MIRROR STABILIZERS FOR LATTICE COMPLEX HYPERBOLIC TRIANGLE GROUPS

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ABSTRACT. For each lattice complex hyperbolic triangle group, we study the Fuchsian stabilizer of a reprentative of each group orbit of mirrors of complex reflections. We give explicit generators for the stabilizers, and compute their signature in the sense of Fuchsian groups. For some of the triangle groups, we also find explicit pairs of complex lines such that the union of their stabilizers generate the ambient lattice.

1. Introduction

Recall that the complex hyperbolic plane $H^2_{\mathbb{C}}$ is a Hermitian symmetric space of constant negative holomorphic sectional curvature -1. Its real sectional curvatures satisfy $-1 \leq K_{sect} \leq -1/4$, where the lower and upper bounds are realized by tangent planes of totally geodesic copies of $H^1_{\mathbb{C}}$ (complex lines) and $H^2_{\mathbb{R}}$ (Lagrangians), respectively. Its group of holomorphic isometries is isomorphic to $G = PU(2,1)$ (and this has index two in the full isometry group, which includes antiholomorphic transformations).

For each complex line L, there is an obvious embedding of $U(1,1)$ into $U(2,1)$ whose image preserves L; this gives a description of $Stab_G(L)$ as a central extension of $PU(1,1)$ by the fixed point stabilizer, which is the subgroup of complex reflections fixing L.

Each complex line is the fixed point set a 1-parameter family of isometries, called complex reflections (beware these may have arbitrary order), and we call a subgroup $\Gamma \subset G$ a *complex reflection group* if it is generated by complex reflections. Complex reflection groups that can be generated by 3 complex reflections are called *complex triangle groups*.

We call a subgroup Γ of G a lattice if it is discrete, and if the quotient $\Gamma \backslash H^2_{\mathbb{C}}$ has finite volume. In a sense, the discreteness assumption says Γ cannot be too large, whereas the finite volume assumption says that Γ cannot be too small... which makes it tricky to find groups satisfying both conditions at the same time. Lattices do exist though, due to very general results of Borel [3]; the basic idea is to use "arithmetic methods", which we think of as a souped up version of the fact that $\mathbb{Z} \subset \mathbb{R}$ is discrete, and $\mathbb{R}/\mathbb{Z}$ has finite Lebesgue measure.

Another important (but difficult) method to construct lattices is to use (very wisely chosen) complex reflection groups. This was the approach taken by Mostow [20] to construct the first examples of non-arithmetic lattices (lattices that cannot be obtained from the arithmetic construction). There are many other approaches, using various ideas around uniformization (see [1], [7], [5] for instance), but in this paper we focus only on the groups

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constructed in [14]; the latter paper can be thought of as one possible generalization of the basic ideas developed by Mostow in [20].

The techniques that are used in [14] in order to generalize Mostow's construction are very intricate, and we will not go into almost any of them. We freely use the results [14] as a black box, keeping technical details to a bare minimum.

For convenience of the exposition, we refer to the lattices constructed in [14] as *the lattice complex hyperbolic triangle groups* (the terminology may be a bit misleading, because there may actually be more lattice complex hyperbolic triangle groups, but we expect our search in [12] to be fairly exhaustive).

The basic point is that the lattice complex hyperbolic triangle groups come in three families of groups

$$\mathcal{S}(p,\tau), \quad \mathcal{T}(p,\mathbf{T}), \quad \Gamma(p,t)$$

where $p\in\mathbb{N}$, $p\geq 2$, $\tau\in\mathbb{C}$, $\mathbf{T}\in\mathbb{C}^3$, $t\in\mathbb{Q}$ are wisely chosen parameters (see the list in the appendix of [14]).

The lattices $\Gamma(p,t)$ are those that were studied by Mostow in [20], and we do not consider them in this paper since they have been given many other descriptions (see [7], [22] for instance).

For each lattice complex hyperbolic triangle group, the reader can find explicit matrices generating the group, a description of a fundamental domain for its action on $H^2_{\mathbb{C}}$ and a presentation in terms of generators in relations (see [14], [8]; see also [9] for more recent development).

In what follows, let Γ denote a lattice complex hyperbolic triangle group, and let $X=\Gamma\backslash H^2_{\mathbb{C}}$ denote the quotient. The main goal of the present paper is to study some detailed properties of the (1-dimensional part of the) branch locus of the quotient map $H^2_{\mathbb{C}}\to X$. Indeed, from the results in [14], one can extract a list $A_1,\dots,A_r$ of representatives of conjugacy classes of complex reflections in the group. For each j, $Stab_\Gamma(A_j)$ is then a central extension of a lattice $F_j\subset PU(1,1)$. Our goal is to describe these groups F_j; for each lattice complex hyperbolic triangle group, we list the group orbits of mirror stabilizers, and for each of them, we

- find an explicit generating set and
- determine the signature of the corresponding Fuchsian group, its trace field and its arithmeticity

From the results in the paper, one can write explicit presentations for the mirror stabilizers (see equation (7) for one example of such a presentation).

We hope that our results give useful information for several purposes. One purpose could be to try and understand the quotient $X=\Gamma\backslash H^2_{\mathbb{C}}$ and branching behavior of the quotient map $H^2_{\mathbb{C}}\to X$; in some cases the quotient map is completely understood (see [11] and [10]). Another purpose is to study possible hybrid structures for complex hyperbolic lattice triangle groups (see [23], [16]); we will give some explicit examples in section 11. Finally, this information is useful in order to study the counting and equidistribution properties of closed geodesic that are reflecting on mirrors in $X=\Gamma\backslash H^2_{\mathbb{C}}$, see [15, §5.2], which was the initial motivation for writing this paper.

Part of the analysis of Fuchsian subgroups of lattice complex hyperbolic triangle group was achieved by Sun in [21]. In that paper, only some of the groups were treated, and only some of the conjugacy classes of complex reflections, but we use the same method. Information on the arithmeticity of the subgroups listed by Sun can be found in a very recent paper by Jiang, Wang and Yang [18].

2. Basic complex hyperbolic geometry

Let V be an (n+1)-dimensional complex vector space and $h : V \times V \to \mathbb{C}$ be a Hermitian form of signature $(n, 1)$. We write $\langle X, Y\rangle$ for $h(X, Y)$ and $||X||^2$ for $h(X, X)$ (which is real, but not necessarily positive). Denote by V^- (resp. V^0, V^+) the set vectors $X \in V$ with $||X||^2 < 0$ (resp. $= 0$, > 0).

The isometry group of the form is given by

$$U(h) = \{A \in GL(V) : \forall X, Y \in V, h(AX, AY) = h(X, Y)\},$$

which we think of as a real Lie group. By Sylvester's law of inertia, that Lie group is independent of the choice of the Hermitian form (up to isomorphism of Lie groups).

Consider the projectivization map $\pi : V \setminus \{0\} \to \mathbb{P}(V)$, and define $H^2_{\mathbb{C}} = p(V^-)$, $\partial_\infty H^n_{\mathbb{C}} = p(V^0 \setminus \{0\})$. It is a standard fact that that $PU(h)$ acts transitively on $H^n_{\mathbb{C}}$, and also on $\partial H^n_{\mathbb{C}}$ (in fact it also acts transitively on $p(V^+)$); these homogeneous spaces are very different, since the stabilizer of a point is compact only in the case of $H^n_{\mathbb{C}}$.

It is a standard fact that $H^n_{\mathbb{C}}$ carries a $PU(h)$-invariant Riemannian metric (whereas $\partial_\infty H^n_{\mathbb{C}}$ does not, nor does $p(V^+)$), which makes it a Hermitian symmetric space. The interested reader can find an expression for that metric in [19] for instance. The corresponding integrated distance formula is easy to write down:

$$\cosh(\frac{1}{2}d(\mathbf{X}, \mathbf{Y})) = \frac{|\langle X, Y\rangle|}{\sqrt{||X||^2||Y||^2}}, \tag{1}$$

where $X \in V$ (resp. $Y \in V$) is any lift of $\mathbf{X} \in H^n_{\mathbb{C}}$ (resp. $\mathbf{Y} \in H^n_{\mathbb{C}}$). The factor $\frac{1}{2}$ is included so that the metric is scaled to have holomorphic sectional curvature identically equal to -1.

There are totally geodesic copies of $H^k_{\mathbb{C}}$ inside $H^n_{\mathbb{C}}$ for any $k \le n$, obtained by choosing a $(k+1)$-dimensional complex subspace $W \subset V$ such that $h|_W$ has signature $(k, 1)$. Similary, there are totally geodesic copies of $H^k_{\mathbb{R}}$, obtained by taking a totally real (k+1)-dimensional subspace $R \subset V$, i.e. the $\mathbb{R}$-span of $k + 1$ vectors $v_0, \dots, v_k$ with $\langle v_j, v_k\rangle \in \mathbb{R}$ for all j, k. It is a well known fact that every complete totally geodesic submanifold of $H^n_{\mathbb{C}}$ is among the ones just described (see section 2.5 of [4]).

Note that $H^n_{\mathbb{C}}$ has non-constant curvature for all $n \ge 2$. More precisely, we have the following (see [4] or [17]).

Proposition 2.1. *The real sectional curvatures of $H^n_{\mathbb{C}}$ are contained in $[-1, -1/4]$. The real 2-planes with curvature -1 are the tangent planes to totally geodesic copies of $H^1_{\mathbb{C}}$, whereas the ones with curvature $-1/4$ are the tangent planes to totally geodesic copies of $H^2_{\mathbb{R}}$.*

Since h is non-degenerate, it defines a bijection between the Grassmannians of complex k-planes and $(n-k)$-planes, where W is in correspondence with $W^{\perp} = \{X \in V : \forall Y \in W, \langle X, Y\rangle = 0\}$. As an important special case, when $k = n-1$, we get a parametrization of totally geodesic copies of $H^{n-1}_{\mathbb{C}}$ by positive lines, i.e. by $p(V^+)$. For $X \in V^+$, we will often abuse notation and write $X^{\perp}$ for $p(V^- \cap X^{\perp})$.

Proposition 2.2. *Let $X, Y \in V^+$. The complex hyperplanes $X^{\perp}$ and $Y^{\perp}$ intersect in $H^2_{\mathbb{C}}$ if and only if*

$$c := \frac{|\langle X, Y\rangle|}{\sqrt{||X||^2||Y||^2}} < 1.$$

If they intersect in $H^2_{\mathbb{C}}$, they do so at a constant angle α, and we have $c = \cos(\alpha)$.

Given a positive vector $U \in V$, it is easy to see that, for any $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, the formula

$$R_{U,\zeta}(X) = X + (\zeta - 1)\frac{\langle X, U\rangle}{\langle U, U\rangle}U, \tag{2}$$

defines an element of $U(h)$. It fixes pointwise $U^{\perp}$, and rotates by $\arg(\zeta)$ in the complex directions orthogonal to its fixed point set. In terms of linear algebra, it is a diagonalizable linear transformation with eigenvalues $(\zeta, 1, \dots, 1)$, such that h restricts to a form of signature $(n-1, 1)$ on the 1-eigenspace.

We call the corresponding isometry $R_{U,\zeta} \in PU(h)$ a *complex reflection* with mirror $U^{\perp}$. Replacing U by λU for $\lambda \in \mathbb{C}$, leaves $R_{U,\zeta}$ unchanged, so we may assume $\langle U, U\rangle = 1$. The complex number ζ is called the *multiplier* of the complex reflection, $arg(\zeta)$ is called the *angle* of the rotation.

Formula (2) makes sense if U is a negative vector, in which case we call $R_{U,\zeta}$ a *complex reflection in a point*; in that case $p(U)$ is its only fixed point in $H^n_{\mathbb{C}}$ (in particular, for $\zeta = -1$, we get isometric involutions with a single fixed point).

Remark 2.3. It is a standard fact that two complex reflections R_{U_1,ζ_1}, R_{U_2,ζ_2} commute if and only if $U_1^{\perp} = U_2^{\perp}$ or $\langle U_1, U_2\rangle = 0$.

We will use the usual classification of isometries of $PU(h)$ into elliptic, parabolic and loxodromic elements (see section 6.2 of [17]). Among elliptic elements, we distinguish regular ones whose matrix representatives have $(n+1)$ distinct eigenvalues. Non-regular elliptic elements are sometimes called special elliptic. There are also fine classifications for parabolic and loxodromic elements, but we will not need that in the present paper.

From now on, we restrict to the case $n = 2$. In this case, every elliptic isometry is either regular elliptic, a complex reflection, or a complex reflection in a point.

Now let $A \in PU(h)$ be regular elliptic. Choose a basis $\mathcal{E} = e_0, e_1, e_2$ of V that diagonalizes A; write $\zeta_0, \zeta_1, \zeta_2$ for the corresponding (distinct) eigenvalues. Since A is elliptic, we must have $|\zeta| = 1$ for all $j = 1, 2, 3$. Moreover, since eigenvectors with distinct eigenvalues are orthogonal, $\mathcal{E}$ is h-orthogonal, and because of the signature of h, one and only one of the basis vectors is h-negative. By reordering and rescaling, we may assume $||e_0||^2 = -1$

and $||e_1||^2 = ||e_2||^2 = 1$. Then $\mathbf{e_0} = p(e_0)$ is the isolated fixed point of A in $H^2_{\mathbb{C}}$. The complex 2-planes $L_1 = \mathbb{C}\{e_0, e_1\}$ and $L_2 = \mathbb{C}\{e_0, e_2\}$ project to complex lines $\mathbf{L_1}$ and $\mathbf{L_2}$, each being A-invariant (and these are the only complex lines through $\mathbf{e_0}$ that are A-invariant). In $\mathbf{L_j}$, A acts as a rotation by $arg(\zeta_j/\zeta_0)$.

Conversely, let $\mathbf{L_j} = p(V^- \cap U_j^{\perp})$ for positive vectors U_j. We assume $\mathbf{L_1}$ and $\mathbf{L_2}$ are orthogonal, i.e. $\langle U_1, U_2\rangle = 0$ (cf. formula (2.2)). Then for any ζ_j with $|\zeta_j| = 1$, the product

$$R_{U_1,\zeta_1}R_{U_2,\zeta_2}$$

of commuting complex reflection is elliptic. It is regular elliptic if and only if $\zeta_1 \neq \zeta_2$.

Note that if A^k is special elliptic for some $k \in \mathbb{N}^*$, then A^k is either a complex reflection in a point, or a complex reflection. In the latter case, the mirror of A^k must be either $\mathbf{L_1}$ or $\mathbf{L_2}$.

3. Lattice complex hyperbolic triangle groups

We refer the reader to [14] for a detailed description of the lattices $\mathcal{S}(p, \tau)$, $\mathcal{T}(p, \mathbf{T})$. Alternatively, the reader can download matrices in Magma format from [9].

Here we only give explicit generators R_1, R_2, R_3 for $\mathcal{T}(p, \mathbf{T})$, where $p \in \mathbb{N}^*$ and $\mathbf{T} = (\rho, \sigma, \tau) \in \mathbb{C}^3$ (these specialize to generators for $\mathcal{S}(p, \tau)$ by taking $\rho = \sigma = \tau$).

Definition 3.1. *For $\mathbf{T} = (\rho, \sigma, \tau) \in \mathbb{C}^3$, the group $\mathcal{T}(p, \mathbf{T})$ is the subgroup of $U(H)$ generated by R_1, R_2 and R_3 (see equations* (3) *and* (4)*), where $u = e^{2pi/3p}$, $\alpha = 2 - u^3 - \bar{u}^3$, $\beta_1 = (\bar{u}^2 - u)\rho$, $\beta_2 = (\bar{u}^2 - u)\sigma$, $\beta_3 = (\bar{u}^2 - u)\tau$.*

$$H = \begin{pmatrix} \alpha & \beta_1 & \overline{\beta}_3 \\ \overline{\beta}_1 & \alpha & \beta_2 \\ \beta_3 & \overline{\beta}_2 & \alpha \end{pmatrix} \tag{3}$$

$$R_1 = \begin{pmatrix} u^2 & \rho & -u\overline{\tau} \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}; \; R_2 = \begin{pmatrix} \bar{u} & 0 & 0 \\ -u\bar{\rho} & u^2 & \sigma \\ 0 & 0 & \bar{u} \end{pmatrix}; \; R_3 = \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & \bar{u} & 0 \\ \tau & -u\bar{\sigma} & u^2 \end{pmatrix} \tag{4}$$

Remark 3.2. (1) If e_1, e_2, e_3 denotes the standard basis for $\mathbb{C}^3$, then in terms of the notation of formula (2), we have $R_j = \bar{u}R_{e_j,u^3}$, so the matrices R_j can easily be reconstructed from the data of H.

(2) The factor $\bar{u}$ is included to get matrices in $SU(H)$ rather than just $U(H)$.

When $\rho = \sigma = \tau$, the map

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is a regular elliptic element of order 3 in $U(H)$, and we have

$$JR_1J^{-1} = R_2, \quad JR_2J^{-1} = R_3.$$

τ or $\mathbf{T}$	p
$\sigma_1 = -1 + i\sqrt{2}$	3,4,6
$\bar{\sigma}_4 = \frac{-1-i\sqrt{7}}{2}$	3,4,5,6,8,12
$\sigma_5 = e^{-\pi i/9}\frac{\sqrt{5}+i\sqrt{3}}{2}$	2,3,4
$\sigma_{10} = \frac{1+\sqrt{5}}{2}$	3,4,5,10
$\mathbf{S_2} = (1 + \frac{-1+i\sqrt{3}}{2}\frac{1+\sqrt{5}}{2}, 1, 1)$	3,4,5
$\mathbf{E_2} = (\sqrt{2}, \frac{-1+i\sqrt{3}}{2}, \sqrt{2})$	3,4,6
$\mathbf{H_1} = (\frac{-1+i\sqrt{7}}{2}, e^{-4\pi i/7}, e^{-4\pi i/7})$	2
$\mathbf{H_2} = (-1 - e^{-2\pi i/5}, e^{4\pi i/5}, e^{4\pi i/5})$	2,3,5

Table 1. Values of the parameters for the known lattice complex hyperbolic triangle groups

Definition 3.3. *For $\tau \in \mathbb{C}$, $\mathcal{S}(p, \tau)$ is the subgroup of $U(H)$ generated by R_1 and J, where $u = e^{2pi/3p}$, $\alpha = 2 - u^3 - \bar{u}^3$, $\beta_1 = \beta_2 = \beta_3 = (\bar{u}^2 - u)\tau$.*

Remark 3.4. For an arbitrary choice of the parameters p and $\mathbf{T}$, the group $\mathcal{T}(p, \mathbf{T})$ is usually not discrete; moreover, the Hermitian form on $\mathbb{C}^3$ given by the Hermitian matrix H is not even of signature $(2, 1)$ in general.

We will only consider the groups with parameters as in Table 3, which gives representatives of all the lattices that are known to be complex hyperbolic triangle groups, apart from Mostow groups. Note that some more parameters are known to give lattices (in other words, some lattices can have several triangle group descriptions, see section 7 of [14]).

4. Braid relations between complex reflections

The structure of each group (and of a fundamental domain for its action on $H^2_{\mathbb{C}}$) is conveniently encoded by *braid relations* between suitable conjugates of the generators. Recall that two elements a, b of a group satisfy a braid relation of length $n \in \mathbb{N}^*$ if

$$(ab)^{n/2} = (ba)^{n/2}. \tag{5}$$

When n is odd, by convention $(ab)^{n/2} = a \cdot b \cdot a \cdot b \cdot a \cdots b \cdot a$ (n factors). When equation (5) holds, we write $\mathrm{br}_n(a, b)$; if n is the smallest n such that $\mathrm{br}_n(a, b)$, we say the braid length of a and b is n, and we write $\mathrm{br}(a, b) = n$.

We will use braid relations only between complex reflections. An important result is the following (see Proposition 2.5 of [14]).

Proposition 4.1. *Suppose R_1 and R_2 are complex reflections in $PU(2, 1)$ with distinct mirrors, and multipliers $\zeta_1 = e^{2\pi i/p_1}$ and $\zeta_2 = e^{2\pi i/p_2}$ respectively ($p_j \in \mathbb{N}$, $p_j \geq 2$). Suppose $br_n(R_1, R_2)$ for some $n \in \mathbb{N}$, $n \geq 2$. Let $Z = (R_1R_2)^n$ if n is odd, $Z = (R_1R_2)^{n/2}$ if n is even.*

1. *Z commutes with both R_1 and R_2.*
2. *If $br_n(R_1, R_2)$ with n odd, then R_1 and R_2 are conjugate, in particular $p_1 = p_2$.*

It is easy to see that Z as defined in 4.1 has a repeated eigenvalue, hence it must be parabolic, a complex reflection or a complex reflection in a point (recall that we work with $H^2_{\mathbb{C}}$, so our homogeneous coordinates live in a complex vector space of dimension 3). The three cases are distinguished by looking at the position with respect to $H^2_{\mathbb{C}}$ of the intersection point in projective space of the mirrors of R_1 and R_2. In fact Z is a complex reflection in a point (resp. parabolic, resp. a complex reflection) if the mirrors intersect in $H^2_{\mathbb{C}}$ (resp. in $\partial_\infty H^2_{\mathbb{C}}$, resp. outside $\overline{H_{\mathbb{C}}}^2$).

Writing $R_j = R_{U_j,\zeta_j}$ (see formula (2)), the three cases can be conveniently characterized computationally as follows. Consider $X = U_1 \boxtimes U_2$, which is the usual cross product of U_1^*H and U_2^*H; by construction $\langle X, U_1\rangle = \langle X, U_2\rangle = 0$, so $\mathbb{C}X$ is the intersection of the mirrors in the projective plane. Then we have

Proposition 4.2. *Z is a complex reflection if $||X||^2 > 0$, parabolic if $||X||^2 = 0$, and a complex reflection in a point if $||X||^2 < 0$.*

When Z is a complex reflection, part (1) of Proposition 4.1 says that it commutes with both R_1 and R_2, so the mirror of Z is the common perpendicular complex line between the mirrors of R_1 and R_2.

The relevant braid relations for each group of section 3 appear in the Appendix of [14]. One important point is that the relations depend only on $\mathbf{T}$, not on p. In fact the fundamental domains for lattices with a given value of $\mathbf{T}$ (but different values of p) are almost the same, they differ only by a simple truncation process (see the discussion in section 5).

5. Sides of our fundamental domain

In this section, we briefly recall the construction of the sides of the fundamental domains described in [14].

It is well known that there are no totally geodesic real hypersurfaces in $H^2_{\mathbb{C}}$ (this can be seen as a consequence of the fact that it has non-constant real sectional curvature). In particular, the sides of the fundamental domains constructed in [14] are not totally geodesic.

On the other hand, part of their skeleton is totally geodesic, namely the 1-skeleton consists only of geodesic arcs, and some facets in the 2-skeleton are also totally geodesic.

Recall that each side of our domains is constructed inside a bisector, i.e. the locus of points equidistant of two given points. The geometric structure of bisectors (and even bisector intersections) is by now well understood, see chapter 5 of [17]. Each bisector $\mathcal{B}$ is topologically a 3-ball, and it has a foliation by complex lines (called the complex slices of $\mathcal{B}$), and a singular foliation by real planes (called the real slices) all intersecting in a fixed real geodesic of $H^2_{\mathbb{C}}$ (called the real spine of $\mathcal{B}$). The unique complex line containing that real geodesic is called the complex spine of $\mathcal{B}$.

If $\mathcal{B}$ is a bisector and $p, q \in \mathcal{B}$, the real geodesic through p and q is contained in $\mathcal{B}$ if and only if p, q are either in a real or in a complex slice of $\mathcal{B}$. This is always true if one of the two points is on the real spine. Our domains use the latter property, namely each side is constructed as a (possibly truncated) geodesic cone inside a bisector $\mathcal{B}$, the base of the cone being a polygon in a complex slice of $\mathcal{B}$, and its apex is on the real spine of $\mathcal{B}$ (see Figure 4.2 of [14] and also the pictures in the Appendix of [14] for examples of truncated cones).

The complex slices containing the base of the cones are given by well chosen mirrors of complex reflections in the triangle group (in fact suitable conjugates of the generating reflections R_1, R_2 and R_3). The vertices of the base polygons are either intersections of mirrors of reflections, or intersections with the common perpendicular of two mirrors of reflections when some truncation is needed (see section 4).

For example, in Figure 1, we reproduce some figures from the appendix of [14]. Let

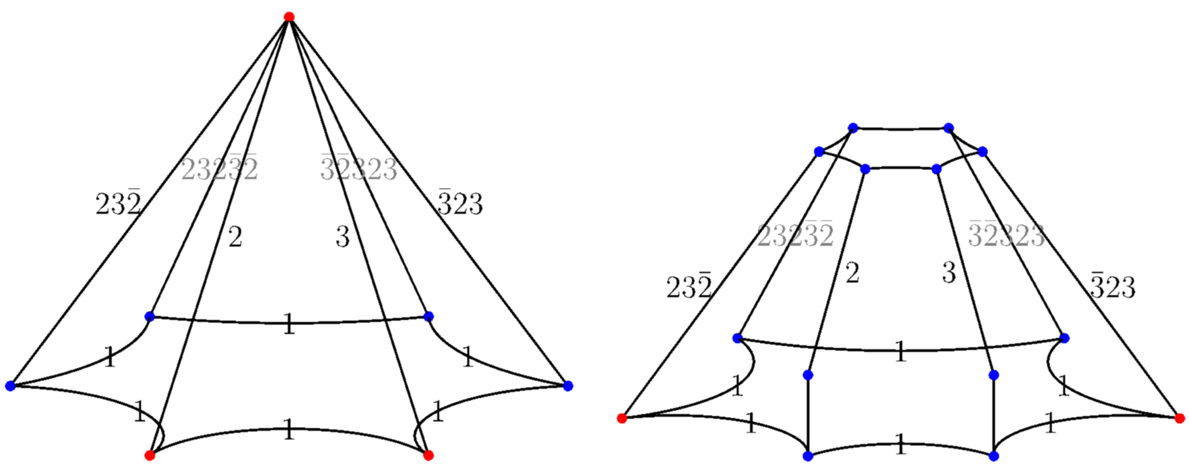


FIGURE 1. Pictures of some sides for our fundamental domain for $\mathcal{S}(p, \sigma_1)$, $p = 3$ (left) or 4 (right)

us focus first on the left half of the picture, which corresponds to the case $p = 3$, i.e. $R_j = R_{e_j,\zeta_3}$ with $\zeta_3 = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$. The labels indicate that the base is a hexagon in the mirror of R_1 (which is $e_1^\perp$), with vertices given by

$$e_1 \boxtimes e_2, \quad e_1 \boxtimes e_3, \quad e_1 \boxtimes R_3^{-1} e_2, \quad e_1 \boxtimes R_3^{-1} R_2^{-1} e_3, \quad e_1 \boxtimes R_3^{-1} R_2^{-1} e_3, \quad e_1 \boxtimes R_2 e_3.$$

Note for example that the mirror of $R_2 R_3 R_2^{-1}$ is the image under R_2 of the mirror of $R_3 = R_{e_3,\zeta_3}$, so it is $R_2(e_3)^\perp$. The apex of the cone is given by $e_2 \boxtimes e_3$.

Now consider the case $p = 4$ (right half of the picture), which correspond to generating reflections of the form $R_j = R_{e_j,i}$ ($i^2 = -1$). The corresponding side is similar (combinatorially it is the same up to truncation). In that case, the base is still a hexagon in $e_1^\perp$, but its vertices are now given by

$$e_1 \boxtimes (e_1 \boxtimes e_2), \quad e_1 \boxtimes (e_1 \boxtimes e_3), \quad e_1 \boxtimes R_3^{-1} e_2, \quad e_1 \boxtimes R_3^{-1} R_2^{-1} e_3, \quad e_1 \boxtimes R_3^{-1} R_2^{-1} e_3, \quad e_1 \boxtimes R_2 e_3.$$

The first vertex correspond to a truncation using the common perpendicular complex line of $e_1^\perp$ and $e_2^\perp$, which is $(e_1 \boxtimes e_2)^\perp$. Note that due to truncation, there is also an extra vertex, which is given by $e_2 \boxtimes (e_1 \boxtimes e_2)$.

The (right half of the) picture also indicates that the apex of the cone $e_2 \boxtimes e_3$ is not a vertex of that side, in fact the apex also needs to be truncated; the truncation is done with the common perpendicular complex line of $e_2^\perp$ and $e_3^\perp$, which is $(e_2 \boxtimes e_3)^\perp$. The truncation facet is then a hexagon on that complex line, with vertices given by

$$m \boxtimes e_2, \quad m \boxtimes e_3, \quad m \boxtimes R_3^{-1} e_2, \quad m \boxtimes R_3^{-1} R_2^{-1} e_3, \quad m \boxtimes R_3^{-1} R_2^{-1} e_3, \quad m \boxtimes R_2 e_3,$$

where $m = e_2 \boxtimes e_3$.

The 1-skeleton of the sides can be constructed from the 0-skeleton if we know the combinatorics, since the 1-skeleton consists of geodesic arcs. The 2-skeleton is more complicated to describe (see section 4.2 of [14] for details); here we simply mention that the only 2-facets that lie in complex lines are the facets on the base of the cone, as well as the truncation 2-facets that replace the apex of the cone.

Note that in the group $\mathcal{S}(p, \sigma_1)$ corresponding to the pictures in Figure 1, we have $\mathrm{br}_6(R_2, R_3)$ (which is the reason why we choose a hexagon in the base, see [14]), and for $p = 6$ $(R_2 R_3)^3$ is a complex reflection fixing the top truncation facet.

In order to refer to sides, we will use the same notation as in [14] and use the notation

$$[n]a; b, c$$

where a, b and c are complex reflections such that $\mathrm{br}_n(b, c)$. For the side written $[n]a; b, c$, a stands for a reflection fixing the base, and b, c should be thought of as defining two of the vertical edges of the cone. The full list of vertical edges is then uniquely determined by b and c; the number of elements in the list is given by n, and the list of edges is determined by the following list of reflections

$$\ldots, bcbc^{-1}b^{-1}, bcb^{-1}, b, c, c^{-1}bc, c^{-1}b^{-1}cbc, \ldots$$

given that this seemingly infinite list actually is n-periodic.

In order to describe a reflection in the group, we will use word notation, writing 1 for R_1, 2 for R_2, 3 for R_3, $\bar{1}$ for R_1^{-1}, $(123\bar{2})^3$ for $(R_1 R_2 R_3 R_2^{-1})^3$, etc.

For example the side described in Figure 1 is simply written as $[6]1; 2, 3$.

The Appendix of [14] contains a picture of (isometry types of) all sides of a fundamental domains, for every triangle group with parameter in 3. All pictures are drawn so that 2-facets that are in complex lines are drawn in a horizontal plane (either as the base of the side, or as the top side when there is truncation at the top). The base sides are always fixed by some complex reflection (in fact by a conjugate of one of the generating triple of reflections). The top sides are usually also fixed by a reflection (see Proposition 4.1), but not always; in some rare cases, the "reflection" actually degenerates to the identity.

For convenience, we reproduce some of the information in the tables from the appendix of [14], see Tables 2 and 3. For each family of groups in the list, we give a description of side representatives, and mention values of the order p of the generating reflections where the corresponding side has top truncation.

$\mathcal{S}(p,\sigma_1)$ groups

Triangle	#(P-orb)	Top trunc.
[6] 1; 2, 3	8	$p=4,6$
[4] 2; 1, $23\bar{2}$	8	$p=6$
[3] $23\bar{2}$; 1, $2323\bar{3}\bar{2}$	8	
[3] $2323\bar{3}\bar{2}$; 1, $\bar{3}\bar{2}323$	8	

$\mathcal{S}(p,\bar{\sigma}_4)$ groups

Triangle	#(P-orb)	Top trunc.
[4] 1; 2, 3	7	$p=5,6,8,12$
[3] 2; 1, $23\bar{2}$	7	$p=8,12$

$\mathcal{S}(p,\sigma_5)$ groups

Triangle	#(P-orb)	Top trunc.
[4] 1; 2, 3	30	
[5] 2; 1, $23\bar{2}$	30	$p=4$
[6] $2\bar{3}\bar{2}123\bar{2}$; 2, $\bar{3}\bar{2}\bar{1}23\bar{2}123$	5	$p=4$

$\mathcal{S}(p,\sigma_{10})$ groups

Triangle	#(P-orb)	Top trunc.
[5] 1; 2, 3	5	$p=4,5,10$
[3] 2; 1, $23\bar{2}$	5	$p=10$

TABLE 2. Side representatives for $\mathcal{S}(p,\tau)$ groups

6. Orbits of complex 2-facets

In this section, we discuss how to list the conjugacy classes of mirrors of complex reflections in all triangle groups with parameters in Table 3.

The first observation is that the list can be read off any "reasonable" fundamental domain; indeed let Γ be a complex hyperbolic lattice, and let Π be a closed, piecewise smooth, finite-sided fundamental domain for Γ, in particular

(1) $H^2_{\mathbb{C}} = \cup_{\gamma\in\Gamma}\gamma\Pi$ and
(2) for every $\gamma\in\Gamma$, $\gamma(\Pi^\circ)\cap\Pi^\circ=\emptyset$.

where Π° denotes the interior of Π. We assume moreover that Π has a side pairing in the sense of the Poincaré polyhedron theorem (see section 4.3 of [14], or section 3.2 of [13]).

Now assume $R\in\Gamma$ is a complex reflection with mirror m. Pick $x\in m$, and let $\gamma\in\Gamma$ be such that $\gamma(x)\in\Pi$ (see condition (1)). By replacing R by $\gamma R\gamma^{-1}$, we may assume $m\cap\Pi\neq\emptyset$. Because of (2), we must have $m\cap\pi\subset\partial\Pi$. In particular, we have the following.

Proposition 6.1. *Every reflection in Γ is conjugate to one fixing a complex 2-facet of Π.*

$\mathcal{T}(p, \mathbf{S_2})$ groups

Triangle	$\#(P\text{-orb})$	Top trunc.
$[3]\ 1;\ 2,\ 3$	5	
$[3]\ 23\bar{2};\ 1,\ 3$	5	
$[4]\ 3;\ 1,\ 2$	5	$p = 5$
$[5]\ 2;\ 1,\ 23\bar{2}$	5	$p = 4, 5$

$\mathcal{T}(p, \mathbf{E_2})$ groups

Triangle	$\#(P\text{-orb})$	Top trunc.
$[3]\ 1;\ 2,\ 3$	6	
$[4]\ 23\bar{2};\ 1,\ 3$	6	$p = 6$
$[4]\ 3;\ 1,\ 2$	6	$p = 6$
$[4]\ 2;\ 1,\ 23\bar{2}$	6	$p = 6$
$[6]\ \bar{3}13;\ 12\bar{1},\ 3$	3	$p = 4, 6$

$\mathcal{T}(2, \mathbf{H_1})$ groups

Triangle	$\#(P\text{-orb})$
$[3]\ 1;\ 2,\ 3$	42
$[3]\ 23\bar{2};\ 1,\ 3$	42
$[4]\ 3;\ 1,\ 2$	42
$[7]\ 2;\ 1,\ 23\bar{2}$	42
$[14]\ \bar{2}123\bar{2}\bar{1}2;\ \bar{2}12,\ \bar{3}12\bar{1}3$	3

$\mathcal{T}(p, \mathbf{H_2})$ groups

Triangle	$\#(P\text{-orb})$	Top trunc.
$[3]\ 1;\ 2,\ 3$	15	
$[3]\ 23\bar{2};\ 1,\ 3$	15	
$[5]\ 2;\ 1,\ 23\bar{2}$	15	$p = 5$
$[5]\ 3;\ 1,\ 2$	15	$p = 5$
$[10]\ (123)^2\bar{2}12(\bar{3}\bar{2}\bar{1})^2;\ 121\bar{2}\bar{1},\ 3$	3	$p = 3, 5$

TABLE 3. Side representatives for $\mathcal{T}(p, \mathbf{T})$ groups

Moreover, we can easily check whether reflections fixing two different complex 2-facets of Π are conjugate in Γ, by tracking cycles in the sense of the Poincaré; moreover, the Poincaré polyhedron theorem allows us to determine the full stabilizer in Γ of every facet (in particular complex 2-facets). This is part of what the computer program available at [8] does systematically, and the results are summarized in the computer output files available with the source code.

A slight subtlety is that the domains we construct in [14] are not fundamental domains, but only fundamental domains for coset decompositions, with respect to a finite cyclic group generated by a regular elliptic element ($P = R_1 J$ for $\mathcal{S}(p, \tau)$ groups, $Q = R_1 R_2 R_3$ for $\mathcal{T}(p, \mathbf{T})$ groups). In the next few paragraphs, we only discuss the case of $\mathcal{S}$ groups, everything goes through for $\mathcal{T}$ groups, as long as we replace P by Q).

The domains used in [14] are of the form

$$\cup_{j=0}^{n} P^j(\Pi),$$

where Π is a true fundamental domain, n is the order of P. For future reference, we write $\Pi^{(P)} = \cup_{j=0}^{n} P^j(\Pi)$.

When all powers of P are regular elliptic, this is inconsequential, but when some power is a complex reflection, we need to add extra conjugacy classes of complex reflections.

Proposition 6.2. *Every reflection in a lattice $\Gamma = \mathcal{S}(p, \tau)$ is conjugate to one fixing a complex 2-facet of $\Pi^{(P)}$ or to one fixing the mirror of some complex reflection power of P (and similarly for the groups $\mathcal{T}(p, \mathbf{T})$ with P replaced by Q).*

It follows from the previous paragraphs (and the discussion of section 5) that the mirror of every complex reflection in the group can be mapped via a suitable group element to one in the following three families.

- The complex lines containing base polygons of various sides (all of these are mirrors of conjugates of the three generating reflections);
- The complex lines containing the top 2-facets of the sides with truncated apex;
- Mirrors of complex reflections obtained as powers of $P = R_1 J$ (for groups in the $\mathcal{S}$-family) or $Q = R_1 R_2 R_3$ (for groups in the $\mathcal{T}$-family) that are complex reflections.

We call the corresponding complex 2-facets in these three families base 2-facets, top 2-facets and central 2-facets, respectively. The list of base/top/central facets can easily be deduced from the tables and pictures in the appendix of [14].

There are some obvious identifications between sides, coming from

- The side-pairing maps; these are all complex reflections fixing the base side.
- The action of $P = R_1 J$ (for $\mathcal{S}$ groups) or $Q = R_1 R_2 R_3$ (for $\mathcal{T}$ groups).

Accordingly, the lists of the appendix in [14] only list one representative for each P-orbit (resp. Q-orbit) of paired sides.

Definition 6.3. *For future reference, we refer to the sides that are listed for a given $\Gamma = \mathcal{T}(p, \mathbf{T})$ in the appendix of* [14] *as* side representatives *for* Γ.

Moreover, when two sides s_1 and s_2 appear in the appendix of [14], the only identifications between s_1 and s_2 can occur on their boundary (here by the boundary of a side s_j contained in a bisector $\mathcal{B}$, mean boundary with respect to the topology on $\mathcal{B}$ induced by the topology on $H^2_{\mathbb{C}}$). The identifications on the boundary can be read off by tracking orbits of all side-pairing transformations.

The details are difficult to write down on paper because there are many groups and each polytope has many facets, so we only summarize the important points in Proposition 6.4 (which follows by case by case analysis, tracking the cycles of 2-facets systematically for all fundamental domains).

Proposition 6.4. (1) *Let r_1 and r_2 be distinct base or top complex 2-facets of (possibly different) side representatives, and let x_j be in the open cell given by the interior of r_j. Then x_1 and x_2 are not in the same Γ-orbit.*

(2) *Let r be a base or top 2-facet of a side representative S. Assume S is not invariant under any non-trivial power P^j (for $\mathcal{S}$ groups) or Q^j (for $\mathcal{T}$ groups), and x_1, x_2 are distinct points in the open cell given by the interior of r. Then x_1 and x_2 are not in the same Γ-orbit.*

(3) *Let r be a base or top 2-facet of a side representative S contained in a bisector $\mathcal{B}$. Assume S is invariant under $R = P^j$ or Q^j (and let $j > 0$ be minimal with that property); then R is a complex reflection whose mirror is the complex spine of $\mathcal{B}$. Points in r are Γ-equivalent if and only if they are invariant under the action of the cyclic group generated by R.*

We think of these statements as giving injectivity for the quotient map on various (unions of) complex 2-facets. Point (2) says that the interior of base and top 2-facets inject, unless the side is invariant under a power of P (or Q). Point (3) says that if the side is invariant under a power or P (or Q), then the side is rotationally symmetric, and one gets a sector (of angle given by the rotation angle of P^j) that injects under the quotient map.

The exceptions to the injectivity statements as in (3) do occur for some of the groups we handle in [14], they can be spotted by listing the sides whose P-orbit contains less than $\mathrm{Ord}(P)$ elements (see the column headed $\#(P - orb)$) for $\mathcal{S}$-groups, and similary with P replaced by Q for $\mathcal{T}$ groups (see Tables 2 and 3). For example, the side $2\bar{3}\bar{2}123\bar{2}; 2, Q^{-1}23\bar{2}Q$ is depicted in Figure 2; a fundamental domain for the action of the rotation is a sector with angle $\pi/3$.

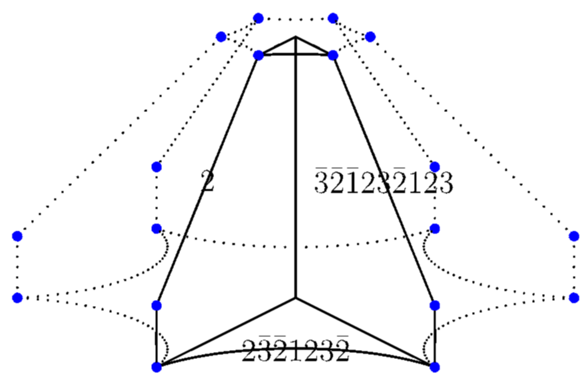


FIGURE 2. One of the sides for $\mathcal{S}(p, \sigma_5)$ is invariant under P^5, with rotational symmetry of angle $\pi/3$.

7. Tiling mirrors with images of complex 2-facets

In this section, we study the mirror stabilizers for a representative of every orbit of a complex 2-facet of Π, and also of the mirror of complex reflection powers of P or Q (if any). Proposition 6.2 says that this will give us the list of all mirrors of complex reflections up to conjugation.

The basic observation is the following.

Theorem 7.1. *If m is the mirror of any complex reflection in R in Γ, then $Stab_\Gamma(m)$ is a lattice in $Stab_{PU(H)}(m)$.*

Proof: Every point on m can be mapped to $\partial\Pi$ by a suitable element of Γ, so m is tiled by Γ-images of complex 2-facets in $\partial\Pi$. If s is a complex 2-facet of π and $\gamma_1(s)$ and $\gamma_2(s)$ are both contained in m, then $\gamma_1\gamma_2^{-1} \in Stab_\Gamma(m)$; Π has finitely many complex 2-facets, and each has a finite area, so $Stab_\Gamma(m)\backslash m$ has finite area. □

Our goal is to make the corresponding lattices in $PU(1,1)$ explicit. The proof of the theorem suggests a method to make the stabilizer explicit; we will list complex 2-facets of Π that are in the same group orbit, map them to lie in the same complex line m, in such a way that their images in m glue nicely to give a fundamental domain for a lattice in $PU(1,1)$, with a well defined side-pairing.

Note that the difficulty is to find an explicit side-pairing, the fact that the images give a (possibly disconnected) fundamental domain for the action of the stabilizer is automatic (see Proposition 6.4 and the proof of Theorem 7.1).

We now give a list of explicit maps that bring the complex 2-facets back to a common complex line. For top and central complex 2-facets, the stabilizers are all triangle groups, and it is very easy to obtain a fundamental domain consisting of two adjacent triangles.

We treat the mirrors of the generating set of reflections R_1, R_2, R_3 by a case by case analysis for all families of groups in the list of Table 3 (the maps are given by the same formula for various values of p within a fixed family $\mathcal{T}(p, \mathbf{T})$ of groups, even though the facets themselves possibly depend on p via truncation).

For $\mathcal{S}(p,\tau)$ groups, we have $JR_jJ^{-1} = R_{j+1}$, so it is enough to handle the mirror of R_1. For most groups $\mathcal{T}(p, \mathbf{T})$, the mirrors of R_1, R_2 and R_3 are still images of each other under a suitable group elements, because of the fact that $\mathrm{br}_n(a,b)$ with n odd implies that a and b are conjugate, because

$$b = (ab)^{-\frac{n-1}{2}} a (ab)^{\frac{n-1}{2}},$$

see also Proposition 4.1(2).

Indeed, the only family in the list of Table 3 where all braid lengths $\mathrm{br}(R_j, R_k)$ with $j \neq k$ are even is the family of $\mathcal{T}(p, \mathbf{E_2})$ groups (see also Table 3 for the braid lengths between generating reflections). For that family, we will handle both the mirrors of R_1 and R_2.

We list the results in the form of pictures in Figures 3-8. Note that the pictures are intended to be only combinatorial, not metric (but coordinates/parametrizations for the actual vertices/sides can be recovered from the labels on the pictures).

8. Side-pairing and fundamental domains

Our goal is now to check how the puzzle pieces described in section 7 fit together to form a fundamental domain for the stabilizer, and to find an explicit side-pairing; from this, one

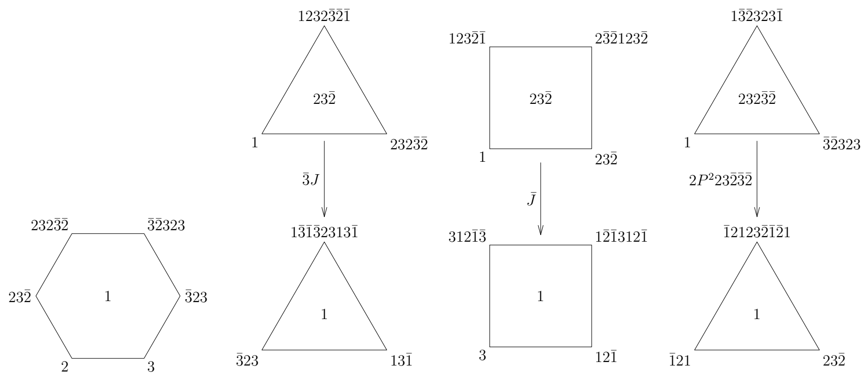


Figure 3. Maps to the mirror of R_1 for groups $\mathcal{S}(p,\sigma_1)$

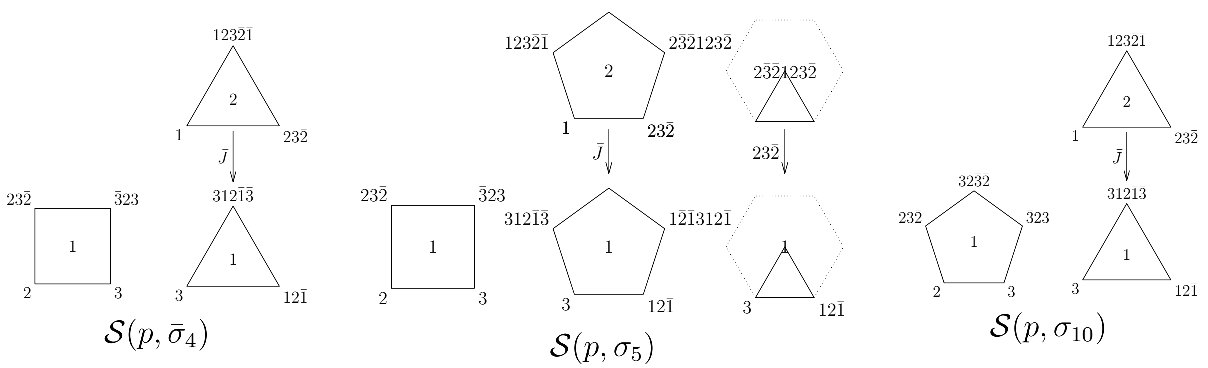


Figure 4. Maps to the mirror of R_1 for groups $\mathcal{S}(p,\bar{\sigma}_4)$, $\mathcal{S}(p,\sigma_5)$ and $\mathcal{S}(p,\sigma_{10})$

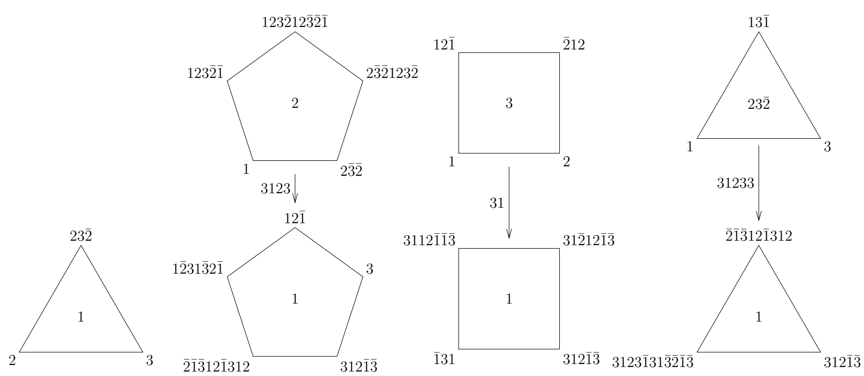


Figure 5. Maps to the mirror of R_1 for groups $\mathcal{T}(p,\mathbf{S_2})$

easily deduces a presentation (via application of the Poincaré polyhedron theorem), as well as the signature of Fuchsian groups.

We do this only for the mirror of base facets, the other ones are much easier (in fact, for top and central complex 2-facets, the jigsaw puzzle has only two pieces) A fundamental

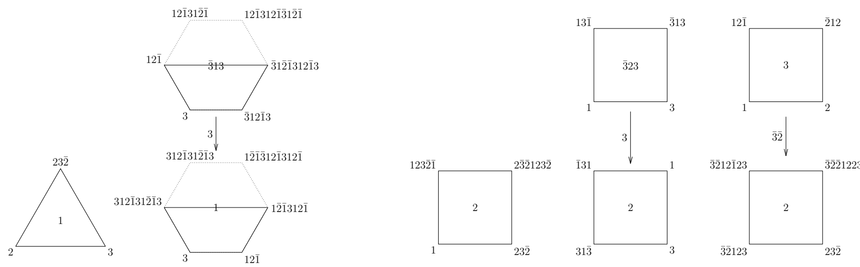


Figure 6. Maps to the mirror of R_1 for groups $\mathcal{T}(p, \mathbf{S_2})$

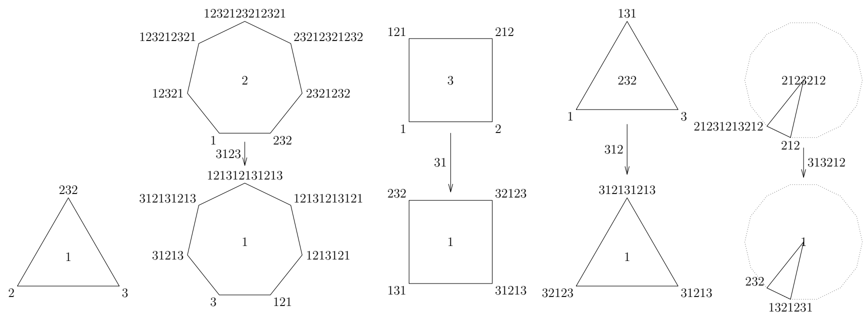


Figure 7. Maps to the mirror of R_1 for groups $\mathcal{T}(p, \mathbf{H_1})$

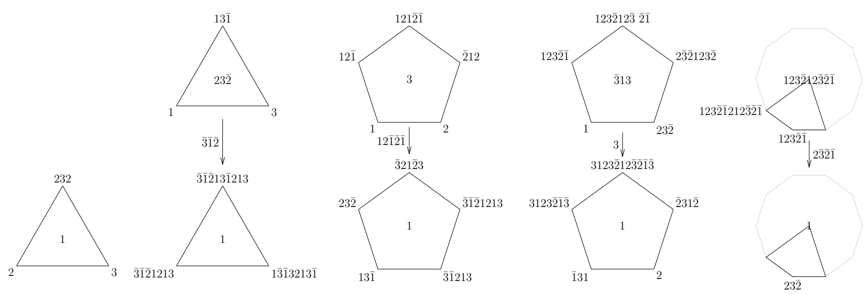


Figure 8. Maps to the mirror of R_1 for groups $\mathcal{T}(p, \mathbf{H_2})$

domain (as well as side-pairing transformations) for each of the groups in Table 3 is given in Figures 9- 17.

In a given family $\mathcal{S}(p, \tau)$ with fixed τ, the pictures are very similar, note that the labels do not indicate whether or not the polytope is truncated at that vertex. For example, for groups $\mathcal{S}(p, \sigma_1)$, the vertex fixed by $(R_1R_2)^3$ is either $e_1 \boxtimes e_2$ (for $p = 3$) or $e_1 \boxtimes (e_1 \boxtimes e_2)$ (for $p = 4$ or 6), see section 5.

9. Applying the Poincaré polyhedron theorem

In order to determine the stabilizer of a mirror (in restriction to the mirror), we use the Poincaré polyhedron theorem; in fact only the 2-dimensional special case of that theorem is needed (for a nice exposition of the special case, see [6]; see also [2]).

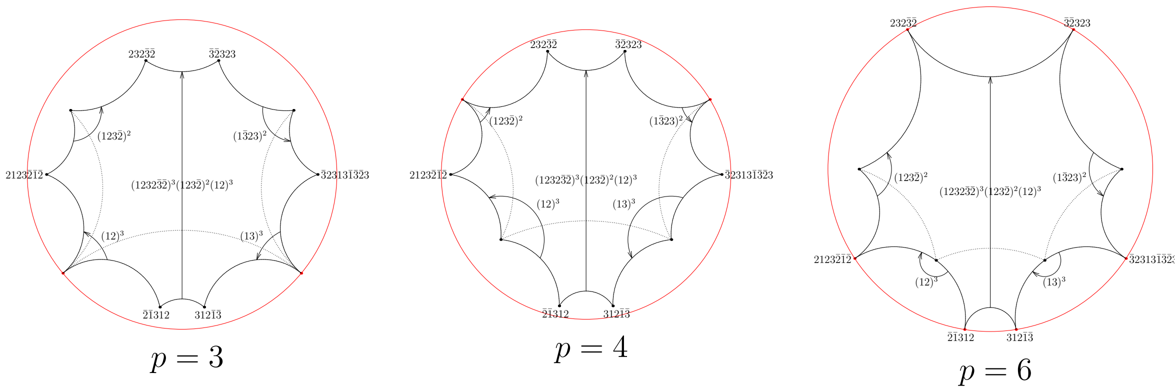


FIGURE 9. Pairing for the stabilizer of the mirror of R_1, for $\mathcal{S}(p, \sigma_1)$ groups.

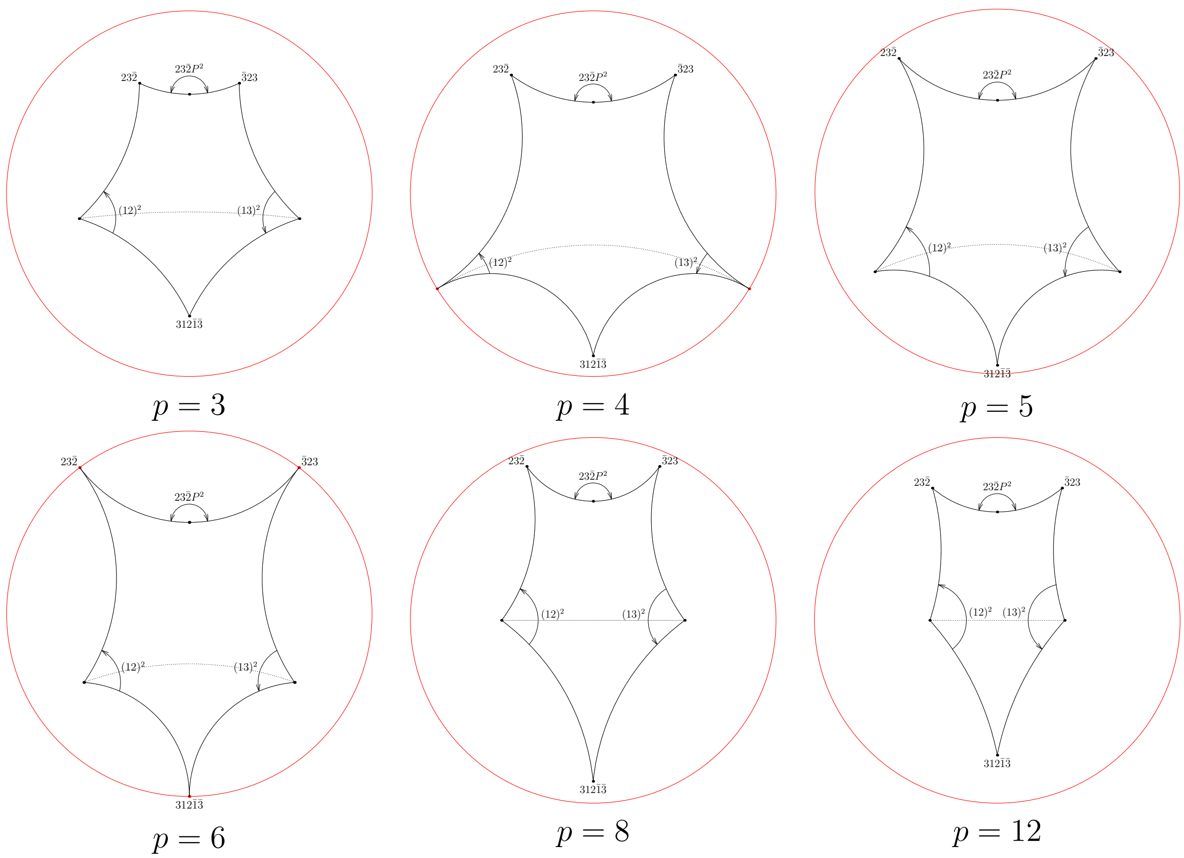


FIGURE 10. Pairing for the stabilizer of the mirror of R_1, for $\mathcal{S}(p, \bar{\sigma}_4)$ groups.

We work through a basic example in detail; the statement we prove in the next few paragraphs is given in Proposition 9.1. In the end of this section, we briefly comment on specific difficulties that come about in treating other cases.

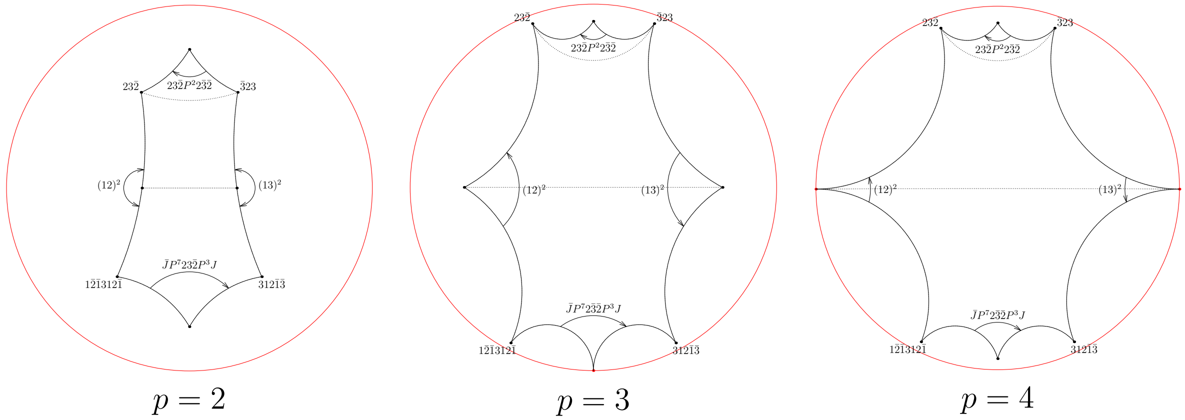


FIGURE 11. Pairing for the stabilizer of the mirror of R_1, for $\mathcal{S}(p, \sigma_5)$ groups.

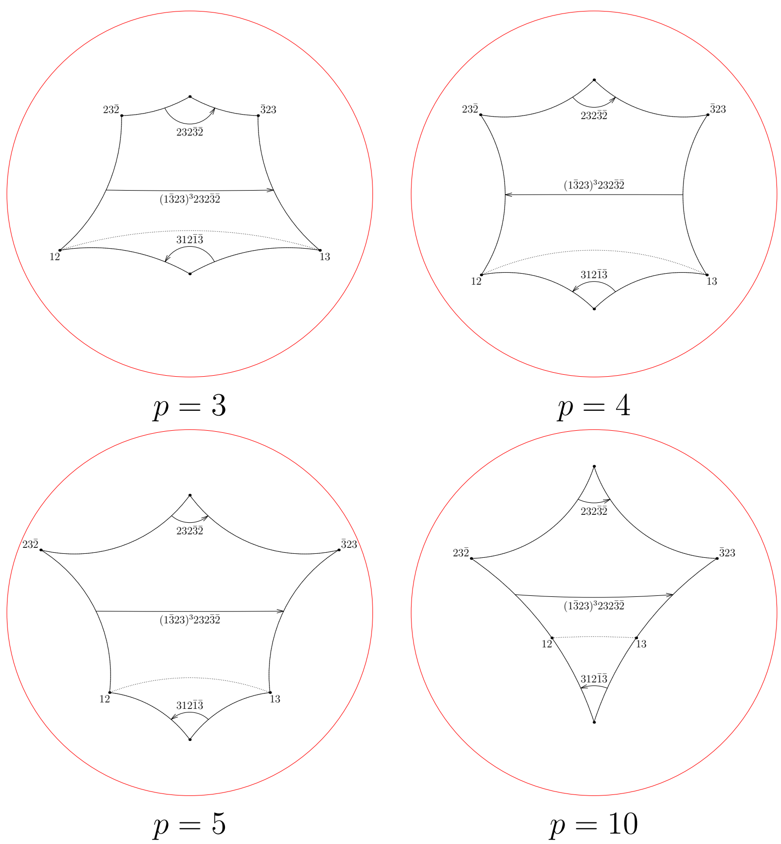


FIGURE 12. Pairing for the stabilizer of the mirror of R_1, for $\mathcal{S}(p, \sigma_{10})$ groups.

Proposition 9.1. *The stabilizer of the mirror of R_1 in $\mathcal{S}(p, \sigma_1)$ is a central extension of a Fuchsian group of signature $(0; \frac{p}{p-3}, \frac{p}{p-3}, \frac{2p}{|p-4|}, \frac{2p}{|p-4|}, \frac{2p}{|p-6|}, \frac{2p}{|p-6|})$, with center of order p (generated by R_1).*

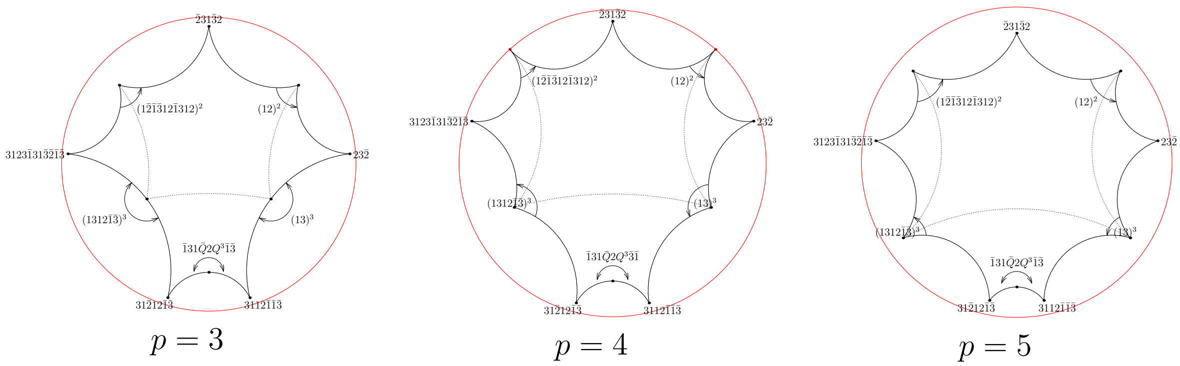


FIGURE 13. Pairing for the stabilizer of the mirror of R_1, for $\mathcal{T}(p, \mathbf{S_2})$ groups.

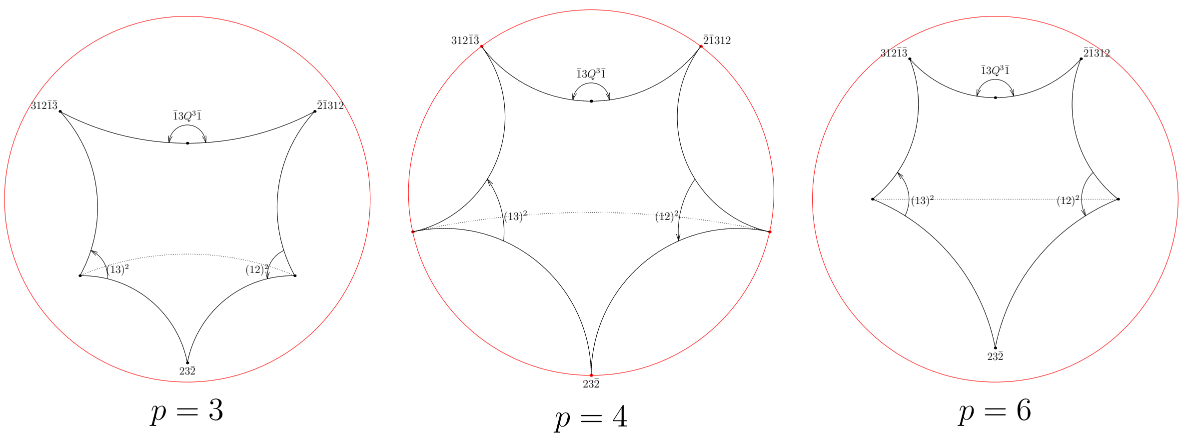


FIGURE 14. Pairing for the stabilizer of the mirror of R_1, for $\mathcal{T}(p, \mathbf{E_2})$ groups.

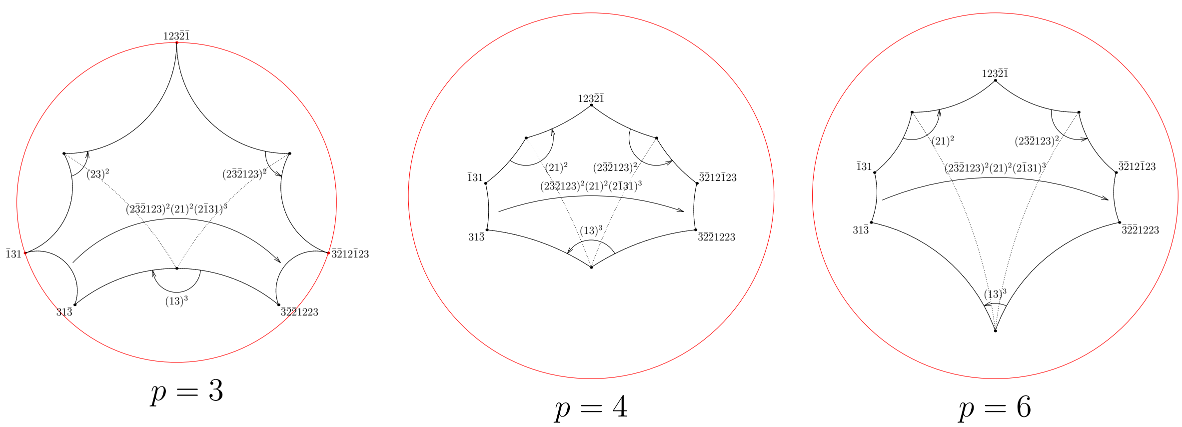


FIGURE 15. Pairing for the stabilizer of the mirror of R_2, for $\mathcal{T}(p, \mathbf{E_2})$ groups.

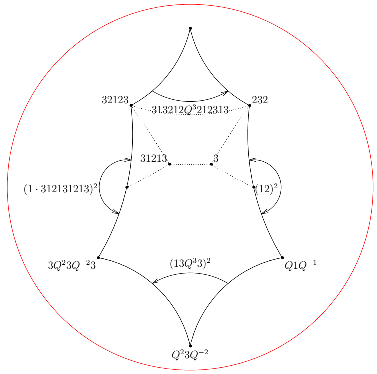
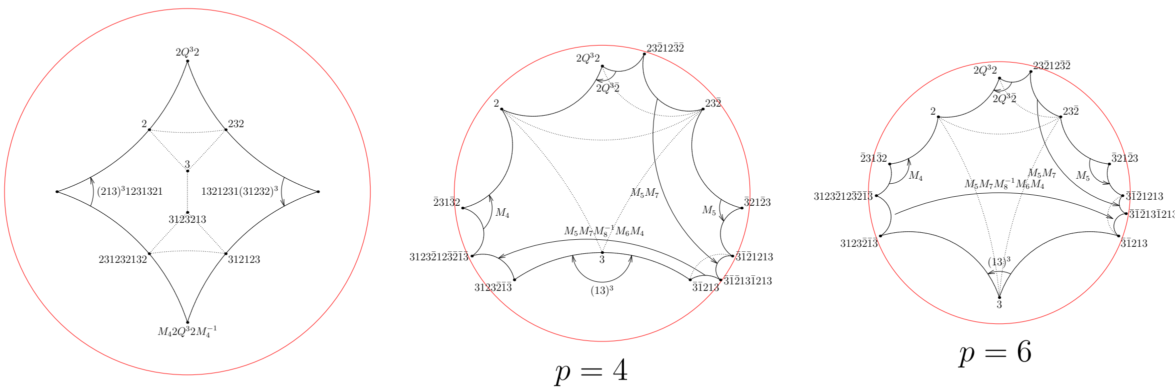


FIGURE 16. Pairing for the stabilizer of the mirror of R_1, for $\mathcal{T}(2, \mathbf{H_1})$.



$p = 3$ $\quad$ $p = 4$ $\quad$ $p = 6$

FIGURE 17. Pairing for the stabilizer of the mirror of R_1, for $\mathcal{T}(p, \mathbf{H_2})$ groups.

Recall that the statement about the signature means that the quotient of the mirror is topologically a compact complex curve of genus 0 (i.e. a copy of $P^1_{\mathbb{C}}$), with 6 cone points of multiplicity given by $\frac{p}{p-3}, \frac{p}{p-3}, \frac{2p}{|p-4|}, \frac{2p}{|p-4|}, \frac{2p}{|p-6|}, \frac{2p}{|p-6|}$) respectively. The fact that these numbers are (either ∞ or) integers comes from the fact that the conditions of the Poincaré polyhedron theorem for $\mathcal{S}(p, \sigma_1)$ are satisfied (see [14]). Note also that a cone point of ∞ multiplicity should actually be removed to get a punctured $P^1_{\mathbb{C}}$.

Proof: (of Proposition 9.1)

We invite the reader to have a look at Figure 9 (page 17). For each relevant value of p, the picture describes a geodesic decagon, with vertices that can easily be made explicit (see the discussion in section 5, in particular for the issue of truncation).

The arrows in the figure indicate an explicit side-pairing for that polygon; the quotient of the mirror of R_1 by the action of its stabilizer is then the quotient of the decagon under the equivalence relation generated by the identifications of the side-pairing.

The generic points on sides of the decagon are identified with one and only one other point on a side, but vertices are usually identified with more than two vertices; to make these identifications explicit, one needs to track the cycles of vertices.

It is easy to see from the identifications in Figure 9 that the fundamental domain for the stabilizer of the mirror of R_1 gives a cell-decomposition of the image of the mirror in the quotient with 6 vertices, 5 edges and a 1 two-cell, so we get the the topological Euler characteristic of the image of the mirror (possibly compactified when some vertices are cusps) is

$$6 - 5 + 1 = 2,$$

so the image has genus 0, and it is isomorphic to $P^1_{\mathbb{C}}$. To figure out the signature of the corresponding Fuchsian group, we need to compute the order of the vertex stabilizers for all vertices of the polygon.

For example, let us describe how to treat the vertex labelled $\bar{2}\bar{1}312$. Recall that this has homogeneous coordinates given by $e_1 \boxtimes R_2^{-1} R_1^{-1} e_3$ (this is a null vector in the case $p = 6$). It cycle looks like the following:

$$1, \bar{2}\bar{3}1312 \xrightarrow{(12)^3} 1, 2123\bar{2}\bar{1}\bar{2} \xrightarrow{(123\bar{2})^2} 1, 232\bar{3}\bar{2} \xrightarrow{((1232\bar{3}\bar{2})^3(123\bar{2})^2(12)^3)^{-1}} 1, \bar{2}\bar{3}1312 \tag{6}$$

Hence the cycle transformation at that vertex is

$$(12)^{-3}(123\bar{2})^{-2}(1232\bar{3}\bar{2})^{-3}(123\bar{2})^2(12)^3,$$

which is conjugate to

$$(1232\bar{3}\bar{2})^{-3},$$

and the latter is a complex reflection in a point of order 2 (resp. 4) if $p = 3$ (resp. 4), or a parabolic element if $p = 6$. It acts on the mirror as a rotation by angle π (resp. $\pi/2$) if $p = 3$ (resp. $p = 4$).

For this polygon, there is another cycle of the same nature (obtained by the automorphism of the lattice induced by $(1, 2, 3) \mapsto (\bar{1}, \bar{3}, \bar{2})$), as well as four cycles consisting of a single point, namely those for $1, 2$, for $1, 3$, for $1, 23\bar{2}$ and for $1, \bar{3}23$.

The maps $(12)^3$ and $(13)^3$ have the same order, they are parabolic for $p = 3$, and they have order 4 (resp. 2) for $p = 4$ (resp. $p = 6$). The maps $(123\bar{2})^2$ and $(1\bar{3}23)^2$ have the same order, they are parabolic for $p = 4$, and have order 6 for $p = 4$ or 6.

For $p = 3$, we get a Fuchsian group of signature $(0; 2, 2, 6, 6, \infty, \infty)$; more generally, the signature for other values of p (and also other group orbits of mirrors) are listed in the table of page 24. □

From the above discussion, we can also get a presentation for the stabilizer, which is a central extension (with center of order p) of the corresponding subgroup of $PU(1, 1)$. The resulting presentation for the stabilizer of the mirror R_1 in $\mathcal{S}(p, \sigma_1)$ is given in equation (7):

$$\begin{gathered} \langle\ z, a_{(12)^3}, a_{(13)^3}, a_{(123\bar{2})^2}, a_{(1\bar{3}\bar{2}3)^2}, a_{(1232\bar{3}\bar{2})^3}, a_{(1\bar{3}\bar{2}323)^3}\ |\ z \text{ central }, z^p, \\ a_{(12)^3}^{\frac{p}{p-3}}, a_{(13)^3}^{\frac{p}{p-3}}, a_{(123\bar{2})^2}^{\frac{2p}{p-4}}, a_{(1\bar{3}23)^2}^{\frac{2p}{p-4}}, a_{(1232\bar{3}\bar{2})^2}^{\frac{2p}{p-6}}, z a_{(1\bar{3}\bar{2}323)^2}^{\frac{2p}{p-6}}\ \rangle \end{gathered} \tag{7}$$

Note that the exponents that occur in the presentation are written as fractions, and the denominator of that fraction vanishes for some value of p; when this happens, the corresponding exponent should be thought of as being infinite, and the corresponding relation should be removed from the presentation. Note also that, when finite, the fractions actually define integers, because the hypotheses of the Poincaré polyhedron theorem hold (see [14] for more details).

We now briefly sketch the diffulties that are to be overcome for other families of groups. The general method is the following, where Γ refers to a given lattice triangle group (see 3), and m is the mirror of a given reflection a. We list a number of ressources we have for finding maps that define a well-defined side-pairing for the polygon. For the simplest cases, ressource (1) suffice (e.g. the example treated above).

(1) Find the obvious complex reflections fixing each vertex of the polygon (for the mirror of a and vertex labelled b, consider $(ab)^k$ for the smallest $k \in \mathbb{N}^*$ such that $(ab)^k$ commutes with both a and b, see Proposition 4.1).
(2) For each vertex v of the polygon, find $Stab_\Gamma(v)$ by tracking vertex cycles in the fundamental polytope for Γ described in [14], and deduce $Stab_{Stab_\Gamma(m)}(v)$ by taking the subgroup of $Stab_\Gamma(v)$ stabilizing m.
(3) Same as the previous ressource, replacing v by an edge e of the polygon. In some cases, $Stab_\Gamma(e)$ contains a flip, i.e. an element of order 2 exchanging the edges of e. In that case, subdivide the edge e and add an extra vertex in its middle.
(4) If the above still does not give a well-defined side-pairing, we consider sides stabilized by a complex reflection spower of P or Q as in Proposition 6.4 and replace the corresponding complex 2-facets by sectors.

In the next few paragraphs, we sketch the specific difficulties of each family of groups in Table 3.

- For the groups $\mathcal{S}(p, \sigma_1)$, ressource (1) gives the side-pairing transformations $(12)^3$, $(13)^3$, $(123\bar{2})^2$, $(1\bar{3}23)^2$; the reflections $(12323\bar{3}\bar{2})^3$, $(1\bar{3}\bar{2}323)^3$, $(1\bar{2}\bar{1}312)^3$, $(13121\bar{1}\bar{3})^3$ are not used (directly) as side-pairing transformations, since for these complex reflections, the image of the polygon meets the original polygon only in a vertex. However, the side-pairing transformation $(12323\bar{3}\bar{2})^4(123\bar{2})^2(12)^3$, which is a product of well-chosen complex reflection as above, was found by guessing (based on a mix of explicit computation and drawing pictures).
- For the groups $\mathcal{S}(p, \bar{\sigma}_4)$, ressource (1) produces $(12)^2$ and $(13)^2$, $(123\bar{2})^3$, $(1\bar{3}23)^3$, $(1312\bar{1}\bar{3})^3$. The first two give side-pairing transformations, while the last three do not (the image of the polygon intersects the polygon only in a point). Ressource (3) gives the element $23\bar{2}P^2$, hence we subdivide the top edges in Figure 10.
- For the groups $\mathcal{S}(p, \sigma_5)$, ressource (1) produces $(12)^2$ and $(13)^2$; ressource (4) gives $23\bar{2}P^2 2\bar{3}\bar{2}$, while $J^{-1}P^7 23\bar{2}P^3 J$ was obtained by combining (2) and (4).
- For the groups $\mathcal{S}(p, \sigma_{10})$ we use ressource (1), and combine the complex reflections to identify the vertical sides via $(1\bar{3}23)^2 2323\bar{3}\bar{2}$.
- For the groups $\mathcal{T}(p, \mathbf{S_2})$ we use (1) and (2) (which gives the sides-pairing transformation for the bottom edge).

- For the mirror of R_1 for groups $\mathcal{T}(p, \mathbf{E_2})$, we use (1) and (2) (which gives the sides-pairing transformation for the top edge). For the mirror of R_2, we use (1) and combine the corresponding complex reflections in an intricate manner.
- For the group $\mathcal{T}(2, \mathbf{H_1})$ we use (1) and (2) and (4).
- For the groups $\mathcal{T}(p, \mathbf{H_2})$, which are by far the most complicated to handle, we use all ressources.

10. Summary of the results

In this section, we gather information on the mirror stabilizers for all conjugacy classes of complex reflections in lattice complex hyperbolic triangle groups. For each group and each orbit of complex reflection mirror, we list

- Generators for the stabilizer;
- The order of the fix point stabilizer of the mirror (see the column headed "Order fix pt stab");
- The signature of the corresponding Fuchsian subgroup, its orbifold Euler characteristic and its area;
- The field $\mathbb{Q}(\mathrm{Tr}\Gamma^2)$ generated by $\mathrm{tr}(\gamma^2)$ for γ in the Fuchsian group (seen as a subgroup of $SU(1,1)$);
- The arithmeticity of the Fuchsian group (A=arithmetic, NA = non-arithmetic).

Note that for the mirrors of R_1 (or R_2 for groups $\mathcal{T}(p, \mathbf{E_2})$), the trace field $\mathbb{Q}(\mathrm{Tr}\Gamma^2)$ is the same as the adjoint trace field of the ambient lattice, and the (non-)arithmeticity is inherited from the ambient lattice.

This can easily be verified by a case by case analysis, for example by using the method explained in [18].

Groups $\mathcal{S}(p, \sigma_1)$

Reflection	Values of order p	Generators
1	3,4,6	$(12)^3, (13)^3, (123\bar{2})^2, (1\bar{3}\,23)^2, (1232\bar{3}\,\bar{2})^3, (1\bar{3}\bar{2}323)^3$
$(12)^3$	4,6	$1, 2$
$(123\bar{2})^2$	6	$1, 23\bar{2}$

Mirror of $R = R_1$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
3	3	$(0; 2, 2, 6, 6, \infty, \infty)$	$-8/3$	$16\pi/3$	$\mathbb{Q}(\sqrt{6})$	NA
4	4	$(0; 4, 4, 4, 4, \infty, \infty)$	-3	6π	$\mathbb{Q}(\sqrt{2})$	NA
6	6	$(0; 2, 2, 6, 6, \infty, \infty)$	$-8/3$	$16\pi/3$	$\mathbb{Q}(\sqrt{6})$	NA

Mirror of $R = (R_1R_2)^3$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
4	4	$(0; 3, 4, 4)$	$-1/6$	$\pi/3$	$\mathbb{Q}$	A
6	2	$(0; 3, 6, 6)$	$-1/3$	$2\pi/3$	$\mathbb{Q}$	A

Mirror of $(R_1R_2R_3R_2^{-1})^2$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
6	6	$(0; 2, 6, 6)$	$-1/6$	$\pi/3$	$\mathbb{Q}$	A

11. Hybrids

Using the results of section 10, we can obtain a description of some lattices in the list as hybrids (in the sense of [23], [16]). By this, we mean that we find two orthogonal $\mathbb{C}$-Fuchsian subgroups Σ_1 and Σ_2 of Γ such that $\Sigma_1 \cup \Sigma_2$ generates Γ. Here "orthogonal" means that the complex lines preserved by the groups are orthogonal.

In this paper, we do not address the issue of arithmeticity or commensurability of the Fuchsian subgroups of complex hyperbolic lattice triangle groups, but we do exhibit several pairs of orthogonal $\mathbb{C}$-Fuchsian subgroup that generate the ambient lattice.

Theorem 11.1. *For $p = 8$ and 12, the group $\mathcal{S}(p, \bar{\sigma}_4)$ is generated by the union of two mirror stabilizers; one can take the following pairs of reflections:*

- *R_1 and $(123\bar{2})^3$,*
- *R_1 and $(1\bar{3}23)^3$,*
- *R_1 and $(1312\bar{1}\bar{3})^3$.*

Proof: Given the table on page 25, the first statement is that $\mathcal{S}(p, \bar{\sigma}_4)$ is generated by 1, $23\bar{2}$, $(12)^3$, $(13)^3$, $(1\bar{3}23)^2$, $23\bar{2}(1J)^2$. This can be checked using computational group theory software (GAP or Magma), using the command to compute the index of a subgroup.

The second statement is similar.

The third one is slightly different, but similar as well; we check that 1, $312\bar{1}\bar{3}$, $(12)^2$, $(13)^2$, $(123\bar{2})^2$, $(1\bar{3}23)^2$ and $23\bar{2}(1J)^2$ generate $\mathcal{S}(p, \bar{\sigma}_4)$. Once again, we check this using Magma.

Groups $\mathcal{S}(p, \overline{\sigma}_4)$

Reflection	Values of order p	Generators
1	3,4,5,6,8,12	$(12)^2$, $(13)^2$, $(123\bar{2})^3$, $(1\bar{3}23)^3$,$23\bar{2}P^2$
$(12)^2$	5,6,8,12	$1, 2$
$(123\bar{2})^2$	8,12	$1, 23\bar{2}$

Mirror of $R = R_1$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
3	3	$(0; 2, 6, 6, 6)$	-1	2π	$\mathbb{Q}(\sqrt{21})$	A
4	4	$(0; 4, 8, \infty, \infty)$	$-13/8$	$13\pi/4$	$\mathbb{Q}(\sqrt{7})$	NA
5	5	$(0; 10, 10, 10, 10)$	$-8/5$	$16\pi/5$	$\mathbb{Q}(\sqrt{\frac{5+\sqrt{5}}{14}})$	NA
6	6	$(0; 6, 6, 12, \infty)$	$-19/12$	$19\pi/6$	$\mathbb{Q}(\sqrt{21})$	NA
8	8	$(0; 4, 4, 8, 16)$	$-21/16$	$21\pi/8$	$\mathbb{Q}(\sqrt{2}, \sqrt{7})$	NA
12	12	$(0; 3, 3, 4, 24)$	$-25/24$	$25\pi/12$	$\mathbb{Q}(\sqrt{3}, \sqrt{7})$	NA

Mirror of $R = (R_1R_2)^2$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
5	10	$(0; 2, 5, 5)$	$-1/10$	$\pi/5$	$\mathbb{Q}(\sqrt{5})$	A
6	6	$(0; 2, 6, 6)$	$-1/6$	$\pi/3$	$\mathbb{Q}$	A
8	4	$(0; 2, 8, 8)$	$-1/4$	$\pi/2$	$\mathbb{Q}(\sqrt{2})$	A
12	3	$(0; 2, 12, 12)$	$-1/3$	$2\pi/3$	$\mathbb{Q}(\sqrt{3})$	A

Mirror of $(R_1R_2R_3R_2^{-1})^3$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
8	8	$(0; 2, 3, 8)$	$-1/24$	$\pi/12$	$\mathbb{Q}(\sqrt{2})$	A
12	4	$(0; 2, 3, 12)$	$-1/12$	$\pi/6$	$\mathbb{Q}(\sqrt{3})$	A

Note that these generation statements are actually true for $p = 3, 4, 5, 6$ as well, but in those cases, the elements $(123\bar{2})^3$, $(1\bar{3}23)^3$, $(1312\bar{1}\bar{3})^3$ are complex reflections in points rather than lines. □

With a similar proof, we get the following generation results.

Theorem 11.2. *The group $\mathcal{S}(2, \sigma_5)$ is generated by the union of stabilizers of the mirrors of P^5 and $2\bar{3}\bar{2}123\bar{2}$. For $p = 4$, it is also generated by union of the stabilizers of the following pairs of reflections:*

- *R_1 and $(123\bar{2})^5$;*
- *R_1 and $(1\bar{3}23)^5$;*
- *R_1 and $(1\bar{2}\bar{1}312)^5$;*
- *R_1 and $(1312\bar{1}\bar{3})^5$;*

Theorem 11.3. *For $p = 4$, 5 and 10, the group $\mathcal{S}(p, \sigma_{10})$ generated by the union of the stabilizers of the mirrors of*

Groups $\mathcal{S}(p,\sigma_5)$

Reflection	Values of order p	Generators
1	2,3,4	$(12)^2, (13)^2, 23\bar{2}P^5 2\bar{3}\bar{2}, J^{-1}P^7 23\bar{2}P^3 J$
$(123\bar{2})^5$	4	$1, 23\bar{2}$
P^5	2,3,4	$P^7, 2\bar{3}\bar{2}123\bar{2}$

Mirror of $R = R_1$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
2	2	$(0;2,2,6,6)$	$-2/3$	$4\pi/3$	$\mathbb{Q}(\sqrt{5})$	A
3	3	$(0;6,6,6,6,\infty)$	$-7/3$	$14\pi/3$	$\mathbb{Q}(\sqrt{5})$	NA
4	4	$(0;4,6,12,\infty,\infty)$	$-5/2$	5π	$\mathbb{Q}(\sqrt{3},\sqrt{5})$	NA

Mirror of $(R_1R_2R_3R_2^{-1})^5$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
4	4	$(0;2,4,5)$	$-1/20$	$\pi/10$	$\mathbb{Q}(\sqrt{5})$	A

Mirror of $R = P^5$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
2	6	$(0;2,5,6)$	$-2/15$	$4\pi/15$	$\mathbb{Q}(\sqrt{5})$	A
3	6	$(0;3,5,\infty)$	$-7/15$	$14\pi/15$	$\mathbb{Q}(\sqrt{5})$	NA
4	6	$(0;4,5,12)$	$-7/15$	$14\pi/15$	$\mathbb{Q}(\sqrt{3},\sqrt{5})$	NA

- R_1 *and* $(12)^5$*;*
- R_1 *and* $(13)^5$.

For $p = 10$*, it is also generated by the stabilizers of*

- R_1 *and* $(123\bar{2})^3$*;*
- R_1 *and* $(1\bar{3}23)^3$.

Theorem 11.4. *For* $p = 4$ *or* 5*, the group* $\mathcal{T}(p,\mathbf{S_2})$ *is generated by the union of the stabilizers of* R_1 *and* $(123\bar{2})^5$.

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Groups $\mathcal{S}(p, \sigma_{10})$

Reflection	Values of order p	Generators
1	3,4,5,10	$312\bar{1}\bar{3}$, $232\bar{3}\bar{2}$, $(123\bar{2})^3$, $(1\bar{3}23)^3$
$(12)^5$	4,5,10	$1, 2$
$(123\bar{2})^3$	10	$1, 23\bar{2}$

Mirror of $R = R_1$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
3	3	$(0; 2, 3, 3, 6)$	$-2/3$	$4\pi/3$	$\mathbb{Q}(\sqrt{5})$	A
4	4	$(0; 4, 4, 4, 4)$	-1	2π	$\mathbb{Q}(\sqrt{5})$	A
5	5	$(0; 2, 5, 5, 10)$	-1	2π	$\mathbb{Q}(\sqrt{5})$	A
10	10	$(0; 5, 10, 10)$	$-3/5$	$6\pi/5$	$\mathbb{Q}(\sqrt{5})$	A

Mirror of $R = (R_1R_2)^5$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
4	4	$(0; 2, 4, 5)$	$-1/20$	$\pi/10$	$\mathbb{Q}(\sqrt{5})$	A
5	2	$(0; 2, 5, 5)$	$-1/10$	$\pi/5$	$\mathbb{Q}(\sqrt{5})$	A
10	1	$(0; 2, 5, 10)$	$-1/5$	$2\pi/5$	$\mathbb{Q}(\sqrt{5})$	A

Mirror of $(R_1R_2R_3R_2^{-1})^3$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
10	5	$(0; 2, 3, 10)$	$-1/15$	$2\pi/15$	$\mathbb{Q}(\sqrt{5})$	A

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Groups $\mathcal{T}(p, \mathbf{S_2})$

Reflection	Values of order p	Generators
1	3,4,5	$(131\bar{2}12\bar{1}\bar{3})^5, (13112\bar{1}\bar{1}\bar{3})^5, (1312\bar{1}\bar{3})^3, (13)^3,$
		$(13123\bar{1}31\bar{3}\bar{2}\bar{1}\bar{3})^5, (123\bar{2})^5, (1\bar{2}\bar{1}\bar{3}12\bar{1}312)^2, (12)^2$
$(12)^2$	5	$1, 2$
$(123\bar{2})^5$	4,5	$1, 23\bar{2}$

Mirror of $R = R_1$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
3	3	$(0; 2, 2, 2, 6, 6, 6)$	-2	4π	$\mathbb{Q}(\sqrt{5})$	A
4	4	$(0; 2, 4, 4, 4, \infty, \infty)$	$-11/4$	$11\pi/2$	$\mathbb{Q}(\sqrt{3}, \sqrt{5})$	NA
5	5	$(0; 2, 2, 10, 10, 10, 10)$	$-13/5$	$26\pi/5$	$\mathbb{Q}(\cos\frac{2\pi}{15})$	NA

Mirror of $R = (R_1R_2)^2$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
5	10	$(0; 2, 5, 5)$	$-1/10$	$\pi/5$	$\mathbb{Q}(\sqrt{5})$	A

Mirror of $(R_1R_2R_3R_2^{-1})^5$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
4	4	$(0; 2, 4, 5)$	$-1/20$	$\pi/10$	$\mathbb{Q}(\sqrt{5})$	A
5	2	$(0; 2, 5, 5)$	$-1/10$	$\pi/5$	$\mathbb{Q}(\sqrt{5})$	A

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Groups $\mathcal{T}(p, \mathbf{E_2})$

Reflection	Values of order p	Generators
1	3,4,6	$(13)^2, (12)^2, (1312\bar{1}\bar{3})^2, (1\bar{2}\bar{1}312)^2$
2	3,4,6	$(21)^2, (2\bar{3}\bar{2}123)^2, (2\bar{1}31)^3, (23)^3$
$(12)^2$	6	$1, 2$
$(13)^2$	6	$1, 3$
$(123\bar{2})^2$	6	$1, 23\bar{2}$
$(312\bar{1})^3$	4,6	$3, 12\bar{1}$
Q^3	3,4,6	$Q, Q^{-1}2\bar{3}\bar{2}123\bar{2}$

Mirror of $R = R_1$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
3	3	$(0; 2, 6, 6, 6)$	-1	2π	$\mathbb{Q}$	A
4	4	$(0; 2, \infty, \infty, \infty)$	$-3/2$	3π	$\mathbb{Q}(\sqrt{3})$	NA
6	6	$(0; 2, 6, 6, 6)$	-1	2π	$\mathbb{Q}$	A

Mirror of $R = R_2$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
3	3	$(0; 2, 6, 6, 6, \infty)$	-2	4π	$\mathbb{Q}$	A
4	4	$(0; 4, 4, \infty, \infty, \infty)$	$-5/2$	5π	$\mathbb{Q}(\sqrt{3})$	NA
6	6	$(0; 2, 6, 6, 6, \infty)$	-2	4π	$\mathbb{Q}$	A

Mirror of $R = (R_1R_2)^2$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
6	4	$(0; 2, 6, 6)$	$-1/6$	$\pi/3$	$\mathbb{Q}$	A

Mirror of $R = (R_1R_3)^2$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
6	4	$(0; 2, 6, 6)$	$-1/6$	$\pi/3$	$\mathbb{Q}$	A

Mirror of $(R_1R_2R_3R_2^{-1})^2$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
6	6	$(0; 2, 6, 6)$	$-1/6$	$\pi/3$	$\mathbb{Q}$	A

Mirror of $(R_3R_1R_2R_1^{-1})^3$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
4	4	$(0; 3, 4, 4)$	$-1/6$	$\pi/3$	$\mathbb{Q}$	A
6	2	$(0; 3, 6, 6)$	$-1/3$	$2\pi/3$	$\mathbb{Q}$	A

Mirror of Q^3

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
3	2	$(0; 3, 3, \infty)$	$-1/3$	$2\pi/3$	$\mathbb{Q}$	A
4	2	$(0; 3, 4, 12)$	$-1/3$	$2\pi/3$	$\mathbb{Q}(\sqrt{3})$	A
6	2	$(0; 3, 6, 6)$	$-1/3$	$2\pi/3$	$\mathbb{Q}$	A

Groups $\mathcal{T}(p, \mathbf{H_1})$

Reflection	Values of order p	Generators
1	2	$(12)^2, (1312131213)^2, 313212Q^3212313, (13Q^33)^2$
Q^3	2	$Q, 2Q^32$

Mirror of $R = R_1$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
2	2	$(0; 2, 2, 14, 14)$	$-6/7$	$12\pi/7$	$\mathbb{Q}(\cos\frac{2\pi}{7})$	A

Mirror of Q^3

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
2	14	$(0; 2, 3, 14)$	$-2/21$	$4\pi/21$	$\mathbb{Q}(\cos\frac{2\pi}{7})$	A

Groups $\mathcal{T}(p, \mathbf{H_2})$

Reflection	Values of order p	Generators
1	2,3,5	$(1\bar{2}31\bar{3}2)^5, (1\bar{3}21\bar{2}3)^5, (13)^3, (123\bar{2})^5, (12)^5, 2Q^3\bar{2}$
$(12)^5$	5	$1, 23\bar{2}$
$(123\bar{2})^5$	5	$1, 23\bar{2}$
$(3121\bar{2}\bar{1})^5$	3,5	$3, 121\bar{2}\bar{1}$
Q^3	2	$Q^{-2}\bar{2}\bar{1}2, 123\bar{2}12\bar{3}\bar{2}\bar{1}$

Mirror of $R = R_1$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
2	2	$(0; 5, 10, 10)$	$-8/5$	$16\pi/5$	$\mathbb{Q}(\sqrt{5})$	A
3	3	$(0; 2, 2, 5, 6, 6, 15, 15)$	$-10/3$	$20\pi/3$	$\mathbb{Q}(\cos\frac{2\pi}{15})$	NA
5	5	$(0; 2, 2, 5, 5, 5, 10, 10)$	$-16/5$	$32\pi/5$	$\mathbb{Q}(\sqrt{3})$	A

Mirror of $(R_1R_2)^5$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
5	2	$(0; 2, 5, 5)$	$-1/10$	$\pi/5$	$\mathbb{Q}(\sqrt{5})$	A

Mirror of $(R_1R_2R_3R_2^{-1})^5$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
5	2	$(0; 2, 5, 5)$	$-1/10$	$\pi/5$	$\mathbb{Q}(\sqrt{5})$	A

Mirror of $(R_3R_1R_2R_1R_2^{-1}R_1^{-1})^5$

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
3	3	$(0; 3, 3, 5)$	$-2/15$	$4\pi/15$	$\mathbb{Q}(\sqrt{5})$	A
5	1	$(0; 5, 5, 5)$	$-2/5$	$4\pi/5$	$\mathbb{Q}(\sqrt{5})$	A

Mirror of Q^3

p	Order fix pt. stab	Signature	χ^{orb}	Area	$\mathbb{Q}(\mathrm{Tr}\Gamma^2)$	A/NA
2	14	$(0; 2, 3, 10)$	$-1/15$	$2\pi/15$	$\mathbb{Q}(\sqrt{5})$	A
3	14	$(0; 3, 3, 15)$	$-4/15$	$8\pi/15$	$\mathbb{Q}(\cos\frac{2\pi}{15})$	A
5	14	$(0; 3, 5, 5)$	$-4/15$	$8\pi/15$	$\mathbb{Q}(\sqrt{5})$	A