that $\Gamma_4 \cong \Delta(1; 2^{12})$ with natural generating set t_1^2, t_2^2, and all the appropriate conjugates of each s_j. For instance, one can take $t_1 s_j t_1^{-1}$, $t_2 s_j t_2^{-1}$, $t_1 t_2 s_j (t_1 t_2)^{-1}$ for $j = 1, 2, 3$. This leads to the diagram of orbifold coverings depicted in Figure 3.

The above leads to the commutative diagram of immersions

$$\begin{tikzcd}
\mathbb{H}^2/\Gamma_0 \arrow[r, hook] & \mathbb{H}^2/\Gamma_0 \times \mathbb{H}^2/\Gamma_0 \\
\mathbb{H}^2/\Gamma_3 \arrow[u] \arrow[r, looseness=1, "" , hook, no head, -{>}] & \mathbb{H}^2/\Gamma_1 \times \mathbb{H}^2/\Gamma_2 \arrow[u] \\
\mathbb{H}^2/\Gamma_4 \arrow[u] \arrow[r] & Y \arrow[u]
\end{tikzcd}$$

associated with projections of the diagonal $\widetilde{D}$ of $\mathbb{H}^2 \times \mathbb{H}^2$. While the diagonal of $\mathbb{H}^2/\Gamma_0 \times \mathbb{H}^2/\Gamma_0$ is certainly embedded, the coverings under consideration

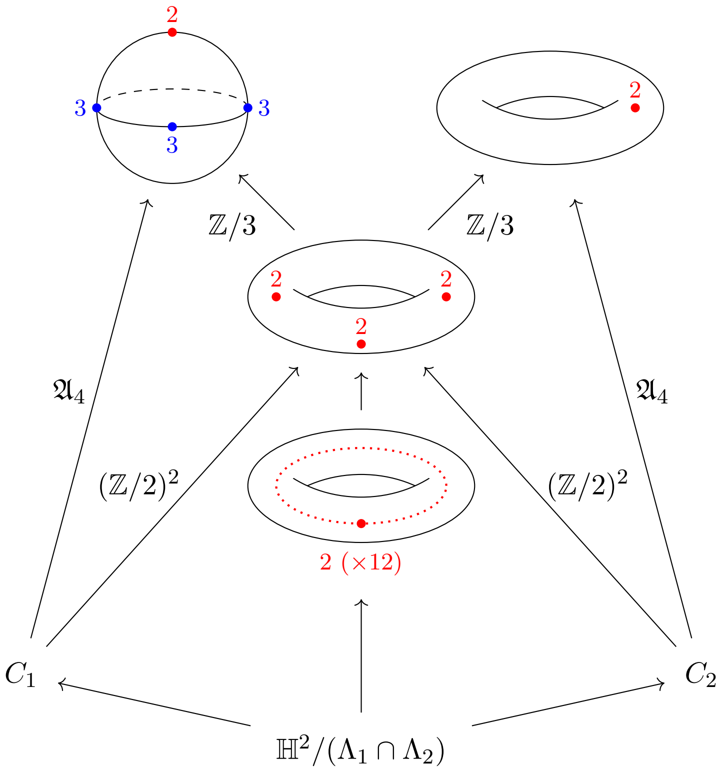

Figure 3: Further coverings of hyperbolic orbifolds

are orbifold covers, hence it does not follow that $\mathbb{H}^2/\Gamma_j$ embeds for $j = 3, 4$. In fact, this is not the case.

Proposition 3.4. *The image* $\widehat{E}$ *of* $\Gamma_4 \backslash \mathbb{H}^2$ *in* Y *meets each of the two* A_1 *singularities with multiplicity six. Away from those points, the map is an embedding. Consequently, the proper transform* E *of* $\widehat{E}$ *to the resolution* Z *of* Y *is a smooth curve of genus one with self-intersection* -12.

The proof of Proposition 3.4 is quite involved, and requires some preliminary lemmas (Lemmas 3.5 through 3.7). Notation established in each step will be used freely in each subsequent step. For $n = 2$, 3 or 12, let $\widetilde{z}_n \in \widetilde{D}$ be the point fixed by the diagonal action of p, q, and r on $\mathbb{H}^2 \times \mathbb{H}^2$, respectively. Then the image z_n of $\widetilde{z}_n$ on $\widetilde{D}/\Gamma_0$ is its cone point of order n.

Lemma 3.5. *All intersections of $\widetilde{D}$ with its orbit under $\Gamma_0 \times \Gamma_0$ arise from translates under the diagonal action of Γ_0 of the three configurations depicted in Figure 4.*

Proof. Consider the diagonal embedding of $D_0 = \mathbb{H}^2/\Gamma_0$ in

$$X_0 = \mathbb{H}^2/\Gamma_0 \times \mathbb{H}^2/\Gamma_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

At a cone point z_n on D_0 of order n ($n = 2$, 3 or 12) with lift $\widetilde{z}_n \in \widetilde{D}$ as above, the orbifold covering by $\mathbb{H}^2 \times \mathbb{H}^2$ has local group $(\mathbb{Z}/n)^2$ and there are n preimages of $\mathbb{H}^2/\Gamma_0$ through $\widetilde{z}_n$, namely the graphs of the various powers of the associated elliptic element of $\mathrm{PSL}_2(\mathbb{R})$. This gives the configurations in Figure 4.

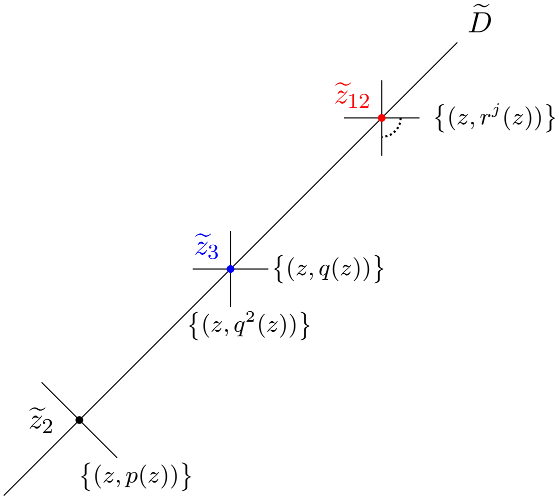

Figure 4: The $\Gamma_0 \times \Gamma_0$ orbits of $\widetilde{D}$ that meet $\widetilde{D}$

Moreover, if $(\alpha, \beta) \in \Gamma_0 \times \Gamma_0$ with $\alpha \neq \beta$ and $(\alpha, \beta)(\widetilde{D}) \cap \widetilde{D}$ is nonempty, then the intersection is a point

$$(w, w) = (\alpha(z), \beta(z)),$$

and thus $\beta^{-1}\alpha(z) = z$ and so $\beta^{-1}\alpha$ is torsion in Γ_0. Since the conjugacy classes of torsion elements of Γ_0 are represented by powers of conjugates of p, q, r, it follows that (z, z) is a Γ_0-translate of some $\widetilde{z}_j$. Thus all intersections of $\widetilde{D}$ with its orbit under $\Gamma_0 \times \Gamma_0$ arise from translates of the three configurations depicted in Figure 4 under the diagonal action of Γ_0. This proves the lemma. □

Next, consider $X_3 = \mathbb{H}^2/\Gamma_1 \times \mathbb{H}^2/\Gamma_2$ and $D_3 = \mathbb{H}^2/\Gamma_3$.

Lemma 3.6. *The immersion $D_3 \looparrowright X_3$ is an embedding away from the three cone points of D_3 and the cone points all map to a single point of X_3, as depicted in Figure 5.*

Proof. The map from X_3 to X_0 has degree 36, since $[\Gamma_0 : \Gamma_j] = 6$ for $j = 1, 2$, and the map from D_3 to D_0 has degree 18; see Figure 3. It follows that the preimage of $D_0 \subset X_0$ in X_3 has two irreducible components, namely the image of D_3 and another curve D_3', where the map $D_3 \looparrowright X_3$ is induced by the orbifold covering projection $\Gamma_3 \backslash \mathbb{H}^2 \to \Gamma_j \backslash \mathbb{H}^2$ in each coordinate $j = 1, 2$. Since the $\Gamma_0 \times \Gamma_0$ orbit of $\widetilde{D}$ only intersects $\widetilde{D}$ nontrivially at lifts of orbifold points of D_0, the irreducible components of the preimage of D_0 and their intersection properties can be understood by understanding the preimages in X_3 of the cone points on D_0.

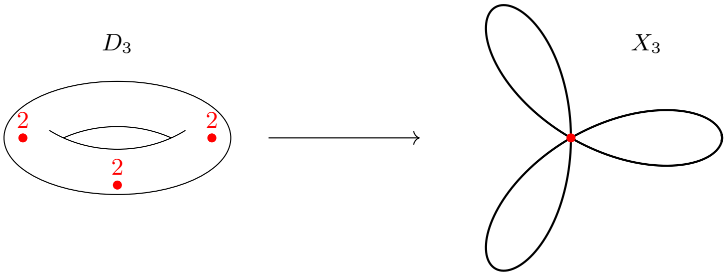

Figure 5: The immersion of D_3 in X_3

Since $p \notin \Gamma_1, \Gamma_2$, the cone point $z_2 \in D_0$ of order two, whose image in X_0 has orbifold weight four, has nine preimages in X_3 that are all smooth points for the orbifold structure. Since z_2 also has exactly nine preimages in D_3, it follows that $D_3 \looparrowright X_3$ is an embedding on these points and thus D_3 meets D_3' at each of these points, and each intersection is transversal since this is the case in the universal cover. Since $q \in \Gamma_1$ but $q \notin \Gamma_2$, the cone point z_3 of order three has $\frac{36}{3} = 12$ lifts to X_3 and $\frac{18}{3} = 6$ lifts to D_3. The action of (q, Id) at $\widetilde{z}_3$ in Figure 4 identifies the three embedded copies of $\mathbb{H}^2$. It follows that D_3 embeds in X_3 at the six lifts of z_3 and that D_3 does not meet D_3' at these points. This proves that D_3 embeds in X_3 away from its three cone points of order two.

The cone point z_{12} on D_0 of order twelve has preimage the three cone points of order two on D_3. The image of z_{12} on X_0 has orbifold weight

$12^2 = 144$, and it lifts to a single orbifold point of weight four with local group $(\mathbb{Z}/2)^2$ on X_3 associated with the conjugacy class of the subgroup of $\Gamma_1 \times \Gamma_2$ generated by (r^6, Id) and (Id, r^6). Indeed, this is the point (w_1, w_2) on X_3 where w_j is the unique cone point of order two on $\mathbb{H}^2/\Gamma_j$. The cone points of order two on D_3 then must all map to this point, hence the image of D_3 passes through this point with multiplicity three. This completely determines the image of D_3 in X_3, which fails to be an embedding at the three cone points of order two, which are all identified. Thus the image of D_3 is as depicted in Figure 5, proving the lemma. □

Lemma 3.7. *Ramification of the orbifold cover $Y \to X_3$ is:*

- *degree n ramification above vertical curves of the form $\{x_n\} \times \mathbb{H}^2/\Gamma_2$, where $x_n \in \mathbb{H}^2/\Gamma_1$ is a cone point of order n;*
- *degree two ramification over $\mathbb{H}^2/\Gamma_1 \times \{w_2\}$, where w_2 is the cone point of order two on $\mathbb{H}^2/\Gamma_2$.*

If p_j denotes projection of D_3 onto $\mathbb{H}^2/\Gamma_j$ and the image $(p_1(z), p_2(z))$ of a point $z \in D_3$ on X_3 intersects the ramification locus, then $p_j(z)$ is a cone point of $\mathbb{H}^2/\Gamma_j$ for at least one $j \in \{1, 2\}$.

Proof. The last statement is an immediate consequence of the first and Lemma 3.6. For the first statement, note that ramification comes from elements of the form (t, Id) or (Id, t) not contained in the subgroup Λ of $\Gamma_1 \times \Gamma_2$ associated with Y, where t is finite order. Since Λ contains none of the elements of $\Gamma_1 \times \Gamma_2$ of this form by definition, as ρ_1 and ρ_2 are injective on torsion, the lemma follows. □

Proof of Proposition 3.4. Set $D_4 = \mathbb{H}^2/\Gamma_4$ and let $\widehat{E}$ denote its image in Y. The map $D_4 \to D_3$ has degree four, and $Y \to X_3$ has degree 12 and is unramified over D_3, so D_3 has three preimages in Y.

The image of D_3 in X_3 meets each vertical curve $\{z_3\} \times \mathbb{H}^2/\Gamma_2$, over which the map $Y \to X_3$ has degree three ramification by Lemma 3.7, in a single point that has four preimages in Y. Since $D_4 \to D_3$ is of degree four and unramified above the relevant points, $\widehat{E}$ must pass through all four preimages. Thus $\widehat{E}$ is smooth through each lift.

The image of D_3 meets $\{x_2\} \times \mathbb{H}^2/\Gamma_2$ and $\mathbb{H}^2/\Gamma_1 \times \{w_2\}$ precisely in the point (x_2, w_2) where the immersion of D_3 fails to be an embedding. This point on X_3 has preimage containing the two A_1 singularities of Y. Locally around an A_1 singularity the cover is the map from a neighborhood V of an A_1 singularity to a neighborhood U of the origin in $\mathbb{C}^2$ given by

$$U \longrightarrow V \longrightarrow U$$

where the composition $U \to U$ is the action of $(\mathbb{Z}/2)^2$ by reflections of order two in each coordinate and the map $U \to V$ is the quotient by the diagonal subgroup generated by $(-1,-1)$. The map $V \to U$ is two-to-one on lines through the singular point. It follows that the preimage of D_3 in Y passes through each A_1 singularity with multiplicity six.

On the other hand, each of the twelve cone points on D_4 must map to one of the A_1 singularities, since these are the only orbifold points on Y for its $\mathbb{H}^2 \times \mathbb{H}^2$ orbifold structure, and orbifold points of D_4 must map to orbifold points of Y. It follows that $\widehat{E}$ must pass through the A_1 singularities with total multiplicity twelve, and therefore it passes through each singularity with multiplicity six. This proves that $\widehat{E}$ is embedded in Y away from the A_1 singularities and passes through each singular point with multiplicity six.

Now consider the proper transform E of $\widehat{E}$ to Z under the minimal resolution $\varphi : Z \to Y$, and let F_1, F_2 be the (-2) curves resolving the singularities. The above implies that

$$E \cdot F_j = 6$$

for $j = 1, 2$. Write E as $\varphi^* \widehat{E} + \alpha_1 F_1 + \alpha_2 F_2$ for $\alpha_j \in \mathbb{Q}$. Then $K_Z \cdot F_j = 0$, so

$$\begin{aligned} K_Z \cdot E &= \varphi^* K_Y \cdot \varphi^* \widehat{E} \\ &= K_Y \cdot \widehat{E} \\ &= 2|e^{\mathrm{orb}}(D_4)| \end{aligned}$$

by relative Hirzebruch proportionality [8, §4], where e^{orb} denotes the orbifold Euler characteristic. Since

$$e^{\mathrm{orb}}(D_4) = 0 - 12 \times \frac{1}{2} = -6$$

it follows that $K_Z \cdot E = 12$. Since E has genus one, adjunction implies that $E^2 = -12$, which completes the proof of the proposition. □

3.4 Proof that (Z, E) is a ball quotient pair

Let (Z, E) be the pair constructed in §3.3. First, there are the simple calculations

$$c_1^2(Z, E) = (K_Z + E)^2 = K_Z^2 - E^2 = 18$$

$$c_2(Z, E) = c_2(Z) = 6$$

that give $c_1^2(Z, E) = 3c_2(Z, E)$. To prove that (Z, E) is a ball quotient pair, it suffices to prove that $K_Z + E$ is both nef [16, Def. 1.4.1], big [16, Def. 2.2.1], and that every (-2) curve of X meets E nontrivially.

Consider an irreducible curve C on Z. If C is not E or one of the (-2) curves F_j, then $C \cdot E \geq 0$ and $\varphi(C)$ is an irreducible curve on Y. Then

$$(K_Z + E) \cdot C = K_Y \cdot \varphi(C) + \widehat{E} \cdot C > 0$$

since K_Y is ample. To see that K_Y is ample, note that Y is a compact quotient of $\mathbb{H}^2 \times \mathbb{H}^2$ by a group action with only isolated fixed points. Since there is no branching divisor, the orbifold canonical class for Y is the same as the canonical class [2, Prop. 4.4.15].

Then

$$\begin{aligned}(K_Z + E) \cdot E &= 0 \\ (K_Z + E) \cdot F_j &= 6\end{aligned}$$

so $(K_Z + E) \cdot C \geq 0$ for all irreducible curves C on Z, and therefore $K_Z + E$ is nef. Then $(K_Z + E)^2 > 0$, so it is big by [16, Thm. 2.2.16]. Moreover, note that ampleness of K_Y implies that there is no (-2) curve on Y disjoint from the singular points, since such a curve C would have $K_Y \cdot C = 0$ by adjunction, contradicting ampleness of K_Y. Therefore there is no (-2) curve on Z that does not meet E. This combined with Chern–Gauss–Bonnet completes the proof of the following result:

Theorem 3.8. *The pair (Z, E) constructed in this section is a ball quotient pair. Specifically, $Z \smallsetminus E$ is a smooth one-cusped ball quotient of volume $16\pi^2$.*

4 The proof of Theorem 1.3

Recall that Theorem 1.1 is an immediate consequence of Theorem 1.3 and Chern–Gauss–Bonnet. This section proves Theorem 1.3. To start, the pair (Z_1, E_1) for Theorem 1.3 is the surface Z from §3 and elliptic curve E on Z, so it suffices to construct (Z_d, E_d) for all odd $d > 1$.

Lemma 4.1. *To prove Theorem 1.3, it suffices to prove that there is an étale cover $Z_d \to Z_1$ of every odd degree d so that E_1 has irreducible preimage E_d in Z_d.*

Proof. Note that $E_d \to E_1$ is étale, so E_d has genus one. Given the hypotheses of the lemma, there is an étale covering

$$Z_d \smallsetminus E_d \longrightarrow Z_1 \smallsetminus E_1$$

of degree d. Since $Z_1 \smallsetminus E_1$ is a ball quotient, so is $Z_d \smallsetminus E_d$. Since E_d is a single genus one curve, $Z_d \smallsetminus E_d$ is a one-cusped ball quotient and hence (Z_d, E_d) is a ball quotient pair. Moreover, characteristic classes multiply in covers, so $c_1^2(Z_d) = c_2(Z_d) = 6d$. Similarly, self-intersection multiplies, so E_d has self-intersection $-12d$. Thus the surfaces (Z_d, E_d) satisfy the hypotheses of Theorem 1.3 and hence their existence would prove that result. ☐

Proposition 4.2. *For each odd $d \geq 1$ there is an étale cover $Z_d \to Z_1$ of every odd degree d so that E_1 has irreducible preimage E_d in Z_d.*

Proof. Recall from Proposition 3.3 that the Albanese map of Z_1 is given by projection onto the curve $D_2 = \mathbb{H}^2/\Gamma_2$ of genus one. It follows from standard facts about the Albanese map that $\pi_1(Z_1)^{\mathrm{ab}}$ modulo torsion is isomorphic to $\mathbb{Z}^2$ induced by the Albanese map (e.g., see [1, §I.13]). By construction, the map $E_1 \to D_2$ is an étale cover of degree 12, hence the image of $\pi_1(E_1)$ in $\mathbb{Z}^2$ has index 12. Thus for each odd $d \geq 1$ there is a finite abelian quotient A_d of $\mathbb{Z}^2$ with order d so that the induced homomorphism $\mu_d : \pi_1(Z_1) \to A_d$ has the property that the restriction to $\pi_1(E_1)$ is surjective. If Z_d is the covering of Z_1 associated with μ_d, then E_1 has exactly one preimage in Z_d, which is an irreducible curve of genus one. This proves the proposition. ☐

This completes the proof of the main result of this paper.

Proof of Theorem 1.3. Take the surfaces (Z_d, E_d) provided by Proposition 4.2 and apply Lemma 4.1. ☐

References

[1] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, second edition, 2004. URL: `https://doi.org/10.1007/978-3-642-57739-0`.

[2] Charles P. Boyer and Krzysztof Galicki. *Sasakian geometry.* Oxford Mathematical Monographs. Oxford University Press, 2008.

[3] Karel Dekimpe. *Almost-Bieberbach groups: affine and polynomial structures*, volume 1639 of *Lecture Notes in Mathematics*. Springer-Verlag, 1996. URL: `https://doi.org/10.1007/BFb0094472`.

[4] Martin Deraux. On Subgroups Finite Index in Complex Hyperbolic Lattice Triangle Groups. *Exp. Math.*, 33(3):456–481, 2024. URL: `https://doi.org/10.1080/10586458.2022.2158969`.

[5] Luca F. Di Cerbo and Matthew Stover. Bielliptic ball quotient compactifications and lattices in $\mathrm{PU}(2,1)$ with finitely generated commutator subgroup. *Ann. Inst. Fourier (Grenoble)*, 67(1):315–328, 2017. URL: `https://doi.org/10.5802/aif.3083`.

[6] Luca F. Di Cerbo and Matthew Stover. Classification and arithmeticity of toroidal compactifications with $3\overline{c}_2 = \overline{c}_1^2 = 3$. *Geom. Topol.*, 22(4):2465–2510, 2018. URL: `https://doi.org/10.2140/gt.2018.22.2465`.

[7] Friedrich Hirzebruch. Automorphe Formen und der Satz von Riemann-Roch. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 129–144. Universidad Nacional Autónoma de México and UNESCO, México, 1958.

[8] Friedrich Hirzebruch. Hilbert modular surfaces. *Enseign. Math. (2)*, 19:183–281, 1973.

[9] Rolf-Peter Holzapfel. *Ball and surface arithmetics.* Aspects of Mathematics, E29. Friedr. Vieweg & Sohn, 1998. URL: `https://doi.org/10.1007/978-3-322-90169-9`.

[10] Yoshinobu Kamishima. Nonexistence of cusp cross-section of one-cusped complete complex hyperbolic manifolds. II. *Int. Math. Forum*, 2(25-28):1251–1258, 2007. URL: `https://doi.org/10.12988/imf.2007.07112`.

[11] Svetlana Katok. *Fuchsian groups.* Chicago Lectures in Mathematics. University of Chicago Press, 1992.

[12] Ryoichi Kobayashi. Einstein-Kaehler metrics on open algebraic surfaces of general type. *Tohoku Math. J. (2)*, 37(1):43–77, 1985. URL: `https://doi.org/10.2748/tmj/1178228722`.

[13] Alexander Kolpakov and Bruno Martelli. Hyperbolic four-manifolds with one cusp. *Geom. Funct. Anal.*, 23(6):1903–1933, 2013. URL: `https://doi.org/10.1007/s00039-013-0247-2`.

[14] Alexander Kolpakov and Leone Slavich. Hyperbolic 4-manifolds, colourings and mutations. *Proc. Lond. Math. Soc. (3)*, 113(2):163–184, 2016. URL: `https://doi.org/10.1112/plms/pdw025`.

[15] Alexander Kolpakov and Leone Slavich. Symmetries of hyperbolic 4-manifolds. *Int. Math. Res. Not. IMRN*, (9):2677–2716, 2016. URL: `https://doi.org/10.1093/imrn/rnv210`.

[16] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, 2004. Classical setting: line bundles and linear series. URL: `https://doi.org/10.1007/978-3-642-18808-4`.

[17] D. D. Long and A. W. Reid. On the geometric boundaries of hyperbolic 4-manifolds. *Geom. Topol.*, 4:171–178, 2000. URL: `https://doi.org/10.2140/gt.2000.4.171`.

[18] D. D. Long and A. W. Reid. All flat manifolds are cusps of hyperbolic orbifolds. *Algebr. Geom. Topol.*, 2:285–296, 2002. URL: `https://doi.org/10.2140/agt.2002.2.285`.

[19] D. B. McReynolds. Peripheral separability and cusps of arithmetic hyperbolic orbifolds. *Algebr. Geom. Topol.*, 4:721–755, 2004. URL: `https://doi.org/10.2140/agt.2004.4.721`.

[20] D. B. McReynolds. Controlling manifold covers of orbifolds. *Math. Res. Lett.*, 16(4):651–662, 2009. URL: `https://doi.org/10.4310/MRL.2009.v16.n4.a8`.

[21] Francesco Polizzi. Standard isotrivial fibrations with $p_g = q = 1$. *J. Algebra*, 321(6):1600–1631, 2009. URL: `https://doi.org/10.1016/j.jalgebra.2008.10.028`.

[22] John G. Ratcliffe and Steven T. Tschantz. Hyperbolic 24-cell 4-manifolds with one cusp. *Exp. Math.*, 32(2):269–279, 2023. URL: `https://doi.org/10.1080/10586458.2021.1926010`.

[23] V. A. Rohlin. A three-dimensional manifold is the boundary of a four-dimensional one. *Doklady Akad. Nauk SSSR (N.S.)*, 81:355–357, 1951.

[24] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983. URL: `https://doi.org/10.1112/blms/15.5.401`.

[25] Leone Slavich. Some hyperbolic 4-manifolds with low volume and number of cusps. *Topology Appl.*, 191:1–9, 2015. URL: `https://doi.org/10.1016/j.topol.2015.05.004`.

[26] Matthew Stover. On the number of ends of rank one locally symmetric spaces. *Geom. Topol.*, 17(2):905–924, 2013. URL: `https://doi.org/10.2140/gt.2013.17.905`.

[27] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982. URL: `https://doi.org/10.1090/S0273-0979-1982-15003-0`.

[28] G. Tian and S.-T. Yau. Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 574–628. World Sci. Publishing, 1987.

[29] Gang Tian. Kähler-Einstein metrics on algebraic manifolds. In *Transcendental methods in algebraic geometry (Cetraro, 1994)*, volume 1646 of *Lecture Notes in Math.*, pages 143–185. Springer, 1996. URL: `https://doi.org/10.1007/BFb0094304`.