# ON SUBGROUPS OF FINITE INDEX IN COMPLEX HYPERBOLIC LATTICE TRIANGLE GROUPS

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Abstract. We study several explicit finite index subgroups in the known complex hyperbolic lattice triangle groups, and show some of them are neat, some of them have positive first Betti number, some of them have a homomorphisms onto a non-Abelian free group. For some lattice triangle groups, we determine the minimal index of a neat subgroup. Finally, we answer a question raised by Stover and describe an infinite tower of neat ball quotients all with a single cusp.

## 1. Introduction

The goal of this paper is to study some explicit finite index subgroups of the known complex hyperbolic lattice triangle groups. Recall that the first examples of such lattices were studied by Mostow [33] (see also [7], [43] and the references given there). New examples were discovered quite a bit later in joint work of the author with Parker aud Paupert, see [13], [14].

Even though triangle groups are of course very special among all lattices in $PU(2,1)$, they allow us to describe all commensurability classes of non-arithmetic lattices in $PU(2,1)$ that are known to this day, so we consider it an important and interesting class.

In the papers cited above, explicit fundamental domains for the action of complex hyperbolic triangle groups are described, which allows us to describe

- a presentation for these groups in terms of generators and relations;
- the conjugacy classes of isotropy groups;
- the conjugacy classes of cusps.

Even though most of that information can be gathered in [14], that task would require using the output of our computer program Spocheck [9], which is probably too much to ask from the average reader.

In this paper we explain some details on how to achieve the last two items, we give enough information for the above data to be recontructed fairly convieniently, and we list the results in the form of a computer file [10]. We also give some applications to the construction of explicit finite index subgroups of (some of) the complex hyperbolic lattice triangle groups.

We focus on three different kinds of subgroups, namely

- torsion-free/neat subgroups;
- subgroups with positive first Betti number;

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- subgroups that surject onto a non-abelian free group.

Recall that a lattice $\Gamma \subset PU(2,1)$ is torsion-free (i.e. contains is no non-trivial element of finite order) if and only if the quotient map $\mathbb{H}^2_{\mathbb{C}} \to \mathbb{H}^2_{\mathbb{C}}/\Gamma$ is an unramified covering. Equivalently, one requires that the quotient $\mathbb{H}^2_{\mathbb{C}}/\Gamma$ is a manifold with charts given by local inverses of the quotient map from $\mathbb{H}^2_{\mathbb{C}}$.

A lattice is neat (see [4]) if it is torsion-free, and every parabolic element in the group can be realized by a unipotent matrix. Neat lattices are also important, because when a lattice $\Gamma \subset PU(2,1)$ is neat, the quotient $X = \mathbb{H}^2_{\mathbb{C}}/\Gamma$ admits a smooth toroidal compactification, obtained by adding to each end of $X$ an elliptic curve with negative self-intersection (see [2] or [32]).

A well-known result of Selberg, usually referred to as Selberg's lemma (see [39], [1]), says that every finitely generated matrix group admits a neat subgroup of finite index. Selberg's argument relies on taking a principal congruence subgroup modulo a well-chosen ideal, and this tends to produce subgroups of very large index (see the tables of section 7.3 for examples). We will attempt to get torsion-free subgroups of reasonably small index, so that one can hope to get a presentation, compute the abelianization, count and describe the cusps.

In the following discussion, let $\Gamma$ stand for a lattice triangle group. By construction $\Gamma$ is not torsion-free, it is in fact generated by a special kind of torsion element (namely complex reflections, which are elliptic transformations with a repeated eigenvalue, see section 2.1).

It is an standard consequence of the orbit-stabilizer theorem that the index of a torsion-free subgroup $H \subset \Gamma$ must be a multiple of the least common multiple $L$ of the orders of isotropy groups in $\Gamma$ (see [17] for example). Moreover, if $\Gamma$ (hence also $H$) is cocompact, a standard consequence of Noether's formula (see Proposition 3.3) implies that $\chi(H\backslash\mathbb{H}^2_{\mathbb{C}})$ must be a multiple of 3, which in some cases gives a slightly larger lower bound for the index $[\Gamma : H]$.

We call the *obvious lower bound* the number

$$L^{opt} = \begin{cases} 3L \text{ if } \Gamma \text{ is cocompact and } 3\chi^{orb}(\mathbb{H}^2_{\mathbb{C}}/\Gamma) \notin 3\mathbb{Z} \\ L \text{ otherwise} \end{cases} \tag{1}$$

If there exists a neat subgroup of index $L^{opt}$, we will say that **the obvious lower bound is realized**.

For lattices in $PU(1,1) \cong PSL(2,\mathbb{R})$, the analogue of the obvious lower bound (with a factor of 2 instead of 3) is realized for every lattice, by a result of Edmonds, Ewing and Kulkarni [17]. For $PSL(2,\mathbb{C})$, Jones and Reid [25] have shown that there are lattices where the obvious lower bound is arbitrarily far from being realized.

About lattice triangle groups in $PU(2,1)$, we prove the following using the results in this paper in conjunction with Magma calculations, see section 7.3.

**Theorem 1.1.** *The obvious lower bound is realized for* $\mathcal{S}(4,\sigma_1)$, $\mathcal{S}(3,\sigma_5)$, $\mathcal{S}(5,\sigma_{10})$, $\mathcal{S}(10,\sigma_{10})$, $\mathcal{T}(3,\mathbf{S_2})$, $\mathcal{T}(3,\mathbf{E_2})$, $\mathcal{T}(4,\mathbf{E_2})$, $\mathcal{T}(5,\mathbf{H_2})$, $\Gamma(4,5/12)$, $\Gamma(6,1/3)$, $\Gamma(3,1/3)$, $\Gamma(4,1/4)$, $\Gamma(5,1/10)$, $\Gamma(10,0)$, $\Gamma(3,1/6)$, $\Gamma(4,1/12)$.

| $\mathcal{S}(4,\sigma_1)$ | $\mathcal{S}(3,\sigma_5)$ | $\mathcal{S}(5,\sigma_{10})$ | $\mathcal{S}(10,\sigma_{10})$ | $\mathcal{T}(3,\mathbf{S_2})$ | $\mathcal{T}(3,\mathbf{E_2})$ | $\mathcal{T}(4,\mathbf{E_2})$ | $\mathcal{T}(5,\mathbf{H_2})$ |
|---|---|---|---|---|---|---|---|
| 96 | 360 | 600 | 300 | 360 | 72 | 96 | 600 |

| $\Gamma(4,\frac{5}{12})$ | $\Gamma(6,\frac{1}{3})$ | $\Gamma(3,\frac{1}{3})$ | $\Gamma(4,\frac{1}{4})$ | $\Gamma(5,\frac{1}{10})$ | $\Gamma(10,0)$ | $\Gamma(3,\frac{1}{6})$ | $\Gamma(4,\frac{1}{12})$ |
|---|---|---|---|---|---|---|---|
| 864 | 18 | 864 | 96 | 600 | 150 | 72 | 864 |

TABLE 1. Minimal index of a torsion-free subgroup

**Corollary 1.2.** *The minimal index of torsion-free subgroups in the lattice triangle groups in the statement of Theorem 1.1 are as in Table 1.*

For the other lattice triangle groups, we were not able to prove that the obvious lower bound cannot be realized (but we suspect that this is the case). Note that for some lattices, we get a neat finite index subgroup of index reasonably close to $L^{opt}$; for others, the only neat subgroups we know are principal congruence subgroups, and in some cases these congruence subgroups seem inaccessible to computation using current computer technology (see Tables 11 through 72 for the list of subgroups we were able to obtain).

We point out that some of the results in Theorem 1.1 are not new. For the group $\Gamma(3,1/3)$, which is also the group Deligne-Mostow group $\Gamma_{\mu,S_3}$ for $\mu=(2,2,2,7,11)/12$, the subgroup of index 864 with Abelianization $\mathbb{Z}^2$ is the fundamental group of the Cartwright-Steger surface.

For the groups $\Gamma(6,1/3)$ and $\Gamma(3,1/6)$, the subgroups of optimal index give a neat subgroup $H$ with $\chi(H\backslash\mathbb{H}^2_{\mathbb{C}})=1$, 4 cusps and Abelianization $\mathbb{Z}^4$, which correspond to Hirzebruch's surface (see [23], [15]).

Note also that there are 4 non-arithmetic lattices in the list given in Theorem 1.1, namely $\mathcal{S}(3,\sigma_1)$, $\mathcal{S}(3,\sigma_5)$, $\mathcal{T}(4,\mathbf{E_2})$, $\Gamma(4,1/12)$. For these lattices, getting a subgroup with optimal index yields a ball quotient with fairly large Euler characteristic (namely $\chi=42,98,102,39$ respectively).

We point out a nice by-product of our search for neat subgroups in lattice triangle groups, which answers a question raised by Stover in [41] (the analogous question for real hyperbolic 4-manifolds was raised by Long-Reid [31], and answered by Kolpakov and Martelli [29]).

**Theorem 1.3.** *The Mostow group $G=\Gamma(6,0)$ has a neat subgroup $H$ of index 72 such that $X=H\backslash\mathbb{H}^2_{\mathbb{C}}$ has exactly one cusp.*

As pointed out to us by Matthew Stover, this result can be improved to the following statement.

**Theorem 1.4.** *There exists an infinite tower $H=H_1\supset H_2\supset H_3\supset\cdots$ of neat subgroups with $[H:H_n]\to\infty$ such that for every $n$, $X_n=H_n\backslash\mathbb{H}^2_{\mathbb{C}}$ has exactly one cusp.*

A similar infinite tower of neat subgroup with 2 cusps was constructed by Di Cerbo and Stover [16].

About subgroups with positive first Betti numbers (see section 7.1), first note that we record the first Betti number of the torsion-free subgroups in the last column of Tables 11 through 72. A lot of these have $b_1 > 0$ (this is equivalent to saying that their Abelianization is infinite). In cases where all the torsion-free subgroups we have found have $b_1 = 0$, we try computing all (normal) subgroups of "small" index, and check if these subgroups have $b_1 > 0$; the word "small" has no precise meaning here, we simply mean that the Magma command for computing the corresponding subgroups should take less than a day, say. For several lattice triangle groups, both of these methods fail, and we do not know any explicit subgroup of finite index with $b_1 > 0$.

Finally we review some facts (that are probably well-known to experts) that show that many Mostow groups are large, i.e. they admit a finite index subgroup that maps onto a non-Abelian free group. The subgroups in question are obtained by constructing explicit subgroups that admit a homomorphism onto a hyperbolic triangle group. Largeness then follows from largeness of every Fuchsian group (see Lemma 7.1).

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## 2. Background on lattice triangle groups

2.1. **The complex hyperbolic plane.** We briefly review basic facts about the complex hyperbolic plane and complex hyperbolic triangle groups. Recall that the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ is the only Hermitian symmetric space of complex dimension 2 with constant negative holomorphic sectional curvature (see [28]). One possible model is the unit ball in $\mathbb{B} \subset \mathbb{C}^2$ equipped with the Bergman metric, but it is convenient to see $\mathbb{C}^2$ as an affine chart of $\mathbb{P}^2_{\mathbb{C}}$, and to work in homogeneous coordinates. As a set, $\mathbb{H}^2_{\mathbb{C}}$ is the set of complex lines that are negative for a fixed Hermitian inner product $\langle \cdot, \cdot \rangle$ of signature $(2, 1)$ on $\mathbb{C}^3$, which we can describe as $\langle V, W \rangle = W^* H V$ for some Hermitian matrix $H$. Recall that $U(H)$ denotes the group

$$U(H) = \{A \in GL_3(\mathbb{C}) : A^* H A = H\},$$

and $PU(H)$ is the quotient of $U(H)$ by the subgroup of scalar matrices. Note that by Sylvester's law of inertia, up to isomorphism, the real Lie group $PU(H)$ is independent of the choice of Hermitian form $H$, a common choice being the diagonal matrix $\mathrm{diag}(-1, 1, 1)$.

Up to scale, there is a unique $PU(H)$-invariant Riemannian metric on $\mathbb{H}^2_{\mathbb{C}}$, which is Kähler and has constant holomorphic sectional curvature. We refer to $\mathbb{H}^2_{\mathbb{C}}$, equipped with the unique $PU(H)$-invariant Riemannian metric of holomorphic sectional curvature $-1$, as the complex hyperbolic plane. When we talk about volume, we mean volume with respect to the corresponding Riemannian volume form.

The stabilizer of a point $x = [V]$ in $\mathbb{H}^2_{\mathbb{C}}$ in $PU(H)$ is isomorphic to group of isometries of the restriction of the Hermitian form to $V^\perp = \{W \in \mathbb{C}^3 : \langle V, W \rangle = 0\}$. Because of the fact that $V$ is a negative vector and $H$ has signature $(2, 1)$, $V^\perp$ is a positive definite subspace, so the stabililizer of a point is isomorphic to $U(2)$.

One checks that $PU(H)$ is the full group of homomorphic isometries, and that it has index two in the full group of isometries (an isometry not in $PU(H)$ is induced by $V \mapsto Q\bar{V}$ where $Q$ is a matrix such that $Q^*\bar{H}Q = H$ (such a matrix exists because $H$ and $\bar{H}$ have the same signature).

An explicit expression for the invariant Riemannian metric can be found in [28], see also [20]. We will not need any expression for that metric; for concreteness, we describe the corresponding distance function, which is given by

$$\cosh\left(\frac{1}{2}d(\mathbb{C}V,\mathbb{C}W)\right) = \frac{|\langle V,W\rangle|}{\sqrt{\langle V,V\rangle\langle W,W\rangle}}.$$

where $V, W \in \mathbb{C}^3$ are any two negative vectors. The factor $\frac{1}{2}$ is included for the constant holomorphic sectional curvature to be equal to $-1$.

Given two distinct points $\mathbb{C}V, \mathbb{C}W \in \mathbb{H}^2_{\mathbb{C}}$, i.e. $V, W$ are linearly independent over $\mathbb{C}$, the restriction of the Hermitian form to the complex span $\mathbb{C}\{V, W\}$ has signature $(1, 1)$, so the set of negative vectors in the span gives a copy of $\mathbb{H}^1_{\mathbb{C}}$, which is totally geodesic. We call this **the complex line through** $V$ **and** $W$.

2.2. **Point stabilizers.** Recall that the isometric action of $PU(H)$ on the open ball $\mathbb{H}^2_{\mathbb{C}}$ extends to an action on the closed ball $\mathbb{H}^2_{\mathbb{C}} \cup \partial_\infty \mathbb{H}^2_{\mathbb{C}}$ by homeomorphisms. Every homeomorphism of a closed ball has a fixed point, and the position of the corresponding fixed points (in $\mathbb{H}^2_{\mathbb{C}}$ or in $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$) gives a classification of isometries into elliptic, parabolic, loxodromic isometries (see section 6.2 in [20]).

Elliptic isometries are the ones that have a fixed point in $\mathbb{H}^2_{\mathbb{C}}$. The ones without any fixed point in $\mathbb{H}^2_{\mathbb{C}}$ are either parabolic (unique fixed point in $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$) or loxodromic (two distinct fixed points in $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$).

Suppose $A \in PU(H)$ is elliptic. Then $A$ has an eigenvector $V_0$ with $\langle V_0, V_0\rangle < 0$, and by multiplying $A$ by a constant, we may assume the corresponding eigenvalue is 1. Moreover, we may normalize $V_0$ so that $\langle V_0, V_0\rangle = -1$. The orthogonal complement $V_0^\perp = \{W \in \mathbb{C}^3 : \langle V_0, W\rangle = 0\}$ is invariant under $A$ and, because of the signature assumption, the restriction of the Hermitian form to $V^\perp$ is positive definite. Pick a $V_1 \in V_0^\perp$, which once again we may normalize so that $\langle V_1, V_1\rangle = 1$, and then take $V_2$ to be the Hermitian box product $V_0 \boxtimes V_1$ (i.e. the usual cross-product of $V_0^*H$ with $V_1^*H$). Normalizing $V_2$, we get $A$ to be diagonal in the basis $V_0, V_1, V_2$, which is standard Lorentzian in the sense that the matrix of the Hermitian form in that basis is

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

As mentioned before, this identifies the stabilizer of $\mathbb{C}V_0$ in $PU(H)$ with the unitary group $U(2)$ (which is a compact group).

When $A$ has distinct eigenvalues, $A$ is called a **regular elliptic** isometry. In that case, $A$ has three linearly independent eigenvectors, and the Lorentzian character of the diagonalizing basis is essentially automatic because eigenvectors with different eigenvalues are orthogonal with respect to $\langle\cdot,\cdot\rangle$ (one simply normalizes any triple of eigenvectors).

When $A$ is elliptic and has a double eigenvalue, the above argument shows that $A$ remains diagonalizable (because every matrix in $U(2)$ is diagonalizable). We can write

$$\mathbb{C}^3 = \mathbb{C}V_0 \oplus V_0^\perp$$

for some $V_0 \in \mathbb{C}^3$ with $\langle V_0, V_0\rangle \neq 0$ (if the inner product were 0, the above decomposition would not be a direct sum). Note that $A$ acts projectively as the identity on the projectivization of $V_0^\perp$, because it is an eigenspace for $A$.

If $\langle V_0, V_0\rangle > 0$, we call $A$ a **complex reflection**, and we call (the projectivization of) $V_0^\perp$ its **mirror**, which is a complex line, i.e. a totally geodesic copy of $\mathbb{H}^1_{\mathbb{C}}$ in $\mathbb{H}^2_{\mathbb{C}}$.

If $\langle V_0, V_0\rangle < 0$, then $\mathbb{C}V_0$ gives an isolated fixed point in $\mathbb{H}^2_{\mathbb{C}}$ for the action of $A$, and we say that $A$ is a **complex reflection in a point**.

In both cases there is a simply formula for $A$ in terms of the Hermitian inner product. Indeed, let $\zeta \in \mathbb{C}$ be such that $|\zeta| = 1$ and consider the linear map given for every $X \in \mathbb{C}^3$ by

$$R(X) = X + (\zeta - 1)\frac{\langle X, V_0\rangle}{\langle V_0, V_0\rangle}V_0. \tag{2}$$

It is easy to see that $V_0^\perp$ and $\mathbb{C}V_0$ are eigenspaces of $R$, with eigenvalue 1 and $\zeta$ respectively. Depending on the sign of $\langle V_0, V_0\rangle$, we get either a complex reflection (with mirror $V_0^\perp$) or a complex reflection in a point. The complex number $\zeta$ is called the **multiplier** of the complex reflection.

*Remark* 2.1.
(1) It follows from the above discussion that if a holomorphic isometry $A \in PU(H)$ fixes two distinct points in $\mathbb{H}^2_{\mathbb{C}}$, then it fixes the complex lines through these two points, hence it is a complex reflection.
(2) Regular elliptic elements can have non-trivial powers that are no longer regular.

2.3. **Ideal point stabilizers.** We now review some facts about the stabilizer in $G = PU(H)$ of an ideal point, i.e. a complex line spanned by a null vector. Recall that all Hermitian forms of signature $(2,1)$ on $\mathbb{C}^3$ are equivalent over $\mathbb{R}$, so we may assume by choosing a suitable basis of $\mathbb{C}^3$ that the matrix of the Hermitian form in the standard basis $e_1, e_2, e_3$ of $\mathbb{C}^3$ is the antidiagonal matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

With such normalization, $e_1$ is a null vector, so $\mathbb{C}e_1 \in \partial_\infty \mathbb{H}^2_{\mathbb{C}}$.

The unipotent stabilizer of $e_1$ in $G$ forms a subgroup of $Stab_G(e_1)$, which is given by the matrices of the form

$$T(z,t) = \begin{pmatrix} 1 & -\bar{z} & \frac{-|z|^2+it}{2} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

where $z \in \mathbb{C}$, $t \in \mathbb{R}$. Let $V = (v_1, v_2, v_3) \in \mathbb{C}^3$ be any null vector for the form $\langle V, W\rangle = W^*JV$, i.e. $2\Re(v_1\bar{v}_3) + |v_2|^2 = 0$. If $v_3 = 0$, then $v_2 = 0$ and $V$ is a multiple of $e_1$; if $v_3 \neq 0$, we can normalize the homogeneous coordinates so that $v_3 = 1$, and then we have $2\Re(v_1) + |v_2|^2 = 0$. This implies that $v$ is in the image of $e_3 = (0, 0, 1)$ under some transformation $T(z,t)$ (take $z = v_2$, $t = (2v_1 + |v_2|^2)/i$), i.e. the unipotent stabilizer acts transitively on $\partial_\infty\mathbb{H}^2_{\mathbb{C}} \setminus \mathbb{C}e_1$.

In fact, one verifies that $T(z,t)T(z',t') = T((z,t) \star (z',t'))$, where $\star$ denotes the Heisenberg group law, namely

$$(z,t) \star (z',t') = (z + z', t + t' + 2\Im(z\bar{z}')).$$

The group $\mathbb{C} \times \mathbb{R}$ equipped with the Heisenberg group law is the (3-dimensional real) Heisenberg group, and we denote it by $\mathcal{H}$. Note that the center of $\mathcal{H}$ is the subgroup $\{(0,t), t \in \mathbb{R}\}$.

Equation (3) can be thought of as embedding $\mathcal{H}$ as the unipotent stabilizer of $e_1$, or as an identification between $\partial_\infty\mathbb{H}^2_{\mathbb{C}} \setminus \mathbb{C}e_1$ and $\mathcal{H}$ (the point $(z,t)$ corresponds to the complex line spanned by $(\frac{-|z|^2+it}{2}, z, 1)$). Since $T(z,t)$ acts on $\mathcal{H}$ via left $\star$-multiplication by $(z,t)$, we often refer to the transformations $T(z,t)$ as a **Heisenberg translations**. The elements with $z = 0$, which are central in the unipotent stabilizer, are called **vertical translations**.

There are complex reflections in the stabilizer, given by the diagonal matrices $R_\zeta = \mathrm{diag}(1, \zeta, 1)$, $|\zeta| = 1$. These act on $\mathcal{H}$ as $(z,t) \mapsto (\zeta z, t)$, and we refer to them as **Heisenberg rotations**. Finally, **Heisenberg dilations** are given by $\mathrm{diag}(\lambda, 1, 1/\lambda)$ for $\lambda \in \mathbb{R}^*$.

The full stabilizer of $e_1$ in $G$ is generated by these three classes (see section 4.2.2 of [20] for instance). In terms of the rough classification of isometries into elliptic, parabolic and loxodromic transformation, Heisenberg translations (resp. rotations, dilations) are parabolic (res. elliptic, loxodromic).

Note also that for arbitrary $z \in \mathbb{C}$, $t \in \mathbb{R}$ and $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $R_\zeta T(z,t)$ is parabolic, but it is unipotent if and only if $\zeta = 1$. The transformations $R_\zeta T(z,t)$ with $\zeta \neq 1$ are often called **screw-parabolic elements**.

## 3. Lattices in $PU(2,1)$

Throughout this section, $\Gamma$ denotes a lattice in $PU(H)$, in the sense of Definition 3.1.

**Definition 3.1.** *A subgroup $\Gamma \subset PU(H)$ is called a lattice if it is discrete and $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ has finite volume. A lattice is called cocompact (or uniform) if $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ is compact.*

To simplify notation, we write $G = PU(H)$. Even though we will not use this in the present paper, we point out that the lattice condition is equivalent to the requirement that

$\Gamma\backslash G$ has a finite Haar measure (this is a consequence of the compactness of $K = Stab_G(x)$, $x \in \mathbb{H}^2_{\mathbb{C}}$).

Since the stabilizer in $G$ of a point $x \in \mathbb{H}^2_{\mathbb{C}}$ is compact, $Stab_\Gamma(x)$ is a finite group, for every $x \in \mathbb{H}^2_{\mathbb{C}}$. It follows that the quotient $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ acquires the structure of a complex hyperbolic orbifold.

In order for the quotient map to induce a complex hyperbolic manifold structure on the quotient, one needs the point-stabilizers to be trivial; since any element of finite order in $G$ fixes at least one point in $\mathbb{H}^2_{\mathbb{C}}$, we have the following.

**Proposition 3.1.** *The local inverses of the quotient map $\mathbb{H}^2_{\mathbb{C}} \to \Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ define a manifold structure on $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ if and only if $\Gamma$ is torsion-free.*

When the conditions of Proposition 3.1 are satisfied, we say that $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ is a smooth ball quotient (of finite volume).

*Remark* 3.2. The quotient $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ can be a smooth even in cases where $\Gamma$ has non-trivial torsion; indeed, some of the ball quotients constructed by Hirzebruch (see [3], [45] or [7]) give lattices such that $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ biholomorphic to $\mathbb{P}^2_{\mathbb{C}}$ (possibly blown up at some points), but the corresponding quotient map $\mathbb{H}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ is then a branched covering.

If $\Gamma$ is torsion-free and cocompact, $X = \Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ is a compact complex surface, and Noether's formula (see [21], p.438) says that

$$\mathbb{Z} \ni \chi(\mathcal{O}_X) = \frac{c_1(X)^2 + c_2(X)}{12}. \tag{4}$$

Moreover, by Hirzebruch proportionality [22], smooth compact ball quotients satisfy the same equality of Chern numbers as $\mathbb{P}^2_{\mathbb{C}}$, namely $c_1(X)^2 = 3c_2(X)$. Since $c_2(X)$ is the topological Euler characteristic $\chi(X)$, equation (4) implies the following.

**Proposition 3.3.** *Let $X$ be a compact smooth ball quotient. Then $\chi(X) \in 3\mathbb{Z}$.*

There exist many compact smooth ball quotients with $\chi(X) = 3$, namely fake projective planes (these are compact ball quotients of dimension 2 with the same Betti numbers as $\mathbb{P}^2_{\mathbb{C}}$, in particular they have $b_1 = 0$). Klingler [27] has shown that every fake projective plane must be arithmetic (see also [47]), and this sets ground for a classification, since arithmetic groups can be described by explicit number-theoretical data (see [46], [44]). Fake projective planes were indeed enumerated by Prasad, Yeung [37], Cartwright and Steger [6]. As a by-product of the work needed for the classification, Cartwright and Steger also found a compact smooth ball quotient of dimension 2 with minimal volume (this is equivalent to having Euler characteristic 3) but which is not a fake projective plane (in fact it has positive first Betti number). The latter is listed in our paper, its fundamental group is (conjugate to) a subgroup of a specific Mostow ball quotient (see Table 54).

Note also that every element in $3\mathbb{Z}$ is the Euler characteristic of some smooth compact ball quotient. Indeed, the fundamental group of the Cartwright-Steger $\Gamma$ has Abelianization $\mathbb{Z} \oplus \mathbb{Z}$, hence a surjective morphism $\Gamma \to \mathbb{Z}$; the preimage of $n\mathbb{Z}$ under this morphism gives a subgroup of index $n$ in $\Gamma$, which has Euler characteristic $3n$.

For non-compact ball quotients, the Euler characteristic can be equal to any natural number $n \in \mathbb{N}^*$. This follows from the fact that the surface constructed by Hirzebruch in [23] has Euler characteristic 1 and the Abelianization of its fundamental group is $\mathbb{Z}^4$, so the corresponding lattice has a homomorphism onto $\mathbb{Z}$ (and the latter has subgroups of any arbitrary finite index). In fact a stronger result was proved by Di Cerbo and Stover [16], namely for every $n \in \mathbb{N}^*$, there exists a non-cocompact ball quotient *with two cusps* and Euler characteristic 1. In this paper, we will improve this to get a tower of smooth ball quotients with a single cusp, see Theorem 1.4.

It is natural to look for a condition analogous to the torsion-free condition but for stabilizers of ideal points. Of course a finite covering of a non-compact ball quotient will remain non-compact, so one cannot hope to get rid of cusps alltogether in a subgroup of finite index (see Proposition 3.4); the relevant condition for cusps to be torsion-free is slightly more subtle.

We start by observing that if $\Gamma \subset G$ is discrete and $x \in \partial_\infty \mathbb{H}^2_{\mathbb{C}}$, then $Stab_\Gamma(x)$ cannot contain both parabolic and loxodromic elements, which simplifies the description of stabilizers of ideal points in discrete groups, see section 2.3.

Because of the lattice assumption, the thick-thin decomposition (see section 4 of [26] for instance) implies that the quotient $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ has finitely many ends, each being homeomorphic to $\mathcal{N}\times]0,\infty[$, where $\mathcal{N}$ is a compact quotient of the Heisenberg group, which is isomorphic to the stabilizer in $\Gamma$ of a given ideal point $x \in \partial_\infty \mathbb{H}^2_{\mathbb{C}}$ (see also [19]). The end of the quotient is usually referred to as a *cusp*; cusps are in 1-1 correspondence with $\Gamma$-conjugacy classes of non-trivial isotropy groups of ideal boundary points that contain at least one parabolic element. By extension, these stabilizers are often called cusps as well.

In particular, we have the following.

**Proposition 3.4.** *Let $\Gamma$ be a lattice in $G = PU(H)$. The quotient $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ is cocompact if and only if $\Gamma$ contains no parabolic element.*

Using the description of $Stab_G(x)$ for $x \in \partial_\infty \mathbb{H}^2_{\mathbb{C}}$ (see section 2.1), we can say a little more about cusp groups. Once again, if $Stab_\Gamma(x)$ is a cusp group, then it consists of screw-parabolic elements, i.e. matrices of the form

$$\begin{pmatrix} 1 & -\bar{z} & \frac{-|z|^2+it}{2} \\ 0 & \zeta & z \\ 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

$z, \zeta \in \mathbb{C}, t \in \mathbb{R}$, $|\zeta| = 1$.

The projection

$$\begin{pmatrix} 1 & -\bar{z} & \frac{-|z|^2+it}{2} \\ 0 & \zeta & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \zeta & z \\ 0 & 1 \end{pmatrix}. \tag{6}$$

is a group homomorphism, which can be interpreted as a map

$$\Phi : Stab_G^{par}(x) \to \mathrm{Isom}(\mathbb{C})$$

of the parabolic stabilizer onto the group of Euclidean isometries of $\mathbb{C}$, with central kernel.

Since we assume that $\Gamma$ is a lattice, $Stab_\Gamma(x)$ must act dicretely and cocompactly on the Heisenberg group $\mathcal{H}$, from which is follows that $\Phi(Stab_\Gamma(x))$ must act dicretely and cocompactly on $\mathbb{C}$ (and the kernel must be an infinite cyclic group).

In those terms, the analogue of the torsion-free condition for cusps is quite natural; one requires that $\Phi(Stab_\Gamma(x))$ be a torsion-free (cocompact) group of isometries of $\mathbb{C}$. It is a standard fact that these are just free Abelian groups generated by two translations in different directions, hence the quotient $\mathbb{C}/\Phi(Stab_\Gamma(x))$ is an elliptic curve.

**Definition 3.2.** *A lattice* $\Gamma \subset PU(2,1)$ *is* **neat** *if every matrix representative* $A$ *of an element* $\gamma \in \Gamma$ *that has a root of unity as an eigenvalue is actually (a multiple of) a unipotent matrix.*

One can push the above discussion to prove the following, see [2] for the arithmetic case, [32] for the general case.

**Theorem 3.5.** *Let* $\Gamma \subset PU(2,1)$ *be a neat lattice. Then* $X = \Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ *is smooth, and it admits a smooth compactification* $\bar{X}$*, where* $\bar{X}\setminus X$ *is a disjoint union of elliptic curves with negative self-intersection.*

One can also give a simple interpretation of the self-intersection of the elliptic curves in the compactification $\bar{X}$.

Indeed, let $\Gamma$ be a neat lattice and let $P$ be a cusp subgroup of $\Gamma$. It follows from the previous discussion that $P$ is a central extension (with infinite cyclic center) of sugroup of $\text{Isom}(\mathbb{C})$ generated by two translations. In terms of the Heisenberg group, we get two Heisenberg translations $A, B$ (such that the entries $A_{1,2}$ and $B_{1,2}$ linearly independent over $\mathbb{R}$) such that $\Phi(A)$ and $\Phi(B)$ generate $\Phi(Stab_\Gamma(x))$, and a non-trivial vertical translation $Z$ that generates the kernel.

Since the commutator of two Heisenberg translations must be a vertical translation, we can write

$$ABA^{-1}B^{-1} = [A,B] = Z^k$$

for some integer $k$. Proposition 4.2.12 and equation (4.2.15) of [24] say that the self-intersection of the elliptic curve corresponding is $-|k|$.

This self-intersection can be computed very efficiently from a presentation for $C$ in terms of generators and relations. Indeed, given the above description, $C$ must be isomorphic to

$$\left\langle\ a,b,z \mid [a,b]z^{-k},\ z \text{ central }\ \right\rangle,$$

whose abelianization is $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}^{|k|}$.

*Remark* 3.6. When $\Gamma$ is a non-neat lattice, one can still compactify the quotient, but the ends are then filled with quotients of elliptic curves by a finite group, see recent work of Eyssidieux [18].

3.1. **Lattice triangle groups.** Recall that complex reflections are elliptic elements whose matrix representative has a repeated eigenvalue, see equation (2). The lattices we consider in this paper are all triangle groups in the following sense.

**Definition 3.3.** *A complex hyperbolic triangle group is a subgroup of $PU(H)$ generated by three complex reflections $R_1, R_2, R_3$.*

It turns out that several triangle groups are lattices, as was first observed by Mostow [33]. More examples were found later [11], [42], [13] and [14].

We refer to the lattice triangle groups listed in [14] as *the known lattice triangle groups*. The list is reproduced here in Tables 11 through 72.

We briefly review notation, a detailed parametrization for these groups can be found in section 3 of [14]. We list groups of three kinds, see Table 3.1. Sporadic groups and Mostow

| Name | Notation | Parameters |
|---|---|---|
| Sporadic groups | $\mathcal{S}(p,\tau)$ | $p \in \mathbb{N}^*, \tau \in \mathbb{C}$ |
| Thompson groups | $\mathcal{T}(p,\mathbf{T})$ | $p \in \mathbb{N}^*, \mathbf{T} \in \mathbb{C}^3$ |
| Mostow groups | $\Gamma(p,t)$ | $p \in \mathbb{N}^*, t \in \mathbb{Q}$ |

groups are generated by $R_1$ and $J$, where $R_1$ is a complex reflection with multiplier $e^{2\pi i/p}$, and $J$ is a regular elliptic element of order 3. One gets two other complex reflections by setting $R_2 = JR_1J^{-1}$ and $R_3 = J^{-1}R_1J$.

Sporadic groups are characterized by this data and the fact that $\mathrm{tr}(R_1J) = \tau$. The corresponding sporadic group $\mathcal{S}(p,\tau)$ is a lattice only for wisely chosen pairs $(p,\tau)$.

Mostow groups $\Gamma(p,t)$ are special cases of sporadic groups, where we take

$$\tau = e^{\pi i(\frac{3}{2}+\frac{1}{3p}-\frac{t}{3})}.$$

Thompson groups are generated by three complex reflections $R_1, R_2, R_3$ with the same multiplier $e^{2\pi i/p}$, but where there is no isometry $J$ as above such that $R_{k+1} = JR_kJ^{-1}$. These are parametrized by a triplet $\mathbf{T} = (\rho,\sigma,\tau) \in \mathbb{C}^3$ that generalizes the trace parameter of sporadic groups (see section 3 of [14] for details).

As discussed in [14], the commensurability classes of the lattices in these three classes contain all known non-arithmetic commensurability classes of lattices in $PU(2,1)$ (there are currently 22 known commensurability classes).

## 4. Conjugacy classes of isotropy groups

In this section we briefly recall how to use a fundamental domain $F$ for a lattice $\Gamma \subset PU(2,1)$ to obtain the list of conjugacy classes of

- isotropy groups in $\Gamma$ of points in $\mathbb{H}^2_{\mathbb{C}}$;
- cusps in $\Gamma$.

The isotropy groups of points in $\mathbb{H}^2_{\mathbb{C}}$ are precisely the maximal finite subgroups in $\Gamma$, and the cusps are isotropy groups of ideal boundary points that contain at least one parabolic element (the latter are infinite, in fact they act cocompactly on horospheres).

Note that the finite isotropy groups were already listed in the tables of [14]; either they are generated by complex reflections (in which case they occur in the tables giving vertex stabilizers), or they are not (in which case they can be deduced from the information about singular points of the quotient).

For cusps, the information given in [14] only says whether the group is generated by reflections, and when it is not, we did not give actual generators. In order to get generators, we use the computer output of Spocheck [9].

What we do in Spocheck uses the fundamental domains constructed in [14] and tracks cycles of (ideal or finite) vertices, as well as facets of higher-dimension. The point is that every isotropy group is conjugate to one with fixed point in the boundary of the fundamental domain, hence it is enough to compute the stabilizers of all facets of the fundamental domain. We use the same method as Mostow in section 18.2 of [33].

In the following paragraphs, we let $F$ be (finite-sided) a fundamental domain for $\Gamma$, and let $v$ denote a facet of $F$. In order to compute $Stab_\Gamma(v)$ we build a graph whose vertices are given by facets of the fundamental domain $F$. For every such facet $v$, we list all sides of $F$ containing $v$. For every such side $s$, we consider the side-pairing map $\gamma_s$, and draw an edge from $v$ to $\gamma_s(v)$. Let us denote by $T$ the corresponding directed graph; it is a finite graph since we assume $F$ has finitely many sides.

By construction, the $\Gamma$-orbits of ideal vertices are in 1-1 correspondence with the connected components of $T$. We have a well-defined representation $\rho_v : \pi_1(T, v) \to \Gamma$, and the stabilizer of $v$ in $\Gamma$ is precisely the image $Im(\rho_v)$. In particular, in order to get a generating set for $Stab_\Gamma(v)$, it is enough to construct explicit generators of $\pi_1(T, v)$ (which can be done by constructing a maximal subtree containing $v$ in $T$).

The computations (and even the end results of these computations) are too long to be included in a paper, we will list them in the form of a Magma file in [10]. In this paper, we give the details only for a couple examples that illustrate the method, see sections 4.1 and 4.2.

The general result for ideal vertices is given in the Tables of section 7.3 (Tables 11 through 72), sixth (cusp generators) and seventh column (cusp relations).

4.1. **Isotropy groups for $\Gamma = \mathcal{S}(3, \sigma_1)$.** In this section, we use word notation used in [14], so that 1, 2, 3 and 4 stand for $R_1$, $R_2$, $R_3$ and $J$ respectively, and $\bar{1}$ stands for $R_1^{-1}$, etc. Consider the group $\Gamma = \mathcal{S}(3, \sigma_1)$. The Spocheck output [9] gives us the stabilizers listed in Table 2. It turns out all non Abelian finite stabilizers are complex reflection groups, so they can be described in terms of the Shephard-Todd classification [40]. In the tables, we write $G_k$ for (a group isomorphic to) the $k$-th imprimitive group in the Shephard-Todd list.

Recall that $\mathrm{br}_n(a, b)$ means $(ab)^{n/2} = (ba)^{n/2}$, which when $n$ is odd means

$$aba \cdots ba = bab \cdots ab \tag{7}$$

where both sides of equation (7) are products of $n$ factors.

For the cusp, we know from [14] or [9] that $R_1$ and $R_2$ braid with length 6. We then use the fact that $(ab)^3$ is central in the braid group $\langle\, a, b \mid \mathrm{br}_6(a, b) \,\rangle$ and the basic geometry of braiding complex reflections (see section 2.3 of [33] or section 2.3 of [14]) to identify the relevant subgroup of $\mathrm{Isom}(\mathbb{C})$ (see section 3) as a $(3, 3, 3)$-triangle group.

| Ridge stab. | Edge stab. | Vertex stab. |
|---|---|---|
| $\langle J\rangle$, order 3 | $\langle R_1\rangle$, order 3 | $\langle 1,2\rangle$, Cusp |
| $\langle R_1\rangle$, order 3 | | $\langle 1,2\bar{3}\bar{2}\rangle$, order 72 ($G_5$) |
| $\langle R_1 J\rangle$, order 8 | | $\langle 1,232\bar{3}\bar{2}\rangle$, order 24 ($G_4$) |
| | | $\langle 1,\bar{3}\bar{2}323\rangle$, order 24 ($G_4$) |

TABLE 2. Facet stabilizers for $\mathcal{S}(3,\sigma_1)$

From this and the fact that the braid relation $\mathrm{br}_6(a,b)$ implies that $(ab)^3$ is central, one easily sees that the cusp has a presentation of the form

$$\left\langle\ r_1,r_2 \mid r_1^6, r_2^6, \mathrm{br}_6(r_1,r_2)\ \right\rangle.$$

4.2. **Isotropy groups for $\Gamma = \mathcal{S}(3,\sigma_5)$.** A more complicated example is given by the group $\Gamma = \mathcal{S}(3,\sigma_5)$. The Spocheck output gives us the information in Table 3, where $\Xi$ is a complicated generating set for the corresponding cusp. Explicitly $\Xi$ consists of the

| Ridge stab. | Edge stab. | Vertex stab. |
|---|---|---|
| $\langle J\rangle$, order 3 | $\langle R_1\rangle$, order 3 | $\langle 1,2\rangle$, order 72 ($G_5$) |
| $\langle R_1\rangle$, order 3 | | $\langle 1,2\bar{3}\bar{2}\rangle$, order 360 ($G_{20}$) |
| $\langle R_1 J\rangle$, order 30 | | $\langle \Xi\rangle$, Cusp |
| $\langle 2\bar{3}\bar{2}123\bar{2},(R_1J)^5\rangle$, order 18 | | |

TABLE 3. Facet stabilizers for $\mathcal{S}(3,\sigma_5)$

following elements:

$$\begin{gathered} x_0 = 2 \\ x_1 = 123\bar{2}123\bar{3}\,\bar{2}\,\bar{1} \\ x_2 = (123)^2 12\bar{1}312\bar{2}\,\bar{1}(\bar{3}\,\bar{2}\,\bar{1})^2 \\ x_3 = (123)^4 1\bar{3}23\bar{1}(\bar{3}\,\bar{2}\,\bar{1})^4 \\ x_4 = (\bar{3}\,\bar{2}\,\bar{1})^2\bar{3}\,\bar{2}\,31\bar{3}23(123)^2 \\ x_5 = (\bar{3}\,\bar{2}\,\bar{1})23\bar{2}(123) \\ x_6 = \bar{4}^2 121\bar{2}\,\bar{1}\,\bar{3}\,\bar{2}\,\bar{1}\,2 \end{gathered}$$

By constructing all words of length $\leq 5$ in $x_0$ and $x_1$, one checks that $x_2,x_3,x_4,x_5 \in \langle x_0,x_1\rangle$, in fact we have: $x_2 = x_1x_0x_1^{-1}$, $x_3 = x_1x_0x_1x_0^{-1}x_1^{-1}$, $x_4 = x_0^{-1}x_1^{-1}x_0x_1x_0$, $x_5 = x_0^{-1}x_1x_0$.

Computing short words in $x_0,x_6$, we also have $x_1 = x_6x_0x_6^{-1}$, so we get the following.

**Proposition 4.1.** *The group generated by $\Xi$ is generated by $x_0$ and $x_6$.*

In order to work out a presentation for the group generated by $x_0$ and $x_6$, we need to work a little more. We use the discussion given in section 2.3 of the stabilizer of an ideal point in $PU(2,1)$.

We start by giving matrices for the group, with integral entries in the smallest possible number field, i.e. $\mathbb{K} = \mathbb{Q}(\omega, \varphi)$, where $\omega = \frac{-1+i\sqrt{3}}{2}$, $\varphi = \frac{1+\sqrt{5}}{2}$. The lattice can be written as the group generated by

$$R_1 = \begin{pmatrix} \omega & \omega+\varphi & \bar{\omega}\varphi+\omega \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} J = \begin{pmatrix} 0 & 0 & -\omega \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

which preserve the Hermitian form

$$H = \begin{pmatrix} \alpha & \beta & -\omega\bar{\beta} \\ \bar{\beta} & \alpha & \beta \\ -\bar{\omega}\beta & \bar{\beta} & \alpha \end{pmatrix},$$

where $\alpha = 3$ and $\beta = -(2+\omega)\varphi + (1-\omega)$.

It follows from Proposition 4.1 that that the cusp is generated by

$$A = \bar{2}\,123121\bar{1}\,\bar{2}\,\bar{1}\,P^2, B = R_2$$

By using a suitable matrix $Q \in GL(3, \mathcal{O}_{\mathbb{K}})$, for example

$$Q = \begin{pmatrix} \omega(\varphi+1) & \bar{\omega}(\varphi+1) & \omega+1 \\ -\bar{\omega}(2\varphi+1) & -(\varphi+1) & \varphi+\bar{\omega} \\ \varphi+1 & \omega(\varphi+1) & 1+\bar{\omega}\varphi \end{pmatrix},$$

we can write $A$ and $B$ in upper triangular form, namely we have (in the projective group)

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & -\bar{\omega} \\ 0 & -\bar{\omega} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q^{-1}BQ = \begin{pmatrix} 1 & 1+2\omega & -\omega \\ 0 & \omega & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $Q^*HQ$ is (a real multiple) of

$$\begin{pmatrix} 0 & 0 & \omega-1 \\ 0 & 3 & 0 \\ \bar{\omega}-1 & 0 & 0 \end{pmatrix}.$$

As discussed in section 2.3, the action on the complex line tangent to the ideal boundary at the ideal fixed point is described projectively by the lower-right $2\times 2$ submatrix, i.e. if we denote by $\tilde{A}, \tilde{B}$ the corresponding Euclidean isometries of $\mathbb{C}$, we have

$$\tilde{A}(z) = -\bar{\omega}z, \quad \tilde{B}(z) = \omega z + 1$$

which are rotations about 0 (resp. $\frac{1-\bar{\omega}}{3}$) and order 6 (resp. 3).

Note that the product $\tilde{B}\tilde{A}$ is a rotation of order 2, and the relevant Euclidean group of isometries is a $(2,3,6)$ triangle group.

Going back to the original matrices $A, B$ and $BA$, note that (still in the projective group)

$$A^6 = \begin{pmatrix} 1 & 0 & -6\bar{\omega} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^3 = Id, \quad (BA)^2 = \begin{pmatrix} 1 & 0 & -\bar{\omega} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

The last equality shows that $(BA)^2$ is central.

In fact we have the following

**Proposition 4.2.** *The cusp of $\mathcal{S}(3,\sigma_5)$ can be represented as the group generated by $A$ and $B$. Its center is generated by $(AB)^2$, and it is isomorphic to*

$$\langle a, b | a^6(ba)^{-12}, b^3, [a,(ab)^2], [b,(ab)^2]\rangle.$$

**Proof:** The above discussion shows that the group of isometries of $\mathbb{C}$ obtained as the action on the complex line tangent to the ball at the fixed point of the cusp is a $(2,3,6)$-triangle group.

It is well known that this triangle group has a presentation of the form $\langle\ \alpha, \beta \mid \alpha^6, \beta^3, (\alpha\beta)^2\ \rangle$, in particular, any word in $\alpha, \beta, \gamma$ that is trivial in the triangle group can be written as a product of conjugates of $\alpha^6$, $\beta^3$ or $(\alpha\beta)^2$. This implies that every central element in the cusp must be a product of powers of $A^6$, $B^3$ and $(AB)^2$. The computation of equation (8) shows that $(AB)^2$ generates the center. $\square$

Note that explicit computation shows that $BA^{-1}$ has order has order 6, and that the braid relation $\mathrm{br}(B, BA^{-1})$ holds in the group. One checks (for example using Magma) that the group

$$\langle\ c, d \mid c^3, d^6, \mathrm{br}_4(c,d)\ \rangle$$

is isomorphic to the above presentation (where the isomorphism maps $c \leftrightarrow b$, $d \leftrightarrow ba^{-1}$).

## 5. Methods for finding subgroups

In order to get subgroups, we use four basic methods, listed

- The `LowIndexSubgroups` gives us list of subgroups of small index (usually reasonable for index bounds of about 30);
- The `LowIndexNormalSubgroups` gives us list of *normal* subgroups of small index, and it works for much larger index than the previous one (usually reasonable for index bounds of 30,000 to 100,000);
- The `SimpleQuotients` allows us to study only simple quotients, and can give homomorphisms to very large finite (simple) groups;
- Congruence subgroups (either we choose an explicit prime ideal and reduce the matrices modulo that ideal, or we use the `CongruenceImage` command in Magma).

The cost of all these methods grows exponentially as the number of generators increase. What makes them usable in the context of subgroups of small index in this context is that lattice triangle groups have few generators (most are generated by 2 elements, some by 3 elements).

There is also a slightly more subtle method to build non-normal (neat) subgroups containing a given normal (neat) subgroup. This is well known to experts, but we give some details in section 5.5.

5.1. **Low Index Subgroups (LIS).** This method works only for subgroups of fairly small index; the indices accessible by this method depend to a great extend on the lattice triangle group we are considering (generally index 20 is already a lot to ask).

5.2. **Low Index Normal Subgroups (LINS).** For normal subgroups, the allowed indices are much larger, but once again, the reasonable values depend on the group (for some, Magma gives reasonable access to the index $\leq 100\,000$, while for some others index $1\,000$ is already a lot to ask).

5.3. **Simple Quotients (SQ).** The search for simple quotients depends on choices of parameters (lower and upper bounds for the order of the group, and an upper bound on the degree of the permutation group, i.e. $N$ such that the group embeds in the symmetric group $S_N$). For more details, see the README file in the computer code [10].

5.4. **Congruence Subgroups (CS).** We say a word about how we compute congruence subgroups. Each triangle group comes with an explicit generating set, given by a finite set $A$ of matrices in $U(H)$, with algebraic integer entries. Denote by $\widetilde{\Gamma}$ the group generated by $A$, and by $\Gamma$ the corresponding subgroup of $PU(H)$.

Here $H$ is a Hermitian form over a number field $\mathbb{K}$, with ring of integers $\mathcal{O}_{\mathbb{K}}$. Given an ideal $I \subset \mathcal{O}_{\mathbb{K}}$, we consider the finite ring $R = \mathcal{O}_{\mathbb{K}}/I$, and consider the group homomorphism

$$\varphi : \widetilde{\Gamma} \to GL(3, R)$$

obtained by reducing all entries modulo $I$.

**Definition 5.1.** *The principal congruence subgroup of $\Gamma$ mod $I$ is the projectivization of the kernel of $\varphi$.*

In order for these principal congruence subgroups to be accessible to computation, we need to define the above data in Magma, which is a little subtle.

Recall that we are given a presentation for $\Gamma$ in terms of generators and relations, and we would like to compute a presentation for a principal congruence subgroup. In order to do this, we follow the next steps.

- Define $\widetilde{F} = \mathrm{Im}(\varphi)$ as the `MatrixGroup` over the finite ring $R = \mathcal{O}_{\mathbb{K}}/I$ generated by $\varphi(a)$, $a \in A$.
- Convert $\widetilde{F}$ to a permutation group via `PermutationGroup(FPGroup( ...));`, and compute the permutation group $F = \widetilde{F}/Z_{\widetilde{F}}$, where $Z_{\widetilde{F}}$ is the group of scalar matrices in $\widetilde{F}$.
- Define a homomorphism from $\Gamma$ to $F$, and compute its kernel.

The above steps are quite heavy computationally, and they will only succeed when the order of $\widetilde{F}$ is not too large.

Note that in order to use the above method, we need to choose an ideal $I$ in $\mathcal{O}_{\mathbb{K}}$. Recall that $\mathcal{O}_{\mathbb{K}}$ is not a unique factorization domain, but there is a unique factorization of ideals as a product of prime ideals, see standard texts on algebraic number theory, e.g. [36]. This factorization is implemented in Magma.

In order to select ideals, we factor a the ideal $n\mathcal{O}_{\mathbb{K}} = I_1 \cdot I_2 \cdots I_r$ for some rational integer $n \in \mathbb{Z}$, and pick one of its prime ideal factors $I_1$ (or sometimes the product of several such prime ideal factors).

5.5. **Promoting normal subgroups to non-normal subgroups of smaller index.** The last three methods (LINS, QS, SC) give normal subgroups of fairly large index, and we now explain how to improve this to get (non-normal) subgroups of smaller index.

Suppose $\varphi : \Gamma \to F$ is a surjective morphism onto a finite group, let $K = \mathrm{Ker}(h)$. The basic observation is that for every subgroup $S \subset F$, $H = \varphi^{-1}(S)$ is a subgroup of $\Gamma$ that contains $K$. Converserly, any subgroup $H$ with $K \subset H \subset \Gamma$ is obtained in this way (as $\varphi^{-1}(\varphi(H))$). Note also that with the above notation, the indices $[\Gamma : H]$ and $[F : S]$ are equal.

Moreover, the preimage $\varphi^{-1}(S)$ is a finitely presented group and, at least when the index is not too large, a presentation can be obtained via Magma.

Now suppose that we have a list $I_1, \dots, I_r$ of finite subgroups of $\Gamma$ such that every torsion element in $\Gamma$ is conjugate to an element of some $I_k$ (in other words, the subgroups $I_k$ give a list of the non-trivial isotropy groups for the action of $\Gamma$ on $\mathbb{H}^2_{\mathbb{C}}$).

**Proposition 5.1.** (1) *$K$ is torsion-free if and only if $|\varphi(I_j)| = |I_j|$ for all $j = 1, \dots, r$.*
(2) *If $K$ is torsion-free, then $H = \varphi^{-1}(S)$ is torsion-free if and only if $x^{-1}Sx \cap I_j = \{Id\}$ for every $x \in F$ and every $j$.*

In part (2), instead of taking all the elements $x \in F$, it is of course enough to check all the elements in a right-transversal for $S$ in $F$. Note that all the verifications of Proposition 5.1 take place in a finite group, so in a sense they can be considered easy... but they can take a long time if the index $[F : S]$ is big.

There is also a variant of this method that allows us to check whether $K$ and $H$ are neat (see Definition 3.2), but this is quite a bit more complicated. We now summarize its main steps.

- Let $C$ denote a cusp of $G$ (we discussed how to get generators and relations for $C$ in section 4.2).
- Let $C_F$ denote $h(C)$ (generators for this group are given by the $h$-images of generators of $C$), and $K_C = \mathrm{Ker}(h|C)$. Note that this has finite index in $C$, so Magma can find a presentation for $K_C$. Let $k_1, \dots, k_n$ denote a generating set for $K_C$.
- The cusps of $K \backslash \mathbb{H}^2_{\mathbb{C}}$ are in 1-1 correspondence with right cosets of $C_F$ in $F$, and the corresponding cusp groups are all conjugate to $K_C$. Denote by $f_1, \dots, f_m$ a right transversal for $C_F$.

- Study the action of $S = h(H)$ on the set of the right coset of $C_F$, find its orbits and stabilizers. The cusps of $H$ are in 1-1 correspondence with these orbits.
- In order to get cusp generators, take a right coset $x = C_F f_j$ and denote by $I_x$ its stabilizer under the $S$-action. Find a generating set for $I_x$, and choose lifts $i_1, \dots, i_r$ to $H$ for this generating set. Cusp generators are given by the elements
$$i_1, \dots, i_r, k_1^{f_j}, \dots, k_n^{f_j}.$$
- Use a suitable linear change of coordinates to make all cusp generators upper triangular, and multiply each of them by a scalar to get the upper left entry to be 1 (recall we are working in $PU(2,1)$ rather than $U(2,1)$). The corresponding cusp is neat if and only if every generator is unipotent.
- Since every cusp is isomorphic to a finite index subgroup of $K_C$, we can use Magma to get a presentation for every cusp group.
- If the cusp is neat, we compute the self-intersection of corresponding the elliptic curve in the toroidal compactification by computing the abelianization of the corresponding cusp group, which is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}_q$ for some $q \in \mathbb{N}^*$. In that case the self-intersection is given by $-q$ (see Proposition 4.2.12 in [24]).

A Magma implementation of these methods can be found in our computer code [10].

## 6. A tower of one-cusped smooth ball quotients

In this section we describe prove Theorem 1.3. As mentioned in the introduction, this gives a positive answer to a question raised by Stover in [41].

The group is a subgroup of index 72 in the Mostow group $G = \Gamma(6, 0)$, which has the presentation
$$\left\langle\ r_1, r_2, r_3, j \mid r_2^{-1} j r_1 j^{-1}, r_3^{-1} j^{-1} r_1 j, r_1^6, j^3, (r_1 j)^{12}, (r_2 r_1 j)^6, \mathrm{br}_3(r_1, r_2)\ \right\rangle.$$
This group has a single cusp, represented by $C = \langle r_1, r_2 \rangle$.

Every non-trivial isotropy group for the action of $G$ on $\mathbb{H}^2_{\mathbb{C}}$ is in the list of table 4.

| Isotropy group | $\langle j \rangle$ | $\langle r_1 j \rangle$ | $\langle r_1, r_2 r_1 j \rangle$ | $\langle r_2 j^{-1} \rangle$ | $\langle r_2, (r_1 j)^2 \rangle$ | $\langle r_2 r_1 j, (r_1 j)^2 \rangle$ |
|---|---|---|---|---|---|---|
| Order | 3 | 12 | 36 | 12 | 36 | 36 |

Table 4. Isotropy groups of $G$

The subgroup alluded to in the statement of Theorem 1.3 is
$$H = \langle r_1 r_2 r_1 j r_2^{-1}, r_1 r_2 r_3 j r_1^{-1}, j r_1^{-1} r_3 r_2 r_3, (r_1^2 r_2)^2 \rangle.$$
Alternatively, the reader can reconstruct the group by loading the code given in [10] into Magma, then running the commands

```
A:=MostowGroup(6,0);
FindTorsionFreeSubgroups(~A,864,[2]);
```

The field `A‘Subgroups` should then have a seven elements, and `A‘Subgroups[7]` contains a Magma description of the subgroup in question. We write $X = H\backslash\mathbb{H}^2_{\mathbb{C}}$.

From the above pieces of information, Magma allows us to check that $[\Gamma : H] = 72$, hence

$$\chi(X) = 72 \cdot \chi^{orb}(\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}) = 72 \cdot \frac{1}{12} = 6.$$

Let $K = \mathrm{Core}_G(H)$, and let $F = \Gamma/K$, which has order 864. Using Magma, it is straighforward to check that the isotropy groups of Table 4 inject in $F$, so $K$ is torsion-free.

Let us denote by $S \subset F$ the image of $H$. A simple computer check shows that $S$ has trivial intersection with every conjugate of the image in $F$ of every isotropy group in Table 4, so $H$ is torsion-free.

We write $X = H\backslash\mathbb{H}^2_{\mathbb{C}}$ and $Y = K\backslash\mathbb{H}^2_{\mathbb{C}}$, and $\bar{X}$, $\bar{Y}$ for their respective toroidal compactifications.

In order to study the cusps of $X$ and $Y$, we let $C_F$ denote the image in $F$ of the cusp $\langle r_1, r_2\rangle$ in $G$; we have $[F : C_F] = 12$, so $Y$ has 12 cusps. Computing the kernel $CK$ of the homomorphism $h_C : C \to C_F$ using Magma, one computes generators, and checks that each generator is unipotent element, so $K$ is neat. The Abelianization of $CK$ is $\mathbb{Z}_{12} \oplus \mathbb{Z}^2$, so the 12 elliptic curves compactifying $Y$ to $\bar{Y}$ have self-intersection $-12$.

In order to prove that $X$ has a single cusp, simply check that $S$ acts transitively on the set of right cosets of $C_F$ in $F$. It also follows that the 12 cusp groups of $X$ are all isomorphic to the cusp group $CK$ of $X$. Using Magma, one can check that the cusp $C$ of $H$ is represented by the group generated by

$$(r_2 r_1^2)^2, \quad r_2^2 r_1^{-1} r_2 r_1^{-2} r_2 r_1^{-1}, \quad r_2 r_1^{-2} r_2 r_1^{-1} r_2^2 r_1^{-1}. \tag{9}$$

We finish this section by proving Theorem 1.4. Let $\varphi : H \to \mathbb{Z}^2$ be obtained from the abelianization map by projecting onto the $\mathbb{Z}^2$-factor of $\mathbb{Z}_3 \oplus \mathbb{Z}^2$. By explicit calculation of the image of the generating set for $C$ of equation (9), we check that $\varphi(C)$ is a lattice in of index 12 in $\mathbb{Z}^2$.

By projecting onto one factor of $\mathbb{Z}^2$, it is easy to get a homomorphism $\psi : H \to \mathbb{Z}$ such that $\psi(C) = 2\mathbb{Z}$. Now for $n \in \mathbb{N}^*$, consider $H_n = \psi^{-1}(n\mathbb{Z})$. For any odd $n$, the action of $n\mathbb{Z}$ is transitive on the cosets of $2\mathbb{Z}$, so $H_n$ has exactly one cusp. The tower of Theorem 1.4 is then obtained by choosing an increasing sequence of odd integers $n_1 < n_2 < \dots$ such that for all $j$, $n_j$ divides $n_{j+1}$.

## 7. Summary of the results for all groups

In this section, we list the subgroups we were able to find inside the known lattice triangle groups. We focus on three different aspects, namely

- neat subgroups (section 7.3)
- subgroups with positive $b_1$ (section 7.1)
- large subgroups (section 7.2)

7.1. **Positive $b_1$.** In order to find subgroups $\Gamma_0 \subset G$ with $b_1(\Gamma_0) > 0$, we study whichever neat subgroups we can find in $G$, ask Magma to compute the Abelizanition using the command `AbelianQuotientInvariants`.

When getting a homomorphism by using `SimpleQuotients` or `CongruenceImage`, the large order of the group often seems to make it too difficult for Magma to compute a presentation for $K = Ker(h)$, it is not clear how to estimate $b_1(K)$.

In Tables 5 through 7, we list the groups where one of these methods yields an explicit (torsion-free or not) subgroup with positive $b_1$. In case we have such an example we mention how the reader can find it, writing

- "LIS(N)" where $N$ is the index of the subgroup;
- "LINS(N)" where $N$ is the (bound on the) index of the normal subgroup or "LINS(N,G)" with an concise description of the quotient if we have one;
- "SQ(G)" where $G$ is a description of the Simple Quotient;
- "mod $p$" if reduction mod $p$ gives a kernel with positive $b_1$.
- "Map(N)" if we know a homomorphism to a triangle group having a torsion-free subgroup of index $N$ (see section 7.2).
- nothing if we were unable to find any subgroup with positive $b_1$.

For non-cocompact groups, in the last column of the tables, we give information as to whether or not the positive first Betti number comes from the cusps; more precisely, we mention whether (for a suitable subgroup in the list) all cusp groups map have infinite image in the abelianization.

7.2. **Large subgroups.** We call a group $G$ large if it has a finite index subgroup $H$ with a surjective homomorphism $\varphi : H \to F_n$, where $n > 1$ and $F_n$ is the non-Abelian free group on $n$ generators. Of course, we may assume $n = 2$, since every $F_n, n > 2$ maps onto $F_2$. If this is the case, then by abelianizing the free group, we get a morphism onto $\mathbb{Z}^2$, hence the first Betti number of $H$ is positive.

Even though there are nice sufficient conditions of largeness due to Button [5], there is no general algorithm for determining largeness. The sufficient condition for largeness used in this paper is based on the following simple observation.

**Lemma 7.1.** (1) *Every torsion-free lattice in $PSL(2,\mathbb{R})$ maps onto a non-Abelian free group;*
(2) *Every lattice in $PSL(2,\mathbb{R})$ is large.*

**Proof:** Part (2) follows from part (1) because of Selberg's lemma. Part (1) is obvious for non cocompact lattices, since the fundamental group of a non-compact hyperbolic Riemann surface is itself a non-Abelian free group. For cocompact lattices, we can write the corresponding closed surface as the boundary of a handlebody, which retracts onto a bouquet of circles. □

As an immediate consequence, we see that if $G$ maps onto a lattice in $PSL(2,\mathbb{R})$, then $G$ is large.

Note that $PSL(2,\mathbb{R}) \cong PU(1,1)$, but when we change this group to $PU(2,1)$, we have no reasonable analogue of Lemma 7.1 at hand. It is a folklore conjecture that every lattice

| | | | | | |
|---|---|---|---|---|---|
| NA | NC | $\mathcal{S}(3,\sigma_1)$ | OK | LINS(18144), SQ | ? |
| NA | NC | $\mathcal{S}(4,\sigma_1)$ | OK | LINS(96) | $\infty$ cusp images |
| NA | NC | $\mathcal{S}(6,\sigma_1)$ | OK | LINS(6) | $\infty$ cusp images |
| | | $\mathcal{S}(3,\bar{\sigma}_4)$ | OK | LINS(18144) | |
| NA | NC | $\mathcal{S}(4,\bar{\sigma}_4)$ | ? | | ? |
| NA | | $\mathcal{S}(5,\bar{\sigma}_4)$ | ? | | |
| NA | NC | $\mathcal{S}(6,\bar{\sigma}_4)$ | ? | | ? |
| NA | | $\mathcal{S}(8,\bar{\sigma}_4)$ | ? | | |
| NA | | $\mathcal{S}(12,\bar{\sigma}_4)$ | ? | | |
| | | $\mathcal{S}(2,\sigma_5)$ | ? | | |
| NA | NC | $\mathcal{S}(3,\sigma_5)$ | OK | LINS(360) | $\infty$ cusp images |
| NA | NC | $\mathcal{S}(4,\sigma_5)$ | ? | | ? |
| | | $\mathcal{S}(3,\sigma_{10})$ | OK | LINS(2160) | |
| | | $\mathcal{S}(4,\sigma_{10})$ | ? | | |
| | | $\mathcal{S}(5,\sigma_{10})$ | OK | LINS(600) | |
| | | $\mathcal{S}(10,\sigma_{10})$ | OK | LINS(18000) | |

TABLE 5. Subgroups of Sporadic groups with $b_1 > 0$

| | | | | | |
|---|---|---|---|---|---|
| | | $\mathcal{T}(3,\mathbf{S}_2)$ | OK | LINS(360) | |
| NA | NC | $\mathcal{T}(4,\mathbf{S}_2)$ | OK | LINS(23040) | not all cusps have $\infty$ image |
| | | $\mathcal{T}(5,\mathbf{S}_2)$ | ? | | |
| | NC | $\mathcal{T}(3,\mathbf{E}_2)$ | OK | LINS(24) | $\infty$ cusp images |
| NA | NC | $\mathcal{T}(4,\mathbf{E}_2)$ | OK | LINS(24) | $\infty$ cusp images |
| | NC | $\mathcal{T}(6,\mathbf{E}_2)$ | OK | LINS(6) | $\infty$ cusp images |
| | | $\mathcal{T}(2,\mathbf{H}_1)$ | OK | LIS(56) | |
| | | $\mathcal{T}(2,\mathbf{H}_2)$ | OK | LIS(30) | |
| | | $\mathcal{T}(3,\mathbf{H}_2)$ | ? | | |
| | | $\mathcal{T}(5,\mathbf{H}_2)$ | OK | LINS(600) | |

TABLE 6. Subgroups of Thompson groups with $b_1 > 0$

in $PU(2,1)$ of Kazhdan type (i.e. such that its ambient algebraic $\mathbb{Q}$-group is the group of a Hermitian form over a number field, rather than a more complicated division algebra) should have a large subgroup of finite index. Every complex reflection group is of Kazhdan type, so all the groups we consider in this paper fall in the scope of this conjecture, and we expect them to be large; but for most known lattice triangle groups, largeness is not known.

| | | | | | |
|---|---|---|---|---|---|
| | | $\Gamma(5,7/10)$ | OK | LINS(15000) | |
| | NC | $\Gamma(6,2/3)$ | OK | LINS(6) | $\infty$ cusp images |
| | | $\Gamma(7,9/14)$ | OK | Map(84) | |
| | | $\Gamma(8,5/8)$ | OK | LIS(12), Map(48) | |
| | | $\Gamma(9,11/18)$ | OK | LIS(9) | |
| | | $\Gamma(10,3/5)$ | OK | LIS(10), Map(30) | |
| | | $\Gamma(12,7/12)$ | OK | LIS(6), Map(72) | |
| | | $\Gamma(18,5/9)$ | OK | LIS(6), Map(54) | |
| | | $\Gamma(4,5/12)$ | OK | LINS(18144) | |
| | | $\Gamma(5,11/30)$ | ? | | |
| | NC | $\Gamma(6,1/3)$ | OK | LINS(6) | $\infty$ cusp images |
| | | $\Gamma(7,13/42)$ | OK | Map(84) | |
| | | $\Gamma(8,7/24)$ | OK | LIS(12), Map(48) | |
| | | $\Gamma(9,5/18)$ | OK | LIS(9) | |
| | | $\Gamma(10,4/15)$ | OK | LIS(10), Map(30) | |
| | | $\Gamma(12,1/4)$ | OK | LIS(6), Map(72) | |
| | | $\Gamma(18,2/9)$ | OK | LIS(6), Map(18) | |
| | | $\Gamma(3,1/3)$ | OK | LINS(18144) | |
| | NC | $\Gamma(4,1/4)$ | OK | LINS(96) | $\infty$ cusp images |
| | | $\Gamma(5,1/5)$ | ? | | |
| NA | NC | $\Gamma(6,1/6)$ | OK | LIS(6), Map(18) | $\infty$ cusp images |
| | | $\Gamma(8,1/8)$ | OK | LINS(1536), Map(72) | |
| | | $\Gamma(12,1/12)$ | OK | LIS(6), Map(72) | |
| | | $\Gamma(3,7/30)$ | ? | | |
| | | $\Gamma(4,3/20)$ | OK | Map(120) | |
| | | $\Gamma(5,1/10)$ | OK | LINS(600) | |
| | | $\Gamma(10,0)$ | OK | LINS(600), Map(30) | |
| | NC | $\Gamma(3,1/6)$ | OK | LINS(72) | $\infty$ cusp images |
| | | $\Gamma(4,1/12)$ | OK | LINS(864), Map(18) | |
| | NC | $\Gamma(6,0)$ | OK | LINS(6) | $\infty$ cusp images |
| | | $\Gamma(3,5/42)$ | ? | | |
| | | $\Gamma(3,1/12)$ | OK | LIS(48) | |
| | | $\Gamma(4,0)$ | OK | LIS(24), Map(48) | |
| | | $\Gamma(3,1/18)$ | OK | LIS(24) | |
| | | $\Gamma(3,1/30)$ | OK | LIS(40) | |
| | | $\Gamma(3,0)$ | OK | LIS(24) | |

TABLE 7. Subgroups of Mostow groups with $b_1 > 0$

We now discuss how to use Lemma 7.1 in order to find explicit subgroups of finite index in some Mostow groups that map to non-Abelian free groups. We do not know how to generalize this to other lattice triangle groups.

It is well known to experts that Livne's thesis [30] gives maps from *some* Mostow lattices to surface groups, which shows that some specific Mostow lattices are large. For a description of Livne's lattices in relation to Mostow's groups, see §16 in the book by Deligne and Mostow [8]. Recall that Livne constructs smooth ball quotients $X_n$ indexed by $n = 5, 6, 7, 8, 9, 10, 12, 18$, whose automorphism group $A$ fits is an extension

$$1 \to \mathbb{Z}/d\mathbb{Z} \to A \to SL(2, \mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})^2 \to 1,$$

and the quotient $A\backslash X_n$ is $\Gamma_{\mu,\Sigma}\backslash\mathbb{H}^2_{\mathbb{C}}$, where $d = gcd(6, n)$. The order of these automorphism groups gives the index of the corresponding subgroup of the relevant Mostow group; we list these in the last column of Table 8. For some Mostow groups, we will give subgroups

| Mostow group | $n$ | $d$ | $\lvert SL(2,n)\rvert$ | Index |
|---|---|---|---|---|
| $\Gamma(5, \frac{7}{10})$ | 5 | 5 | 120 | 15 000 |
| $\Gamma(6, \frac{2}{3})$ | 6 | 1 | 144 | 5 184 |
| $\Gamma(7, \frac{9}{14})$ | 7 | 7 | 336 | 115 248 |
| $\Gamma(8, \frac{5}{8})$ | 8 | 4 | 384 | 98 304 |
| $\Gamma(9, \frac{11}{18})$ | 9 | 3 | 648 | 157 464 |
| $\Gamma(10, \frac{3}{5})$ | 10 | 5 | 720 | 360 000 |
| $\Gamma(12, \frac{7}{12})$ | 12 | 2 | 1 152 | 331 776 |
| $\Gamma(18, \frac{5}{9})$ | 18 | 3 | 3 888 | 3 779 136 |

TABLE 8. Index of torsion-free subgroups coming from Livne's construction

of much smaller index that map onto a non-Abelian free group, see Proposition 7.1 and Table 10.

Note that the maps from Livne groups to Fuchsian groups are actually induced by holomorphic fibrations of the corresponding ball quotients over suitable Riemann surfaces (actually $X_n$ fibers over the quotient of the upper half plane under the principal congruence subgroup modulo $n$, see [8]). The maps we construct are also induced by holomorphic maps to curves, as can be seen from their interpretation as forgetful maps of moduli spaces of points on $\mathbb{P}^1_{\mathbb{C}}$, see [12].

Rather than going into the details of Deligne-Mostow theory and forgetful maps, we give a more down to earth approach that requires only group theory.

**Proposition 7.1.** *The Mostow groups* $\Gamma(7, 9/14)$, $\Gamma(8, 5/8)$, $\Gamma(9, 11/18)$, $\Gamma(10, 3/5)$, $\Gamma(12, 7/12)$, $\Gamma(18, 5/9)$, $\Gamma(7, 13/42)$, $\Gamma(8, 7/24)$, $\Gamma(9, 5/18)$, $\Gamma(10, 4/15)$, $\Gamma(12, 1/4)$, $\Gamma(18, 2/9)$, $\Gamma(6, 1/6)$, $\Gamma(8, 1/8)$, $\Gamma(12, 1/12)$, $\Gamma(4, 3/20)$, $\Gamma(10, 0)$, $\Gamma(4, 1/12)$, $\Gamma(6, 0)$, $\Gamma(4, 0)$ *have a surjective homomorphism onto a lattice in* $PSL(2, \mathbb{R})$.

| $p$ | 7 | 8 | 9 | 10 | 12 | 18 |
|---|---|---|---|---|---|---|
| Index | 84 | 48 | 36 | 30 | 24 | 18 |

TABLE 9. Optimal index of torsion-free subgroups in the triangle group $T_{2,3,p}$.

The basis of the construction of these homorphisms is simply to combine Mostow's braid group description of the lattices $\Gamma(p,t)$ with the results in [12] about forgetful maps of Deligne-Mostow moduli spaces.

We briefly sketch how this is done; the reader without any knowledge of Deligne-Mostow theory can skip this part. Each group $\Gamma(p,t)=\langle R_1,J\rangle$ is conjugate in $PU(2,1)$ to a specific Deligne-Mostow group from [7], [34], namely $\Gamma_{\mu,\Sigma}$ where

$$\mu=\left(\frac{1}{2}-\frac{1}{p},\frac{1}{2}-\frac{1}{p},\frac{1}{2}-\frac{1}{p},\frac{1}{4}+\frac{3}{2p}-\frac{t}{2},\frac{1}{4}+\frac{3}{2p}+\frac{t}{2}\right)$$

and $\Sigma=S_3$ permutes the first three weights.

If $\mu$ satisfies the Deligne-Mostow INT condition, then the corresponding Deligne-Mostow group $\Gamma_\mu$ is contained in $\Gamma_{\mu,\Sigma}$ as a subgroups of index $|\Sigma|=3!=6$. Mostow explains in [35] how to write explicit generators for $\Gamma_\mu$, namely $A_j,B_j$, $j=1,2,3$ with

$$B_j=R_j^2,\quad A_j^{-1}=J^{-1}R_jR_{j+1}. \tag{10}$$

Recall that $R_{j+1}=JR_jJ^{-1}$ index $j$ taken mod 3 ($J^3=Id$). From this information, it is easy to ask Magma for a presentation for each $\Gamma_\mu$.

Note also that when $\mu$ satisfies Mostow's $\Sigma$-INT condition for $\Sigma=S_3$ but not the INT condition, we have $\Gamma_\mu=\Gamma_{\mu,\Sigma}$, see [38] for instance.

Recall the presentation for $\Gamma(p,t)$ given in Appendix A.9 of[14], namely

$$\Gamma(p,t)=\Big\langle\, R_1,R_2,R_3,J\ \Big|\ R_1^p,\ J^3,\ (R_1J)^{2k},\ JR_1J^{-1}=R_2,\ JR_2J^{-1}=R_3,\\ \mathrm{br}_3(R_1,R_2),\ (R_1R_2)^{\frac{6p}{p-6}},\ (JR_2R_1)^{\frac{4kp}{(2k-4)p-4k}}\ \Big\rangle$$

where the last two relations are omitted when the corresponding exponent is negative.

We wish to construct a homomorphism of $\Gamma(p,t)$ onto the $(2,3,p)$ triangle group

$$T_{2,3,p}=\langle\ \beta,\gamma\ |\ (\beta\gamma)^2,\beta^3,\gamma^p\ \rangle.$$

For convenience, recall that the optimal index of a torsion-free subgroup in the hyperbolic triangle groups are as in Table 9 (cf. [17]).

**Proposition 7.2.** *Suppose $p>6$ and $\Gamma(p,t)$ satisfies the Mostow $\Sigma$-INT condition. Then there is a unique group homomorphism $\varphi:\Gamma(p,t)\to T_{2,3,p}$ such that $\varphi(J)=\beta$ and $\varphi(R_1)=\gamma$.*

**Proof:** If the map exists, then we must have $\varphi(R_2)=\beta\gamma\beta^{-1}$ and $\varphi(R_3)=\beta^{-1}\gamma\beta$. We then check that, under the hypotheses of Proposition 7.2, for every relator $w_1\dots w_n$ in the presentation of $\Gamma(p,t)$, the relation $\varphi(w_1)\dots\varphi(w_n)=id$ holds in $T_{2,3,p}$ (in that case, we say that the relation is compatible).

For example, $\varphi(R_1)\varphi(R_2)\varphi(R_1) = \gamma\beta\gamma\beta^{-1}\gamma = \beta^{-2}\gamma = \beta\gamma$, and $\varphi(R_2)\varphi(R_1)\varphi(R_2) = \beta\gamma\beta^{-1}\gamma\beta\gamma\beta^{-1} = \beta\gamma\beta^2\gamma\beta\gamma\beta^2 = \gamma^{-1}\beta^{-1} = \beta\gamma$, so the braid relation $R_1R_2R_1(R_2R_1R_2)^{-1}$ is compatible.

For the other relations, one checks that $\varphi(R_1)\varphi(R_2) = \beta$ and $\varphi(R_1)\varphi(R_2)\varphi(J) = \gamma^{-2}$. It is then easy to verify that for the relevant values of $k = 2, 3, 4$ and 5, $6p/(p-6)$ is a multiple of 3 and $4kp/((k-2)p-2k)$ is a multiple of $p$. □

Proposition 7.2 implies Proposition 7.1 for the Mostow groups $\Gamma(7, \frac{9}{14})$, $\Gamma(8, \frac{5}{8})$, $\Gamma(9, \frac{11}{18})$, $\Gamma(10, \frac{3}{5})$, $\Gamma(12, \frac{7}{12})$, $\Gamma(18, \frac{5}{9})$, $\Gamma(7, \frac{13}{42})$, $\Gamma(8, \frac{7}{24})$, $\Gamma(9, \frac{5}{18})$, $\Gamma(10, \frac{4}{15})$, $\Gamma(12, \frac{1}{4})$, $\Gamma(18, \frac{12}{9})$, $\Gamma(8, \frac{1}{8})$, $\Gamma(12, \frac{1}{12})$, $\Gamma(10, 0)$. Note that the first 6 groups in this list are Livne lattices; one can check that the corresponding maps are the same as the ones that come from the Livné construction (the construction is conveniently reviewed in [8], §16).

For the other cases of Proposition 7.1, we use $\Gamma_\mu$ instead of $\Gamma_{\mu,\Sigma}$. We define two kinds of subgroups of $\Gamma_\mu$, namely

$$K = \langle\langle A_1, A_2, A_3\rangle\rangle, \quad L = \langle\langle A_1, B_2, B_3\rangle\rangle, \tag{11}$$

see the notation in equation (10).

We then use group theory software to compute a simplified presentation for $\Gamma_\mu/K$ and $\Gamma_\mu/L$, and find that in each of the cases listed in Proposition 7.1, the corresponding quotient is a $p, q, r$ triangle group.

The details are listed in Table 10. In the last column of the table, we list the index of the subgroups we obtain that map to a non-Abelian free group. The factor 6 comes from the index of $\Gamma_\mu$ in $\Gamma_{\mu,\Sigma}$, the other factor comes from the minimal index of a torsion-free subgroup in a Fuchsian group [17].

Recall that when $1/p+1/q+1/r$, the $(p, q, r)$ is a lattice in $PSL(2, \mathbb{R})$, and this is satisfied for at least one triple $(p, q, r)$ of each row of table 10, so the proof of Proposition 7.1 is complete.

Now Lemma 7.1 implies the following.

**Theorem 7.3.** *Let $\Gamma$ be one of the Mostow lattices listed in Proposition 7.1. Then $\Gamma$ is large.*

Note also that our proof give a way to describe explicit subgroups, and explicit morphisms to $F_2$.

7.3. **Neat subgroups.** We summarize the results of our search for neat subgroups of small index in Tables 11 through 72. In these tables, for each triangle group, we give

- The Euler characteristic and the least common multiple (LCM) of the orders of isotropy groups; this gives a lower bound for the index of torsion-free subgroups. Note that by Noether's formula (see section 3), compact ball quotients have Euler characteristic an integer multiple of 3. In some cases, this yields a slightly higher lower bound (given by 3·LCM);
- The arithmeticity (A/NA) and cocompactness (C/NC) of the group;
- Its Abelianization;

| $\Gamma(p,t)$ | Cocompact, Arithmetic? | $\Gamma_\mu/K$ | $\Gamma_\mu/L$ | Index |
|---|---|---|---|---|
| $\Gamma(8,5/8)$ | C,A | 4,4,4 | 2,2,4 | $6\cdot 8=48$ |
| $\Gamma(10,3/5)$ | C,A | 5,5,5 | 2,2,5 | $6\cdot 5=30$ |
| $\Gamma(12,7/12)$ | C,A | 6,6,6 | 2,2,2 | $6\cdot 12=72$ |
| $\Gamma(18,5/9)$ | C,A | 9,9,9 | 2,2,3 | $6\cdot 9=54$ |
| $\Gamma(8,7/24)$ | C,NA | 4,4,4 | 3,3,4 | $6\cdot 8=48$ |
| $\Gamma(10,4/15)$ | C,NA | 5,5,5 | 3,3,5 | $6\cdot 5=30$ |
| $\Gamma(12,1/4)$ | C,A | 6,6,6 | 2,3,3 | $6\cdot 12=72$ |
| $\Gamma(18,2/9)$ | C,A | 9,9,9 | 3,3,3 | $6\cdot 3=18$ |
| $\Gamma(6,1/6)$ | NC,NA | 3,3,3 | 3,4,4 | $6\cdot 3=18$ |
| $\Gamma(8,1/8)$ | C,A | 4,4,4 | 4,4,4 | $6\cdot 12=72$ |
| $\Gamma(12,1/12)$ | C,A | 6,6,6 | 2,2,2 | $6\cdot 12=72$ |
| $\Gamma(4,3/20)$ | C,A | 2,2,2 | 2,5,5 | $6\cdot 20=120$ |
| $\Gamma(10,0)$ | C,A | 5,5,5 | 5,5,5 | $6\cdot 5=30$ |
| $\Gamma(4,1/12)$ | C,NA | 2,2,2 | 2,6,6 | $6\cdot 12=72$ |
| $\Gamma(6,0)$ | NC,A | 3,3,3 | 3,6,6 | $6\cdot 3=18$ |
| $\Gamma(4,0)$ | C,A | 2,2,2 | 2,8,8 | $6\cdot 8=48$ |

TABLE 10. List of maps to triangle groups; in the last column, we list the index of the subgroups we find that map to a non-Abelian free group.

- Generators for (representatives of) its cusps (inside one given box corresponding to the description of the cusps, each line corresponds to different conjugacy classes of cusps).
- The order of the smallest congruence image such that the corresponding principal congruence subgroup is torsion-free (we also an explicit description of the congruence image group, if we know one). In some cases, the congruence image seems too large to compute anything about it, in which case we simply write "?" in the corresponding box of the table. In the last column, we list the rational prime whose prime ideal factor was used to get a torsion-free congruence subgroup (in some rare cases, one can get a smaller congruence by reduction modulo a non-prime ideal).
- If we found any neat subgroup with index smaller than the order of the congruence image, we list some basic invariants for the corresponding subgroups (index, index of the normal core, quotient by the normal core, abelianization, self-intersections of the elliptic curves in the toroidal compactification, first Betti number).

*Remark* 7.4. (1) When the Congruence image column of the following tables contains a question mark "?", we mean that the corresponding finite group seems too large to compute anything. We suspect that one could probably work a little more and compute the order of these groups and identify them as explicit projectivized linear groups over finite fields, but it is probably hopeless to try and compute anything about the corresponding principal congruence subgroups (presentation, Abelianization).

(2) In Tables 11 through 72, we write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$, and $PU(m,n)$ denotes $PU(m,\mathbb{F}_n)$, $PGL(m,n)$ denotes $PGL(m,\mathbb{F}_n)$ where $F_n$ is the field with $n$ elements ($n$ is a power of a single prime).

$\mathcal{S}(3,\sigma_1)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{2}{9}$ | 72 | NA | NC | $\mathbb{Z}_3$ | $R_1, R_2$ | $\mathrm{br}_6(R_1,R_2)$ | 378 000, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $432 = 6\cdot 72$ | 16868 | $\mathbb{Z}_3 \times PGL(3,3)$ | $\mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{18}$ | $(-3)^6, (-9)^{14}$ | 0 |
| $864 = 12\cdot 72$ | 18144 | $\mathbb{Z}_3 \times PSU(3,3)$ | $\mathbb{Z}_2^2 \oplus \mathbb{Z}^2$ | $(-2)^{36}$ | 2 |

TABLE 11. Neat subgroups of $\mathcal{S}(3,\sigma_1)$ with core index $\leq 20000$

$\mathcal{S}(4,\sigma_1)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{7}{16}$ | 96 | NA | NC | $\mathbb{Z}_4$ | $1, 23\bar{2}$ | $\mathrm{br}_4(1,23\bar{2})$ | 42 456 960, PGL(3,9) |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 96 | 96 | | $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}^4$ | $(-8)^{24}$ | 4 |
| 96 | 384 | | $\mathbb{Z}_2^5 \oplus \mathbb{Z}_8^2$ | $(-8)^6$ | 0 |
| 96 | 384 | | $\mathbb{Z}_2^5 \oplus \mathbb{Z}_4^3$ | $(-4)^{12}$ | 0 |
| 96 | 384 | | $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_8^2$ | $(-4)^{12}$ | 0 |
| 96 | 384 | | $\mathbb{Z}_2^5 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_8^2$ | $(-2)^8, (-4)^8$ | 0 |
| 96 | 384 | | $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}^2$ | $(-2)^8, (-4)^8$ | 2 |

TABLE 12. Neat subgroups of $\mathcal{S}(3,\sigma_1)$ with core index $\leq 384$

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[2] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai. *Smooth compactifications of locally symmetric varieties*. Cambridge: Cambridge University Press, 2010.

[3] Gottfried Barthel, Friedrich Hirzebruch, and Thomas Höfer. *Geradenkonfigurationen und algebraische Flächen*, volume D4 of *Aspects Math.* Springer, Wiesbaden, 1987.

[4] Armand Borel. Introduction aux groupes arithmétiques. Paris: Hermann & Cie, 1969.

[5] J. O. Button. Proving finitely presented groups are large by computer. *Exp. Math.*, 20(2):153–168, 2011.

$\mathcal{S}(6,\sigma_1)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{43}{72}$ | 72 | NA | NC | $\mathbb{Z}_6$ | $1, 2323\bar{3}\bar{2}$<br>$1, \bar{3}\bar{2}323$ | $\mathrm{br}_3(1, 2323\bar{3}\bar{2})$<br>$\mathrm{br}_3(1, \bar{3}\bar{2}323)$ | $378\,000$, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $1728 = 24 \cdot 72$ | 36288 | $PSU(3,3) \times \mathbb{Z}_3$ | $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^4$ | $(-2)^{72}$ | 4 |

Table 13. Neat subgroups of $\mathcal{S}(6,\sigma_1)$.

$\mathcal{S}(3,\bar{\sigma}_4)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{2}{63}$ | 504 | A | C | $\mathbb{Z}_3$ | | | $378\,000$, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $6048 = 12 \cdot 504$ | 18144 | $PSU(3,3) \times \mathbb{Z}_3$ | $\mathbb{Z}^{10}$ | | 10 |

Table 14. Neat subgroups of $\mathcal{S}(3,\bar{\sigma}_4)$.

$\mathcal{S}(4,\bar{\sigma}_4)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{25}{224}$ | 672 | NA | NC | $\{0\}$ | $R_1, R_2$ | $\mathrm{br}_4(R_1, R_2)$ | $6\,048$, $PGU(3,3)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $6048 = 9 \cdot 672$ | 6048 | $PSU(3,3)$ | $\mathbb{Z}_3^8$ | $(-6)^{56}$ | 0 |

Table 15. Neat subgroups of $\mathcal{S}(3,\bar{\sigma}_4)$.

$\mathcal{S}(5,\bar{\sigma}_4)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{47}{280}$ | 4200 | NA | C | $\{0\}$ | | | ? |

Table 16. Invariants for $\mathcal{S}(5,\bar{\sigma}_4)$

$\mathcal{S}(6,\bar{\sigma}_4)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{25}{126}$ | 252 | NA | NC | $\mathbb{Z}_3$ | $1, 23\bar{2}$ | $\mathrm{br}_3(1, 23\bar{2})$ | $378\,000$, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $3024 = 12 \cdot 252$ | $378\,000$ | $PGU(3,5)$ | $\mathbb{Z}_5^3$ | $(-1)^4, (-5)^4$ | 0 |

Table 17. Torsion-free subgroups of $\mathcal{S}(6,\bar{\sigma}_4)$.

$\mathcal{S}(8, \bar{\sigma}_4)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{99}{448}$ | 1344 | NA | C | $\{0\}$ | | | 42 456 960, PGL(3,9) |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $131712 = 98 \cdot 1344$ | 5663616 | $PGU(3,7)$ | ? | | ? |

TABLE 18. Torsion-free subgroups of $\mathcal{S}(8, \bar{\sigma}_4)$.

$\mathcal{S}(12, \bar{\sigma}_4)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{221}{1008}$ | 1008 | NA | C | $\mathbb{Z}_3$ | | | ? |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $6048 = 6 \cdot 1008$ | 18144 | $PGU(3,3) \times \mathbb{Z}_3$ | $\mathbb{Z}_3^4$ | | 0 |

TABLE 19. Torsion-free subgroups of $\mathcal{S}(12, \bar{\sigma}_4)$ with core index $\leq 20000$.

$\mathcal{S}(2, \sigma_5)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{45}$ | 360 | A | C | $\mathbb{Z}_6$ | | | 378 000, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $54,000 = 50 \cdot 1080$ | 378 000 | $PGU(3,5)$ | $\mathbb{Z}_2^{10} \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{70}$ | | 0 |

TABLE 20. Torsion-free subgroups of $\mathcal{S}(2, \sigma_5)$

$\mathcal{S}(3, \sigma_5)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{49}{180}$ | 360 | NA | NC | $\mathbb{Z}_3^2$ | $c_1 = 2, c_2 = 2(\bar{J}\bar{1})^2 121\bar{2}\bar{1}(\bar{J}\bar{1})^3 2$ | $c_1^3, c_2^6, \mathrm{br}_4(c_1, c_2)$ | 378 000, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 360 | 360 | | $\mathbb{Z}_3 \oplus \mathbb{Z}^8$ | $(-1)^{20}$ | 8 |
| 360 | 360 | | $\mathbb{Z}_3^9$ | $(-1)^{20}$ | 0 |
| 360 | 1080 | | $\mathbb{Z}_3^2 \oplus \mathbb{Z}^6$ | $(-1)^{20}$ | 6 |

TABLE 21. Neat subgroups of $\mathcal{S}(3, \sigma_5)$ with core index $\leq 10,000$

$\mathcal{S}(4, \sigma_5)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{17}{36}$ | 720 | NA | NC | $\mathbb{Z}_6$ | $R_1, R_2$ | $\mathrm{br}_4(R_1, R_2)$ | 152 334 000 000, $PGL(3,25)$ |

TABLE 22. Invariants for $\mathcal{S}(4, \sigma_5)$

$\mathcal{S}(3, \sigma_{10})$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{45}$ | 360 | A | C | $\mathbb{Z}_3$ | | | 378 000, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $2160 = 2 \cdot 1080$ | 2160 | | $\mathbb{Z}^{16}$ | | 16 |
| $2160 = 2 \cdot 1080$ | 34560 | | $\mathbb{Z}_2^8 \oplus \mathbb{Z}_8$ | | 0 |
| $2160 = 2 \cdot 1080$ | 34560 | | $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_{16}$ | | 0 |

TABLE 23. Neat subgroups of $\mathcal{S}(3, \sigma_{10})$ with core index $\leq 40,000$

$\mathcal{S}(4, \sigma_{10})$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{3}{32}$ | 480 | A | C | $\{1\}$ | | | 42 456 960, $PGL(3,9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $12,000 = 25 \cdot 480$ | $372,000$ | $PGL(3,5)$ | $\mathbb{Z}_5 \oplus \mathbb{Z}_{155}$ | | 0 |

TABLE 24. Neat subgroups of $\mathcal{S}(4, \sigma_{10})$

$\mathcal{S}(5, \sigma_{10})$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{8}$ | 600 | A | C | $\mathbb{Z}_5$ | | | 42 573 600, $PGU(3,9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 600 | 600 | | $\mathbb{Z}^{20}$ | | 20 |
| 600 | 15 000 | | $\mathbb{Z}_5^7$ | | 0 |
| 600 | 15 000 | | $\mathbb{Z}_5 \oplus \mathbb{Z}^8$ | | 8 |

TABLE 25. Neat subgroups of $\mathcal{S}(5, \sigma_{10})$ with core index $\leq 20,000$

$\mathcal{S}(10, \sigma_{10})$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{13}{100}$ | 300 | A | C | $\mathbb{Z}_5$ | | | 42 573 600, $PGU(3,9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 300 | 18,000 | | $\mathbb{Z}_5^2 \oplus \mathbb{Z}_{15}$ | | 0 |

TABLE 26. Neat subgroups of $\mathcal{S}(10, \sigma_{10})$ with core index $\leq 20,000$

[6] Donald I. Cartwright and Tim Steger. Enumeration of the 50 fake projective planes. *C. R., Math., Acad. Sci. Paris*, 348(1-2):11–13, 2010.

$\mathcal{T}(3, \mathbf{S}_2)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{2}{15}$ | 360 | A | C | $\mathbb{Z}_3$ | | | 378 000, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 360 | 360 | | $\mathbb{Z}^{16}$ | | 16 |

TABLE 27. Neat subgroups of $\mathcal{T}(3, \mathbf{S}_2)$ with core index $\leq 2,000$

$\mathcal{T}(4, \mathbf{S}_2)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{3}$ | 480 | NA | NC | $\{1\}$ | $R_1, R_2$ | $\mathrm{br}_4(R_1, R_2)$ | 42 456 960, $PGL(3,9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $155,520 = 324 \cdot 480$ | 42456960 | | ? | | ? |

TABLE 28. Neat subgroups of $\mathcal{T}(4, \mathbf{S}_2)$

$\mathcal{T}(5, \mathbf{S}_2)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{133}{300}$ | 600 | NA | C | $\mathbb{Z}_5$ | | | 23 619 600 |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $3600 = 2 \cdot 1800$ | 3600 | | $\mathbb{Z}_3^8$ | | 0 |

TABLE 29. Neat subgroups of $\mathcal{T}(5, \mathbf{S}_2)$

$\mathcal{T}(3, \mathbf{E}_2)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{4}$ | 72 | A | NC | $\mathbb{Z}_3^2$ | $c_1 = 2, c_2 = (\bar{3}\bar{2}\bar{1})^2\bar{2}\bar{1}2$ | $c_1^3, c_2^6(c_1c_2)^{-4}, (c_1c_2)^2$ central | 378 000, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 72 | 72 | | $\mathbb{Z}^8$ | $(-1)^{12}$ | 8 |
| 72 | 648 | | $\mathbb{Z}_3^2 \oplus \mathbb{Z}^4$ | $(-1)^3, (-3)^3$ | 4 |
| 72 | 648 | | $\mathbb{Z}_3^3 \oplus \mathbb{Z}^2$ | $(-3)^4$ | 2 |
| 72 | 648 | | $\mathbb{Z}_3^4 \oplus \mathbb{Z}^2$ | $(-1)^3, (-3)^3$ | 2 |

TABLE 30. Neat subgroups of $\mathcal{T}(3, \mathbf{E}_2)$ of core index $\leq 1000$

[7] P. Deligne and G. D. Mostow. Monodromy of hypergeometric functions and non-lattice integral monodromy. *Publ. Math., Inst. Hautes Étud. Sci.*, 63:5–89, 1986.
[8] Pierre Deligne and George Daniel Mostow. *Commensurabilities among lattices in $PU(1,n)$*, volume 132 of *Ann. Math. Stud.* Princeton, NJ: Princeton University Press, 1993.

## $\mathcal{T}(4, \mathbf{E}_2)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{17}{32}$ | 96 | NA | NC | $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ | $R_1, R_2$<br>$R_1, R_3$<br>$R_1, R_2R_3R_2^{-1}$ | $\mathrm{br}_4(R_1, R_2)$<br>$\mathrm{br}_4(R_1, R_3)$<br>$\mathrm{br}_4(R_1, R_2R_3R_2^{-1})$ | 50 778 000 000, $PSL(3, 25)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 96 | 192 | | $\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}^4$ | $(-2)^{18}$ | 4 |
| 96 | 768 | | $\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}^4$ | $(-2)^{10}, (-4)^4$ | 4 |
| 96 | 768 | | $\mathbb{Z}_2^6 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_8$ | $(-2)^{10}, (-4)^4$ | 0 |

TABLE 31. Neat subgroups of $\mathcal{T}(4, \mathbf{E}_2)$ of core index $\leq 1000$

## $\mathcal{T}(6, \mathbf{E}_2)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{3}{4}$ | 36 | A | NC | $\mathbb{Z}_6^2$ | $R_2, R_3$ | $\mathrm{br}_3(R_2, R_3)$ | 378 000, $PGU(3, 5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 72 | 72 | | $\mathbb{Z}_2^3 \oplus \mathbb{Z}^8$ | $(-1)^{12}$ | 8 |

TABLE 32. Neat subgroups of $\mathcal{T}(6, \mathbf{E}_2)$ of core index $\leq 200$

## $\mathcal{T}(2, \mathbf{H}_1)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{49}$ | 1176 | A | C | $\mathbb{Z}_2$ | | | ? |

TABLE 33. Invariants for $\mathcal{T}(2, \mathbf{H}_1)$

## $\mathcal{T}(2, \mathbf{H}_2)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{100}$ | 300 | A | C | $\{1\}$ | | | 42 573 600, $PGU(3, 9)$ |

TABLE 34. Invariants for $\mathcal{T}(2, \mathbf{H}_2)$

## $\mathcal{T}(3, \mathbf{H}_2)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 7) |
|---|---|---|---|---|---|---|---|
| $\frac{26}{75}$ | 1800 | NA | C | $\mathbb{Z}_3$ | | | ? |

TABLE 35. Invariants for $\mathcal{T}(3, \mathbf{H}_2)$

## $\mathcal{T}(5, \mathbf{H}_2)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{73}{100}$ | 600 | A | C | $\mathbb{Z}_5$ | | | 23 619 600 |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 600 | 600 | | $\mathbb{Z}^{20}$ | | 20 |

TABLE 36. Neat subgroups of $\mathcal{T}(5, \mathbf{H}_2)$

## $\Gamma(5, 7/10)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{200}$ | 600 | A | C | $\{1\}$ | | | 42 573 600, $PGU(3, 9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $3000 = 5 \cdot 600$ | 15 000 | | $\mathbb{Z}_5 \oplus \mathbb{Z}^4$ | | 4 |

TABLE 37. Neat subgroups of $\Gamma(5, 7/10)$ of core index $\leq 40,000$

## $\Gamma(6, 2/3)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{72}$ | 72 | A | NC | $\mathbb{Z}_6$ | $R_1, R_2$ | $\mathrm{br}_3(R_1, R_2)$ | 378 000, $PGU(3, 5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 72 | 1944 | | $\mathbb{Z}_3 \oplus \mathbb{Z}^2$ | $-1, -3$ | 2 |
| 72 | 1944 | | $\mathbb{Z}^4$ | $(-1)^4$ | 4 |

TABLE 38. Neat subgroups of $\Gamma(6, 2/3)$ of core index $\leq 2000$

## $\Gamma(7, 9/14)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{49}$ | 588 | A | C | $\{1\}$ | | | ? |

TABLE 39. Invariants for $\Gamma(7, 9/14)$

## $\Gamma(8, 5/8)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{3}{128}$ | 384 | A | C | $\mathbb{Z}_4$ | | | 42 456 960, $PGL(3, 9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $466560 = 1215 \cdot 384$ | 42456960 | | ? | | ? |

TABLE 40. Neat subgroups of $\Gamma(8, 5/8)$

$\Gamma(9, 11/18)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{2}{81}$ | 324 | NA | C | $\mathbb{Z}_3$ | | | ? |

TABLE 41. Invariants for $\Gamma(9, 11/18)$

$\Gamma(10, 3/5)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{40}$ | 600 | A | C | $\mathbb{Z}_2$ | | | 42 573 600, $PGU(3,9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $583200 = 972 \cdot 600$ | 42573600 | | ? | | ? |

TABLE 42. Neat subgroups of $\Gamma(10, 3/5)$

$\Gamma(12, 7/12)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{7}{288}$ | 288 | A | C | $\mathbb{Z}_{12}$ | | | 152 334 000 000 |

TABLE 43. Invariants for $\Gamma(12, 7/12)$

$\Gamma(18, 5/9)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{5}{9}$ | 648 | A | C | $\mathbb{Z}_6$ | | | ? |

TABLE 44. Invariants for $\Gamma(18, 5/9)$

$\Gamma(4, 5/12)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{72}$ | 288 | A | C | $\mathbb{Z}_6$ | | | 152 334 000 000 |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 864 | 18144 | $PGU(3,3) \times \mathbb{Z}_3$ | $\mathbb{Z}_2^3 \oplus \mathbb{Z}^2$ | | 2 |

TABLE 45. Neat subgroups of $\Gamma(4, 5/12)$ of core index $\leq 40,000$

$\Gamma(5, 11/30)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 7) |
|---|---|---|---|---|---|---|---|
| $\frac{8}{225}$ | 1800 | A | C | $\mathbb{Z}_3$ | | | ? |

TABLE 46. Invariants for $\Gamma(5, 11/30)$

## $\Gamma(6, 1/3)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{18}$ | 18 | A | NC | $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ | $R_1, R_2$<br>$c_1 = R_1, c_2 = J\bar{2}\bar{1}$ | $\mathrm{br}_3(R_1, R_2)$<br>$c_1^6, c_2^3, \mathrm{br}_4(c_1, c_2)$ | 378 000, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 18 | 18 | | $\mathbb{Z}^4$ | $(-1)^4$ | 4 |
| 18 | 54 | | $\mathbb{Z}_3 \oplus \mathbb{Z}^2$ | $-1, -3$ | 2 |
| 18 | 162 | | $\mathbb{Z}^2$ | $-1, -3$ | 2 |

Table 47. Neat subgroups of $\Gamma(6, 1/3)$ of core index $\leq 1,000$

## $\Gamma(7, 13/42)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{61}{882}$ | 1764 | NA | C | $\mathbb{Z}_3$ | | | ? |

Table 48. Invariants for $\Gamma(7, 13/42)$

## $\Gamma(8, 7/24)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{11}{144}$ | 576 | NA | C | $\mathbb{Z}_6$ | | | ? |

Table 49. Invariants for $\Gamma(8, 7/24)$

## $\Gamma(9, 5/18)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{13}{162}$ | 324 | A | C | $\mathbb{Z}_3^2$ | | | ? |

Table 50. Invariants for $\Gamma(9, 5/18)$

## $\Gamma(10, 4/15)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 7) |
|---|---|---|---|---|---|---|---|
| $\frac{37}{450}$ | 450 | NA | C | $\mathbb{Z}_6$ | | | ? |

Table 51. Invariants for $\Gamma(10, 4/15)$

## $\Gamma(12, 1/4)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{12}$ | 144 | A | C | $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ | | | 152 334 000 000 |

Table 52. Invariants for $\Gamma(12, 1/4)$

$\Gamma(18, 2/9)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{13}{162}$ | 162 | A | C | $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ | | | |

TABLE 53. Invariants for $\Gamma(18, 2/9)$

$\Gamma(3, 1/3)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{288}$ | 288 | A | C | $\mathbb{Z}_3$ | | | 152 334 000 000 |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 864 | 18,144 | $\mathbb{Z}_3 \times PSU(3,3)$ | $\mathbb{Z}^2$ | | 2 |

TABLE 54. Neat subgroups of $\Gamma(3, 1/3)$ of core index $\leq 30,000$

$\Gamma(4, 1/4)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{32}$ | 96 | A | NC | $\mathbb{Z}_4$ | $c_1 = 1, c_2 = J\bar{2}\bar{1}$ | $c_1^4, c_2^4, \mathrm{br}_4(c_1, c_2)$ | 6 048 |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 96 | 96 | | $\mathbb{Z}_2^2 \oplus \mathbb{Z}^4$ | $(-2)^6$ | 4 |
| 96 | 384 | | $\mathbb{Z}_2 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}^2$ | $(-2)^2, (-4)^2$ | 2 |

TABLE 55. Neat subgroups of $\Gamma(4, 1/4)$ of core index $\leq 1,000$

$\Gamma(5, 1/5)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{23}{400}$ | 1200 | NA | C | $\{1\}$ | | | 1 852 734 273 062 400 |

TABLE 56. Invariants for $\Gamma(5, 1/5)$


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## $\Gamma(6, 1/6)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{11}{144}$ | 144 | NA | NC | $\mathbb{Z}_6$ | $R_1, R_2$ | $\mathrm{br}_3(R_1, R_2)$ | 152 334 000 000 |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $864 = 6 \cdot 144$ | 20736 | | $\mathbb{Z}_3 \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}^2$ | $-1, -2, (-3)^5, (-6)^5$ | 2 |

TABLE 57. Neat subgroup of $\Gamma(6, 1/6)$ obtained from reduction mod 6

## $\Gamma(8, 1/8)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{3}{32}$ | 192 | A | C | $\mathbb{Z}_8$ | | | 42 456 960, $PGL(3, 9)$ |

TABLE 58. Invariants for $\Gamma(8, 1/8)$

## $\Gamma(12, 1/12)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{7}{72}$ | 144 | A | C | $\mathbb{Z}_{12}$ | | | 152 334 000 000 |

TABLE 59. Invariants for $\Gamma(12, 1/12)$

## $\Gamma(3, 7/30)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 7) |
|---|---|---|---|---|---|---|---|
| $\frac{2}{225}$ | 1800 | A | C | $\mathbb{Z}_3$ | | | ? |

TABLE 60. Invariants for $\Gamma(3, 7/30)$

## $\Gamma(4, 3/20)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{33}{800}$ | 2400 | NA | C | $\mathbb{Z}_2$ | | | 1 852 734 273 062 400 |

TABLE 61. Invariants for $\Gamma(4, 3/20)$

## $\Gamma(5, 1/10)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{13}{200}$ | 600 | A | C | $\mathbb{Z}_5$ | | | 42 573 600, $PGU(3, 9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 600 | 600 | | $\mathbb{Z}^8$ | | 8 |
| 600 | 15 000 | | $\mathbb{Z}_5^2 \oplus \mathbb{Z}^4$ | | 4 |
| 600 | 15 000 | | $\mathbb{Z}_5^5$ | | 0 |

TABLE 62. Neat subgroups of $\Gamma(5, 1/10)$ of core index $\leq 30,000$

## $\Gamma(10,0)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{10}$ | 150 | A | C | $\mathbb{Z}_{10}$ | | | 42 573 600, $PGU(3,9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 150 | 600 | | $\mathbb{Z}_2^4 \oplus \mathbb{Z}^4$ | | 4 |
| 150 | 750 | | $\mathbb{Z}_5^2 \oplus \mathbb{Z}^4$ | | 4 |
| 150 | 1800 | | $\mathbb{Z}_5^2 \oplus \mathbb{Z}^4$ | | 4 |

TABLE 63. Neat subgroups of $\Gamma(10,0)$ of core index $\leq 2,000$

## $\Gamma(3,1/6)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{72}$ | 72 | A | NC | $\mathbb{Z}_3^2$ | $c_1 = 1, c_2 = J\bar{2}\bar{1}$ | $c_1^3, c_2^6, \mathrm{br}_4(c_1,c_2)$ | 378 000, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 72 | 72 | | $\mathbb{Z}^4$ | $(-1)^4$ | 4 |
| 72 | 648 | | $\mathbb{Z}_3 \oplus \mathbb{Z}^2$ | $-1, -3$ | 2 |

TABLE 64. Neat subgroups of $\Gamma(3,1/6)$ of core index $\leq 2,000$

## $\Gamma(4,1/12)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{13}{288}$ | 288 | NA | C | $\mathbb{Z}_{12}$ | | | 152 334 000 000 |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 864 | 864 | | $\mathbb{Z}_2^6 \oplus \mathbb{Z}^4$ | | 4 |
| 864 | 3456 | | $\mathbb{Z}_2^8 \oplus \mathbb{Z}^2$ | | 2 |

TABLE 65. Neat subgroups of $\Gamma(4,1/12)$ of core index $\leq 10,000$

## $\Gamma(6,0)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{12}$ | 36 | A | NC | $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ | $R_1, R_2$ | $\mathrm{br}_3(R_1,R_2)$ | 378 000, $PGU(3,5)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| 72 | 216 | | $\mathbb{Z}_3 \oplus \mathbb{Z}^4$ | $(-1)^3, (-3)^3$ | 4 |
| 72 | 216 | | $\mathbb{Z}_3^2 \oplus \mathbb{Z}^2$ | $(-3)^4$ | 2 |
| 72 | 648 | | $\mathbb{Z}_3 \oplus \mathbb{Z}^2$ | $-3, -9$ | 2 |
| 72 | 864 | | $\mathbb{Z}_3 \oplus \mathbb{Z}^2$ | $-12$ | 2 |

TABLE 66. Neat subgroups of $\Gamma(6,0)$ of core index $\leq 1,000$

$\Gamma(3, 5/42)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{61}{3528}$ | 3528 | NA | C | $\mathbb{Z}_3$ | | | ? |

TABLE 67. Invariants for $\Gamma(3, 5/42)$

$\Gamma(3, 1/12)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{11}{576}$ | 576 | NA | C | $\mathbb{Z}_3$ | | | ? |

TABLE 68. Invariants for $\Gamma(3, 1/12)$

$\Gamma(4, 0)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 3) |
|---|---|---|---|---|---|---|---|
| $\frac{3}{64}$ | 192 | A | C | $\mathbb{Z}_4$ | | | 42 456 960, $PGL(3, 9)$ |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $16512 = 86 \cdot 192$ | 5663616 ($PGU(3, 7)$) | | ? | | ? |

TABLE 69. Neat subgroups of $\Gamma(4, 0)$

$\Gamma(3, 1/18)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{13}{648}$ | 648 | A | C | $\mathbb{Z}_3^2$ | | | ? |

TABLE 70. Invariants for $\Gamma(3, 1/18)$

$\Gamma(3, 1/30)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 7) |
|---|---|---|---|---|---|---|---|
| $\frac{37}{1800}$ | 1,800 | NA | C | $\mathbb{Z}_3$ | | | ? |

TABLE 71. Invariants for $\Gamma(3, 1/30)$

$\Gamma(3, 0)$

| $\chi$ | LCM | A? | C? | Ab | Cusps | Cusp relations | Congr. image (mod 5) |
|---|---|---|---|---|---|---|---|
| $\frac{1}{48}$ | 144 | A | C | $\mathbb{Z}_3^2$ | | | ? |

| Index | Core Index | Quotient | Ab | Self-int | $b_1$ |
|---|---|---|---|---|---|
| $2016 = 14 \cdot 144$ | 54,432 | $PGU(3, 3) \times \mathbb{Z}_3^2$ | $\mathbb{Z}_3^4 \oplus \mathbb{Z}^4$ | | 4 |
| 2016 | 54,432 | $PGU(3, 3) \times \mathbb{Z}_3^2$ | $\mathbb{Z}_3^6 \oplus \mathbb{Z}^2$ | | 2 |
| 2016 | 54,432 | $PGU(3, 3) \times \mathbb{Z}_3^2$ | $\mathbb{Z}_3^8$ | | 0 |

TABLE 72. Neat subgroups of $\Gamma(3, 0)$

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