Positivity in Kähler geometry

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Chapitre 1

Introduction (Français)

L'objectif de cette thèse est d'étudier divers concepts de positivité en géométrie kählerienne. En particulier, pour une variété kählerienne compacte de dimension n, nous étudions la positivité des classes transcendantes de type $(1, 1)$ et $(n-1, n-1)$ – ces classes comprennent donc en particulier les classes de diviseurs et les classes de courbes. Les résultats principaux présentés dans ce mémoire sont basés sur les articles suivants que j'ai rédigés au cours de mes études de doctorat :

1. (avec Brian Lehmann) *Zariski decomposition of curves on algebraic varieties.* arXiv preprint 2015, arXiv : 1507.04316, soumis.
2. *A remark on the convergence of inverse σ_k-flow*, arXiv preprint 2015, arXiv : 1505.04999, Comptes Rendus Mathématique 354 (2016) 395-399.
3. *Characterizing volume via cone duality*, arXiv preprint 2015, arXiv : 1502.06450, soumis.
4. (avec Jixiang Fu) *Teissier's problem on proportionality of nef and big classes over a compact Kähler manifold*, arXiv preprint 2014, arXiv : 1410.4878, soumis.
5. *Movable intersections and bigness criterion*, arXiv preprint 2014, arXiv : 1405.1582.
6. *Weak transcendental holomorphic Morse inequalities on compact Kähler manifolds*, arXiv preprint 2013, arXiv : 1308.2878, Annales de l'Institut Fourier (Grenoble) 65 (2015) 1367-1379.
7. (avec Jixiang Fu) *Relations between Kähler cone and balanced cone of a Kähler manifold*, arXiv preprint 2012, arXiv : 1203.2978, Advances in Mathematics 263 (2014) 230-252.

Le mémoire de thèse tente de réorganiser les articles ci-dessus pour en rendre la lecture plus fluide. Indiquons-en brièvement le plan. Le Chapitre 3 porte sur le problème de proportionnalité de Teissier pour les classes nef transcendantes sur les variétés kählerienne compactes, et sur ses applications. Il repose sur les travaux [FX14a] et [FX14b]. Le Chapitre 4 traite de la conjecture de Demailly sur les inégalités de Morse transcendantes, et il fait la synthèse des travaux [Xia13], [Xia14] et [Xia15b]. Le Chapitre 5 étudie la caractérisation du volume via la dualité des cônes ; il est principalement basé sur le manuscrit [Xia15a]. Le Chapitre 6 développe la théorie de la décomposition de Zariski des courbes sur les variétés algébriques. Il est basé sur le travail [LX15].

Le présent Chapitre 1 est introductif et présente un résumé des principaux résultats de cette thèse ; voir le Chapitre 2 pour une introduction en Anglais. Pour les notions de base de la géométrie kählerienn et de la géométrie algébrique complexe, en particulier les définitions des divers concepts de positivité, nous avons préféré ne pas trop nous étendre dans cette introduction. Cependant, les définitions utiles seront données les sections suivantes auxquelles nous référons fréquemment. (Voir aussi les excellents livres [Dem12a], [Dem12b] ou [GH94].)

Le problème de proportionnalité de Teissier

Dans le chapitre 3, nous résolvons d'abord le problème de proportionnalité de Teissier pour les classes nef transcendantes sur une variété kählerienne compacte. Il s'agit de montrer l'égalité dans les inégalités

Khovanskii-Teissier a lieu pour un couple de classes nefs et grosses si et seulement si les deux classes sont proportionnelles. Ce résultat recouvre celui de Boucksom-Favre-Jonsson [BFJ09] pour le cas des diviseurs nefs et gros sur une variété algébrique complexe.

Théorème 1.0.1. *(= Théorème 3.1.1) Supposons que X soit une variété kählerienne compacte de dimension n. Soient $\alpha, \beta \in \overline{\mathcal{K}} \cap \mathcal{E}^\circ$ deux classes nefs et grosses. Notons $s_k := \alpha^k \cdot \beta^{n-k}$. Alors, les assertions suivantes sont équivalentes :*

1. $s_k^2 = s_{k-1} \cdot s_{k+1}$ *pour* $1 \leq k \leq n-1$;
2. $s_k^n = s_0^{n-k} \cdot s_n^k$ *pour* $0 \leq k \leq n$;
3. $s_{n-1}^n = s_0 \cdot s_n^{n-1}$;
4. $\mathrm{vol}(\alpha+\beta)^{1/n} = \mathrm{vol}(\alpha)^{1/n} + \mathrm{vol}(\beta)^{1/n}$;
5. α *et* β *sont proportionnelles ;*
6. α^{n-1} *et* β^{n-1} *sont proportionnelles.*

En conséquence, l'application $\gamma \mapsto \gamma^{n-1}$ est injective du cône nef et gros $\overline{\mathcal{K}} \cap \mathcal{E}^\circ$ vers le cône mobile $\overline{\mathcal{M}}$.

Comme application du théorème de proportionnalité de Teissier, nous discutons une question de géométrie non-kählerienne – à savoir l'étude du cône équilibré – sur les variétés kählerienne compactes. Nous considérons l'application naturelle du cône de Kähler vers le cône équilibré :

$$\overline{\mathbf{b}} : \overline{\mathcal{K}} \to \overline{\mathcal{B}}, \quad \alpha \mapsto \alpha^{n-1}.$$

Comme corollaire immédiat du théorème de proportionnalité ci-dessus, nous déduisons l'injectivité de la restriction de $\overline{\mathbf{b}}$ sur $\overline{\mathcal{K}} \cap \mathcal{E}^\circ$ (voir la Section 3.3.2). Nous étudions également la surjectivité, en donnant quelques exemples intéressants où l'application envoie la frontière du cône kählerien dans l'intérieur du cône équilibré (voir la Section 3.3.3). En particulier, pour les classes nef rationnelles sur les variétés de Calabi-Yau projectives, nous donnons une critère pour qu'une classe de la frontière soit envoyée dans l'intérieur du cône équilibré - ce résultat est le point de départ des résultats obtenus dans le Chapitre 6.

Théorème 1.0.2. *(= Théorème 3.3.3) Soit X une variété de Calabi-Yau projective de dimension n.*

1. *Si $\alpha \in \partial\mathcal{K}$ est une classe dans la frontière de $\mathcal{K}$, alors $\overline{\mathbf{b}}(\alpha) \in \mathcal{B}$ implique que α est une classe grosse.*
2. *Si $\alpha \in \partial\mathcal{K}$ est une classe grosse et rationnelle, alors $\overline{\mathbf{b}}(\alpha) \in \mathcal{B}$ si et seulement si F_α est une petite contraction, ou de manière équivalente, si l'ensemble exceptionnel $\mathrm{Exc}(F_\alpha)$ de la contraction F_α induite par la classe α est de* $\mathrm{codim} \geq 2$.

Par des arguments similaires à ceux de notre preuve du théorème de proportionnalité de Teissier, on peut donner une critère analytique pour qu'une classe nef soit une classe kählerienne. Comme ce résultat est intéressant en lui-même, nous présenterons quelques détails de sa preuve dans ce chapitre.

Théorème 1.0.3. *(= Théorème 3.3.17) Soit X une variété kählerienne compacte de dimension n et soit η une forme de volume lisse sur X satisfaisant* $\mathrm{vol}(\eta) = 1$. *Supposons que α soit une classe nef et que α^{n-1} soit une classe équilibrée, i.e. α^{n-1} est la classe de certaine $(n-1, n-1)$-forme lisse strictement positive. S'il existe une métrique équilibrée $\tilde{\omega}$ dans α^{n-1} telle que*

$$\tilde{\omega}^n \geq \mathrm{vol}(\alpha)\eta$$

ponctuellement sur X, alors α est une classe kählerienne.

Le résultat ci-dessus est lié à la résolubilité des équations de Monge-Ampère de "type forme" pour une classe intéressante de métriques équilibrées – à savoir celles données par les puissances α^{n-1} des formes $\alpha \in \partial\mathcal{K}$.

Inégalités de Morse transcendantes

Dans le Chapitre 4, nous étudions la conjecture de Demailly sur les inégalités de Morse transcendantes.

Conjecture 1.0.4. *(voir [BDPP13, Conjecture 10.1]) Soit X une variété complexe compacte de dimension n.*

1. *Soit θ une $(1,1)$-forme d-fermé réelle représentant la classe α. Soit $X(\theta, \leq 1)$ le lieu au-dessus duquel θ a au plus une valeur propre négative. Si $\int_{X(\theta,\leq 1)} \theta^n > 0$, alors la classe de Bott-Chern α contient un courant kählerien et*
$$\mathrm{vol}(\alpha) \geq \int_{X(\theta,\leq 1)} \theta^n.$$
2. *Soient α et β deux $(1,1)$-classes nef sur X satisfaisant $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$. Alors la classe de Bott-Chern $\alpha - \beta$ contient un courant kählerien et*
$$\mathrm{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

Inspiré par la méthode de [Chi13], nous avons pu d'abord prouver une version faible de la conjecture de Demailly sur les inégalités de Morse transcendantes sur des variétés kähleriennes compactes.

Théorème 1.0.5. *(= Théorème 4.1.4) Soit X une variété complexe compacte de dimension n, munie d'une métrique hermitienne ω satisfaisant $\bar{\partial}\partial\omega^k = 0$ pour $k = 1, 2, ..., n-1$. Supposons que α, β soient deux classes nefs sur X satisfaisant*
$$\alpha^n - 4n\alpha^{n-1} \cdot \beta > 0.$$
Alors il existe un courant kählerien dans la classe de Bott-Chern $\alpha - \beta$.

On notera que ce résultat couvre le cas kählerien et améliore un résultat de [BDPP13]. En outre, un point remarquable est que les classes de cohomologie α, β peuvent être transcendantes.

Récemment, en prolongeant la méthode de [Xia13, Chi13] et en utilisant de nouvelles estimations des équations de Monge-Ampère, D. Popovici [Pop14] a prouvé que la constante $4n$ de notre Théorème 4.1.4 peut être améliorée en la valeur naturelle et optimale n. On obtient ainsi un critère de type Morse pour la "grosseur" de la différence de deux classes nefs transcendantes. Il est naturel de se demander si le critère :
$$\alpha^n - n\alpha^{n-1} \cdot \beta > 0 \Rightarrow \mathrm{vol}(\alpha - \beta) > 0$$
valable pour les classes nefs, peut être généralisé à des classes pseudo-effectives. Pour une telle généralisation, nous avons besoin des produits d'intersection mobiles (désignés par $\langle - \rangle$) des classes pseudo-effectives (voir par exemple [Bou02a, BDPP13]). Le problème peut alors être énoncé comme suit :

Question 1.0.6. Soit X une variété compacte kählerienne de dimension n, et soient $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ soit deux classes pseudo-effectives. Est-ce que la condition $\mathrm{vol}(\alpha) - n\langle \alpha^{n-1} \rangle \cdot \beta > 0$ implique qu'il existe un courant kählerien dans la classe $\alpha - \beta$?

Malheureusement, un exemple très simple donné dans [Tra95] montre que la généralisation ci-dessus n'est pas toujours vraie.

Exemple 1.0.7. (voir [Tra95, Example 3.8]) Soit $\pi : X \to \mathbb{P}^2$ l'éclatement de d'un point p dans $\mathbb{P}^2$. Soit $R = \pi^* H$, où H est le fibré tautologique des sections hyperplanes sur $\mathbb{P}^2$. Soit $E = \pi^{-1}(p)$ le diviseur exceptionnel. Alors pour tout entier positif k, l'espace des sections holomorphes globales de $k(R - 2E)$ est l'espace des polynômes homogènes en trois variables de degré au plus k qui s'annulent de l'ordre $2k$ à p. Donc $k(R - 2E)$ n'a pas de sections holomorphes globales, ce qui implique $R - 2E$ ne peut pas être gros. Cependant, nous avons $R^2 - R \cdot 2E > 0$, puisque $R^2 = 1$ et $R \cdot E = 0$.

Cependant, en utilisant quelques propriétés de base des intersections mobiles, nous généralisons le résultat principal de [Pop14] aux classes pseudo-effectives. Nous montrons que ce résultat est toujour vrai si β est mobile. Rappelons ici que "β mobile" signifie que la partie négative de β dans la décomposition divisorielle de Zariski (voir [Bou04]) est nulle.

Théorème 1.0.8. *(= Théorème 4.1.10) Soit X une variété kählerienne compacte de dimension n, et soient $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ deux classes pseudo-effectives, avec β mobile. Alors*

$$\mathrm{vol}(\alpha) - n\langle\alpha^{n-1}\rangle \cdot \beta > 0$$

implique qu'il existe un courant kählerien dans la classe $\alpha - \beta$.

Comme application, nous donnons un critère de type Morse pour la "grosseur" des $(n-1, n-1)$-classes mobiles. Ce résultat sera appliqué pour étudier la positivité des classes de courbes dans le Chapitre 6.

Théorème 1.0.9. *(= Théorème 4.1.15) Soit X une variété kählerienne compacte de dimension n, et soient $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ deux classes pseudo-effectives. Alors la condition*

$$\mathrm{vol}(\alpha) - n\alpha \cdot \langle\beta^{n-1}\rangle > 0$$

implique qu'il existe un $(n-1, n-1)$-courant strictement positif dans la classe $\langle\alpha^{n-1}\rangle - \langle\beta^{n-1}\rangle$.

A la fin du Chapitre 4, nous appliquons nos résultats au problème de la caractérisation numérique de la convergence des σ_k-flots inverses, et donnons des résponses positives partielles à la conjecture de Lejmi et Székelyhidi [LS15].

Adoptant un point de vue qui relie l'existence de métriques kähleriennes canoniques à des conditions de stabilité algébro-géométrique, Lejmi et Székelyhidi [LS15] ont proposé un critère numérique pour que les σ_k-flots inverses convergent. Notre objectif est étudier la positivité des classes de cohomologies apparaissant dans cet énoncé conjectural. Nous généralisons celui-ci en affaiblissant la condition numérique sur X.

Conjecture 1.0.10. *(voir [LS15, Conjecture 18]) Soit X une variété kählerienne compacte de dimension n, et soient ω, α deux métriques kähleriennes sur X satisfaisant la condition*

$$\int_X \omega^n - \frac{n!}{k!(n-k)!}\omega^{n-k} \wedge \alpha^k \geq 0. \tag{1.1}$$

Alors il existe une métrique kählerienne $\omega' \in \{\omega\}$ telle que

$$\omega'^{n-1} - \frac{(n-1)!}{k!(n-k-1)!}\omega'^{n-k-1} \wedge \alpha^k > 0 \tag{1.2}$$

(condition de positivité ponctuelle comme $(n-1, n-1)$-forme lisse), si et seulement si

$$\int_V \omega^p - \frac{p!}{k!(p-k)!}\omega^{p-k} \wedge \alpha^k > 0 \tag{1.3}$$

pour chaque sous-variété irréductible de dimension p avec $k \leq p \leq n-1$.

Nous nous somme concentrés sur les cas où $k = 1$ et $k = n-1$. Dans ce cas, nous avons obtenu les résultats suivants :

Théorème 1.0.11. *(= Théorème 4.4.2) Soit X une variété kählerienne compacte de dimension n, et soient ω, α deux métriques kähleriennes sur X qui vérifient les conditions numériques de la conjecture ci-dessus pour $k = 1$. Alors $\{\omega - \alpha\}$ est une classe kählerienne.*

Théorème 1.0.12. *(= Théorème 4.4.3) Soit X une variété kählerienne compacte de dimension n, et soient ω, α deux métriques kähleriennes sur X satisfaisant les conditions numériques de la conjecture ci-dessus pour $k = n-1$. Alors la classe $\{\omega^{n-1} - \alpha^{n-1}\}$ appartient à l'adhérence du cône de Gauduchon, i.e. elle a un nombre d'intersection positif avec chaque $(1,1)$-classe pseudo-effective.*

Compte tenu des résultats ci-dessus et de l'énoncé conjectural 1.0.10, nous proposons la question suivante sur la positivité des (k,k)-classes, en lien étroit avec les singularités des (k,k)-courants positifs.

Question 1.0.13. Soit X une variété kählerienne compacte (ou même une variété complexe compacte générale) de dimension n. Soit $\Omega \in H^{k,k}(X, \mathbb{R})$ une (k,k)-classe grosse, c'est-à-dire une classe pouvant être représentée par un (k,k)-courant strictement positif sur X. Supposons que la classe de restriction $\Omega_{|V}$ soit également grosse sur toute sous-variété irréductible V telle que $k \leq \dim V \leq n-1$. Alors est-ce que Ω contient une (k,k)-forme lisse strictement positive dans sa classe de Bott-Chern ? Ou est-ce que la classe de Bott-Chern de Ω contient au moins un (k,k)-courant strictement positif avec singularités analytiques de codimension au moins $n-k+1$?

Caractérisation du volume via la dualité des cônes

Le Chapitre 5 est consacré à l'étude de la fonction volume via la propriété de dualité des cônes - qui est la première étape du Chapitre 6. Tout d'abord, pour les diviseurs sur des variétés projectives lisses, nous montrons que le volume peut être caractérisé par la dualité entre le cône des diviseurs pseudo-effectifs et le cône des courbes mobiles. Rappelons que le volume d'un diviseur sur une variété projective est un nombre qui mesure la positivité des diviseurs. Soit X une variété projective lisse de dimension n, et soit D un diviseur sur X. Par définition, le volume de D est défini comme étant

$$\mathrm{vol}(D) := \limsup_{m\to\infty} \frac{h^0(X, mD)}{m^n/n!}.$$

Grâce au travail remarquable de Boucksom-Demailly-Paun-Peternell (voir [BDPP13]), nous savons qu'il existe une dualité entre le cône des diviseurs pseudo-effectifs et le cône convexe fermé engendré par les courbes mobiles :

$$\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X).$$

En utilisant cette dualité de cônes et un invariant approprié des classes de courbes mobiles, nous donnons une nouvelle caractérisation du volume.

Définition 1.0.14. (voir Définition 5.2.6) Soit X une variété projective lisse de dimension n, et soit γ une classe de courbe mobile. On définit un l'invariant $\mathfrak{M}(\gamma)$ par

$$\mathfrak{M}(\gamma) := \inf_{\beta \in \overline{\mathrm{Eff}}(X)^\circ} \left(\frac{\beta \cdot \gamma}{\mathrm{vol}(\beta)^{1/n}} \right)^{\frac{n}{n-1}}.$$

Théorème 1.0.15. *(= Théorème 5.1.1) Soit X une variété projective lisse de dimension n et soit $\alpha \in \overline{\mathrm{Eff}}^1(X)$ une classe de diviseurs pseudo-effective. Alors, le volume de α peut être caractérisée comme suit :*

$$\mathrm{vol}(\alpha) = \inf_{\gamma \in \mathrm{Mov}_1(X)^\circ} \left(\frac{\alpha \cdot \gamma}{\mathfrak{M}(\gamma)^{n-1/n}} \right)^n$$

En outre, dans cette écriture, on peut remplacer le cône mobile $\mathrm{Mov}_1(X)$ par le cône de Gauduchon $\mathcal{G}$ (ou le cône équilibré $\mathcal{B}$) qui est engendré par les métriques hermitiennes spéciales.

Remarque 1.0.16. L'invariant $\mathfrak{M}$ peut être défini pour les classes mobiles transcendantes sur des variétés kähleriennes. De plus, on peut généraliser cette formule de volume pour des classes transcendantes si on suppose la conjecture de Demailly sur les inégalités Morse holomorphes transcendantes.

Compte tenu de la formule précédente relative aux classes de diviseurs, et en utilisant les dualités de cônes, nous introduisons un volume fonctionnel pour les 1-cycles pseudo-effectifs – à savoir les $(n-1,n-1)$-classes représentées par les courants positifs – sur les variétés compactes kähleriennes.

Pour une variété projective lisse X, par le critère de Kleiman, on a la dualité de cônes

$$\mathrm{Nef}^1(X)^* = \overline{\mathrm{Eff}}_1(X)$$

où Nef^1 est le cône engendré par les classes de diviseurs nef et $\overline{\mathrm{Eff}}_1$ est le cône engendré par les classes de courbes pseudo-effectives. Pour une variété compacte kählerienne, par la caractérisation numérique de Demailly-Paun du cône de Kähler (voir [DP04]), on a la dualité de cônes

$$\mathcal{K}^* = \mathcal{N}$$

où $\mathcal{K}$ est le cône de Kähler engendré par les classes kähleriennes et $\mathcal{N}$ est le cône engendré par les $(n-1,n-1)$-courants positifs fermés.

Définition 1.0.17. 1. Soit X une variété projective lisse de dimension n, et soit $\gamma \in \overline{\mathrm{Eff}}_1(X)$ une classe de courbe pseudo-effective. Alors, le volume de γ est défini par

$$\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) = \inf_{\beta \in \mathrm{Nef}^1(X)^\circ} \left(\frac{\beta \cdot \gamma}{\mathrm{vol}(\beta)^{1/n}} \right)^{\frac{n}{n-1}}.$$

2. Soit X une variété compacte kählerienne de dimension n, et soit $\gamma \in H^{n-1,n-1}_{BC}(X,\mathbb{R})$ une $(n-1,n-1)$-classe pseudo-effective. Alors, le volume de γ est défini par

$$\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \inf_{\beta \in \mathcal{K}(X)} \left(\frac{\beta \cdot \gamma}{\mathrm{vol}(\beta)^{1/n}} \right)^{\frac{n}{n-1}}.$$

Il est bien connu que le volume $\mathrm{vol}(\bullet)$ pour une classe de diviseur ne dépend que de la classe numérique, que $\mathrm{vol}^{1/n}$ est homogène de degré un, concave sur le cône pseudo-effectif et s'étend en une fonction continue sur l'espace vectoriel réel de Néron-Severi qui est strictement positive exactement sur l'ensemble des classes grosses. Nous montrons que notre fonction volume $\widehat{\mathrm{vol}}$ jouit de propriétés similaires. Par simplicité, nous énonçons seulement le résultat pour $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$.

Théorème 1.0.18. *(= Théorème 5.1.4) Soit X une variété projective lisse de dimension n. Alors*

1. $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}^{n-1/n}$ *est une fonction concave homogène de degré un.*
2. $\gamma \in \overline{\mathrm{Eff}}_1(X)^\circ$ *si et seulement si* $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) > 0$.
3. $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ *peut s'étendre en une fonction continue sur tout l'espace vectoriel* $N_1(X,\mathbb{R})$ *en posant* $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}} = 0$ *à l'extérieur de* $\overline{\mathrm{Eff}}_1(X)$.

Pour les variétés projectives, la fonction $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ est étroitement liée à la mobilité fonctionnelle récemment introduite par Lehmann (voir [Leh13b]).

Définition 1.0.19. (voir [Leh13b, Definition 1.1]) Soit X une variété projective de dimension n et soit $\alpha \in N_k(X)$ soit une classe de cycle à coefficients entiers. La mobilité de α est définie comme étant

$$\mathrm{mob}(\alpha) := \limsup_{m\to\infty} \frac{\max \left\{ b \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{c} \text{Tous les } b \text{ points généraux sont contenus} \\ \text{dans un cycle effectif de classe } m\alpha \end{array} \right\}}{m^{\frac{n}{n-k}}/n!}.$$

Le numérateur sera appelé "coefficient de mobilité" et sera désigné par $\mathrm{mc}(m\alpha)$. La mobilité fonctionnelle pour les cycles a été suggérée dans [DELV11] comme un analogue de la fonction de volume des diviseurs. La motivation est que l'on peut interpréter le volume d'un diviseur D comme une mesure asymptotique du nombre de points généraux contenus dans les membres de $|mD|$ lorsque m tend vers l'infini. En particulier, nous pouvons définir la mobilité pour les classes numériques de courbes. Lehmann a prouvé que la mobilité fonctionnelle distingue aussi les points intérieurs et les points à la frontière, et s'étend en une fonction continue homogène sur l'espace $N_1(X)$. Ainsi, dans la situation des courbes, en tenant compte du Théorème 5.1.4, nous avons deux fonctionnelles possédant cette propriété. Il est donc intéressant de comparer mob et $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ sur $\overline{\mathrm{Eff}}_1$. Nous aimerions proposer la conjecture suivante.

Conjecture 1.0.20. *Soit X une variété projective lisse de dimension n, alors nous avons*

$$\mathrm{mob} = \widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}.$$

Dans cette direction, il est au moins clair qu'il existe deux constantes positives c_1, c_2 ne dépendant que de la dimension de la variété sous-jacente, de telle sorte que

$$c_1\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) \leq \mathrm{mob}(\gamma) \leq c_2\widehat{\mathrm{vol}}_{\overline{NE}}(\gamma)$$

pour toute $\gamma \in \overline{\mathrm{Eff}}_1(X)$. Dans le Chapitre 5, nous avons observé que la constante positive c_2 peut être obtenue en utilisant les estimations de Lehmann du coefficient de mobilité mc. Dans le Chapitre 6, qui décrit la théorie développée dans le travail avec Lehmann [LX15] (en plus d'autres résultats), nous obtiendrons une valeur de la constante c_1 et une meilleure borne pour c_2.

Théorème 1.0.21. *(voir Théorème 5.1.5) Soit X une variété projective lisse de dimension n, et soit $\overline{\mathrm{Eff}}_1(X)$ l'adhérence du cône engendré par les classes courbes effectives. Alors pour tout $\gamma \in \overline{\mathrm{Eff}}_1(X)$, nous avons*

$$\mathrm{mob}(\gamma) \leq n!2^{4n+1}\widehat{\mathrm{vol}}_{\overline{NE}}(\gamma).$$

Une approximation de type Fujita pour les classes de courbes a été donnée dans [FL13] en relation avec la mobilité fonctionnelle. Notre objectif présent est d'étudier certains résultats d'approximation de type Fujita pour les $(n-1, n-1)$-classes pseudo-effectives sur les variétés compactes kähleriennes, en relation cette fois avec la fonction volume "transcendante" $\widehat{\mathrm{vol}}_{\mathcal{N}}$. Après avoir rappelé la version analytique de la décomposition de Zariski dûe à Boucksom [Bou04,Bou02a] (pour l'approche algébrique, voir [Nak04]), nous étudions la décomposition de Zariski pour les $(n-1, n-1)$-classes pseudo-effectives dans le sens de Boucksom.

Dans la décomposition de Zariski divisorielle, la partie négative est un diviseur effectif de dimension de Kodaira zéro, et la classe de cette partie négative ne contient qu'un seul $(1,1)$-courant positif. Dans notre contexte, nous pouvons prouver que ce fait est valable aussi pour les 1-cycles gros. Si on compare ceci avec d'autres définitions antérieures de la décomposition de Zariski pour les 1-cycles (voir par exemple [FL13]), l'avantage de notre décomposition est que la partie négative est effective. En utilisant sa caractérisation du volume par la masse de Monge-Ampère, Boucksom a montré que la "projection de Zariski" sur la partie positive préserve la volume. Il était également naturel d'espérer que notre "projection de Zariski" ait la propriété de préserver $\widehat{\mathrm{vol}}_{\mathcal{N}}$: cela résulte en effet de la décomposition de Zariski pour les 1-cycles développée dans le Chapitre 6 (voir [LX15]), qui est plus étroitement liée à $\widehat{\mathrm{vol}}_{\mathcal{N}}$.

Théorème 1.0.22. *(= Théorème 5.1.6) Soit X une variété compacte kählerienne de dimension n et soit $\gamma \in \mathcal{N}^{\circ}$ un point intérieur. Soit $\gamma = Z(\gamma) + \{N(\gamma)\}$ la décomposition de Zariski dans le sens de Boucksom, alors :*

1. *$N(\gamma)$ est une courbe effective et c'est l'unique courant positif contenu dans la partie négative $\{N(\gamma)\}$.*
2. *En outre, on a $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma))$.*

La décomposition de Zariski pour les classes de courbes

Dans le Chapitre 6, nous introduisons une décomposition de Zariski pour les classes de courbes, et nous l'utilisons pour développer la théorie correspondante de la fonction volume pour les courbes, telle que définie dans le Chapitre 5. Plus généralement, nous développons une théorie formelle de la décomposition de Zariski par rapport aux fonctions concaves homogènes de degré $s > 1$ définies sur un cône. Pour les variétés toriques et les variétés hyperkählériennes, la décomposition de Zariski admet une interprétation géométrique intéressante. Grâce à cette décomposition, nous prouvons quelques résultats de positivité pour les classes de courbes, en particulier une inégalité de type Morse. Nous comparons le volume d'une classe de courbes avec sa mobilité, ce qui donne quelques résultats inattendus pour le comptage asymptotique des points. Enfin, nous donnons un certain nombre d'applications à la géométrie birationnelle, y compris un théorème de structure raffiné pour le cône mobile des courbes.

Dans ce chapitre, nous nous concentrons sur l'étude de la fonction $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ sur les variétés projectives ; c'est la raison pour laquelle nous allons la désigner simplement par $\widehat{\mathrm{vol}}$.

Dans [Zar62] Zariski a introduit un outil fondamental pour l'étude des séries linéaires sur une surface, maintenant connu sous le nom de "décomposition de Zariski". Au cours des 50 dernières années, la décomposition de Zariski et ses généralisations aux diviseurs de dimensions supérieures ont joué un rôle central dans la géométrie birationnelle. Nous introduisons une décomposition analogue pour les classes de courbes sur les variétés de dimension arbitraire. Notre décomposition est définie pour les classes grosses, qui sont les éléments de l'intérieur du cône $\overline{\mathrm{Eff}}_1(X)$. Tout au long de cette partie, nous travaillons sur $\mathbb{C}$, mais les principaux résultats resteraient vrais également sur un corps algébriquement clos ou dans la situation kählerienne (voir la section 6.1.5).

Définition 1.0.23. (voir Définition 6.1.1) Soit X une variété projective de dimension n et soit $\alpha \in N_1(X)$ une classe grosse de courbe. Alors une décomposition de Zariski pour α est une décomposition

$$\alpha = B^{n-1} + \gamma$$

où B est une classe de diviseur $\mathbb{R}$-Cartier grosse et nef, γ est pseudo-effectif, et $B \cdot \gamma = 0$. Nous appelons B^{n-1} la "partie positive" et γ la "partie négative" de cette décomposition.

Cette définition généralise directement la définition originale de Zariski, qui (pour les classes grosses) est donnée par des critères d'intersection similaires. Elle généralise aussi la σ-decomposition de [Nak04], et reflète la décomposition de Zariski de [FL13] dans le sens suivant. La caractéristique fondamentale d'une décomposition de Zariski est que la partie positive doit conserver toute la "positivité" de la classe d'origine. Dans notre contexte, nous allons mesurer la positivité d'une classe de courbe en utilisant la fonction $\widehat{\mathrm{vol}}$ définie dans [Xia15a] (voir le Chapitre 5).

En effet, la fonction $\widehat{\mathrm{vol}}$ est une sorte de "transformée polaire" de la fonction volume pour les diviseurs. Sa définition est motivée par l'observation que le volume d'un diviseur a une description en théorie de l'intersection similaire à celle de [Xia15a, Theorem 2.1]. [Xia15a] prouve que $\widehat{\mathrm{vol}}$ satisfait des nombreuses caractéristiques analytiques souhaitées du volume pour les diviseurs.

D'après [FL13, Proposition 5.3], nous savons que la σ-décomposition $L = P_\sigma(L) + N_\sigma(L)$ est la décomposition unique de L en une partie mobile et une partie pseudo-effective telle que $\mathrm{vol}(L) = \mathrm{vol}(P_\sigma(L))$. De la même manière, la décomposition de la définition 6.1.1 est compatible avec la fonction de volume pour les courbes :

Théorème 1.0.24. *(= Théorème 6.1.3) Soit X une variété projective de dimension n et soit $\alpha \in \overline{\mathrm{Eff}}_1(X)^\circ$ une classe grosse de courbe. Alors α admet une décomposition de Zariski unique $\alpha = B^{n-1}+\gamma$. De plus,*

$$\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(B^{n-1}) = \mathrm{vol}(B)$$

et B est la classe de diviseur grosse et nef unique avec cette propriété satisfaisant $B^{n-1} \preceq \alpha$. Toute classe de diviseur grosse et nef obtenant $\widehat{\mathrm{vol}}(\alpha)$ est proportionnelle à B.

Nous définissons le cône $\mathrm{CI}_1(X)$, le cône des intersections complètes, comme l'adhérence de l'ensemble des classes de la forme A^{n-1}, où A est un diviseur ample. La partie positive de la décomposition de Zariski prend ses valeurs dans $\mathrm{CI}_1(X)$.

Notre objectif est de développer la théorie de la décomposition de Zariski pour les courbes et la théorie de la fonction $\widehat{\mathrm{vol}}$. En raison de leur étroite relation, nous pensons qu'il est très fructueux de développer les deux théories en parallèle. En particulier, nous pourrons mettre à profit l'intuition originelle de Zariski, selon laquelle le comptage asymptotique des points produit des invariants numériques pour les courbes.

Exemple 1.0.25. Si X est une surface algébrique, la décomposition de Zariski fournie par le Théorème 6.1.3 coïncide (pour les classes grosses) avec la version numérique de la définition classique de [Zar62]. En effet, en utilisant la Proposition 6.5.14, on voit que la partie négative γ est représentée par une courbe N effective. La matrice d'auto-intersection de N doit être définie négative par le Théorème de l'indice de Hodge. (Voir par exemple [Nak04] pour un autre point de vue basé sur la fonction volume.)

Il se trouve que la plupart des propriétés importantes de la fonction volume pour des diviseurs ont des analogues dans le cas des courbes. Tout d'abord, la décomposition de Zariski est continue et satisfait une condition de linéarité (Théorèmes 6.5.3 et 6.5.6). Même si la partie négative d'une décomposition de Zariski pour une courbe n'est pas nécessairement représentée par une courbe effective, la Proposition 6.5.14 prouve une résultat de "rigidité" qui est un analogue pertinent de l'énoncé familier pour les diviseurs. La décomposition de Zariski et la fonction $\widehat{\mathrm{vol}}$ ont des comportements birationnels très agréables, qui sont discutés dans la section 6.5.6.

Des autres propriétés importantes incluent la log concavité stricte de $\widehat{\mathrm{vol}}$ et une inégalité de type Morse.

Théorème 1.0.26. *(= Théorème 6.5.10) Soit X une variété projective de dimension n. Pour deux classes pseudo-effectives des courbes α, β, nous avons*

$$\widehat{\mathrm{vol}}(\alpha+\beta)^{\frac{n-1}{n}} \geq \widehat{\mathrm{vol}}(\alpha)^{\frac{n-1}{n}} + \widehat{\mathrm{vol}}(\beta)^{\frac{n-1}{n}}.$$

En outre, si α et β sont grosses, il y a égalité si et seulement si les parties positives de α et β sont proportionnelles.

Théorème 1.0.27. *(= Théorème 6.5.18) Soit X une variété projective de dimension n. Soit α une classe grosse de courbe et soit β une classe mobile de courbe. Si $\alpha = B^{n-1} + \gamma$ est la décomposition de Zariski pour α, alors*

$$\begin{aligned}\widehat{\mathrm{vol}}(\alpha-\beta)^{n-1/n} &\geq (\widehat{\mathrm{vol}}(\alpha) - nB\cdot\beta)\cdot \widehat{\mathrm{vol}}(\alpha)^{-1/n} \\ &= (B^n - nB\cdot\beta)\cdot (B^n)^{-1/n}.\end{aligned}$$

En particulier, on a

$$\widehat{\mathrm{vol}}(\alpha-\beta) \geq B^n - \frac{n^2}{n-1} B\cdot\beta.$$

Nous avons également la description suivante des dérivées qui reflète les résultats de [BFJ09] et [LM09].

Théorème 1.0.28. *(= Théorème 6.1.7) Soit X une variété projective de dimension n. Alors, la fonction $\widehat{\mathrm{vol}}$ est $\mathcal{C}^1$ sur le cône des classes grosses de courbes. Plus précisément, soit α une classe grosse de courbe et soit $\alpha = B^{n-1} + \gamma$ sa décomposition de Zariski. Alors, pour toute classe de courbe β, on a*

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{\mathrm{vol}}(\alpha + t\beta) = \frac{n}{n-1} B\cdot\beta.$$

La décomposition de Zariski est particulièrement intéressante pour les variétés ayant une riche structure géométrique. Nous discutons deux exemples : les variétés toriques et les variétés hyperkählériennes (voir Section 6.8 et Section 6.9).

Tout d'abord, supposons que X soit une variété torique projective simpliciale de dimension n définie par un éventail Σ. Une classe α dans l'intérieur du cône mobile de courbes correspond à un poids positif de Minkowski sur les rayons de Σ. Un théorème fondamental de Minkowski attache un tel poids à un polytope P_α dont les normales des facettes sont les rayons de Σ, et dont les volumes des facettes sont déterminés par les poids.

Théorème 1.0.29. *Le cône des intersections complètes de X est l'adhérence des poids positifs de Minkowski α dont le polytope correspondant P_α a un éventail normal Σ. Pour ces classes, nous avons $\widehat{\mathrm{vol}}(\alpha) = n!\,\mathrm{vol}(P_\alpha)$.*

En fait, pour tout poids positif de Minkowski, l'éventail normal du polytope P_α construit par le théorème de Minkowski décrit le modèle birationnel associé à α (voir l'exemple 6.1.6).

Dans ce cadre, nous discutons la décomposition de Zariski et le volume d'un poids positif de Minkowski α. Le calcul du volume revient alors à la résolution d'un problème isopérimétrique : étant donné P_α fixé parmi tous les polytopes dont l'éventail normal raffine Σ, il y a un unique Q (à homothétie près) minimisant le quotient

$$\frac{V(P_\alpha^{n-1}, Q)}{\mathrm{vol}(Q)^{1/n}}.$$

Si nous laissons Q varier dans l'ensemble de tous les polytopes, alors l'inégalité de Brunn-Minkowski montre que le minimum est donné par $Q = cP_\alpha$, mais l'hypothèse de normalité de l'éventail sur Q conduit à une nouvelle version de ce problème classique.

De ce point de vue, la compatibilité avec la décomposition de Zariski correspond au fait que la solution d'un problème isopérimétrique doit être traité par une condition sur la dérivée. Nous montrons dans la section 6.8 que ce problème isopérimétrique peut être résolu (sans avoir besoin d'effectuer une minimisation) en utilisant la décomposition de Zariski.

Nous passons ensuite au cas des variétés hyperkählériennes. Les résultats de [Bou04, Section 4] montrent que le volume et la σ-decomposition des diviseurs satisfont une compatibilité naturelle avec la forme de Beauville-Bogomolov. Nous démontrons des propriétés analogues pour les classes des courbes. Le théorème suivant est formulé dans le cadre kählérien.

Théorème 1.0.30. *Soit X une variété hyperkählérienne de dimension n et soit q la forme bilinéaire sur $H^{n-1,n-1}(X)$ induite par la forme de Beauville-Bogomolov sur $H^{1,1}(X)$ via la dualité de Serre. Alors,*

1. *Le cône des intersections complètes de $(n-1, n-1)$-classes est q-dual au cône des $(n-1, n-1)$-classes pseudo-effectives.*
2. *Si α est une $(n-1, n-1)$-classe d'intersection complète, alors $\widehat{\mathrm{vol}}(\alpha) = q(\alpha, \alpha)^{n/2(n-1)}$.*
3. *Supposons que α réside dans l'intérieur du cône des $(n-1, n-1)$-classes pseudo-effectives et soit $\alpha = B^{n-1} + \gamma$ sa décomposition de Zariski. Alors $q(B^{n-1}, \gamma) = 0$, et si γ est non nulle, alors $q(\gamma, \gamma) < 0$.*

La principale caractéristique de la décomposition de Zariski pour les surfaces est qu'elle clarifie la relation entre les propriétés asymptotiques des sections d'un diviseur et ses propriétés dans la théorie de l'intersection. Par analogie avec le travail de [Zar62], il est naturel de se demander comment la fonction volume $\widehat{\mathrm{vol}}$ d'une classe de courbe est liée à la géométrie asymptotique des courbes représentées par la classe. Nous allons analyser cette question en comparant $\widehat{\mathrm{vol}}$ avec deux fonctions de type volume pour les courbes : la fonction de mobilité et la fonction de mobilité pondéré de [Leh13b]. Cela va aussi nous permettre de comparer notre définition de la décomposition de Zariski avec la notion de [FL13].

Rappelons que la définition de la mobilité est très parallèle à la définition du volume d'un diviseur via la croissance asymptotique des sections.

Dans [Leh13b], Lehmann montre que la mobilité s'étend en une fonction continue homogène sur l'ensemble $N_1(X)$. Le théorème suivant poursuit une étude commencée dans [Xia15a] (voir [Xia15a, Conjecture 3.1 and Theorem 3.2]). La Proposition 6.1.22 ci-dessous fournit une énoncé voisin.

Théorème 1.0.31. *(= Théorème 6.1.11) Soit X une variété projective lisse de dimension n et soit $\alpha \in \overline{\mathrm{Eff}}_1(X)$ une classe pseudo-effective de courbe. Alors :*

1. $\widehat{\mathrm{vol}}(\alpha) \leq \mathrm{mob}(\alpha) \leq n!\widehat{\mathrm{vol}}(\alpha)$.
2. *Si une certaine conjecture énoncée ci-dessous est vraie, alors* $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$.

Le point clé qui se situe derrière le Théorème 6.1.11 est une comparaison entre la décomposition de Zariski pour la fonction mob construite dans [FL13] et la décomposition de Zariski pour $\widehat{\mathrm{vol}}$. La deuxième partie de ce théorème repose sur la description conjecturale suivante de la mobilité d'une classe d'intersection complète :

Conjecture 1.0.32. *(voir [Leh13b, Question 7.1]) Soit X une variété projective lisse de dimension n et soit A un diviseur ample sur X. Alors*

$$\mathrm{mob}(A^{n-1}) = A^n.$$

Le Théorème 6.1.11 ci-dessus est assez surprenant : il suggère que le coefficient de mobilité de *toute* classe de courbes est réalisé de manière optimale par des courbes intersections complètes.

Exemple 1.0.33. Soit α la classe d'une droite projective de $\mathbb{P}^3$. Le coefficient de mobilité de α est déterminé par la question énumérative suivante : quel est le degré minimal d'une courbe passant par b points généraux de $\mathbb{P}^3$? La réponse est inconnue, même seulement en un sens asymptotique.

Perrin [Per87] conjecture que les courbes "optimales" (qui maximisent le nombre de points par rapport à leur degré à la puissance 3/2) sont des intersections complètes de deux diviseurs de même degré. Le Théorème 6.1.11 constitue en quelque sorte une vaste généralisation de la conjecture de Perrin, relative à toutes les classes grosses de courbes sur toutes les variétés projectives lisses.

Bien que la mobilité pondérée de [Leh13b] soit légèrement plus compliquée, elle nous permet de prouver un énoncé de manière inconditionnelle. La mobilité pondérée est similaire à la mobilité, mais elle compte les points singuliers d'un cycle avec un poids plus grand ; nous donnons la définition précise dans la section 6.10.1.

Théorème 1.0.34. *Soit X une variété projective lisse et soit $\alpha \in \overline{\mathrm{Eff}}_1(X)$ une classe de courbe. Alors $\widehat{\mathrm{vol}}(\alpha) = \mathrm{wmob}(\alpha)$.*

Ainsi la fonction $\widehat{\mathrm{vol}}$ prend en compte certains aspects fondamentaux du comportement géométrique asymptotique de courbes.

Selon la philosophie de [FL13], il faut interpréter la décomposition de Zariski (ou la σ-decomposition pour diviseurs) comme le défaut de log concavité stricte de la fonction volume. Ceci suggère que l'on devrait utiliser les outils d'analyse convexe – en particulier une certaine version de la transformée de Legendre-Fenchel – pour analyser les décompositions de Zariski. Nous allons montrer que la plupart des propriétés analytiques de base de la fonction $\widehat{\mathrm{vol}}$ et de la décomposition de Zariski peuvent en effet être déduites d'un cadre de dualité beaucoup plus général pour des fonctions concaves arbitraires. Dans cette perspective, la caractéristique la plus surprenante de $\widehat{\mathrm{vol}}$ est qu'elle incorpore des informations géométriques précise sur les courbes situées dans la classe correspondante.

Soit $\mathcal{C}$ un cône fermé saillant convexe de dimension pleine dans un espace vectoriel de dimension finie. Pour tout $s > 1$, soit $\mathrm{HConc}_s(\mathcal{C})$ l'ensemble des fonctions $f : \mathcal{C} \to \mathbb{R}$ qui sont semi-continues supérieurement, homogènes de poids $s > 1$, strictement positives à l'intérieur de $\mathcal{C}$, et qui sont s-concaves dans le sens que

$$f(v)^{1/s} + f(x)^{1/s} \leq f(x+v)^{1/s}$$

pour tous $v, x \in \mathcal{C}$. Dans ce contexte, l'analogue correct de la transformée de Legendre-Fenchel est la transformée polaire. Pour toute $f \in \mathrm{HConc}_s(\mathcal{C})$, la polaire $\mathcal{H}f$ est un élément de $\mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$ pour le cône dual $\mathcal{C}^*$ défini comme

$$\mathcal{H}f(w^*) = \inf_{v \in \mathcal{C}^\circ} \left(\frac{w^* \cdot v}{f(v)^{1/s}} \right)^{s/s-1} \qquad \forall w^* \in \mathcal{C}^*.$$

Nous définissons ce que cela signifie pour $f \in \mathrm{HConc}_s(\mathcal{C})$ d'avoir une "structure de décomposition de Zariski" et montrons que cette propriété résulte de la dérivabilité de $\mathcal{H}f$ (voir la section 6.4). Ceci est l'analogue dans notre situation de la façon dont la transformée de Legendre-Fenchel relie la différentiabilité et la convexité stricte. En outre, cette structure permet de transformer systématiquement les inégalités géométriques d'un cadre à l'autre. Beaucoup d'inégalités géométriques de base en géométrie algébrique – et donc pour les polytopes ou corps convexes liés aux variétés toriques (comme dans [Tei82] et [Kho89] et les références citées) – peuvent être comprises dans ce cadre.

Enfin, nous discutons de certaines connexions avec d'autres domaines de la géométrie birationnelle.

Un objectif important auxiliaire du chapitre correspondant est de montrer quelques nouveaux résultats concernant la fonction volume pour les diviseurs et le cône mobiles de courbes. L'outil clé est un autre invariante de la théorie de l'intersection pour les classes de courbes mobiles, noté $\mathfrak{M}$ et défini dans [Xia15a, Definition 2.2]. Comme les résultats ont un intérêt indépendant, nous rappelons ici certains d'entre eux.

Tout d'abord, nous donnons une version raffinée d'un théorème de [BDPP13] décrivant le cône mobile des courbes. Dans [BDPP13], il est prouvé que le cône mobile $\mathrm{Mov}_1(X)$ est *engendré* par les puissances positives $(n-1)$-ièmes des diviseurs gros. Nous montrons en fait que les points intérieurs de $\mathrm{Mov}_1(X)$ sont *exactement* l'ensemble des puissances positives $(n-1)$-ièmes des diviseurs gros pris dans l'intérieur de $\mathrm{Mov}^1(X)$.

Théorème 1.0.35. *Soit X une variété projective lisse de dimension n et soit α soit un point intérieur de* $\mathrm{Mov}_1(X)$*. Alors, il existe une unique classe grosse et mobile de diviseur L_α située dans l'intérieur de* $\mathrm{Mov}^1(X)$ *et dependant continuement de α, telle que $\langle L_\alpha^{n-1} \rangle = \alpha$.*

Exemple 1.0.36. Ce résultat montre que l'application $\langle \bullet^{n-1} \rangle$ est un homéomorphisme de l'intérieur du cône mobile des diviseurs vers l'intérieur du cône mobile de courbes. Ainsi, toute décomposition en chambres du cône mobile des courbes induit naturellement une décomposition du cône mobile des diviseurs et vice versa. Cette relation pourrait être utile dans l'étude des conditions de stabilité géométriques (comme dans [Neu10]).

Comme corollaire intéressant, nous obtenons :

Corollaire 1.0.37. *Soit X une variété projective de dimension n. Alors, l'ensemble des rayons de classes de courbes irréductibles dont la déformation domine X sont denses dans* $\mathrm{Mov}_1(X)$.

Nous pouvons aussi décrire la frontière de $\mathrm{Mov}_1(X)$.

Théorème 1.0.38. *Soit X une variété projective lisse et soit α une classe de courbe située à la frontière de* $\mathrm{Mov}_1(X)$*. Alors on est exactement dans l'une des situations alternatives suivantes :*
- *$\alpha = \langle L^{n-1} \rangle$ pour une classe grosse et mobile de diviseur L située à la frontière de* $\mathrm{Mov}^1(X)$.
- *$\alpha \cdot M = 0$ pour une classe mobile de diviseur M.*

L'homéomorphisme de $\mathrm{Mov}^1(X)^\circ \to \mathrm{Mov}_1(X)^\circ$ *s'étend en une application envoyant les classes grosses et mobiles de diviseurs sur la frontière de* $\mathrm{Mov}^1(X)$*, de manière bijective pour les classes du premier type.*

Nous généralisons également [BFJ09, Theorem D] à une classe plus large de diviseurs.

Théorème 1.0.39. *Soit X une variété projective lisse de dimension n. Pour des classes grosses de diviseurs L_1, L_2, on a*

$$\mathrm{vol}(L_1 + L_2)^{1/n} \geq \mathrm{vol}(L_1)^{1/n} + \mathrm{vol}(L_2)^{1/n}$$

avec égalité si et seulement si les parties positives $P_\sigma(L_1)$, $P_\sigma(L_2)$ sont proportionnelles. Ainsi, la fonction $L \mapsto \mathrm{vol}(L)^{1/n}$ est strictement concave sur le cône des diviseurs gros et mobiles.

Une technique de base en géométrie birationnelle est de majorer la positivité d'un diviseur en utilisant ses intersections avec des courbes spécifiées. Ces résultats peuvent être réinterprétés de façon utile en invoquant la fonction de volume des courbes. On a ainsi par example :

Proposition 1.0.40. *(= Proposition 6.1.22) Soit X une variété projective lisse de dimension n, et soient $\{k_i\}_{i=1}^r$ des entiers positifs. Supposons que $\alpha \in \mathrm{Mov}_1(X)$ soit représentée par une famille de courbes irréductibles telles que pour toute collection de points généraux $x_1, x_2, \ldots, x_r, y$ de X, il y ait une courbe de la famille qui contienne y et contienne chaque x_i avec multiplicité $\geq k_i$. Alors*

$$\widehat{\mathrm{vol}}(\alpha)^{n-1/n} \geq \frac{\sum_i k_i}{r^{1/n}}.$$

Nous pouvons ainsi appliquer les volumes de courbes pour étudier des questions telles que les constantes de Seshadri, les bornes de volumes de diviseurs et d'autres sujets connexes. Nous renvoyons le lecteur à la section 6.11 pour une discussion plus approfondie.

Exemple 1.0.41. Si X est rationnellement connexe, il est intéressant d'analyser les volumes des classes de courbes rationnelles spéciales dans X. Lorsque X est une variété de Fano de nombre de Picard 1, ces invariants sont étroitement liés aux invariants classiques, comme la longueur et le degré.

Par exemple, nous disons que $\alpha \in N_1(X)$ est une classe de connexion rationnelle si, pour deux points généraux de X, il existe une chaîne de courbes rationnelles de la classe α reliant ces deux points. Existe-il une borne supérieure uniforme (ne dépendant que de la dimension) pour le volume minimal d'une classe de courbes rationnelles réalisant cette chaîne sur une variété X rationnellement connexe ? [KMM92] et [Cam92] montrent que ceci est vrai pour les variétés de Fano lisses. Nous discutons cette question brièvement dans la section 6.11.2.

Chapitre 2

Introduction

The goal of this thesis is to study various positivity concepts in Kähler geometry. In particular, for a compact Kähler manifold of dimension n, we study the positivity of transcendental $(1,1)$ and $(n-1, n-1)$ classes. These objects include the divisor classes and curve classes over smooth projective varieties over $\mathbb{C}$. The main results presented in this thesis are mainly based on the following papers which I finished during my PhD studies :

1. (joint with Brian Lehmann) *Zariski decomposition of curves on algebraic varieties.* arXiv preprint 2015, arXiv : 1507.04316, submitted.
2. *A remark on the convergence of inverse σ_k-flow*, arXiv preprint 2015, arXiv : 1505.04999, Comptes Rendus Mathématique 354 (2016) 395-399.
3. *Characterizing volume via cone duality*, arXiv preprint 2015, arXiv : 1502.06450, submitted.
4. (joint with Jixiang Fu) *Teissier's problem on proportionality of nef and big classes over a compact Kähler manifold*, arXiv preprint 2014, arXiv : 1410.4878, submitted.
5. *Movable intersections and bigness criterion*, arXiv preprint 2014, arXiv : 1405.1582.
6. *Weak transcendental holomorphic Morse inequalities on compact Kähler manifolds*, arXiv preprint 2013, arXiv : 1308.2878, Annales de l'Institut Fourier 65 (2015) 1367-1379.
7. (joint with Jixiang Fu) *Relations between Kähler cone and balanced cone of a Kähler manifold*, arXiv preprint 2012, arXiv : 1203.2978, Advances in Mathematics 263 (2014) 230-252.

Thus this thesis can be seen as a reorganization of the above papers : Chapter 3 is on Teissier's proportionality problem for transcendental nef $(1,1)$-classes over compact Kähler manifolds and its applications, and it is based on the papers [FX14a] and [FX14b] ; Chapter 4 is on Demailly's conjecture on transcendental Morse inequalities which is a combination of [Xia13], [Xia14] and [Xia15b] ; Chapter 5 discusses the characterization of volume via cone duality, which is mainly based on the paper [Xia15a] ; Chapter 6 develops the theory of Zariski decomposition of curves on algebraic varieties and is based on the paper [LX15].

Next, let us summarize the main results in this thesis. For the basic facts of Kähler geometry and complex algebraic geometry, especially the definitions of various positivity, we will not spread in this introduction. Instead, we will briefly recall them in related sections. (We also refer the readers to the excellent books [Dem12a], [Dem12b] or [GH94].)

Teissier's proportionality problem in Kähler geometry

Around the year 1979, inspired by the Aleksandrov-Fenchel inequalities in convex geometry, Khovanskii and Teissier discovered independently deep inequalities in algebraic geometry which now is called Khovanskii-Teissier inequalities. These inequalities present a nice relationship between the theory of mixed volumes and algebraic geometry. Their proofs are based on the usual Hodge-Riemmann bilinear relations. A natural problem is how to characterize the equality case in these inequalities for a pair of big and nef line bundles, which was first considered by Bernard Teissier [Tei82, Tei88].

In Chapter 3, we first solve Teissier's proportionality problem for transcendental nef $(1,1)$-classes over a compact Kähler manifold, which says that the equality in the Khovanskii-Teissier inequalities holds for a pair of big and nef classes if and only if the two classes are proportional.

Theorem 2.0.42. *(= Theorem 3.1.1) Assume X is an n-dimensional compact Kähler manifold. Let $\alpha, \beta \in \overline{\mathcal{K}} \cap \mathcal{E}^\circ$ be two big and nef classes. Denote $s_k := \alpha^k \cdot \beta^{n-k}$. Then the following statements are equivalent :*

1. $s_k^2 = s_{k-1} \cdot s_{k+1}$ *for* $1 \leq k \leq n-1$ *;*
2. $s_k^n = s_0^{n-k} \cdot s_n^k$ *for* $0 \leq k \leq n$ *;*
3. $s_{n-1}^n = s_0 \cdot s_n^{n-1}$ *;*
4. $\mathrm{vol}(\alpha+\beta)^{1/n} = \mathrm{vol}(\alpha)^{1/n} + \mathrm{vol}(\beta)^{1/n}$ *;*
5. α *and* β *are proportional;*
6. α^{n-1} *and* β^{n-1} *are proportional.*

As a consequence, the map $\gamma \mapsto \gamma^{n-1}$ is injective from the big and nef cone $\overline{\mathcal{K}} \cap \mathcal{E}^\circ$ to the movable cone $\overline{\mathcal{M}}$.

This result recovers the previous one of Boucksom-Favre-Jonsson [BFJ09] for the case of big and nef line bundles over a (complex) projective algebraic manifold.

As an application of Teissier's proportionality theorem, we study non-Kähler geometry – the balanced cone – over compact Kähler manifolds. Recall that the balanced cone $\mathcal{B}$ is an open cone in $H^{n-1,n-1}(X,\mathbb{R})$, which is generated by d-closed strictly positive $(n-1,n-1)$-forms.

We consider a natural map from the closure of the Kähler cone of a compact Kähler manifold to the closure of its balanced cone :

$$\overline{\mathbf{b}} : \overline{\mathcal{K}} \to \overline{\mathcal{B}}, \quad \alpha \mapsto \alpha^{n-1}.$$

As an immediate corollary of the above proportionality theorem, we get its injectivity when restricted on the big and nef subcone $\overline{\mathcal{K}} \cap \mathcal{E}^\circ$; see Section 3.3.2. We also study its surjectivity, giving some interesting examples where the map takes some boundary points of the Kähler cone into the interior of the balanced cone ; see Section 3.3.3. In particular, for rational nef classes on projective Calabi-Yau manifolds, we characterize when a boundary class is mapped into the interior of the balanced cone – this result can be seen as an inspiration of related results in Chapter 6.

Theorem 2.0.43. *(see Theorem 3.3.3) Let X be a projective Calabi-Yau manifold of dimension n. Then we have :*

1. *If $\alpha \in \partial\mathcal{K}$ is a boundary class, then $\overline{\mathbf{b}}(\alpha) \in \mathcal{B}$ implies that α is a big class.*
2. *If $\alpha \in \partial\mathcal{K}$ is a big class given by some $\mathbb{Q}$-divisor, then $\overline{\mathbf{b}}(\alpha) \in \mathcal{B}$ if and only if F_α is a small contraction, or equivalently, the exceptional set $\mathrm{Exc}(F_\alpha)$ of the contraction map F_α induced by the class α is of* $\mathrm{codim} \geq 2$.

By similar arguments as in our proof of Teissier's proportionality theorem, using non-Kähler metrics, we could give an analytic characterization on when a nef class is a Kähler class.

Theorem 2.0.44. *(= Theorem 3.3.17) Let X be a compact n-dimensional Kähler manifold and let η be a smooth volume form on X satisfying* $\mathrm{vol}(\eta) = 1$*. Assume that α is a nef class such that α^{n-1} is a balanced class (i.e. the class α^{n-1} contains some strictly positive smooth $(n-1,n-1)$-form). If there exists a balanced metric $\tilde{\omega}$ in α^{n-1} such that*

$$\tilde{\omega}^n \geq \mathrm{vol}(\alpha)\eta$$

pointwise on X, then α must be a Kähler class.

The above result is related to the solvability of "form-type" Monge-Ampère equations for an interesting class of balanced metrics, that is, the balanced class given by α^{n-1} with $\alpha \in \partial\mathcal{K}$. In particular, using the notations of the above theorem, for any positive constant $c < \mathrm{vol}(\alpha)$ and $\alpha \in \partial\mathcal{K}$, there exist no balanced metrics $\tilde{\omega}^{n-1} \in \alpha^{n-1}$ such that

$$\tilde{\omega}^n = c\eta.$$

Demailly's conjecture on transcendental Morse inequalities

We first recall Demailly's conjecture on transcendental Morse inequalities on compact complex manifolds.

Conjecture 2.0.45. *(see [BDPP13, Conjecture 10.1]) Let X be a compact complex manifold of dimension n.*

1. *Let θ be a real d-closed $(1,1)$-form representing the class α and let $X(\theta, \leq 1)$ be the set where θ has at most one negative eigenvalue. If $\int_{X(\theta,\leq 1)} \theta^n > 0$, then the Bott-Chern class α contains a Kähler current and*
$$\mathrm{vol}(\alpha) \geq \int_{X(\theta,\leq 1)} \theta^n.$$
2. *Let α and β be two nef $(1,1)$-classes on X satisfying $\alpha^n - n\alpha^{n-1}\cdot\beta > 0$. Then the Bott-Chern class $\alpha - \beta$ contains a Kähler current and*
$$\mathrm{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1}\cdot\beta.$$

In Chapter 4, by reconsidering the main ideas of [Chi13], we first prove a weak version of Demailly's conjecture on transcendental Morse inequalities on compact Kähler manifolds.

Theorem 2.0.46. *(= Theorem 4.1.4) Let X be an n-dimensional compact complex manifold with a hermitian metric ω satisfying $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2, ..., n-1$. Assume α, β are two nef classes on X satisfying*
$$\alpha^n - 4n\alpha^{n-1}\cdot\beta > 0,$$
then there exists a Kähler current in the Bott-Chern class $\alpha - \beta$.

Thus, our result covers the Kähler case and improves a result of [BDPP13]. Moreover, the key point is that the cohomology classes α, β can be transcendental.

Recently, by keeping the same method of [Xia13, Chi13] and with the new estimates of Monge-Ampère equations, [Pop14] proved that the constant $4n$ in our Theorem 4.1.4 can be improved to be the natural and optimal constant n. Thus we have a Morse-type bigness criterion for the difference of two transcendental nef classes. It is natural to ask whether the Morse-type bigness criterion

$$\alpha^n - n\alpha^{n-1}\cdot\beta > 0 \Rightarrow \mathrm{vol}(\alpha - \beta) > 0$$

for nef classes can be generalized to pseudo-effective $(1,1)$-classes. Towards this generalization, we need the movable intersection products (denoted by $\langle - \rangle$) of pseudo-effective $(1,1)$-classes (see e.g. [Bou02a, BDPP13]). Then our problem can be stated as following :

Question 2.0.47. Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes. Does $\mathrm{vol}(\alpha) - n\langle\alpha^{n-1}\rangle\cdot\beta > 0$ imply that there exists a Kähler current in the class $\alpha - \beta$?

Unfortunately, a very simple example due to [Tra95] implies that the above generalization does not always hold.

Example 2.0.48. (see [Tra95, Example 3.8]) Let $\pi : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ along a point p. Let $R = \pi^* H$, where H is the hyperplane line bundle on $\mathbb{P}^2$. Let $E = \pi^{-1}(p)$ be the exceptional divisor. Then for every positive integer k, the space of global holomorphic sections of $k(R-2E)$ is the space of homogeneous polynomials in three variables of degree at most k and vanishes of order $2k$ at p; hence $k(R-2E)$ does not have any global holomorphic sections. The space $H^0(X, \mathcal{O}(k(R-2E))) = \{0\}$ implies that $R-2E$ can not be big. However, we have $R^2 - R \cdot 2E > 0$, as $R^2 = 1$ and $R \cdot E = 0$.

However, with some basic properties of movable intersections, we can generalize the main result of [Pop14] to pseudo-effective $(1,1)$-classes. We show that it holds if β is movable. Here β being movable means that the negative part of β vanishes in its divisorial Zariski decomposition (see [Bou04]).

Theorem 2.0.49. *(= Theorem 4.1.10) Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes with β movable. Then $\mathrm{vol}(\alpha) - n\langle \alpha^{n-1}\rangle \cdot \beta > 0$ implies that there exists a Kähler current in the class $\alpha - \beta$.*

As an application, we give a Morse-type bigness criterion for movable $(n-1, n-1)$-classes which will be applied to study the positivity of curve classes in the subsequent Chapter 6.

Theorem 2.0.50. *(= Theorem 4.1.15) Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes. Then $\mathrm{vol}(\alpha) - n\alpha \cdot \langle \beta^{n-1}\rangle > 0$ implies that there exists a strictly positive $(n-1, n-1)$-current in the class $\langle \alpha^{n-1}\rangle - \langle \beta^{n-1}\rangle$.*

At the end of Chapter 4, we apply the above results to the numerical characterization problem on the convergence of inverse σ_k-flow, giving some partial positivity results towards the conjecture of Lejmi and Székelyhidi [LS15].

From the point of view that relates the existence of canonical Kähler metrics with algebro-geometric stability conditions, Lejmi and Székelyhidi [LS15] proposed a numerical characterization on when the inverse σ_k-flow converges. We aim to study the positivity of related cohomology classes in their conjecture. We generalize their conjecture by weakening the numerical condition on X a little bit.

Conjecture 2.0.51. *(see [LS15, Conjecture 18]) Let X be a compact Kähler manifold of dimension n, and let ω, α be two Kähler metrics over X satisfying*

$$\int_X \omega^n - \frac{n!}{k!(n-k)!}\omega^{n-k} \wedge \alpha^k \geq 0. \tag{2.1}$$

Then there exists a Kähler metric $\omega' \in \{\omega\}$ such that

$$\omega'^{n-1} - \frac{(n-1)!}{k!(n-k-1)!}\omega'^{n-k-1} \wedge \alpha^k > 0 \tag{2.2}$$

as a smooth $(n-1, n-1)$-form if and only if

$$\int_V \omega^p - \frac{p!}{k!(p-k)!}\omega^{p-k} \wedge \alpha^k > 0 \tag{2.3}$$

for every irreducible subvariety of dimension p with $k \leq p \leq n-1$.

We focus on the cases when $k = 1$ and $k = n-1$, we get the following results. For $k = 1$, we have :

Theorem 2.0.52. *(= Theorem 4.4.2) Let X be a compact Kähler manifold of dimension n, and let ω, α be two Kähler metrics over X satisfying the numerical conditions in the above conjecture for $k = 1$. Then $\{\omega - \alpha\}$ is a Kähler class.*

For $k = n-1$, we have the following similar result.

Theorem 2.0.53. *(= Theorem 4.4.3) Let X be compact Kähler manifold of dimension n, and let ω, α be two Kähler metrics over X satisfying the numerical conditions in the above conjecture for $k = n-1$. Then the class $\{\omega^{n-1} - \alpha^{n-1}\}$ lies in the closure of the Gauduchon cone, i.e. it has nonnegative intersection number with every pseudo-effective $(1,1)$-class.*

Inspired by the above results and the prediction of Conjecture 4.4.1, we propose the following question on the positivity of (k,k)-classes which is closely related to the singularities of positive (k,k)-currents.

Question 2.0.54. Let X be a compact Kähler manifold (or general compact complex manifold) of dimension n. Let $\Omega \in H^{k,k}(X,\mathbb{R})$ be a big (k,k)-class, i.e. it can be represented by a strictly positive (k,k)-current over X. Assume that the restriction class $\Omega_{|V}$ is also big over every irreducible subvariety V with $k \leq \dim V \leq n-1$, then does Ω contain a smooth strictly positive (k,k)-form in its Bott-Chern class ? Or does Ω at least contain a strictly positive (k,k)-current with analytic singularities of codimension at least $n-k+1$ in its Bott-Chern class ?

Characterizing volume via cone dualities

Recall that the volume of a divisor on a projective variety is a non-negative number measuring the positivity of the divisor. Let X be a smooth projective variety of dimension n, and let D be a divisor on X. By definition, the volume of D is defined to be

$$\mathrm{vol}(D) := \limsup_{m\to\infty} \frac{h^0(X, mD)}{m^n/n!}.$$

Chapter 5 is devoted to study the volume function via cone duality – which is the first step towards Chapter 6. Firstly, for divisors over smooth projective varieties we show that the volume can be characterized by the duality between the pseudo-effective cone of divisors and the movable cone of curves. From the seminal work of Boucksom-Demailly-Paun-Peternell (see [BDPP13]), we know that there exists a duality between the pseudo-effective cone of divisors and the cone generated by movable curves :

$$\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X).$$

Using this cone duality and a suitable invariant of movable curve classes, we give the following new volume characterization of divisors by the infimum of intersection numbers between the pairings of $\overline{\mathrm{Eff}}^1(X)$ and $\mathrm{Mov}_1(X)$.

We define an intersection-theoretic invariant of movable curve classes :

Definition 2.0.55. (see Definition 5.2.6) Let X be a smooth projective variety of dimension n, and let γ be a movable curve class. Then the invariant $\mathfrak{M}(\gamma)$ is defined as following :

$$\mathfrak{M}(\gamma) := \inf_{\beta\in\overline{\mathrm{Eff}}(X)^\circ} \left(\frac{\beta\cdot\gamma}{\mathrm{vol}(\beta)^{1/n}}\right)^{\frac{n}{n-1}}.$$

With this invariant, we have :

Theorem 2.0.56. *(= Theorem 5.1.1) Let X be an n-dimensional smooth projective variety and let $\alpha \in \overline{\mathrm{Eff}}^1(X)$ be a pseudo-effective divisor class. Then the volume of α can be characterized as following :*

$$\mathrm{vol}(\alpha) = \inf_{\gamma\in\mathrm{Mov}_1(X)^\circ} \left(\frac{\alpha\cdot\gamma}{\mathfrak{M}(\gamma)^{n-1/n}}\right)^n.$$

Furthermore, we can also replace the movable cone $\mathrm{Mov}_1(X)$ by the Gauduchon cone $\mathcal{G}$ (or the balanced cone $\mathcal{B}$), which is generated by special Hermitian metrics.

Remark 2.0.57. The invariant $\mathfrak{M}$ can be defined for transcendental movable classes over compact Kähler manifolds. And similar volume characterization holds true for any transcendental $(1,1)$-class under Demailly's conjecture on transcendental holomorphic Morse inequalities.

Inspired by the above volume characterization for divisor classes, using cone dualities, we introduce a volume functional for pseudo-effective 1-cycles – $(n-1,n-1)$-classes represented by positive $(n-1,n-1)$-currents – over compact Kähler manifolds.

For smooth projective variety, by Kleiman's criterion, we have the cone duality

$$\mathrm{Nef}^1(X)^* = \overline{\mathrm{Eff}}_1(X)$$

where Nef^1 is the cone generated by nef divisor classes and $\overline{\mathrm{Eff}}_1$ is the cone generated by pseudo-effective curve classes. For compact Kähler manifold, by Demailly-Paun's numerical characterization of Kähler cone (see [DP04]), we have the cone duality

$$\mathcal{K}^* = \mathcal{N}$$

where $\mathcal{K}$ is the Kähler cone generated by Kähler classes and $\mathcal{N}$ is the cone generated by d-closed positive $(n-1,n-1)$-currents.

Definition 2.0.58. 1. Let X be an n-dimensional smooth projective variety, and let $\gamma \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then the volume of γ is defined to be

$$\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) = \inf_{\beta \in \mathrm{Nef}^1(X)^\circ} \left(\frac{\beta \cdot \gamma}{\mathrm{vol}(\beta)^{1/n}} \right)^{\frac{n}{n-1}}.$$

2. Let X be an n-dimensional compact Kähler manifold, and let $\gamma \in H_{BC}^{n-1,n-1}(X,\mathbb{R})$ be a pseudo-effective $(n-1,n-1)$-class. Then the volume of γ is defined to be

$$\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \inf_{\beta \in \mathcal{K}(X)} \left(\frac{\beta \cdot \gamma}{\mathrm{vol}(\beta)^{1/n}} \right)^{\frac{n}{n-1}}.$$

It is well known that the volume $\mathrm{vol}(\bullet)$ for divisor class depends only on the numerical class of the divisor, and $\mathrm{vol}^{1/n}$ is homogeneous of degree one, concave on the pseudo-effective cone and extends to a continuous function on the whole real Néron-Severi space which is strictly positive exactly on big classes. We show that our volume function $\widehat{\mathrm{vol}}$ enjoys similar properties. For simplicity, we state the result for $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$.

Theorem 2.0.59. *(= Theorem 5.1.4) Let X be a smooth projective variety of dimension n. Then we have :*

1. $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}^{n-1/n}$ *is a homogeneous concave function of degree one.*
2. $\gamma \in \overline{\mathrm{Eff}}_1(X)^\circ$ *if and only if* $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) > 0$.
3. $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ *can be extended to be a continuous function on the whole vector space* $N_1(X,\mathbb{R})$ *by setting* $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}} = 0$ *outside* $\overline{\mathrm{Eff}}_1(X)$.

For projective varieties, the functional $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ is closely related to the mobility functional recently studied by Lehmann (see [Leh13b]).

Definition 2.0.60. (see [Leh13b, Definition 1.1]) Let X be a projective variety of dimension n and let $\alpha \in N_k(X)$ be a k-cycle class with integer coefficients. The mobility of α is defined to be

$$\mathrm{mob}(\alpha) := \limsup_{m\to\infty} \frac{\max \left\{ b \in \mathbb{Z}_{\geq 0} \,\middle|\, \begin{array}{c} \text{Any } b \text{ general points are contained} \\ \text{in an effective cycle of class } m\alpha \end{array} \right\}}{m^{\frac{n}{n-k}}/n!}.$$

The numerator is called mobility count which is denoted by $\mathrm{mc}(m\alpha)$. Mobility functional for cycles was suggested in [DELV11] as an analogue of the volume function for divisors. The motivation is that one can interpret the volume of a divisor D as an asymptotic measurement of the number of general points contained in members of $|mD|$ as m tends to infinity.

In particular, we can define the mobility for numerical classes of curves. Lehmann proved that the mobility functional also distinguishes interior points and boundary points, and extends to a continuous homogeneous function on all of $N_1(X)$. Thus, in the situation of curves, combining with Theorem 5.1.4, we have two functionals with this property. It is interesting to compare mob and $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ over $\overline{\mathrm{Eff}}_1$. We would like to propose the following conjecture.

Conjecture 2.0.61. *Let X be a smooth projective varieties of dimension n, then we have*

$$\mathrm{mob} = \widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}.$$

At least there should exist two positive constants c_1, c_2, depending only on the dimension of the underlying manifold, such that

$$c_1 \widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) \leq \mathrm{mob}(\gamma) \leq c_2 \widehat{\mathrm{vol}}_{\overline{NE}}(\gamma)$$

for any $\gamma \in \overline{\mathrm{Eff}}_1(X)$. In Chapter 5, we observed that the positive constant c_2 can be obtained easily by using Lehamnn's estimates of mobility count functional mc.

Theorem 2.0.62. *(see Theorem 5.1.5) Let X be an n-dimensional smooth projective variety, and let $\overline{\mathrm{Eff}}_1(X)$ be the closure of the cone generated by effective curve classes. Then for any $\gamma \in \overline{\mathrm{Eff}}_1(X)$, we have*

$$\mathrm{mob}(\gamma) \leq n! 2^{4n+1} \widehat{\mathrm{vol}}_{\overline{NE}}(\gamma).$$

In the subsequent Chapter 6, based on the theory developed in the joint work [LX15], besides other results, we will obtain the positive constant c_1 (and a better bound c_2).

Inspired by the Fujita type approximation for curve classes with respect to mobility functional in [FL13], we aim to study some Fujita type approximation results for pseudo-effective $(n-1, n-1)$-classes over compact Kähler manifolds with respect to our volume functional $\widehat{\mathrm{vol}}_{\mathcal{N}}$. Following Boucksom's analytical version of divisorial Zariski decomposition [Bou04, Bou02a] (for the algebraic approach, see [Nak04]), we study Zariski decomposition for pseudo-effective $(n-1, n-1)$-classes in the sense of Boucksom.

In divisorial Zariski decomposition, the negative part is an effective divisor of Kodaira dimension zero, and indeed it contains only one positive $(1,1)$-current. In our setting, we can prove that this fact also holds for big 1-cycles. Comparing with other definitions of Zariski decomposition for big 1-cycles (see e.g. [FL13]), the negative part is always effective. Using his characterization of volume by Monge-Ampère mass, Boucksom showed that the "Zariski projection" preserves volume. It is also expected that in our setting the Zariski projection preserves $\widehat{\mathrm{vol}}_{\mathcal{N}}$. Indeed, this follows from the Zariski decomposition for 1-cycles developed in Chapter 6 (see [LX15]), which is more closely related to $\widehat{\mathrm{vol}}_{\mathcal{N}}$.

Theorem 2.0.63. *(= Theorem 5.1.6) Let X be a compact Kähler manifold of dimension n and let $\gamma \in \mathcal{N}^\circ$ be an interior point. Let $\gamma = Z(\gamma) + \{N(\gamma)\}$ be the Zariski decomposition in the sense of Boucksom, then we have*

1. *$N(\gamma)$ is an effective curve and it is the unique positive current contained in the negative part $\{N(\gamma)\}$.*
2. *Moreover, $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma))$.*

Thus our volume function $\widehat{\mathrm{vol}}$ is compatible with the Zariski decomposition in the sense of Boucksom.

Zariski decomposition of curves

In Chapter 6, we introduce a Zariski decomposition for curve classes and use it to develop the theory of the volume function for curves defined in the previous Chapter 5. More generally, we develop a theory of formal Zariski decomposition with respect to homogeneous concave function of degree $s > 1$ defined on a cone. For toric varieties and for hyperkähler manifolds the Zariski decomposition admits an interesting geometric interpretation. With the decomposition, we prove some fundamental positivity results for curve classes, such as a Morse-type inequality. We continue to compare the volume of a curve class with its mobility, yielding some surprising results about asymptotic point counts. Finally, we give a number of applications to birational geometry, including a refined structure theorem for the movable cone of curves.

Since in this chapter we focus on studying the function $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ over projective varieties, we will simply denote it by $\widehat{\mathrm{vol}}$.

In [Zar62] Zariski introduced a fundamental tool for studying linear series on a surface now known as a Zariski decomposition. Over the past 50 years the Zariski decomposition and its generalizations to divisors in higher dimensions have played a central role in birational geometry. We introduce an analogous decomposition for curve classes on varieties of arbitrary dimension. Our decomposition is defined for big curve classes – elements of the interior of the pseudo-effective cone of curves $\overline{\mathrm{Eff}}_1(X)$. Throughout we work over $\mathbb{C}$, but the main results also hold over an algebraically closed field or in the Kähler setting (see Section 6.1.5).

Definition 2.0.64. (see Definition 6.1.1) Let X be a projective variety of dimension n and let $\alpha \in N_1(X)$ be a big curve class. Then a Zariski decomposition for α is a decomposition

$$\alpha = B^{n-1} + \gamma$$

where B is a big and nef $\mathbb{R}$-Cartier divisor class, γ is pseudo-effective, and $B \cdot \gamma = 0$. We call B^{n-1} the "positive part" and γ the "negative part" of the decomposition.

This definition directly generalizes Zariski's original definition, which (for big classes) is given by similar intersection criteria. It also generalizes the σ-decomposition of [Nak04], and mirrors the Zariski decomposition of [FL13], in the following sense. The basic feature of a Zariski decomposition is that the positive part should retain all the "positivity" of the original class. In our setting, we will measure the positivity of a curve class using an interesting new volume-type function $\widehat{\mathrm{vol}}$ defined in [Xia15a] (see Chapter 5).

Indeed, the function $\widehat{\mathrm{vol}}$ is a kind of polar transformation of the volume function for divisors. It is motivated by the realization that the volume of a divisor has a similar intersection-theoretic description against curves as in Chapter 5. In Chapter 5, we prove that $\widehat{\mathrm{vol}}$ satisfies many of the desirable analytic features of the volume for divisors.

By [FL13, Proposition 5.3], we know that the σ-decomposition $L = P_\sigma(L) + N_\sigma(L)$ is the unique decomposition of L into a movable piece and a pseudo-effective piece such that $\mathrm{vol}(L) = \mathrm{vol}(P_\sigma(L))$. In the same way, our decomposition for curves is compatible with the volume function for curves :

Theorem 2.0.65. *(= Theorem 6.1.3) Let X be a projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)^\circ$ be a big curve class. Then α admits a unique Zariski decomposition $\alpha = B^{n-1} + \gamma$. Furthermore,*

$$\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(B^{n-1}) = \mathrm{vol}(B)$$

and B is the unique big and nef divisor class with this property satisfying $B^{n-1} \preceq \alpha$. Any big and nef divisor class computing $\widehat{\mathrm{vol}}(\alpha)$ is proportional to B.

We define the complete intersection cone $\mathrm{CI}_1(X)$ to be the closure of the set of classes of the form A^{n-1} for an ample divisor A on X. The positive part of the Zariski decomposition takes values in $\mathrm{CI}_1(X)$.

Our goal is to develop the theory of Zariski decompositions of curves and the theory of $\widehat{\mathrm{vol}}$. Due to their close relationship, we will see that is very fruitful to develop the two theories in parallel. In particular, we recover Zariski's original intuition that asymptotic point counts coincide with numerical invariants for curves.

Example 2.0.66. If X is an algebraic surface, then the Zariski decomposition provided by Theorem 6.1.3 coincides (for big classes) with the numerical version of the classical definition of [Zar62]. Indeed, using Proposition 6.5.14 one sees that the negative part γ is represented by an effective curve N. The self-intersection matrix of N must be negative-definite by the Hodge Index Theorem. (See e.g. [Nak04] for another perspective focusing on the volume function.)

It turns out that most of the important properties of the volume function for divisors have analogues in the curve case. First of all, Zariski decompositions are continuous and satisfy a linearity condition (Theorems 6.5.3 and 6.5.6). While the negative part of a Zariski decomposition need not be represented by an effective curve, Proposition 6.5.14 proves a "rigidity" result which is a suitable analogue of the familiar statement for divisors. Zariski decompositions and $\widehat{\mathrm{vol}}$ exhibit very nice birational behavior, discussed in Section 6.5.6.

Other important properties include the strict log concavity of $\widehat{\mathrm{vol}}$ and a Morse-type inequality for curves.

Theorem 2.0.67. *(= Theorem 6.5.10) Let X be a projective variety of dimension n. For any two pseudo-effective curve classes α, β we have*

$$\widehat{\mathrm{vol}}(\alpha+\beta)^{\frac{n-1}{n}} \geq \widehat{\mathrm{vol}}(\alpha)^{\frac{n-1}{n}} + \widehat{\mathrm{vol}}(\beta)^{\frac{n-1}{n}}.$$

Furthermore, if α and β are big, then we obtain an equality if and only if the positive parts of α and β are proportional.

Theorem 2.0.68. *(= Theorem 6.5.18) Let X be a projective variety of dimension n. Let α be a big curve class and let β be a movable curve class. Write $\alpha = B^{n-1} + \gamma$ for the Zariski decomposition of α. Then*

$$\begin{aligned}\widehat{\mathrm{vol}}(\alpha-\beta)^{n-1/n} &\geq (\widehat{\mathrm{vol}}(\alpha) - nB\cdot\beta)\cdot\widehat{\mathrm{vol}}(\alpha)^{-1/n}\\ &= (B^n - nB\cdot\beta)\cdot(B^n)^{-1/n}.\end{aligned}$$

In particular, we have

$$\widehat{\mathrm{vol}}(\alpha-\beta) \geq B^n - \frac{n^2}{n-1}B\cdot\beta.$$

We also have the following description of the derivative which mirrors the results of [BFJ09] and [LM09].

Theorem 2.0.69. *(= Theorem 6.1.7) Let X be a projective variety of dimension n. Then the function $\widehat{\mathrm{vol}}$ is $\mathcal{C}^1$ on the big cone of curves. More precisely, let α be a big curve class on X and write $\alpha = B^{n-1}+\gamma$ for its Zariski decomposition. For any curve class β, we have*

$$\left.\frac{d}{dt}\right|_{t=0}\widehat{\mathrm{vol}}(\alpha+t\beta) = \frac{n}{n-1}B\cdot\beta.$$

The Zariski decomposition is particularly striking for varieties with a rich geometric structure. We discuss two examples : toric varieties and hyperkähler manifolds. See Section 6.8 and Section 6.9.

First, suppose that X is a simplicial projective toric variety of dimension n defined by a fan Σ. A class α in the interior of the movable cone of curves corresponds to a positive Minkowski weight on the rays of Σ. A fundamental theorem of Minkowski attaches to such a weight a polytope P_α whose facet normals are the rays of Σ and whose facet volumes are determined by the weights.

Theorem 2.0.70. *The complete intersection cone of X is the closure of the positive Minkowski weights α whose corresponding polytope P_α has normal fan Σ_α. For such classes we have $\widehat{\mathrm{vol}}(\alpha) = n! \operatorname{vol}(P_\alpha)$.*

In fact, for any positive Minkowski weight the normal fan of the polytope P_α constructed by Minkowski's Theorem describes the birational model associated to α as in Example 6.1.6.

We next discuss the Zariski decomposition and volume of a positive Minkowski weight α. In this setting, the calculation of the volume is the solution of an isoperimetric problem : fixing P_α, amongst all polytopes whose normal fan refines Σ there is a unique Q (up to homothety) minimizing the mixed volume calculation

$$\frac{V(P_\alpha^{n-1}, Q)}{\operatorname{vol}(Q)^{1/n}}.$$

If we let Q vary over all polytopes then the Brunn-Minkowski inequality shows that the minimum is given by $Q = cP_\alpha$, but the normal fan condition on Q yields a new version of this classical problem.

From this viewpoint, the compatibility with the Zariski decomposition corresponds to the fact that the solution of an isoperimetric problem should be given by a condition on the derivative. We show in Section 6.8 that this isoperimetric problem can be solved (with no minimization necessary) using the Zariski decomposition.

We next turn to hyperkähler manifolds. The results of [Bou04, Section 4] show that the volume and σ-decomposition of divisors satisfy a natural compatibility with the Beauville-Bogomolov form. We prove the analogous properties for curve classes. The following theorem is phrased in the Kähler setting. (Of course, the analogous statements in the projective setting are also true.)

Theorem 2.0.71. *Let X be a hyperkähler manifold of dimension n and let q denote the bilinear form on $H^{n-1,n-1}(X)$ induced via duality from the Beauville-Bogomolov form on $H^{1,1}(X)$.*

1. *The cone of complete intersection $(n-1, n-1)$-classes is q-dual to the cone of pseudo-effective $(n-1, n-1)$-classes.*
2. *If α is a complete intersection $(n-1, n-1)$-class then $\widehat{\mathrm{vol}}(\alpha) = q(\alpha, \alpha)^{n/2(n-1)}$.*
3. *Suppose α lies in the interior of the cone of pseudo-effective $(n-1, n-1)$-classes and write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. Then $q(B^{n-1}, \gamma) = 0$ and if γ is non-zero then $q(\gamma, \gamma) < 0$.*

The main feature of the Zariski decomposition for surfaces is that it clarifies the relationship between the asymptotic sectional properties of a divisor and its intersection-theoretic properties. By analogy with the work of [Zar62], it is natural to wonder how the volume function $\widehat{\mathrm{vol}}$ of a curve class is related to the asymptotic geometry of the curves represented by the class. We will analyze this question by comparing $\widehat{\mathrm{vol}}$ with two "volume-type" functions for curves : the mobility function and the weighted mobility function of [Leh13b]. This will also allow us to contrast our definition of Zariski decompositions with the notion from [FL13].

Recall that the definition of the mobility is a close parallel to the definition of the volume of a divisor via asymptotic growth of sections.

In [Leh13b], Lehmann showed that the mobility extends to a continuous homogeneous function on all of $N_1(X)$. The following theorem continues a project begun by [Xia15a] (see [Xia15a, Conjecture 3.1 and Theorem 3.2]). Proposition 6.1.22 below gives a related statement.

Theorem 2.0.72. *(= Theorem 6.1.11) Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then :*

1. $\widehat{\mathrm{vol}}(\alpha) \leq \operatorname{mob}(\alpha) \leq n! \widehat{\mathrm{vol}}(\alpha)$.
2. *Assume the conjecture below. Then* $\operatorname{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$.

The driving force behind Theorem 6.1.11 is a comparison of the Zariski decomposition for mob constructed in [FL13] with the Zariski decomposition for $\widehat{\mathrm{vol}}$ defined above. The second part of this theorem relies on the following (difficult) conjectural description of the mobility of a complete intersection class :

Conjecture 2.0.73. *(see [Leh13b, Question 7.1]) Let X be a smooth projective variety of dimension n and let A be an ample divisor on X. Then*

$$\mathrm{mob}(A^{n-1}) = A^n.$$

Theorem 6.1.11 is quite surprising : it suggests that the mobility count of *any* curve class is optimized by complete intersection curves.

Example 2.0.74. Let α denote the class of a line on $\mathbb{P}^3$. The mobility count of α is determined by the following enumerative question : what is the minimal degree of a curve through b general points of $\mathbb{P}^3$? The answer is unknown, even in an asymptotic sense.

Perrin [Per87] conjectures that the "optimal" curves (which maximize the number of points relative to their degree to the 3/2) are complete intersections of two divisors of the same degree. Theorem 6.1.11 supports a vast generalization of Perrin's conjecture to all big curve classes on all smooth projective varieties.

While the weighted mobility of [Leh13b] is slightly more complicated, it allows us to prove an unconditional statement. The weighted mobility is similar to the mobility, but it counts singular points of the cycle with a higher "weight" ; we give the precise definition in Section 6.10.1.

Theorem 2.0.75. *Let X be a smooth projective variety and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then $\widehat{\mathrm{vol}}(\alpha) = \mathrm{wmob}(\alpha)$.*

Thus $\widehat{\mathrm{vol}}$ captures some fundamental aspects of the asymptotic geometric behavior of curves.

According to the philosophy of [FL13], one should interpret the Zariski decomposition (or the σ-decomposition for divisors) as capturing the failure of strict log concavity of the volume function. This suggests that one should use the tools of convex analysis – in particular some version of the Legendre-Fenchel transform – to analyze Zariski decompositions. We will show that many of the basic analytic properties of $\widehat{\mathrm{vol}}$ and Zariski decompositions can in fact be deduced from a much more general duality framework for arbitrary concave functions. From this perspective, the most surprising feature of $\widehat{\mathrm{vol}}$ is that it captures actual geometric information about curves representing the corresponding class.

Let $\mathcal{C}$ be a full dimensional closed proper convex cone in a finite dimensional vector space. For any $s > 1$, let $\mathrm{HConc}_s(\mathcal{C})$ denote the collection of functions $f : \mathcal{C} \to \mathbb{R}$ that are upper-semicontinuous, homogeneous of weight $s > 1$, strictly positive on the interior of $\mathcal{C}$, and which are s-concave in the sense that

$$f(v)^{1/s} + f(x)^{1/s} \leq f(x+v)^{1/s}$$

for any $v, x \in \mathcal{C}$. In this context, the correct analogue of the Legendre-Fenchel transform is the (concave homogeneous) polar transform. For any $f \in \mathrm{HConc}_s(\mathcal{C})$, the polar $\mathcal{H}f$ is an element of $\mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$ for the dual cone $\mathcal{C}^*$ defined as

$$\mathcal{H}f(w^*) = \inf_{v \in \mathcal{C}^\circ} \left(\frac{w^* \cdot v}{f(v)^{1/s}} \right)^{s/s-1} \qquad \forall w^* \in \mathcal{C}^*.$$

We define what it means for $f \in \mathrm{HConc}_s(\mathcal{C})$ to have a "Zariski decomposition structure" and show that it follows from the differentiability of $\mathcal{H}f$; see Section 6.4. This is the analogue in our situation of how the Legendre-Fenchel transform relates differentiability and strict convexity. Furthermore, this structure allows one to systematically transform geometric inequalities from one setting to the other. Many of the basic geometric inequalities in algebraic geometry – and hence for polytopes or convex bodies via toric varieties (as in [Tei82] and [Kho89] and the references therein) – can be understood in this framework.

Finally, we discuss some connections with other areas of birational geometry.

An important ancillary goal of the paper is to prove some new results concerning the volume function of divisors and the movable cone of curves. The key tool is another intersection-theoretic invariant $\mathfrak{M}$ of nef curve classes from [Xia15a, Definition 2.2]. Since the results seem likely to be of independent interest, we recall some of them here.

First of all, we give a refined version of a theorem of [BDPP13] describing the movable cone of curves. In [BDPP13], it is proved that the movable cone $\mathrm{Mov}_1(X)$ is *generated* by $(n-1)$-self positive products of big divisors. We show that the interior points in $\mathrm{Mov}_1(X)$ are *exactly* the set of $(n-1)$-self positive products of big divisors on the interior of $\mathrm{Mov}^1(X)$.

Theorem 2.0.76. *Let X be a smooth projective variety of dimension n and let α be an interior point of* $\mathrm{Mov}_1(X)$. *Then there is a unique big movable divisor class L_α lying in the interior of* $\mathrm{Mov}^1(X)$ *and depending continuously on α such that $\langle L_\alpha^{n-1}\rangle = \alpha$.*

Example 2.0.77. This result shows that the map $\langle -^{n-1}\rangle$ is a homeomorphism from the interior of the movable cone of divisors to the interior of the movable cone of curves. Thus, any chamber decomposition of the movable cone of curves naturally induces a decomposition of the movable cone of divisors and vice versa. This relationship could be useful in the study of geometric stability conditions (as in [Neu10]).

As an interesting corollary, we obtain :

Corollary 2.0.78. *Let X be a projective variety of dimension n. Then the rays over classes of irreducible curves which deform to dominate X are dense in* $\mathrm{Mov}_1(X)$.

We can describe the boundary of $\mathrm{Mov}_1(X)$.

Theorem 2.0.79. *Let X be a smooth projective variety and let α be a curve class lying on the boundary of* $\mathrm{Mov}_1(X)$. *Then exactly one of the following alternatives holds :*
– $\alpha = \langle L^{n-1}\rangle$ *for a big movable divisor class L on the boundary of* $\mathrm{Mov}^1(X)$.
– $\alpha \cdot M = 0$ *for a movable divisor class M.*
The homeomorphism given by $\langle -^{n-1}\rangle$, from $\mathrm{Mov}^1(X)^\circ \to \mathrm{Mov}_1(X)^\circ$, *extends to map the big movable divisor classes on the boundary of* $\mathrm{Mov}^1(X)$ *bijectively to the classes of the first type.*

We also extend [BFJ09, Theorem D] to a wider class of divisors.

Theorem 2.0.80. *Let X be a smooth projective variety of dimension n. For any two big divisor classes L_1, L_2, we have*

$$\mathrm{vol}(L_1 + L_2)^{1/n} \geq \mathrm{vol}(L_1)^{1/n} + \mathrm{vol}(L_2)^{1/n}$$

with equality if and only if the (numerical) positive parts $P_\sigma(L_1)$, $P_\sigma(L_2)$ are proportional. Thus the function $L \mapsto \mathrm{vol}(L)^{1/n}$ is strictly concave on the cone of big and movable divisors.

A basic technique in birational geometry is to bound the positivity of a divisor using its intersections against specified curves. These results can profitably be reinterpreted using the volume function of curves. For example :

Proposition 2.0.81. *(= Proposition 6.1.22) Let X be a smooth projective variety of dimension n. Choose positive integers $\{k_i\}_{i=1}^r$. Suppose that $\alpha \in \mathrm{Mov}_1(X)$ is represented by a family of irreducible curves such that for any collection of general points $x_1, x_2, \ldots, x_r, y$ of X, there is a curve in our family which contains y and contains each x_i with multiplicity $\geq k_i$. Then*

$$\widehat{\mathrm{vol}}(\alpha)^{n-1/n} \geq \frac{\sum_i k_i}{r^{1/n}}.$$

We can thus apply volumes of curves to study Seshadri constants, bounds on volume of divisors, and other related topics. We defer a more in-depth discussion to Section 6.11, contenting ourselves with a fascinating example.

Example 2.0.82. If X is rationally connected, it is interesting to analyze the possible volumes for classes of special rational curves on X. When X is a Fano variety of Picard rank 1, these invariants will be closely related to classical invariants such as the length and degree.

For example, we say that $\alpha \in N_1(X)$ is a rationally connecting class if for any two general points of X there is a chain of rational curves of class α connecting the two points. Is there a uniform upper bound (depending only on the dimension) for the minimal volume of a rationally connecting class on a rationally connected X ? [KMM92] and [Cam92] show that this is true for smooth Fano varieties. We discuss this question briefly in Section 6.11.2.

Chapitre 3

Teissier's proportionality problem over compact Kähler manifolds

We first solve Teissier's proportionality problem for transcendental nef classes over a compact Kähler manifold, which says that the equality in the Khovanskii-Teissier inequalities holds for a pair of big and nef classes if and only if the two classes are proportional. This result recovers one result of Boucksom-Favre-Jonsson [BFJ09] for the case of big and nef line bundles over a (complex) projective algebraic manifold.

We then consider a natural map from the Kähler cone of a compact Kähler manifold to its balanced cone. As an immediate corollary of the main theorem, we show its injectivity. We also study its surjectivity. By similar arguments as our proof of Teissier's proportionality theorem, using non-Kähler metrics, we give an analytic characterization on a nef class being Kähler.

3.1 Introduction

Around the year 1979, inspired by the Aleksandrov-Fenchel inequalities in convex geometry, Khovanskii and Teissier discovered independently deep inequalities in algebraic geometry which now is called Khovanskii-Teissier inequalities. These inequalities present a nice relationship between the theory of mixed volumes and algebraic geometry. Their proofs are based on the usual Hodge-Riemmann bilinear relations. A natural problem is how to characterize the equality case in these inequalities for a pair of big and nef line bundles, which was first considered by Bernard Teissier [Tei82, Tei88].

In their nice paper [BFJ09], besides other results, Boucksom, Favre and Jonsson solved this problem and the answer is that the equality holds if and only if two line bundles are (numerically) proportional. In their paper, they proved an algebro-geometric version of the Diskant inequality in convex geometry following the same strategy of Diskant which is based on the differentiability of the volume function of convex bodies. To obtain their Diskant inequality, they developed an algebraic construction of the positive intersection products of pseudo-effective classes and used them to prove that the volume function on the big cone of a projective variety is $\mathcal{C}^1$-differentiable, expressing its differential as a positive intersection product. Note that their results hold on any complete algebraic variety over an algebraically closed field of characteristic zero. Later, Cutkosky [Cut13] extended these remarkable results to a complete variety over an arbitrary field.

On the other hand, the Khovanskii-Teissier inequalities for transcendental nef classes follow from either the work [Dem93, DP03], [DN06] or [Gro90]. So a natural question is how to characterize the equality case in this situation. We give the same answer of this question as in the algebro-geometric case.

In [BFJ09] and [Cut13], a key ingredient, in the proof of the differentiability theorem of the volume of big line bundles over a projective variety, and thus in the proof of the algebro-geometric version of the Diskant inequality, is the algebraic Morse inequality

$$\mathrm{vol}(A-B) \geq A^n - nA^{n-1}\cdot B$$

for any nef line bundles A and B. Hence, if one would like to use their methods to extend the results to transcendental classes, the main missing part is the weak transcendentally holomorphic Morse inequality. However, up to now, it is not completely solved yet (see [Xia13, Pop14]). In this part, without using the transcendental version of Diskant inequality, we can still solve Teissier's proportionality problem for transcendental classes. Thus, our result covers the previous one of Boucksom-Favre-Jonsson. Indeed, the key idea in the proof of our main result has been hidden in our previous work [FX14a], and we will present it in details.

Theorem 3.1.1. *Assume X is an n-dimensional compact Kähler manifold. Let $\alpha, \beta \in \overline{\mathcal{K}} \cap \mathcal{E}^\circ$ be two big and nef classes. Denote $s_k := \alpha^k \cdot \beta^{n-k}$. Then the following statements are equivalent :*

1. $s_k^2 = s_{k-1}\cdot s_{k+1}$ *for* $1 \leq k \leq n-1$;
2. $s_k^n = s_0^{n-k}\cdot s_n^k$ *for* $0 \leq k \leq n$;
3. $s_{n-1}^n = s_0 \cdot s_n^{n-1}$;
4. $\mathrm{vol}(\alpha+\beta)^{1/n} = \mathrm{vol}(\alpha)^{1/n} + \mathrm{vol}(\beta)^{1/n}$;
5. α *and* β *are proportional ;*
6. α^{n-1} *and* β^{n-1} *are proportional.*

As a consequence, the map $\gamma \mapsto \gamma^{n-1}$ is injective from the big and nef cone $\overline{\mathcal{K}} \cap \mathcal{E}^\circ$ to the movable cone $\overline{\mathcal{M}}$.

For its applications to non-Kähler geometry – the balanced cone – over compact Kähler manifolds and more other results on balanced metrics, we leave them in Section 3.3.

3.2 Proof of the main theorem

Let us first recall the definition of nefness and bigness for $(1,1)$-classes on a compact Kähler manifold.

Assume that X is an n-dimensional compact Kähler manifold with a Kähler metric ω. Let $\alpha \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ be a $(1,1)$ Bott-Chern class. Then α is called *nef* if for any $\varepsilon > 0$, there exists a smooth representative $\alpha_\varepsilon \in \alpha$ such that $\alpha_\varepsilon > -\varepsilon\omega$. This definition is equivalent to say that, α belongs to the closure of the Kähler cone of X which is denoted as $\overline{\mathcal{K}}$. And α is called *big* if there exist a positive number δ and a positive current $T \in \alpha$ such that $T > \delta\omega$ (such a current T is called a *Kähler current*). This is equivalent to say that, α belongs to the interior of pseudo-effective cone which is denoted as $\mathcal{E}^\circ$. For more notions, such as the movable cone $\overline{\mathcal{M}}$ in the following theorem, one can see e.g. [BDPP13].

Let α be a big class. Recall that the ample locus $\mathrm{Amp}(\alpha)$ is the set of points $x \in X$ such that, there is a Kähler current $T_x \in \alpha$ with analytic singularities which is smooth near x. Indeed, by [Bou04] there exists a Kähler current T_α with analytic singularities such that the complement of $\mathrm{Amp}(\alpha)$ is exactly the singularities of T_α. This implies that $\mathrm{Amp}(\alpha)$ must be a Zariski open set of X.

Theorem 3.1.1

Now we give the proof of Theorem 3.1.1.

Proof. For a projective algebraic manifold, the usual Khovanskii-Teissier inequalities imply

$$s_k^2 \geq s_{k-1} \cdot s_{k+1} \quad \text{for } 1 \leq k \leq n-1, \tag{3.1}$$

if α and β are two nef divisor classes. We remark that it also holds if α and β are two transcendental nef classes on a compact Kähler manifold (see e.g. [Dem93, Section 5], [DP03, Proposition 2.5] or [DN06, Theorem A and Theorem C]).

For the reader's convenience, we include a proof here (see also [Cao13, Proposition 6.2.1]), which follows from the result on mixed Hodge-Riemann bilinear relations for compact Kähler manifolds [DN06] as in the projective algebraic manifold situation. To this end, let $\omega_1, ..., \omega_{n-2}$ be $n-2$ Kähler classes on X. Consider the following quadratic form Q on $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$:

$$Q(\lambda,\mu) := \int_X \lambda \wedge \mu \wedge \omega_1 \wedge ... \wedge \omega_{n-2}.$$

According to [DN06, Theorem A and Theorem C], Q is of signature $(1, h^{1,1})$. For any α, $\beta \in \overline{\mathcal{K}}$ and $t \in \mathbb{R}$, consider the function

$$Q(\alpha + t\beta, \alpha + t\beta).$$

As a function of t, we claim that $Q(\alpha + t\beta, \alpha + t\beta) = 0$ has at least a real solution. We only need to consider the case when α and β are linearly independent and thus, α and β span a 2-dimensional subspace of $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$. In view of the signature of Q, it can not be positive on this 2-dimensional subspace. Now our claim follows from this easily. The existence of real solutions is equivalent to

$$\Big(\int_X \alpha \wedge \beta \wedge \omega_1 \wedge ... \wedge \omega_{n-2}\Big)^2 \geq \Big(\int_X \alpha^2 \wedge \omega_1 \wedge ... \wedge \omega_{n-2}\Big) \cdot \Big(\int_X \beta^2 \wedge \omega_1 \wedge ... \wedge \omega_{n-2}\Big). \tag{3.2}$$

Since $\omega_1, ..., \omega_{n-2}$ are arbitrary, choosing appropriate ω_i and then taking limits, we obtain the inequalities (3.1) for any two transcendental nef classes.

We commence to prove the main result. Indeed, it is easy to see the equivalences of (1)-(4) (see e.g. [Cut13]). Since α, $\beta \in \overline{\mathcal{K}} \cap \mathcal{E}^\circ$, it is clear that $s_k > 0$ for $0 \leq k \leq n$.

We first prove the equivalence (1) $\Leftrightarrow$ (3). It is trivial that (1) implies (3). On the other hand, the inequalities (3.1) imply

$$\begin{aligned}\frac{s_{n-1}}{s_0} &= \frac{s_{n-1}}{s_{n-2}} \cdot \frac{s_{n-2}}{s_{n-3}} \cdot ... \cdot \frac{s_1}{s_0} \\ &\geq \frac{s_{n-1}}{s_{n-2}} \cdot \frac{s_{n-2}}{s_{n-3}} \cdot ... \cdot \Big(\frac{s_2}{s_1}\Big)^2 \geq \cdots \geq \Big(\frac{s_n}{s_{n-1}}\Big)^{n-1}.\end{aligned}$$

Thus if (3) holds, then all the above inequalities must be equalities, and hence (1) holds.

Next let us prove (1) $\Leftrightarrow$ (2). This also follows from (3.1). By (3.1) we have

$$\left(\frac{s_k}{s_{k-1}}\right)^{n-k}\cdot\ldots\cdot\left(\frac{s_1}{s_0}\right)^{n-k}\geq\left(\frac{s_n}{s_{n-1}}\right)^{k}\cdot\ldots\cdot\left(\frac{s_{k+1}}{s_k}\right)^{k}, \tag{3.3}$$

which clearly implies the equivalence (1) $\Leftrightarrow$ (2).

Now we prove (2) $\Leftrightarrow$ (4). The inequality (3.3) can be rewritten as

$$s_k^n \geq s_0^{n-k}\cdot s_n^k \quad \text{for} \quad 0\leq k\leq n\ .$$

These inequalities yield

$$\begin{aligned}\mathrm{vol}(\alpha+\beta)=(\alpha+\beta)^n&=\sum_k\frac{n!}{k!(n-k)!}s_k\\&\geq\sum_k\frac{n!}{k!(n-k)!}s_0^{n-k/n}\cdot s_n^{k/n}\\&=\left(\mathrm{vol}(\alpha)^{1/n}+\mathrm{vol}(\beta)^{1/n}\right)^n,\end{aligned}$$

which clearly implies (2) $\Leftrightarrow$ (4).

Next we prove the equivalence (3) $\Leftrightarrow$ (5). The implication (5) $\Rightarrow$ (3) is trivial. The real difficulty is to prove the implication (3) $\Rightarrow$ (5). The proof is inspired by our previous work [FX14a]. To make the proof of the implication (3) $\Rightarrow$ (5) clear, let us first give a sketch of the proof. Without loss of generality, we assume $\mathrm{vol}(\alpha)=\mathrm{vol}(\beta)$ in the following, then we need to prove the classes α,β are equal. If (3) holds, we will construct two equal positive $(1,1)$-currents in α and β respectively. To prove this, we first construct two positive $(1,1)$-currents in α and β respectively, which are equal on a Zariski open set. The construction heavily depends on the main theorem in [BEGZ10], which solves Monge-Ampère equations in big cohomology classes. Then, by the support theorem of currents, the difference of these two currents can only be a combination of some prime divisors. By showing that all the coefficients in the combination vanish, we deduce that these two currents are equal. Hence this implies (5). All is all, the key elements in the proof of (3) $\Rightarrow$ (5) are to solve Monge-Ampère equations in big and nef cohomology classes and to use some basic facts in pluripotential theory. In the following, we will carry out the details.

For simplicity, we will use the same symbol α (resp. β) to denote a smooth representation in the cohomology class α (resp. β). Fix a Kähler metric ω and a smooth volume form Φ with

$$\int_X\Phi=1.$$

As a starting point, to see how the above ideas work, we first give a proof of the implication (3) $\Rightarrow$ (5) when both α and β are Kähler classes. In this simple case, we can construct two equal Kähler metrics easily. By [Yau78], we can solve the following two Monge-Ampère equations :

$$\begin{aligned}(\alpha+i\partial\bar\partial\varphi)^n&=c_\alpha\Phi,\\(\beta+i\partial\bar\partial\psi)^n&=c_\beta\Phi,\end{aligned}$$

where $\alpha_\varphi:=\alpha+i\partial\bar\partial\varphi$ and $\beta_\psi:=\beta+i\partial\bar\partial\psi$ are two Kähler metrics, and $c_\alpha=c_\beta$ by our assumption $\mathrm{vol}(\alpha)=\mathrm{vol}(\beta)$. We claim that the assumption (3) implies $\alpha_\varphi=\beta_\psi$. To this end, let us consider the two $(n-1,n-1)$-forms α_φ^{n-1} and β_ψ^{n-1}. We write

$$\alpha_\varphi^{n-1}=\beta_\psi^{n-1}+\Theta$$

for some $(n-1,n-1)$-form Θ. If we can prove $\Theta=0$, then we get $\alpha_\varphi=\beta_\psi$. We consider the $(1,1)$-form or $(n-1,n-1)$-form as a matrix, then we have the equality

$$\frac{\det \alpha_\varphi^{n-1}}{\det \beta_\psi^{n-1}}=\frac{\det(\beta_\psi^{n-1}+\Theta)}{\det \beta_\psi^{n-1}}=\left(\frac{\det \alpha_\varphi}{\det \beta_\psi}\right)^{n-1}=1.$$

Note that we have the following elementary pointwise inequality

$$1=\left(\frac{\det(\beta_\psi^{n-1}+\Theta)}{\det \beta_\psi^{n-1}}\right)^{1/n}\leq 1+\frac{1}{n}\sum_{ij}(\beta_\psi^{n-1})^{i\bar{j}}\Theta_{i\bar{j}}, \tag{3.4}$$

where $[(\beta_\psi^{n-1})^{i\bar{j}}]$ is the inverse of the matrix β_ψ^{n-1}. After multiplying both sides of (3.4) by β_ψ^n, it is easy to see (3.4) is equivalent to

$$\beta_\psi^n\leq \beta_\psi^n+\beta_\psi\wedge\Theta.$$

Thus $\beta_\psi\wedge\Theta$ is a positive (n,n)-form on X. By the assumption (3), we have

$$\int_X \beta_\psi\wedge\Theta=\int_X \beta_\psi\wedge(\alpha_\varphi^{n-1}-\beta_\psi^{n-1})=0.$$

So the inequality (3.4) must be an equality everywhere, which forces $\Theta=0$. This finishes the proof when α,β are Kähler classes.

Next we begin the proof when α and β are big and nef classes. Comparing with the proof of the Kähler classes case, we need to solve a family of Monge-Ampère equations and analysis the behaviour of the family of solutions.

By [BEGZ10, Theorem C], we can solve the following two degenerate complex Monge-Ampère equations :

$$\left\langle(\alpha+i\partial\bar{\partial}\varphi)^n\right\rangle=c_{\alpha,0}\Phi, \tag{3.5}$$

$$\left\langle(\beta+i\partial\bar{\partial}\psi)^n\right\rangle=c_{\beta,0}\Phi, \tag{3.6}$$

where $\langle - \rangle$ denotes the non-pluripolar products of positive currents, and

$$c_{\alpha,0}=\mathrm{vol}(\alpha)=\mathrm{vol}(\beta)=c_{\beta,0}.$$

Moreover, φ (resp. ψ) has minimal singularities and is smooth on the ample locus $\mathrm{Amp}(\alpha)$ (resp. $\mathrm{Amp}(\beta)$), which is a Zariski open set of X depending only on the cohomology class of α (resp. β).

Let us first briefly recall how the solutions φ and ψ are obtained, which is needed in our proof. Indeed, based on Yau's seminal work [Yau78] on the Calabi conjecture, the above two degenerate complex Monge-Ampère equations can be solved by approximations. By Yau's theorem, for $0<t<1$, we can solve the following two families of Monge-Ampère equations :

$$(\alpha+t\omega+i\partial\bar{\partial}\varphi_t)^n=c_{\alpha,t}\Phi, \tag{3.7}$$

$$(\beta+t\omega+i\partial\bar{\partial}\psi_t)^n=c_{\beta,t}\Phi, \tag{3.8}$$

where $c_{\alpha,t}=\int_X(\alpha+t\omega)^n$, $c_{\beta,t}=\int_X(\beta+t\omega)^n$ and $\sup_X\varphi_t=\sup_X\psi_t=0$. Denote

$$\alpha_t:=\alpha+t\omega+i\partial\bar{\partial}\varphi_t \quad \text{and} \quad \beta_t:=\beta+t\omega+i\partial\bar{\partial}\psi_t\ .$$

We consider the limits of α_t and β_t as t tends to zero. By the basic properties of plurisubharmonic functions, the family of solutions $\{\varphi_t\}_t$ (resp. $\{\psi_t\}_t$) is compact in $L^1(X)$-topology since

$$\sup_X\varphi_t=\sup_X\psi_t=0.$$

Thus there exists a convergent subsequence which we still denote by the same symbol $\{\varphi_t\}_t$ (resp. $\{\psi_t\}_t$), and there exists an α-psh function φ (resp. a β-psh function ψ) such that, when t tends to zero, we have the following limits *in the sense of currents* on X :

$$\alpha_t \to \alpha + i\partial\bar{\partial}\varphi, \tag{3.9}$$

and

$$\beta_t \to \beta + i\partial\bar{\partial}\psi. \tag{3.10}$$

Moreover, by the theory developed in [BEGZ10] and the basic estimates in [Yau78], φ_t (resp. ψ_t) is compact in $C^\infty_{\mathrm{loc}}(\mathrm{Amp}(\alpha))$ (resp. $C^\infty_{\mathrm{loc}}(\mathrm{Amp}(\beta))$). Therefore there exist convergent subsequences such that the convergence (3.9) and the convergence (3.10) are *in the topology of* $C^\infty_{\mathrm{loc}}(\mathrm{Amp}(\alpha))$ *and* $C^\infty_{\mathrm{loc}}(\mathrm{Amp}(\beta))$. Hence φ (resp. ψ) is smooth on $\mathrm{Amp}(\alpha)$ (resp. $\mathrm{Amp}(\beta)$) respectively. Moreover, since Φ is a smooth volume form, $\alpha + i\partial\bar{\partial}\varphi$ (resp. $\beta + i\partial\bar{\partial}\psi$) must be a Kähler metric on $\mathrm{Amp}(\alpha)$ (resp. $\mathrm{Amp}(\beta)$).

Denote the Zariski open set $\mathrm{Amp}(\alpha) \cap \mathrm{Amp}(\beta)$ by $\mathrm{Amp}(\alpha, \beta)$, and denote

$$\begin{aligned} \alpha_0 &:= \alpha + i\partial\bar{\partial}\varphi, \\ \beta_0 &:= \beta + i\partial\bar{\partial}\psi. \end{aligned}$$

We first show that $\alpha_0 = \beta_0$ on $\mathrm{Amp}(\alpha, \beta)$. Let $c_t = c_{\alpha,t}/c_{\beta,t}$. By our assumption $\mathrm{vol}(\alpha) = \mathrm{vol}(\beta)$, it is clear that

$$\lim_{t\to 0} c_t = 1 \ . \tag{3.11}$$

Assume $\alpha^{n-1} = \beta^{n-1} + \Theta(\alpha, \beta)$ for some smooth $(n-1, n-1)$-form $\Theta(\alpha, \beta)$. Then

$$\alpha_t^{n-1} = \beta_t^{n-1} + \Theta_t \tag{3.12}$$

for some smooth $(n-1, n-1)$-form Θ_t. Pointwisely, α_t, β_t, α_t^{n-1}, β_t^{n-1} and Θ_t can be viewed as matrixes. In this sense, we have

$$\frac{\det \alpha_t^{n-1}}{\det \beta_t^{n-1}} = \left(\frac{\det \alpha_t}{\det \beta_t}\right)^{n-1} . \tag{3.13}$$

Hence, as in the Kähler classes case, we have

$$\begin{aligned} c_t^{n-1/n} &= \left(\frac{\det \alpha_t^{n-1}}{\det \beta_t^{n-1}}\right)^{1/n} = \left(\frac{\det(\beta_t^{n-1} + \Theta_t)}{\det \beta_t^{n-1}}\right)^{1/n} \\ &\le 1 + \frac{1}{n}\sum_{ij} (\beta_t^{n-1})^{i\bar{j}} (\Theta_t)_{i\bar{j}}, \end{aligned} \tag{3.14}$$

where the matrix $[(\beta_t^{n-1})^{i\bar{j}}]$ is the inverse of β_t^{n-1}. Equivalently, multiplying both sides of (3.14) by β_t^n, we get

$$c_t^{n-1/n} \beta_t^n \le \beta_t^n + \beta_t \wedge \Theta_t. \tag{3.15}$$

Note that we have

$$\beta_t \wedge \Theta_t = \alpha_t^{n-1} \wedge \beta_t - \beta_t^n.$$

Consider $\{\alpha_t^{n-1} \wedge \beta_t\}_t$ (resp. $\{\beta_t^n\}_t$) as a family of positive measures, then it is of bounded mass. Thus there exist convergent subsequences, which we still denote by $\{\alpha_t^{n-1} \wedge \beta_t\}_t$ and $\{\beta_t^n\}_t$, and positive measures μ_1 and μ_2 such that

$$\alpha_t^{n-1} \wedge \beta_t \to \mu_1, \tag{3.16}$$

$$\beta_t^n \to \mu_2 \tag{3.17}$$

in the sense of measures. If denote $\mu = \mu_1 - \mu_2$, then we get

$$\beta_t \wedge \Theta_t \to \mu.$$

We claim that μ is a zero measure. It is not hard to see from (3.11) and (3.15) that μ is a positive measure on X. Indeed, let f be any positive continuous function over X, we have

$$\begin{aligned}\int_X f\mu &= \lim_{t\to 0}\int_X f(\beta_t \wedge \Theta_t)\\ &\geq \liminf_{t\to 0}\int_X f(c_t^{n-1/n}\beta_t^n - \beta_t^n) = 0.\end{aligned}$$

Meanwhile, the assumption (3) implies

$$\begin{aligned}\int_X \mu &= \lim_{t\to 0}\int_X (\beta_t \wedge \alpha_t^{n-1} - \beta_t^n)\\ &= \int_X (\beta \wedge \alpha^{n-1} - \beta^n) = 0.\end{aligned}$$

Hence μ must be a zero measure. In particular, since $\mathrm{Amp}(\alpha, \beta)$ is a Zariski open set (thus a Borel measurable set), we have

$$\beta_t \wedge \Theta_t \to 0 \tag{3.18}$$

in the sense of measures on $\mathrm{Amp}(\alpha, \beta)$.

Using the convergence (3.9) and (3.10) in the topology of $C^\infty_{\mathrm{loc}}(\mathrm{Amp}(\alpha))$ and $C^\infty_{\mathrm{loc}}(\mathrm{Amp}(\beta))$, it is clear that there exists some smooth form Θ_0, which is only defined on $\mathrm{Amp}(\alpha, \beta)$, such that

$$\Theta_t \to \Theta_0$$

in the topology of $C^\infty_{\mathrm{loc}}(\mathrm{Amp}(\alpha, \beta))$. This implies that in the same topology

$$\beta_t \wedge \Theta_t \to \beta_0 \wedge \Theta_0\ . \tag{3.19}$$

Combining (3.18) and (3.19), and using uniqueness of the limit, we obtain

$$\beta_0 \wedge \Theta_0 = 0 \tag{3.20}$$

on $\mathrm{Amp}(\alpha, \beta)$. The above equality (3.20) implies that, if we take the limits on $\mathrm{Amp}(\alpha, \beta)$ of both sides of (3.14), we have

$$1 = \Big(\frac{\det \alpha_0^{n-1}}{\det \beta_0^{n-1}}\Big)^{\frac{1}{n}} = \Big(\frac{\det(\beta_0^{n-1} + \Theta_0)}{\det \beta_0^{n-1}}\Big)^{\frac{1}{n}} \tag{3.21}$$

$$\leq 1 + \frac{1}{n}\sum_{i,j}(\beta_0^{n-1})^{i\bar{j}}(\Theta_0)_{i\bar{j}} \tag{3.22}$$

$$= 1. \tag{3.23}$$

This forces $\Theta_0 = 0$, and hence $\alpha_0^{n-1} = \beta_0^{n-1}$ on $\mathrm{Amp}(\alpha, \beta)$. Since α_0 and β_0 are Kähler metrics, we have $\alpha_0 = \beta_0$ on $\mathrm{Amp}(\alpha, \beta)$.

We claim $\alpha_0 = \beta_0$ on X. Before going on, we need the following two lemmas.

Lemma 3.2.1. *(see [Dem12b, Page 142-143]) Let T be a d-closed (p, p)-current. Suppose that the support of T (i.e. $\mathrm{supp}T$) is contained in an analytic subset A. If $\dim A < n - p$, then $T = 0$; if T is of order zero and A is of pure dimension $n - p$ with $(n - p)$-dimensional irreducible components $A_1, ..., A_k$, then $T = \sum c_j[A_j]$ with $c_j \in \mathbb{C}$.*

Lemma 3.2.2. *(see [Bou04, Proposition 3.2 and Proposition 3.6]) Let α be a big and nef class, and let $T_{\min}$ be a positive current in α with minimal singularities. Then the Lelong number $\nu(T_{\min}, x) = 0$ for any point $x \in X$.*

It is clear that $S := X \setminus \mathrm{Amp}(\alpha, \beta)$ is a proper analytic subset of X. Let $T = \alpha_0 - \beta_0$, then T is a real d-closed (1,1)-current and supp$T \subset S$. If codim$S \geq 2$, then $T = 0$ according to Lemma 3.2.1. This implies $\alpha_0 = \beta_0$ on X. If codim$S = 1$ and S has only irreducible components $D_1, \cdots, D_k$ of pure dimension one, then Lemma 3.2.1 implies

$$\alpha_0 - \beta_0 = \sum c_j [D_j].$$

If codim$S = 1$ and S has also components of codimension more than one, we just repeat the proof of Lemma 3.2.1 as in [Dem12b], and still get

$$\alpha_0 - \beta_0 = \sum c_j [D_j].$$

Since α_0 and β_0 are real, all c_j can be chosen to be real numbers. If there exists at least one $c_j > 0$, we write this equality as

$$\alpha_0 - \sum c_{j'} [D_{j'}] = \beta_0 + \sum c_{j''} [D_{j''}] \tag{3.24}$$

with $c_{j'} \leq 0$ and $c_{j''} > 0$.

Fix a j'' which we denote as j''_0. We take a generic point $x \in D_{j''_0}$, for example, we can take such a point x with $\nu([D_{j''_0}], x) = 1$ and $x \notin \bigcup_{j \neq j''_0} D_j$. Then taking the Lelong number at the point x on both sides of (3.24), we have

$$\nu(\alpha_0, x) - \sum c_{j'} \nu([D_{j'}], x) = \nu(\beta_0, x) + \sum c_{j''} \nu([D_{j''}], x).$$

Since α_0 and β_0 are positive currents with minimal singularities in big and nef classes, Lemma 3.2.2 tells us that $\nu(\alpha_0, x) = 0$ and $\nu(\beta_0, x) = 0$. The property of x also implies $\nu([D_{j'}], x) = 0$ and $\nu([D_{j''}], x) = 0$ for all j' and all $j'' \neq j''_0$. All these force $c_{j''_0} = 0$, which contradicts to our assumption $c_{j''_0} > 0$. Thus we have

$$\alpha_0 - \sum c_{j'} [D_{j'}] = \beta_0.$$

By the same reason, we can also prove $c_{j'} = 0$. Hence we finish the proof of $\alpha_0 = \beta_0$ over X, and the proof of the implication (3) $\Rightarrow$ (5).

The implication (5) $\Rightarrow$ (6) is trivial. For the implication of (6) $\Rightarrow$ (3), suppose $\alpha^{n-1} = c\beta^{n-1}$ for some $c > 0$, then we have

$$\alpha^n = c\beta^{n-1} \cdot \alpha \geq c(\beta^n)^{n-1/n} (\alpha^n)^{1/n},$$

and

$$\alpha^{n-1} \cdot \beta = c\beta^n \geq (\alpha^n)^{n-1/n} (\beta^n)^{1/n}.$$

This yields $(\alpha^n)^{n-1/n} = c(\beta^n)^{n-1/n}$. And as a consequence, we get

$$\alpha^{n-1} \cdot \beta = c\beta^n = (\alpha^n)^{n-1/n} (\beta^n)^{1/n},$$

which is just (3). By the implication (3) $\Rightarrow$ (5), we then have the equivalence (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (3).

Summarizing all the above arguments, we have finished the proof of the equivalence of the statements (1) – (6). And as a consequence, we get that the map $\gamma \mapsto \gamma^{n-1}$ is injective from the big and nef cone $\overline{\mathcal{K}} \cap \mathcal{E}^\circ$ to the movable cone $\overline{\mathcal{M}}$. $\square$

3.3 Applications to the balanced cone

3.3.1 Background and main applications

A balanced metric on a complex n-dimensional manifold is a Hermitian metric such that its associated fundamental form ω satisfies $d(\omega^{n-1}) = 0$. We also call such ω a balanced metric. It is easy to see that the existence of a balanced metric ω is equivalent to the existence of a d-closed strictly positive $(n-1, n-1)$-form Ω such that $\Omega = \omega^{n-1}$ (see [Mic82]). Hence, for convenience, such Ω will also be called a balanced metric.

Before going on, we fixed some notations in this subsection. In the following, as we will deal with several different cohomology groups, we denote a cohomology class by $[\cdot]$. We also use some subscripts, e.g. $[\cdot]_{bc}$ or $[\cdot]_a$, to indicate the corresponding cohomology groups.

Assume that X is a compact complex manifold, recall that the (real) (p, p)-th Bott-Chern cohomology group of X is defined as

$$H^{p,p}_{\mathrm{BC}}(X, \mathbb{R}) = \{\text{real } d\text{-closed } (p, p)\text{-forms}\}/i\partial\bar{\partial}\{\text{real } (p-1, p-1)\text{-forms}\}.$$

We remark that Bott-Chern cohomology groups can be also defined by currents. Its elements will be denoted by $[\cdot]_{bc}$. It is easy to see that the cohomology classes of all balanced metrics as real $(n-1, n-1)$-forms form an open convex cone in $H^{n-1,n-1}_{\mathrm{BC}}(X, \mathbb{R})$. And we denote it by

$$\mathcal{B} \quad = \quad \{[\Omega]_{bc} \in H^{n-1,n-1}_{\mathrm{BC}}(X, \mathbb{R}) | \Omega \text{ is a balanced metric}\}.$$

It is called the *balanced cone* of X. Note that the zero cohomology class may be in $\mathcal{B}$. For example, Fu-Li-Yau [FLY12] constructed a balanced metric ω on the connected sum Y of $k(\geq 2)$ copies of $S^3 \times S^3$. Since $H^{2,2}_{\mathrm{BC}}(Y, \mathbb{R}) = 0$, $[\omega^2]_{bc} = 0 \in \mathcal{B}$. Clearly if 0 is in $\mathcal{B}$, then $\mathcal{B} = H^{n-1,n-1}_{\mathrm{BC}}(X, \mathbb{R})$. However, if X is a compact Kähler manifold, then 0 can not be in $\mathcal{B}$.

Now we assume that X is a compact Kähler manifold. In this case, by the $\partial\bar{\partial}$-lemma, it is well known that $H^{p,p}_{\mathrm{BC}}(X, \mathbb{R})$ is the same as the cohomology group $H^{p,p}_{dR}(X, \mathbb{R})$, the set of the de Rham classes represented by a real d-closed (p, p)-form, see e.g. [Voi07]. (The cohomology group $H^{p,p}_{dR}(X, \mathbb{R})$ is also usually denoted by $H^{p,p}(X, \mathbb{R})$.) Recall that the Kähler cone $\mathcal{K}$ of X is defined to be

$$\mathcal{K} \quad = \quad \{[\omega] \in H^{1,1}_{dR}(X, \mathbb{R}) | \omega \text{ is a Kähler metric}\}.$$

It is an open convex cone in $H^{1,1}_{dR}(X, \mathbb{R})$. The Kähler cone was studied thoroughly by Demailly and Paun in [DP04]. Since on a Kähler surface, the balanced cone and the Kähler cone coincide by their definitions. In the following, we will always assume that $n \geq 3$.

The balanced cone $\mathcal{B}$ of a compact Kähler manifold is related to its movable cone $\mathcal{M}$. In [Tom10], Toma observed that every movable curve on a projective manifold can be represented by a balanced metric under the conjectural cone duality $\mathcal{E}^* = \overline{\mathcal{M}}$ which is [BDPP13, Conjecture 2.3]. Indeed, Toma's result holds for all movable classes on compact Kähler manifolds. And along the lines of [Tom10], one can obtain the equivalence of $\mathcal{B}$ and $\mathcal{M}$ under $\mathcal{E}^* = \overline{\mathcal{M}}$ (see the appendix).

Motivated by the papers [FWW10, FWW15], we consider the map

$$\mathbf{b} : \mathcal{K} \to \mathcal{B}$$

which maps $[\omega]$ to $[\omega^{n-1}]$. Clearly it is well-defined and can be extended to the map

$$\overline{\mathbf{b}} : \overline{\mathcal{K}} \to \overline{\mathcal{B}},$$

where $\overline{\mathcal{K}}$ and $\overline{\mathcal{B}}$ are the closures of the corresponding cones. We want to study the properties $\overline{\mathbf{b}}$.

Let α be a smooth d-closed $(1, 1)$-form. Recall that, in the Kähler case, a cohomology class $[\alpha] \in H^{1,1}_{dR}(X, \mathbb{R})$ is nef if $[\alpha] \in \overline{\mathcal{K}}$. And $[\alpha]$ is called big if $[\alpha]$ contains a Kähler current. Indeed, for compact Kähler manifolds, Demailly and Paun [DP04] proved that a nef class $[\alpha]$ is big if and only if $\int_X \alpha^n > 0$. An immediate consequence of Theorem 3.1.1 implies the injectivity of $\overline{\mathbf{b}}$.

Theorem 3.3.1. *Let X be an n-dimensional compact Kähler manifold. Then the map $\overline{\mathbf{b}}$ is injective when $\overline{\mathbf{b}}$ is restricted to the subcone generated by nef and big classes.*

Remark 3.3.2. The bigness condition is necessary for injectivity. For example, the complex torus T^n with $n \geq 3$ shows that the bigness condition in the above Theorem 3.3.1 can not be omitted.

In general $\mathbf{b}$ is not surjective. It is interesting to study the image of the boundary $\partial\mathcal{K}$. In fact, We will show that $\overline{\mathbf{b}}(\partial\mathcal{K}) \cap \mathcal{B}$ need not to be empty. Let $\mathcal{K}_{\mathrm{NS}} = \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}$ where $\mathrm{NS}_{\mathbb{R}}$ is the real Neron-Severi group of X, that is,

$$\mathrm{NS}_{\mathbb{R}} = \left(H^{1,1}_{\mathrm{BC}}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})_{\mathrm{free}}\right) \otimes_{\mathbb{Z}} \mathbb{R}.$$

If X is a projective Calabi-Yau manifold – a projective manifold with $c_1 = 0$, then we can characterize when a nef class $[\alpha] \in \partial\mathcal{K}_{\mathrm{NS}}$ can be mapped into $\mathcal{B}$ by the map $\overline{\mathbf{b}}$. In fact, inspired by the method in [Tom10] – which goes back to [Sul76], we can give some sufficient conditions when a d-closed non-negative $(n-1, n-1)$-form is a balanced class. Then applying these criterions to [Tos09, Proposition 4.1], we get

Theorem 3.3.3. *Let X be a projective Calabi-Yau manifold. If $[\alpha] \in \partial\mathcal{K}$, then $\overline{\mathbf{b}}([\alpha]) \in \mathcal{B}$ implies $[\alpha]$ is a big class. On the other hand, if $[\alpha] \in \partial\mathcal{K}_{\mathrm{NS}}$ is a big class, then $\overline{\mathbf{b}}([\alpha]) \in \mathcal{B}$ if and only if the exceptional set $\mathrm{Exc}(F_{[\alpha]})$ of the contraction map $F_{[\alpha]}$ induced by $[\alpha]$ is of codimension more than one, or equivalently, $F_{[\alpha]}$ is a small contraction.*

Remark 3.3.4. For the contraction map in Theorem 3.3.3, if $[\alpha] \in \mathrm{NS}(X)_{\mathbb{Q}}$, then the map $F_{[\alpha]}$ is induced by the usual Kodaira map ; for general $[\alpha] \in \mathrm{NS}(X)_{\mathbb{R}}$, the map $F_{[\alpha]}$ is described in [Tos09] (see Lemma 3.3.14).

We will give some examples satisfying the conditions in the above theorem, which then show that the balanced cone can be bigger than the image of the Kähler cone under the map $\mathbf{b}$. We believe that it will be very interesting if one can clearly describe $\overline{\mathbf{b}}(\overline{\mathcal{K}}) \cap \mathcal{B}$ on compact Kähler manifolds.

By a similar method as in the proof of Theorem 3.1.1 and using balanced metrics, we can characterize when a nef class is Kähler.

Theorem 3.3.5. *Let X be a compact n-dimensional Kähler manifold and let η be a smooth volume form of X. Assume that $[\alpha]$ is a nef class such that $[\alpha^{n-1}]$ is a balanced class. If there exists a balanced metric $\tilde{\omega}$ in $[\alpha^{n-1}]$ such that $c_{\tilde{\omega}} \geq c_\alpha$ where*

$$c_{\tilde{\omega}} = \min_X \frac{\tilde{\omega}^n}{\eta}, \qquad c_\alpha = \frac{\int_X \alpha^n}{\int_X \eta},$$

then $[\alpha]$ is a Kähler class.

3.3.2 Injectivity : application of Theorem 3.1.1

By Theorem 3.1.1, it is clear that the map $\overline{\mathbf{b}} : \overline{\mathcal{K}} \to \overline{\mathcal{B}},\ [\alpha] \mapsto [\alpha^{n-1}]$ is injective when restricted to the subcone generated by nef and big classes.

3.3.3 Surjectivity : the image of the boundary of Kähler cone

In this section, we study the sujectivity of $\mathbf{b}$, focusing on the image of the boundary of the Kähler cone under the map $\overline{\mathbf{b}}$.

Sometimes it is convenient to consider the Aeppli cohomology groups $H^{p,q}_A(X, \mathbb{C})$. Since we are interested in the real case, we give the following definition.

Definition 3.3.6. If we denote by $\mathcal{D}^{p,p}_{\mathbb{R}}(X)$ the space of the smooth $\mathbb{R}$-valued (p,p)-forms, then

$$H^{p,p}_A(X, \mathbb{R}) := \{\phi \in \mathcal{D}^{p,p}_{\mathbb{R}}(X) | \partial\bar{\partial}\phi = 0\} / \{\partial\mathcal{D}^{p-1,p}(X) + \bar{\partial}\mathcal{D}^{p,p-1}(X)\} \cap \mathcal{D}^{p,p}_{\mathbb{R}}(X).$$

We denote the space of (p,q)-currents by $\mathcal{D}'^{p,q}(X)$. Then it is well known that we can also replace $\mathcal{D}^{p,q}$ by $\mathcal{D}'^{p,q}$ in the above definition. We denote an element of the Aeppli cohomology groups by $[\cdot]_a$.

We need the following lemma due to Bigolin [Big69].

Lemma 3.3.7. *Let X be a compact complex n-dimensional manifold. The dual space of the (p,p)-th Aeppli group is just the $(n-p,n-p)$-th Bott-Chern group.*

In particular, $H_A^{p,p}(X,\mathbb{R})$ is a finite dimensional vector space. The following lemma is inspired by the method in [Sul76] (see also [Tom10]), which is an easy consequence of the Hahn-Banach theorem.

Lemma 3.3.8. *Let X be a compact complex n-dimensional manifold. Suppose that Ω_0 is a real d-closed $(n-1,n-1)$-form satisfying that, for any positive $\partial\bar{\partial}$-closed $(1,1)$-current T we have*

$$\int_X \Omega_0 \wedge T \geq 0$$

and

$$\int_X \Omega_0 \wedge T = 0$$

if and only if $T=0$, then $[\Omega_0]_{bc}$ is a balanced class.

Proof. Fix a Hermitian metric ω on X. We define the following two subsets of $\mathcal{D}'^{1,1}_{\mathbb{R}}(X)$:

$$D_1 = \{T \in \mathcal{D}'^{1,1}_{\mathbb{R}}(X) | \partial\bar{\partial}T = 0, \int_X \Omega_0 \wedge T = 0\},$$
$$D_2 = \{T \in \mathcal{D}'^{1,1}_{\mathbb{R}}(X) | T \geq 0, \int_X \omega^{n-1} \wedge T = 1\}.$$

Then D_1 is a closed subspace and D_2 is a compact convex subset under the weak topology of currents.

Since Ω_0 is d-closed, D_1 contains the subset $\{\partial\bar{S} + \bar{\partial}S | S \in \mathcal{D}'^{1,0}(X,\mathbb{C})\}$. It is clear that $D_1 \cap D_2$ is empty. By the Hahn-Banach theorem, there exists a smooth real $(n-1,n-1)$-form Ω such that

$$\Omega|_{D_1} = 0, \qquad \Omega|_{D_2} > 0$$

Note that $\Omega|_{D_1} = 0$ and $D_1 \supseteq \{\partial\bar{S} + \bar{\partial}S | S \in \mathcal{D}'^{1,0}(X,\mathbb{C})\}$ imply $d\Omega = 0$, and $\Omega|_{D_2} > 0$ implies Ω is strictly positive. Hence Ω is a balanced metric.

On the other hand, Lemma 3.3.7 says that $[\Omega_0]_{bc}$ and $[\Omega]_{bc}$ are linear functionals on $H_A^{1,1}(X,\mathbb{R})$. We have a natural quotient map

$$\pi : \{T \in \mathcal{D}'^{1,1}_{\mathbb{R}}(X) | \partial\bar{\partial}T = 0\} \to H_A^{1,1}(X,\mathbb{R})$$

with $\pi(T) = [T]_a$. Then the definition of D_1 implies $\pi(D_1) = \ker[\Omega_0]_{bc}$, and $\Omega|_{D_1} = 0$ implies $\pi(D_1) \subseteq \ker[\Omega]_{bc}$. Thus we have

$$\ker[\Omega_0]_{bc} \subseteq \ker[\Omega]_{bc} \subseteq H_A^{1,1}(X,\mathbb{R}).$$

If $\ker[\Omega]_{bc}$ is the whole Aeppli group, then $[\Omega]_{bc}$=0. Since X is compact, there exists an $\varepsilon > 0$ small enough such that $\Omega + \varepsilon\Omega_0 > 0$, i.e., $[\Omega] + \varepsilon[\Omega_0] = \varepsilon[\Omega_0]$ is balanced. If $\ker[\Omega]_{bc}$ is a proper subspace, since $H_A^{1,1}(X,\mathbb{R})$ is a finite dimensional vector space, we must have $\ker[\Omega_0]_{bc} = \ker[\Omega]_{bc}$. Hence there exists some constant c such that $[\Omega_0]_{bc} = c[\Omega]_{bc}$. In this case, if there exists some non-trivial positive $\partial\bar{\partial}$-closed $(1,1)$-current T, we get the constant c must be positive, and this implies $[\Omega_0]_{bc}$ is balanced. Otherwise, if there are no non-trivial positive $\partial\bar{\partial}$-closed $(1,1)$-currents, then the zero class $[0]_{bc}$ satisfies our assumption in the lemma and we can repeat our procedure above. We can use the the zero class $[0]_{bc}$ to define the space D_1, and get $[0]_{bc}$ is a balanced class. This implies every class in $H_{\mathrm{BC}}^{n-1,n-1}(X,\mathbb{R})$ is balanced. Then we finish the proof of Lemma 3.3.8. $\square$

Remark 3.3.9. Let X be a compact balanced manifold. If we denote $\mathcal{E}_{dd^c} \subseteq H_A^{1,1}(X, \mathbb{R})$ the convex cone generated by dd^c-closed positive $(1,1)$-currents, then the above lemma just implies the cone duality $\mathcal{E}_{dd^c}^* = \overline{\mathcal{B}}$.

The above lemma implies the following two interesting propositions. Let Ω_0 be a semi-positive $(n-1, n-1)$-form on X and strictly positive on $X \setminus V$ for a subvariety V of X. If $\operatorname{codim} V > 1$, we first recall [AB92, Theorem 1.1].

Lemma 3.3.10. *Let X be a complex n-dimensional manifold. Assume T is a $\partial\bar{\partial}$-closed positive (p,p)-current on X such that the Hausdorff $2(n-p)$-measure of* $\operatorname{supp} T$ *vanishes. Then $T = 0$.*

Proposition 3.3.11. *Suppose X is a compact complex n-dimensional manifold. If Ω_0 is a d-closed semi-positive $(n-1, n-1)$-form on X and is strictly positive outside a subvariety V with* $\operatorname{codim} V > 1$, *then $[\Omega_0]_{bc}$ is a balanced class.*

Proof. Fix a $\partial\bar{\partial}$-closed positive $(1,1)$-current T. Then $\Omega_0 \geq 0$ implies $\int_X \Omega_0 \wedge T \geq 0$. And $\Omega_0 > 0$ on $X \setminus V$ implies that $\int_X \Omega_0 \wedge T = 0$ if and only if $\operatorname{supp} T \subset V$. Hence according to the above lemma, since T is $\partial\bar{\partial}$-closed and $\operatorname{codim} V > 1$, we have $T = 0$. Thus Ω_0 satisfies the conditions of Lemma 3.3.8 and therefore $[\Omega_0]_{bc}$ a balanced class. $\square$

If $\operatorname{codim} V = 1$ and $[\Omega_0]_{bc}$ is a balanced class, then we have $\int_V \Omega_0 > 0$. We prove that is also a sufficient condition when Ω_0 is semi-positive on X and is strictly positive on $X \setminus V$. We need [AB92, Theorem 1.5].

Lemma 3.3.12. *Let X be a complex n-dimensional manifold and let E a compact analytic subset. Let $E_1, ..., E_k$ be the irreducible p-dimensional components of E. Assume T is a positive $\partial\bar{\partial}$-closed $(n-p, n-p)$-current such that* $\operatorname{supp} T \subset E$. *Then there exist constants $c_j \geq 0$ such that $T - \sum_1^k c_j [E_j]$ is a positive $\partial\bar{\partial}$-closed $(n-p, n-p)$-current on X, supported on the union of the irreducible components of E of* $\dim > p$.

Then we have

Proposition 3.3.13. *Let X be a compact complex n-dimensional manifold. If Ω_0 is a d-closed semi-positive $(n-1, n-1)$-form on X such that it is strictly positive outside a codimension one subvariety V with irreducible components $E_1, ..., E_k$ and $[\Omega_0]_{bc} \cdot [E_j]_a > 0$ for $j = 1, ..., k$, then $[\Omega_0]_{bc}$ is a balanced class.*

Proof. Since Ω_0 is a semi-positive form on X, for any $\partial\bar{\partial}$-closed positive $(1,1)$-current T on X we have $\int_X \Omega_0 \wedge T \geq 0$. And $\int_X \Omega_0 \wedge T = 0$ implies $\operatorname{supp} T \subset V$. We need to prove $T = 0$. By the above lemma, there exist constants $c_j \geq 0$ such that

$$T \;=\; \sum_{j=1}^{k} c_j [E_j].$$

Hence $[\Omega_0]_{bc} \cdot [T]_a = 0$ implies that, if $[\Omega_0]_{bc} \cdot [E_j]_a > 0$ then the constants c_j must be zero. This implies $T = 0$. Thus by Lemma 3.3.8, $[\Omega_0]_{bc}$ is a balanced class. $\square$

Now we apply Proposition 3.3.11 and Proposition 3.3.13 to a nef and big class on a projective Calabi-Yau manifold. We need the following lemma given by Tosatti [Tos09].

Lemma 3.3.14. *Let X be a projective Calabi-Yau n-dimensional manifold and let $[\alpha] \in \partial\mathcal{K}_{\mathrm{NS}}$ be a big class. Then there exists a smooth form $\alpha_0 \in [\alpha]$ which is nonnegative and strictly positive outside a proper subvariety of X.*

For reader's convenience, we present some details how [Tos09] proved the above lemma. Firstly, we assume $\alpha = c_1(L)$ for some holomorphic line bundle L, which means that α lies in the space $\mathrm{NS}(X)_{\mathbb{Z}}$. Now L is nef and big, and the base point free theorem implies that L is semiample, so there exists some positive integer k such that kL is globally generated. This gives a holomorphic map

$$F_{[\alpha]} : X \to \mathbb{P}(H^0(X, \mathcal{O}(kL))^*)$$

such that $F_{[\alpha]}^* \mathcal{O}(1) = kL$. If $\alpha \in \mathrm{NS}(X)_{\mathbb{Q}}$, then $k\alpha \in \mathrm{NS}(X)_{\mathbb{Z}}$ for some positive integer k, and we can also define a holomorphic map $F_{[\alpha]}$ similarly. Finally, if $\alpha \in \mathrm{NS}(X)_{\mathbb{R}}$ then by [Kaw88, Theorem 5.7] or [Kaw97, Theorem 1.9] we know that the subcone of nef and big classes is locally rational polyhedral. Hence α lies on a face of this cone which is cut out by linear equations with rational coefficients. It follows that rational points on this face are dense, and it is then possible to write α as a linear combination of classes in $\mathrm{NS}(X)_{\mathbb{Q}}$ which are nef and big, with nonnegative coefficients. Notice that all of these classes give the same contraction map, because they lie on the same face. We then also denote this map by $F_{[\alpha]}$.

Recall that the exceptional set $\mathrm{Exc}(F_{[\alpha]})$ is defined to be the complement of points where $F_{[\alpha]}$ is a local isomorphism. It is now clear that we can represent α by a smooth nonnegative form which is the pull back of Fubini-Study metric (modulo a positive constant). And it is strictly positive outside the exceptional set $\mathrm{Exc}(F_{[\alpha]})$.

In birational geometry (see e.g. [KMM87]), we call $F_{[\alpha]}$ is a divisorial contraction if $\mathrm{Exc}(F_{[\alpha]})$ is of codimension 1 and a small contraction if the exceptional set $\mathrm{Exc}(F_{[\alpha]})$ is of codimension more than 1. We remark that if $\mathrm{Exc}(F_{[\alpha]})$ is of codimension 1, i.e., $F_{[\alpha]}$ is a divisorial contraction, then the image of $\mathrm{Exc}(F_{[\alpha]})$ under $F_{[\alpha]}$ is of dimension less than $n-1$. In our situation, X is smooth, thus under divisorial contractions, its image is Q-factorial and has only weak log-terminal singularities (see [KMM87, Proposition 5-1-6]). Thus, its image is Q-factorial and normal. Then the image of $\mathrm{Exc}(F_{[\alpha]})$ under $F_{[\alpha]}$ has codimension at least 2 (see e.g. [Deb01, page 28]). In this case, $[\alpha^{n-1}]$ can not be a balanced class. Indeed, if E_j is any codimension one component of $\mathrm{Exc}(F_{[\alpha]})$ then we must have $[\alpha^{n-1}] \cdot [E_j] = 0$. Write $\mathrm{Exc}(F_{[\alpha]}) = F \cup_j E_j$ where all irreducible components of F have codimension at least 2. For a fixed j and for any $p \in E_j \backslash (F \cup_{l \neq j} E_l)$, let $S = F_{[\alpha]}^{-1}(F_{[\alpha]}(p))$ be the fiber over $F_{[\alpha]}(p)$. Since the image of $F_{[\alpha]}$ is a normal variety, Zariski's Main Theorem shows that all irreducible components of S are positive-dimensional, so there is at least one such component $S' \subset E_j$ which contains p. Then α is a smooth semipositive form in the class $[\alpha]$ and $\alpha|_{S'} \equiv 0$ since S' is contained in a fiber of $F_{[\alpha]}$ and α is the pull back of Fubini-Study metric. But this means that $(\alpha|_{E_j})^{n-1}(p) = 0$, since $\alpha|_{E_j}$ has zero eigenvalues in all directions tangent to S, and since this is true for all p in a Zariski open subset of E_j, we conclude that $[\alpha^{n-1}] \cdot [E_j] = \int_{E_j} (\alpha|_{E_j})^{n-1} = 0$.

Now we can prove Theorem 3.3.3.

Proof. By Lemma 3.3.14, there exists a semipositive $(1,1)$-form $\alpha_0 \in [\alpha]$ such that α_0 is strictly positive outside a subvariety V. If V is of codimension bigger than one, then Proposition 3.3.11 implies $[\alpha^{n-1}] = [\alpha_0^{n-1}]$ is a balanced metric. If V is of codimension one with irreducible components $E_1, ..., E_k$, then $[\alpha^{n-1}] \cdot [E_j] = 0$ for all $1 \leq j \leq k$, thus $[\alpha^{n-1}] \notin \mathcal{B}$. On the other hand, the converse is obvious.

Next, let us prove $[\alpha^{n-1}] \in \mathcal{B}$ implies $[\alpha]$ is a big class. Otherwise, we would have $\int_X \alpha^n = 0$. Since $[\alpha]$ is nef, there exists a positive current $T \in [\alpha]$. Hence

$$\int_X \alpha^{n-1} \wedge T = \int_X \alpha^n = 0.$$

Then $[\alpha^{n-1}] \in \mathcal{B}$ implies $T = 0$. Thus $[\alpha] = [T] = 0$. It is a contradiction. □

We are going to give some examples where holomorphic maps contract high codimensional subvarieties to points. The first one is known as a conifold in the physics literature [GMS95] (see also [Ros06]). We knew this from [Tos09]. Let X_0 be a nodal quintic in $\mathbb{P}^4$ which has 16 nodal points. Then a smooth Calabi-Yau manifold X is given by a small resolution $f : X \to X_0$, that is a birational morphism such that it is an isomorphism outside the preimages of the nodes, which are 16 rational curves. Thus we get a contracting map from X to $\mathbb{P}^4$. It is easy to see that the pullback of the Fubini-Study metric is our desired form.

There are also other examples from algebraic geometry (see e.g. [Deb01, page 24-26]). Let r and s be positive integers, let E be the vector bundle on $\mathbb{P}^s$ associated with the locally free sheaf $\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1)^{r+1}$, and let $Y_{r,s}$ be the smooth $(r+s+1)$-dimensional variety $\mathbb{P}(E^*)$. The projection $\pi : Y_{r,s} \to \mathbb{P}^s$ has a section $P_{r,s}$ corresponding to the trivial quotient of E. The linear system $|\mathcal{O}_{Y_{r,s}}(1)|$ is base point free. Hence it induces a holomorphic map

$$C_{r,s} : Y_{r,s} \to \mathbb{P}^{(r+1)(s+1)}.$$

Moreover, $C_{r,s}$ contracts $P_{r,s}$ to a point and is an immersion on its complement. And its image is the cone over the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$.

Thus, the pull-back of the Fubini-Study metric of $\mathbb{P}^{(r+1)(s+1)}$ is a smooth $(1,1)$-form $\alpha = C_{r,s}^* \omega_{FS}$. Clearly α is pointwise nonnegative on the whole space $Y_{r,s}$ and is strictly positive outside $P_{r,s}$ of codimension $r+1$. Thus $[\alpha^{r+s}]$ is a balanced class on $\mathbb{P}(E^*)$. Furthermore, $\int_{P_{r,s}} \alpha^s = 0$ implies $\alpha \in \partial\mathcal{K}(Y_{r,s})$.

Indeed, there are a lot of such examples in the Minimal Model Program, encountered when dealing with contraction maps of flipping type ([KMM87]).

Remark 3.3.15 (V. Tosatti)**.** In order to produce more examples of birational contraction morphisms as in Lemma 3.3.14, one can take more general X to be any smooth projective variety with $-K_X$ nef. For example, this includes Calabi-Yau but also Fano manifolds. Under this assumption, if L is any line bundle on X which is nef and big, then Kawamata's base-point-free theorem again gives us that L is semi-ample and so there is a birational contraction F_L exactly as in Lemma 3.3.14.

It also works for $\mathbb{R}$-linear combinations of line bundles (i.e. big classes on the boundary of $\mathcal{K}_{\mathrm{NS}}$), because again the big points on the boundary of $\mathcal{K}_{\mathrm{NS}}$ are locally rational polyhedral (if X is Fano, then the whole boundary of $\mathcal{K}_{\mathrm{NS}}$ is rational polyhedral). Thus if X has nef anticanonical bundle, we can still apply Theorem 3.3.3.

Remark 3.3.16 (S. Boucksom)**.** Indeed, Theorem 3.3.3 can be generalized in the following way. Let X be a compact Kähler manifold of dimension n, and let α be a big and nef class. Then α^{n-1} is in the interior of the dual cone $\mathcal{E}^*$ if and only if the non-ample locus of α is of codim ≥ 2. We only need to verify the "only if" part : as α^{n-1} is in the interior of the dual cone $\mathcal{E}^*$, for any prime divisor D we have $\alpha^{n-1} \cdot D > 0$. By [CT13], it is clear that the non-ample locus of α is of codim ≥ 2.

3.3.4 Characterization on a nef class being Kähler

Using a similar method as in the proof of Theorem 3.1.1, we can characterize when a nef class $[\alpha]$ is Kähler under the assumption that $[\alpha^{n-1}]$ is a balanced class. As this result seems interesting in itself, we presents the details of its proof.

Theorem 3.3.17. *Let X be a compact n-dimensional Kähler manifold and let η be a smooth volume form of X. Assume that $[\alpha]$ is a nef class such that $[\alpha^{n-1}]$ is a balanced class. If there exists a balanced metric $\tilde{\omega}$ in $[\alpha^{n-1}]$ such that $c_{\tilde{\omega}} \geq c_\alpha$ where*

$$c_{\tilde{\omega}} = \min_X \frac{\tilde{\omega}^n}{\eta}, \qquad c_\alpha = \frac{\int_X \alpha^n}{\int_X \eta},$$

then $[\alpha]$ is a Kähler class.

Proof. Since $\tilde{\omega}^{n-1} \in [\alpha^{n-1}]$, there exists a smooth $(n-2,n-2)$-form ϕ such that

$$\tilde{\omega}^{n-1} = \alpha^{n-1} + i\partial\bar{\partial}\phi > 0.$$

Fix a Kähler metric ω on X. Then for $0 < t \ll 1$,

$$(\alpha + t\omega)^{n-1} + i\partial\bar{\partial}\phi = \tilde{\omega}^{n-1} + O(t) > 0.$$

Thus there exists a balanced metric $\tilde{\omega}_t$ such that

$$\tilde{\omega}_t^{n-1} = (\alpha + t\omega)^{n-1} + i\partial\bar{\partial}\phi \tag{3.25}$$

and $\tilde{\omega}_0 = \tilde{\omega}$. Clearly, as $t \to 0$, $\tilde{\omega}_t \to \tilde{\omega}$. Then if we let $F_{\tilde{\omega}_t} := \frac{\tilde{\omega}_t^n}{\eta}$, as $t \to 0$, we have

$$F_{\widetilde{\omega}_t} \to F_{\widetilde{\omega}}$$

in $C^\infty(X)$.

On the other hand, since $[\alpha + t\omega]$ is a Kähler class, by [Yau78] there exists a family of smooth functions u_t such that $\alpha + t\omega + i\partial\bar{\partial}u_t$ is Kähler and

$$(\alpha + t\omega + i\partial\bar{\partial}u_t)^n = c_t\eta, \qquad c_t = \frac{\int_X (\alpha + t\omega)^n}{\int_X \eta}.$$

Moreover, by [BEGZ10] there also exists an α-psh function u_0 such that

$$\left\langle (\alpha + i\partial\bar{\partial}u_0)^n \right\rangle = c_\alpha \eta.$$

Such u_t and u_0 satisfy the following relations

$$\alpha + t\omega + i\partial\bar{\partial}u_t \to \alpha + i\partial\bar{\partial}u_0 \quad \text{as currents on } X$$

and

$$\alpha + t\omega + i\partial\bar{\partial}u_t \to \alpha + i\partial\bar{\partial}u_0 \quad \text{in } C^\infty_{\mathrm{loc}}(\mathrm{Amp}(\alpha)). \tag{3.26}$$

We denote $\alpha_t = \alpha + t\omega + i\partial\bar{\partial}u_t$ and $\alpha_0 = \alpha + i\partial\bar{\partial}u_0$. Then from (3.25), we have

$$\tilde{\omega}_t^{n-1} = \alpha_t^{n-1} + i\partial\bar{\partial}\phi_t \tag{3.27}$$

for some smooth $(n-2, n-2)$-form ϕ_t on X.

By the above notations, we have

$$\frac{F_{\widetilde{\omega}_t}}{c_t} = \frac{\widetilde{\omega}_t^n}{\alpha_t^n}.$$

We apply the arithmetic-geometric mean inequality to obtain

$$\begin{aligned}\Big(\frac{F_{\widetilde{\omega}_t}}{c_t}\Big)^{\frac{n-1}{n}} =& \Big(\frac{\det(\alpha_t^{n-1} + i\partial\bar{\partial}\phi_t)}{\det \alpha_t^{n-1}}\Big)^{\frac{1}{n}} \\ \leq& 1 + \frac{1}{n}\sum_{k,l}(\alpha_t^{n-1})^{k\bar{l}}(i\partial\bar{\partial}\phi_t)_{k\bar{l}}.\end{aligned}$$

Equivalently, we have

$$\Big(\frac{F_{\widetilde{\omega}_t}}{c_t}\Big)^{\frac{n-1}{n}} \alpha_t^n \leq \alpha_t^n + \alpha_t \wedge i\partial\bar{\partial}\phi_t. \tag{3.28}$$

We deal with the second term in the above equality, namely

$$\alpha_t \wedge i\partial\bar{\partial}\phi_t = \alpha_t \wedge \tilde{\omega}_t^{n-1} - \alpha_t^n.$$

As discussed in the proof of Theorem 3.1.1, there exists a convergent subsequence $\alpha_{t_k}\wedge\tilde{\omega}_{t_k}^{n-1}$ of measures $\alpha_t\wedge\tilde{\omega}_t^{n-1}$ and a convergent sequence $\alpha_{t_k}^n$ of measures α_t^n. If we denote their limits by μ_1 and μ_2, and denote $\mu_0=\mu_1-\mu_2$, then we have

$$\alpha_{t_k}\wedge i\partial\bar{\partial}\phi_{t_k}\to\mu_0 \quad \text{as currents.}$$

Letting $t=t_k$ in (3.28), integrating with respect to any positive smooth function, and letting t_k go to zero, we find that the condition $c_{\tilde{\omega}}\geq c_0$ implies μ_0 is a positive measure.

Meanwhile, since

$$\begin{aligned}\int_X\mu_0 &= \lim_{t\to 0}\int_X \alpha_t\wedge\tilde{\omega}_t^{n-1}-\alpha_t^n\\ &= \int_X \alpha\wedge(\tilde{\omega}^{n-1}-\alpha^{n-1}),\end{aligned}$$

and since α is nef and $\tilde{\omega}^{n-1}\in[\alpha^{n-1}]$, we have $\int_X\mu_0=0$. Thus $\mu_0=0$ and $F_{\tilde{\omega}}=c_\alpha$ pointwise.

On $\mathrm{Amp}(\alpha)$, we define a smooth $(1,1)$-form

$$\Psi_0=\lim_{t\to 0} i\partial\bar{\partial}\phi_t.$$

Then from (3.27), (3.26) and (3.25), we have

$$\Psi_0=\lim_{t\to 0}(\tilde{\omega}_t^{n-1}-\alpha_t^{n-1})=\tilde{\omega}^{n-1}-\alpha_0^{n-1}.$$

Hence by uniqueness of the limit, we have on $\mathrm{Amp}(\alpha)$

$$\alpha_0\wedge\Psi_0=0.$$

Since $F_{\tilde{\omega}}=c_\alpha$, this implies that on $\mathrm{Amp}(\alpha)$

$$1=\Big(\frac{\det\tilde{\omega}^{n-1}}{\det\alpha_0^{n-1}}\Big)^{\frac{1}{n}}\leq 1+\frac{1}{n}\sum_{k,l}(\alpha_0^{n-1})^{k\bar{l}}(\Psi_0)_{k\bar{l}}=1.$$

Thus $\Psi_0=0$. Therefore $\tilde{\omega}^{n-1}=\alpha_0^{n-1}$ on $\mathrm{Amp}(\alpha)$, which implies $\tilde{\omega}=\alpha_0$ on $\mathrm{Amp}(\alpha)$.

Since $\tilde{\omega}$ is smooth on X and $d\tilde{\omega}=d\alpha_0=0$ on $\mathrm{Amp}(\alpha)$, by continuity $d\tilde{\omega}=0$ on X. Thus $\tilde{\omega}$ is a Kähler metric on X. By Theorem 3.1.1 and $[\tilde{\omega}^{n-1}]=[\alpha^{n-1}]$, we get $[\tilde{\omega}]=[\alpha]$. Thus $[\alpha]$ must be a Kähler calss. □

3.3.5 Appendix

In this subsection, we show that the conjectured cone duality $\mathcal{E}^*=\overline{\mathcal{M}}$ in [BDPP13] implies that the movable cone $\mathcal{M}$ coincides with the balanced cone $\mathcal{B}$. In [Tom10], Toma observed that every movable curve on a projective manifold can be represented by a balanced metric under $\mathcal{E}^*=\overline{\mathcal{M}}$. We observe that Toma's result holds for all movable classes on compact Kähler manifolds. As we shall see the proof is along the lines of [Tom10] and his arguments carry over mutatis mutandis.

Theorem 3.3.18. *Let X be an n-dimensional compact Kähler manifold, then $\mathcal{E}^*=\overline{\mathcal{M}}$ implies $\mathcal{M}=\mathcal{B}$*

Proof. As we have proved the cone duality $\mathcal{E}_{dd^c}^*=\overline{\mathcal{B}}$. Combining $\mathcal{E}^*=\overline{\mathcal{M}}$, we only need to show every dd^c-closed positive current $T\in\mathcal{E}_{dd^c}$ can be modified to be a d-closed positive current. We consider the natural homomorphism

$$j: H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})\to H^{1,1}_A(X,\mathbb{R}).$$

Then j is well defined, moreover, it is an isomorphism. On one hand, $j([T]_{bc})=0$ implies $T=\partial\bar{S}+\bar{\partial}S$. Then $dT=0$ yields $\partial\bar{\partial}S=0$. Now, since X is Kähler, the $\partial\bar{\partial}$-lemma tells us that $\bar{\partial}S=\partial\bar{\partial}\psi$

for some ψ. So we must have $[T]_{bc} = 0$ in $H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$. On the other hand, we also have $\dim H^{1,1}_{\mathrm{BC}}(X, \mathbb{R}) = \dim H^{1,1}_{A}(X, \mathbb{R})$. Thus the injective linear map j must be an isomorphism.

When j is restricted on $\mathcal{E}$, we then also get an injective map (which we also denotes it by j)

$$j : \mathcal{E} \to \mathcal{E}_{dd^c}.$$

We want to show that this map j is surjective. For any $[T]_a \in \mathcal{E}_{dd^c}$ with T positive, there exists some current S such that $d(T + \partial\bar{S} + \bar{\partial}S) = 0$. We claim that the class $[T + \partial\bar{S} + \bar{\partial}S]_{bc}$ is pseudo-effective, i.e., $[T+\partial\bar{S}+\bar{\partial}S]_{bc} \in \mathcal{E}$. Here, we need a result of [AB95] : [AB95] guarantees that for any modification $\mu : \widetilde{X} \to X$ and any positive dd^c-closed $(1,1)$-current T on X, there exists an unique positive dd^c-closed $(1,1)$-current $\widetilde{T}$ on $\widetilde{X}$ such that $\mu_*\widetilde{T} = T$ and $\widetilde{T} \in \mu^*[T]_a$. Now, take a smooth $(1,1)$-form $\alpha \in [T + \partial\bar{S} + \bar{\partial}S]_{bc}$ (which will also be a representative of $[T]_a$), $\widetilde{T} \in \mu^*[T]_a$ implies that there exists some current $\widetilde{S}$ such that $\widetilde{T} = \mu^*\alpha + \partial\bar{\widetilde{S}} + \bar{\partial}\widetilde{S}$. Thus, for any modification $\mu : \widetilde{X} \to X$ with $\widetilde{X}$ being Kähler, we have

$$\begin{aligned}
\int_X \alpha \wedge \mu_*(\widetilde{\omega}_1 \wedge ... \wedge \widetilde{\omega}_{n-1}) &= \int_{\widetilde{X}} \mu^*\alpha \wedge \widetilde{\omega}_1 \wedge ... \wedge \widetilde{\omega}_{n-1} \\
&= \int_{\widetilde{X}} (\mu^*\alpha + \partial\bar{\widetilde{S}} + \bar{\partial}\widetilde{S}) \wedge \widetilde{\omega}_1 \wedge ... \wedge \widetilde{\omega}_{n-1} \\
&= \int_{\widetilde{X}} \widetilde{T} \wedge \widetilde{\omega}_1 \wedge ... \wedge \widetilde{\omega}_{n-1} \\
&\geq 0.
\end{aligned}$$

By the arbitrariness of μ and $\widetilde{\omega}_i$'s, the duality $\mathcal{E}^* = \overline{\mathcal{M}}$ indicates that $[T + \partial\bar{S} + \bar{\partial}S]_{bc} \in \mathcal{E}$. And this confirms the surjectivity of $j : \mathcal{E} \to \mathcal{E}_{dd^c}$.

Now, it is easy to see that $\mathcal{M} = \mathcal{B}$. On one hand, since any balanced metric takes positive values on $\mathcal{E}\backslash\{0\}$, $\mathcal{B}$ is obviously contained in the interior of $\mathcal{E}^*$, thus $\mathcal{B} \subseteq \mathcal{M}$. On the other hand, $j(\mathcal{E}) = \mathcal{E}_{dd^c}$ yields any movable class takes positive values on $\mathcal{E}_{dd^c}\backslash\{0\}$, thus $\mathcal{E}^*_{dd^c} = \overline{\mathcal{B}}$ implies $\mathcal{M} \subseteq \mathcal{B}$. Finally, we obtain $\mathcal{M} = \mathcal{B}$. □

Inspired by the above theorem, we naturally propose the following problem concerning the balanced cone of a general compact balanced manifold.

Conjecture 3.3.19. *Let X be a compact balanced manifold, then we have $\mathcal{E}^* = \overline{\mathcal{B}}$.*

Remark 3.3.20. If X is a compact complex surface, then balanced cone is just Kähler cone, so the above conjecture holds for compact Kähler surfaces. Moreover, [BDPP13] has observed that their conjectural cone duality is true for hyperkähler manifolds or Kähler manifolds which are the limits of projective manifolds with maximal Picard number under holomorphic deformations, thus our conjecture also holds in these cases.

Chapitre 4

Transcendental Morse inequalities over compact Kähler manifolds

By reconsidering the main ideas of [Chi13], we first prove a weak version of Demailly's conjecture on transcendental Morse inequalities on compact Kähler manifolds. As a consequence, we partially improve a result of [BDPP13]. With the recent improvement of [Pop14] and some basic properties of movable intersections, we generalize the main result of [Pop14] to pseudo-effective $(1, 1)$-classes. As an application, we give a Morse-type bigness criterion for movable $(n-1, n-1)$-classes.

Finally we apply these results to study the numerical characterization problem on the convergence of the inverse σ_k-flow, giving some partial positivity results towards the conjecture of Lejmi and Székelyhidi [LS15].

4.1 Introduction

There are many beautiful results on holomorphic Morse inequalities for rational cohomology classes of type $(1,1)$ over smooth projective varieties. For rational cohomology $(1,1)$-classes which are the first Chern classes of holomorphic $\mathbb{Q}$-line bundles, these inequalities are related to the holomorphic sections of line bundles. Demailly [Dem85] has applied his holomorphic Morse inequalities for rational cohomology classes to reprove a stronger statement of Grauert-Riemenschneider conjecture. Recently, these inequalities are also applied to the Green-Griffiths-Lang conjecture (see [Dem11b]).

However, if the cohomology class is not rational, which we also call *transcendental* class, we do not have holomorphic sections for these cohomology classes, it is hard to prove the associated holomorphic Morse inequalities. In the nice paper of Boucksom-Demailly-Paun-Peternell [BDPP13], the authors formulated the following conjecture on transcendental holomorphic Morse inequalities – which has been proposed by Demailly in his earlier works.

Conjecture 4.1.1. *(see [BDPP13, Conjecture 10.1]) Let X be an n-dimensional compact complex manifold.*

1. *Let θ be a real d-closed $(1,1)$-form representing the class α and let $X(\theta,\leq 1)$ be the set where θ has at most one negative eigenvalue. If $\int_{X(\theta,\leq 1)}\theta^n > 0$, then the Bott-Chern class α contains a Kähler current and*
$$\mathrm{vol}(\alpha) \geq \int_{X(\theta,\leq 1)} \theta^n.$$
2. *Let α and β be two nef $(1,1)$-classes on X satisfying $\alpha^n - n\alpha^{n-1}\cdot\beta > 0$. Then the Bott-Chern class $\alpha-\beta$ contains a Kähler current and*
$$\mathrm{vol}(\alpha-\beta) \geq \alpha^n - n\alpha^{n-1}\cdot\beta.$$

In this section, all the cohomology classes are in the Bott-Chern cohomology groups $H^{\bullet,\bullet}_{BC}(X,\mathbb{R})$. Note that Bott-Chern cohomology groups coincide with the usual cohomology groups $H^{\bullet,\bullet}(X,\mathbb{R})$ over compact Kähler manifolds. First let us recall some definitions about the positivity of $(1,1)$-forms on general compact complex manifolds. For convenience, we use the same symbols to denote the smooth forms representing the cohomology classes. Let X be a compact complex manifold, and fix a hermitian metric ω on X. A cohomology class $\alpha\in H^{1,1}_{BC}(X,\mathbb{R})$ is called a nef (numerically effective) class if for any $\varepsilon>0$, there exists a smooth function ψ_ε such that $\alpha+\varepsilon\omega+i\partial\bar\partial\psi_\varepsilon$ is strictly positive. And a cohomology class α is called pseudo-effective if there exists a positive current $T\in\alpha$. A positive $(1,1)$-current T is called a Kähler current if T is d-closed and $T>\delta\omega$ for some $\delta>0$. Then if α contains a Kähler current, we call α a big class. We remark that we can also define similar positivity for (k,k)-classes.

Remark 4.1.2. For any pseudo-effective $(1,1)$-class α, we can define its volume as following (see e.g. [Bou02a])
$$\mathrm{vol}(\alpha) := \sup_T \int_X T_{ac}^n,$$
where T ranges over all the positive currents in α and T_{ac} is the absolutely continuous part of T. Indeed, for holomorphic line bundle L over a compact Kähler manifold, the above analytical definition of volume coincides with its volume in algebraic geometry, i.e.,
$$\mathrm{vol}(L) = \limsup_{k\to\infty}\frac{n!}{k^n}h^0(X,kL).$$
And Conjecture 4.1.1 holds true for holomorphic line bundles (see [Dem85, Dem91]).

In their paper [BDPP13], the authors observed that in Conjecture 4.1.1, (1) implies (2). Thus we will call the second statement (2) weak transcendental holomorphic Morse inequalities. Indeed, the authors proved the following theorem.

Theorem 4.1.3. *(see [BDPP13, Theorem 10.4]) Let X be a projective manifold of dimension n. Then*

$$\mathrm{vol}(\omega - c_1(A)) \geq \omega^n - \frac{(n+1)^2}{4}\omega^{n-1}\cdot c_1(A)$$

holds for every Kähler class ω and every ample line bundle A on X, where $c_1(A)$ is the first Chern class of A. In particular, if $\omega^n - \frac{(n+1)^2}{4}\omega^{n-1}\cdot c_1(A) > 0$, then $\omega - c_1(A)$ contains a Kähler current.

We can improve the second part of Theorem 4.1.3 and get rid of the projective and rational conditions. For part (2) of Conjecture 4.1.1, we get some partial results for Kähler manifolds and even for some a priori non-Kähler manifolds. For general compact complex manifolds, we do not know how to prove the transcendental holomorphic Morse inequalities unless the underlying manifolds admit some special metrics.

Theorem 4.1.4. *Let X be an n-dimensional compact complex manifold with a hermitian metric ω satisfying $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2, ..., n-1$. Assume α, β are two nef classes on X satisfying*

$$\alpha^n - 4n\alpha^{n-1}\cdot\beta > 0,$$

then there exists a Kähler current in the Bott-Chern class $\alpha - \beta$.

Thus, our result covers the Kähler case and improves theorem 4.1.3 for n large enough. Moreover, the key point is that the cohomology classes α, β can be transcendental.

Remark 4.1.5. Indeed, when $n \leq 3$, we can slightly weaken the metric hypothesis. In this situation, a hermitian metric ω just satisfying $\partial\bar{\partial}\omega = 0$ is sufficient (see the Appendix).

For any n-dimensional compact complex manifold X, the existence of Gauduchon metric implies that there always exists a metric ω such that $\partial\bar{\partial}\omega^{n-1} = 0$. In particular, if $n = 2$, then there always exists a metric ω such that $\partial\bar{\partial}\omega = 0$. Thus Theorem 4.1.4 holds on any compact complex surfaces. And as a consequence, these compact complex surfaces must be Kähler surfaces. Indeed, this is already known thanks to the work of Buchdahl [Buc99, Buc00] and Lamari [Lam99a, Lam99b].

Remark 4.1.6. A priori, a compact complex manifold admitting a special hermitian metric described in Theorem 4.1.4 need not be Kählerian. However, I. Chiose [Chi13] has proved that if a compact complex manifold X admits a nef class with positive top self-intersection and a hermitian metric ω with $\partial\bar{\partial}\omega^k = 0$ for every k, then X must be a Kähler manifold. In our proof, we do not need this fact and we will prove Theorem 4.1.4 directly.

Now let X be a compact complex manifold in the Fujiki class $\mathcal{C}$, then there exists a proper modification $\mu : \widetilde{X} \to X$ such that $\widetilde{X}$ is Kähler. This yields the following direct corollary for compact complex manifolds in the Fujiki class $\mathcal{C}$.

Corollary 4.1.7. *Let X be a compact complex manifold in the Fujiki class $\mathcal{C}$ with $\dim X = n$. Assume α, β are two nef classes on X satisfying $\alpha^n - 4n\alpha^{n-1}\cdot\beta > 0$, then $\alpha - \beta$ contains a Kähler current.*

Indeed, the proof of our theorem is inspired by I. Chiose. In [Chi13, Section 3], the author cleverly applied a result of Lamari (see [Lam99a, Lemma 3.3]) on characterization of positive $(1,1)$-currents and the ideas on mass concentration of [DP04] to simplify the proof of the main theorem of Demailly-Paun. However, just as I. Chiose said, the proof of [Chi13] is not independent of the proof of Demailly-Paun. The paper [Chi13] replaced the explicit and involved construction of the metrics ω_ε in [DP04] by the abstract sequence of Gauduchon metrics given by the Hahn-Banach theorem, via Lamari's lemma. We remark that Lamari's lemma uses the technique introduced by Sullivan in [Sul76].

We find Chiose's method is useful to prove positivity of the difference of cohomology classes, at least in our case. Indeed, in addition to solving a different family of complex Monge-Ampère equations, our proof almost follows the argument of [Chi13]. However, our result seems not easily reachable by the mass concentration method.

Recently, by keeping the same method of [Xia13, Chi13] and with the new estimates in [Pop14], D. Popovici proved that the constant $4n$ in our Theorem 4.1.4 can be improved to be the natural and optimal constant n.

Theorem 4.1.8. *(see [Pop14]) Under the same conditions and notations of Theorem 4.1.4, if* $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$*, then there exists a Kähler current in the class* $\alpha - \beta$*.*

Thus we have a Morse-type criterion for the difference of two transcendental nef classes. It is natural to ask whether the above Morse-type bigness criterion "$\alpha^n - n\alpha^{n-1} \cdot \beta > 0 \Rightarrow \mathrm{vol}(\alpha - \beta) > 0$" for nef classes can be generalized to pseudo-effective $(1,1)$-classes. Towards this generalization, we need the movable intersection products (denoted by $\langle - \rangle$) of pseudo-effective $(1,1)$-classes (see e.g. [Bou02a, BDPP13]). Then our problem can be stated as following :

- *Let* X *be a compact Kähler manifold of dimension* n*, and let* $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ *be two pseudo-effective classes. Does* $\mathrm{vol}(\alpha) - n\langle \alpha^{n-1} \rangle \cdot \beta > 0$ *imply that there exists a Kähler current in the class* $\alpha - \beta$ *?*

Unfortunately, a very simple example due to [Tra95] implies that the above generalization does not always hold.

Example 4.1.9. (see [Tra95, Example 3.8]) Let $\pi : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ along a point p. Let $R = \pi^* H$, where H is the hyperplane line bundle on $\mathbb{P}^2$. Let $E = \pi^{-1}(p)$ be the exceptional divisor. Then for every positive integral k, the space of global holomorphic sections of $k(R - 2E)$ is the space of homogeneous polynomials in three variables of degree at most k and vanishes of order $2k$ at p ; hence $k(R - 2E)$ does not have any global holomorphic sections. The space $H^0(X, \mathcal{O}(k(R - 2E))) = \{0\}$ implies $R - 2E$ can not be big. However, we have $R^2 - R \cdot 2E > 0$ as $R^2 = 1$ and $R \cdot E = 0$.

However, we show that it holds if β is movable. Here β being movable means the negative part of β vanishes in its divisorial Zariski decomposition (see [Bou04]). In particular, if $\beta = c_1(L)$ for some pseudo-effective line bundle, then β being movable is equivalent to that the base locus of $mL + A$ is of codimension at least two for a fixed ample line bundle A and for large m.

Theorem 4.1.10. *Let* X *be a compact Kähler manifold of dimension* n*, and let* $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ *be two pseudo-effective classes with* β *movable. Then* $\mathrm{vol}(\alpha) - n\langle \alpha^{n-1} \rangle \cdot \beta > 0$ *implies that there exists a Kähler current in the class* $\alpha - \beta$*.*

Remark 4.1.11. In the case when $\beta = 0$, Theorem 4.1.10 recovers [Bou02b, Theorem 4.7], and when α is also nef, it is [DP04, Theorem 0.5].

An ancillary goal is to explain the fact that Demailly's conjecture on weak transcendental holomorphic Morse inequality over compact Kähler manifolds is equivalent to the $\mathcal{C}^1$ differentiability of the volume function for transcendental $(1,1)$-classes. Though not stated explicitly, this fact is already contained in [BFJ09] and the key ingredients are also implicitly contained in [BDPP13].

Proposition 4.1.12. *Let* X *be a compact Kähler manifold of dimension* n*. Then the following statements are equivalent :*

1. *Let* $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ *be two nef classes, then we have*
$$\mathrm{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$
2. *Let* $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ *be two pseudo-effective classes with* β *movable, then*
$$\mathrm{vol}(\alpha - \beta) \geq \langle \alpha^n \rangle - n\langle \alpha^{n-1} \rangle \cdot \beta.$$
3. *Let* $\alpha, \gamma \in H^{1,1}(X, \mathbb{R})$ *be two* $(1,1)$*-classes with* α *big, then we have*
$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(\alpha + t\gamma) = n\langle \alpha^{n-1} \rangle \cdot \gamma.$$

As an application of Proposition 4.1.12 and the $\mathcal{C}^1$ differentiability of the volume function for line bundles (see [BFJ09, Theorem A]), the algebraic Morse inequality can be generalized as following. It generalizes the previous result [Tra11, Corollary 3.2].

Theorem 4.1.13. *Let X be a smooth projective variety of dimension n, and let α, β be the first Chern classes of two pseudo-effective line bundles with β movable. Then we have*

$$\mathrm{vol}(\alpha - \beta) \geq \mathrm{vol}(\alpha) - n\langle \alpha^{n-1} \rangle \cdot \beta.$$

Remark 4.1.14. In particular, if α is nef and β is movable then we have $\mathrm{vol}(\alpha-\beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta$ which is just [Tra11, Corollary 3.2].

Finally, as an application of Theorem 4.1.10, we give a Morse-type bigness criterion for the difference of two movable $(n-1, n-1)$-classes.

Theorem 4.1.15. *Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes. Then $\mathrm{vol}(\alpha) - n\alpha \cdot \langle \beta^{n-1} \rangle > 0$ implies that there exists a strictly positive $(n-1, n-1)$-current in the class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$.*

Finally, as an application of the main results above, we give some positivity results towards the conjecture of [LS15] concerning the convergence of inverse σ_k-flow ; we leave the main results and applications in Section 4.4.

4.2 Technical preliminaries

4.2.1 Monge-Ampère equation on compact Hermitian manifolds

In order to apply the method of [DP04] on a general compact complex manifold which may be a priori non-Kähler, we need Tosatti-Weinkove's result on the solvability of complex Monge-Ampère equation on Hermitian manifolds.

Lemma 4.2.1. *(see [TW10, Corollary 1]) Let X be an n-dimensional compact complex manifold with a hermitian metric ω. Then for any smooth real-valued function F on X, there exist a unique real number b and a unique smooth real-valued function φ on X solving*

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^{F+b}\omega^n,$$

where $\omega + i\partial\bar{\partial}\varphi > 0$ and $\sup_X \varphi = 0$.

4.2.2 Lamari's lemma

Next we state [Lam99a, Lemma 3.3] on the characterization of positive currents. Lamari's result is stated for positive $(1,1)$-currents, and it also can be generalized for positive (k,k)-currents for any k – where the proof for general k is the same as [Lam99a]. And for the reader's convenience and the potential applications for the positivity of (k,k)-classes, we include Lamari's proof in the Appendix.

Lemma 4.2.2. *(see [Lam99a, Lemma 3.3]) Let X be an n-dimensional compact complex manifold and let Φ be a real (k,k)-form, then there exists a real $(k-1,k-1)$-current Ψ such that $\Phi + i\partial\bar{\partial}\Psi$ is positive if and only if for any strongly positive $\partial\bar{\partial}$-closed $(n-k, n-k)$-form Υ we have $\int_X \Phi \wedge \Upsilon \geq 0$.*

4.2.3 Resolution of singularities of positive currents

Let X be a compact complex manifold, and let T be a d-closed almost positive $(1,1)$-current on X, that is, there exists a smooth $(1,1)$-form γ such that $T \geq \gamma$. Demailly's regularization theorem (see [Dem92]) implies that we can always approximate the almost positive $(1,1)$-current T by a family of almost positive closed $(1,1)$-currents T_k with analytic singularities such that $T_k \geq \gamma - \varepsilon_k \omega$, where $\varepsilon_k \downarrow 0$ is a sequence of positive constants and ω is a fixed hermitian metric. In particular, when T is a Kähler current, it can be approximated by a family of Kähler currents with analytic singularities.

When T has analytic singularities along an analytic subvariety $V(\mathcal{I})$ where $\mathcal{I} \subset \mathcal{O}_X$ is a coherent ideal sheaf, by blowing up along $V(\mathcal{I})$ and then resolving the singularities, we get a modification

$\mu : \widetilde{X} \to X$ such that $\mu^* T = \widetilde{\theta} + [D]$ where $\widetilde{\theta}$ is an almost positive smooth $(1,1)$-form with $\widetilde{\theta} \geq \mu^* \gamma$ and D is an effective $\mathbb{R}$-divisor; see e.g. [BDPP13, Theorem 3.1]. In particular, if T is positive, then $\widetilde{\theta}$ is a smooth positive $(1,1)$-form. We call such a modification the log-resolution of singularities of T.

For almost positive $(1,1)$-current T, we can always decompose T with respect to the Lebesgue measure; see e.g. [Bou02b, Section 2.3]. We write $T = T_{ac} + T_{sg}$ where T_{ac} is the absolutely continuous part and T_{sg} is the singular part. The absolutely part T_{ac} can be seen as a form with L^1_{loc} coefficients, and the wedge product $T^k_{ac}(x)$ makes sense for almost every point x. We always have $T_{ac} \geq \gamma$ since γ is smooth. If T has analytic singularities along V, then $T_{ac} = \mathbf{1}_{X \setminus V} T$. However, in general T_{ac} is not closed even if T is closed. We have the following proposition.

Proposition 4.2.3. *Let $T_1, ..., T_k$ be k almost positive closed $(1,1)$-currents with analytic singularities on X and let ψ be a smooth $(n-k, n-k)$-form. Let $\mu : \widetilde{X} \to X$ be a simultaneous log-resolution with $\mu^* T_i = \widetilde{\theta}_i + [D_i]$. Then*

$$\int_X T_{1,ac} \wedge ... \wedge T_{k,ac} \wedge \psi = \int_{\widetilde{X}} \widetilde{\theta}_1 \wedge ... \wedge \widetilde{\theta}_k \wedge \mu^* \psi.$$

Proof. This is obvious since μ is an isomorphism outside a proper analytic subvariety and $T_{1,ac} \wedge ... \wedge T_{k,ac}$ puts no mass on such subset and $\widetilde{\theta}_i$ is smooth on $\widetilde{X}$. □

4.2.4 Movable cohomology classes

We first briefly recall the definition of divisorial Zariski decomposition and the definition of movable $(1,1)$-class on compact complex manifold; see [Bou04], see also [Nak04] for the algebraic approach.

Let X be a compact complex manifold of dimension n and let α be a pseudo-effective $(1,1)$-class over X, then one can always associate an effective divisor $N(\alpha) := \sum \nu(\alpha, D) D$ to α where the sum ranges among all prime divisors on X. The class $\{N(\alpha)\}$ is called the negative part of α. And $Z(\alpha) = \alpha - \{N(\alpha)\}$ is called the positive part of α. The decomposition $\alpha = Z(\alpha) + \{N(\alpha)\}$ then is the divisorial Zariski decomposition of α.

Definition 4.2.4. Let X be a compact complex manifold of dimension n, and let α be a pseudo-effective $(1,1)$-class. Then α is called movable if $\alpha = Z(\alpha)$.

Proposition 4.2.5. *(see [Bou04, Proposition 2.3]) Let α be a movable $(1,1)$-class and let ω be a Kähler class, then for any $\delta > 0$ there exist a modification $\mu : Y \to X$ and a Kähler class $\widetilde{\omega}$ over Y such that $\alpha + \delta\omega = \mu_* \widetilde{\omega}$.*

Remark 4.2.6. In [Bou04], α is called modified nef if $\alpha = Z(\alpha)$ (see [Bou04, Definition 2.2 and Proposition 3.8]). Here we call it movable in order to keep the same notation as the algebraic geometry situation. Let L be a big line bundle over a smooth projective variety and let $\alpha = c_1(L)$. Then α is modified nef if and only if L is movable, that is, its base locus is of codimension at least two.

Inspired by [BDPP13, Definition 1.3, Theorem 1.5 and Conjecture 2.3], the definition of movable $(n-1, n-1)$-classes in the Kähler setting can be formulated as following.

Definition 4.2.7. Let X be a compact Kähler manifold of dimension n, and let $\gamma \in H^{n-1,n-1}(X, \mathbb{R})$. Then γ is called a movable $(n-1, n-1)$-class if it is in the closure of the convex cone generated by cohomology classes of the form $\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle$ with every α_i pseudo-effective.

Remark 4.2.8. When X is a smooth projective variety of dimension n, [BDPP13, Theorem 1.5] implies that the rational movable $(n-1, n-1)$-classes are the same with the classes of movable curves.

4.2.5 Movable intersections

In this section, we take the opportunity to briefly explain the well known fact that the several definitions of movable intersections of pseudo-effective $(1,1)$-classes over compact Kähler manifold coincide ; see [Bou02a, BDPP13, BEGZ10] for the analytic constructions over compact Kähler manifold and [BFJ09] for the algebraic construction on smooth projective variety. We remark that it is helpful to know the definition of movable intersections of pseudo-effective $(1,1)$-classes can be interpreted in several equivalent ways. And one can see [Pri13, Proposition 1.10] for the detailed proof.

Let $\alpha_1, ..., \alpha_k \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes on a compact Kähler manifold of dimension n. By the common basic property of these definitions of movable intersection products, we only need to verify the respectively defined positive (k,k) cohomology classes $\langle \alpha_1 \cdot ... \cdot \alpha_k \rangle$ coincide when all the classes are big. Firstly, by the definition of Riemann-Zariski space, it is clear from [BDPP13, Theorem 3.5] and [BFJ09, Definition 2.5] that the two definitions of movable intersection products are the same for $k = 1$ or $k = n-1$ when X is a smooth projective variety defined over $\mathbb{C}$ and all the classes α_i are in the Néron-Severi space. Next, by testing on $\partial\bar{\partial}$-closed smooth positive $(n-k, n-k)$-forms, [BDPP13, Theorem 3.5], [BEGZ10, Definition 1.17, Proposition 1.18 and Proposition 1.20] and [Bou02a, Definition 3.2.1 and Lemma 3.2.5] imply these three definitions give the same positive cohomology class in $H^{k,k}(X, \mathbb{R})$.

4.3 Proof of the main results

4.3.1 Theorem 4.1.4

Now we first prove our Theorem 4.1.4 on weak transcendental Morse inequalities. Though the a priori non-Kähler manifolds satisfying the conditions in Theorem 4.1.4 are actually Kähler, we still hope Tosatti-Weinkove's Hermitian version of Calabi-Yau theorem could apply to general compact complex manifolds (with some new ideas). Therefore, we give the proof for the special possibly non-Kähler metrics described in the statement of Theorem 4.1.4.

Proof. Firstly, fix a special hermitian metric ω satisfying $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2, ..., n-1$. In the following, we use the same symbols to denote the smooth $(1,1)$-forms in the corresponding cohomology classes. Since α, β are nef classes, for any $\varepsilon > 0$, there exist smooth functions $\varphi_\varepsilon, \psi_\varepsilon$ such that $\alpha_\varepsilon := \alpha + \varepsilon\omega + i\partial\bar{\partial}\varphi_\varepsilon > 0$ and $\beta_\varepsilon := \beta + \varepsilon\omega + i\partial\bar{\partial}\psi_\varepsilon > 0$. There is no doubt we can always assume $\sup \varphi_\varepsilon = \sup \psi_\varepsilon = 0$. And we have $\alpha - \beta = \alpha_\varepsilon - \beta_\varepsilon$ as cohomology classes, thus $\alpha - \beta$ is a big class if and only if there exists a positive constant $\delta > 0$ and a $(\alpha_\varepsilon - \beta_\varepsilon)$-PSH function θ_δ, such that

$$\alpha_\varepsilon - \beta_\varepsilon + i\partial\bar{\partial}\theta_\delta \geq \delta\alpha_\varepsilon. \tag{4.1}$$

Now let us first fix ε. Then Lemma 4.2.2 implies (4.1) is equivalent to

$$\int_X (\alpha_\varepsilon - \beta_\varepsilon - \delta\alpha_\varepsilon) \wedge G \geq 0 \tag{4.2}$$

for any strictly positive $\partial\bar{\partial}$-closed $(n-1, n-1)$-form G. Then G is $(n-1)$-th power of a Gauduchon metric. Now (4.2) is equivalent to

$$\int_X (1-\delta)\alpha_\varepsilon \wedge G \geq \int_X \beta_\varepsilon \wedge G. \tag{4.3}$$

Thus the class $\alpha - \beta = \alpha_\varepsilon - \beta_\varepsilon$ is not big is equivalent to that, for any $\delta_m \downarrow 0$ there exists a Gauduchon metric $G_{m,\varepsilon}$ such that

$$\int_X (1-\delta_m)\alpha_\varepsilon \wedge G_{m,\varepsilon} < \int_X \beta_\varepsilon \wedge G_{m,\varepsilon}. \tag{4.4}$$

Without loss of generality, we can normalize $G_{m,\varepsilon}$ such that

$$\int_X \beta_\varepsilon \wedge G_{m,\varepsilon} = 1.$$

By the Calabi-Yau theorem on Hermitian manifold given by Lemma 4.2.1, we can solve the following family of Monge-Ampère equations

$$\widetilde{\alpha_\varepsilon}^n = (\alpha_\varepsilon + i\partial\bar{\partial}u_\varepsilon)^n = c_\varepsilon \beta_\varepsilon \wedge G_{m,\varepsilon} \tag{4.5}$$

with $\widetilde{\alpha_\varepsilon} = \alpha_\varepsilon + i\partial\bar{\partial}u_\varepsilon$, $\sup_X(\varphi_\varepsilon + u_\varepsilon) = 0$ and $c_\varepsilon = \int_X (\alpha_\varepsilon + i\partial\bar{\partial}u_\varepsilon)^n$. Then $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2, ..., n-1$ implies

$$c_\varepsilon = \int_X (\alpha + \varepsilon\omega)^n \downarrow c_0 = \int_X \alpha^n > 0. \tag{4.6}$$

We define

$$M_\varepsilon = \int_X (\alpha_\varepsilon + i\partial\bar{\partial}u_\varepsilon)^{n-1} \wedge \beta_\varepsilon,$$

then $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2, ..., n-1$ also implies

$$M_\varepsilon = \int_X (\alpha + \varepsilon\omega)^{n-1} \wedge (\beta + \varepsilon\omega) \downarrow M_0 = \int_X \alpha^{n-1} \wedge \beta. \tag{4.7}$$

We define

$$E_\gamma := \{x \in X | \frac{\widetilde{\alpha_\varepsilon}^{n-1} \wedge \beta_\varepsilon}{G_{m,\varepsilon} \wedge \beta_\varepsilon}(x) > \gamma M_\varepsilon\}$$

for some $\gamma > 1$. The condition $\gamma > 1$ implies E_γ is a proper open subset in X, since we have assumed

$$\int_X \beta_\varepsilon \wedge G_{m,\varepsilon} = 1$$

and

$$\int_{E_\gamma} G_{m,\varepsilon} \wedge \beta_\varepsilon = \int_{E_\gamma} \frac{G_{m,\varepsilon} \wedge \beta_\varepsilon}{\widetilde{\alpha_\varepsilon}^{n-1} \wedge \beta_\varepsilon} \cdot \widetilde{\alpha_\varepsilon}^{n-1} \wedge \beta_\varepsilon < \frac{1}{\gamma M_\varepsilon} M_\varepsilon = \frac{1}{\gamma} < 1. \tag{4.8}$$

On the closed subset $X \backslash E_\gamma$, the definition of E_γ tells us that

$$\widetilde{\alpha_\varepsilon}^{n-1} \wedge \beta_\varepsilon \leq \gamma M_\varepsilon \cdot G_{m,\varepsilon} \wedge \beta_\varepsilon. \tag{4.9}$$

For any fixed point $p \in X \backslash E_\gamma$, choose some holomorphic coordinates around p such that

$$\beta_\varepsilon(p) = \sum_i \sqrt{-1} dz_i \wedge d\bar{z}_i, \quad \widetilde{\alpha}_\varepsilon(p) = \sum_i \sqrt{-1}\lambda_i dz_i \wedge d\bar{z}_i$$

where $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. At the point p, if we denote $dV(p) := (\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n$, then (4.5) is just

$$n!\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_n dV(p) = c_\varepsilon \beta_\varepsilon \wedge G_{m,\varepsilon}, \tag{4.10}$$

and (4.9) is

$$(n-1)! \sum \lambda_{i_1} \cdot \lambda_{i_2} \cdot ... \cdot \lambda_{i_{n-1}} dV(p) \leq \gamma M_\varepsilon \cdot G_{m,\varepsilon} \wedge \beta_\varepsilon. \tag{4.11}$$

The above two inequalities (4.10) and (4.11) yield

$$\lambda_1(p) \geq \frac{c_\varepsilon}{n\gamma M_\varepsilon}.$$

Since $p \in X \backslash E_\gamma$ is arbitrary, we get

$$\widetilde{\alpha_\varepsilon} \geq \frac{c_\varepsilon}{n\gamma M_\varepsilon} \cdot \beta_\varepsilon \tag{4.12}$$

on $X \backslash E_\gamma$.

Now let us estimate the integral

$$\int_X \widetilde{\alpha_\varepsilon} \wedge G_{m,\varepsilon} = \int_X (\alpha + \varepsilon\omega) \wedge G_{m,\varepsilon}.$$

The inequality (4.12) implies

$$\begin{aligned}\int_X \widetilde{\alpha_\varepsilon} \wedge G_{m,\varepsilon} &\geq \int_{X\backslash E_\gamma} \widetilde{\alpha_\varepsilon} \wedge G_{m,\varepsilon} \\ &\geq \int_{X\backslash E_\gamma} \frac{c_\varepsilon}{n\gamma M_\varepsilon} \cdot \beta_\varepsilon \wedge G_{m,\varepsilon} \\ &= \frac{c_\varepsilon}{n\gamma M_\varepsilon}\Big(\int_X \beta_\varepsilon \wedge G_{m,\varepsilon} - \int_{E_\gamma} \beta_\varepsilon \wedge G_{m,\varepsilon}\Big) \\ &> \frac{c_\varepsilon}{n\gamma M_\varepsilon}\Big(1 - \frac{1}{\gamma}\Big).\end{aligned}$$

Take $\gamma = 2$, we get

$$c_\varepsilon - 4nM_\varepsilon \int_X \widetilde{\alpha_\varepsilon} \wedge G_{m,\varepsilon} = c_\varepsilon - 4nM_\varepsilon \int_X (\alpha + \varepsilon\omega) \wedge G_{m,\varepsilon} < 0. \tag{4.13}$$

On the other hand, (4.4) implies

$$\int_X \alpha_\varepsilon \wedge G_{m,\varepsilon} = \int_X (\alpha + \varepsilon\omega) \wedge G_{m,\varepsilon} < \frac{1}{1-\delta_m}. \tag{4.14}$$

Fix a small ε to be determined. Since $G_{m,\varepsilon}$ is normalized, by compactness of the sequence $\{G_{m,\varepsilon}\}$, there exists a weakly convergent subsequence – which we also denote by $\{G_{m,\varepsilon}\}$ – such that

$$\lim_{m\to\infty} G_{m,\varepsilon} = G_{\infty,\varepsilon},$$

where the convergence is in the weak topology of currents and $G_{\infty,\varepsilon}$ is a $\partial\bar{\partial}$-closed positive $(n-1,n-1)$-current with

$$0 \leq \int_X (\alpha + \varepsilon\omega) \wedge G_{\infty,\varepsilon} \leq 1. \tag{4.15}$$

Now our assumption

$$\alpha^n - 4n\alpha^{n-1} \cdot \beta > 0$$

implies

$$c_0 - 4nM_0 > 0.$$

Then after taking the limit of m in (4.13) and (4.14), (4.15) implies

$$c_\varepsilon - 4nM_\varepsilon \leq c_\varepsilon - 4nM_\varepsilon \int_X (\alpha + \varepsilon\omega) \wedge G_{\infty,\varepsilon} < 0.$$

It is clear that the contradiction is obtained in the limit when we let ε go to zero.

Thus the assumption that $\alpha - \beta$ is not a big class is not true. In other words, $\alpha^n - 4n\alpha^{n-1} \cdot \beta > 0$ implies there exists a Kähler current in the class $\alpha - \beta$. $\square$

After proving Theorem 4.1.4, Corollary 4.1.7 follows easily.

Proof. (of Corollary 4.1.7) Since X is in the Fujiki class $\mathcal{C}$, there exists a proper modification $\mu : \widetilde{X} \to X$ such that $\widetilde{X}$ is Kähler. Pull back α, β to $\widetilde{X}$, the class $\mu^*\alpha, \mu^*\beta$ are still nef classes on $\widetilde{X}$ and $(\mu^*\alpha)^n - 4n(\mu^*\alpha)^{n-1} \cdot \mu^*\beta > 0$. Theorem 4.1.4 yields there exists a Kähler current

$$\widetilde{T} \in \mu^*(\alpha - \beta).$$

Then $T := \mu_*\widetilde{T}$ is our desired Kähler current in the class $\alpha - \beta$. □

Remark 4.3.1. We point out that, for the Bott-Chern cohomology classes $\alpha^k - \beta^k$ on compact Kähler manifolds it is not hard to prove a similar positivity result analogous to Theorem 4.1.4 :

⋆ *Let α and β be two nef cohomology classes of type $(1,1)$ on an n-dimensional compact Kähler manifold X satisfying the inequality $\alpha^n - 4\frac{n!}{k!(n-k)!}\alpha^{n-k} \cdot \beta^k > 0$, then $\alpha^k - \beta^k$ contains a "strictly positive" (k,k)-current.*

Its proof is almost a copy and paste of that in the $(1,1)$-classes case. And one can get the natural constant $\frac{n!}{k!(n-k)!}$ by using [Pop14].

Here we call such a (k,k)-current T "strictly positive" if there exist a positive constant δ and a hermitian metric ω such that $T \geq \delta\omega^k$, and call a (k,k)-cohomology class big if it can be represented by such a positive current. Fix a Kähler metric ω, since α and β are nef, for any $\varepsilon > 0$ there exist functions $\varphi_\varepsilon, \psi_\varepsilon$ such that $\alpha_\varepsilon := \alpha + \varepsilon\omega + i\partial\bar{\partial}\varphi_\varepsilon$ and $\beta_\varepsilon := \beta + \varepsilon\omega + i\partial\bar{\partial}\psi_\varepsilon$ are Kähler metrics. In general, unlike the $(1,1)$-case, we should note that as classes

$$\alpha_\varepsilon^k - \beta_\varepsilon^k \neq \alpha^k - \beta^k.$$

Thus the bigness of $\alpha_\varepsilon^k - \beta_\varepsilon^k$ does not imply the bigness of $\alpha^k - \beta^k$. However, we can still apply the ideas of the proof of Theorem 4.1.4 by the following observation. It is obvious that $\alpha^n - 4\frac{n!}{k!(n-k)!}\alpha^{n-k} \cdot \beta_\varepsilon^k > 0$ for ε small enough. We fix such a ε_0, then we claim that the bigness of $\alpha^k - \beta_{\varepsilon_0}^k$ implies the bigness of $\alpha^k - \beta^k$. The bigness of $\alpha^k - \beta_{\varepsilon_0}^k$ yields the existence of some current θ_{ε_0} and some positive constant δ_{ε_0} such that

$$\alpha^k - \beta_{\varepsilon_0}^k + i\partial\bar{\partial}\theta_{\varepsilon_0} \geq \delta_{\varepsilon_0}\omega^k.$$

Then we have $\alpha^k - \beta^k + i\partial\bar{\partial}\widetilde{\theta}_{\varepsilon_0} \geq \delta_{\varepsilon_0}\omega^k + \gamma_{\varepsilon_0}$, where

$$i\partial\bar{\partial}\widetilde{\theta}_{\varepsilon_0} = i\partial\bar{\partial}\theta_{\varepsilon_0} - \sum_{l=1}^{k} C_k^l \sum_{p=1}^{l} C_l^p (i\partial\bar{\partial}\psi_{\varepsilon_0})^p \wedge (\varepsilon_0\omega)^{l-p} \wedge \beta^{k-l}$$

and

$$\gamma_{\varepsilon_0} = \sum_{l=1}^{k} C_k^l (\varepsilon_0\omega)^l \wedge \beta^{k-l}.$$

Since β is nef, it is clear that the class $\{\gamma_{\varepsilon_0}\}$ contains a positive current $\Upsilon_{\varepsilon_0} := \gamma_{\varepsilon_0} + i\partial\bar{\partial}\Phi_{\varepsilon_0}$. Then $\alpha^k - \beta^k + i\partial\bar{\partial}(\widetilde{\theta}_{\varepsilon_0} + \Phi_{\varepsilon_0})$ is a "strictly positive" (k,k)-current in $\alpha^k - \beta^k$.

Thus we can assume β is a Kähler metric in the beginning. With this assumption, we only need to show that the class $\alpha^k - \beta^k$ contains a (k,k)-current $T := \alpha^k - \beta^k + i\partial\bar{\partial}\theta$ such that $T \geq \delta\beta^k$ for some positive constant δ. This can be done as in the proof of Theorem 4.1.4.

4.3.2 Theorem 4.1.10 and Theorem 4.1.13

Now let us begin to prove Theorem 4.1.10 and Theorem 4.1.13. We first give a Morse-type bigness criterion for the difference of two pseudo-effective $(1,1)$-classes by using movable intersections. To this end, we need some properties of movable intersections.

Proposition 4.3.2. *Let X be a compact Kähler manifold of dimension n, and let $\alpha_1, ..., \alpha_k \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes. Let $\mu : Y \to X$ be a modification with Y Kähler, then we have*

$$\mu^* \langle \alpha_1 \cdot ... \cdot \alpha_k \rangle = \langle \mu^* \alpha_1 \cdot ... \cdot \mu^* \alpha_k \rangle.$$

Proof. By taking limits, we only need to verify the case when all the classes α_i are big. By [BEGZ10, Definition 1.17], the movable intersections can be defined by positive currents of minimal singularities, that is,

$$\langle \alpha_1 \cdot ... \cdot \alpha_k \rangle := \{ \langle T_{1,\min} \wedge ... \wedge T_{k,\min} \rangle \},$$

where $\langle T_{1,\min} \wedge ... \wedge T_{k,\min} \rangle$ is the non-pluripolar product of positive currents and $T_{i,\min}$ is a positive current in the big class α_i with minimal singularities. And if $\alpha_1, ..., \alpha_k$ are merely pseudo-effective, we set

$$\langle \alpha_1 \cdot ... \cdot \alpha_k \rangle := \lim_{\epsilon \to 0} \langle (\alpha_1 + \epsilon\omega) \cdot ... \cdot (\alpha_k + \epsilon\omega) \rangle$$

where ω is an arbitrary Kähler class on X.

To prove the desired equality, using Poincaré duality, we need to verify

$$\mu^* \langle \alpha_1 \cdot ... \cdot \alpha_k \rangle \cdot \{\eta\} = \langle \mu^* \alpha_1 \cdot ... \cdot \mu^* \alpha_k \rangle \cdot \{\eta\}$$

for an arbitrary d-closed smooth $(n-k, n-k)$-form η. Let $T_{i,\min} \in \alpha_i$ be a positive current with minimal singularities, then [BEGZ10, Proposition 1.12] implies $\mu^* T_{i,\min} \in \mu^* \alpha_i$ is also a positive current with minimal singularities. Thus we have

$$\langle \mu^* \alpha_1 \cdot ... \cdot \mu^* \alpha_k \rangle = \{ \langle \mu^* T_{1,\min} \wedge ... \wedge \mu^* T_{k,\min} \rangle \}.$$

By the definition of non-pluripolar products of d-closed positive $(1,1)$-currents, these products do not put mass on pluripolar subsets. In particular, they do not put mass on proper analytic subvarieties. Indeed, by Demailly's regularization theorem, there exists an analytic Zariski open set where μ is an isomorphism and all the currents $T_{i,\min}$ are of locally bounded potentials. Integrating over this set, we get

$$\int_Y \mu^* \langle T_{1,\min} \wedge ... \wedge T_{k,\min} \rangle \wedge \eta = \int_Y \langle \mu^* T_{1,\min} \wedge ... \wedge \mu^* T_{k,\min} \rangle \wedge \eta.$$

Since η is arbitrary, this proves the equality $\mu^* \langle \alpha_1 \cdot ... \cdot \alpha_k \rangle = \langle \mu^* \alpha_1 \cdot ... \cdot \mu^* \alpha_k \rangle$. □

Corollary 4.3.3. *Let X be a compact Kähler manifold of dimension n, and let $\alpha_1, ..., \alpha_{n-1}, \beta \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes with β nef. Then we have*

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \beta \rangle = \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta.$$

Proof. By taking limits, we can assume $\alpha_1, ..., \alpha_{n-1}$ are big and β is Kähler.

First, by [BDPP13, Theorem 3.5], there exists a sequence of simultaneous log-resolutions $\mu_m : X_m \to X$ with $\mu_m^* \alpha_i = \omega_{i,m} + [D_{i,m}]$ and $\mu_m^* \beta = \gamma_m + [E_m]$ such that

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \beta \rangle = \limsup_{m \to \infty} \, (\omega_{1,m} \cdot ... \cdot \omega_{n-1,m} \cdot \gamma_m).$$

By the definition of $\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle$, we always have

$$\limsup_{m \to \infty} \, (\mu_m)_* (\omega_{1,m} \cdot ... \cdot \omega_{n-1,m}) \leq \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle$$

in the sense of integrating against smooth $\partial\bar{\partial}$-closed positive $(1,1)$-forms. In particular, since β can be represented by a Kähler metric and $\mu_m^* \beta = \gamma_m + [E_m]$, we get

$$\begin{aligned} \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \beta \rangle &\leq \limsup_{m \to \infty} \, (\mu_m)_* (\omega_{1,m} \cdot ... \cdot \omega_{n-1,m}) \cdot \beta \\ &\leq \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta. \end{aligned}$$

On the other hand, we claim that

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \beta \rangle \geq \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta$$

if we merely assume β is movable. Without loss of generality, we can assume that $\beta = \pi_* \widetilde{\omega}$ for some modification $\pi : Y \to X$ and some Kähler class $\widetilde{\omega}$ on Y. Let $T_{i,\min} \in \alpha_i$ be the positive current with minimal singularities, and denote a Kähler metric in the Kähler class $\widetilde{\omega}$ by the same symbol $\widetilde{\omega}$. By Proposition 4.3.2 we have

$$\begin{aligned} \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \pi_* \widetilde{\omega} &= \langle \pi^* \alpha_1 \cdot ... \cdot \pi^* \alpha_{n-1} \rangle \cdot \widetilde{\omega} \\ &= \int_Y \langle \pi^* T_{1,\min} \cdot ... \cdot \pi^* T_{n-1,\min} \rangle \wedge \widetilde{\omega} \\ &= \int_X \langle T_{1,\min} \cdot ... \cdot T_{n-1,\min} \wedge \pi_* \widetilde{\omega} \rangle \\ &\leq \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \pi_* \widetilde{\omega} \rangle, \end{aligned}$$

where the third line follows by integrating $\langle \pi^* T_{1,\min} \cdot ... \cdot \pi^* T_{n-1,\min} \rangle \wedge \widetilde{\omega}$ outside a pluripolar subset (including the center of π) and the last line follows from the definition of $\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \pi_* \widetilde{\omega} \rangle$ and [BEGZ10, Proposition 1.20].

In conclusion, if β is nef then we have the desired equality

$$\langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \cdot \beta \rangle = \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta.$$

□

Corollary 4.3.4. *Let X be a compact Kähler manifold of dimension n. Let $\alpha_1', \alpha_1, ..., \alpha_{n-1}$ and β be pseudo-effective $(1,1)$-classes such that $\alpha_1' - \alpha_1$ is pseudo-effective and β is movable, then we have*

$$\langle \alpha_1' \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta \geq \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot \beta.$$

Proof. Fix a Kähler class ω. By taking limits, we only need to verify

$$\langle {\alpha'}_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot (\beta + \delta\omega) \geq \langle \alpha_1 \cdot ... \cdot \alpha_{n-1} \rangle \cdot (\beta + \delta\omega)$$

for any $\delta > 0$. Note that as β is movable there exists some modification $\mu : Y \to X$ and some Kähler class $\widehat{\omega}$ such that $\mu_* \widehat{\omega} = \beta + \delta\omega$. Then the result follows directly from Proposition 4.3.2 and Corollary 4.3.3. □

Now we can give the proof of Theorem 4.1.10.

Proof. (of Theorem 4.1.10.) Fix a Kähler metric ω on X, and denote the Kähler class by the same symbol. By continuity and the definition of movable intersections, we have

$$\lim_{\delta \to 0} \langle (\alpha + \delta\omega)^n \rangle - n \langle (\alpha + \delta\omega)^{n-1} \rangle \cdot (\beta + \delta\omega) = \langle \alpha^n \rangle - n \langle \alpha^{n-1} \rangle \cdot \beta.$$

So $\langle (\alpha + \delta\omega)^n \rangle - n \langle (\alpha + \delta\omega)^{n-1} \rangle \cdot (\beta + \delta\omega) > 0$ for small $\delta > 0$. Note also that $\alpha - \beta = (\alpha + \delta\omega) - (\beta + \delta\omega)$. Thus to prove the bigness of the class $\alpha - \beta$, we can assume α is big, and assume $\beta = \mu_* \widetilde{\omega}$ for some modification $\mu : Y \to X$ and some Kähler class $\widetilde{\omega}$ on Y at the beginning.

By Proposition 4.3.2 and Corollary 4.3.3, our assumption then implies

$$\langle (\mu^* \alpha)^n \rangle - n \langle (\mu^* \alpha)^{n-1} \cdot \widetilde{\omega} \rangle > 0.$$

We claim that this implies there exists a Kähler current in the class $\mu^* \alpha - \widetilde{\omega}$, which then implies the bigness of the class $\alpha - \beta = \mu_* (\mu^* \alpha - \widetilde{\omega})$.

Now it is reduced to prove the case when α is big and β is Kähler. Let ω be a Kähler metric in the class β. The definition of movable intersections implies there exists some Kähler current $T \in \alpha$ with analytic singularities along some subvariety V such that

$$\int_{X\backslash V} T^n - n\int_{X\backslash V} T^{n-1} \wedge \omega > 0.$$

Let $\pi : Z \to X$ be the log-resolution of the current T with $\pi^* T = \theta + [D]$ such that θ is a smooth positive $(1,1)$-form on Z. By Proposition 4.2.3 we have

$$\int_Z \theta^n - n\int_Z \theta^{n-1} \wedge \pi^*\omega > 0.$$

The result of [Pop14] then implies that there exists a Kähler current in the class $\{\theta - \pi^*\omega\}$. As $\pi^*\alpha = \{\theta + [D]\}$, this proves the bigness of the class $\alpha - \beta$.

Thus we finish the proof that there exists a Kähler current in the general case when α is pseudo-effective and β is movable. ☐

Remark 4.3.5. By the proof of Corollary 4.3.3, we know that $\langle\alpha^{n-1}\rangle \cdot \beta \leq \langle\alpha^{n-1} \cdot \beta\rangle$ if β is movable. So we have

$$\langle\alpha^n\rangle - n\langle\alpha^{n-1}\rangle \cdot \beta \geq \langle\alpha^n\rangle - n\langle\alpha^{n-1} \cdot \beta\rangle.$$

In particular, since $\langle\alpha^n\rangle$ and $\langle\alpha^{n-1} \cdot \beta\rangle$ depends only on the positive parts of α, β (see [Bou02a, Proposition 3.2.10]), we get the following weaker bigness criterion :

$$\langle\alpha^n\rangle - n\langle\alpha^{n-1} \cdot \beta\rangle > 0 \Rightarrow \mathrm{vol}(Z(\alpha) - Z(\beta)) > 0.$$

In the case of Example 4.1.9, since R is nef and E is exceptional, we have $\langle R^2\rangle - 2\langle R \cdot 2E\rangle = R^2 > 0$. We then get the bigness of $Z(R) - Z(2E) = R$.

The algebraic Morse inequality tells us that if L and F are two nef line bundles, then

$$\mathrm{vol}(L - F) \geq L^n - nL^{n-1} \cdot F.$$

Recently, [Tra11] generalizes this result to the case when F is only movable. Assume that L is nef and F is pseudo-effective, and let $F = Z(F) + N(F)$ be the divisorial Zariski decomposition of F. Then [Tra11, Corollary 3.2] shows that

$$\mathrm{vol}(L - Z(F)) \geq L^n - nL^{n-1} \cdot Z(F).$$

Moreover, if we write the negative part $N(F) = \sum_j \nu_j D_j$ where $\nu_j > 0$ and let u be a nef class on X such that $c_1(\mathcal{O}_{TX}(1)) + \pi^* u$ is a nef class on $\mathbb{P}(T^*X)$. Then [Tra11, Theorem 3.3] also gives a lower bound for $\mathrm{vol}(L - F)$:

$$\mathrm{vol}(L - F) \geq L^n - nL^{n-1} \cdot Z(F) - n\sum_j (L + \nu_j u)^{n-1} \cdot \nu_j D_j.$$

In particular, if $\mathcal{O}_{TX}(1)$ is nef, then we can take $u = 0$ and we have

$$\mathrm{vol}(L - F) \geq L^n - nL^{n-1} \cdot F.$$

Our next result shows that L can be any pseudo-effective line bundle, which is just Theorem 4.1.13.

Theorem 4.3.6. *Let X be a smooth projective variety of dimension n, and let L, M be two pseudo-effective line bundles with M movable. Then we have*

$$\mathrm{vol}(L - M) \geq \mathrm{vol}(L) - n\langle L^{n-1}\rangle \cdot M.$$

Proof. This follows from Theorem 4.1.10, Proposition 4.1.12 (see below for the proof) and [BFJ09, Theorem A]. □

Remark 4.3.7. When M is nef and L is pseudo-effective, Theorem 4.3.6 can be proved by using the singular Morse inequalities for line bundles (see [Bon98]). Without loss of generality, we can assume that L is big and M is ample. Let $\omega \in c_1(M)$ be a Kähler metric. For any Kähler current $T \in c_1(L)$ with analytic singularities, $T - \omega$ is an almost positive curvature current of $L - M$ with analytic singularities. With the elementary pointwise inequality

$$\mathbf{1}_{X(\alpha-\beta,\leq 1)}(\alpha-\beta)^n \geq \alpha^n - n\alpha^{n-1}\wedge\beta$$

for positive $(1,1)$-forms, Theorem 4.3.6 then follows easily from [Bon98].

Towards the transcendental version of Theorem 4.3.6, we give the proof of Proposition 4.1.12 which is essentially contained in [BFJ09].

Proof. It is obvious (2)⇒(1). We will show that (1)⇒(3)⇒(2), this then proves the equivalence of the above statements.

Firstly, we prove (3)⇒(2). To prove (2), we only need to consider the case when $\langle\alpha^n\rangle - n\langle\alpha^{n-1}\rangle\cdot\beta > 0$. By Theorem 4.1.10 we know $\alpha - \beta$ is big, thus (3) implies that the volume function vol is $\mathcal{C}^1$ differentiable at the points $\alpha - t\beta$ for $t \in [0,1]$. And we have

$$\frac{d}{dt}\bigg|_{t=t_0} \mathrm{vol}(\alpha - t\beta) = -n\langle(\alpha - t_0\beta)^{n-1}\rangle\cdot\beta.$$

This implies

$$\mathrm{vol}(\alpha-\beta) = \mathrm{vol}(\alpha) - \int_0^1 n\langle(\alpha - t\beta)^{n-1}\rangle\cdot\beta dt.$$

By Corollary 4.3.4 we have the inequality $\langle(\alpha - t\beta)^{n-1}\rangle\cdot\beta \leq \langle\alpha^{n-1}\rangle\cdot\beta$, then we get

$$\mathrm{vol}(\alpha-\beta) \geq \mathrm{vol}(\alpha) - n\langle\alpha^{n-1}\rangle\cdot\beta.$$

Next, the implication (1)⇒(3) is essentially [BFJ09, Section 3.2]. For reader's convenience, we briefly recall and repeat the arguments of [BFJ09, Section 3.2]. By [BFJ09, Corollary 3.4] (or the proof of [BDPP13, Theorem 4.1]), (1) implies

$$\mathrm{vol}(\beta + t\gamma) \geq \beta^n + tn\beta^{n-1}\cdot\gamma - Ct^2$$

for an arbitrary nef class β, an arbitrary $(1,1)$-class γ and $t \in [0,1]$. Here the constant C depends only on the class β, γ; more precisely, the constant C depends on the volume of a big and nef class ω such that $\omega - \beta$ is pseudo-effective and $\omega \pm \gamma$ is nef.

Now take a log-resolution $\mu^*\alpha = \beta + [E]$, then we have

$$\begin{aligned}\mathrm{vol}(\alpha + t\gamma) &\geq \mathrm{vol}(\beta + t\mu^*\gamma)\\ &\geq \beta^n + tn\beta^{n-1}\cdot\mu^*\gamma - Ct^2\\ &= \beta^n + tn\mu_*(\beta^{n-1})\cdot\gamma - Ct^2.\end{aligned}$$

Note that the constant C does not depend on the resolution μ, since $\mu^*\omega - \beta$ is pseudo-effective and $\mu^*\omega \pm \mu^*\gamma$ is nef if ω has similar property with respect to α, γ. And we have $\mathrm{vol}(\mu^*\omega) = \mathrm{vol}(\omega)$. By taking limits of some sequence of log-resolutions, we get

$$\mathrm{vol}(\alpha + t\gamma) \geq \mathrm{vol}(\alpha) + tn\langle\alpha^{n-1}\rangle\cdot\gamma - Ct^2.$$

Replace γ by $-\gamma$, we then get

$$\mathrm{vol}(\alpha) \geq \mathrm{vol}(\alpha + t\gamma) - tn\langle(\alpha + t\gamma)^{n-1}\rangle \cdot \gamma - Ct^2.$$

Since α is big, by the concavity of movable intersections (see e.g. [BDPP13, Theorem 3.5]) we have

$$\lim_{t\to 0}\langle(\alpha + t\gamma)^{n-1}\rangle = \langle\alpha^{n-1}\rangle.$$

Then (3) follows easily from the above inequalities. $\square$

Remark 4.3.8. It is proved in [Den15] that the $\mathcal{C}^1$ differentiability of the volume function for transcendental $(1,1)$-classes holds on compact Kähler surfaces. And it is used to construct the Okounkov bodies of transcendental $(1,1)$-classes over compact Kähler surfaces.

4.3.3 Theorem 4.1.15

Finally, inspired by the method in [Chi13], we show that Theorem 4.1.10 gives a Morse-type bigness criterion of the difference of two movable $(n-1,n-1)$-classes, thus finishing the proof of Theorem 4.1.15.

Proof. Denote the Kähler cone of X by $\mathcal{K}$, and denote the cone generated by cohomology classes represented by positive $(n-1,n-1)$-currents by $\mathcal{N}$. Then by the numerical characterization of Kähler cone of [DP04] (see also [BDPP13, Theorem 2.1]) we have the cone duality relation

$$\overline{\mathcal{K}}^* = \mathcal{N}.$$

Without loss of generality, we can assume that α, β are big. Then the existence of a strictly positive $(n-1,n-1)$-current in the class $\langle\alpha^{n-1}\rangle - \langle\beta^{n-1}\rangle$ is equivalent to the existence of some positive constant $\delta > 0$ such that

$$\langle\alpha^{n-1}\rangle - \langle\beta^{n-1}\rangle \succeq \delta\langle\beta^{n-1}\rangle,$$

or equivalently,

$$\langle\alpha^{n-1}\rangle \succeq (1+\delta)\langle\beta^{n-1}\rangle.$$

Here we denote $\gamma \succeq \eta$ if $\gamma - \eta$ contains a positive current.

In the following, we will argue by contradiction. By the cone duality relation $\overline{\mathcal{K}}^* = \mathcal{N}$, the class $\langle\alpha^{n-1}\rangle - \langle\beta^{n-1}\rangle$ does not contain any strictly positive $(n-1,n-1)$-current is then equivalent to the statement : for any $\epsilon > 0$ there exists some non-zero class $N_\epsilon \in \overline{\mathcal{K}}$ such that

$$\langle\alpha^{n-1}\rangle \cdot N_\epsilon \leq (1+\epsilon)\langle\beta^{n-1}\rangle \cdot N_\epsilon.$$

On the other hand, we claim Theorem 4.1.10 implies that

$$n(N \cdot \langle\alpha^{n-1}\rangle)(\alpha \cdot \langle\beta^{n-1}\rangle) \geq \langle\alpha^n\rangle(N \cdot \langle\beta^{n-1}\rangle)$$

for any nef $(1,1)$-class N. First note that both sides of the above inequality are of the same degree of each cohomology class. After scaling, we can assume

$$\alpha \cdot \langle\beta^{n-1}\rangle = N \cdot \langle\beta^{n-1}\rangle.$$

Then we need to prove $nN \cdot \langle\alpha^{n-1}\rangle \geq \langle\alpha^n\rangle$. Otherwise, we have $nN \cdot \langle\alpha^{n-1}\rangle < \langle\alpha^n\rangle$. And Theorem 4.1.10 implies that there must exist a Kähler current in the class $\alpha - N$. Then we must have

$$\langle\beta^{n-1}\rangle \cdot (\alpha - N) > 0,$$

which contradicts with our scaling equality $\langle\beta^{n-1}\rangle \cdot (\alpha - N) = 0$.

Let $N = N_\epsilon$, we get

$$\begin{aligned}(1+\epsilon)n(N_\epsilon \cdot \langle \beta^{n-1}\rangle)(\alpha \cdot \langle \beta^{n-1}\rangle) &\geq n(N_\epsilon \cdot \langle \alpha^{n-1}\rangle)(\alpha \cdot \langle \beta^{n-1}\rangle)\\ &\geq \langle \alpha^n\rangle (N_\epsilon \cdot \langle \beta^{n-1}\rangle).\end{aligned}$$

This implies

$$(1+\epsilon)n\alpha \cdot \langle \beta^{n-1}\rangle \geq \langle \alpha^n\rangle.$$

Since $\epsilon > 0$ is arbitrary, this contradicts with our assumption $\langle \alpha^n\rangle - n\alpha \cdot \langle \beta^{n-1}\rangle > 0$. Thus there must exist a strictly positive $(n-1, n-1)$-current in the class $\langle \alpha^{n-1}\rangle - \langle \beta^{n-1}\rangle$. □

Remark 4.3.9. Let X be a smooth projective variety of dimension n and let $\mathrm{Mov}_1(X)$ be the closure of the cone generated by movable curve classes. In the sequel Chapter 6 (see also [LX15]), we show that any interior point of $\mathrm{Mov}_1(X)$ is the form $\langle L^{n-1}\rangle$ for a unique big and movable divisor class. And under Demailly's conjecture on transcendental Morse inequality, this also extends to transcendental movable $(n-1, n-1)$-classes over compact Kähler manifold. In particular, this extends to compact hyperkähler manifolds.

4.4 Applications to a conjecture of Lejmi and Székelyhidi

4.4.1 Background and main applications

From the point of view that relates the existence of canonical Kähler metrics with algebro-geometric stability conditions, Lejmi and Székelyhidi [LS15] proposed a numerical characterization on when the inverse σ_k-flow converges. We aim to study the positivity of related cohomology classes in their conjecture. We generalize their conjecture by weakening the numerical condition on X a little bit.

Conjecture 4.4.1. *(see [LS15, Conjecture 18]) Let X be a compact Kähler manifold of dimension n, and let ω, α be two Kähler metrics over X satisfying*

$$\int_X \omega^n - \frac{n!}{k!(n-k)!}\omega^{n-k}\wedge \alpha^k \geq 0. \tag{4.16}$$

Then there exists a Kähler metric $\omega' \in \{\omega\}$ such that

$$\omega'^{n-1} - \frac{(n-1)!}{k!(n-k-1)!}\omega'^{n-k-1}\wedge \alpha^k > 0 \tag{4.17}$$

as a smooth $(n-1, n-1)$-form if and only if

$$\int_V \omega^p - \frac{p!}{k!(p-k)!}\omega^{p-k}\wedge \alpha^k > 0 \tag{4.18}$$

for every irreducible subvariety of dimension p with $k \leq p \leq n-1$.

For the previous works closely related to this conjecture, we refer the reader to [Don99], [Che00, Che04], [SW08] and [FLM11]. And in this note we mainly concentrate on the case when $k = 1$ and $k = n-1$.

For $k = 1$, [CS14, Theorem 3] confirmed this conjecture for toric manifolds. Over a general compact Kähler manifold, it is not hard to see the implication (4.17) ⇒ (4.18) holds. In the reverse direction, we prove $\{\omega - \alpha\}$ must be a Kähler class under the numerical conditions in Conjecture 4.4.1 for $k = 1$; indeed, this is a necessary condition of (4.17) and [LS15, Proposition 14] proved this over Kähler surfaces.

Theorem 4.4.2. *Let X be a compact Kähler manifold of dimension n, and let ω, α be two Kähler metrics over X satisfying the numerical conditions in Conjecture 4.4.1 for $k = 1$. Then $\{\omega - \alpha\}$ is a Kähler class.*

For $k = n-1$, we have the following similar result.

Theorem 4.4.3. *Let X be compact Kähler manifold of dimension n, and let ω, α be two Kähler metrics over X satisfying the numerical conditions in Conjecture 4.4.1 for $k = n-1$. Then the class $\{\omega^{n-1} - \alpha^{n-1}\}$ lies in the closure of the Gauduchon cone, i.e. it has nonnegative intersection number with every pseudo-effective $(1,1)$-class.*

4.4.2 Proof of the main applications

Theorem 4.4.2

Proof. The first observation is that, when $k = 1$, the inequalities in the numerical conditions are just the right hand side in weak transcendental holomorphic Morse inequalities. Recall that Demailly's conjecture on weak transcendental holomorphic Morse inequalities (see e.g. [BDPP13, Conjecture 10.1]) is stated as following :

Let X be a compact complex manifold of dimension n, and let γ, β be two nef classes over X. Then we have

$$\mathrm{vol}\,(\gamma - \beta) \geq \gamma^n - n\gamma^{n-1} \cdot \beta.$$

In particular, $\gamma^n - n\gamma^{n-1} \cdot \beta > 0$ implies the class $\gamma - \beta$ is big, that is, $\gamma - \beta$ contains a Kähler current.

Note that the last statement has been proved for Kähler manifolds [Pop14], that is, if X is a compact Kähler manifold then $\gamma^n - n\gamma^{n-1} \cdot \beta > 0$ implies there exists a Kähler current in the class $\gamma - \beta$.

We apply this bigness criterion to the classes $\{\omega\}$ and $\{\alpha\}$, then the numerical condition (4.18) implies $\{\omega - \alpha\}_{|V}$ is a big class on every proper irreducible subvariety V. More precisely, if V is singular then by some resolution of singularities we have a proper modification $\pi : \widehat{V} \to V$ with $\widehat{V}$ smooth, and by (4.18) we know

$$\pi^*\{\omega\}^p_{|V} - p\pi^*\{\omega\}^{p-1}_{|V} \cdot \pi^*\{\alpha\}_{|V} > 0,$$

thus the class $\pi^*\{\omega - \alpha\}_{|V}$ contains a Kähler current over $\widehat{V}$. So by the push-forward map π_* we obtain that the class $\{\omega - \alpha\}_{|V}$ is big over V.

In particular, by (4.16) and (4.18) the restriction of the class $\{\omega - (1-\epsilon)\alpha\}$ is big on every irreducible subvariety (including X itself) for any sufficiently small $\epsilon > 0$.

We claim this yields $\{\omega - (1-\epsilon)\alpha\}$ is a Kähler class over X for any $\epsilon > 0$ small. Indeed, our proof implies the following fact.

- Assume β is a big class over a compact complex manifold (or compact complex space) and its restriction to every irreducible subvariety is also big, then β is a Kähler class over X.

To this end, we will argue by induction on the dimension of X. If X is a compact complex curve, then this is obvious. For the general case, we need a result of Mihai Păun (see [Pău98b, Pău98a]) :

Let X be a compact complex manifold (or compact complex space), and let $\beta = \{T\}$ be the cohomology class of a Kähler current T over X. Then β is a Kähler class over X if and only if the restriction $\beta_{|Z}$ is a Kähler class on every irreducible component Z of the Lelong sublevel set $E_c(T)$.

As $\{\omega - (1-\epsilon)\alpha\}$ is a big class on X, by Demailly's regularization theorem [Dem92] we can choose a Kähler current $T \in \{\omega - (1-\epsilon)\alpha\}$ such that T has analytic singularities on X. Then the singularities of T are just the Lelong sublevel set $E_c(T)$ for some positive constant c. For every irreducible component Z of $E_c(T)$, by (4.18) the restriction $\{\omega - (1-\epsilon)\alpha\}_{|Z}$ is a big class. After resolution of singularities of Z if necessary, we obtain a Kähler current $T_Z \in \{\omega - (1-\epsilon)\alpha\}_{|Z}$ over Z with its analytic singularities contained in a proper subvariety of Z, and for every irreducible subvariety $V \subseteq Z$ the restriction $\{\omega - (1-\epsilon)\alpha\}_{|V}$ is also a big class. By induction on the dimension, we get that $\{\omega - (1-\epsilon)\alpha\}_{|Z}$ is a

Kähler class over Z. So the above result of [Pău98b, Pău98a] implies $\{\omega - (1-\epsilon)\alpha\}$ is a Kähler class over X, finishing the proof our claim.

By the arbitrariness of $\epsilon > 0$, we get $\{\omega - \alpha\}$ is a nef class on X. Next we prove $\{\omega - \alpha\}$ is a big class. By [DP04, Theorem 2.12], we only need to show

$$\mathrm{vol}(\{\omega - \alpha\}) = \int_X (\omega - \alpha)^n > 0.$$

Since $\{\omega - \alpha\}$ is nef, we can compute the derivative of the function $\mathrm{vol}(\omega - t\alpha)$ for any $t \in [0, 1)$. Thus we have

$$\begin{aligned} \mathrm{vol}(\{\omega\} - \{\alpha\}) - \mathrm{vol}(\{\omega\}) &= \int_0^1 \frac{d}{dt} \mathrm{vol}(\{\omega\} - t\{\alpha\}) dt \\ &= -\int_0^1 n\{\omega - t\alpha\}^{n-1} \cdot \{\alpha\} dt, \end{aligned}$$

which implies

$$\begin{aligned} \mathrm{vol}(\{\omega\} - \{\alpha\}) &= \mathrm{vol}(\{\omega\}) - \int_0^1 n\{\omega - t\alpha\}^{n-1} \cdot \{\alpha\} dt \\ &\geq \int_0^1 n(\{\omega\}^{n-1} - \{\omega - t\alpha\}^{n-1}) \cdot \{\alpha\} dt. \end{aligned}$$

Here the last line follows from the equality (4.16). Since ω, α are Kähler metrics, this shows $\mathrm{vol}(\{\omega - \alpha\}) > 0$. Thus $\{\omega - \alpha\}$ is a big and nef class on X with its restriction to every irreducible subvariety being big and nef. By the arguments before, we know $\{\omega - \alpha\}$ must be a Kähler class.

Finally, we give an alternative proof of the fact that the class $\{\omega - \alpha\}$ is nef using the main result of [CT13] instead of using [Pău98b, Pău98a]. (I would like to thank Tristan C. Collins who pointed out this to me.) Since $\{\omega\}$ is a Kähler class, the class $\{\omega - t\alpha\}$ is Kähler for $t > 0$ small. Let s be the largest number such that $\{\omega - s\alpha\}$ is nef. We prove that $s \geq 1$. Otherwise, suppose $s < 1$. Then by the numerical conditions (4.16) and (4.18), the bigness criterion given by transcendental holomorphic Morse inequalities implies that the class $\{\omega - s\alpha\}$ is big if $s < 1$, and furthermore, this holds for all irreducible subvarieties in X. Thus $\{\omega - s\alpha\}$ is big and nef on every irreducible subvariety V in X. This means the null locus of the big and nef class $\{\omega - s\alpha\}$ is empty, and then the main result of [CT13] implies that $\{\omega - s\alpha\}$ is a Kähler class. This contradicts with the definition of s, so we get $s \geq 1$, or equivalently, $\{\omega - \alpha\}$ must be a nef class. Then by the estimate of the volume $\mathrm{vol}(\{\omega - \alpha\})$ as above, we know $\{\omega - \alpha\}$ is also big and nef over every irreducible subvariety of X. By applying [CT13] again, this proves that $\{\omega - \alpha\}$ must be a Kähler class. □

Remark 4.4.4. If X is a smooth projective variety of dimension n and $\{\omega\}$ and $\{\alpha\}$ are the first Chern classes of holomorphic line bundles, then the nefness of the class $\{\omega - \alpha\}$ just follows from Kleiman's ampleness criterion, since the numerical condition (4.18) for $p = 1$ implies the divisor class $\{\omega - \alpha\}$ has non-negative intersection against every irreducible curve.

Theorem 4.4.3

Next we give the proof of Theorem 4.4.3.

Proof. The proof mainly depends on Boucksom's divisorial Zariski decomposition for pseudoeffective $(1,1)$-classes [Bou04] and the bigness criterion for the difference of two movable $(n-1, n-1)$-classes [Xia14].

Through a sufficiently small perturbation of the Kähler metric α, e.g. replace α by

$$\alpha_\epsilon = (1-\epsilon)\alpha$$

with $\epsilon \in (0,1)$, we can obtain that the inequality in (4.16) is strict for the classes $\{\omega\}$ and $\{\alpha_\epsilon\}$. We claim that in this case the $(n-1,n-1)$-class $\{\omega^{n-1} - \alpha_\epsilon^{n-1}\}$ has nonnegative intersections with all pseudoeffective $(1,1)$-classes. Then let ϵ tends to zero, we conclude the desired result for the class $\{\omega^{n-1} - \alpha^{n-1}\}$. Thus we can assume the inequality in (4.16)is strict for the classes $\{\omega\}$ and $\{\alpha\}$ at the beginning.

Let β be a pseudoeffective $(1,1)$-class over X. By [Bou04, Section 3], β admits a divisorial Zariski decomposition

$$\beta = Z(\beta) + N(\beta).$$

Note that $N(\beta)$ is the class of some effective divisor (may be zero) and $Z(\beta)$ is a modified nef class. In particular, we have

$$\{\omega^{n-1} - \alpha^{n-1}\} \cdot N(\beta) \geq 0. \tag{4.19}$$

For any $\delta > 0$, we have

$$Z(\beta) + \delta\{\omega\} = \pi_*\{\widehat{\omega}\}$$

for some modification $\pi : \widehat{X} \to X$ and some Kähler metric $\widehat{\omega}$ on $\widehat{X}$ (see [Bou04, Proposition 2.3]).

By our assumption on (4.16), we have

$$\int_{\widehat{X}} \pi^*\omega^n - n\pi^*\omega \wedge \pi^*\alpha^{n-1} > 0. \tag{4.20}$$

By Theorem 4.1.15, the inequality (4.20) implies that the class $\{\pi^*\omega^{n-1} - \pi^*\alpha^{n-1}\}$ contains a strictly positive $(n-1,n-1)$-current. This implies that

$$\begin{aligned}
&\{\omega^{n-1} - \alpha^{n-1}\} \cdot (Z(\beta) + \delta\{\omega\}) \\
&= \{\omega^{n-1} - \alpha^{n-1}\} \cdot \pi_*\{\widehat{\omega}\} \\
&= \pi^*\{\omega^{n-1} - \alpha^{n-1}\} \cdot \{\widehat{\omega}\} \\
&> 0.
\end{aligned}$$

By the arbitrariness of δ, we get $\{\omega^{n-1} - \alpha^{n-1}\} \cdot Z(\beta) \geq 0$. With (4.19), we show that

$$\{\omega^{n-1} - \alpha^{n-1}\} \cdot \beta \geq 0.$$

Since β can be any pseudoeffective $(1,1)$-class, this implies $\{\omega^{n-1} - \alpha^{n-1}\}$ lies in the closure of the Gauduchon cone by [Lam99a, Lemma 3.3] (see also [Xia15a, Proposition 2.1]). □

Remark 4.4.5. We expect $\{\omega^{n-1} - \alpha^{n-1}\}$ should have strictly positive intersection numbers with nonzero pseudoeffective $(1,1)$-classes. To show this, one only need to verify this for modified nef classes.

Remark 4.4.6. Let X be a smooth projective variety, and assume $\{\omega^{n-1} - \alpha^{n-1}\}$ is a curve class. Then the numerical condition (4.18) in Theorem 4.4.3 implies that $\{\omega^{n-1} - \alpha^{n-1}\}$ is a movable class by [BDPP13, Theorem 2.2].

4.4.3 Further discussions

In analogue with Theorem 4.4.2 and Theorem 4.4.3, one would like to prove similar positivity of the class $\{\omega^k - \alpha^k\}$. To generalize our results in this direction, one can apply Remark 4.3.1 (see [Xia13, Remark 3.1]). By Remark 4.3.1, we know that the condition

$$\int_V \omega^p - \frac{p!}{k!(p-k)!}\omega^{p-k} \wedge \alpha^k > 0$$

implies that the class $\{\omega^k - \alpha^k\}_{|V}$ contains a strictly positive (k,k)-current over every irreducible subvariety V of dimension p with $k < p \leq n-1$. However, the difficulties appear as we know little about the singularities of positive (k,k)-currents for $k > 1$. We have no analogues of Demailly's regularization theorem for such currents.

Inspired by the prediction of Conjecture 4.4.1, we propose the following question on the positivity of positive (k,k)-currents.

Question 4.4.7. Let X be a compact Kähler manifold (or general compact complex manifold) of dimension n. Let $\Omega \in H^{k,k}(X,\mathbb{R})$ be a big (k,k)-class, i.e. it can be represented by a strictly positive (k,k)-current over X. Assume the restriction class $\Omega_{|V}$ is also big over every irreducible subvariety V with $k \leq \dim V \leq n-1$, then does Ω contain a smooth strictly positive (k,k)-form in its Bott-Chern class ? Or does Ω contain a strictly positive (k,k)-current with analytic singularities of codimension at least $n-k+1$ in its Bott-Chern class ?

4.5 Appendix

Proof of Lamari's lemma

In this section, we include the proof of lemma 4.2.2 due to Lamari (see Lemma 3.3 of [Lam99a]). The proof is an application of Hahn-Banach theorem.

Lemma 4.5.1. *Let X be a compact complex manifold of dimension n and let Φ be a real (k,k)-form, then there exists a real $(k-1,k-1)$-current Ψ such that $\Phi + i\partial\bar{\partial}\Psi$ is positive if and only if for any strictly positive $\partial\bar{\partial}$-closed $(n-k,n-k)$-forms Υ, we have $\int_X \Phi \wedge \Upsilon \geq 0$.*

Proof. It is obvious that if there exists a $(k-1,k-1)$-current Ψ such that $\Phi + i\partial\bar{\partial}\Psi$ is positive, then for any strictly positive $\partial\bar{\partial}$-closed $(n-k,n-k)$-form Υ, we have $\int_X \Phi \wedge \Upsilon \geq 0$.

In the other direction, assume $\int_X \Phi \wedge \Upsilon \geq 0$ for any strictly positive $\partial\bar{\partial}$-closed $(n-k,n-k)$-form Υ. Firstly, let us define some subspaces in the real vector space $\mathcal{D}_{\mathbb{R}}^{n-k,n-k}$ consisting of real smooth $(n-k,n-k)$-forms with Fréchet topology. We denote

$$\begin{aligned} E &= \{\Upsilon \in \mathcal{D}_{\mathbb{R}}^{n-k,n-k} | \partial\bar{\partial}\Upsilon = 0\}, \\ C_1 &= \{\Upsilon \in E | \Upsilon \text{ is strictly positive}\}, \\ C_2 &= \{\Upsilon \in \mathcal{D}_{\mathbb{R}}^{n-k,n-k} | \Upsilon \text{ is strictly positive}\}. \end{aligned}$$

Then if we consider Φ as a linear functional on $\mathcal{D}_{\mathbb{R}}^{n-k,n-k}$, we have $\Phi|_{C_1} \geq 0$.

If there exists a $\Upsilon_0 \in C_1$ such that $\Phi(\Upsilon_0) = 0$. Then we consider the affine function $f(t) = \Phi(t\alpha + (1-t)\Upsilon_0)$, where $\alpha \in E$ is fixed. The function $f(t)$ satisfies $f(0) = 0$, moreover, since $\Upsilon_0 \in C_1$ is strictly positive and X is compact, for ε small enough, $f(\pm\varepsilon) \geq 0$ by the assumption. This implies $f(t) \equiv 0$, in particular, $f(1) = \Phi(\alpha) = 0$. By the arbitrariness of $\alpha \in E$, we get $\Phi|_E = 0$, thus $\Phi = i\partial\bar{\partial}\Psi$ for some current Ψ. So in this case, we have $\Phi + i\partial\bar{\partial}(-\Psi) = 0$.

Otherwise, for any $\Upsilon_0 \in C_1$, we have $\Phi(\Upsilon_0) > 0$, i.e., $\Phi|_{C_1} > 0$. Since Φ can be seen as a linear functional on $\mathcal{D}_{\mathbb{R}}^{n-k,n-k}$, we can define its kernel space $ker\Phi$, it's a linear subspace. We denote $F = E \cap ker\Phi$, then $F \cap C_2 = \emptyset$. Next, we need the following geometric Hahn-Banach theorem or Mazur's theorem.

- *Let M be a vector subspace of the topological vector space V. Suppose K is a non-empty convex open subset of V with $K \cap M = \emptyset$. Then there is a closed hyperplane N in V containing M with $K \cap N = \emptyset$.*

The above theorem yields there exists a real (k,k)-current T such that $T|_F = 0$ and $T|_{C_2} > 0$. Take $\Upsilon \in C_1$, then $\Phi(\Upsilon), T(\Upsilon)$ are both positive. So there exists a positive constant λ such that $(\Phi - \lambda T)(\Upsilon) = 0$. Observe that F is codimension one in E and $\Upsilon \in E \backslash F$, thus $\Phi - \lambda T$ is identically zero on E. This fact yields there exists a current Ψ such that $\Phi + i\partial\bar{\partial}\Psi = \lambda T \geq 0$. $\square$

Proof of Remark 4.1.5

Let X be an n-dimensional compact complex manifold, for every real $(1,1)$-form α, we have the space $PSH(X,\alpha)$ consisting of all α-PSH functions. A function u is called α-PSH (α-plurisubharmonic) if u is an upper semi-continuous and locally integrable function such that $\alpha + i\partial\bar{\partial}u \geq 0$ in the sense of currents. We have the following uniform L^1 bound for α-PSH functions.

Lemma 4.5.2. *Let X be an n-dimensional compact complex manifold with a hermitian metric ω and let α be a real $(1,1)$-form, then there exists a positive constant c such that $||u||_{L^1(\omega^n)} = \int_X |u|\omega^n \leq c$ for any $u \in PSH(X,\alpha)$ with $sup_X u = 0$.*

Proof. Since X is compact and α is smooth, there exists a constant B such that $B\omega > \alpha$, then $B\omega + i\partial\bar{\partial}u \geq 0$ for $u \in PSH(X,\alpha)$. Then the above result follows from Proposition 2.1 of [DK09]. □

Proof. From the proof of theorem 4.1.4, we know that a key ingredient is the dependence of $c_\varepsilon, M_\varepsilon$ on ε as ε tends to zero. These constants come from the following family of Monge-Ampère equations :

$$\widetilde{\alpha_\varepsilon}^n = (\alpha_\varepsilon + i\partial\bar{\partial}u_\varepsilon)^n = c_\varepsilon \beta_\varepsilon \wedge G_{m,\varepsilon}.$$

In this case, the uniform L^1 bound in lemma 4.5.2 plays an important role. For c large enough, we have $\psi_\varepsilon, \varphi_\varepsilon + u_\varepsilon$ are all $c\omega$-PSH. Since $sup\psi_\varepsilon = sup(\varphi_\varepsilon + u_\varepsilon) = 0$, if we denote $\varphi_\varepsilon + u_\varepsilon$ by η_ε, we have

$$||\psi_\varepsilon||_{L^1(\omega^n)} + ||\eta_\varepsilon||_{L^1(\omega^n)} < C \tag{4.21}$$

for a uniform constant C.

Firstly, assume $n = 3$, then by (4.21) and $\partial\bar{\partial}\omega = 0$

$$\begin{aligned} c_\varepsilon &= \int_X (\alpha + \varepsilon\omega + i\partial\bar{\partial}\eta_\varepsilon)^3 \\ &= \int_X (\alpha + i\partial\bar{\partial}\eta_\varepsilon)^3 + \varepsilon^3\omega^3 \\ &+ 3\varepsilon\omega \wedge (\alpha + i\partial\bar{\partial}\eta_\varepsilon)^2 + 3\varepsilon^2\omega^2 \wedge (\alpha + i\partial\bar{\partial}\eta_\varepsilon) \\ &= \int_X \alpha^3 + O(\varepsilon). \end{aligned}$$

Thus, $c_\varepsilon > 0$ for ε small and $\lim_{\varepsilon\to 0} c_\varepsilon = c_0$. Similarly, by the definition of M_ε, we have

$$\begin{aligned} M_\varepsilon &= \int_X (\alpha + \varepsilon\omega + i\partial\bar{\partial}\eta_\varepsilon)^2 \wedge (\beta + \varepsilon\omega + i\partial\bar{\partial}\psi_\varepsilon) \\ &= \int_X ((\alpha + i\partial\bar{\partial}\eta_\varepsilon)^2 + \varepsilon^2\omega^2 + 2(\alpha + i\partial\bar{\partial}\eta_\varepsilon) \wedge \varepsilon\omega) \wedge \beta \\ &+ \int_X (\cdots) \wedge \varepsilon\omega + \int_X (\cdots) \wedge i\partial\bar{\partial}\psi_\varepsilon \\ &= r_\varepsilon + s_\varepsilon + t_\varepsilon. \end{aligned}$$

By (4.21) and $\partial\bar{\partial}\omega = 0$ again, it is easy to see that

$$\begin{aligned} r_\varepsilon &= \int_X \alpha^2 \wedge \beta + 2\varepsilon\alpha \wedge \beta \wedge \omega + O(\varepsilon^2), \\ s_\varepsilon &= \varepsilon \int_X \alpha^2 \wedge \omega + O(\varepsilon^2), \\ t_\varepsilon &= O(\varepsilon^2). \end{aligned}$$

So by the above calculation, we get $\lim_{\varepsilon\to 0} M_\varepsilon = M_0 = \int_X \alpha^2 \wedge \beta$. *A priori, it is not obvious whether we have $M_\varepsilon > 0$ for $\varepsilon > 0$ small enough. We claim $M_\varepsilon > 0$ and this depends on $c_0 = \int_X \alpha^3 > 0$.*

Since α and β are nef, we only need to verify $\int_X \alpha^2 \wedge \omega > 0$. Firstly, inspired by [Dem93], we solve the following family of complex Monge-Ampère equations

$$(\alpha + \varepsilon\omega + i\partial\bar{\partial}u_\varepsilon)^3 = U_\varepsilon \omega^3 \tag{4.22}$$

where $sup u_\varepsilon = 0$ and $U_\varepsilon = \int_X (\alpha + \varepsilon\omega + i\partial\bar{\partial}u_\varepsilon)^3 / \int_X \omega^3$ is a positive constant. By the above estimate of c_ε, we know

$$U_\varepsilon = \frac{\int_X (\alpha + \varepsilon\omega + i\partial\bar{\partial}u_\varepsilon)^3}{\int_X \omega^3} = \frac{c_0 + O(\varepsilon)}{\int_X \omega^3}. \tag{4.23}$$

It is easy to see that

$$\int_X (\alpha + \varepsilon\omega + i\partial\bar{\partial}u_\varepsilon)^2 \wedge \omega = \int_X \alpha^2 \wedge \omega + O(\varepsilon), \tag{4.24}$$

or equivalently,

$$\int_X \alpha^2 \wedge \omega = \int_X (\alpha + \varepsilon\omega + i\partial\bar{\partial}u_\varepsilon)^2 \wedge \omega - O(\varepsilon). \tag{4.25}$$

Then the pointwise inequality

$$\frac{(\alpha + \varepsilon\omega + i\partial\bar{\partial}u_\varepsilon)^2 \wedge \omega}{\omega^3} \geq \Big(\frac{(\alpha + \varepsilon\omega + i\partial\bar{\partial}u_\varepsilon)^3}{\omega^3}\Big)^{\frac{2}{3}} \cdot \Big(\frac{\omega^3}{\omega^3}\Big)^{\frac{1}{3}}$$

implies

$$\int_X \alpha^2 \wedge \omega \geq U_\varepsilon^{\frac{2}{3}} \int_X \omega^3 - O(\varepsilon) = (c_0 + O(\varepsilon))^{\frac{2}{3}} \Big(\int_X \omega^3\Big)^{\frac{1}{3}} - O(\varepsilon). \tag{4.26}$$

Then $c_0 > 0$ yields the existence of some positive constant c' such that

$$\int_X \alpha^2 \wedge \omega \geq c'.$$

And this concludes our claim that $M_\varepsilon > 0$ for ε small enough. With these preparations, the proof of Remark 4.1.5 when $n = 3$ is the same as theorem 4.1.4. Similarly, we can also prove the case when $n < 3$. □

Chapitre 5

Characterizing volume via cone duality

For divisors over smooth projective varieties, we show that the volume can be characterized by the duality between the pseudo-effective cone of divisors and the movable cone of curves. Inspired by this result, we give and study a natural intersection-theoretic volume functional for 1-cycles over compact Kähler manifolds. In particular, for numerical equivalence classes of curves over projective varieties, it is closely related to the mobility functional studied by B. Lehmann.

5.1 Introduction

In this section, all projective varieties are defined over $\mathbb{C}$. The volume of a divisor on a projective variety is a non-negative number measuring the positivity of the divisor. Let X be an n-dimensional smooth projective variety, and let D be a divisor on X. By definition, the volume of D is defined to be

$$\mathrm{vol}(D) := \limsup_{m\to\infty} \frac{h^0(X, mD)}{m^n/n!}.$$

Thus $\mathrm{vol}(D)$ measures the asymptotic growth of the dimensions of the section space of multiplied divisors mD. We call D a big divisor if $h^0(X, mD)$ has growth of order m^n as m tends to infinity, that is, D is big if and only if $\mathrm{vol}(D) > 0$. The pseudo-effective cone of divisors (denoted by $\overline{\mathrm{Eff}}^1$) is the closure of the cone generated by numerical classes of big divisors. It contains the cone of ample divisors as a subcone. It is well known that the volume vol depends only on the numerical class of the divisor, and $\mathrm{vol}^{1/n}$ is homogeneous of degree one, concave on the pseudo-effective cone and extends to a continuous function on the whole real Néron-Severi space which is strictly positive exactly on big classes.

In the analytical context, from the work [Bou02b], we know that the volume can be characterized by Monge-Ampère mass ; and from the work [Dem11a], it can even be characterized by Morse-type integrals. In this paper, the starting point is to give a new characterization of the volume of divisors by using cone duality. From the seminal work of Boucksom-Demailly-Paun-Peternell (see [BDPP13]), we know there exists a duality between the pseudo-effective cone of divisors and the cone generated by movable curves, that is,

$$\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X).$$

Using this cone duality and a suitable invariant of movable curve classes, we give the following new volume characterization of divisors by the infimum of intersection numbers between the pairings of $\overline{\mathrm{Eff}}^1$ and Mov_1.

Theorem 5.1.1. *Let X be an n-dimensional smooth projective variety and let $\alpha \in N^1(X, \mathbb{R})$ be a numerical class of divisor. Then the volume of α can be characterized as following :*

$$\mathrm{vol}(\alpha) = \inf_{\gamma \in \mathrm{Mov}_1(X)_1} \max(\alpha \cdot \gamma, 0)^n$$

where $\mathrm{Mov}_1(X)_1$ is a subset of the movable cone $\mathrm{Mov}_1(X)$ (see Definition 5.2.13). Conversely, this volume characterization implies the cone duality $\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X)$. Furthermore, we can also replace the movable cone Mov_1 by the Gauduchon cone $\mathcal{G}$ or balanced cone $\mathcal{B}$ which is generated by special hermitian metrics.

Remark 5.1.2. Under the conjecture on weak transcendental holomorphic Morse inequalities (see [BDPP13]), the above result also holds true for any Bott-Chern $(1,1)$-class over compact Kähler manifolds. In particular, even without this assumption, for any $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ over a hyperkähler manifold X, we have

$$\mathrm{vol}(\alpha) = \inf_{\gamma \in \mathcal{M}_1} \max(\alpha \cdot \gamma, 0)^n.$$

Inspired by the above volume characterization for divisors, using cone dualities, we introduce a volume functional for 1-cycles over compact Kähler manifolds. For smooth projective variety, by Kleiman's criterion, we have the cone duality $\mathrm{Nef}^{1*} = \overline{\mathrm{Eff}}_1$ where Nef^1 is the nef cone generated by nef divisors and $\overline{\mathrm{Eff}}_1$ is the cone generated by irreducible curves. For n-dimensional compact Kähler manifold, by Demailly-Paun's numerical characterization of Kähler cone (see [DP04]), we have the cone duality $\mathcal{K}^* = \mathcal{N}$ where $\mathcal{K}$ is the Kähler cone generated by Kähler classes and $\mathcal{N}$ is the cone generated by d-closed positive $(n-1, n-1)$-currents.

Definition 5.1.3. (1) Let X be an n-dimensional smooth projective variety, and let $\gamma \in N_1(X,\mathbb{R})$ be a numerical equivalence class of curve. Let $\mathrm{Nef}^1(X)_1$ be the set containing all numerical classes of nef divisors of volume one. Then the volume of γ is defined to be

$$\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) = \inf_{\beta \in \mathrm{Nef}^1(X)_1} \max(\beta \cdot \gamma, 0)^{\frac{n}{n-1}}.$$

(2) Let X be an n-dimensional compact Kähler manifold, and let $\gamma \in H^{n-1,n-1}_{BC}(X,\mathbb{R})$ be a Bott-Chern $(n-1,n-1)$-class. Let $\mathcal{K}_1$ be the set containing all Kähler classes of volume one. Then the volume of γ is defined to be

$$\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \inf_{\gamma \in \mathcal{K}_1} \max(\beta \cdot \gamma, 0)^{\frac{n}{n-1}}.$$

From its definition, it is clear $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}^{n-1/n}$ (resp. $\widehat{\mathrm{vol}}_{\mathcal{N}}^{n-1/n}$) is a concave function. It also has other nice properties.

Theorem 5.1.4. *Let X be an n-dimensional smooth projective variety (resp. compact Kähler manifold). Then $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ (resp. $\widehat{\mathrm{vol}}_{\mathcal{N}}$) is a continuous function on the whole vector space $N_1(X,\mathbb{R})$ (resp. $H^{n-1,n-1}_{BC}(X,\mathbb{R})$). Furthermore, $\gamma \in \overline{\mathrm{NE}}^{\circ}$ (resp. $\mathcal{N}^{\circ}$) if and only if $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) > 0$ (resp. $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) > 0$).*

The functional $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ is closely related to the mobility functional recently introduced by Lehmann (see [Leh13b]). Mobility functional for cycles was suggested in [DELV11] as an analogue of the volume function for divisors. The motivation is that one can interpret the volume of a divisor D as an asymptotic measurement of the number of general points contained in members of $|mD|$ as m tends to infinity. Let γ be a numerical equivalence class of k-cycles over an n-dimensional integral projective variety X, following [DELV11], Lehmann defined the mobility of γ as following :

$$\mathrm{mob}(\gamma) := \limsup_{m\to\infty} \frac{\mathrm{mc}(m\gamma)}{m^{\frac{n}{n-k}}/n!},$$

where $\mathrm{mc}(m\gamma)$ is the mobility count of the cycle class $m\gamma$, which is the maximal non-negative integer b such that any b general points of X are contained in a cycle of class $m\gamma$. In particular, we can define the mobility for numerical classes of curves. Lehmann proved that the mobility functional also distinguishes interior points and boundary points. Thus, in the situation of curves, combining with Theorem 5.1.4, we have two functionals with this property. It is interesting to compare mob and $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ over $\overline{\mathrm{Eff}}_1$. The optimistic expectation is that there are two positive constants c_1, c_2 depending only on the dimension of the underlying manifold such that

$$c_1\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) \leq \mathrm{mob}(\gamma) \leq c_2\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma)$$

for any $\gamma \in \overline{\mathrm{Eff}}_1$. Moreover, we expect $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) = \mathrm{mob}(\gamma)$. In this paper, we obtain the positive constant c_2 by using Lehamnn's estimates of mobility count functional *mc*. In the sequel Chapter 6, based on the joint work [LX15] with Lehmann, besides other results, we will obtain the positive constant c_1. Indeed, for any fixed ample divisor A and boundary point $\gamma \in \partial\,\overline{\mathrm{Eff}}_1$, it is not hard to obtain the asymptotic behaviour of the quotient $\mathrm{mob}(\gamma+\varepsilon A^{n-1})/\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma+\varepsilon A^{n-1})$ as ε tends to zero.

Theorem 5.1.5. *Let X be an n-dimensional smooth projective variety, and let $\overline{\mathrm{Eff}}_1$ be the closure of the cone generated by effective 1-cycles. Then for any $\gamma \in \overline{\mathrm{Eff}}_1$, we have*

$$\mathrm{mob}(\gamma) \leq n!2^{4n+1}\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma).$$

And for any fixed ample divisor A and boundary point $\gamma \in \partial\,\overline{\mathrm{Eff}}_1$, there is a positive constant $c(A,\gamma)$ such that $\mathrm{mob}(\gamma+\varepsilon A^{n-1}) \geq c(A,\gamma)\varepsilon\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma+\varepsilon A^{n-1})$. In particular, we have

$$\liminf_{\varepsilon\to 0} \frac{\mathrm{mob}(\gamma+\varepsilon A^{n-1})}{\varepsilon\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma+\varepsilon A^{n-1})} \geq c(A,\gamma).$$

With respect to our volume functional $\widehat{\mathrm{vol}}_{\mathcal{N}}$, we want to study Fujita type approximation results for 1-cycles over compact Kähler manifolds. In this paper, following Boucksom's analytical version of divisorial Zariski decomposition (see [Bou04], [Bou02a]) (for the algebraic approach, see [Nak04]), we study Zariski decomposition for 1-cycles in the sense of Boucksom. In divisorial Zariski decomposition, the negative part is an effective divisor of Kodaira dimension zero, and indeed it contains only one positive $(1,1)$-current. In our setting, we can prove this fact also holds for big 1-cycles. Comparing with other definitions of Zariski decomposition for 1-cycles (see e.g. [FL13]), effectiveness of the negative part is one of its advantage in the sense that it can be seen as a high codimensional analogy of the divisor situation. Using his characterization of volume by Monge-Ampère mass, Boucksom showed that the Zariski projection preserves volume. It is also expected that in our setting the Zariski projection preserves $\widehat{\mathrm{vol}}_{\mathcal{N}}$. Indeed, this follows from our another kind of Zariski decomposition for 1-cycles developed in the sequel Chapter (see also [LX15]), which is more closely related to $\widehat{\mathrm{vol}}_{\mathcal{N}}$.

Theorem 5.1.6. *Let X be an n-dimensional compact Kähler manifold and let $\gamma \in \mathcal{N}^\circ$ be an interior point. Let $\gamma = Z(\gamma) + \{N(\gamma)\}$ be the Zariski decomposition in the sense of Boucksom, then $N(\gamma)$ is an effective curve and it is the unique positive current contained in the negative part $\{N(\gamma)\}$. As a consequence, this implies $\widehat{\mathrm{vol}}_{\mathcal{N}}(\{N(\gamma)\}) = 0$. Moreover, we have $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma))$.*

5.2 Characterizing volume for divisors

5.2.1 Technical preliminaries

Smoothing movable classes

Besides the well known cone duality $\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X)$, we also have cone dualities between the cone defined by positive currents and the cone defined by positive forms. They provide a method to smooth movable classes, which will be useful in volume characterization by using special metrics.

Let X be an n-dimensional compact complex manifold, then we have Bott-Chern cohomology groups $H^{\bullet,\bullet}_{BC}(X,\mathbb{K})$ and Aeppli cohomology groups $H^{\bullet,\bullet}_{A}(X,\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Recall that we have canonical duality between $H^{\bullet,\bullet}_{BC}(X,\mathbb{K})$ and $H^{n-\bullet,n-\bullet}_{A}(X,\mathbb{K})$ (see e.g. [AT13]).

Definition 5.2.1. Let X be an n-dimensional compact complex manifold.
(1) The cone $\mathcal{E}$ is defined to be the convex cone in $H^{1,1}_{BC}(X,\mathbb{R})$ generated by d-closed positive $(1,1)$-currents;
(2) The cone $\mathcal{E}_A$ is defined to be the convex cone in $H^{1,1}_{A}(X,\mathbb{R})$ generated by dd^c-closed positive $(1,1)$-currents;
(3) The balanced cone $\mathcal{B}$ is defined to be the convex cone in $H^{n-1,n-1}_{BC}(X,\mathbb{R})$ generated by d-closed strictly positive $(n-1,n-1)$-forms;
(4) The Gauduchon cone $\mathcal{G}$ is defined to be the convex cone in $H^{n-1,n-1}_{A}(X,\mathbb{R})$ generated by dd^c-closed strictly positive $(n-1,n-1)$-forms.

Under the duality of $H^{1,1}_{BC}(X,\mathbb{R})$ and $H^{n-1,n-1}_{A}(X,\mathbb{R})$ and the duality of $H^{1,1}_{A}(X,\mathbb{R})$ and $H^{n-1,n-1}_{BC}(X,\mathbb{R})$, we have the following cone dualities between the above positive cones.

Proposition 5.2.2. *Let X be an n-dimensional compact complex manifold, then we have $\mathcal{E}^* = \overline{\mathcal{G}}$ and $\mathcal{E}_A^* = \overline{\mathcal{B}}$*

Proof. Indeed, the above cone duality properties are consequences of the geometric form of the Hahn-Banach theorem, for example, see [Sul76], [Lam99a] or [Tom10]. For reader's convenience, let us sketch its proof. Firstly, we prove $\mathcal{E}^* = \overline{\mathcal{G}}$. Let α be a real smooth $(1,1)$-form. Applying Lamari's characterization of positive $(1,1)$-currents, we know that there exists a distribution ψ such that $\alpha + dd^c\psi$ is a positive $(1,1)$-current if and only if

$$\int \alpha \wedge G \geq 0$$

for any dd^c-closed strictly positive $(n-1,n-1)$-form G (thus $G=\omega^{n-1}$ for some Gauduchon metric ω). Under the natural duality of $H^{1,1}_{BC}(X,\mathbb{R})$ and $H^{n-1,n-1}_A(X,\mathbb{R})$, it is clear this implies the cone duality $\mathcal{E}^*=\overline{\mathcal{G}}$. Using the same technique (Hahn-Banach theorem), one can also give a characterization of dd^c-closed positive $(1,1)$-currents. More precisely, there exists a $(0,1)$-current θ such that $\alpha+\partial\theta+\overline{\partial\theta}$ is a positive $(1,1)$-current if and only if

$$\int \alpha\wedge B\geq 0$$

for any d-closed strictly positive $(n-1,n-1)$-form B (thus $B=\omega^{n-1}$ for some balanced metric ω). Under the natural duality of $H^{1,1}_A(X,\mathbb{R})$ and $H^{n-1,n-1}_{BC}(X,\mathbb{R})$, this implies the cone duality $\mathcal{E}^*_A=\overline{\mathcal{B}}$. □

Recall that the cone of movable curves Mov_1 is generated by numerical equivalence classes of curves of the form $\mu_*(\tilde{A}_1\wedge ...\wedge\tilde{A}_{n-1})$, where $\mu:\tilde{X}\to X$ ranges among all modifications with $\tilde{X}$ smooth projective and $\tilde{A}_1,...,\tilde{A}_{n-1}$ range among all ample divisors over $\tilde{X}$. And its transcendental version is the movable cone $\mathcal{M}\subseteq H^{n-1,n-1}_{BC}(X,\mathbb{R})$ over a compact Kähler manifold X. $\mathcal{M}$ is the cone generated by all the Bott-Chern classes of the form $[\mu_*(\tilde{\omega}_1\wedge ...\wedge\tilde{\omega}_{n-1})]_{bc}$, where $\mu:\tilde{X}\to X$ ranges among all modifications with $\tilde{X}$ Kähler and $\tilde{\omega}_1,...,\tilde{\omega}_{n-1}$ range among all Kähler metrics over $\tilde{X}$.

Our first observation is that any current $\mu_*(\tilde{\omega}_1\wedge ...\wedge\tilde{\omega}_{n-1})$ can be smoothed to be a Gauduchon metric G such that $[\mu_*(\tilde{\omega}_1\wedge ...\wedge\tilde{\omega}_{n-1})]_a=[G]_a$.

Proposition 5.2.3. *Let $\mu:\tilde{X}\to X$ be a modification between compact complex manifold, and let $\tilde{G}$ be a Gauduchon metric on $\tilde{X}$. Then $\mu_*\tilde{G}$ can be smoothed to be a Gauduchon metric G such that $[\mu_*\tilde{G}]_a=[G]_a$.*

Proof. From the cone duality $\mathcal{E}^*=\overline{\mathcal{G}}$, in order to prove $[\mu_*\tilde{G}]_a\in\mathcal{G}$, we only need to verify that $[\mu_*\tilde{G}]_a$ is an interior point of $\mathcal{E}^*(=\overline{\mathcal{G}})$. For any $\alpha\in\mathcal{E}\setminus\{[0]_{bc}\}$, since the pull-back $\mu^*\alpha$ is also pseudo-effective, we have

$$[\mu_*\tilde{G}]_a\cdot\alpha=[\tilde{G}]_a\cdot\mu^*\alpha\geq 0.$$

Take a positive current $\tilde{T}\in\mu^*\alpha$, then we have

$$[\tilde{G}]_a\cdot\mu^*\alpha=\int\tilde{G}\wedge\tilde{T}.$$

By the strictly positivity of $\tilde{G}$, $\int\tilde{G}\wedge\tilde{T}=0$ if and only if $\tilde{T}=0$, and this contradicts to our assumption $\alpha=[\mu_*\tilde{T}]_{bc}\in\mathcal{E}\setminus\{[0]_{bc}\}$. Thus $[\mu_*\tilde{G}]_a\cdot\alpha>0$ for any $\alpha\in\mathcal{E}\setminus\{[0]_{bc}\}$, and this implies $[\mu_*\tilde{G}]_a$ is an interior point of $\overline{\mathcal{G}}$, which means that there exists a Gauduchon metric G such that $[\mu_*\tilde{G}]_a=[G]_a$. □

Indeed, the current $\mu_*(\tilde{\omega}_1\wedge ...\wedge\tilde{\omega}_{n-1})$ can not only be smoothed to be a Gauduchon class, it can also smoothed to be a balanced metric B such that $[\mu_*(\tilde{\omega}_1\wedge ...\wedge\tilde{\omega}_{n-1})]_{bc}=[B]_{bc}$. From the proof of Proposition 5.2.3, we see that a key ingredient is that the pull-back of cohomology class in $\mathcal{E}$ contains positive currents. Analogue to this fact, due to a result of [AB95], one can also always pull back Aeppli class in $\mathcal{E}_A$ and get dd^c-closed positive $(1,1)$-currents on the manifold upstairs.

Lemma 5.2.4. *(see [AB95]) Let $\mu:\tilde{X}\to X$ be a modification between compact complex manifold, and let T be a dd^c-closed positive $(1,1)$-current on X. Then there exists a unique dd^c-closed positive $(1,1)$-current $\tilde{T}\in\mu^*[T]_a$ such that $\mu_*\tilde{T}=T$.*

We remark that the above fact is already used by Toma (see [Tom10]), and the following proposition is essentially due to Toma.

Proposition 5.2.5. *Let $\mu:\tilde{X}\to X$ be a modification between compact balanced manifold, and let $\tilde{B}$ be a balanced metric on $\tilde{X}$. Then $\mu_*\tilde{B}$ can be smoothed to be a balanced metric B such that $[\mu_*\tilde{B}]_{bc}=[B]_{bc}$.*

Proof. Similar to the proof of Proposition 5.2.3, we only need to show

$$[\mu_*\tilde{B}]_{bc}\cdot\alpha>0$$

for any $\alpha \in \mathcal{E}_A \setminus \{[0]_a\}$. Now, by using Lemma 5.2.4, for any $\alpha = [T]_a \in \mathcal{E}_A \setminus \{[0]_a\}$, one can find a non-zero dd^c-closed positive $(1,1)$-current $\tilde{T} \in \mu^*\alpha$, so we have

$$[\mu_*\tilde{B}]_{bc} \cdot \alpha = [\tilde{B}]_{bc} \cdot \mu^*\alpha = \int \tilde{B} \wedge \tilde{T} > 0.$$

And as a consequence, there exists a balanced metric B such that $[\mu_*\tilde{B}]_{bc} = [B]_{bc}$. □

An invariant of movable classes

In this subsection, we introduce an (universal) invariant $\mathfrak{M}$ for movable, balanced or Gauduchon classes. This invariant is defined by cone duality and intersection numbers. We will see that they coincide with the volume of Kähler classes if the cohomology classes are given by the $(n-1)$-power of Kähler classes.

Definition 5.2.6. Let X be an n-dimensional compact Kähler manifold, and let γ be a movable (or balanced, or Gauduchon) class. Let $\mathcal{E}_1$ be the set of pseudo-effective classes of volume one. Then the invariant $\mathfrak{M}(\gamma)$ is defined as following :

$$\mathfrak{M}(\gamma) := \inf_{\beta \in \mathcal{E}_1} (\beta \cdot \gamma)^{\frac{n}{n-1}}.$$

Remark 5.2.7. In the case when X is a smooth projective variety, we can also define $\mathfrak{M}(\gamma)$ for $\gamma \in \mathrm{Mov}_1(X)$. In this situation, the pairings $\beta \cdot \gamma$ are the pairings of numerical equivalence classes of divisors and curves.

Remark 5.2.8. Recall that we have the cone duality relations $\mathcal{E}^* = \overline{\mathcal{G}}$ and $\mathcal{E}_A^* = \overline{\mathcal{B}}$. Indeed, under the assumption of the conjectured transcendental cone duality $\mathcal{E}^* = \overline{\mathcal{M}}$ (see [BDPP13]), the movable cone $\mathcal{M}$, the balanced cone $\mathcal{B}$ and the Gauduchon cone $\mathcal{G}$ should be the same, that is, $\mathcal{E}^* = \overline{\mathcal{M}} = \overline{\mathcal{B}} = \overline{\mathcal{G}}$ (see e.g. [FX14a]). This is why we call $\mathfrak{M}$ is an universal invariant associated to movable, balanced and Gauduchon classes over compact Kähler manifolds.

It is clear from the definition of $\mathfrak{M}$ we have

$$\mathfrak{M}(\gamma_1 + \gamma_2)^{\frac{n-1}{n}} \geq \mathfrak{M}(\gamma_1)^{\frac{n-1}{n}} + \mathfrak{M}(\gamma_2)^{\frac{n-1}{n}}.$$

Proposition 5.2.9. *Let X be an n-dimensional compact Kähler manifold, and let $\gamma = \omega^{n-1}$ for some Kähler class ω, then we have $\mathfrak{M}(\gamma) = \mathrm{vol}(\omega)$.*

Proof. Firstly, let $\beta = \frac{\omega}{\mathrm{vol}(\omega)^{\frac{1}{n}}}$, then it is clear that

$$\beta \cdot \omega^{n-1} = \mathrm{vol}(\omega)^{\frac{n-1}{n}},$$

which implies $\mathfrak{M}(\gamma) \leq \mathrm{vol}(\omega)$. On the other hand, we claim that, for any $\beta \in \mathcal{E}$ with $\mathrm{vol}(\beta) = 1$, we have

$$(\beta \cdot \gamma)^{\frac{n}{n-1}} \geq \mathrm{vol}(\omega).$$

This is just the Khovanskii-Teissier inequality which follows from the singular version of Calabi-Yau theorem (see [Bou02b]) : there exists a positive $(1,1)$-current $T \in \beta$ such that

$$T_{ac}^n = \Phi$$

almost everywhere, where $\Phi = \omega^n/\mathrm{vol}(\omega)$ and T_{ac} is the absolutely continuous part of T with respect to Lebesgue measure. Here we use the same symbol ω to denote a Kähler metric in the Kähler class ω. Then we have

$$\beta \cdot \gamma = \int T \wedge \omega^{n-1} \geq \int T_{ac} \wedge \omega^{n-1} \geq \int (\frac{T_{ac}^n}{\Phi})^{\frac{1}{n}} (\frac{\omega^n}{\Phi})^{\frac{n-1}{n}} \Phi = \mathrm{vol}(\omega)^{\frac{n-1}{n}}.$$

This implies the claim, thus finishing our proof. □

An easy corollary is the strict positivity of $\mathfrak{M}$.

Corollary 5.2.10. *Let X be an n-dimensional compact Kähler manifold, and let $\gamma \in \mathcal{G}$ (resp. $\mathcal{M}$ or $\mathcal{B}$) be an interior point, then we have $\mathfrak{M}(\gamma) > 0$.*

Next let $\mu : \tilde{X} \to X$ be a modification between compact Kähler manifolds, we want to study the behaviour of $\mathfrak{M}$ under μ. Firstly, we need the following elementary fact on the transform of the volume of pseudo-effective $(1,1)$-classes under bimeromorphic maps.

Lemma 5.2.11. *Let $\mu : \tilde{X} \to X$ be a modification between n-dimensional compact Kähler manifolds. Assume $\beta \in \mathcal{E}$ is a pseudo-effective class on X, then $\mathrm{vol}(\beta) = \mathrm{vol}(\mu^*\beta)$; Assume $\tilde{\beta} \in \widetilde{\mathcal{E}}$ is a pseudo-effective class on $\tilde{X}$, then $\mathrm{vol}(\tilde{\beta}) \leq \mathrm{vol}(\mu_*\tilde{\beta})$.*

Proof. Recall that the volume of β is defined to be the supremum of Monge-Ampère mass, that is,

$$\mathrm{vol}(\beta) = \sup_T \int T_{ac}^n,$$

where T ranges among all positive $(1,1)$-currents in the class β. For any positive current $T \in \beta$, we obtain a positive current $\mu^*T \in \mu^*\beta$. By the definition of the absolutely part with respect to Lebesgue measure, we have $(\mu^*T)_{ac} = \mu^*T_{ac}$. And T_{ac} is a $(1,1)$-form with L^1_{loc} coefficients. In particular, analytic subset is of zero measure with respect to the measure T_{ac}^n, which yields

$$\int (\mu^*T)_{ac}^n = \int T_{ac}^n.$$

This implies $\mathrm{vol}(\mu^*\beta) \geq \mathrm{vol}(\beta)$. On the other hand, for any positive current $\tilde{T} \in \mu^*\beta$, we get a positive current $\mu_*\tilde{T} \in \beta$. By $(\mu_*\tilde{T})_{ac} = \mu_*\tilde{T}_{ac}$, we obtain $\mathrm{vol}(\mu^*\beta) \leq \mathrm{vol}(\beta)$. All in all we have $\mathrm{vol}(\beta) = \mathrm{vol}(\mu^*\beta)$. Similarly, it is also easy to see $\mathrm{vol}(\tilde{\beta}) \leq \mathrm{vol}(\mu_*\tilde{\beta})$ for any $\tilde{\beta} \in \widetilde{\mathcal{E}}$. □

Now we can show that $\mathfrak{M}$ has the same property as vol under bimeromorphic maps. We only state the result for Kähler manifolds. It is clear that $\mathfrak{M}$ admits an extension to the closure of $\mathcal{G}$ (resp. $\mathcal{M}$ or $\mathcal{B}$).

Proposition 5.2.12. *Let $\mu : \tilde{X} \to X$ be a modification between n-dimensional compact Kähler manifolds. Assume $\gamma \in \overline{\mathcal{G}}$ (resp. $\overline{\mathcal{M}}$ or $\overline{\mathcal{B}}$), then $\mathfrak{M}(\gamma) = \mathfrak{M}(\mu^*\gamma)$; Assume $\tilde{\gamma} \in \overline{\widetilde{\mathcal{G}}}$ (resp. $\overline{\widetilde{\mathcal{M}}}$ or $\overline{\widetilde{\mathcal{B}}}$), then $\mathfrak{M}(\tilde{\gamma}) \leq \mathfrak{M}(\mu_*\tilde{\gamma})$.*

Proof. We firstly consider the pull-back case. By Lemma 5.2.11, for any fixed $\tilde{\beta} \in \tilde{\mathcal{E}}$ with $\mathrm{vol}(\tilde{\beta}) = 1$, we have

$$\tilde{\beta} \cdot \mu^*\gamma = \mu_*\tilde{\beta} \cdot \gamma \geq \frac{\mu_*\tilde{\beta}}{\mathrm{vol}(\mu_*\tilde{\beta})^{1/n}} \cdot \gamma \geq \mathfrak{M}(\gamma)^{\frac{n-1}{n}}.$$

This clearly implies $\mathfrak{M}(\mu^*\gamma) \geq \mathfrak{M}(\gamma)$. For the other direction, for any fixed $\beta \in \mathcal{E}$ with $\mathrm{vol}(\beta) = 1$, using Lemma 5.2.11 again, we have

$$\beta \cdot \gamma = \mu_*(\mu^*\beta) \cdot \gamma = \mu^*\beta \cdot \mu^*\gamma \geq \mathfrak{M}(\mu^*\gamma)^{\frac{n-1}{n}}.$$

Thus $\mathfrak{M}(\mu^*\gamma) \leq \mathfrak{M}(\gamma)$. And as a consequence, we finish the proof of $\mathfrak{M}(\mu^*\gamma) = \mathfrak{M}(\gamma)$. For the push-forward case, the proof of $\mathfrak{M}(\tilde{\gamma}) \leq \mathfrak{M}(\mu_*\tilde{\gamma})$ is the same. □

We remark that the inequality $\mathfrak{M}(\tilde{\gamma}) \leq \mathfrak{M}(\mu_*\tilde{\gamma})$ is important in the characterization of the volume of divisors in the following section.

5.2.2 Volume characterization

In this section, using the invariant $\mathfrak{M}$ introduced in the previous section, we show the volume of divisors can be characterized by cone duality.

Definition 5.2.13. Let X be an n-dimensional smooth projective variety, and let $\mathrm{Mov}_1(X)$ be the cone of movable curves. Then $\mathcal{M}_{NS,1}$ is defined to be the subset containing all $\gamma \in \mathrm{Mov}_1(X)$ with $\mathfrak{M}(\gamma) = 1$. Similarly, for compact Kähler manifolds, we can define $\mathcal{G}_1, \mathcal{M}_1$ and $\mathcal{B}_1$ in the same way.

Theorem 5.2.14. *Let X be an n-dimensional smooth projective variety and let $\alpha \in N^1(X, \mathbb{R})$ be a numerical equivalence class of divisor. Then the volume of α can be characterized as following :*

$$(\star)\ \ \mathrm{vol}(\alpha)^{\frac{1}{n}} = \inf_{\gamma \in \mathrm{Mov}_1(X)_1} \max(\alpha \cdot \gamma, 0).$$

Conversely, this volume characterization implies the cone duality $\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X)$. Moreover, we can also replace the cone of movable curves by Gauduchon or balanced cone, that is,

$$\mathrm{vol}(\alpha)^{\frac{1}{n}} = \inf_{\gamma \in \mathcal{G}_1} \max(\alpha \cdot \gamma, 0) = \inf_{\gamma \in \mathcal{B}_1} \max(\alpha \cdot \gamma, 0).$$

Proof. We first consider the case when α is not pseudo-effective, by the definition of volume of divisors, it is clear $vol(\alpha) = 0$. On the other hand, the cone duality $\overline{\mathrm{Eff}}^{1*} = \mathrm{Mov}_1$ implies there exists some interior point $\gamma \in \mathrm{Mov}_1$ such that $\alpha \cdot \gamma < 0$. Furthermore, using Corollary 5.2.10, we can even normalize γ such that $\mathfrak{M}(\gamma) = 1$. Thus $\inf_{\gamma \in \mathrm{Mov}_1(X)_1} \max(\alpha \cdot \gamma, 0) = 0 = \mathrm{vol}(\alpha)^{\frac{1}{n}}$.

Next consider the case when α is given by a big divisor. By the very definition of $\mathfrak{M}$, for any $\gamma \in \mathrm{Mov}_1$, it is clear that

$$\frac{\alpha}{\mathrm{vol}(\alpha)^{1/n}} \cdot \gamma \geq \mathfrak{M}(\gamma)^{\frac{n-1}{n}},$$

or equivalently,

$$\alpha \cdot \gamma \geq \mathrm{vol}(\alpha)^{\frac{1}{n}} \mathfrak{M}(\gamma)^{\frac{n-1}{n}}.$$

In particular, for any $\gamma \in \mathrm{Mov}_1(X)_1$, this yields $\alpha \cdot \gamma \geq \mathrm{vol}(\alpha)^{\frac{1}{n}}$. Thus we have

$$\mathrm{vol}(\alpha)^{\frac{1}{n}} \leq \inf_{\gamma \in \mathrm{Mov}_1(X)_1} (\alpha \cdot \gamma).$$

In order to prove the equality, we need to show that, for any $\varepsilon > 0$, there exists a movable class $\gamma_\varepsilon \in \mathrm{Mov}_1(X)_1$ such that

$$\alpha \cdot \gamma_\varepsilon \leq \mathrm{vol}(\alpha)^{\frac{1}{n}} + \varepsilon.$$

This mainly depends on approximating Zariski decomposition of Kähler currents and orthogonality estimates of the decomposition (see [BDPP13]). Since α is given by a big divisor, for any $\delta > 0$, there exists a modification $\mu_\delta : X_\delta \to X$ such that $\mu_\delta^* \alpha = \beta_\delta + [E_\delta]$ with β_δ given by an ample divisor and E_δ given by an effective divisor. Moreover, we also have

$$\mathrm{vol}(\alpha) - \delta \leq \mathrm{vol}(\beta_\delta) \leq \mathrm{vol}(\alpha) \tag{5.1}$$

and

$$[E_\delta] \cdot \beta_\delta^{n-1} \leq c(\mathrm{vol}(\alpha) - \mathrm{vol}(\beta_\delta))^{1/2} \tag{5.2}$$

where c is a positive constant depending only on the class α and dimension n. Applying (5.1) and (5.2) to $\alpha \cdot \mu_{\delta *} \beta_\delta^{n-1}$, we get

$$\begin{aligned} \alpha \cdot \mu_{\delta *}(\beta_\delta^{n-1}) &= \mu_\delta^* \alpha \cdot \beta_\delta^{n-1} && (5.3)\\ &= \mathrm{vol}(\beta_\delta) + [E_\delta] \cdot \beta_\delta^{n-1} && (5.4)\\ &\leq \mathrm{vol}(\alpha) + \mathbf{O}(\delta^{1/2}). && (5.5) \end{aligned}$$

Next by Proposition 5.2.9 and Proposition 5.2.12, we know that

$$\mathfrak{M}(\mu_{\delta*}(\beta_\delta^{n-1})) \geq \mathfrak{M}(\beta_\delta^{n-1}) = \mathrm{vol}(\beta_\delta). \tag{5.6}$$

We claim that $\gamma_\delta := \mu_{\delta*}(\beta_\delta^{n-1})/\mathfrak{M}(\mu_{\delta*}(\beta_\delta^{n-1}))^{\frac{n-1}{n}}$ is our desired movable class. Firstly, by the definition of $\mathfrak{M}$, it is obvious that $\mathfrak{M}(\gamma_\delta) = 1$. Secondly, by using (5.1) and (5.6), we can estimate $\alpha\cdot\gamma_\delta$ as following :

$$\alpha \cdot \gamma_\delta \leq \mu_\delta^* \alpha \cdot \frac{\beta_\delta^{n-1}}{\mathrm{vol}(\beta_\delta)^{n-1/n}} \tag{5.7}$$

$$\leq \mathrm{vol}(\alpha)^{1/n} + [\frac{\mathrm{vol}(\alpha)}{(\mathrm{vol}(\alpha)-\delta)^{\frac{n-1}{n}}} - \mathrm{vol}(\alpha)^{1/n}] + \mathbf{O}(\delta^{1/2}) \tag{5.8}$$

Thus, for any $\varepsilon > 0$, we can choose some $\delta(\varepsilon) > 0$, such that $\gamma_{\delta(\varepsilon)}$ is our desired movable class. In summary, we have finished the proof of the equality

$$\mathrm{vol}(\alpha)^{\frac{1}{n}} = \inf_{\gamma\in \mathrm{Mov}_1(X)_1} (\alpha\cdot\gamma)$$

for big class α.

In the case when α lies on the boundary of $\mathcal{E}_{NS}$, for any $\varepsilon > 0$ and ample divisor A, apply the above proved equality for $\alpha + \varepsilon A$, we have

$$\mathrm{vol}(\alpha+\varepsilon A)^{\frac{1}{n}} = \inf_{\gamma\in \mathrm{Mov}_1(X)_1} (\alpha+\varepsilon A)\cdot\gamma.$$

Take inf on both sides with respect to $\varepsilon > 0$, we get the equality for boundary class.

Now we show that $(\star)$ implies $\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X)$. It is obvious $\overline{\mathrm{Eff}}^1(X) \subseteq \mathrm{Mov}_1(X)^*$. In order to prove the converse inclusion, we only need to show : if α is an interior point of $\mathrm{Mov}_1(X)^*$, then α is also an interior point of $\overline{\mathrm{Eff}}^1(X)$ (or equivalently, $\mathrm{vol}(\alpha) > 0$). Fix an ample divisor A. Since α is an interior point of $\mathrm{Mov}_1(X)^*$, for $\varepsilon > 0$ small, $\alpha - \varepsilon A$ also lies in the interior of $\mathrm{Mov}_1(X)^*$. In particular, we have $\alpha\cdot\gamma > \varepsilon A\cdot\gamma$ for any $\gamma \in \mathrm{Mov}_1(X)\setminus[0]$. Then $(\star)$ implies

$$\mathrm{vol}(\alpha)^{\frac{1}{n}} = \inf_{\gamma\in \mathrm{Mov}_1(X)_1} \max(\alpha\cdot\gamma, 0) \geq \varepsilon\,\mathrm{vol}(A)^{\frac{1}{n}} > 0.$$

For the volume characterization by Gauduchon or balanced cone, from the proof for movable cone, one can see that if we can show $\gamma_{\delta(\varepsilon)}$ can be smoothed to be a Gauduchon or balanced class, then we have the desired equality. And this just follows from the results of Proposition 5.2.3 and Proposition 5.2.5. □

Remark 5.2.15. Let X be an n-dimensional compact Kähler manifold. Under the assumption of the conjectured weak transcendental holomorphic Morse inequality, that is,

$$\mathrm{vol}(\alpha-\beta) \geq \alpha^n - n\alpha^{n-1}\cdot\beta$$

for any nef classes α, β (for recent progress of this problem, see [Xia13], [Pop14]), then we will also have orthogonality estimates for interior points of $\mathcal{E}$ (see [BDPP13]). By the arguments above, we will have volume characterization for any Bott-Chern $(1,1)$-class α, that is,

$$\mathrm{vol}(\alpha)^{\frac{1}{n}} = \inf_{\gamma\in\mathcal{M}_1} \max(\alpha\cdot\gamma, 0).$$

Moreover, this implies the cone duality $\mathcal{E}^* = \overline{\mathcal{M}}$. Thus it is natural to ask whether one can prove this volume characterization without using orthogonality estimates of approximation Zariski decomposition. And this also provides new perspectives to prove the conjectured cone duality $\mathcal{E}^* = \overline{\mathcal{M}}$.

5.3 Volume functional for 1-cycles

5.3.1 Definition and properties

Inspired by Theorem 5.2.14, using cone dualities, we introduce a volume functional for the numerical equivalence class of curves over smooth projective varieties and a volume functional for Bott-Chern $(n-1,n-1)$-classes over compact Kähler manifolds. For smooth projective variety, we have the nef cone $\mathrm{Nef}^1(X)$ generated by nef divisors and the cone $\overline{\mathrm{Eff}}_1(X)$ generated by irreducible curves. Then we have the cone duality

$$\mathrm{Nef}^1(X)^* = \overline{\mathrm{Eff}}_1(X)$$

which is just Kleiman's criterion. For n-dimensional compact Kähler manifold, we have Kähler cone $\mathcal{K}$ generated by Kähler classes and the cone $\mathcal{N}$ generated by d-closed positive $(n-1,n-1)$-currents. Then we have the cone duality

$$\mathcal{K}^* = \mathcal{N}$$

which follows from Demailly-Paun's numerical characterization of Kähler cone (see [DP04]).

Now we can give the following definition.

Definition 5.3.1. (1) Let X be an n-dimensional smooth projective variety, and let $\gamma \in N_1(X,\mathbb{R})$ be a numerical equivalence class of curve. Let $\mathrm{Nef}^1(X)_1$ be the set containing all numerical classes of nef divisors of volume one. Then the volume of γ is defined to be

$$\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma) = \inf_{\beta \in \mathrm{Nef}^1(X)_1} \max(\beta\cdot\gamma, 0)^{\frac{n}{n-1}}.$$

(2) Let X be an n-dimensional compact Kähler manifold, and let $\gamma \in H^{n-1,n-1}_{BC}(X,\mathbb{R})$ be a Bott-Chern $(n-1,n-1)$-class. Let $\mathcal{K}_1$ be the set containing all Kähler classes of volume one. Then the volume of γ is defined to be

$$\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \inf_{\gamma \in \mathcal{K}_1} \max(\beta\cdot\gamma, 0)^{\frac{n}{n-1}}.$$

Remark 5.3.2. In the volume characterization of divisors, using the cone duality $\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X)$ (or the conjectured $\mathcal{E}^* = \overline{\mathcal{M}}$), we introduce an invariant $\mathfrak{M}$ for movable classes (see Definition 5.2.6). Now $\widehat{\mathrm{vol}}_{\mathcal{N}}$ gives another invariant of movable classes when it restricts on $\overline{\mathcal{M}}$. From their definitions, it is clear that we have $\mathfrak{M}(\gamma) \leq \widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)$ for any $\gamma \in \overline{\mathcal{M}}$. Unlike vol giving an uniform volume functional on $\mathcal{E}$ and $\overline{\mathcal{K}}$, *we do not know whether they would coincide on the movable cone*. In general, the nef cone $\overline{\mathcal{K}}$ can be strictly contained in $\mathcal{E}$, it seems possible that $\mathfrak{M}$ may be smaller than $\widehat{\mathrm{vol}}_{\mathcal{N}}$ – we will study this problem in details in Chapter 6. However, if X is a projective or compact Kähler surface, both our volume functional $\widehat{\mathrm{vol}}_{\mathcal{N}}$ (or $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$) and $\mathfrak{M}$ coincide with the usual volume for pseudo-effective classes.

Example 5.3.3. To illustrate the definition of volume functional for 1-cycles, we propose to do some concrete calculations on an example similar to the one due to Cutkosky [Cut86] (we learnt this from [Bou04]). Let Y be a smooth projective surface, and let D, H be two very ample divisors over Y. Let $X = \mathbb{P}(\mathcal{O}(D) \oplus \mathcal{O}(-H))$ with its canonical projection $\pi : X \to Y$. Denote by $L = \mathcal{O}_X(1)$ the tautological bundle of X, then the nef cone $\mathcal{K}_X$ of X is generated by $\pi^*\mathcal{K}_Y$ and $\pi^*H + L$. In Cutkosky's example, Y is an Abelian surface (or more generally, a projective surface with $\overline{\mathcal{K}}_Y = \mathcal{E}_Y$). For simplicity, we consider the very simple case $Y = \mathbb{P}^2$ with $D = \mathcal{O}(d), H = \mathcal{O}(1)$, then we have

$$L^3 = (d-1)^2 + d, \pi^*H^2 \cdot L = 1, \pi^*H \cdot L^2 = d-1, \pi^*H^3 = 0.$$

Let $\alpha = a\pi^*H + b(\pi^*H + L)$ with $a, b \in \mathbb{R}_+$ be a nef class, then the volume of α is as following

$$\mathrm{vol}(\alpha) = b^3((d-1)^2) + d) + 3b^2(a+b)(d-1) + 3(a+b)^2 b.$$

Consider the 1-cycle $\gamma(x,y) = x\pi^*H^2 + y\pi^*H \cdot L$ with $x, y \geq 0$, then we have

$$\alpha \cdot \gamma(x,y) = (a+b)y + bx + by(d-1).$$

From the above expressions, we have an explicit formula of $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$. In particular, if we take $d=1$, then

$$\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma(x,y)) = \underset{\substack{b^3+3(a+b)^2b=1 \\ a,b\geq 0}}{inf} (by+(a+b)x)^{\frac{3}{2}}.$$

The volume functionals $\widehat{\mathrm{vol}}_{\mathcal{N}}$ and $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ have many nice properties. For simplicity, we only state the result for $\widehat{\mathrm{vol}}_{\mathcal{N}}$. The argument for $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ is similar.

Theorem 5.3.4. *Let X be an n-dimensional compact Kähler manifold. Then $\widehat{\mathrm{vol}}_{\mathcal{N}}$ has the following properties :*
(1) $\widehat{\mathrm{vol}}_{\mathcal{N}}^{\frac{n-1}{n}}$ is concave and homogeneous of degree one.
(2) $\widehat{\mathrm{vol}}_{\mathcal{N}}$ is continuous on the whole vector space $H^{n-1,n-1}_{BC}(X,\mathbb{R})$.
(3) $\gamma\in\mathcal{N}^\circ$ if and only if $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)>0$.

Proof. Property (1) just follows from the definition of $\widehat{\mathrm{vol}}_{\mathcal{N}}$. Now let us first prove property (3). Let $\gamma\in\mathcal{N}^\circ$ be an interior point, we want to show that $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)>0$. $\gamma\in\mathcal{N}^\circ$ means that there exists some Kähler class ω such that $\gamma-\omega^{n-1}\in\mathcal{N}$, this implies $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)\geq\widehat{\mathrm{vol}}_{\mathcal{N}}(\omega^{n-1})$. We claim that

$$\widehat{\mathrm{vol}}_{\mathcal{N}}(\omega^{n-1})=\mathrm{vol}(\omega),$$

which yields $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)\geq\mathrm{vol}(\omega)>0$. The proof of this claim is the same with Proposition 5.2.9, so we omit it. Conversely, we need to show that if $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)>0$ then $\gamma\in\mathcal{N}^\circ$. Otherwise, $\gamma\in\partial\mathcal{N}\setminus\{[0]_{BC}\}$. And the cone duality $\mathcal{K}^*=\mathcal{N}$ implies there exists some $\theta\in\overline{\mathcal{K}}\setminus\{[0]_{BC}\}$ such that $\theta\cdot\gamma=0$. Fix a Kähler class ω. For any $\varepsilon>0$, we consider the Kähler class $\theta+\varepsilon\omega$ and the following intersection number

$$\rho_\varepsilon := \frac{\theta+\varepsilon\omega}{\mathrm{vol}(\theta+\varepsilon\omega)^{1/n}}\cdot\gamma.$$

Since $\theta\in\overline{\mathcal{K}}\setminus\{[0]_{BC}\}$, the class θ contains at least one non-zero positive current, then we have $\theta\cdot\omega^{n-1}>0$. And we have

$$\mathrm{vol}(\theta+\varepsilon\omega)^{\frac{1}{n}}\geq n\theta\cdot\omega^{n-1}\varepsilon^{\frac{n-1}{n}}=\mathbf{O}(\varepsilon^{\frac{n-1}{n}}).$$

Using $\theta\cdot\gamma=0$, we get $\rho_\varepsilon\leq\mathbf{O}(\varepsilon^{1/n})$. Thus, $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)=0$. In conclusion, we have proved that $\gamma\in\mathcal{N}^\circ$ if and only if $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)>0$.

Next we consider the continuity of $\widehat{\mathrm{vol}}_{\mathcal{N}}$, thus proving property (2). Since concave function defined in a convex set is continuous in the interior. In order to show the continuity of $\widehat{\mathrm{vol}}_{\mathcal{N}}$, we need to verify

$$\lim_{\varepsilon\to 0}\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma+\varepsilon\omega^{n-1})=0$$

for any $\gamma\in\partial\mathcal{N}\setminus\{[0]_{BC}\}$ and any Kähler class ω. Indeed, for $\gamma\in\partial\mathcal{N}\setminus\{[0]_{BC}\}$, we will prove

$$\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma+\varepsilon\omega^{n-1})\leq\mathbf{O}(\varepsilon^{\frac{1}{n-1}}). \tag{5.9}$$

The arguments are similar with the estimation of ρ_ε, but with little modification. Once again, using the fact $\gamma\in\partial\mathcal{N}\setminus\{[0]_{BC}\}$, there exists some $\theta\in\overline{\mathcal{K}}\setminus\{[0]_{BC}\}$ such that $\theta\cdot\gamma=0$. We consider the following intersection number

$$\rho_{\delta,\varepsilon} := \frac{\theta+\delta\omega}{\mathrm{vol}(\theta+\delta\omega)^{1/n}}\cdot(\gamma+\varepsilon\omega^{n-1}) \tag{5.10}$$

with δ positive to be determined. Using $\theta\cdot\gamma=0$ and $\theta\cdot\omega^{n-1}>0$ again, it is easy to see that

$$\rho_{\delta,\varepsilon}\leq\mathbf{O}(\delta^{\frac{1}{n}}+\delta^{\frac{1}{n}}\varepsilon+\delta^{-\frac{n-1}{n}}\varepsilon). \tag{5.11}$$

Take $\delta=\varepsilon$, we get $\rho_{\delta,\varepsilon}\leq\mathbf{O}(\varepsilon^{1/n})$, which implies

$$\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma+\varepsilon\omega^{n-1})\leq\mathbf{O}(\varepsilon^{\frac{1}{n-1}}), \tag{5.12}$$

thus finishing the proof of continuity. $\square$

We give a new interpretation of our volume functional as the infinimum of a family of geometric norms. We only work for $\widehat{\mathrm{vol}}_{\mathcal{N}}$, and the arguments go through mutatis mutandis for the volume functional $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$.

Lemma 5.3.5. *(see also Corollary 2.8 of [FL13]) Let X be an n-dimensional compact Kähler manifold. Then any Kähler class α gives a norm $||\cdot||_\alpha$ over $H_{BC}^{n-1,n-1}(X,\mathbb{R})$. Moreover, for $\gamma \in \mathcal{N}$, we have $||\gamma||_\alpha = \alpha\cdot\gamma$.*

Proof. For any fixed Kähler class α, there exist $d = h^{1,1}$ Kähler classes $\alpha_1, ..., \alpha_d$ such that $\alpha_1, ..., \alpha_d$ constitute a basis of the real vector space $H_{BC}^{1,1}(X,\mathbb{R})$, and $\alpha = \sum_{1\leq i\leq d} \alpha_i$. Then for any $\eta \in H_{BC}^{n-1,n-1}(X,\mathbb{R})$, we define $||\eta||_\alpha$ as following :

$$||\eta||_\alpha = \sum_{1\leq i\leq d} |\alpha_i\cdot\eta|.$$

It is clear that the above $||\cdot||_\alpha$ is a norm, since it is just the sum of absolute values of the coordinates with respect to the basis $\alpha_1, ..., \alpha_d$. Now, for $\gamma\in\mathcal{N}$, we have $\alpha_i\cdot\gamma\geq 0$. And this implies

$$||\gamma||_\alpha = \sum_{1\leq i\leq d} \alpha_i\cdot\gamma = \alpha\cdot\gamma.$$

□

Now by the definition of $\widehat{\mathrm{vol}}_{\mathcal{N}}$, we have the following proposition.

Proposition 5.3.6. *Let X be an n-dimensional compact Kähler manifold, then for $\gamma\in\mathcal{N}$ we have*

$$\widehat{\mathrm{vol}}_{\mathcal{N}}^{\frac{n-1}{n}}(\gamma) = \inf_{\alpha\in\mathcal{K}_1} ||\gamma||_\alpha.$$

5.3.2 Relation with mobility

In this section, we focus on comparing $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$ and Lehmann's mobility functional mob for 1-cycles over smooth projective variety. Firstly, let us recall the definition of mobility of numerical equivalence classes of curves. Let γ be a 1-cycle class over X of dimension n, the mobility of γ is defined as following :

$$\mathrm{mob}(\gamma) := \limsup_{m\to\infty}\frac{\mathrm{mc}(m\gamma)}{m^{\frac{n}{n-1}}/n!},$$

where $\mathrm{mc}(m\gamma)$ is the mobility count of the 1-cycle class $m\gamma$ defined as the maximal non-negative integer b such that any b general points of X are contained in a 1-cycle of class $m\gamma$. From Theorem 5.3.4 and Theorem A in [Leh13b], both functionals take positive values exactly in $\overline{\mathrm{Eff}}_1^\circ$ and are continuous over $\overline{\mathrm{Eff}}_1$. Moreover, both of them are homogeneous over $\overline{\mathrm{Eff}}_1$, it is natural to propose the following question.

Conjecture 5.3.7. *Let X be a smooth projective variety, then* $\mathrm{mob} = \widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$*, or at least there exist two positive constants c_1 and c_2 depending only on the dimension of X such that*

$$c_1\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}} \leq \mathrm{mob} \leq c_2\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}.$$

We observe that the constant c_2 is provided by the upper bound estimation of mobility count. For any fixed ample divisor A and boundary point $\gamma\in\partial\overline{\mathrm{Eff}}_1$, it is clear that if we can find a positive constant $c(A,\gamma)$ such that

$$\liminf_{\varepsilon\to 0}\frac{\mathrm{mob}(\gamma+\varepsilon A^{n-1})}{\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma+\varepsilon A^{n-1})} \geq c(A,\gamma),$$

then we can obtain the desired uniform constant c_1. In this direction, we can easily get a weaker asymptotic behaviour as ε tends to zero.

Theorem 5.3.8. *Let X be an n-dimensional smooth projective variety. Then for any $\gamma \in \overline{\mathrm{Eff}}_1$, we have*

$$\mathrm{mob}(\gamma) \leq n! 2^{4n+1} \widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma).$$

And for any fixed ample divisor A and boundary point $\gamma \in \partial \overline{\mathrm{Eff}}_1$, there is a positive constant $c(A, \gamma)$ such that

$$\mathrm{mob}(\gamma + \varepsilon A^{n-1}) \geq c(A, \gamma) \varepsilon \widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma + \varepsilon A^{n-1}).$$

In particular, we have

$$\liminf_{\varepsilon \to 0} \frac{\mathrm{mob}(\gamma + \varepsilon A^{n-1})}{\varepsilon \widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma + \varepsilon A^{n-1})} \geq c(A, \gamma).$$

Proof. The upper bound c_2 relies on the estimations of mobility counts. By homogeneity and continuity, we only need to consider the case when γ is given by a 1-cycle with $\mathbb{Z}$-coefficients. We need Lehmann's upper bound estimation (see [Leh13b, Theorem 6.24]) : let A be a very ample divisor and let s be a positive integer such that $A \cdot \gamma \leq s \,\mathrm{vol}(A)$, then

$$\mathrm{mc}(\gamma) \leq 2^{4n+1} s^{\frac{n}{n-1}} \mathrm{vol}(A).$$

Indeed, by inspection of the proof of [Leh13b, Theorem 6.24], any real number $s \geq 1$ is sufficient for the above estimation of $\mathrm{mc}(\gamma)$. Fix a $\mathbb{Q}$-ample divisor α, then there exists a positive integer m_α such that $m_\alpha \alpha$ is very ample. And for this very ample divisor $m_\alpha \alpha$, there exists a positive integer k_α such that

$$\frac{m_\alpha \alpha \cdot k\gamma}{\mathrm{vol}(m_\alpha \alpha)} \geq 1 \tag{5.13}$$

for all positive integer $k \geq k_\alpha$ Applying Lehmann's mobility count estimation to $k\gamma$ when $A = m_\alpha \alpha$ and $s = \frac{m_\alpha \alpha \cdot k\gamma}{\mathrm{vol}(m_\alpha \alpha)}$, we get

$$\begin{aligned} \mathrm{mc}(k\gamma) &\leq 2^{4n+1} \Big(\frac{m_\alpha \alpha \cdot k\gamma}{\mathrm{vol}(m_\alpha \alpha)}\Big)^{\frac{n}{n-1}} \mathrm{vol}(m_\alpha \alpha) && (5.14)\\ &= 2^{4n+1} \Big(\frac{\alpha}{\mathrm{vol}(\alpha)^{1/n}} \cdot k\gamma\Big)^{\frac{n}{n-1}}. && (5.15) \end{aligned}$$

This yields the upper bound of $\mathrm{mob}(\gamma)$:

$$\mathrm{mob}(\gamma) = \limsup_{k \to \infty} \frac{\mathrm{mc}(k\gamma)}{k^{n/n-1}/n!} \leq n! 2^{4n+1} \Big(\frac{\alpha}{\mathrm{vol}(\alpha)^{1/n}} \cdot \gamma\Big)^{\frac{n}{n-1}}. \tag{5.16}$$

Since any point of the ample cone can be approximated by $\mathbb{Q}$-ample divisors, we obtain the desired equality

$$\mathrm{mob}(\gamma) \leq n! 2^{4n+1} \widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma). \tag{5.17}$$

Now let us consider the lower bound. In the proof of Theorem 5.3.4 (see (5.9)-(5.12)), we obtain the estimation of $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma + \varepsilon \omega^{n-1})$. Using similar argument, we can get the same estimation of $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma + \varepsilon A^{n-1})$ with $\gamma \in \partial \overline{\mathrm{Eff}}_1$ and A ample, that is,

$$\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma + \varepsilon A^{n-1}) \leq \mathbf{O}(\varepsilon^{\frac{1}{n-1}}). \tag{5.18}$$

By the basic property of mobility functional (see Lemma 6.17 of [Leh13b]), we have

$$\mathrm{mob}(\gamma + \varepsilon A^{n-1}) \geq \mathrm{mob}(\varepsilon A^{n-1}) = \mathbf{O}(\varepsilon^{\frac{n}{n-1}}). \tag{5.19}$$

Thus we get

$$\mathrm{mob}(\gamma + \varepsilon A^{n-1}) \geq c(A, \gamma) \varepsilon \widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma + \varepsilon A^{n-1})$$

for some positive constant $c(A,\gamma)$. In particular, we have

$$\liminf_{\varepsilon\to 0}\frac{\mathrm{mob}(\gamma+\varepsilon A^{n-1})}{\varepsilon\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(\gamma+\varepsilon A^{n-1})}\geq c(A,\gamma).$$

□

Remark 5.3.9. In order to obtain such a uniform lower bound c_1, one possible way is to find a better estimation of $\mathrm{mob}(\gamma+\varepsilon A^{n-1})$, that is,

$$\mathrm{mob}(\gamma+\varepsilon A^{n-1})\geq \mathbf{O}(\varepsilon^{\frac{1}{n-1}}),$$

as ε tends to zero. To obtain this, we need a deeper understanding of the functional mob.

Remark 5.3.10. Just from its definition, the mobility functional mob seems very hard to compute. For example, even in the case of complete intersection of ample divisor (see Question 7.1 of [Leh13b]), we do not know how to calculate its mobility. However, using our volume functional, we have seen that $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}(A^{n-1})=\mathrm{vol}(A)$ for any ample divisor A. For the concavity of mob, it is conjectured (see Conjecture 6.20 of [Leh13b]) that

$$\mathrm{mob}(\gamma_1+\gamma_2)^{\frac{n-1}{n}}\geq \mathrm{mob}(\gamma_1)^{\frac{n-1}{n}}+\mathrm{mob}(\gamma_2)^{\frac{n-1}{n}}.$$

For our $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$, concavity just follows from its definition (see Theorem 5.3.4). Thus concavity of mob will follow if we can prove $\mathrm{mob}=\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$.

5.3.3 Towards Fujita approximation for 1-cycles

In the work of [FL13], Fulger and Lehmann proved the existence of Zariski decomposition for big cycles with respect to mobility functional. Moreover, they also proved a Fujita type approximation for numerical class of curves. Our goal is to give such a Fujita type approximation for Bott-Chern classes of d-closed positive $(n-1,n-1)$-currents over compact Kähler manifolds with respect to our volume functional $\widehat{\mathrm{vol}}_{\mathcal{N}}$, thus also give a Fujita type approximation for numerical class of curves over projective variety with respect to $\widehat{\mathrm{vol}}_{\overline{\mathrm{NE}}}$. Analogue to Fujita approximation for Kähler currents (see inequality (5.1)), one may conjecture the following :

Let X be an n-dimensional compact Kähler manifold and let $\gamma\in\mathcal{N}^\circ$. Then for any $\varepsilon>0$, there exists a proper modification $\mu:\widetilde{X}\to X$ with $\widetilde{X}$ Kähler such that $\mu^*\gamma=\beta_\varepsilon+[C_\varepsilon]$ and $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)-\varepsilon\leq\widehat{\mathrm{vol}}_{\mathcal{N}}(\beta_\varepsilon)\leq\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)$, where β_ε is an interior point of movable cone $\widetilde{\mathcal{M}}$ (or balanced cone $\widetilde{\mathcal{B}}$) and C_ε is an effective curve.

Indeed, if we have the decomposition $\mu^*\gamma=\beta_\varepsilon+[C_\varepsilon]$, then we have $\gamma-\mu_*\beta_\varepsilon\in\mathcal{N}$. This implies $\widehat{\mathrm{vol}}_{\mathcal{N}}(\mu_*\beta_\varepsilon)\leq\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)$. Now similar to Proposition 5.2.12, it is easy to see $\widehat{\mathrm{vol}}_{\mathcal{N}}(\beta_\varepsilon)\leq\widehat{\mathrm{vol}}_{\mathcal{N}}(\mu_*\beta_\varepsilon)$. Thus the above expected decomposition automatically implies $\widehat{\mathrm{vol}}_{\mathcal{N}}(\beta_\varepsilon)\leq\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)$. Unfortunately, the pull-back $\mu^*\gamma$ need not to be a pseudo-effective class in general. Note that $\mu^*\gamma$ is pseudo-effective over $\widetilde{X}$ if and only if $\mu^*\gamma\cdot\tilde{\alpha}\geq 0$ for any Kähler class $\tilde{\alpha}$, which is equivalent to $\gamma\cdot\mu_*\tilde{\alpha}\geq 0$. In general, $\mu_*\tilde{\alpha}$ is not a nef class on X. By the cone duality $\overline{\mathcal{K}}^*=\mathcal{N}$, we have $\gamma\cdot\mu_*\tilde{\alpha}<0$ if $\mu_*\tilde{\alpha}\notin\overline{\mathcal{K}}$. Anyhow, if $\gamma\in\overline{\mathcal{M}}$ is movable, then its pull-back $\mu^*\gamma$ is also movable (thus pseudo-effective). For movable classes, it is possible to obtain the conjectured decomposition $\mu^*\gamma=\beta_\varepsilon+[C_\varepsilon]$ with all desired properties.

To prove Fujita approximation for γ with respect to our volume functional, the first step of our strategy is to decompose γ over the underlying manifold X into some "good" part with its volume near the volume of γ. We also call it the positive part, and call the difference the negative part. Here "good" means that we can find a positive current in the class with less singularities, then we may get a movable or balanced class from its pull-back on some Kähler manifold $\widetilde{X}$ such that its volume is as near $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)$ as possible (this will be developed in our subsequent chapter). Besides the desired positive part, we also want to obtain some effective curve from such a decomposition. For the Zariski decomposition of Fulger and Lehmann, in general the negative part is not the class of an effective curve (see Example 5.18 of [FL13]). In the work [Bou04], Boucksom defined a beautiful divisorial Zariski decomposition for any pseudo-effective $(1, 1)$-class over compact complex manifolds. Boucksom's definition is totally analytic which depends on Siu decomposition of positive currents (see [Siu74]). And it can be seen as a cohomology version of Siu decomposition. As Siu decomposition holds for d-closed positive currents of any bidegree, the method of Boucksom provides a possible Zariski decomposition for pseudo-effective $(n-1, n-1)$-classes. However, unlike the $(1, 1)$-classes, we do not have an analogue of Demailly's regularization theorem (see [Dem92]) for d-closed positive $(n-1, n-1)$-currents. We know little about the singularities of such currents. Thus we can not expect too much about such decompositions. Following Boucksom's method of divisorial Zariski decomposition, we give such a decomposition for pseudo-effective $(n-1, n-1)$-classes. It shares many nice properties with divisorial Zariski decomposition.

Firstly, we give the definition of minimal multiplicity.

Definition 5.3.11. Let X be an n-dimensional compact Kähler manifold with a Kähler metric ω, and let $\gamma \in \mathcal{N}$ be a pseudo-effective $(n-1, n-1)$-class.
(1) The minimal multiplicity of γ at the point x is defined to be

$$\nu(\gamma, x) := \sup_{\varepsilon > 0} \inf_{T_\varepsilon} \nu(T_\varepsilon, x),$$

where $T_\varepsilon \in \gamma$ ranges among all currents such that $T_\varepsilon \geq -\varepsilon\omega^{n-1}$ (we also denote this set by $\gamma[-\varepsilon\omega^{n-1}]$) and $\nu(T_\varepsilon, x)$ is the Lelong number of T_ε at x.
(2) For any irreducible curve C, the minimal multiplicity of γ along C is defined to be

$$\nu(\gamma, C) := \inf_{x \in C} \nu(\gamma, x).$$

Remark 5.3.12. It is easy to see that $\nu(\gamma, x)$ is finite. And $\nu(\gamma, C) = \nu(\gamma, x)$ for a generic point $x \in C$, here generic means outside at most countable union of analytic subsets.

Definition 5.3.13. Let $\gamma \in \mathcal{N}$ be a pseudo-effective $(n-1, n-1)$-class, the negative part $N(\gamma)$ of γ is defined to be $N(\gamma) := \sum \nu(\gamma, C)[C]$, where C ranges among all irreducible curves on X. And the positive part $Z(\gamma)$ of γ is defined to be $Z(\gamma) := \gamma - \{N(\gamma)\}$. And we call $\gamma = Z(\gamma) + \{N(\gamma)\}$ the Zariski decomposition of γ.

Intuitively, the positive part $Z(\gamma)$ should share almost all positivity of γ and the negative part should have very little positivity. Indeed, in the divisorial Zariski decomposition case, using his volume characterization by Monge-Ampère mass, Boucksom showed that $\mathrm{vol}(\alpha) = \mathrm{vol}(Z(\alpha))$ for any $\alpha \in \mathcal{E}$ over compact Kähler manifolds. In our setting, one way to compare the positivity of $Z(\gamma)$ and γ is to compare their respective volumes $\widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma))$ and $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)$. For the negative part $\{N(\gamma)\}$, like the one in divisorial Zariski decomposition, we find $N(\gamma)$ is an effective curve which is very rigidly embedded in X if we assume γ is an interior point. This is an advantage compared with the other decompositions (e.g. the decompositions in [FL13] and [LX15]).

Remark 5.3.14. From Theorem 5.3.4, it is clear that $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma)) = 0$ if $\gamma \in \partial\mathcal{N}$. And by the concavity of $\widehat{\mathrm{vol}}_{\mathcal{N}}$, the equality $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma))$ will imply $\widehat{\mathrm{vol}}_{\mathcal{N}}(\{N(\gamma)\}) = 0$.

Theorem 5.3.15. *Let X be an n-dimensional compact Kähler manifold and let $\gamma \in \mathcal{N}^\circ$ be an interior point. Let $\gamma = Z(\gamma) + \{N(\gamma)\}$ be the Zariski decomposition in the sense of Boucksom, then $N(\gamma)$ is an effective curve and it is the unique positive current contained in the negative part $\{N(\gamma)\}$. As a consequence, this implies $\widehat{\mathrm{vol}}_{\mathcal{N}}(\{N(\gamma)\}) = 0$. Moreover, we have $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma))$.*

Proof. We first prove the first part of the above theorem. Indeed, as the Zariski decomposition here is an $(n-1, n-1)$-analogue of Boucksom's divisorial Zariski decomposition, the statement concerning

$N(\gamma)$ can be proved using almost the same arguments as in [Bou04]. In [Bou04], some arguments use Demailly's regularization theorem. As we do not have such a regularization theorem for $(n-1,n-1)$-currents, for reader's convenience, we present the details here. The assumption $\gamma \in \mathcal{N}^\circ$ will play the role as Demailly's regularization theorem in the divisorial Zariski decomposition situation.

We first show the claim $(*) : N(\gamma) = \sum \nu(\gamma, C)[C]$ is the unique positive current in the class $\{N(\gamma)\}$ if $\gamma \in \mathcal{N}^\circ$. We remark that claim $(*)$ implies $\widehat{\mathrm{vol}}_{\mathcal{N}}(\{N(\gamma)\}) = 0$ (or equivalently, $\{N(\gamma)\} \in \partial\mathcal{N}$). Otherwise, $\{N(\gamma)\} \in \mathcal{N}^\circ$. Fix a Kähler class ω, then there exists a positive constant $\delta > 0$ such that $\{N(\gamma)\} - \delta\omega^{n-1} \in \mathcal{N}^\circ$. In particular, there exists a positive current $\Theta \in \{N(\gamma)\}$ such that $\Theta \geq \delta\omega^{n-1}$. Here we use the same symbol ω to represent a Kähler metric in the Kähler class ω. Note that $H_A^{n-1,n-1}(X,\mathbb{R}) \neq \{[0]_A\}$ over compact Kähler manifolds, thus there exists some smooth $(n-2,n-2)$-form ψ such that $i\partial\bar{\partial}\psi \neq 0$. For $\varepsilon > 0$ small enough,

$$\Theta_\varepsilon := \Theta + \varepsilon i\partial\bar{\partial}\psi \in \{N(\gamma)\}$$

is a positive current and $\Theta_\varepsilon \neq \Theta$, contradicting our claim $(*)$.

Now let us begin the proof of the claim $(*)$. The proof is divided into several steps.

Lemma 5.3.16. *Let $\gamma \in \mathcal{N}^\circ$, then $\nu(\gamma, C) = \inf_{0 \leq T \in \gamma} \nu(T, C)$ for any irreducible curve C.*

Proof. To prove this, we only need to verify $\nu(\gamma, x) = \inf_{0 \leq T \in \gamma} \nu(T, x)$ for any point x, then we will have

$$\nu(\gamma, C) = \inf_{x \in C} \nu(\gamma, x) = \inf_{x \in C} \inf_{0 \leq T \in \gamma} \nu(T, x) = \inf_{0 \leq T \in \gamma} \nu(T, C).$$

From the definition of $\nu(\gamma, x)$, we only need to prove

$$\nu(\gamma, x) \geq \inf_{0 \leq T \in \gamma} \nu(T, x). \tag{5.20}$$

As $\gamma \in \mathcal{N}^\circ$, there exists a positive current $T \in \gamma$ such that $T \geq \beta^{n-1}$ for some Kähler metric β. Fix $\varepsilon > 0$, for any $\delta > 0$ there exists a current $T_{\varepsilon,\delta} \in \gamma[-\varepsilon\beta^{n-1}]$ such that

$$\nu(T_{\varepsilon,\delta}, x) - \delta < \inf_{T_\varepsilon} \nu(T_\varepsilon, x), \tag{5.21}$$

where T_ε ranges among $\gamma[-\varepsilon\beta^{n-1}]$. Since $T \geq \beta^{n-1}$, we have $(1-\varepsilon)T_{\varepsilon,\delta} + \varepsilon T \geq \varepsilon^2\beta^{n-1}$ which is a positive current in γ, thus

$$\inf_{0 \leq T \in \gamma} \nu(T, x) \leq \nu((1-\varepsilon)T_{\varepsilon,\delta} + \varepsilon T, x) \tag{5.22}$$

$$\leq (1-\varepsilon)\inf_{T_\varepsilon} \nu(T_\varepsilon, x) + (1-\varepsilon)\delta + \varepsilon\nu(T, x). \tag{5.23}$$

Now let $\delta \to 0$ and then let $\varepsilon \to 0$, we get the desired inequality $\nu(\gamma, x) \geq \inf_{0 \leq T \in \gamma} \nu(T, x)$. □

Lemma 5.3.17. *(compare with Proposition 3.8 of [Bou04]) Let $\gamma \in \mathcal{N}^\circ$, then $Z(\gamma) \in \mathcal{N}^\circ$ and $\nu(Z(\gamma), C) = 0$.*

Proof. Once again, $\gamma \in \mathcal{N}^\circ$ implies there exists a positive current $T \in \gamma$ such that $T \geq \beta^{n-1}$ for some Kähler metric β. Apply Siu decomposition to the d-closed positive current $T - \beta^{n-1}$:

$$T - \beta^{n-1} = R + \sum \nu(T - \beta^{n-1}, C)[C] = R + \sum \nu(T, C)[C]$$

for some residue positive current R. Then the definition of $N(\gamma)$ implies

$$T - \beta^{n-1} - N(\gamma) \geq 0,$$

which yields $T-N(\gamma)\geq\beta^{n-1}$. This implies $Z(\gamma)=\{T-N(\gamma)\}\in\mathcal{N}^{\circ}$. Indeed, by the above arguments, Siu decomposition also shows that any positive current in $Z(\gamma)$ are of the form $T-N(\gamma)$ for some positive current $T\in\gamma$. With Lemma 5.3.16 and this fact, we get

$$\nu(Z(\gamma),C)=\inf_{0\leq\Gamma\in Z(\gamma)}\nu(\Gamma,C) \tag{5.24}$$

$$=\inf_{0\leq T\in\gamma}\nu(T-N(\gamma),C) \tag{5.25}$$

$$=\nu(\gamma,C)-\nu(\gamma,C)=0. \tag{5.26}$$

□

Lemma 5.3.18. *Let $\gamma\in\mathcal{N}^{\circ}$, then $\{N(\gamma)\}=\{N(\{N(\gamma)\})\}$.*

Proof. By the definition of $Z(\cdot)$, it is easy to see that $Z(\gamma_1+\gamma_2)-Z(\gamma_1)-Z(\gamma_2)\in\mathcal{N}$ for any two $\gamma_1,\gamma_2\in\mathcal{N}$. In particular, we have $Z(\gamma)-Z(Z(\gamma))-Z(\{N(\gamma)\})\in\mathcal{N}$. Now Lemma 5.3.17 implies $N(Z(\gamma))=\sum\nu(Z(\gamma),C)[C]=0$, so we have $Z(Z(\gamma))=Z(\gamma)-\{N(Z(\gamma))\}=Z(\gamma)$. And this yields $Z(\{N(\gamma)\})=0$, which is equivalent to the equality $\{N(\gamma)\}=\{N(\{N(\gamma)\})\}$. □

Now we can finish the proof of claim $(*)$. Firstly, by the definition of $N(\{N(\gamma)\})$ and Siu decomposition, we have $N(\gamma)\geq N(\{N(\gamma)\})$. As Lemma 5.3.18 shows they lie in the same Bott-Chern class, we must have $N(\gamma)=N(\{N(\gamma)\})$. For any positive current $T\in\{N(\gamma)\}$, using Siu decomposition and the definition of $N(\{N(\gamma)\})$ again, we have

$$T\geq\sum\nu(T,C)[C]\geq N(\{N(\gamma)\})=N(\gamma).$$

Thus $T=N(\gamma)$, and $N(\gamma)$ is the unique positive current in the class $\{N(\gamma)\}$.

Next we show $N(\gamma)$ is an effective curve, that is, it is a finite sum of irreducible curves. Indeed, we will show $N(\gamma)$ is a sum of at most $\rho=\dim_{\mathbb{R}}N_1(X,\mathbb{R})$ irreducible curves. This follows from the following lemma.

Lemma 5.3.19. *(compare with Proposition 3.11 of [Bou04]) Let $\gamma\in\mathcal{N}^{\circ}$, and let S the set of irreducible curves C satisfying $\nu(\gamma,C)>0$, then $\#S\leq\rho$.*

Proof. Take finite curves $C_1,...,C_k\in S$, and let $\Gamma=\sum_{i=1}^k a_i[C_i]$ with $a_i\in\mathbb{R}$. We claim that if the class $\{\Gamma\}=0$ then all $a_i=0$. This of course yields $\#S\leq\rho$. Write $\Gamma=\Gamma_+-\Gamma_-$ such that both Γ_+ and Γ_- are positive. Since we have assumed $\{\Gamma\}=0$, we have $N(\{\Gamma_+\})=N(\{\Gamma_-\})$. By the definition of $N(\gamma)$, we can take a positive constant c large enough such that $\{cN(\gamma)-\Gamma_+\}\in\mathcal{N}$. By Lemma 5.3.18 we know $Z(\{cN(\gamma)\})=cZ(\{N(\gamma)\})=0$, which implies $Z(\{\Gamma_+\})=0$. So we have $\{\Gamma_+\}=\{N(\{\Gamma_+\})\}$, and this implies $\Gamma_+=N(\{\Gamma_+\})$. This also holds for Γ_-. Combining with $N(\{\Gamma_+\})=N(\{\Gamma_-\})$, we get $\Gamma=\Gamma_+-\Gamma_-=0$, which proves our claim. □

Finally let us prove Zariski projection preserves $\widehat{vol}_{\mathcal{N}}$. By the Zariski decomposition developed in the next chapter (see Theorem 6.5.4), we know $\gamma\in\mathcal{N}^{\circ}$ can be uniquely decomposed as following :

$$\gamma=B_\gamma^{n-1}+\zeta_\gamma \tag{5.27}$$

with B_γ big and nef, $B_\gamma\cdot\zeta_\gamma=0$, $\widehat{\mathrm{vol}}_{\mathcal{N}}(B_\gamma^{n-1})=\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma)$ and $\zeta_\gamma\in\partial\mathcal{N}$. Denote by the same symbol B_γ a smooth $(1,1)$-form in the class B_γ. Since B_γ is nef, for any $\varepsilon>0$ there exists a smooth function ψ_ε such that $B_\gamma+\varepsilon\omega+i\partial\bar{\partial}\psi_\varepsilon>0$. From this, it is easy to see for any $\varepsilon>0$ there exists a smooth $(n-2,n-2)$-form Ψ_ε such that

$$\Omega_\varepsilon:=B_\gamma^{n-1}+i\partial\bar{\partial}\Psi_\varepsilon\geq-\varepsilon\omega^{n-1}.$$

Denote by T_γ a positive $(n-1,n-1)$-current in the class ζ_γ, then $\Omega_\varepsilon + T_\gamma \in \gamma[-\varepsilon\omega^{n-1}]$. And by the definition of minimal multiplicity (see Definition 5.3.11), we get

$$\nu(\gamma,x) = \sup_{\varepsilon>0} \inf_{T_\varepsilon} \nu(T_\varepsilon, x) \tag{5.28}$$

$$\leq \sup_{\varepsilon>0} \nu(\Omega_\varepsilon + T_\gamma, x) \tag{5.29}$$

$$= \nu(T_\gamma, x). \tag{5.30}$$

The last line follows because Ω_ε is smooth. By Siu decomposition, the above inequality implies $\zeta_\gamma - \{N(\gamma)\} \in \mathcal{N}$. Thus $Z(\gamma) - B_\gamma^{n-1} = \zeta_\gamma - \{N(\gamma)\} \in \mathcal{N}$, which yields

$$\widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma)) \geq \widehat{\mathrm{vol}}_{\mathcal{N}}(B_\gamma^{n-1}).$$

Combining with $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) \geq \widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma))$ and $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \widehat{\mathrm{vol}}_{\mathcal{N}}(B_\gamma^{n-1})$, we finish the proof of the equality $\widehat{\mathrm{vol}}_{\mathcal{N}}(\gamma) = \widehat{\mathrm{vol}}_{\mathcal{N}}(Z(\gamma))$. $\square$

Remark 5.3.20. It will be interesting to know whether the statement for $N(\gamma)$ in Theorem 5.3.15 is still true for $\gamma \in \partial\mathcal{N}$. Our above arguments show that the assumption $\gamma \in \mathcal{N}^\circ$ is important in Lemma 5.3.16. And we need Lemma 5.3.16 to prove the other lemmas.

Remark 5.3.21. One may expect that $Z(\gamma)$ could be represented by some positive smooth $(n-1,n-1)$-form, more precisely, one may expect $Z(\gamma) \in \overline{\mathcal{M}}$. Thus, by Proposition 5.2.3 and Proposition 5.2.5, there exists a smooth positive $(n-1,n-1)$-form in the class $Z(\gamma)$ if $Z(\gamma)$ is an interior point of $\overline{\mathcal{M}}$. However, in general, $Z(\gamma)$ could not be a movable class. Let $\pi : X \to \mathbb{P}^3$ be the blow-up along a point, and let $E = \mathbb{P}^2$ be the exceptional divisor. Let ω_{FS} be the Fubini-Study metric of $\mathbb{P}^3$ and let $\mathbb{P}^1 \subseteq E$ be a line of E, then we claim that

$$Z(\{\pi^*(\omega_{FS}^2) + [\mathbb{P}^1]\}) = \{\pi^*(\omega_{FS}^2) + [\mathbb{P}^1]\} \in \mathcal{N}^\circ \setminus \overline{\mathcal{M}}.$$

Firstly, it is easy to see $\{\pi^*(\omega_{FS}^2) + [\mathbb{P}^1]\} \in \mathcal{N}^\circ$ which of course implies $Z(\{\pi^*(\omega_{FS}^2) + [\mathbb{P}^1]\}) \in \mathcal{N}^\circ$. For any point x, we can always choose an integration current in the class $[\mathbb{P}^1]$ but with its support avoiding x. Then we have $\nu(\{\pi^*(\omega_{FS}^2) + [\mathbb{P}^1]\}, x) = 0$, which yields the equality

$$Z(\{\pi^*(\omega_{FS}^2) + [\mathbb{P}^1]\}) = \{\pi^*(\omega_{FS}^2) + [\mathbb{P}^1]\}.$$

Since we have $\{\pi^*(\omega_{FS}^2) + [\mathbb{P}^1]\} \cdot E = -1$, the class $\{\pi^*(\omega_{FS}^2) + [\mathbb{P}^1]\}$ can not be movable. Comparing with the Zariski decompositions developed in [FL13] and [LX15], $Z(\gamma)$ not always being movable is its disadvantage in the sense that this is not analogous to the "usual" definitions. Anyhow, if $\gamma \in \mathcal{N}^\circ$, then Lemma 5.3.16 and Lemma 5.3.17 show that we can always choose a positive current in the class $Z(\gamma)$ with its Lelong number along any curve being arbitrarily small. In some sense, this means that $Z(\gamma)$ is less singular than γ. Indeed, $Z(\gamma) \in \overline{\mathcal{K}}$ if X is a Kähler surface.

At the end of this section, we show that Zariski decomposition for 1-cycles is trivial for compact Kähler manifold with nef tangent bundle.

Proposition 5.3.22. *Let X be a compact Kähler manifold with nef tangent bundle, then $\gamma = Z(\gamma)$ for any $\gamma \in \mathcal{N}$. Indeed, we will have $\gamma = Z(\gamma) \in \overline{\mathcal{M}}$.*

Proof. This follows from Demailly's regularization theorem of positive $(1,1)$-currents and the previous work on transcendental holomorphic Morse inequality.

If TX is nef, then $\overline{\mathcal{K}} = \mathcal{E}$ (see Corollary 1.5 of [Dem92]). Now let $\alpha, \beta \in \overline{\mathcal{K}}$ be two nef classes such that $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$, then $\alpha - \beta$ must be an interior point of $\mathcal{E}$ (see [Pop14]). By $\overline{\mathcal{K}} = \mathcal{E}$, $\alpha - \beta$ must be a Kähler class. In particular, $\alpha - t\beta \in \overline{\mathcal{K}}$ for $t \in [0,1]$.

Consider the difference $\mathrm{vol}(\alpha-\beta)-\mathrm{vol}(\alpha)$, we have

$$\begin{aligned}\mathrm{vol}(\alpha-\beta)-\mathrm{vol}(\alpha) &= \int_0^1 \frac{d}{dt}\mathrm{vol}(\alpha-t\beta)dt\\ &= \int_0^1 -n(\alpha-t\beta)^{n-1}\cdot\beta dt\\ &\geq -n\alpha^{n-1}\cdot\beta,\end{aligned}$$

thus $\mathrm{vol}(\alpha-\beta)\geq\alpha^n-n\alpha^{n-1}\cdot\beta$. Using the same arguments as [BDPP13], this of course implies the cone duality $\mathcal{E}^*=\overline{\mathcal{M}}$. And this yields $\mathcal{E}^*=\overline{\mathcal{M}}=\overline{\mathcal{B}}$ (see e.g. [FX14a]). Using $\overline{\mathcal{K}}=\mathcal{E}$ again, $\overline{\mathcal{K}}^*=\mathcal{N}$ implies $\mathcal{N}=\overline{\mathcal{B}}$. Since $\gamma\in\mathcal{N}=\overline{\mathcal{B}}$, for any $\varepsilon>0$ there exists a smooth $(n-1,n-1)$-form $\Omega_\varepsilon\in\gamma$ such that $\Omega_\varepsilon\geq-\varepsilon\omega^{n-1}$. Now by the definition of minimal multiplicity (see Definition 5.3.11), we get $\nu(\gamma,x)=0$ for every point, yielding $N(\gamma)=0$. This implies $\gamma=Z(\gamma)$. □

5.4 Further discussions

5.4.1 Another invariant of movable class

As remarked in the previous section, under the assumption of the conjecture on transcendental holomorphic Morse inequality, we would have $\overline{\mathcal{M}}=\overline{\mathcal{G}}=\overline{\mathcal{B}}$. Thus, any invariant of Gauduchon or balanced classes would be an invariant of movable class. Inspired by our previous work [FX14a], we introduce another invariant $\mathfrak{M}_{CY}$ of Gauduchon class by using form-type Calabi-Yau equations (or complex Monge-Ampère equations for $(n-1)$-plurisubharmonic functions) (see e.g. [FWW10], [TW13b], [TW13a]).

Definition 5.4.1. Let X be an n-dimensional compact Kähler manifold, and let γ be a Gauduchon class. Then we define $\mathfrak{M}_{CY}(\gamma)$ as following :

$$\mathfrak{M}_{CY}(\gamma):=\sup_{\Phi,\omega}\{c_{\Phi,\omega}\}$$

where $c_{\Phi,\omega}$ is a positive constant satisfying $\omega^n=c_{\Phi,\omega}\Phi$ such that Φ is a smooth volume form with $\int\Phi=1$ and $\omega^{n-1}\in\gamma$ is a Gauduchon metric.

Assume $\gamma=\alpha^{n-1}$ for some $\alpha\in\mathcal{K}$, we prove that $\mathfrak{M}_{CY}(\gamma)=\mathrm{vol}(\alpha)$.

Proposition 5.4.2. *Let X be an n-dimensional compact Kähler manifold, and let $\gamma=\alpha^{n-1}$ for some Kähler class α. Then we have $\mathfrak{M}_{CY}(\gamma)=\mathrm{vol}(\alpha)$.*

Proof. Firstly, since α is a Kähler class, by Calabi-Yau theorem (see [Yau78]) there exists an unique Kähler metric $\alpha_u\in\alpha$ such that $\alpha_u^n=\mathrm{vol}(\alpha)\Phi$. In particular, $c_{\Phi,\alpha_u}=\mathrm{vol}(\alpha)$, thus $\mathrm{vol}(\alpha)\leq\mathfrak{M}_{CY}(\gamma)$. We claim that, for any Φ,ω in the definition of $\mathfrak{M}_{CY}(\gamma)$, we have

$$c_{\Phi,\omega}\leq\mathrm{vol}(\alpha).$$

For any fixed such Φ,ω, we first apply Calabi-Yau theorem to find a Kähler metric α_ψ such that

$$\alpha_\psi^n=\frac{\mathrm{vol}(\alpha)}{c_{\Phi,\omega}}\omega^n.$$

Using the following pointwise inequality

$$\omega^{n-1}\wedge\alpha_\psi\geq(\frac{\alpha_\psi^n}{\omega^n})^{\frac{1}{n}}\omega^n$$

and $\omega^{n-1}\in\gamma=\alpha^{n-1}$ being Gauduchon, we estimate $\mathrm{vol}(\alpha)$ as following :

$$\mathrm{vol}(\alpha)=\int\gamma\wedge\alpha=\int\omega^{n-1}\wedge\alpha_\psi\geq\int(\frac{\alpha_\psi^n}{\omega^n})^{\frac{1}{n}}\omega^n=\mathrm{vol}(\alpha)^{\frac{1}{n}}c_{\Phi,\omega}^{\frac{n-1}{n}}.$$

This of course implies $\mathfrak{M}_{CY}(\gamma)\leq\mathrm{vol}(\alpha)$. Combining with $\mathrm{vol}(\alpha)\leq\mathfrak{M}_{CY}(\gamma)$, we get the desired equality $\mathfrak{M}_{CY}(\gamma)=\mathrm{vol}(\alpha)$. □

Note that $\mathfrak{M}_{CY}$ is an analytical invariant by solving non-linear PDEs, and $\mathfrak{M}$ is an intersection-theoretic invariant. It will be very interesting to compare $\mathfrak{M}_{CY}$ and $\mathfrak{M}$, and we have the following proposition.

Proposition 5.4.3. *Let X be an n-dimensional compact Kähler manifold, and let γ be a Gauduchon class. Then we always have $\mathfrak{M}_{CY}(\gamma) \leq \mathfrak{M}(\gamma)$. Moreover, they coincide over Kähler classes, that is, $\mathfrak{M}_{CY}(\alpha^{n-1}) = \mathfrak{M}(\alpha^{n-1})$ for any Kähler class α.*

Proof. For any smooth volume form Φ with $\int \Phi = 1$ and any $\beta \in \mathcal{E}$ with $\mathrm{vol}(\beta) = 1$, by the singular version of Calabi-Yau theorem (see [Bou02b]), there exists a positive $(1,1)$-current $T \in \beta$ such that $T_{ac}^n = \Phi$ almost everywhere. Now for any Gauduchon metric $\omega^{n-1} \in \gamma$ in the definition of $c_{\Phi,\omega}$, we get

$$\beta \cdot \gamma = \int T \wedge \omega^{n-1} \geq \int T_{ac} \wedge \omega^{n-1} \geq \int (\frac{T_{ac}^n}{\Phi})^{\frac{1}{n}} (\frac{\omega^n}{\Phi})^{\frac{n-1}{n}} \Phi = c_{\Phi,\omega}^{\frac{n-1}{n}}.$$

Since β, ω^{n-1} and Φ are (conditionally) arbitrary, we get $\mathfrak{M}_{CY}(\gamma) \leq \mathfrak{M}(\gamma)$.

By Proposition 5.2.9 and Proposition 5.4.2, we have $\mathfrak{M}_{CY}(\alpha^{n-1}) = \mathfrak{M}(\alpha^{n-1})$ for any Kähler class α. $\square$

Remark 5.4.4. The above proposition also implies that $\mathfrak{M}_{CY}(\gamma)$ is always well defined over compact Kähler manifolds, that is, $\mathfrak{M}_{CY}(\gamma) < \infty$. This is not immediately obvious from its definition.

Remark 5.4.5. Let X be an n-dimensional compact Kähler manifold, *we do not know whether $\mathfrak{M}_{CY}(\gamma) = \mathfrak{M}(\gamma)$ for any $\gamma \in \mathcal{G}$*. As we always have $\mathfrak{M}_{CY}(\gamma) \leq \mathfrak{M}(\gamma)$, we only need to show $\mathfrak{M}_{CY}(\gamma) \geq \mathfrak{M}(\gamma)$.

We also want to know the behaviour of $\mathfrak{M}_{CY}$ under bimeromorphic maps (compare with Proposition 5.2.12). In particular, *we do not know whether we have $\mathfrak{M}_{CY}(\mu_*\tilde{\gamma}) \geq \mathfrak{M}_{CY}(\tilde{\gamma})$*. If this would be true, then we can use this invariant in Theorem 5.2.14. It will also be very interesting to study the concavity of $\mathfrak{M}_{CY}$. To study these problems, we need know more about the family of constants $c_{\Phi,\omega}$ in the definition of $\mathfrak{M}_{CY}$.

5.4.2 A general approach

This section comes from a suggestion of Mattias Jonsson – more results will be developed in Chapter 6. Let $\mathcal{C} \subseteq V$ be a proper convex cone of a real vector space. Let $u : \overline{\mathcal{C}} \to \mathbb{R}_+$ be a continuous function. Let $p > 1$ be a constant. Let $\mathcal{C}^* \subseteq V^*$ be the dual of $\mathcal{C}$. In general, we can define the dual of u in the following way :

$$\widehat{u}(x^*) := \inf_{y \in \mathcal{C}_1} (x^* \cdot y)^q,$$

where $\mathcal{C}_1 = \{y \in \mathcal{C} |\ u(y) = 1\}$ and $\frac{1}{p} + \frac{1}{q} = 1$. This is similar to some kind of Legendre-Fenchel transform. It is easy to see $\widehat{u}^{\frac{1}{q}}$ is concave and homogeneous of degree one over $\mathcal{C}^*$. Let us assume $u^{1/p}$ is concave and homogeneous of degree one. Then we have

$$\widehat{u}(x^*) := \inf_{y \in \mathcal{C}_{\geq 1}} (x^* \cdot y)^q,$$

where $\mathcal{C}_{\geq 1} = \{y \in \mathcal{C} |\ u(y) \geq 1\}$. Since $u^{1/p}$ is concave, $\mathcal{C}_{\geq 1}$ is a convex closed subset of $\mathcal{C}$.

In our definition of $\widehat{\mathrm{vol}}_{\mathcal{N}}$ for 1-cycles over compact Kähler manifold, we have $\mathcal{C} = \mathcal{K}$, $u = \mathrm{vol}$ and $p = 1/n$. For $1 < k < n-1$, let $\mathcal{N}_k \in H_{BC}^{k,k}(X, \mathbb{R})$ be the cone generated by d-closed positive (k,k)-currents. It will be interesting if one can generalize this kind of construction of volume to $\mathcal{N}_k$, thus define a volume functional for general k-cycles. Principally, we first need to define a function u on some kind of smooth positive $(n-k, n-k)$-forms. However, unlike the case for the cone $\mathcal{N}$, the structure of the dual of $\mathcal{N}_k$ is not clear (and indeed this problem is still widely open). As a starting point, it will be very interesting to carry out the above general approach over toric varieties.

Chapitre 6

Zariski decomposition of curves on algebraic varieties

We introduce a Zariski decomposition for curve classes and use it to develop the theory of the volume function for curves defined in the previous Chapter 5. For toric varieties and for hyperkähler manifolds the Zariski decomposition admits an interesting geometric interpretation. With the decomposition, we prove some fundamental positivity results for curve classes, such as a Morse-type inequality. We compare the volume of a curve class with its mobility, yielding some surprising results about asymptotic point counts. Finally, we give a number of applications to birational geometry, including a refined structure theorem for the movable cone of curves.

6.1 Introduction

In [Zar62] Zariski introduced a fundamental tool for studying linear series on a surface now known as a Zariski decomposition. Over the past 50 years the Zariski decomposition and its generalizations to divisors in higher dimensions have played a central role in birational geometry. We introduce an analogous decomposition for curve classes on varieties of arbitrary dimension. Our decomposition is defined for big curve classes – elements of the interior of the pseudo-effective cone of curves $\overline{\mathrm{Eff}}_1(X)$. Throughout we work over $\mathbb{C}$, but the main results also hold over an algebraically closed field or in the Kähler setting (see Section 6.1.5).

Definition 6.1.1. Let X be a projective variety of dimension n and let $\alpha \in N_1(X)$ be a big curve class. Then a Zariski decomposition for α is a decomposition

$$\alpha = B^{n-1} + \gamma$$

where B is a big and nef $\mathbb{R}$-Cartier divisor class, γ is pseudo-effective, and $B \cdot \gamma = 0$. We call B^{n-1} the "positive part" and γ the "negative part" of the decomposition.

This definition directly generalizes Zariski's original definition, which (for big classes) is given by similar intersection criteria. It also generalizes the σ-decomposition of [Nak04], and mirrors the Zariski decomposition of [FL13], in the following sense. The basic feature of a Zariski decomposition is that the positive part should retain all the "positivity" of the original class. In our setting, we will measure the positivity of a curve class using an interesting new volume-type function defined in [Xia15a] (see also Chapter 5).

Definition 6.1.2. (see [Xia15a, Definition 1.1]) Let X be a projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then the volume of α is defined to be

$$\widehat{\mathrm{vol}}(\alpha) = \inf_{A \text{ big and nef divisor class}} \left(\frac{A \cdot \alpha}{\mathrm{vol}(A)^{1/n}} \right)^{\frac{n}{n-1}}.$$

We say that a big and nef divisor class A computes $\widehat{\mathrm{vol}}(\alpha)$ if this infimum is achieved by A. When α is a curve class that is not pseudo-effective, we set $\widehat{\mathrm{vol}}(\alpha) = 0$.

This is a kind of polar transformation of the volume function for divisors. It is motivated by the realization that the volume of a divisor has a similar intersection-theoretic description against curves as in [Xia15a, Theorem 2.1]. [Xia15a] proves that $\widehat{\mathrm{vol}}$ satisfies many of the desirable analytic features of the volume for divisors.

By [FL13, Proposition 5.3], we know that the σ-decomposition $L = P_\sigma(L) + N_\sigma(L)$ is the unique decomposition of L into a movable piece and a pseudo-effective piece such that $\mathrm{vol}(L) = \mathrm{vol}(P_\sigma(L))$. In the same way, the decomposition of Definition 6.1.1 is compatible with the volume function for curves :

Theorem 6.1.3. *Let X be a projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)^\circ$ be a big curve class. Then α admits a unique Zariski decomposition $\alpha = B^{n-1} + \gamma$. Furthermore,*

$$\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(B^{n-1}) = \mathrm{vol}(B)$$

and B is the unique big and nef divisor class with this property satisfying $B^{n-1} \preceq \alpha$. Any big and nef divisor class computing $\widehat{\mathrm{vol}}(\alpha)$ is proportional to B.

We define the complete intersection cone $\mathrm{CI}_1(X)$ to be the closure of the set of classes of the form A^{n-1} for an ample divisor A on X. The positive part of the Zariski decomposition takes values in $\mathrm{CI}_1(X)$.

Our goal is to develop the theory of Zariski decompositions of curves and the theory of $\widehat{\mathrm{vol}}$. Due to their close relationship, we will see that is very fruitful to develop the two theories in parallel. In particular, we recover Zariski's original intuition that asymptotic point counts coincide with numerical invariants for curves.

Example 6.1.4. If X is an algebraic surface, then the Zariski decomposition provided by Theorem 6.1.3 coincides (for big classes) with the numerical version of the classical definition of [Zar62]. Indeed, using Proposition 6.5.14 one sees that the negative part γ is represented by an effective curve N. The self-intersection matrix of N must be negative-definite by the Hodge Index Theorem. (See e.g. [Nak04] for another perspective focusing on the volume function.)

Example 6.1.5. An important feature of Zariski decompositions and $\widehat{\mathrm{vol}}$ for curves is that they can be calculated via intersection theory directly on X once one has identified the nef cone of divisors. (In contrast, the analogous divisor constructions may require passing to birational models of X to admit an interpretation via intersection theory.) This is illustrated by Example 6.5.5 where we calculate the Zariski decomposition of any curve class on the projective bundle over $\mathbb{P}^1$ defined by $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)$.

Example 6.1.6. If X is a Mori Dream Space, then the movable cone of divisors admits a chamber structure defined via the ample cones on small $\mathbb{Q}$-factorial modifications. This chamber structure behaves compatibly with the σ-decomposition and the volume function for divisors.

By the results of [HK00], for curves we obtain a complementary picture. The movable cone of curves admits a "chamber structure" defined via the complete intersection cones on small $\mathbb{Q}$-factorial modifications. However, the Zariski decomposition and volume of curves are no longer invariant under small $\mathbb{Q}$-factorial modifications but instead exactly reflect the changing structure of the pseudo-effective cone of curves. Thus the Zariski decomposition is the right tool to understand the birational geometry of movable curves on X. See Example 6.7.5 for more details.

It turns out that most of the important properties of the volume function for divisors have analogues in the curve case. First of all, Zariski decompositions are continuous and satisfy a linearity condition (Theorems 6.5.3 and 6.5.6). While the negative part of a Zariski decomposition need not be represented by an effective curve, Proposition 6.5.14 proves a "rigidity" result which is a suitable analogue of the familiar statement for divisors. Zariski decompositions and $\widehat{\mathrm{vol}}$ exhibit very nice birational behavior, discussed in Section 6.5.6.

Other important properties include a Morse inequality (Corollary 6.5.19), the strict log concavity of $\widehat{\mathrm{vol}}$ (Theorem 6.5.10), and the following description of the derivative which mirrors the results of [BFJ09] and [LM09].

Theorem 6.1.7. *Let X be a projective variety of dimension n. Then the function $\widehat{\mathrm{vol}}$ is $\mathcal{C}^1$ on the big cone of curves. More precisely, let α be a big curve class on X and write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. For any curve class β, we have*

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{\mathrm{vol}}(\alpha + t\beta) = \frac{n}{n-1} B \cdot \beta.$$

6.1.1 Examples

The Zariski decomposition is particularly striking for varieties with a rich geometric structure. We discuss two examples : toric varieties and hyperkähler manifolds.

First, suppose that X is a simplicial projective toric variety of dimension n defined by a fan Σ. A class α in the interior of the movable cone of curves corresponds to a positive Minkowski weight on the rays of Σ. A fundamental theorem of Minkowski attaches to such a weight a polytope P_α whose facet normals are the rays of Σ and whose facet volumes are determined by the weights.

Theorem 6.1.8. *The complete intersection cone of X is the closure of the positive Minkowski weights α whose corresponding polytope P_α has normal fan Σ_α. For such classes we have $\widehat{\mathrm{vol}}(\alpha) = n! \,\mathrm{vol}(P_\alpha)$.*

In fact, for any positive Minkowski weight the normal fan of the polytope P_α constructed by Minkowski's Theorem describes the birational model associated to α as in Example 6.1.6.

We next discuss the Zariski decomposition and volume of a positive Minkowski weight α. In this setting, the calculation of the volume is the solution of an isoperimetric problem : fixing P_α, amongst all polytopes whose normal fan refines Σ there is a unique Q (up to homothety) minimizing the mixed volume calculation

$$\frac{V(P_\alpha^{n-1}, Q)}{\mathrm{vol}(Q)^{1/n}}.$$

If we let Q vary over all polytopes then the Brunn-Minkowski inequality shows that the minimum is given by $Q = cP_\alpha$, but the normal fan condition on Q yields a new version of this classical problem.

From this viewpoint, the compatibility with the Zariski decomposition corresponds to the fact that the solution of an isoperimetric problem should be given by a condition on the derivative. We show in Section 6.8 that this isoperimetric problem can be solved (with no minimization necessary) using the Zariski decomposition.

We next turn to hyperkähler manifolds. The results of [Bou04, Section 4] show that the volume and σ-decomposition of divisors satisfy a natural compatibility with the Beauville-Bogomolov form. We prove the analogous properties for curve classes. The following theorem is phrased in the Kähler setting, although the analogous statements in the projective setting are also true.

Theorem 6.1.9. *Let X be a hyperkähler manifold of dimension n and let q denote the bilinear form on $H^{n-1,n-1}(X)$ induced via duality from the Beauville-Bogomolov form on $H^{1,1}(X)$.*

1. *The cone of complete intersection $(n-1, n-1)$-classes is q-dual to the cone of pseudo-effective $(n-1, n-1)$-classes.*
2. *If α is a complete intersection $(n-1, n-1)$-class then $\widehat{\mathrm{vol}}(\alpha) = q(\alpha, \alpha)^{n/2(n-1)}$.*
3. *Suppose α lies in the interior of the cone of pseudo-effective $(n-1, n-1)$-classes and write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. Then $q(B^{n-1}, \gamma) = 0$ and if γ is non-zero then $q(\gamma, \gamma) < 0$.*

6.1.2 Volume and mobility

The main feature of the Zariski decomposition for surfaces is that it clarifies the relationship between the asymptotic sectional properties of a divisor and its intersection-theoretic properties. By analogy with the work of [Zar62], it is natural to wonder how the volume function $\widehat{\mathrm{vol}}$ of a curve class is related to the asymptotic geometry of the curves represented by the class. We will analyze this question by comparing $\widehat{\mathrm{vol}}$ with two "volume-type" functions for curves : the mobility function and the weighted mobility function of [Leh13b]. This will also allow us to contrast our definition of Zariski decompositions with the notion from [FL13].

The definition of the mobility is a close parallel to the definition of the volume of a divisor via asymptotic growth of sections.

Definition 6.1.10. (see [Leh13b, Definition 1.1]) Let X be a projective variety of dimension n and let $\alpha \in N_1(X)$ be a curve class with integer coefficients. The mobility of α is defined to be

$$\mathrm{mob}(\alpha) := \limsup_{m \to \infty} \frac{\max \left\{ b \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{c}\text{Any } b \text{ general points are contained} \\ \text{in an effective curve of class } m\alpha\end{array} \right\}}{m^{\frac{n}{n-1}}/n!}.$$

In [Leh13b], Lehmann shows that the mobility extends to a continuous homogeneous function on all of $N_1(X)$. The following theorem continues a project begun by the second named author (see [Xia15a, Conjecture 3.1 and Theorem 3.2]). Proposition 6.1.22 below gives a related statement.

Theorem 6.1.11. *Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then :*

1. *$\widehat{\mathrm{vol}}(\alpha) \leq \mathrm{mob}(\alpha) \leq n!\widehat{\mathrm{vol}}(\alpha)$.*
2. *Assume Conjecture 6.1.12 below. Then $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$.*

The driving force behind Theorem 6.1.11 is a comparison of the Zariski decomposition for mob constructed in [FL13] with the Zariski decomposition for $\widehat{\mathrm{vol}}$ defined above. The second part of this theorem relies on the following (difficult) conjectural description of the mobility of a complete intersection class :

Conjecture 6.1.12. *(see [Leh13b, Question 7.1]) Let X be a smooth projective variety of dimension n and let A be an ample divisor on X. Then*

$$\mathrm{mob}(A^{n-1}) = A^n.$$

Theorem 6.1.11 is quite surprising : it suggests that the mobility count of *any* curve class is optimized by complete intersection curves.

Example 6.1.13. Let α denote the class of a line on $\mathbb{P}^3$. The mobility count of α is determined by the following enumerative question : what is the minimal degree of a curve through b general points of $\mathbb{P}^3$? The answer is unknown, even in an asymptotic sense.

[Per87] conjectures that the "optimal" curves (which maximize the number of points relative to their degree to the 3/2) are complete intersections of two divisors of the same degree. Theorem 6.1.11 supports a vast generalization of Perrin's conjecture to all big curve classes on all smooth projective varieties.

While the weighted mobility of [Leh13b] is slightly more complicated, it allows us to prove an unconditional statement. The weighted mobility is similar to the mobility, but it counts singular points of the cycle with a higher "weight" ; we give the precise definition in Section 6.10.1.

Theorem 6.1.14. *Let X be a smooth projective variety and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then $\widehat{\mathrm{vol}}(\alpha) = \mathrm{wmob}(\alpha)$.*

Thus $\widehat{\mathrm{vol}}$ captures some fundamental aspects of the asymptotic geometric behavior of curves.

6.1.3 Formal Zariski decompositions

According to the philosophy of [FL13], one should interpret the Zariski decomposition (or the σ-decomposition for divisors) as capturing the failure of strict log concavity of the volume function. This suggests that one should use the tools of convex analysis – in particular some version of the Legendre-Fenchel transform – to analyze Zariski decompositions. We will show that many of the basic analytic properties of $\widehat{\mathrm{vol}}$ and Zariski decompositions can in fact be deduced from a much more general duality framework for arbitrary concave functions. From this perspective, the most surprising feature of $\widehat{\mathrm{vol}}$ is that it captures actual geometric information about curves representing the corresponding class.

Let $\mathcal{C}$ be a full dimensional closed proper convex cone in a finite dimensional vector space. For any $s > 1$, let $\mathrm{HConc}_s(\mathcal{C})$ denote the collection of functions $f : \mathcal{C} \to \mathbb{R}$ that are upper-semicontinuous, homogeneous of weight $s > 1$, strictly positive on the interior of $\mathcal{C}$, and which are s-concave in the sense that

$$f(v)^{1/s} + f(x)^{1/s} \leq f(x+v)^{1/s}$$

for any $v, x \in \mathcal{C}$. In this context, the correct analogue of the Legendre-Fenchel transform is the (concave homogeneous) polar transform. For any $f \in \mathrm{HConc}_s(\mathcal{C})$, the polar $\mathcal{H}f$ is an element of $\mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$ for the dual cone $\mathcal{C}^*$ defined as

$$\mathcal{H}f(w^*) = \inf_{v \in \mathcal{C}^\circ} \left(\frac{w^* \cdot v}{f(v)^{1/s}} \right)^{s/s-1} \qquad \forall w^* \in \mathcal{C}^*.$$

We define what it means for $f \in \mathrm{HConc}_s(\mathcal{C})$ to have a "Zariski decomposition structure" and show that it follows from the differentiability of $\mathcal{H}f$. This is the analogue in our situation of how the Legendre-Fenchel transform relates differentiability and strict convexity. Furthermore, this structure allows one to systematically transform geometric inequalities from one setting to the other. Many of the basic geometric inequalities in algebraic geometry – and hence for polytopes or convex bodies via toric varieties (as in [Tei82] and [Kho89] and the references therein) – can be understood in this framework.

Example 6.1.15. Let q be a bilinear form on a vector space V of signature $(1, \dim V - 1)$ and set $f(v) = q(v, v)$. Suppose $\mathcal{C}$ is a closed full-dimensional convex cone on which f is non-negative. Identifying V with V^* under q, we see that $\mathcal{C} \subset \mathcal{C}^*$ and that $\mathcal{H}f|_{\mathcal{C}} = f$ by the Hodge inequality. Then $\mathcal{H}f$ on the entire cone $\mathcal{C}^*$ is controlled by a "Zariski decomposition" projecting onto $\mathcal{C}$. This is of course the familiar picture for surfaces, where f is the self-intersection on the nef cone and $\mathcal{H}f$ is the volume on the pseudo-effective cone.

Example 6.1.16. Fix a spanning set of vectors $\mathcal{Q}$ in $\mathbb{R}^n$. Then the set of all polytopes whose facet normals are (up to rescaling) a subset of $\mathcal{Q}$ are naturally parametrized by a cone $\mathcal{C}$ in a vector space V. The volume function defines a homogeneous non-negative function on $\mathcal{C}$.

The dual space V^* is the set of Minkowski weights on $\mathcal{Q}$. A classical theorem of Minkowski shows that each strictly positive Minkowski weight α defines a polytope P_α whose facet normals are given by $\mathcal{Q}$ and whose facet areas are controlled by the values of α. Such weights define a cone $\mathcal{M}$ contained in $\mathcal{C}^*$. Then the polar of the function vol restricted to $\mathcal{M}$ is again just the volume (after normalizing properly). This is proved in Section 6.8. It would be interesting to see a version which applies to arbitrary convex bodies.

6.1.4 Other applications

Finally, we discuss some connections with other areas of birational geometry.

An important ancillary goal of the paper is to prove some new results concerning the volume function of divisors and the movable cone of curves. The key tool is another intersection-theoretic invariant $\mathfrak{M}$ of nef curve classes from [Xia15a, Definition 2.2]. Since the results seem likely to be of independent interest, we recall some of them here.

First of all, we give a refined version of a theorem of [BDPP13] describing the movable cone of curves. In [BDPP13], it is proved that the movable cone $\mathrm{Mov}_1(X)$ is *generated* by $(n-1)$-self positive products of big divisors. We show that the interior points in $\mathrm{Mov}_1(X)$ are *exactly* the set of $(n-1)$-self positive products of big divisors on the interior of $\mathrm{Mov}^1(X)$.

Theorem 6.1.17. *Let X be a smooth projective variety of dimension n and let α be an interior point of $\mathrm{Mov}_1(X)$. Then there is a unique big movable divisor class L_α lying in the interior of $\mathrm{Mov}^1(X)$ and depending continuously on α such that $\langle L_\alpha^{n-1} \rangle = \alpha$.*

Example 6.1.18. This result shows that the map $\langle -^{n-1} \rangle$ is a homeomorphism from the interior of the movable cone of divisors to the interior of the movable cone of curves. Thus, any chamber decomposition of the movable cone of curves naturally induces a decomposition of the movable cone of divisors and vice versa. This relationship could be useful in the study of geometric stability conditions (as in [Neu10]).

As an interesting corollary, we obtain :

Corollary 6.1.19. *Let X be a projective variety of dimension n. Then the rays over classes of irreducible curves which deform to dominate X are dense in $\mathrm{Mov}_1(X)$.*

We can describe the boundary of $\mathrm{Mov}_1(X)$.

Theorem 6.1.20. *Let X be a smooth projective variety and let α be a curve class lying on the boundary of $\mathrm{Mov}_1(X)$. Then exactly one of the following alternatives holds :*

- *$\alpha = \langle L^{n-1} \rangle$ for a big movable divisor class L on the boundary of $\mathrm{Mov}^1(X)$.*
- *$\alpha \cdot M = 0$ for a movable divisor class M.*

The homeomorphism from $\mathrm{Mov}^1(X)^\circ \to \mathrm{Mov}_1(X)^\circ$ extends to map the big movable divisor classes on the boundary of $\mathrm{Mov}^1(X)$ bijectively to the classes of the first type.

We also extend [BFJ09, Theorem D] to a wider class of divisors.

Theorem 6.1.21. *Let X be a smooth projective variety of dimension n. For any two big divisor classes L_1, L_2, we have*

$$\mathrm{vol}(L_1 + L_2)^{1/n} \geq \mathrm{vol}(L_1)^{1/n} + \mathrm{vol}(L_2)^{1/n}$$

with equality if and only if the (numerical) positive parts $P_\sigma(L_1)$, $P_\sigma(L_2)$ are proportional. Thus the function $L \mapsto \mathrm{vol}(L)^{1/n}$ is strictly concave on the cone of big and movable divisors.

A basic technique in birational geometry is to bound the positivity of a divisor using its intersections against specified curves. These results can profitably be reinterpreted using the volume function of curves. For example :

Proposition 6.1.22. *Let X be a smooth projective variety of dimension n. Choose positive integers $\{k_i\}_{i=1}^r$. Suppose that $\alpha \in \mathrm{Mov}_1(X)$ is represented by a family of irreducible curves such that for any collection of general points $x_1, x_2, \ldots, x_r, y$ of X, there is a curve in our family which contains y and contains each x_i with multiplicity $\geq k_i$. Then*

$$\widehat{\mathrm{vol}}(\alpha)^{\frac{n-1}{n}} \geq \frac{\sum_i k_i}{r^{1/n}}.$$

We can thus apply volumes of curves to study Seshadri constants, bounds on volume of divisors, and other related topics. We defer a more in-depth discussion to Section 6.11, contenting ourselves with a fascinating example.

Example 6.1.23. If X is rationally connected, it is interesting to analyze the possible volumes for classes of special rational curves on X. When X is a Fano variety of Picard rank 1, these invariants will be closely related to classical invariants such as the length and degree.

For example, we say that $\alpha \in N_1(X)$ is a rationally connecting class if for any two general points of X there is a chain of rational curves of class α connecting the two points. Is there a uniform upper bound (depending only on the dimension) for the minimal volume of a rationally connecting class on a rationally connected X ? [KMM92] and [Cam92] show that this is true for smooth Fano varieties. We discuss this question briefly in Section 6.11.2.

6.1.5 Outline

In this part we will work with projective varieties over $\mathbb{C}$ for simplicity of arguments and for compatibility with cited references. However, except for certain results in Section 6.6 through Section 6.11, all the results will extend to smooth varieties over arbitrary algebraically closed fields on the one hand and arbitrary compact Kähler manifolds on the other. We give a general framework for this extension in Sections 6.2.3 and 6.2.4 and then explain the details as we go.

In Section 6.2 we review the necessary background, and make several notes explaining how the proofs can be adjusted to arbitrary algebraically closed fields and compact Kähler manifolds. Sections 6.3 and 6.4 discuss polar transforms and formal Zariski decompositions for log concave functions. In Section 6.5 we construct the Zariski decomposition of curves and study its basic properties and its relationship with $\widehat{\mathrm{vol}}$. In Section 6.6, we give a refined structure of the movable cone of curves and generalize several results on big and nef divisors to big and movable divisors. Section 6.7 compares the complete intersection and movable cone of curves. Section 6.8 discusses toric varieties, and Section 6.9 is devoted to the study of hyperkähler manifolds. In Section 6.10 we compare the mobility function and $\widehat{\mathrm{vol}}$. Section 6.11 outlines some applications to birational geometry. Finally, Appendix A collects some “reverse" Khovanskii-Teissier type results in the analytic setting and a result related to the transcendental holomorphic Morse inequality, and Appendix B gives a toric example where the complete intersection cone of curves is not convex.

6.2 Preliminaries

In this section, we first fix some notations over a projective variety X :

$N^1(X)$: the real vector space of numerical classes of divisors ;

$N_1(X)$: the real vector space of numerical classes of curves ;

$\overline{\mathrm{Eff}}^1(X)$: the cone of pseudo-effective divisor classes.

$\mathrm{Nef}^1(X)$: the cone of nef divisor classes ;

$\overline{\mathrm{Eff}}_1(X)$: the cone of pseudo-effective curve classes ;

$\mathrm{Mov}_1(X)$: the cone of movable curve classes, equivalently by [BDPP13] the dual of $\overline{\mathrm{Eff}}^1(X)$;

$\mathrm{CI}_1(X)$: the closure of the set of all curve classes of the form A^{n-1} for an ample divisor A;

With only a few exceptions, capital letters A, B, D, L will denote $\mathbb{R}$-Cartier divisor classes and greek letters α, β, γ will denote curve classes. For two curve classes α, β, we write $\alpha \succeq \beta$ (resp. $\alpha \preceq \beta$) to denote that $\alpha - \beta$ (resp. $\beta - \alpha$) belongs to $\overline{\mathrm{Eff}}_1(X)$. We will do similarly for divisor classes, or two elements of a cone $\mathcal{C}$ if the cone is understood.

We will use the notation $\langle - \rangle$ for the positive product as in [BDPP13], [BFJ09] and [Bou02a]. We make a few remarks on this construction for singular projective varieties. Suppose that X has dimension n. Then $N_{n-1}(X)$ denotes the vector space of $\mathbb{R}$-classes of Weil divisors up to numerical equivalence as in [Ful84, Chapter 19]. In this setting, the 1st and $(n-1)$st positive product should be interpreted respectively as maps $\overline{\mathrm{Eff}}^1(X) \to N_{n-1}(X)$ and $\overline{\mathrm{Eff}}^1(X)^{\times n-1} \to \mathrm{Mov}_1(X)$. We will also let $P_\sigma(L)$ denote the positive part in this sense – that is, pullback L to closer and closer Fujita approximations, take its positive part, and push the numerical class forward to X as a numerical Weil divisor class. With these conventions, we still have the crucial result of [BFJ09] and [LM09] that the derivative of the volume is controlled by intersecting against the positive part.

We define the movable cone of divisors $\mathrm{Mov}^1(X)$ to be the subset of $\overline{\mathrm{Eff}}^1(X)$ consisting of divisor classes L such that $N_\sigma(L) = 0$ and $P_\sigma(L) = L \cap [X]$. On any projective variety, by [Ful84, Example 19.3.3] capping with X defines an injective linear map $N^1(X) \to N_{n-1}(X)$. Thus if $D, L \in \mathrm{Mov}^1(X)$ have the same positive part in $N_{n-1}(X)$, then by the injectivity of the capping map we must have $D = L$.

To extend our results to arbitrary compact Kähler manifolds, we need to deal with transcendental objects which are not given by divisors or curves. Let X be a compact Kähler manifold of dimension n. By analogue with the projective situation, we need to deal with the following spaces and positive cones :

$H^{1,1}_{BC}(X, \mathbb{R})$: the real Bott-Chern cohomology group of bidegree $(1,1)$;

$H^{n-1,n-1}_{BC}(X, \mathbb{R})$: the real Bott-Chern cohomology group of bidegree $(n-1, n-1)$;

$\mathcal{N}(X)$: the cone of pseudo-effective $(n-1, n-1)$-classes ;

$\mathcal{M}(X)$: the cone of movable $(n-1, n-1)$-classes ;

$\overline{\mathcal{K}}(X)$: the cone of nef $(1,1)$-classes, i.e. the closure of the Kähler cone generated by Kähler classes ;

$\mathcal{E}(X)$: the cone of pseudo-effective $(1,1)$-classes.

Recall that a $(1,1)$ (or $(n-1, n-1)$) class is a pseudo-effective class if it contains a d-closed positive current, and an $(n-1, n-1)$-class is a movable class if it is contained in the closure of the cone generated by the classes of the form $\mu_*(\widetilde{\omega}_1 \wedge ... \wedge \widetilde{\omega}_{n-1})$ where $\mu : \widetilde{X} \to X$ is a modification and $\widetilde{\omega}_1, ..., \widetilde{\omega}_{n-1}$ are Kähler metrics on $\widetilde{X}$. For the basic theory of positive currents, we refer the reader to [Dem12b].

If X is a smooth projective variety over $\mathbb{C}$, then we have the following relations (see e.g. [BDPP13])

$$\mathrm{Nef}^1(X) = \overline{\mathcal{K}}(X) \cap N^1(X), \ \overline{\mathrm{Eff}}^1(X) = \mathcal{E}(X) \cap N^1(X)$$

and

$$\overline{\mathrm{Eff}}_1(X) = \mathcal{N}(X) \cap N_1(X), \ \mathrm{Mov}_1(X) = \mathcal{M}(X) \cap N_1(X).$$

6.2.1 Khovanskii-Teissier inequalities

We collect several results which we will frequently use in our paper. In every case, the statement for arbitrary projective varieties follows from the familiar smooth versions via a pullback argument. Recall the well-known Khovanskii-Teissier inequalities for a pair of nef divisors over projective varieties (see e.g. [Tei79]).

– Let X be a projective variety and let A, B be two nef divisor classes on X. Then we have

$$A^{n-1} \cdot B \geq (A^n)^{n-1/n} (B^n)^{1/n}$$

We also need the characterization of the equality case in the above inequality as in [BFJ09, Theorem D] – see also [FX14b] for the analytic proof for transcendental classes in the Kähler setting. (We call this characterization Teissier's proportionality theorem as it was first proposed and studied by B. Teissier.)

– Let X be a projective variety and let A, B be two big and nef divisor classes on X. Then

$$A^{n-1} \cdot B = (A^n)^{n-1/n} (B^n)^{1/n}$$

if and only if A and B are proportional.

We next prove a more general version of Teissier's proportionality theorem for n big and nef $(1,1)$-classes over compact Kähler manifolds (thus including projective varieties defined over $\mathbb{C}$) which follows easily from the result of [FX14b]. We expect that it can be applied to study the structure of complete intersection curve classes of mixed type.

Theorem 6.2.1. *Let X be a compact Kähler manifold of dimension n, and let $B_1, ..., B_n$ be n big and nef $(1,1)$-classes over X. Then we have*

$$B_1 \cdot B_2 \cdots B_n \geq (B_1^n)^{1/n} \cdot (B_2^n)^{1/n} \cdots (B_n^n)^{1/n},$$

where the equality is obtained if and only if $B_1, ..., B_n$ are proportional.

We include a proof, since we are not aware of any reference in the literature. The proof reduces the global inequalities to the pointwise Brunn-Minkowski inequalities by solving Monge-Ampère equations (see [Dem93], [FX14b]), and then applies the result of Chapter 3 (see also [FX14b]) for a pair of big and nef classes (see also [BFJ09, Theorem D] for divisor classes).

Recall that the ample locus $\mathrm{Amp}(D)$ of a big $(1,1)$-class D is the set of points $x \in X$ such that there is a strictly positive current $T_x \in D$ with analytic singularities which is smooth near x. When L is a big $\mathbb{R}$-divisor class on a smooth projective variety X, then the ample locus $\mathrm{Amp}(L)$ is equal to the complement of the augmented base locus $\mathbb{B}_+(L)$ (see [Bou04]).

Proof. Without loss of generality, we can assume all the $B_i^n = 1$. Then we need to prove

$$B_1 \cdot B_2 \cdots B_n \geq 1,$$

with the equality obtained if and only if $B_1, ..., B_n$ are equal.

To this end, we fix a smooth volume form Φ with $\mathrm{vol}(\Phi) = 1$. We choose a smooth $(1,1)$-form b_j in the class B_j. Then by [BEGZ10, Theorem C], for every class B_j we can solve the following singular Monge-Ampère equation

$$\langle (b_j + i\partial\bar{\partial}\psi_j)^n \rangle = \Phi,$$

where $\langle - \rangle$ denotes the non-pluripolar products of positive currents (see [BEGZ10, Definition 1.1 and Proposition 1.6]).

Denote $T_j = b_j + i\partial\bar{\partial}\psi_j$, then [BEGZ10, Theorem B] implies T_j is a positive current with minimal singularities in the class B_j. Moreover, T_j is a Kähler metric over the ample locus $\mathrm{Amp}(B_j)$ of the big class B_j by [BEGZ10, Theorem C].

Note that $\mathrm{Amp}(B_j)$ is a Zariski open set of X. Denote $\Omega = \mathrm{Amp}(B_1) \cap ... \cap \mathrm{Amp}(B_n)$, which is also a Zariski open set. By [BEGZ10, Definition 1.17], we then have

$$\begin{aligned} B_1 \cdot B_2 \cdots B_n &= \int_X \langle T_1 \wedge ... \wedge T_n \rangle \\ &= \int_\Omega T_1 \wedge ... \wedge T_n, \end{aligned}$$

where the second line follows because the non-pluripolar product $\langle T_1 \wedge ... \wedge T_n \rangle$ puts no mass on the subvariety $X \setminus \Omega$ and all the T_j are Kähler metrics over Ω.

For any point $x \in \Omega$, we have the following pointwise Brunn-Minkowski inequality

$$T_1 \wedge ... \wedge T_n \geq \left(\frac{T_1^n}{\Phi}\right)^{1/n} \cdots \left(\frac{T_n^n}{\Phi}\right)^{1/n} \Phi = \Phi$$

with equality if and only if the Kähler metrics T_j are proportional at x. Here the second equality follows because we have $T_j^n = \Phi$ on Ω. In particular, we get the Khovanskii-Teissier inequality

$$B_1 \cdot B_2 \cdots B_n \geq 1.$$

And we know the equality $B_1 \cdot B_2 \cdots B_n = 1$ holds if and only if the Kähler metrics T_j are pointwise proportional. At this step, we can not conclude that the Kähler metrics T_j are equal over Ω since we can not control the proportionality constants from the pointwise Brunn-Minkowski inequalities. However, for any pair of T_i and T_j, we have the following pointwise equality over Ω :

$$T_i^{n-1} \wedge T_j = \left(\frac{T_i^n}{\Phi}\right)^{n-1/n} \cdot \left(\frac{T_j^n}{\Phi}\right)^{1/n} \Phi,$$

since T_i and T_j are pointwise proportional over Ω. This implies the equality

$$B_i^{n-1} \cdot B_j = 1.$$

Then by the pointwise estimates of Chapter 3 (see also [FX14b]), we know the currents T_i and T_j must be equal over X, which implies $B_i = B_j$.

In conclusion, we get that $B_1 \cdot B_2 \cdots B_n = 1$ if and only if the B_j are equal. □

6.2.2 Complete intersection cone

Since the complete intersection cone plays an important role in the paper, we quickly outline its basic properties. Recall that $\mathrm{CI}_1(X)$ is the closure of the set of all curve classes of the form A^{n-1} for an ample divisor A. It naturally has the structure of a closed pointed cone.

Proposition 6.2.2. *Let X be a projective variety of dimension n. Suppose that $\alpha \in \mathrm{CI}_1(X)$ lies on the boundary of the cone. Then either*

1. *$\alpha = B^{n-1}$ for some big and nef divisor class B, or*
2. *α lies on the boundary of $\overline{\mathrm{Eff}}_1(X)$.*

Proof. We fix an ample divisor class K. Since $\alpha \in \mathrm{CI}_1(X)$ is a boundary point of the cone, we can write α as the limit of classes A_i^{n-1} for some sequence of ample divisor classes A_i.

First suppose that the values of $A_i \cdot K^{n-1}$ are bounded above as i varies. Then the classes of the divisor A_i vary in a compact set, so they have some nef accumulation point B. Clearly $\alpha = B^{n-1}$. Furthermore, if B is not big then α will lie on the boundary of $\overline{\mathrm{Eff}}_1(X)$ since in this case $B^{n-1} \cdot B = 0$. If B is big, then it is not ample, since the map $A \mapsto A^{n-1}$ from the ample cone of divisors to $N_1(X)$ is locally surjective. Thus in this case B is big and nef.

Now suppose that the values of $A_i \cdot K^{n-1}$ do not have any upper bound. Since the A_i^{n-1} limit to α, for i sufficiently large we have

$$2(\alpha \cdot K) > A_i^{n-1} \cdot K \geq \mathrm{vol}(A_i)^{n-1/n}\,\mathrm{vol}(K)^{1/n}$$

by the Khovanskii-Teissier inequality. In particular this shows that $\mathrm{vol}(A_i)$ admits an upper bound as i varies. Note that the classes $A_i/(K^{n-1} \cdot A_i)$ vary in a compact slice of the nef cone of divisors. Without loss of generality, we can assume they limit to a nef divisor class B. Then we have

$$\begin{aligned} B \cdot \alpha &= \lim_{i\to\infty} \frac{A_i}{K^{n-1} \cdot A_i} \cdot A_i^{n-1} \\ &= \lim_{i\to\infty} \frac{\mathrm{vol}(A_i)}{K^{n-1} \cdot A_i} \\ &= 0. \end{aligned}$$

The last equality holds because $\mathrm{vol}(A_i)$ is bounded above but $A_i \cdot K^{n-1}$ is not. So in this case α must be on the boundary of the pseudo-effective cone $\overline{\mathrm{Eff}}_1$. □

The complete intersection cone differs from most cones considered in birational geometry in that it is *not* convex. Since we are not aware of any such example in the literature, we give a toric example from [FS09] in Appendix B. The same example shows that the cone that is the closure of all products of $(n-1)$ ample divisors is also not convex.

Remark 6.2.3. It is still true that $\mathrm{CI}_1(X)$ is "locally convex". Let A, B be two ample divisor classes. If ϵ is sufficiently small, then

$$A^{n-1} + \epsilon B^{n-1} = A_\epsilon^{n-1}$$

for a unique ample divisor A_ϵ. The existence of A_ϵ follows from the Hard Lefschetz theorem. Consider the following smooth map

$$\Phi : N^1(X) \to N_1(X)$$

sending D to D^{n-1}. By the Hard Lefschetz theorem, the derivative $d\Phi$ is an isomorphism at the point A. Thus Φ is local diffeomorphism near A, yielding the existence of A_ϵ. The uniqueness follows from Teissier's proportionality theorem. (See [GT13] for a more in-depth discussion.)

Another natural question is :

Question 6.2.4. Suppose that X is a projective variety of dimension n and that $\{A_i\}_{i=1}^{n-1}$ are ample divisor classes on X. Then is $A_1 \cdot \ldots \cdot A_{n-1} \in \mathrm{CI}_1(X)$?

One can imagine that such a statement could be studied using an "averaging" method. We hope Theorem 6.2.1 can be helpful for this problem.

6.2.3 Fields of characteristic p

Almost all the results in the paper will hold for smooth varieties over an arbitrary algebraically closed field. The necessary technical generalizations are verified in the following references :

- [Laz04, Remark 1.6.5] checks that the Khovanskii-Teissier inequalities hold over an arbitrary algebraically closed field.
- The existence of Fujita approximations over an arbitrary algebraically closed field is proved in [Tak07].
- The basic properties of the σ-decomposition in positive characteristic are considered in [Mus13].
- The results of [Cut13] lay the foundations of the theory of positive products and volumes over an arbitrary field.
- [FL13] describes how the above results can be used to extend [BDPP13] and most of the results of [BFJ09] over an arbitrary algebraically closed field. In particular the description of the derivative of the volume function in [BFJ09, Theorem A] holds for smooth varieties in any characteristic.

6.2.4 Compact Kähler manifolds

The following results enable us to extend most of our results to arbitrary compact Kähler manifolds.

- The Khovanskii-Teissier inequalities for classes in the nef cone $\overline{\mathcal{K}}$ can be proved by the mixed Hodge-Riemann bilinear relations [Gro90,DN06], or by solving complex Monge-Ampère equations [Dem93] ; see also Theorem 6.2.1.
- Teissier's proportionality theorem for transcendental big and nef classes has recently been proved by [FX14b] ; see also Theorem 6.2.1.
- The theory of positive intersection products for pseudo-effective $(1,1)$-classes has been developed by [Bou02a, BDPP13, BEGZ10].
- The cone duality $\overline{\mathcal{K}}^* = \mathcal{N}$ follows from the numerical characterization of the Kähler cone of [DP04].

We remark that we need the cone duality $\overline{\mathcal{K}}^* = \mathcal{N}$ to extend the Zariski decompositions and Morse-type inequality for curves to positive currents of bidimension $(1,1)$.

Comparing with the projective situation, the main ingredient missing is Demailly's conjecture on the transcendental holomorphic Morse inequality, which is in turn implied by the expected identification of the derivative of the volume function on pseudo-effective $(1,1)$-classes as in [BFJ09]. Indeed, it is not hard to see these two expected results are equivalent (see e.g. [Xia14, Proposition 1.1] – which is essentially [BFJ09, Section 3.2]). And they would imply the duality of the cones $\mathcal{M}(X)$ and $\mathcal{E}(X)$. Thus, any of our results which relies on either the transcendental holomorphic Morse inequality, or the results of [BFJ09], is still conjectural in the Kähler setting. However, these conjectures are known if X is a compact hyperkähler manifold (see [BDPP13, Theorem 10.12]), so all of our results extend to compact hyperkähler manifolds.

6.3 Polar transforms

As explained in the introduction, Zariski decompositions capture the failure of the volume function to be strictly log concave. In this section and the next, we use some basic convex analysis to define a formal Zariski decomposition which makes sense for any non-negative homogeneous log concave function on a cone. The main tool is a Legendre-Fenchel type transform for such functions.

6.3.1 Duality transforms

Let V be a finite-dimensional $\mathbb{R}$-vector space of dimension n, and let V^* be its dual. We denote the pairing of $w^* \in V^*$ and $v \in V$ by $w^* \cdot v$. Let $\mathrm{Cvx}(V)$ denote the class of lower-semicontinuous convex functions on V. Then [AAM09, Theorem 1] shows that, up to composition with an additive linear function and a symmetric linear transformation, the Legendre-Fenchel transform is the unique order-reversing involution $\mathcal{L} : \mathrm{Cvx}(V) \to \mathrm{Cvx}(V^*)$. Motivated by this result, the authors define a duality transform to be an order-reversing involution of this type and characterize the duality transforms in many other contexts (see e.g. [AAM11], [AAM08]).

Below we study a duality transform for the set of non-negative homogeneous functions on a cone. This transform is the concave homogeneous version of the well-known polar transform ; see [Roc70, Chapter 15] for the basic properties of this transform in a related context. This transform is also a special case of the generalized Legendre-Fenchel transform studied by [Mor67, Section 14], which is the usual Legendre-Fenchel transform with a "coupling function" – we would like to thank M. Jonsson for pointing this out to us. See also [Sin97, Section 0.6] and [Rub00, Chapter 1] for a brief introduction to this perspective. Finally, it is essentially the same as the transform $\mathcal{A}$ from [AAM11] when applied to homogeneous functions, and is closely related to other constructions of [AAM08]. [Rub00, Chapter 2] and [RD02] work in a different setting which nonetheless has some nice parallels with our situation.

Let $\mathcal{C} \subset V$ be a proper closed convex cone of full dimension and let $\mathcal{C}^* \subset V^*$ denote the dual cone of $\mathcal{C}$, that is,

$$\mathcal{C}^* = \{w^* \in V^* |\ w^* \cdot v \geq 0 \text{ for any } v \in \mathcal{C}\}.$$

We let $\mathrm{HConc}_s(\mathcal{C})$ denote the collection of functions $f : \mathcal{C} \to \mathbb{R}$ satisfying :
- f is upper-semicontinuous and homogeneous of weight $s > 1$;
- f is strictly positive in the interior of $\mathcal{C}$ (and hence non-negative on $\mathcal{C}$) ;
- f is s-concave : for any $v, x \in \mathcal{C}$ we have $f(v)^{1/s} + f(x)^{1/s} \leq f(v+x)^{1/s}$.

Note that since $f^{1/s}$ is homogeneous of degree 1, the definition of concavity for $f^{1/s}$ above coheres with the usual one. For any $f \in \mathrm{HConc}_s(\mathcal{C})$, the function $f^{1/s}$ can extend to a proper upper-semicontinuous concave function over V by letting $f^{1/s}(v) = -\infty$ whenever $v \notin \mathcal{C}$. Thus many tools developed for arbitrary concave functions on V also apply in our case.

Since an upper-semicontinuous function is continuous along decreasing sequences, the following continuity property of f follows immediately from the non-negativity and concavity of $f^{1/s}$.

Lemma 6.3.1. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ and $v \in \mathcal{C}$. For any element $x \in \mathcal{C}$ we have*

$$f(v) = \lim_{t \to 0^+} f(v + tx).$$

In particular, any $f \in \mathrm{HConc}_s(\mathcal{C})$ must vanish at the origin.

In this section we outline the basic properties of the polar transform $\mathcal{H}$ (following a suggestion of M. Jonsson). In contrast to abstract convex transforms, $\mathcal{H}$ retains all of the properties of the classical Lengendre-Fenchel transform. Despite the many mentions of this transform we have been unable to find a comprehensive reference, so we have included the proofs.

Recall that the polar transform $\mathcal{H}$ associates to a function $f \in \mathrm{HConc}_s(\mathcal{C})$ the function $\mathcal{H}f : \mathcal{C}^* \to \mathbb{R}$ defined as

$$\mathcal{H}f(w^*) := \inf_{v \in \mathcal{C}^\circ} \left(\frac{w^* \cdot v}{f^{1/s}(v)} \right)^{s/s-1}.$$

By Lemma 6.3.1 the definition is unchanged if we instead vary v over all elements of $\mathcal{C}$ where f is positive. The following proposition shows that $\mathcal{H}$ defines an order-reversing involution from $\mathrm{HConc}_s(\mathcal{C})$ to $\mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$.

Proposition 6.3.2. *Let $f, g \in \mathrm{HConc}_s(\mathcal{C})$. Then we have*

1. $\mathcal{H}f \in \mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$.
2. *If $f \leq g$ then $\mathcal{H}f \geq \mathcal{H}g$.*
3. $\mathcal{H}^2 f = f$.

The proof is closely related to results in the literature (see e.g. [Roc70, Theorem 15.1]).

Proof. We first show (1). It is clear that $\mathcal{H}f$ is non-negative, homogeneous of weight $s/s-1$, and has a concave $s/s-1$-root. Since $\mathcal{H}f$ is defined as a pointwise infimum of a family of continuous functions, $\mathcal{H}f$ is upper-semicontinuous. So it only remains to show that $\mathcal{H}f$ is positive in the interior of $\mathcal{C}^*$.

Let w^* be an interior point of $\mathcal{C}^*$, we need to verify $\mathcal{H}f(w^*) > 0$. To this end, take a fixed compact slice T of the cone $\mathcal{C}$, e.g. take T to be the intersection of $\mathcal{C}$ with some hyperplane of V. By homogeneity we can compute $\mathcal{H}f(w^*)$ by taking the infimum over $v \in T \cap \mathcal{C}^\circ$. Since w^* is an interior point, for any $v \in T$, $w^* \cdot v$ has a uniform strictly positive lower bound. On the other hand, by the upper semi-continuity of f, we have a uniform upper bound on $f(v)$ as $v \in T$ varies. These then imply $\mathcal{H}f(w^*) > 0$.

The second statement (2) is obvious.

For the third statement (3), we always have $\mathcal{H}^2 f \geq f$. Indeed, by Lemma 6.3.1 we can find a sequence $\{v_k\}$ of points in $\mathcal{C}^\circ$ such that $\lim_k f(v_k) = f(v)$. Then

$$\mathcal{H}^2 f(v) = \inf_{w^* \in \mathcal{C}^{*\circ}} \left(\frac{w^* \cdot v}{\mathcal{H}f(w^*)^{s-1/s}} \right)^s \geq \liminf_k \inf_{w^* \in \mathcal{C}^{*\circ}} \left(\frac{w^* \cdot v}{(w^* \cdot v_k)/f(v_k)^{1/s}} \right)^s = f(v).$$

So we need to show $\mathcal{H}^2 f \leq f$. Note that $f^{1/s}$ is the pointwise infimum of the set of all affine functions which are minorized by f (since $f^{1/s}$ can be extended to a proper upper-semicontinuous concave function on all of V by assigning formally $f^{1/s}(v) = -\infty$ outside of $\mathcal{C}$). Thus f is the pointwise infimum of the set of all functions L which are minorized by f and are the s-th powers of some affine function which is positive on $\mathcal{C}^\circ$. Such functions have the form $L : v \mapsto (w^* \cdot v + b)^s$ for some $w^* \in \mathcal{C}^*$ and $b \geq 0$. In fact it suffices to consider only those L which are s-th powers of linear functions – if the function $v \mapsto (w^* \cdot v + b)^s$ is minorized by f then by homogeneity the smaller function $v \mapsto (w^* \cdot v)^s$ is also minorized by f. For any $v \in \mathcal{C}$ we have

$$f(v) = \inf_L L(v) \qquad \text{and} \qquad \inf_L \mathcal{H}^2 L(v) \geq \mathcal{H}^2 f(v),$$

where the infimum is taken over all functions L of the form $L_{w^*} = (w^* \cdot v)^s$ for some $w^* \in \mathcal{C}^*$ such that $L_{w^*} \geq f$. Thus it suffices to prove the statement for L_{w^*}. Since

$$\mathcal{H}L_{w^*}(w^*)^{s-1/s} = \inf_{v \in \mathcal{C}^\circ} \frac{w^* \cdot v}{w^* \cdot v} = 1,$$

for any $v \in \mathcal{C}$ we get by taking limits as in Lemma 6.3.1

$$\mathcal{H}^2 L_{w^*}(v) \leq \left(\frac{w^* \cdot v}{\mathcal{H}L_{w^*}(w^*)^{s-1/s}} \right)^s = (w^* \cdot v)^s = L_{w^*}(v).$$

This finishes the proof of $\mathcal{H}^2 f = f$. □

It will be crucial to understand which points obtain the infimum in the definition of $\mathcal{H}f$.

Definition 6.3.3. Let $f \in \mathrm{HConc}_s(\mathcal{C})$. For any $w^* \in \mathcal{C}^*$, we define G_{w^*} to be the set of all $v \in \mathcal{C}$ which satisfy $f(v) > 0$ and which achieve the infimum in the definition of $\mathcal{H}f(w^*)$, so that

$$\mathcal{H}f(w^*) = \left(\frac{w^* \cdot v}{f(v)^{1/s}} \right)^{s/s-1}.$$

Remark 6.3.4. The set G_{w^*} is the analogue of supergradients of concave functions. In particular, in the following sections we will see that the differential of $\mathcal{H}f$ at w^* lies in G_{w^*} if $\mathcal{H}f$ is differentiable.

It is easy to see that $G_{w^*} \cup \{0\}$ is a convex subcone of $\mathcal{C}$. Note the symmetry in the definition : if $v \in G_{w^*}$ and $\mathcal{H}f(w^*) > 0$ then $w^* \in G_v$. Thus if $v \in \mathcal{C}$ and $w^* \in \mathcal{C}^*$ satisfy $f(v) > 0$ and $\mathcal{H}f(w^*) > 0$ then the conditions $v \in G_{w^*}$ and $w^* \in G_v$ are equivalent.

The analogue of the Young-Fenchel inequality in our situation is :

Proposition 6.3.5. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Then for any $v \in \mathcal{C}$ and $w^* \in \mathcal{C}^*$ we have*

$$\mathcal{H}f(w^*)^{s-1/s} f(v)^{1/s} \leq v \cdot w^*.$$

Furthermore, equality is obtained only if either $v \in G_{w^*}$ and $w^* \in G_v$, or at least one of $\mathcal{H}f(w^*)$ and $f(v)$ vanishes.

Proof. The statement is obvious if either $\mathcal{H}f(w^*) = 0$ or $f(v) = 0$. Otherwise by Lemma 6.3.1 there is a sequence of $v_k \in \mathcal{C}^\circ$ such that $\lim_{k\to\infty} f(v_k) = f(v)$ and for every k

$$\mathcal{H}f(w^*)^{s-1/s} \leq \frac{v_k \cdot w^*}{f(v_k)^{1/s}}.$$

We obtain the desired inequality by taking limits. The last statement follows from the definition and the symmetry in the definition of G noted above. □

Theorem 6.3.6. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$.*

1. *Fix $v \in \mathcal{C}$. Let $\{w_i^*\}$ be a sequence of elements of $\mathcal{C}^*$ with $\mathcal{H}f(w_i^*) = 1$ such that*

$$f(v) = \lim_i (v \cdot w_i^*)^s > 0.$$

Suppose that the sequence admits an accumulation point w^*. Then $f(v) = (v \cdot w^*)^s$ and $\mathcal{H}f(w^*) = 1$.

2. *For every $v \in \mathcal{C}^\circ$ we have that G_v is non-empty.*
3. *Fix $v \in \mathcal{C}^\circ$. Let $\{v_i\}$ be a sequence of elements of $\mathcal{C}^\circ$ whose limit is v and for each v_i choose $w_i^* \in G_{v_i}$ with $\mathcal{H}f(w_i^*) = 1$. Then the w_i^* admit an accumulation point w^*, and any accumulation point lies in G_v and satisfies $\mathcal{H}f(w^*) = 1$.*

Proof. (1) The limiting statement for $f(v)$ is clear. We have $\mathcal{H}f(w^*) \geq 1$ by upper semicontinuity, so that

$$f(v)^{1/s} = \lim_{i \to \infty} v \cdot w_i^* \geq \frac{v \cdot w^*}{\mathcal{H}f(w^*)^{s-1/s}} \geq f(v)^{1/s}.$$

Thus we have equality everywhere. If $\mathcal{H}f(w^*)^{s-1/s} > 1$ then we obtain a strict inequality in the middle, a contradiction.

(2) Let w_i^* be a sequence of points in $\mathcal{C}^{*\circ}$ with $\mathcal{H}f(w_i^*) = 1$ such that $f(v) = \lim_{i \to \infty} (w_i^* \cdot v)^s$. By (1) it suffices to see that the w_i^* vary in a compact set. But since v is an interior point, the set of points which have intersection with v less than $2f(v)^{1/s}$ is bounded.

(3) By (1) it suffices to show that the w_i^* vary in a compact set. For sufficiently large i we have that $2v_i - v \in \mathcal{C}$. By the log concavity of f on $\mathcal{C}$ we see that f must be continuous at v. Thus for any fixed $\epsilon > 0$, we have for sufficiently large i

$$w_i^* \cdot v \leq 2w_i^* \cdot v_i \leq 2(1+\epsilon) f(v)^{1/s}.$$

Since v lies in the interior of $\mathcal{C}$, this implies that the w_i^* must lie in a bounded set. □

We next identify the collection of points where f is controlled by $\mathcal{H}$.

Definition 6.3.7. Let $f \in \mathrm{HConc}_s(\mathcal{C})$. We define $\mathcal{C}_f$ to be the set of all $v \in \mathcal{C}$ such that $v \in G_{w^*}$ for some $w^* \in \mathcal{C}$ satisfying $\mathcal{H}f(w^*) > 0$.

Since $v \in G_{w^*}$ and $\mathcal{H}f(w^*) > 0$, Proposition 6.3.5 and the symmetry of G show that $w^* \in G_v$. Furthermore, we have $\mathcal{C}^\circ \subset \mathcal{C}_f$ by Theorem 6.3.6 and the symmetry of G.

6.3.2 Differentiability

Definition 6.3.8. We say that $f \in \mathrm{HConc}_s(\mathcal{C})$ is differentiable if it is $\mathcal{C}^1$ on $\mathcal{C}^\circ$. In this case we define the function

$$D : \mathcal{C}^\circ \to V^* \qquad \text{by} \qquad v \mapsto \frac{Df(v)}{s}.$$

The main properties of the derivative are :

Theorem 6.3.9. *Suppose that $f \in \mathrm{HConc}_s(\mathcal{C})$ is differentiable. Then*

1. *D defines an $(s-1)$-homogeneous function from $\mathcal{C}^\circ$ to $\mathcal{C}^*_{\mathcal{H}f}$.*
2. *D satisfies a Brunn-Minkowski inequality with respect to f : for any $v \in \mathcal{C}^\circ$ and $x \in \mathcal{C}$*

$$D(v) \cdot x \geq f(v)^{s-1/s} f(x)^{1/s}.$$

Moreover, we have $D(v) \cdot v = f(v) = \mathcal{H}f(D(v))$.

Proof. For (1), the homogeneity is clear. Note that for any $v \in \mathcal{C}^\circ$ and $x \in C$ we have $f(v+x) \geq f(v)$ by the non-negativity of f and the concavity of $f^{1/s}$. Thus D takes values in $\mathcal{C}^*$. The fact that it takes values in $\mathcal{C}^*_{\mathcal{H}f}$ is a consequence of (2) which shows that $D(v) \in G_v$.

For (2), we start with the inequality $f(v+\epsilon x)^{1/s} \geq f(v)^{1/s} + f(\epsilon x)^{1/s}$. Since we have equality when $\epsilon = 0$, by taking derivatives with respect to ϵ at 0, we obtain

$$\frac{Df(v)}{s} \cdot x \geq f(v)^{s-1/s} f(x)^{1/s}.$$

The equality $\mathcal{H}f(D(v)) = f(v)$ is a consequence of the Brunn-Minkowski inequality, and the equality $D(v) \cdot v = f(v)$ is a consequence of the homogeneity of f. □

We will need the following familiar criterion for the differentiability of f, which is an analogue of related results in convex analysis connecting the differentiability with the uniqueness of supergradient (see e.g. [Roc70, Theorem 25.1]).

Proposition 6.3.10. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Let $U \subset \mathcal{C}^\circ$ be an open set. Then $f|_U$ is differentiable if and only if for every $v \in U$ the set $G_v \cup \{0\}$ consists of a single ray. In this case $D(v)$ is defined by intersecting against the unique element $w^* \in G_v$ satisfying $\mathcal{H}f(w^*) = f(v)$.*

Proof. We first show the forward implication. Let $v \in \mathcal{C}^\circ$ and choose some $w^* \in G_v$ satisfying $\mathcal{H}f(w^*) = f(v)$. We claim that $w^* = D(v)$, which then shows that $G_v \cup \{0\}$ is the ray generated by $D(v)$. To this end, by the symmetry of G we first have $v \in G_{w^*}$. Thus for any $x \in \mathcal{C}^\circ$ and $t > 0$ we get

$$\frac{v+tx}{f(v+tx)^{1/s}} \cdot w^* \geq \frac{v \cdot w^*}{f(v)^{1/s}}$$

with equality at $t = 0$. Taking the derivative at $t = 0$ we have

$$x \cdot w^* \geq \frac{v \cdot w^*}{f(v)} \left(D(v) \cdot x\right).$$

Since by our choice of normalization $v \cdot w^* = f(v)$, we have

$$x \cdot (w^* - D(v)) \geq 0.$$

By the arbitrariness of x, we obtain that

$$w^* - D(v) \in \mathcal{C}^*.$$

Since v is an interior point and $(w^* - D(v)) \cdot v = 0$ by homogeneity, we must have $w^* = D(v)$.

We next show the the reverse implication. Suppose $v \in U$. Fix any $x \in V$, and for each sufficiently small t let w_t^* denote the unique element in G_{v+tx} satisfying $\mathcal{H}f(w_t^*) = 1$. By Theorem 6.3.6 the w_t^* admit an accumulation point $w^* \in G_v$ satisfying $\mathcal{H}f(w^*) = 1$, which indeed is a limit point. For any t we have

$$tx \cdot w_t^* \leq f(v+tx)^{1/s} - f(v)^{1/s} \leq tx \cdot w^*.$$

Thus we see that the derivative of $f^{1/s}$ at v in the direction of x exists and is given by intersecting against w^*. This shows the reverse implication and (after rescaling to derive f instead of $f^{1/s}$) the final statement as well. □

We next discuss the behaviour of the derivative along the boundary.

Definition 6.3.11. We say that $f \in \mathrm{HConc}_s(\mathcal{C})$ is $+$-differentiable if f is $\mathcal{C}^1$ on $\mathcal{C}^\circ$ and the derivative on $\mathcal{C}^\circ$ extends to a continuous function on all of $\mathcal{C}_f$.

It is easy to see that the $+$-differentiability implies continuity.

Lemma 6.3.12. *If $f \in \mathrm{HConc}_s(\mathcal{C})$ is $+$-differentiable then f is continuous on $\mathcal{C}_f$.*

Remark 6.3.13. For $+$-differentiable functions f, we define the function $D : \mathcal{C}_f \to V^*$ by extending continuously from $\mathcal{C}^\circ$. Many of the properties in Theorem 6.3.9 hold for D on all of $\mathcal{C}_f$. By taking limits and applying Lemma 6.3.1 we obtain the Brunn-Minkowski inequality. In particular, for any $x \in \mathcal{C}_f$ we still have

$$D(x) \cdot x = f(x) = \mathcal{H}f(D(x)).$$

Thus it is clear that $D(x) \in \mathcal{C}^*_{\mathcal{H}f}$ for any $x \in \mathcal{C}_f$.

Lemma 6.3.14. *Assume $f \in \mathrm{HConc}_s(\mathcal{C})$ is $+$-differentiable. For any $x \in \mathcal{C}_f$ and $y \in \mathcal{C}^\circ$, we have*

$$\left.\frac{d}{dt}\right|_{t=0^+} f(x+ty)^{1/s} = (D(x) \cdot y) f(x)^{1-s/s}.$$

Proof. Consider the concave function $f(x+ty)^{1/s}$ of t. By [Roc70, Theorem 24.1] we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0^+} f(x+ty)^{1/s} &= \lim_{\epsilon \downarrow 0} \left.\frac{d}{dt}\right|_{t=\epsilon^+} f(x+ty)^{1/s} \\ &= \lim_{\epsilon \downarrow 0} (D(x+\epsilon y) \cdot y) f(x+\epsilon y)^{1-s/s} \\ &= (D(x) \cdot y) f(x)^{1-s/s},\end{aligned}$$

where the second line follows from the differentiability of f, and the third line follows from the $+$-differentiability of f. □

We next analyze what we can deduce about f in a neighborhood of $v \in \mathcal{C}_f$ from the fact that $G_v \cup \{0\}$ is a unique ray.

Lemma 6.3.15. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Let $v \in \mathcal{C}_f$ and assume that $G_v \cup \{0\}$ consists of a single ray. Suppose $\{v_i\}$ is a sequence of elements of $\mathcal{C}_f$ converging to v. Let $w_i^* \in G_{v_i}$ be any point satisfying $\mathcal{H}f(w_i^*) = 1$. Then the w_i^* vary in a compact set. Any accumulation point w^* must be the unique point in G_v satisfying $\mathcal{H}f(w^*) = 1$.*

Proof. By Theorem 6.3.6 it suffices to prove that the w_i^* vary in a compact set. Otherwise, we must have that $w_i^* \cdot m$ is unbounded for some interior point $m \in \mathcal{C}^\circ$. By passing to a subsequence we may suppose that $w_i^* \cdot m \to \infty$. Consider the normalization

$$\widehat{w}_i^* := \frac{w_i^*}{w_i^* \cdot m};$$

note that $\widehat{w}_i^*$ vary in a compact set. Take some convergent subsequence, which we still denote by $\widehat{w}_i^*$, and write $\widehat{w}_i^* \to \widehat{w}_0^*$. Since $\widehat{w}_0^* \cdot m = 1$ we see that $\widehat{w}_0^* \neq 0$.

We first prove $v \cdot \widehat{w}_0^* > 0$. Otherwise, $v \cdot \widehat{w}_0^* = 0$ implies

$$\frac{v \cdot (w^* + \widehat{w}_0^*)}{\mathcal{H}f(w^* + \widehat{w}_0^*)^{s-1/s}} \leq \frac{v \cdot w^*}{\mathcal{H}f(w^*)^{s-1/s}} = f(v)^{1/s}.$$

By our assumption on G_v, we get $w^* + \widehat{w}_0^*$ and w^* are proportional, which implies $\widehat{w}_0^*$ lies in the ray spanned by w^*. Since $\widehat{w}_0^* \neq 0$ and $v \cdot w^* > 0$, we get that $v \cdot \widehat{w}_0^* > 0$. So our assumption $v \cdot \widehat{w}_0^* = 0$ does not hold. On the other hand, $\mathcal{H}f(w_i^*) = 1$ implies

$$\mathcal{H}f(\widehat{w}_i^*)^{s-1/s} = \frac{1}{m \cdot w_i^*} \to 0.$$

By the upper-semicontinuity of f and the fact that $\lim v_i \cdot \widehat{w}_i^* = v \cdot \widehat{w}_0^* > 0$, we get

$$\begin{aligned} f(v)^{1/s} &\geq \limsup_{i \to \infty} f(v_i)^{1/s} \\ &= \limsup_{i \to \infty} \frac{v_i \cdot \widehat{w}_i^*}{\mathcal{H}f(\widehat{w}_i^*)^{s-1/s}} = \infty. \end{aligned}$$

This is a contradiction, thus the sequence w_i^* must vary in a compact set. □

Theorem 6.3.16. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Suppose that $U \subset \mathcal{C}_f$ is a relatively open set and $G_v \cup \{0\}$ consists of a single ray for any $v \in U$. If f is continuous on U then f is $+$-differentiable on U. In this case $D(v)$ is defined by intersecting against the unique element $w^* \in G_v$ satisfying $\mathcal{H}f(w^*) = f(v)$.*

Even if f is not continuous, we at least have a similar statement along the directions in which f is continuous (for example, any directional derivative toward the interior of the cone).

Proof. Theorem 6.3.10 shows that f is differentiable on $U \cap \mathcal{C}^\circ$ and is determined by intersections. By combining Lemma 6.3.15 with the continuity of f, we see that the derivative extends continuously to any point in U. □

Remark 6.3.17. Assume $f \in \mathrm{HConc}_s(\mathcal{C})$ is $+$-differentiable. In general, we can not conclude that $G_v \cup \{0\}$ contains a single ray if $x \in \mathcal{C}_f$ is not an interior point. An explicit example is in Section 6.5. Let X be a smooth projective variety of dimension n, let $\mathcal{C} = \mathrm{Nef}^1(X)$ be the cone of nef divisor classes and let $f = \mathrm{vol}$ be the volume function of divisors. Let B be a big and nef divisor class which is not ample. Then G_B contains the cone generated by all $B^{n-1} + \gamma$ with γ pseudo-effective and $B \cdot \gamma = 0$, which in general is more than a ray.

6.4 Formal Zariski decompositions

The Legendre-Fenchel transform relates the strict concavity of a function to the differentiability of its transform. The transform $\mathcal{H}$ will play the same role in our situation ; however, one needs to interpret the strict concavity slightly differently. We will encapsulate this property using the notion of a Zariski decomposition.

Definition 6.4.1. Let $f \in \mathrm{HConc}_s(\mathcal{C})$ and let $U \subset \mathcal{C}$ be a non-empty subcone. We say that f admits a strong Zariski decomposition with respect to U if :

1. For every $v \in \mathcal{C}_f$ there are unique elements $p_v \in U$ and $n_v \in \mathcal{C}$ satisfying
$$v = p_v + n_v \qquad \text{and} \qquad f(v) = f(p_v).$$
We call the expression $v = p_v + n_v$ the Zariski decomposition of v, and call p_v the positive part and n_v the negative part of v.
2. For any $v, w \in \mathcal{C}_f$ satisfying $v + w \in \mathcal{C}_f$ we have
$$f(v)^{1/s} + f(w)^{1/s} \leq f(v+w)^{1/s}$$
with equality only if p_v and p_w are proportional.

Remark 6.4.2. Note that the vector n_v must satisfy $f(n_v) = 0$ by the non-negativity and log-concavity of f. In particular n_v lies on the boundary of $\mathcal{C}$. Furthermore, any $w^* \in G_v$ is also in G_{p_v} and must satisfy $w^* \cdot n_v = 0$.

Note also that the proportionality of p_v and p_w may not be enough to conclude that $f(v)^{1/s} + f(w)^{1/s} = f(v+w)^{1/s}$. This additional property turns out to rely on the strict log concavity of $\mathcal{H}f$.

The main principle of the section is that when f satisfies a differentiability property, $\mathcal{H}f$ admits some kind of Zariski decomposition. Usually the converse is false, due to the asymmetry of G when f or $\mathcal{H}f$ vanishes. However, the existence of a Zariski decomposition is usually strong enough to determine the differentiability of f along some subcone. We will give a version that takes into account the behavior of f along the boundary of $\mathcal{C}$.

Theorem 6.4.3. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$. Then we have the following results :*

- *If f is $+$-differentiable, then $\mathcal{H}f$ admits a strong Zariski decomposition with respect to the cone $D(\mathcal{C}_f) \cup \{0\}$.*
- *If $\mathcal{H}f$ admits a strong Zariski decomposition with respect to a cone U, then f is differentiable.*

Proof. First suppose f is $+$-differentiable; we must prove the function $\mathcal{H}f$ satisfies properties (1), (2) in Definition 6.4.1.

We first show the existence of the Zariski decomposition in property (1). If $w^* \in \mathcal{C}^*_{\mathcal{H}f}$ then by definition there is some $v \in \mathcal{C}$ satisfying $f(v) > 0$ such that $w^* \in G_v$. In particular, by the symmetry of G we also have $v \in G_{w^*}$, thus $v \in \mathcal{C}_f$. Since $f(v) > 0$ we can define

$$p_{w^*} := \left(\frac{\mathcal{H}f(w^*)}{f(v)}\right)^{s-1/s} \cdot D(v), \qquad n_{w^*} = w^* - p_{w^*}.$$

Then $p_{w^*} \in D(\mathcal{C}_f)$ and

$$\begin{aligned}\mathcal{H}f(p_{w^*}) &= \mathcal{H}\left(\left(\frac{\mathcal{H}f(w^*)}{f(v)}\right)^{s-1/s} \cdot D(v)\right) \\ &= \frac{\mathcal{H}f(w^*)}{f(v)} \cdot \mathcal{H}f\left(D(v)\right) = \mathcal{H}f(w^*)\end{aligned}$$

where the final equality follows from Theorem 6.3.9 and Remark 6.3.13. We next show that $n_{w^*} \in \mathcal{C}^*$. Choose any $x \in \mathcal{C}^\circ$ and note that for any $t > 0$ we have the inequality

$$\frac{v + tx}{f(v+tx)^{1/s}} \cdot w^* \geq \frac{v}{f(v)^{1/s}} \cdot w^*$$

with equality when $t = 0$. By Lemma 6.3.14, taking derivatives at $t = 0$ we obtain

$$\frac{x \cdot w^*}{f(v)^{1/s}} - \frac{(v \cdot w^*)(D(v) \cdot x)}{f(v)^{(s+1)/s}} \geq 0,$$

or equivalently, identifying $v \cdot w^*/f(v)^{1/s} = \mathcal{H}f(w^*)^{s-1/s}$,

$$x \cdot \left(w^* - D(v) \cdot \frac{\mathcal{H}f(w^*)^{s-1/s}}{f(v)^{s-1/s}}\right) \geq 0.$$

Since this is true for any $x \in \mathcal{C}^\circ$, we see that $n_{w^*} \in \mathcal{C}^*$ as claimed.

We next show that p_{w^*} constructed above is the unique element of $D(\mathcal{C}_f)$ satisfying the two given properties. First, after some rescaling we can assume $\mathcal{H}f(w^*) = f(v)$, which then implies $w^* \cdot v = f(v)$. Suppose that $z \in \mathcal{C}_f$ and $D(z)$ is another vector satisfying $\mathcal{H}f(D(z)) = \mathcal{H}f(w^*)$ and $w^* - D(z) \in \mathcal{C}$. Note that by Remark 6.3.13 $f(z) = \mathcal{H}f(D(z)) = f(v)$. By Proposition 6.3.5 we have

$$\mathcal{H}f(D(z))^{s-1/s} f(v)^{1/s} \leq D(z) \cdot v \leq w^* \cdot v = f(v)$$

so we obtain equality everywhere. In particular, we have $D(z) \cdot v = f(v)$. By Theorem 6.3.9, for any $x \in \mathcal{C}$ we have

$$D(z) \cdot x \geq f(z)^{s-1/s} f(x)^{1/s}.$$

Set $x = v + \epsilon q$ where $\epsilon > 0$ and $q \in \mathcal{C}^\circ$. With this substitution, the two sides of the equation above are equal at $\epsilon = 0$, so taking an ϵ-derivative of the above equation and arguing as before, we see that $D(z) - D(v) \in \mathcal{C}^*$.

We claim that $D(z) = D(v)$. First we note that $D(v) \cdot z = f(z)$. Indeed, since $f(z) = f(v)$ and $D(v) \preceq D(z)$ we have

$$f(v)^{s-1/s} f(z)^{1/s} \leq D(v) \cdot z \leq D(z) \cdot z = f(z).$$

Thus we have equality everywhere, proving the equality $D(v) \cdot z = f(z)$. Then we can apply the same argument as before with the roles of v and z switched. This shows $D(v) \succeq D(z)$, so we must have $D(z) = D(v)$.

We next turn to (2). The inequality is clear, so we only need to characterize the equality. Suppose $w^*, y^* \in \mathcal{C}^*_{\mathcal{H}f}$ satisfy

$$\mathcal{H}f(w^*)^{s-1/s} + \mathcal{H}f(y^*)^{s-1/s} = \mathcal{H}f(w^* + y^*)^{s-1/s}$$

and $w^* + y^* \in \mathcal{C}^*_{\mathcal{H}f}$. We need to show they have proportional positive parts. By assumption $G_{w^*+y^*}$ is non-empty, so we may choose some $v \in G_{w^*+y^*}$. Then also $v \in G_{w^*}$ and $v \in G_{y^*}$. Note that by homogeneity v is also in G_{aw^*} and G_{by^*} for any positive real numbers a and b. Thus by rescaling w^* and y^*, we may suppose that both have intersection $f(v)$ against v, so that $\mathcal{H}f(w^*) = \mathcal{H}f(y^*) = f(v)$. Then we need to verify the positive parts of w^* and y^* are equal. But they both coincide with $D(v)$ by the argument in the proof of (1).

Conversely, suppose that $\mathcal{H}f$ admits a strong Zariski decomposition with respect to the cone U. We claim that f is differentiable. By Proposition 6.3.10 it suffices to show that $G_v \cup \{0\}$ is a single ray for any $v \in \mathcal{C}^\circ$.

For any two elements w^*, y^* in G_v we have

$$\mathcal{H}f(w^*)^{1/s} + \mathcal{H}f(y^*)^{1/s} = \frac{w^* \cdot v}{f(v)^{1/s}} + \frac{y^* \cdot v}{f(v)^{1/s}} \geq \mathcal{H}f(w^* + y^*)^{1/s}.$$

Since w^*, y^* and their sum are all in $\mathcal{C}^*_{\mathcal{H}f}$, we conclude by the strong Zariski decomposition condition that w^* and y^* have proportional positive parts. After rescaling so that $\mathcal{H}f(w^*) = f(v) = \mathcal{H}f(y^*)$ we have $p_{w^*} = p_{y^*}$. Thus it suffices to prove $w^* = p_{w^*}$. Note that $\mathcal{H}f(w^*) = \mathcal{H}f(p_{w^*})$ as p_{w^*} is the positive part. If $w^* \neq p_{w^*}$, then $v \cdot w^* > v \cdot p_{w^*}$ since v is an interior point. This implies

$$f(v) = \inf_{y^* \in \mathcal{C}^{*\circ}} \left(\frac{v \cdot y^*}{\mathcal{H}f(y^*)^{s-1/s}} \right)^s < \left(\frac{v \cdot w^*}{\mathcal{H}f(w^*)^{s-1/s}} \right)^s,$$

contradicting with $w^* \in G_v$. Thus $w^* = p_{w^*}$ and $G_v \cup \{0\}$ must be a single ray. □

Remark 6.4.4. It is worth emphasizing that if f is $+$-differentiable and $w^* \in \mathcal{C}^*_{\mathcal{H}f}$, we can construct a positive part for w^* by choosing *any* $v \in G_{w^*}$ with $f(v) > 0$ and taking an appropriate rescaling of $D(v)$.

Remark 6.4.5. It would also be interesting to study some kind of weak Zariski decomposition. For example, one can define a weak Zariski decomposition as a decomposition $v = p_v + n_v$ only demanding $f(v) = f(p_v)$ and the strict log concavity of f over the set of positive parts. Appropriately interpreted, the existence of a weak decomposition for $\mathcal{H}f$ should correspond to the differentiability of f.

Under some additional conditions, we can get the continuity of the Zariski decompositions.

Theorem 6.4.6. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable. Then the function taking an element $w^* \in \mathcal{C}^{*\circ}$ to its positive part p_{w^*} is continuous.*

If furthermore $G_v \cup \{0\}$ is a unique ray for every $v \in \mathcal{C}_f$ and $\mathcal{H}f$ is continuous on all of $\mathcal{C}^*_{\mathcal{H}f}$, then the Zariski decomposition is continuous on all of $\mathcal{C}^*_{\mathcal{H}f}$.

Proof. Fix any $w^* \in \mathcal{C}^{*\circ}$ and suppose that w_i^* is a sequence whose limit is w^*. For each choose some $v_i \in G_{w_i^*}$ with $f(v_i) = 1$. By Theorem 6.3.6, the v_i admit an accumulation point $v \in G_{w^*}$ with $f(v) = 1$. By the symmetry of G, each v_i and also v lies in $\mathcal{C}_f$. The $D(v_i)$ limit to $D(v)$ by the continuity of D. Recall that by the argument in the proof of Theorem 6.4.3 we have $p_{w_i^*} = \mathcal{H}f(w_i^*)^{s-1/s} D(v_i)$ and similarly for w^*. Since $\mathcal{H}f$ is continuous at interior points, we see that the positive parts vary continuously as well.

The last statement follows by a similar argument using Lemma 6.3.15. □

Example 6.4.7. Suppose that q is a bilinear form on V and $f(v) = q(v, v)$. Let $\mathcal{P}$ denote one-half of the positive cone of vectors satisfying $f(v) \geq 0$. It is easy to see that f is 2-concave and non-trivial on

$\mathcal{P}$ if and only if q has signature $(1, \dim V - 1)$. Identifying V with V^* under q, we have $\mathcal{P} = \mathcal{P}^*$ and $\mathcal{H}f = f$ by the usual Hodge inequality argument.

Now suppose $\mathcal{C} \subset \mathcal{P}$. Then $\mathcal{C}^*$ contains $\mathcal{C}$. As discussed above, by the Hodge inequality $\mathcal{H}f|_{\mathcal{C}} = f$. Note that f is everywhere differentiable and $D(v) = v$ for classes in $\mathcal{C}$. Thus on $\mathcal{C}$ the polar transform $\mathcal{H}f$ agrees with f, but outside of $\mathcal{C}$ the function $\mathcal{H}f$ is controlled by a Zariski decomposition involving a projection to $\mathcal{C}$.

This is of course just the familiar picture for curves on a surface identifying f with the self-intersection on the nef cone and $\mathcal{H}f$ with the volume on the pseudo-effective cone. More precisely, for big curve classes the decomposition constructed in this way is the numerical version of Zariski's original construction. Along the boundary of $\mathcal{C}^*$, the function $\mathcal{H}f$ vanishes identically so that Theorem 6.4.3 does not apply. The linear algebra arguments of [Zar62], [Bau09] give a way of explicitly constructing the vector computing the minimal intersection as above.

Example 6.4.8. Fix a spanning set of unit vectors $\mathcal{Q}$ in $\mathbb{R}^n$. Recall that the polytopes whose unit facet normals are a subset of $\mathcal{Q}$ naturally define a cone $\mathcal{C}$ in a finite dimensional vector space V which parametrizes the constant terms of the bounding hyperplanes. One can also consider the cone $\mathcal{C}_\Sigma$ which is the closure of those polytopes whose normal fan is Σ. The volume function vol defines a weight-n homogeneous function on $\mathcal{C}$ and (via restriction) vol_Σ on $\mathcal{C}_\Sigma$, and it is interesting to ask for the behavior of the polar transforms. (Note that this is somewhat different from the link between polar sets and polar functions, which is described for example in [AAM11].)

The dual space V^* consists of the Minkowski weights on $\mathcal{Q}$. We will focus on the subcone $\mathcal{M}$ of strictly positive Minkowski weights, which is contained in the dual of both cones. By Minkowski's theorem, a strictly positive Minkowski weight determines naturally a polytope in $\mathcal{C}$, so we can identify $\mathcal{M}$ with the interior of $\mathcal{C}$. As explained in Section 6.8, the Brunn-Minkowski inequality shows that $\mathcal{H}\,\mathrm{vol}\,|_{\mathcal{M}}$ coincides with the volume function on $\mathcal{M}$. However, calculating $\mathcal{H}\,\mathrm{vol}_\Sigma\,|_{\mathcal{M}}$ is more subtle, and is an interesting special case of an isoperimetric inequality as in the introduction.

It would be very interesting to extend this duality to all convex sets, perhaps by working on an infinite dimensional space.

6.4.1 Teissier proportionality

In this section, we give some conditions which are equivalent to the strict log concavity. The prototype is the volume function of divisors over the cone of big and movable divisor classes.

Definition 6.4.9. Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable and let $\mathcal{C}_T$ be a non-empty subcone of $\mathcal{C}_f$. We say that f satisfies Teissier proportionality with respect to $\mathcal{C}_T$ if for any $v, x \in \mathcal{C}_T$ satisfying

$$D(v) \cdot x = f(v)^{s-1/s} f(x)^{1/s}$$

we have that v and x are proportional.

Note that we do not assume that $\mathcal{C}_T$ is convex – indeed, in examples it is important to avoid this condition. However, since f is defined on the convex hull of $\mathcal{C}_T$, we can (somewhat abusively) discuss the strict log concavity of $f|_{\mathcal{C}_T}$:

Definition 6.4.10. Let $\mathcal{C}' \subset \mathcal{C}$ be a (possibly non-convex) subcone. We say that f is strictly s-concave on $\mathcal{C}'$ if

$$f(v)^{1/s} + f(x)^{1/s} < f(v+x)^{1/s}$$

holds whenever $v, x \in \mathcal{C}'$ are not proportional. Note that this definition makes sense even when $\mathcal{C}'$ is not itself convex.

Theorem 6.4.11. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable. For any non-empty subcone $\mathcal{C}_T$ of $\mathcal{C}_f$, consider the following conditions :*

1. *The restriction $f|_{\mathcal{C}_T}$ is strictly s-concave (in the sense defined above).*
2. *f satisfies Teissier proportionality with respect to $\mathcal{C}_T$.*
3. *The restriction of D to $\mathcal{C}_T$ is injective.*

Then we have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). If $\mathcal{C}_T$ is convex, then we have (2) $\Longrightarrow$ (1). If $\mathcal{C}_T$ is an open subcone, then we have (3) $\Longrightarrow$ (1).

Proof. We first prove (1) $\Longrightarrow$ (2). Let $v, x \in \mathcal{C}_T$ satisfy $D(v)\cdot x = f(v)^{s-1/s} f(x)^{1/s}$ and $f(v) = f(x)$. Assume for a contradiction that $v \neq x$. Since $f|_{\mathcal{C}_T}$ is strictly s-concave, for any two $v, x \in \mathcal{C}_T$ which are not proportional we have

$$f(x)^{1/s} < f(v)^{1/s} + \frac{D(v)\cdot(x-v)}{f(v)^{s-1/s}}.$$

Since we have assumed $D(v)\cdot x = f(v)^{s-1/s} f(x)^{1/s}$ and $f(v) = f(x)$, we must have

$$f(x)^{1/s} = f(v)^{1/s} + \frac{D(v)\cdot(x-v)}{f(v)^{s-1/s}}$$

since $D(v)\cdot v = f(v)$. This is a contradiction, so we must have $v = x$. This then implies f satisfies Teissier proportionality.

We next show (2) $\Longrightarrow$ (3). Let $v_1, v_2 \in \mathcal{C}_T$ with $D(v_1) = D(v_2)$. Then we have

$$\begin{aligned} f(v_1) &= D(v_1)\cdot v_1 = D(v_2)\cdot v_1 \\ &\geq f(v_2)^{s-1/s} f(v_1)^{1/s}, \end{aligned}$$

which implies $f(v_1) \geq f(v_2)$. By symmetry, we get $f(v_1) = f(v_2)$. So we must have

$$D(v_1)\cdot v_2 = f(v_1)^{s-1/s} f(v_2)^{1/s}.$$

By the Teissier proportionality we see that v_1, v_2 are proportional, and since $f(v_1) = f(v_2)$ they must be equal.

We next show that if $\mathcal{C}_T$ is convex then (2) $\Longrightarrow$ (1). Fix y in the interior of $\mathcal{C}_T$ and fix $\epsilon > 0$. Then

$$f(v+x+\epsilon y)^{1/s} - f(v)^{1/s} = \int_0^1 (D(v+t(x+\epsilon y))\cdot x) f(v+t(x+\epsilon y))^{1-s/s} dt.$$

The integrand is bounded by a positive constant independent of ϵ as we let ϵ go to 0 due to the $+$-differentiability of f (which also implies the continuity of f). Using Lemma 6.3.1, the dominanted convergence theorem shows that

$$f(v+x)^{1/s} - f(v)^{1/s} = \int_0^1 (D(v+tx)\cdot x) f(v+tx)^{1-s/s} dt.$$

This immediately shows the strict log concavity.

Finally, we show that if $\mathcal{C}_T$ is open then (3) $\Longrightarrow$ (1). By [Roc70, Corollary 26.3.1], it is clear that for any convex open set $U \subset \mathcal{C}_T$ the injectivity of D over U is equivalent to the strict log concavity of $f|_U$. Using the global log concavity of f, we obtain the conclusion. More precisely, assume $x, y \in \mathcal{C}_T$ are not proportional, then by the strict log concavity of f near x and the global log concavity on $\mathcal{C}$, for $t > 0$ sufficiently small we have

$$\begin{aligned} f^{1/s}(x+y) &\geq f^{1/s}(x+ty) + (1-t)f^{1/s}(y) \\ &> (f^{1/s}(x) + f^{1/s}(x+2ty))/2 + (1-t)f^{1/s}(y) \\ &\geq f^{1/s}(x) + f^{1/s}(y). \end{aligned}$$

$\square$

Another useful observation is :

Proposition 6.4.12. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be differentiable and suppose that f is strictly s-concave on an open subcone $\mathcal{C}_T \subset \mathcal{C}^\circ$. Then $\mathcal{H}f$ is differentiable on $D(\mathcal{C}_T)$ and the derivative is determined by the prescription*

$$D(D(v)) = v.$$

Proof. We first show that $D(\mathcal{C}_T) \subset \mathcal{C}^{*\circ}$. Suppose that there were some $v \in \mathcal{C}_T$ such that $D(v)$ lay on the boundary of $\mathcal{C}^*$. Choose $x \in \mathcal{C}$ satisfying $x \cdot D(v) = 0$. By openness we have $v + tx \in \mathcal{C}_T$ for sufficiently small t. Since $D(v) \in G_{v+tx}$, we must have that $D(v)$ and $D(v + tx)$ are proportional by Proposition 6.3.10. This is a contradiction by Theorem 6.4.11.

Now suppose $w^* = D(v) \in D(\mathcal{C}_T)$. By the strict log concavity of f on $\mathcal{C}_T$ (and the global log concavity), we must have that $G_{w^*} \cup \{0\}$ consists only of the ray spanned by v. Applying Proposition 6.3.10, we obtain the statement. □

Combining all the results above, we obtain a very clean property of D under the strongest possible assumptions.

Theorem 6.4.13. *Assume $f \in \mathrm{HConc}_s(\mathcal{C})$ and its polar transform $\mathcal{H}f \in \mathrm{HConc}_{s/s-1}(\mathcal{C}^*)$ are $+$-differentiable. Let $U = D(\mathcal{C}^*_{\mathcal{H}f}) \cup \{0\}$ and $U^* = D(\mathcal{C}_f) \cup \{0\}$. Then we have :*

- *f and $\mathcal{H}f$ admit a strong Zariski decomposition with respect to the cone U and the cone U^* respectively ;*
- *For any $v \in \mathcal{C}_f$ we have $D(v) = D(p_v)$ (and similarly for $w \in \mathcal{C}^*_{\mathcal{H}f}$) ;*
- *D defines a bijection $D : U^\circ \to U^{*\circ}$ with inverse also given by D. In particular, f and $\mathcal{H}f$ satisfy Teissier proportionality with respect to the open cone U° and $U^{*\circ}$ respectively.*

Proof. Note that $U^* \subset \mathcal{C}^*_{\mathcal{H}f}$ (and $U \subset \mathcal{C}_f$) since for any $v \in \mathcal{C}_f$ we have $D(v) \in G_v$ and $f(v) > 0$.

The first statement is immediate from Theorem 6.4.3.

We next show the second statement. By the definition of positive parts, we have $G_v \subset G_{p_v}$. Since both $v, p_v \in \mathcal{C}_f$, we know by the argument of Theorem 6.4.3 that $D(v)$ and $D(p_v)$ are both proportional to the (unique) positive part of any $w^* \in G_v$ with positive $\mathcal{H}f$.

Finally we show the third statement. We start by proving the Teissier proportionality on U°. By part (2) of the Zariski decomposition condition f is strictly s-concave on U°, and Teissier proportionality follows by Theorem 6.4.11. Furthermore, the argument of Proposition 6.4.12 then shows that $D(U^\circ) \subset \mathcal{C}^{*\circ}$ and $D(D(U^\circ)) = U^\circ$.

We must show that $D(U^\circ) \subset U^{*\circ}$. Suppose that $v \in U^\circ$ had that $D(v)$ was on the boundary of U^*. Since $D(v) \in \mathcal{C}^{*\circ}$, there must be some sequence $w_i^* \in C^{*\circ} - U^*$ whose limit is $D(v)$. We note that each $D(w_i^*)$ lies on the boundary of $\mathcal{C}$, thus must lie on the boundary of U. Indeed, by the second statement we have $D(w_i^*) = D(w_i^* + tn_{w_i^*})$ for any $t > 0$, which would violate the uniqueness of $G_{D(w_i^*)}$ as in Proposition 6.3.10 if it were an interior point. Using the continuity of D we see that $v = D(D(v))$ lies on the boundary of U, a contradiction.

In all, we have shown that $D : U^\circ \to U^{*\circ}$ is an isomorphism onto its image with inverse D. By symmetry we also have $D(U^{*\circ}) \subset U^\circ$, and we conclude after taking D the reverse inclusion $U^{*\circ} \subset D(U^\circ)$. □

6.4.2 Morse-type inequality

The polar transform $\mathcal{H}$ also gives a natural way of translating cone positivity conditions from $\mathcal{C}$ to $\mathcal{C}^*$.

Definition 6.4.14. Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable. We say that f satisfies a Morse-type inequality if for any $v \in \mathcal{C}_f$ and $x \in \mathcal{C}$ satisfying the inequality

$$f(v) - sD(v) \cdot x > 0$$

we have that $v - x \in \mathcal{C}^\circ$.

Note that the prototype of the Morse-type inequality is the well known algebraic Morse inequality for nef divisors.

In order to translate the positivity in $\mathcal{C}$ to $\mathcal{C}^*$, we need the following "reverse" Khovanskii-Teissier inequality.

Proposition 6.4.15. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable and satisfy a Morse-type inequality. Then we have*

$$s(y^* \cdot v)(D(v) \cdot x) \geq f(v)(y^* \cdot x),$$

for any $y^* \in \mathcal{C}^*$, $v \in \mathcal{C}_f$ and $x \in \mathcal{C}$.

Proof. The inequality holds when $y^* = 0$, so we need to deal with the case when $y^* \neq 0$. Since both sides are homogeneous in all the arguments, we may rescale to assume that $y^* \cdot v = y^* \cdot x$. Then we need to show that $sD(v) \cdot x \geq f(v)$. If not, then

$$f(v) - sD(v) \cdot x > 0,$$

so that $v - x \in \mathcal{C}^\circ$ by the Morse-type inequality. But then we conclude that $y^* \cdot v > y^* \cdot x$, a contradiction. $\square$

Theorem 6.4.16. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ be $+$-differentiable and satisfy a Morse-type inequality. Then for any $v \in \mathcal{C}_f$ and $y^* \in \mathcal{C}^*$ satisfying*

$$\mathcal{H}f(D(v)) - sv \cdot y^* > 0,$$

we have $D(v) - y^* \in \mathcal{C}^{*\circ}$. In particular, we have $D(v) - y^* \in \mathcal{C}^*_{\mathcal{H}f}$ and

$$\begin{aligned}\mathcal{H}f(D(v) - y^*)^{s-1/s} &\geq (\mathcal{H}f(D(v)) - sv \cdot y^*)\mathcal{H}f(D(v))^{-1/s}\\ &= (f(v) - sv \cdot y^*)f(v)^{-1/s}.\end{aligned}$$

As a consequence, we get

$$\mathcal{H}f(D(v) - y^*) \geq f(v) - \frac{s^2}{s-1} v \cdot y^*.$$

Proof. Note that $\mathcal{H}f(D(v)) = f(v)$. First we claim that the inequality $f(v) - sv \cdot y^* > 0$ implies $D(v) - y^* \in \mathcal{C}^{*\circ}$. To this end, fix some sufficiently small $y'^* \in \mathcal{C}^{*\circ}$ such that $y^* + y'^*$ still satisfies $f(v) - sv \cdot (y^* + y'^*) > 0$.

Then by the "reverse" Khovanskii-Teissier inequality, for some $\delta > 0$ and any $x \in \mathcal{C}$ we have

$$D(v) \cdot x \geq \left(\frac{f(v)}{s(y^* + y'^*) \cdot v}\right)(y^* + y'^*) \cdot x \geq (1+\delta)(y^* + y'^*) \cdot x.$$

This implies $D(v) - y^* \in \mathcal{C}^{*\circ}$.

By the definition of $\mathcal{H}f$ we have

$$\begin{aligned}\mathcal{H}f(D(v) - y^*) &= \inf_{x \in \mathcal{C}^\circ} \left(\frac{(D(v) - y^*) \cdot x}{f(x)^{1/s}}\right)^{s/s-1}\\ &\geq \left(\frac{f(v) - sy^* \cdot v}{f(v)}\right)^{s/s-1} \inf_{x \in \mathcal{C}^\circ} \left(\frac{D(v) \cdot x}{f(x)^{1/s}}\right)^{s/s-1}\\ &= \mathcal{H}f(D(v)) \left(\frac{f(v) - sy^* \cdot v}{f(v)}\right)^{s/s-1},\end{aligned}$$

where the second line follows from "reverse" Khovanskii-Teissier inequality. To obtain the desired inequality, we only need to use the equality $\mathcal{H}f(D(v)) = f(v)$ again.

To show the last inequality, we only need to note that the function $(1-x)^\alpha$ is convex for $x \in [0,1)$ if $\alpha \geq 1$. This implies $(1-x)^\alpha \geq 1-\alpha x$. Applying this inequality in our situation, we get

$$\begin{aligned}\mathcal{H}f(D(v)-y^*) &\geq \left(1-\frac{sv\cdot y^*}{f(v)}\right)^{s/s-1} f(v)\\ &\geq f(v) - \frac{s^2}{s-1} v\cdot y^*.\end{aligned}$$

□

6.4.3 Boundary conditions

Under certain conditions we can control the behaviour of $\mathcal{H}f$ near the boundary, and thus obtain the continuity.

Definition 6.4.17. Let $f \in \mathrm{HConc}_s(\mathcal{C})$ and let $\alpha \in (0,1)$. We say that f satisfies the sublinear boundary condition of order α if for any non-zero v on the boundary of $\mathcal{C}$ and for any x in the interior of $\mathcal{C}$, there exists a constant $C := C(v,x) > 0$ such that $f(v+\epsilon x)^{1/s} \geq C\epsilon^\alpha$.

Note that the condition is always satisfied at v if $f(v) > 0$. Furthermore, the condition is satisfied for any v, x with $\alpha = 1$ by homogeneity and log-concavity, so the crucial question is whether we can decrease α slightly.

Using this sublinear condition, we get the vanishing of $\mathcal{H}f$ along the boundary.

Proposition 6.4.18. *Let $f \in \mathrm{HConc}_s(\mathcal{C})$ satisfy the sublinear boundary condition of order α. Then $\mathcal{H}f$ vanishes along the boundary. As a consequence, $\mathcal{H}f$ extends to a continuous function over V^* by setting $\mathcal{H}f = 0$ outside $\mathcal{C}^*$.*

Proof. Let w^* be a boundary point of $\mathcal{C}^*$. Then there exists some non-zero $v \in \mathcal{C}$ such that $w^* \cdot v = 0$. Fix $x \in \mathcal{C}^\circ$. By the definition of $\mathcal{H}f$ we get

$$\mathcal{H}f(w^*)^{s-1/s} \leq \frac{w^*\cdot(v+\epsilon x)}{f^{1/s}(v+\epsilon x)} \leq \frac{\epsilon w^* \cdot x}{C\epsilon^\alpha}.$$

Letting ϵ tend to zero, we see $\mathcal{H}f(w^*) = 0$.

To show the continuity, by Lemma 6.3.1 we only need to verify

$$\lim_{\epsilon\to 0} \mathcal{H}f(w^*+\epsilon y^*) = 0$$

for some $y^* \in \mathcal{C}^{*\circ}$ (as any other limiting sequence is dominated by such a sequence). This follows easily from

$$\begin{aligned}\mathcal{H}f(w^*+\epsilon y^*)^{s-1/s} &\leq \frac{(w^*+\epsilon y^*)\cdot(v+\epsilon x)}{f^{1/s}(v+\epsilon x)}\\ &\leq \frac{\epsilon(y^*\cdot v + w^*\cdot x + \epsilon y^* \cdot x)}{C\epsilon^\alpha}.\end{aligned}$$

□

Remark 6.4.19. If f satisfies the sublinear condition, then $\mathcal{C}^*_{\mathcal{H}f} = \mathcal{C}^{*\circ}$. This makes the statements of the previous results very clean. In the following sections, the function $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ both have this nice property.

6.5 Positivity for curves

We now study the basic properties of $\widehat{\mathrm{vol}}$ and of the Zariski decompositions for curves. Some aspects of the theory will follow immediately from the formal theory of Section 6.4 ; others will require a direct geometric argument.

We first outline how to apply the results of Section 6.4. Recall that $\widehat{\mathrm{vol}}$ is the polar transform of the volume function for divisors restricted to the nef cone. More precisely, we are now in the situation :

$$\mathcal{C} = \mathrm{Nef}^1(X), \quad f = \mathrm{vol}, \quad \mathcal{C}^* = \overline{\mathrm{Eff}}_1(X), \quad \mathcal{H}f = \widehat{\mathrm{vol}}.$$

Thus, to understand the properties of $\widehat{\mathrm{vol}}$ we need to recall the basic features of the volume function on the nef cone of divisors. It is an elementary fact that the volume function on the nef cone of divisors is differentiable everywhere (with $D(A) = A^{n-1}$). In the notation of Section 6.3 the cone $\mathrm{Nef}^1(X)_{\mathrm{vol}}$ coincides with the big and nef cone. The Khovanskii-Teissier inequality (with Teissier proportionality) holds on the big and nef cone as recalled in Section 6.2. Finally, the volume for nef divisors satisfies the sublinear boundary condition of order $n-1/n$: this follows from an elementary intersection calculation using the fact that $N \cdot A^{n-1} \neq 0$ for any non-zero nef divisor N and ample divisor A.

Remark 6.5.1. Due to the outline above, the proofs in this section depend only upon elementary facts about intersection theory, the Khovanskii-Teissier inequality and Teissier's proportionality theorem. As discussed in the preliminaries, the arguments in this section thus extend immediately to smooth varieties over an arbitrary algebraically closed field and to the Kähler setting.

6.5.1 Basic properties

The following theorems collect the various analytic consequences for $\widehat{\mathrm{vol}}$.

Theorem 6.5.2. *Let X be a projective variety of dimension n. Then :*

1. *$\widehat{\mathrm{vol}}$ is continuous and homogeneous of weight $n/n-1$ on $\overline{\mathrm{Eff}}_1(X)$ and is positive precisely for the big classes.*
2. *For any big and nef divisor class A, we have $\widehat{\mathrm{vol}}(A^{n-1}) = \mathrm{vol}(A)$.*
3. *For any big curve class α, there is a big and nef divisor class B such that*

$$\widehat{\mathrm{vol}}(\alpha) = \left(\frac{B \cdot \alpha}{\mathrm{vol}(B)^{1/n}} \right)^{n/n-1}.$$

We say that the class B computes $\widehat{\mathrm{vol}}(\alpha)$.

The first two were already proved in the previous chapter ([Xia15a, Theorem 3.1]).

Proof. (1) follows immediately from Propositions 6.3.2 and 6.4.18. Since $D(A) = A^{n-1}$, (2) follows from the computation

$$\widehat{\mathrm{vol}}(A^{n-1}) = D(A) \cdot A = A^n.$$

The existence in (3) follows from Theorem 6.3.6. □

We also note the following easy basic linearity property, which follows immediately from the Khovanskii-Teissier inequalities.

Theorem 6.5.3. *Let X be a projective variety of dimension n and let α be a big curve class. If A computes $\widehat{\mathrm{vol}}(\alpha)$, it also computes $\widehat{\mathrm{vol}}(c_1\alpha + c_2 A^{n-1})$ for any positive constants c_1 and c_2.*

After constructing Zariski decompositions below, we will see that in fact we can choose a possibly negative c_2 so long as $c_1\alpha + c_2 A^{n-1}$ is a big class.

6.5.2 Zariski decompositions for curves

The following theorem is the basic result establishing the existence of Zariski decompositions for curve classes.

Theorem 6.5.4. *Let X be a projective variety of dimension n. Any big curve class α admits a unique Zariski decomposition : there is a unique pair consisting of a big and nef divisor class B_α and a pseudo-effective curve class γ satisfying $B_\alpha \cdot \gamma = 0$ and*

$$\alpha = B_\alpha^{n-1} + \gamma.$$

In fact $\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(B_\alpha^{n-1}) = \mathrm{vol}(B_\alpha)$. In particular B_α computes $\widehat{\mathrm{vol}}(\alpha)$, and any big and nef divisor computing $\widehat{\mathrm{vol}}(\alpha)$ is proportional to B_α.

Proof. The existence of the Zariski decomposition and the uniqueness of the positive part B_α^{n-1} follow from Theorem 6.4.3. The uniqueness of B_α follows from Teissier proportionality for big and nef divisor classes. It is clear that B_α computes $\widehat{\mathrm{vol}}(\alpha)$ by Theorem 6.4.3. The last claim follows from Teissier proportionality and the fact that $\alpha \succeq B_\alpha^{n-1}$. □

As discussed before, conceptually the Zariski decomposition $\alpha = B_\alpha^{n-1} + \gamma$ captures the failure of log concavity of $\widehat{\mathrm{vol}}$: the term B_α^{n-1} captures all the of the positivity encoded by $\widehat{\mathrm{vol}}$ and is positive in a very strong sense, while the negative part γ lies on the boundary of the pseudo-effective cone.

Example 6.5.5. Let X be the projective bundle over $\mathbb{P}^1$ defined by $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)$. There are two natural divisor classes on X : the class f of the fibers of the projective bundle and the class ξ of the sheaf $\mathcal{O}_{X/\mathbb{P}^1}(1)$. Using for example [Ful11, Theorem 1.1] and [FL13, Proposition 7.1], one sees that f and ξ generate the algebraic cohomology classes with the relations $f^2 = 0$, $\xi^2 f = -\xi^3 = 1$ and

$$\overline{\mathrm{Eff}}^1(X) = \mathrm{Mov}^1(X) = \langle f, \xi \rangle \qquad \mathrm{Nef}^1(X) = \langle f, \xi + f \rangle$$

and

$$\overline{\mathrm{Eff}}_1(X) = \langle \xi f, \xi^2 \rangle \qquad \mathrm{Nef}_1(X) = \langle \xi f, \xi^2 + \xi f \rangle$$
$$\mathrm{CI}_1(X) = \langle \xi f, \xi^2 + 2\xi f \rangle.$$

Using this explicit computation of the nef cone of the divisors, we have

$$\widehat{\mathrm{vol}}(x\xi f + y\xi^2) = \inf_{a,b \geq 0} \frac{ay + bx}{(3ab^2 + 2b^3)^{1/3}}$$

This is essentially a one-variable minimization problem due to the homogeneity in a, b. It is straightforward to compute directly that for non-negative values of x, y :

$$\begin{aligned} \widehat{\mathrm{vol}}(x\xi f + y\xi^2) &= \left(\frac{3}{2}x - y\right) y^{1/2} && \text{if } x \geq 2y; \\ &= \frac{x^{3/2}}{2^{1/2}} && \text{if } x < 2y. \end{aligned}$$

Note that when $x < 2y$, the class $x\xi f + y\xi^2$ no longer lies in the complete intersection cone – to obtain $\widehat{\mathrm{vol}}$, Theorem 6.5.4 indicates that we must project α onto the complete intersection cone in the y-direction. This exactly coheres with the calculation above.

The Zariski decomposition for curves is continuous.

Theorem 6.5.6. *Let X be a projective variety of dimension n. The function sending a big class α to its positive part B_α^{n-1} or to the corresponding divisor B_α is continuous.*

Proof. The first statement follows from Theorem 6.4.6. The second then follows from the continuity of the inverse map to the $n-1$-power map. $\square$

It is interesting to study whether the Zariski projection taking α to its positive part is $\mathcal{C}^1$. This is true on the ample cone – the map Φ sending an ample divisor class A to A^{n-1} is a $\mathcal{C}^1$ diffeomorphism by the argument in Remark 6.2.3.

Remark 6.5.7. The continuity of the Zariski decomposition does not extend to the entire pseudo-effective cone, even for surfaces. For example, suppose that a surface S admits a nef class N which is a limit of (rescalings of) irreducible curve classes which each have negative self-intersection. (A well-known example of such a surface is $\mathbb{P}^2$ blown up at 9 general points.) For any $c \in [0,1]$ one can find a sequence of big divisors $\{L_i\}$ whose limit is N but whose positive parts have limit cN.

An important feature of the σ-decomposition for divisors is its concavity : given two big divisors L_1, L_2 we have

$$P_\sigma(L_1 + L_2) \succeq P_\sigma(L_1) + P_\sigma(L_2).$$

However, the analogous property fails for curves :

Example 6.5.8. Let X be a smooth projective variety such that $\mathrm{CI}_1(X)$ is not convex. (An explicit example is given in Appendix B.) Then there are complete intersection classes $\alpha = B_\alpha^{n-1}$ and $\beta = B_\beta^{n-1}$ such that $\alpha+\beta$ is not a complete intersection class. Let $B_{\alpha+\beta}^{n-1}$ denote the positive part of the Zariski decomposition for $\alpha+\beta$. Then

$$B_{\alpha+\beta}^{n-1} \preceq B_\alpha^{n-1} + B_\beta^{n-1}.$$

Furthermore, we can not have equality since the sum is not a complete intersection class. Thus

$$B_{\alpha+\beta}^{n-1} \not\succeq B_\alpha^{n-1} + B_\beta^{n-1}.$$

However, one can still ask :

Question 6.5.9. Fix $\alpha \in \overline{\mathrm{Eff}}_1(X)$. Is there a fixed class $\xi \in \mathrm{CI}_1(X)$ such that for any $\epsilon > 0$ there is a $\delta > 0$ satisfying

$$B_{\alpha+\delta\beta}^{n-1} \preceq B_{\alpha+\epsilon\xi}^{n-1}$$

for every $\beta \in N_1(X)$ of bounded norm ?

This question is crucial for making sense of the Zariski decomposition of a curve class on the boundary of $\overline{\mathrm{Eff}}_1(X)$ via taking a limit.

6.5.3 Strict log concavity

The following theorem is an immediate consequence of Theorem 6.4.3, which gives the strict log concavity of $\widehat{\mathrm{vol}}$.

Theorem 6.5.10. *Let X be a projective variety of dimension n. For any two pseudo-effective curve classes α, β we have*

$$\widehat{\mathrm{vol}}(\alpha+\beta)^{\frac{n-1}{n}} \geq \widehat{\mathrm{vol}}(\alpha)^{\frac{n-1}{n}} + \widehat{\mathrm{vol}}(\beta)^{\frac{n-1}{n}}.$$

Furthermore, if α and β are big, then we obtain an equality if and only if the positive parts of α and β are proportional.

6.5.4 Differentiability

In [BFJ09] the derivative of the volume function was calculated using the positive product : given a big divisor class L and any divisor class E, we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(L+tE) = n\langle L^{n-1}\rangle \cdot E.$$

In this section we prove an analogous statement for curve classes. For curves, the big and nef divisor class B occurring in the Zariski decomposition plays the role of the positive product, and the homogeneity constant $n/n-1$ plays the role of n.

Theorem 6.5.11. *Let X be a projective variety of dimension n, and let α be a big curve class with Zariski decomposition $\alpha = B^{n-1} + \gamma$. Let β be any curve class. Then $\widehat{\mathrm{vol}}(\alpha + t\beta)$ is differentiable at 0 and*

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{\mathrm{vol}}(\alpha+t\beta) = \frac{n}{n-1} B\cdot\beta.$$

In particular, the function $\widehat{\mathrm{vol}}$ is $\mathcal{C}^1$ on the big cone of curves. If C is an irreducible curve on X, then we can instead write

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{\mathrm{vol}}(\alpha+tC) = \frac{n}{n-1}\,\mathrm{vol}(B|_C).$$

Proof. This follows immediately from Proposition 6.3.10 since $G_\alpha \cup \{0\}$ consists of a single ray by the last statement of Theorem 6.5.4. □

Example 6.5.12. We return to the setting of Example 6.5.5 : let X be the projective bundle over $\mathbb{P}^1$ defined by $\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(-1)$. Using our earlier notation we have

$$\overline{\mathrm{Eff}}_1(X) = \langle \xi f, \xi^2\rangle$$

and

$$\begin{aligned}\widehat{\mathrm{vol}}(x\xi f + y\xi^2) &= \left(\frac{3}{2}x - y\right) y^{1/2} && \text{if } x \geq 2y;\\ &= \frac{x^{3/2}}{2^{1/2}} && \text{if } x < 2y.\end{aligned}$$

We focus on the complete intersection region where $x \geq 2y$. Then we have

$$x\xi f + y\xi^2 = \left(\frac{x-2y}{2y^{1/2}} f + y^{1/2}(\xi+f)\right)^2.$$

The divisor in the parentheses on the right hand side is exactly the B appearing in the Zariski decomposition expression for $x\xi f + y\xi^2$. Thus, we can calculate the directional derivative of $\widehat{\mathrm{vol}}$ along a curve class β by intersecting against this divisor.

For a very concrete example, set $\alpha = 3\xi f + \xi^2$, and consider the behavior of $\widehat{\mathrm{vol}}$ for

$$\alpha_t := 3\xi f + \xi^2 - t(2\xi f + \xi^2).$$

Note that α_t is pseudo-effective precisely for $t \leq 1$. In this range, the explicit expression for the volume above yields

$$\begin{aligned}\widehat{\mathrm{vol}}(\alpha_t) &= \left(\frac{7}{2} - 2t\right)(1-t)^{1/2},\\ \frac{d}{dt}\widehat{\mathrm{vol}}(\alpha_t) &= -3(1-t)^{1/2} - \frac{3}{4}(1-t)^{-1/2}.\end{aligned}$$

Note that this calculation agrees with the prediction of Theorem 6.5.11, which states that if B_t is the divisor defining the positive part of α_t then

$$\begin{aligned}\frac{d}{dt}\widehat{\mathrm{vol}}(\alpha_t) &= \frac{3}{2}B_t \cdot (2\xi f + \xi^2) \\ &= \frac{-3}{2}\left(\frac{(3-2t)-2(1-t)}{2(1-t)^{1/2}} + 2(1-t)^{1/2}\right).\end{aligned}$$

In particular, the derivative decreases to $-\infty$ as t approaches 1 (and the coefficients of the divisor B also increase without bound). This is a surprising contrast to the situation for divisors. Note also that $\widehat{\mathrm{vol}}$ is not convex on this line segment, while vol is convex in any pseudo-effective direction in the nef cone of divisors by the Morse inequality.

6.5.5 Negative parts

We next analyze the structure of the negative part of the Zariski decomposition. First we have :

Lemma 6.5.13. *Let X be a projective variety. Suppose α is a big curve class and write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. If $\gamma \neq 0$ then $\gamma \notin \mathrm{Mov}_1(X)$.*

Proof. Since B is big and $B \cdot \gamma = 0$, γ cannot be movable if it is non-zero. □

For the Zariski decomposition under $\widehat{\mathrm{vol}}$, we can not guarantee the negative part is a curve class of effective curve. As in [FL13], it is more reasonable to ask if the negative part is the pushforward of a pseudo-effective class from a proper subvariety. Note that this property is automatic when the negative part is represented by an effective class, and for surfaces it is actually equivalent to asking that the negative part be effective. In general this subtle property of pseudo-effective classes is crucial for inductive arguments on dimension.

Proposition 6.5.14. *Let X be a projective variety of dimension n. Let α be a big curve class and write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. There is a proper subscheme $i : V \subsetneq X$ and a pseudo-effective class $\gamma' \in N_1(V)$ such that $i_*\gamma' = \gamma$.*

Proof. We may choose an effective nef $\mathbb{R}$-Cartier divisor D whose class is B. By resolving the base locus of a sufficiently high multiple of D we obtain a blow-up $\phi : Y \to X$, a birational morphism $\psi : Y \to Z$ and an effective ample divisor A on Z such that after replacing D by some numerically equivalent divisor we have $\phi^* D \geq \psi^* A$. Write E for the difference of these two divisors and set V_Y to be the union of $\mathrm{Supp}(E)$ with the ψ-exceptional locus.

There is a pseudo-effective curve class γ_Y on Y which pushes forward to γ and thus satisfies $\phi^* D \cdot \gamma_Y = 0$. There is an infinite sequence of effective 1-cycles C_i such that $\lim_{i\to\infty}[C_i] = \gamma_Y$. Each effective cycle C_i can be decomposed as a sum $C_i = T_i + T_i'$ where T_i' consists of the components contained in V_Y and T_i consists of the rest.

Note that

$$\lim_{i\to\infty} A \cdot \psi_* T_i \leq \lim_{i\to\infty} \phi^* D \cdot T_i = 0.$$

This shows that $\lim_{i\to\infty}[T_i]$ converges to a pseudo-effective curve class $\beta \in N_1(Y)$ satisfying $\psi_*\beta = 0$.

Clearly $\lim_{i\to\infty}[T_i']$ is the pushforward of a pseudo-effective curve class from V_Y. [DJV13, Theorem 4.1] (which holds in the singular case by the same argument) shows that β is also the pushforward of a pseudo-effective curve class on V_Y. Thus γ_Y is the pushforward of a pseudo-effective curve class on V_Y. Pushing forward to X, we see that γ is the pushforward of a pseudo-effective curve class on $V := \phi(V_Y)$. Note that V is a proper subset of X since ϕ is birational. □

Remark 6.5.15. In contrast, for the Zariski decomposition of curves in the sense of Boucksom (see the previous chapter) the negative part can always be represented by an effective curve which is very rigidly embedded in X. This has a similar feel as the σ-decomposition of [Nak04] for curve classes.

6.5.6 Birational behavior

We next use the Zariski decomposition to analyze the behavior of positivity of curves under birational maps $\phi : Y \to X$. Note that (in contrast to divisors) the birational pullback can only decrease the positivity for curve classes : we have

$$\widehat{\mathrm{vol}}(\alpha) \geq \widehat{\mathrm{vol}}(\phi^*\alpha).$$

In fact pulling back does not preserve pseudo-effectiveness, and even for a movable class we can have a strict inequality of $\widehat{\mathrm{vol}}$ (for example, a big movable class can pull back to a movable class on the pseudo-effective boundary). Again guided by [FL13], the right approach is to consider all ϕ_*-preimages of α at once.

Proposition 6.5.16. *Let $\phi : Y \to X$ be a birational morphism of projective varieties of dimension n. Let α be a big curve class on X with Zariski decomposition $B^{n-1} + \gamma$. Let $\mathcal{A}$ be the set of all pseudo-effective curve classes α' on Y satisfying $\phi_*\alpha' = \alpha$. Then*

$$\sup_{\alpha' \in \mathcal{A}} \widehat{\mathrm{vol}}(\alpha') = \widehat{\mathrm{vol}}(\alpha).$$

This supremum is achieved by an element $\alpha_Y \in \mathcal{A}$.

Proof. Suppose $\alpha' \in \mathcal{A}$. Since $\phi_*\alpha' = \alpha$, it is clear from the projection formula that $\widehat{\mathrm{vol}}(\alpha') \leq \widehat{\mathrm{vol}}(\alpha)$. Conversely, set γ_Y to be any pseudo-effective curve class on Y pushing forward to γ. Define $\alpha_Y = \phi^*B^{n-1} + \gamma_Y$. Since $\phi^*B \cdot \gamma_Y = 0$, by Theorem 6.5.4 this expression is the Zariski decomposition for α_Y. In particular $\widehat{\mathrm{vol}}(\alpha_Y) = \widehat{\mathrm{vol}}(\alpha)$. □

This proposition indicates the existence of some "distinguished" preimages of α with maximum $\widehat{\mathrm{vol}}$. In fact, these distinguished preimages also have a very nice structure.

Proposition 6.5.17. *Let $\phi : Y \to X$ be a birational morphism of projective varieties of dimension n. Let α be a big curve class on X with Zariski decomposition $B^{n-1} + \gamma$. Set $\mathcal{A}'$ to be the set of all pseudo-effective curve class α' on Y satisfying $\phi_*\alpha' = \alpha$ and $\widehat{\mathrm{vol}}(\alpha') = \widehat{\mathrm{vol}}(\alpha)$. Then*

1. *Every $\alpha' \in \mathcal{A}'$ has a Zariski decomposition of the form*
$$\alpha' = \phi^*B^{n-1} + \gamma'.$$
Thus $\mathcal{A}' = \{\phi^*B^{n-1} + \gamma' \mid \gamma' \in \overline{\mathrm{Eff}}_1(Y), \phi_*\gamma' = \gamma\}$ is determined by the set of pseudo-effective preimages of γ.
2. *These Zariski decompositions are stable under adding ϕ-exceptional curves : if ξ is a pseudo-effective curve class satisfying $\phi_*\xi = 0$, then for any $\alpha' \in \mathcal{A}'$ we have*
$$\alpha' + \xi = \phi^*B^{n-1} + (\gamma' + \xi)$$
is the Zariski decomposition for $\alpha' + \xi$.

Proof. To see (1), note that

$$\frac{\phi^*B}{\mathrm{vol}(B)^{1/n}} \cdot \alpha' = \frac{B}{\mathrm{vol}(B)^{1/n}} \cdot \alpha = \widehat{\mathrm{vol}}(\alpha).$$

Thus if $\widehat{\mathrm{vol}}(\alpha') = \widehat{\mathrm{vol}}(\alpha)$ then $\widehat{\mathrm{vol}}(\alpha')$ is computed by ϕ^*B. By Theorem 6.5.4 we obtain the statement.

(2) follows immediately from (1), since

$$\widehat{\mathrm{vol}}(\alpha) = \widehat{\mathrm{vol}}(\alpha') \leq \widehat{\mathrm{vol}}(\alpha' + \xi) \leq \widehat{\mathrm{vol}}(\alpha)$$

by Proposition 6.5.16. □

While there is not necessarily a *uniquely* distinguished ϕ_*-preimage of α, there *is* a uniquely distinguished complete intersection class on Y whose ϕ-pushforward lies beneath α – namely, the positive part of any sufficiently large class pushing forward to α. This is the analogue in our setting of the "movable transform" of [FL13].

6.5.7 Morse-type inequality for curves

In this section we prove a Morse-type inequality for curves under the volume function $\widehat{\mathrm{vol}}$. First let us recall the algebraic Morse inequality for nef divisor classes over smooth projective varieties. If A, B are nef divisor classes on a smooth projective variety X of dimension n, then by [Laz04, Example 2.2.33] (see also [Dem85], [Siu93], [Tra95])

$$\mathrm{vol}(A-B) \geq A^n - nA^{n-1} \cdot B.$$

In particular, if $A^n - nA^{n-1} \cdot B > 0$, then $A-B$ is big. This gives us a very useful bigness criterion for the difference of two nef divisors.

By analogy with the divisor case, we can ask :

– Let X be a projective variety of dimension n, and let $\alpha, \gamma \in \overline{\mathrm{Eff}}_1(X)$ be two nef curve classes. Is there a criterion for the bigness of $\alpha - \gamma \in \overline{\mathrm{Eff}}_1(X)$ using only intersection numbers defined by α, γ ?

We give such a criterion using the $\widehat{\mathrm{vol}}$ function. In Section 6.6, we answer the above question by giving a slightly different criterion which needs the refined structure of the movable cone of curves ; see Theorem 6.6.18. The following results follow from Theorem 6.4.16.

Theorem 6.5.18. *Let X be a projective variety of dimension n. Let α be a big curve class and let β be a movable curve class. Write $\alpha = B^{n-1} + \gamma$ for the Zariski decomposition of α. Then*

$$\begin{aligned}\widehat{\mathrm{vol}}(\alpha-\beta)^{n-1/n} &\geq (\widehat{\mathrm{vol}}(\alpha) - nB\cdot\beta)\cdot\widehat{\mathrm{vol}}(\alpha)^{-1/n}\\ &= (B^n - nB\cdot\beta)\cdot(B^n)^{-1/n}.\end{aligned}$$

In particular, we have

$$\widehat{\mathrm{vol}}(\alpha-\beta) \geq B^n - \frac{n^2}{n-1}B\cdot\beta.$$

Proof. The theorem follows immediately from Theorem 6.4.16 and the fact that $\alpha \succeq B^{n-1}$. □

Corollary 6.5.19. *Let X be a projective variety of dimension n. Let α be a big curve class and let β be a movable curve class. Write $\alpha = B^{n-1} + \gamma$ for the Zariski decomposition of α. If*

$$\widehat{\mathrm{vol}}(\alpha) - nB\cdot\beta > 0$$

then $\alpha - \beta$ is big.

Remark 6.5.20. Superficially, the above theorem appears to differ from the classical algebraic Morse inequality for nef divisors, since α can be any big curve class. However, using the Zariski decomposition one sees that the statement for α is essentially equivalent to the statement for the positive part of α, so that Theorem 6.5.18 is really a claim about nef curve classes.

Example 6.5.21. The constant n is optimal in Corollary 6.5.19. Indeed, for any $\epsilon > 0$ there exists a projective variety X such that

$$\widehat{\mathrm{vol}}(\alpha) - (n-\epsilon)B_\alpha\cdot\gamma > 0,$$

for some $\alpha \in \overline{\mathrm{Eff}}_1(X)$ and $\gamma \in \mathrm{Mov}_1(X)$ but $\alpha - \gamma$ is not a big curve class.

To find such a variety, let E be an elliptic curve with complex multiplication and set $X = E^{\times n}$. The pseudo-effective cone of divisors $\overline{\mathrm{Eff}}^1(X)$ is identified with the cone of constant positive $(1,1)$-forms, while the pseudo-effective cone of curves $\overline{\mathrm{Eff}}_1(X)$ is identified with the cone of constant positive $(n-1,n-1)$-forms. Furthermore, every strictly positive $(n-1,n-1)$-form is a $(n-1)$-self-product of a strictly positive $(1,1)$-form.

Set $B_\alpha = i\sum_{j=1}^n dz^j \wedge d\bar{z}^j$ and $B_\gamma = i\sum_{j=1}^n \lambda_j dz^j \wedge d\bar{z}^j$ with $\lambda_j > 0$. Let $\alpha = B_\alpha^{n-1}$ and $\gamma = B_\gamma^{n-1}$. Then $\widehat{\mathrm{vol}}(\alpha) - (n-\epsilon)B_\alpha\cdot\gamma > 0$ is equivalent to

$$\sum_{j=1}^n \lambda_1 ... \widehat{\lambda_j} ... \lambda_n < \frac{n}{n-\epsilon},$$

and $\alpha - \gamma$ being big is equivalent to

$$\lambda_1 ... \widehat{\lambda_j} ... \lambda_n < 1$$

for every j. Now it is easy to see we can always choose $\lambda_1, ..., \lambda_n$ such that the first inequality holds but the second does not hold.

Remark 6.5.22. Using the cone duality $\overline{\mathcal{K}}^* = \mathcal{N}$ and Theorem 6.12.1 in Appendix A, it is easy to extend the above Morse-type inequality for curves to positive currents of bidimension $(1,1)$ over compact Kähler manifolds.

One wonders if Theorem 6.5.18 can be improved :

Question 6.5.23. Let X be a projective variety of dimension n. Let α be a big curve class and let β be a movable curve class. Write $\alpha = B^{n-1} + \gamma$ for the Zariski decomposition of α. Is

$$\widehat{\mathrm{vol}}(\alpha - \beta) \geq \mathrm{vol}(\alpha) - nB \cdot \beta?$$

Remark 6.5.24. By Theorem 6.5.18, if $\widehat{\mathrm{vol}}(\alpha) - nB \cdot \beta > 0$ then $\widehat{\mathrm{vol}}$ is $\mathcal{C}^1$ at the point $\alpha - s\beta$ for every $s \in [0,1]$. The derivative formula of $\widehat{\mathrm{vol}}$ implies

$$\widehat{\mathrm{vol}}(\alpha - \beta) - \widehat{\mathrm{vol}}(\alpha) = \int_0^1 -\frac{n}{n-1} B_{\alpha - s\beta} \cdot \beta \, ds,$$

where $B_{\alpha - s\beta}$ is the big and nef divisor class defining the Zariski decomposition of $\alpha - s\beta$. To give an affirmative answer to Question 6.5.23, we conjecture the following :

$$B_{\alpha - s\beta} \cdot \beta \leq (n-1) B_\alpha \cdot \beta \text{ for every } s \in [0,1].$$

Without loss of generality, we can assume $B_\alpha \cdot \beta > 0$. Then by continuity of the decomposition, this inequality holds for s in a neighbourhood of 0. At this moment, we do not know how to see this neighbourhood covers $[0,1]$.

6.6 Positive products and movable curves

In this section, we study the movable cone of curves and its relationship to the positive product of divisors. A key tool in this study is the following function of [Xia15a, Definition 2.2] :

Definition 6.6.1 (see [Xia15a] Definition 2.2)**.** Let X be a projective variety of dimension n. For any curve class $\alpha \in \mathrm{Mov}_1(X)$ define

$$\mathfrak{M}(\alpha) = \inf_{L \text{ big } \mathbb{R}\text{-divisor}} \left(\frac{L \cdot \alpha}{\mathrm{vol}(L)^{1/n}} \right)^{n/n-1}.$$

We say that a big class L computes $\mathfrak{M}(\alpha)$ if this infimum is achieved by L. When α is a curve class that is not movable, we set $\mathfrak{M}(\alpha) = 0$.

In other words, $\mathfrak{M}$ is the function on $\mathrm{Mov}_1(X)$ defined as the polar transform of the volume function on $\overline{\mathrm{Eff}}^1(X)$, so we are in the situation :

$$\mathcal{C} = \overline{\mathrm{Eff}}^1(X), \quad f = \mathrm{vol}, \quad \mathcal{C}^* = \mathrm{Mov}_1(X), \quad \mathcal{H}f = \mathfrak{M}.$$

Note that $\mathcal{C}^* = \mathrm{Mov}_1(X)$ follows from the main result of [BDPP13].

While the definition is a close analogue of $\widehat{\mathrm{vol}}$, the function $\mathfrak{M}$ exhibits somewhat different behavior. We will show that $\mathfrak{M}$ measures the volume of the "$(n-1)$st root" of α, in a sense described below. In order to establish some deeper properties of the function $\mathfrak{M}$, we need to better understand the volume function for divisors.

We first extend several well known results on big and nef divisors to big and movable divisors.

6.6.1 The volume function on big and movable divisors

The key will be an extension of Teissier's proportionality theorem for big and nef divisors (see Section 6.2) to big and movable divisors.

Lemma 6.6.2. *Let X be a projective variety of dimension n. Let L_1 and L_2 be big movable divisor classes. Set s to be the largest real number such that $L_1 - sL_2$ is pseudo-effective. Then*

$$s^n \leq \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}$$

with equality if and only if L_1 and L_2 are proportional.

Proof. We first prove the case when X is smooth. Certainly we have $\mathrm{vol}(L_1) \geq \mathrm{vol}(sL_2) = s^n \mathrm{vol}(L_2)$. If they are equal, then since sL_2 is movable and $L_1 - sL_2$ is pseudo-effective we get a Zariski decomposition of

$$L_1 = sL_2 + (L_1 - sL_2)$$

in the sense of [FL13]. By [FL13, Proposition 5.3], this decomposition coincides with the numerical version of the σ-decomposition of [Nak04] so that $P_\sigma(L_1) = sL_2$. Since L_1 is movable, we obtain equality $L_1 = sL_2$.

For arbitrary X, let $\phi : X' \to X$ be a resolution. The inequality follows by pulling back L_1 and L_2 and replacing them by their positive parts. Indeed using the numerical analogue of [Nak04, III.1.14 Proposition] we see that $\phi^* L_1 - sP_\sigma(\phi^* L_2)$ is pseudo-effective if and only if $P_\sigma(\phi^* L_1) - sP_\sigma(\phi^* L_2)$ is pseudo-effective, so that s can only go up under this operation. To characterize the equality, recall that if L_1 and L_2 are movable and $P_\sigma(\phi^* L_1) = sP_\sigma(\phi^* L_2)$, then $L_1 = sL_2$ by the injectivity of the capping map. □

Proposition 6.6.3. *Let X be a projective variety of dimension n. Let L_1, L_2 be big and movable divisor classes. Then*

$$\langle L_1^{n-1} \rangle \cdot L_2 \geq \mathrm{vol}(L_1)^{n-1/n} \mathrm{vol}(L_2)^{1/n}$$

with equality if and only if L_1 and L_2 are proportional.

Proof. We first suppose X is smooth. Set s_L to be the largest real number such that $L_1 - s_L L_2$ is pseudo-effective, and fix an ample divisor H on X.

For any $\epsilon > 0$, by taking sufficiently good Fujita approximations we may find a birational map $\phi_\epsilon : Y_\epsilon \to X$ and ample divisor classes $A_{1,\epsilon}$ and $A_{2,\epsilon}$ such that

- $\phi_\epsilon^* L_i - A_{i,\epsilon}$ is pseudo-effective for $i = 1, 2$;
- $\mathrm{vol}(A_{i,\epsilon}) > \mathrm{vol}(L_i) - \epsilon$ for $i = 1, 2$;
- $\phi_{\epsilon *} A_{i,\epsilon}$ is in an ϵ-ball around L_i for $i = 1, 2$.

Furthermore :

- By applying the argument of [FL13, Theorem 6.22], we may ensure

$$\phi_\epsilon^*(\langle L_1^{n-1} \rangle - \epsilon H^{n-1}) \preceq A_{1,\epsilon}^{n-1} \preceq \phi_\epsilon^*(\langle L_1^{n-1} \rangle + \epsilon H^{n-1}).$$

- Set s_ϵ to be the largest real number such that $A_{1,\epsilon} - s_\epsilon A_{2,\epsilon}$ is pseudo-effective. Then we may ensure that $s_\epsilon < s_L + \epsilon$.

By the Khovanskii-Teissier inequality for nef divisor classes, we have

$$(A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{n/n-1} \geq \mathrm{vol}(A_{1,\epsilon}) \mathrm{vol}(A_{2,\epsilon})^{1/n-1}.$$

Note that $\langle L_1^{n-1} \rangle \cdot L_2 = \langle L_1^{n-1} \cdot L_2 \rangle$ as L_1, L_2 are movable, thus $\langle L_1^{n-1} \rangle \cdot L_2$ is approximated by $A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon}$ by the projection formula. Taking a limit as ϵ goes to 0, we see that

$$(\star) \qquad \langle L_1^{n-1} \rangle \cdot L_2 \geq \mathrm{vol}(L_1)^{n-1/n} \mathrm{vol}(L_2)^{1/n}.$$

On the other hand, the Diskant inequality for big and nef divisors in [BFJ09, Theorem F] implies that

$$\begin{aligned}(A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{n/n-1} - &\operatorname{vol}(A_{1,\epsilon}) \operatorname{vol}(A_{2,\epsilon})^{1/n-1} \\ &\geq \left((A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{1/n-1} - s_\epsilon \operatorname{vol}(A_{2,\epsilon})^{1/n-1}\right)^n \\ &\geq \left((A_{1,\epsilon}^{n-1} \cdot A_{2,\epsilon})^{1/n-1} - (s_L + \epsilon) \operatorname{vol}(A_{2,\epsilon})^{1/n-1}\right)^n.\end{aligned}$$

Taking a limit as ϵ goes to 0 again, we see that

$$\begin{aligned}(\langle L_1^{n-1}\rangle \cdot L_2)^{n/n-1} - &\operatorname{vol}(L_1) \operatorname{vol}(L_2)^{1/n-1} \\ &\geq \left((\langle L_1^{n-1}\rangle \cdot L_2)^{1/n-1} - s_L \operatorname{vol}(L_2)^{1/n-1}\right)^n.\end{aligned}$$

Thus we extend the Diskant inequality to big and movable divisor classes. Lemma 6.6.2, equation $(\star)$ and the above Diskant inequality together show that

$$\langle L_1^{n-1}\rangle \cdot L_2 = \operatorname{vol}(L_1)^{n-1/n} \operatorname{vol}(L_2)^{1/n}$$

if and only if L_1 and L_2 are proportional.

Now suppose X is singular. The inequality can be computed by passing to a resolution $\phi : X' \to X$ and replacing L_1 and L_2 by their positive parts, since the left hand side can only decrease under this operation. To characterize the equality, recall that if L_1 and L_2 are movable and $P_\sigma(\phi^* L_1) = sP_\sigma(\phi^* L_2)$, then $L_1 = sL_2$ by the injectivity of the capping map. □

Remark 6.6.4. As a byproduct of the proof above, we get the Diskant inequality for big and movable divisor classes.

Remark 6.6.5. In the analytic setting, applying Proposition 6.6.3 and the same method in the proof of Theorem 6.2.1, it is not hard to generalize Theorem 6.2.1 to big and movable divisor classes provided we have enough regularity of degenerate complex Monge-Ampère equations :

– Let $L_1, ..., L_n$ be n big divisor classes over a smooth complex projective variety X, then we have

$$\langle L_1 \cdot ... \cdot L_n\rangle \geq \operatorname{vol}(L_1)^{1/n} \cdot ... \cdot \operatorname{vol}(L_n)^{1/n}$$

where the equality is obtained if and only if $P_\sigma(L_1), ..., P_\sigma(L_n)$ are proportional.

We only need to characterize the equality situation. To see this, we need the fact that the above positive intersection $\langle L_1 \cdot ... \cdot L_n\rangle$ depends only on the positive parts $P_\sigma(L_i)$, which follows from the analytic construction of positive product [Bou02a, Proposition 3.2.10]. Then by the method in the proof of Theorem 6.2.1 where we apply [BEGZ10] or [DDG+14, Theorem D], we reduce it to the case of a pair of divisor classes, e.g. we get

$$\langle P_\sigma(L_1)^{n-1} \cdot P_\sigma(L_2)\rangle = \operatorname{vol}(L_1)^{n-1/n} \operatorname{vol}(L_2)^{1/n}.$$

By the definition of positive product we always have

$$\langle P_\sigma(L_1)^{n-1} \cdot P_\sigma(L_2)\rangle = \langle P_\sigma(L_1)^{n-1}\rangle \cdot P_\sigma(L_2) \geq \operatorname{vol}(L_1)^{n-1/n} \operatorname{vol}(L_2)^{1/n},$$

this then implies the equality

$$\langle P_\sigma(L_1)^{n-1}\rangle \cdot P_\sigma(L_2) = \operatorname{vol}(L_1)^{n-1/n} \operatorname{vol}(L_2)^{1/n}.$$

By Proposition 6.6.3, we immediately obtain the desired result.

Corollary 6.6.6. *Let X be a smooth projective variety of dimension n. Let $\alpha \in \operatorname{Mov}_1(X)$ be a big movable curve class. All big divisor classes L satisfying $\alpha = \langle L^{n-1}\rangle$ have the same positive part $P_\sigma(L)$.*

Proof. Suppose L_1 and L_2 have the same positive product. We have $\mathrm{vol}(L_1) = \langle L_2^{n-1}\rangle \cdot L_1$ so that $\mathrm{vol}(L_1) \geq \mathrm{vol}(L_2)$. By symmetry we obtain the reverse inequality, hence equality everywhere, and we conclude by Proposition 6.6.3 and the σ-decomposition for smooth varieties. □

As a consequence of Proposition 6.6.3, we show the strict log concavity of the volume function vol on the cone of big and movable divisors.

Proposition 6.6.7. *Let X be a projective variety of dimension n. Then the volume function* vol *is strictly n-concave on the cone of big and movable divisor classes.*

Proof. Since the big and movable cone is convex, this follows from Proposition 6.6.3 and Theorem 6.4.11. □

6.6.2 The function $\mathfrak{M}$

We now return to the study of the function $\mathfrak{M}$. As preparation for using the polar transform theory of Section 6.4, we note the following features of the volume function of divisors on smooth varieties. By [BFJ09] the volume function on the pseudo-effective cone of divisors is differentiable on the big cone (with $D(L) = \langle L^{n-1}\rangle$). In the notation of Section 6.3 the cone $\overline{\mathrm{Eff}}^1(X)_{\mathrm{vol}}$ coincides with the big cone, so that vol is $+$-differentiable. The volume function is n-concave, and is strictly n-concave on the big and movable cone by Proposition 6.6.7. Furthermore, it admits a strong Zariski decomposition with respect to the movable cone of divisors using the σ-decomposition of [Nak04] and Proposition 6.6.7.

Remark 6.6.8. Note that if X is not smooth (or at least $\mathbb{Q}$-factorial), then it is unclear whether vol admits a Zariski decomposition structure with respect to the cone of movable divisors. For this reason, we will focus on smooth varieties in this section. See Remark 6.6.22 for more details.

Note that the sublinearity condition does not hold for the volume function. Thus our first task is to understand the behaviour of $\mathfrak{M}$ on the boundary of the movable cone of curves.

Lemma 6.6.9. *Let X be a smooth projective variety of dimension n and let α be a movable curve class. Then $\mathfrak{M}(\alpha) = 0$ if and only if α has vanishing intersection a non-zero movable divisor class L.*

Proof. We first show that if there exists some nonzero movable divisor class M such that $\alpha \cdot M = 0$ then $\mathfrak{M}(\alpha) = 0$. Fix an ample divisor class A. Note that $M + \epsilon A$ is big and movable for any $\epsilon > 0$. Thus there exists some modification $\mu_\epsilon : Y_\epsilon \to X$ and an ample divisor class A_ϵ on Y_ϵ such that $M + \frac{\epsilon}{2}A = \mu_{\epsilon*}A_\epsilon$. So we can write

$$M + \epsilon A = \mu_{\epsilon*}\left(A_\epsilon + \frac{\epsilon}{2}\mu_\epsilon^* A\right),$$

which implies

$$\begin{aligned}
\mathrm{vol}(M + \epsilon A) &= \mathrm{vol}\left(\mu_{\epsilon*}\left(A_\epsilon + \frac{\epsilon}{2}\mu_\epsilon^* A\right)\right)\\
&\geq \mathrm{vol}\left(A_\epsilon + \frac{\epsilon}{2}\mu_\epsilon^* A\right)\\
&\geq n\left(\frac{\epsilon}{2}\mu_\epsilon^* A\right)^{n-1} \cdot A_\epsilon\\
&\geq c\epsilon^{n-1}A^{n-1} \cdot M.
\end{aligned}$$

Consider the following intersection number

$$\rho_\epsilon = \alpha \cdot \frac{M + \epsilon A}{\mathrm{vol}(M + \epsilon A)^{1/n}}.$$

The above estimate shows that ρ_ϵ tends to zero as ϵ tends to zero, and so $\mathfrak{M}(\alpha) = 0$.

Conversely, suppose that $\mathfrak{M}(\alpha) = 0$. From the definition of $\mathfrak{M}(\alpha)$, we can take a sequence of big divisor classes L_k with $\mathrm{vol}(L_k) = 1$ such that

$$\lim_{k\to\infty} (\alpha \cdot L_k)^{\frac{n}{n-1}} = \mathfrak{M}(\alpha).$$

Moreover, let $P_\sigma(L_k)$ be the positive part of L_k. Then we have $\mathrm{vol}(P_\sigma(L_k)) = 1$ and

$$\alpha \cdot P_\sigma(L_k) \leq \alpha \cdot L_k$$

since α is movable. Thus we can assume the sequence of big divisor classes L_k is movable in the beginning.

Fix an ample divisor A of volume 1, and consider the classes $L_k/(A^{n-1} \cdot L_k)$. These lie in a compact slice of the movable cone, so they must have a non-zero movable accumulation point L, which without loss of generality we may assume is a limit.

Choose a modification $\mu_\epsilon : Y_\epsilon \to X$ and an ample divisor class $A_{\epsilon,k}$ on Y such that

$$A_{\epsilon,k} \leq \mu_\epsilon^* L_k, \quad \mathrm{vol}(A_{\epsilon,k}) > \mathrm{vol}(L_k) - \epsilon$$

Then

$$L_k \cdot A^{n-1} \geq A_{\epsilon,k} \cdot \mu_\epsilon^* A^{n-1} \geq \mathrm{vol}(A_{\epsilon,k})^{1/n}$$

by the Khovanskii-Teissier inequality. Taking a limit over all ϵ, we find $L_k \cdot A^{n-1} \geq \mathrm{vol}(L_k)^{1/n}$. Thus

$$L \cdot \alpha = \lim_{k\to\infty} \frac{L_k \cdot \alpha}{L_k \cdot A^{n-1}} \leq \mathfrak{M}(\alpha)^{n-1/n} = 0.$$

□

Example 6.6.10. Note that a movable curve class α with positive $\mathfrak{M}$ need not lie in the interior of the movable cone of curves. A simple example is when X is the blow-up of $\mathbb{P}^2$ at one point, H denotes the pullback of the hyperplane class. For surfaces the functions $\mathfrak{M}$ and vol coincide, so $\mathfrak{M}(H) = 1$ even though H is on the boundary of $\mathrm{Mov}_1(X) = \mathrm{Nef}^1(X)$.

It is also possible for a big movable curve class α to have $\mathfrak{M}(\alpha) = 0$. This occurs for the projective bundle $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1))$ analyzed in Example 6.5.5. Keeping the notation there, we see that the big and movable curve class $\xi^2 + \xi f$ has vanishing intersection against the movable divisor ξ so that $\mathfrak{M}(\xi^2 + \xi f) = 0$ by Lemma 6.6.9.

Remark 6.6.11. Another perspective on Lemma 6.6.9 is provided by the numerical dimension of [Nak04] and [Bou04]. We recall from [Leh13a] the fact that on a smooth variety the following conditions are equivalent for a class $L \in \overline{\mathrm{Eff}}^1(X)$. (They both correspond to the non-vanishing of the numerical dimension.)

- Fix an ample divisor class A. For any big class D, there is a positive constant C such that $Ct^{n-1} < \mathrm{vol}(L + tA)$ for all $t > 0$.
- $P_\sigma(L) \neq 0$.

In particular, this implies that vol satisfies the sublinear boundary condition of order $n - 1/n$ when restricted to the movable cone, and can be used in the previous proof. A variant of this statement in characteristic p is proved by [CHMS14].

In many ways it is most natural to define $\mathfrak{M}$ using the movable cone of divisors instead of the pseudo-effective cone of divisors. Conceptually, this coheres with the fact that the polar transform can be calculated using the positive part of a Zariski decomposition. Indeed, the argument above (passing to the positive part) shows that when X is smooth, for any $\alpha \in \mathrm{Mov}_1(X)$ we have

$$\mathfrak{M}(\alpha) = \inf_{D \text{ big and movable}} \left(\frac{D \cdot \alpha}{\mathrm{vol}(D)^{1/n}} \right)^{n/n-1}.$$

All in all, for X smooth it is better to consider the following polar transform :

$$\mathcal{C} = \mathrm{Mov}^1(X), \quad f = \mathrm{vol}, \quad \mathcal{C}^* = \mathrm{Mov}^1(X)^*, \quad \mathcal{H}f = \mathfrak{M}'.$$

In particular, since vol satisfies a sublinear condition on $\mathrm{Mov}^1(X)$, the function $\mathfrak{M}'$ is strictly positive exactly in $\mathrm{Mov}^1(X)^{*\circ}$ and extends to a continuous function over $N_1(X)$.

Since this polar function admits a Zariski decomposition onto $\mathrm{Mov}_1(X)$, we continue to focus on the subcone $\mathrm{Mov}_1(X) \subset \mathrm{Mov}^1(X)^*$ where there is interesting behavior and apply $\mathfrak{M}'|_{\mathrm{Mov}_1(X)} = \mathfrak{M}$. Note however an important consequence of this perspective : Lemma 6.6.9 shows that the subcone of $\mathrm{Mov}_1(X)$ where $\mathfrak{M}$ is positive lies in the interior of $\mathrm{Mov}^1(X)^*$. Thus this region agrees with $\mathrm{Mov}_1(X)_{\mathfrak{M}}$ and $\mathfrak{M}$ extends to a differentiable function on an open set containing this cone by applying Theorem 6.4.3. In particular $\mathfrak{M}$ is continuous on $\mathrm{Mov}_1(X)$.

We next prove a refined structure of the movable cone of curves. Recall that by [BDPP13] the movable cone of curves $\mathrm{Mov}_1(X)$ is generated by the $(n-1)$-self positive products of big divisors. In other words, any curve class in the interior of $\mathrm{Mov}_1(X)$ is a *convex combination* of such positive products. We show that $\mathrm{Mov}_1(X)$ actually coincides with the closure of such products (which naturally form a cone).

Theorem 6.6.12. *Let X be a smooth projective variety of dimension n. Then any movable curve class α with $\mathfrak{M}(\alpha) > 0$ has the form*

$$\alpha = \langle L_\alpha^{n-1} \rangle$$

for a unique big and movable divisor class L_α. We then have $\mathfrak{M}(\alpha) = \mathrm{vol}(L_\alpha)$ and any big and movable divisor computing $\mathfrak{M}(\alpha)$ is proportional to L_α.

Proof. Applying Theorem 6.4.3 to $\mathfrak{M}'$, we get

$$\alpha = D(L_\alpha) + n_\alpha$$

where L_α is a big movable class computing $\mathfrak{M}(\alpha)$ and $n_\alpha \in \mathrm{Mov}^1(X)^*$. As D is the differential of $\mathrm{vol}^{1/n}$ on big and movable divisor classes, we have $D(L_\alpha) = \langle L_\alpha^{n-1} \rangle$. Note that $\mathfrak{M}(\alpha) = \langle L_\alpha^{n-1} \rangle \cdot L_\alpha = \mathrm{vol}(L_\alpha)$.

To finish the proof, we observe that $n_\alpha \in \mathrm{Mov}_1(X)$. This follows since α is movable : by the definition of L_α, for any pseudo-effective divisor class E and $t \geq 0$ we have

$$\frac{\alpha \cdot L_\alpha}{\mathrm{vol}(L_\alpha)^{1/n}} \leq \frac{\alpha \cdot P_\sigma(L_\alpha + tE)}{\mathrm{vol}(L_\alpha + tE)^{1/n}} \leq \frac{\alpha \cdot (L_\alpha + tE)}{\mathrm{vol}(L_\alpha + tE)^{1/n}}$$

with equality at $t = 0$. This then implies

$$n_\alpha \cdot E \geq 0.$$

Thus $n_\alpha \in \mathrm{Mov}_1(X)$. Intersecting against L_α, we have $n_\alpha \cdot L_\alpha = 0$. This shows $n_\alpha = 0$ because L_α is an interior point of $\overline{\mathrm{Eff}}^1(X)$ and $\overline{\mathrm{Eff}}^1(X)^* = \mathrm{Mov}_1(X)$.

So we have

$$\alpha = D(L_\alpha) = \langle L_\alpha^{n-1} \rangle.$$

Finally, the uniqueness follows from Corollary 6.6.6. □

We note in passing that we immediately obtain :

Corollary 6.6.13. *Let X be a projective variety of dimension n. Then the rays spanned by classes of irreducible curves which deform to cover X are dense in* $\mathrm{Mov}_1(X)$.

Proof. It suffices to prove this on a resolution of X. By Theorem 6.6.12 it suffices to show that any class of the form $\langle L^{n-1} \rangle$ for a big divisor L is a limit of rescalings of classes of irreducible curves which deform to cover X. Indeed, we may even assume that L is a $\mathbb{Q}$-Cartier divisor. Then the positive product is a limit of the pushforward of complete intersections of ample divisors on birational models, whence the result. □

We can also describe the boundary of $\mathrm{Mov}_1(X)$, in combination with Lemma 6.6.9.

Corollary 6.6.14. *Let X be a smooth projective variety of dimension n. Let α be a movable class with $\mathfrak{M}(\alpha) > 0$ and let L_α be the unique big movable divisor whose positive product is α. Then α is on the boundary of $\mathrm{Mov}_1(X)$ if and only if L_α is on the boundary of $\mathrm{Mov}^1(X)$.*

Proof. Note that α is on the boundary of $\mathrm{Mov}_1(X)$ if and only if it has vanishing intersection against a class D lying on an extremal ray of $\overline{\mathrm{Eff}}^1(X)$. Lemma 6.6.9 shows that in this case D is not movable, so by [Nak04, Chapter III.1] D is (after rescaling) the class of an integral divisor on X which we continue to call D. By [BFJ09, Proposition 4.8 and Theorem 4.9], the equation $\langle L_\alpha^{n-1}\rangle \cdot D = 0$ holds if and only if $D \in \mathbb{B}_+(L_\alpha)$. Altogether, we see that α is on the boundary of $\mathrm{Mov}_1(X)$ if and only if L_α is on the boundary of $\mathrm{Mov}^1(X)$. □

Arguing just as in Section 6.5, we obtain most of the other analytic properties of $\mathfrak{M}$.

Theorem 6.6.15. *Let X be a smooth projective variety of dimension n. For any movable curve class α with $\mathfrak{M}(\alpha) > 0$, let L_α denote the unique big and movable divisor class satisfying $\langle L_\alpha^{n-1}\rangle = \alpha$. As we vary α in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$, L_α depends continuously on α.*

Theorem 6.6.16. *Let X be a smooth projective variety of dimension n. For a curve class $\alpha = \langle L_\alpha^{n-1}\rangle$ in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$ and for an arbitrary curve class $\beta \in N_1(X)$ we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathfrak{M}(\alpha + t\beta) = \frac{n}{n-1} P_\sigma(L_\alpha) \cdot \beta.$$

Theorem 6.6.17. *Let X be a smooth projective variety of dimension n. Let α_1, α_2 be two big and movable curve classes in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$. Then*

$$\mathfrak{M}(\alpha_1 + \alpha_2)^{n-1/n} \geq \mathfrak{M}(\alpha_1)^{n-1/n} + \mathfrak{M}(\alpha_2)^{n-1/n}$$

with equality if and only if α_1 and α_2 are proportional.

Another application of the results in this section is the promised Morse-type bigness criterion for movable curve classes, which is slightly different from Theorem 6.5.18.

Theorem 6.6.18. *Let X be a smooth projective variety of dimension n. Let α, β be two curve classes lying in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$. Write $\alpha = \langle L_\alpha^{n-1}\rangle$ and $\beta = \langle L_\beta^{n-1}\rangle$ for the unique big and movable divisor classes L_α, L_β given by Theorem 6.6.12. Then we have*

$$\begin{aligned}\mathfrak{M}(\alpha - \beta)^{n-1/n} &\geq (\mathfrak{M}(\alpha) - nL_\alpha \cdot \beta) \cdot \mathfrak{M}(\alpha)^{-1/n} \\ &= (\mathrm{vol}(L_\alpha) - nL_\alpha \cdot \beta) \cdot \mathrm{vol}(L_\alpha)^{-1/n}.\end{aligned}$$

In particular, we have

$$\mathfrak{M}(\alpha - \beta) \geq \mathrm{vol}(L_\alpha) - \frac{n^2}{n-1} L_\alpha \cdot \beta$$

and the curve class $\alpha - \beta$ is big whenever $\mathfrak{M}(\alpha) - nL_\alpha \cdot \beta > 0$.

Proof. By Theorem 6.4.16 the above inequality follows if we have a Morse-type bigness criterion for the difference of two movable divisor classes. So we need to prove $L - M$ is big whenever

$$\langle L^n\rangle - n\langle L^{n-1}\rangle \cdot M > 0.$$

This is proved (in the Kähler setting) in [Xia14, Theorem 1.1] (see also Chapter 4). □

Remark 6.6.19. We remark that we can not extend this Morse-type criterion from big and movable divisors to arbitrary pseudo-effective divisor classes. A very simple construction provides the counter examples, e.g. the blow up of $\mathbb{P}^2$ (see [Tra95, Example 3.8]).

Combining Theorem 6.6.12 and Theorem 6.6.15, we obtain :

Corollary 6.6.20. *Let X be a smooth projective variety of dimension n. Then*

$$\Phi : \mathrm{Mov}^1(X)_{\mathrm{vol}} \to \mathrm{Mov}_1(X)_{\mathfrak{M}}, \qquad L \mapsto \langle L^{n-1} \rangle$$

is a homeomorphism.

Remark 6.6.21. Corollary 6.6.20 gives a systematic way of translating between "chamber decompositions" on $\mathrm{Mov}_1(X)$ and $\mathrm{Mov}^1(X)$. This relationship could be exploited to elucidate the geometry underlying chamber decompositions.

One potential application is in the study of stability conditions. For example, [Neu10] studies a decomposition of $\mathrm{Mov}_1(X)$ into chambers defining different Harder-Narasimhan filtrations of the tangent bundle of X with respect to movable curves. Let α be a movable curve class. Denote by $\mathrm{HNF}(\alpha, TX)$ the Harder-Narasimhan filtration of the tangent bundle with respect to the class α. Then we have the following "destabilizing chambers" :

$$\Delta_\alpha := \{\beta \in \mathrm{Mov}_1(X) |\, \mathrm{HNF}(\beta, TX) = \mathrm{HNF}(\alpha, TX)\}.$$

By [Neu10, Theorem 3.3.4, Proposition 3.3.5], the destabilizing chambers are pairwise disjoint and provide a decomposition of the movable cone $\mathrm{Mov}_1(X)$. Moreover, the decomposition is locally finite in $\mathrm{Mov}_1(X)^\circ$ and the destabilizing chambers are convex cones whose closures are locally polyhedral in $\mathrm{Mov}_1(X)^\circ$. In particular, if $\mathrm{Mov}_1(X)$ is polyhedral, then the chamber structure is finite.

For Fano threefolds, [Neu10] shows that the destabilizing subsheaves are all relative tangent sheaves of some Mori fibration on X. See also [Keb13] for potential applications of this analysis. It would be interesting to study whether the induced filtrations on TX are related to the geometry of the movable divisors L in the Φ-inverse of the corresponding chamber of $\mathrm{Mov}_1(X)$.

Remark 6.6.22. Modified versions of many of the results in this section hold for singular varieties as well (see Remark 6.6.8). For example, by similar arguments we can see that any element in the interior of $\mathrm{Mov}_1(X)$ is the positive product of some big divisor class regardless of singularities. Conversely, whenever $\mathfrak{M}$ is $+$-differentiable we obtain a Zariski decomposition structure for vol by Theorem 6.4.3.

Remark 6.6.23. All the results above extend to smooth varieties over algebraically closed fields. However, for compact Kähler manifolds some results rely on Demailly's conjecture on the transcendental holomorphic Morse-type inequality, or equivalently, on the extension of the results of [BFJ09] to the Kähler setting. Since the results of [BFJ09] are used in an essential way in the proof of Theorems 6.6.12 and 6.6.2 (via the proof of [FL13, Proposition 5.3]), the only statement in this section which extends unconditionally to the Kähler setting is Lemma 6.6.9.

6.7 Comparing the complete intersection cone and the movable cone

Consider the functions $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ on the movable cone of curves $\mathrm{Mov}_1(X)$. By their definitions we always have $\widehat{\mathrm{vol}} \geq \mathfrak{M}$ on the movable cone, and [Xia15a, Remark 3.1] asks whether one can characterize when equality holds. In this section we show :

Theorem 6.7.1. *Let X be a smooth projective variety of dimension n and let α be a big and movable class. Then $\widehat{\mathrm{vol}}(\alpha) > \mathfrak{M}(\alpha)$ if and only if $\alpha \notin \mathrm{CI}_1(X)$.*

Thus $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ can be used to distinguish whether a big movable curve class lies in $\mathrm{CI}_1(X)$ or not.

Proof. If $\alpha = B^{n-1}$ is a complete intersection class, then $\widehat{\mathrm{vol}}(\alpha) = \mathrm{vol}(B) = \mathfrak{M}(\alpha)$. By continuity the equality holds true for any big curve class in $\mathrm{CI}_1(X)$.

Conversely, suppose that α is not in the complete intersection cone. The claim is clearly true if $\mathfrak{M}(\alpha)=0$, so by Theorem 6.6.12 it suffices to consider the case when there is a big and movable divisor class L such that $\alpha=\langle L^{n-1}\rangle$. Note that L can not be big and nef since $\alpha\notin \mathrm{CI}_1(X)$.

We prove $\widehat{\mathrm{vol}}(\alpha)>\mathfrak{M}(\alpha)$ by contradiction. First, by the definition of $\widehat{\mathrm{vol}}$ we always have

$$\widehat{\mathrm{vol}}(\langle L^{n-1}\rangle)\geq \mathfrak{M}(\langle L^{n-1}\rangle)=\mathrm{vol}(L).$$

Suppose $\widehat{\mathrm{vol}}(\langle L^{n-1}\rangle)=\mathrm{vol}(L)$. For convenience, we assume $\mathrm{vol}(L)=1$. By Theorem 6.5.2.(3), there exists a big and nef divisor class B with $\mathrm{vol}(B)=1$ computing $\widehat{\mathrm{vol}}(\langle L^{n-1}\rangle)$. For the divisor class B we get

$$\langle L^{n-1}\rangle\cdot B=1=\mathrm{vol}(L)^{n-1/n}\,\mathrm{vol}(B)^{1/n}.$$

By Proposition 6.6.3, this implies L and B are proportional which contradicts the non-nefness of L. Thus we must have $\widehat{\mathrm{vol}}(\langle L^{n-1}\rangle)>\mathrm{vol}(L)=\mathfrak{M}(\langle L^{n-1}\rangle)$. □

Remark 6.7.2. Alternatively, suppose that $\alpha=\langle L^{n-1}\rangle$ for a big movable divisor L that is not nef. Note that there is a pseudo-effective curve class β satisfying $L\cdot\beta<0$. As we subtract a small amount of β from α, Theorems 6.5.11 and 6.6.16 show that $\widehat{\mathrm{vol}}$ decreases but $\mathfrak{M}$ increases. Since we always have an inequality $\widehat{\mathrm{vol}}\geq\mathfrak{M}$ for movable classes, we can not have an equality at α.

We also obtain :

Proposition 6.7.3. *Let X be a smooth projective variety of dimension n and let α be a big and movable curve class. Assume that $\widehat{\mathrm{vol}}(\phi^*\alpha)=\widehat{\mathrm{vol}}(\alpha)$ for any birational morphism ϕ. Then $\alpha\in\mathrm{CI}_1(X)$.*

Proof. We first consider the case when $\mathfrak{M}(\alpha)>0$. Let L be a big movable divisor class satisfying $\langle L^{n-1}\rangle=\alpha$. Choose a sequence of birational maps $\phi_\epsilon:Y_\epsilon\to X$ and ample divisor classes A_ϵ on Y_ϵ defining an ϵ-Fujita approximation for L. Then $\mathrm{vol}(L)\geq\mathrm{vol}(A_\epsilon)>\mathrm{vol}(L)-\epsilon$ and the classes $\phi_{\epsilon*}A_\epsilon$ limit to L. Note that $A_\epsilon\cdot\phi_\epsilon^*\alpha=\phi_{\epsilon*}A_\epsilon\cdot\alpha$. This implies that for any $\epsilon>0$ we have

$$\widehat{\mathrm{vol}}(\alpha)=\widehat{\mathrm{vol}}(\phi_\epsilon^*\alpha)\leq\frac{(\alpha\cdot\phi_{\epsilon*}A_\epsilon)^{n/n-1}}{\mathrm{vol}(L)^{1/n-1}}.$$

As ϵ shrinks the right hand side approaches $\mathrm{vol}(L)=\mathfrak{M}(\alpha)$, and we conclude by Theorem 6.7.1.

Next we consider the case when $\mathfrak{M}(\alpha)>0$. Choose a class ξ in the interior of $\mathrm{Mov}_1(X)$ and consider the classes $\alpha+\delta\xi$ for $\delta>0$. The argument above shows that for any $\epsilon>0$, there is a birational model $\phi_\epsilon:Y_\epsilon\to X$ such that

$$\widehat{\mathrm{vol}}(\phi_\epsilon^*(\alpha+\delta\xi))<\mathfrak{M}(\alpha+\delta\xi)+\epsilon.$$

But we also have $\widehat{\mathrm{vol}}(\phi_\epsilon^*\alpha)\leq\widehat{\mathrm{vol}}(\phi_\epsilon^*(\alpha+\delta\xi))$ since the pullback of the nef curve class $\delta\xi$ is pseudo-effective. Taking limits as $\epsilon\to 0$, $\delta\to 0$, we see that we can make the volume of the pullback of α arbitrarily small, a contradiction to the assumption and the bigness of α. □

Let L be a big divisor class and let $\alpha=\langle L^{n-1}\rangle$ be the corresponding big movable curve class. From the Zariski decomposition $\alpha=B^{n-1}+\gamma$, we get a “canonical" map π from a big divisor class to a big and nef divisor class, that is, $\pi(L):=B$. Note that the map π is continuous and satisfies $\pi^2=\pi$. It is natural to ask whether we can compare L and B. However, if $P_\sigma(L)$ is not nef then L and B can not be compared :

– if $L\succeq B$ then we have $\mathrm{vol}(L)\geq\mathrm{vol}(B)$ which contradicts with Theorem 6.7.1 ;

– if $L\preceq B$ then we have $\langle L^{n-1}\rangle\preceq B^{n-1}$ which contradicts with $\gamma\neq 0$.

If we modify the map π a little bit, we can always get a “canonical" nef divisor class lying below the big divisor class.

Theorem 6.7.4. *Let X be a smooth projective variety of dimension n, and let α be a big movable curve class. Let L be a big divisor class such that $\alpha=\langle L^{n-1}\rangle$, and let $\alpha=B^{n-1}+\gamma$ be the Zariski* *decomposition of α. Define the map $\widehat{\pi}$ from the cone of big divisor classes to the cone of big and nef divisor classes as*

$$\widehat{\pi}(L) := \left(1 - \left(1 - \frac{\mathfrak{M}(\alpha)}{\widehat{\mathrm{vol}}(\alpha)}\right)^{1/n}\right) B.$$

Then $\widehat{\pi}$ is a surjective continuous map satisfying $L \succeq \widehat{\pi}(L)$ and $\widehat{\pi}^2 = \widehat{\pi}$.

Proof. It is clear if L is nef then we have $\widehat{\pi}(L) = L$, and this implies $\widehat{\pi}$ is surjective and $\widehat{\pi}^2 = \widehat{\pi}$. By Theorem 6.5.6 and Theorem 6.6.15, we get the continuity of $\widehat{\pi}$. So we only need to verify $L \succeq \widehat{\pi}(L)$. And this follows from the Diskant inequality for big and movable divisor classes.

Let s be the largest real number such that $L \succeq sB$. By the properties of σ-decompositions, s is also the largest real number such that $P_\sigma(L) \succeq sB$. First, observe that $s \leq 1$ since

$$\mathrm{vol}(L) = \mathfrak{M}(\alpha) \leq \widehat{\mathrm{vol}}(\alpha) = \mathrm{vol}(B).$$

Applying the Diskant inequality to $P_\sigma(L)$ and B, we have

$$\begin{aligned}(\langle P_\sigma(L)^{n-1}\rangle \cdot B)^{n/n-1} &- \mathrm{vol}(L)\,\mathrm{vol}(B)^{1/n-1} \\ &\geq ((\langle P_\sigma(L)^{n-1}\rangle \cdot B)^{1/n-1} - s\,\mathrm{vol}(B)^{1/n-1})^n.\end{aligned}$$

Note that $\widehat{\mathrm{vol}}(\alpha) = (\langle P_\sigma(L)^{n-1}\rangle \cdot \frac{B}{\mathrm{vol}(B)^{1/n}})^{n/n-1}$ and $\mathfrak{M}(\alpha) = \mathrm{vol}(L)$. The above inequality implies

$$s \geq 1 - \left(1 - \frac{\mathfrak{M}(\alpha)}{\widehat{\mathrm{vol}}(\alpha)}\right)^{1/n},$$

which yields the desired relation $L \succeq \widehat{\pi}(L)$. □

Example 6.7.5. Let X be a Mori Dream Space. Recall that a small $\mathbb{Q}$-factorial modification (henceforth SQM) $\phi : X \dashrightarrow X'$ is a birational contraction (i.e. does not extract any divisors) defined in codimension 1 such that X' is projective $\mathbb{Q}$-factorial. [HK00] shows that for any SQM the strict transform defines an isomorphism $\phi_* : N^1(X) \to N^1(X')$ which preserves the pseudo-effective and movable cones of divisors. (More generally, any birational contraction induces an injective pullback $\phi^* : N^1(X') \to N^1(X)$ and dually a surjection $\phi_* : N_1(X) \to N_1(X')$.) By [HK00], the SQM structure induces a chamber decomposition of the pseudo-effective and movable cones of divisors.

One would like to see a "dual picture" in $N_1(X)$ of this chamber decomposition. However, it does not seem interesting to simply dualize the divisor decomposition : the resulting cones are no longer pseudo-effective and are described as intersections instead of unions. Motivated by the Zariski decomposition for curves, we define a chamber structure on the movable cone of curves as a union of the complete intersection cones on SQMs.

Note that for each SQM we obtain by duality an isomorphism $\phi_* : N_1(X) \to N_1(X')$ which preserves the movable cone of curves. The results of [HK00] imply that the strict transforms of the various complete intersection cones define a chamber structure on $\mathrm{Mov}_1(X)$. More precisely, given any birational contraction $\phi : X \dashrightarrow X'$ with X' normal projective, define

$$\mathrm{CI}_\phi^\circ := \bigcup_{A \text{ ample on } X'} \langle \phi^* A^{n-1}\rangle.$$

Then

- $\mathrm{Mov}_1(X)$ is the union over all SQMs $\phi : X \dashrightarrow X'$ of $\overline{\mathrm{CI}_\phi^\circ} = \phi_*^{-1}\,\mathrm{CI}_1(X')$, and the interiors of the $\overline{\mathrm{CI}_\phi^\circ}$ are disjoint.
- The set of classes in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$ is the disjoint union over all birational contractions $\phi : X \dashrightarrow X'$ of the CI_ϕ°.

To see this, first recall that for a pseudo-effective divisor L the σ-decomposition of L and the volume are preserved by ϕ_*. We know that each $\alpha \in \mathrm{Mov}_1(X)_{\mathfrak{M}}$ has the form $\langle L^{n-1} \rangle$ for a unique big and movable divisor L. If $\phi : X \dashrightarrow X'$ denotes the birational canonical model obtained by running the L-MMP, and A denotes the corresponding ample divisor on X', then $\phi_* \alpha = A^{n-1}$ and $\alpha = \langle \phi^* A^{n-1} \rangle$. The various claims now can be deduced from the properties of divisors and the MMP for Mori Dream Spaces as in [HK00, 1.11 Proposition].

Since the volume of divisors behaves compatibly with strict transforms of pseudo-effective divisors, the description of ϕ_* above shows that $\mathfrak{M}$ also behaves compatibly with strict transforms of movable curves under an SQM. However, the volume function can change : we may well have $\widehat{\mathrm{vol}}(\phi_* \alpha) \neq \widehat{\mathrm{vol}}(\alpha)$. The reason is that the pseudo-effective cone of curves is also changing as we vary ϕ. In particular, the set

$$C_{\alpha,\phi} := \{\phi_* \alpha - \gamma | \gamma \in \overline{\mathrm{Eff}}_1(X')\}$$

will look different as we vary ϕ. By the Zariski decomposition of curves, $\widehat{\mathrm{vol}}(\alpha)$ is the same as the maximum value of $\mathfrak{M}(\beta)$ for movable $\beta \in C_{\alpha,\phi}$, the volume and Zariski decomposition for a given model will depend on the exact shape of $C_{\alpha,\phi}$.

Remark 6.7.6. Theorem 6.7.1 and Theorem 6.7.4 also hold for smooth varieties over any algebraically closed field. However, since they rely on the results of Section 6.6 we do not know if they hold in the Kähler setting.

6.8 Toric varieties

In this section X will denote a simplicial projective toric variety of dimension n. In terms of notation, X will be defined by a fan Σ in a lattice N with dual lattice M. We let $\{v_i\}$ denote the primitive generators of the rays of Σ and $\{D_i\}$ denote the corresponding classes of T-divisors. Our goal is to interpret the properties of the functions $\widehat{\mathrm{vol}}$ and $\mathfrak{M}$ in terms of toric geometry.

6.8.1 Positive product on toric varieties

Suppose that L is a big movable divisor class on the toric variety X. Then L naturally defines a (non-lattice) polytope Q_L : if we choose an expression $L = \sum a_i D_i$, then

$$Q_L = \{u \in M_{\mathbb{R}} | \langle u, v_i \rangle + a_i \geq 0\}$$

and changing the choice of representative corresponds to a translation of Q_L. Conversely, suppose that Q is a full-dimensional polytope such that the unit normals to the facets of Q form a subset of the rays of Σ. Then Q uniquely determines a big movable divisor class L_Q on X. The divisors in the interior of the movable cone correspond to those polytopes whose facet normals coincide with the rays of Σ.

Given polytopes $Q_1, \ldots, Q_n$, let $V(Q_1, \ldots, Q_n)$ denote the mixed volume of the polytopes. [BFJ09] explains that the positive product of big movable divisors $L_1, \ldots, L_n$ can be interpreted via the mixed volume of the corresponding polytopes :

$$\langle L_1 \cdot \ldots \cdot L_n \rangle = n! V(Q_1, \ldots, Q_n).$$

6.8.2 The function $\mathfrak{M}$

In this section we use a theorem of Minkowski to describe the function $\mathfrak{M}$. We thank J. Huh for a conversation working out this picture.

Recall that a class $\alpha \in \mathrm{Mov}_1(X)$ defines a non-negative Minkowski weight on the rays of the fan Σ – that is, an assignment of a positive real number t_i to each vector v_i such that $\sum t_i v_i = 0$. From now on we will identify α with its Minkowski weight. We will need to identify which movable curve classes are positive along a set of rays which span $\mathbb{R}^n$.

Lemma 6.8.1. *Suppose $\alpha \in \mathrm{Mov}_1(X)$ satisfies $\mathfrak{M}(\alpha) > 0$. Then α is positive along a spanning set of rays of Σ.*

We will soon see that the converse is also true in Theorem 6.8.2.

Proof. Suppose that there is a hyperplane V which contains every ray of Σ along which α is positive. Since X is projective, Σ has rays on both sides of V. Let D be the effective toric divisor consisting of the sum over all the primitive generators of rays of Σ not contained in V. It is clear that the polytope defined by D has non-zero projection onto the subspace spanned by $V^{\perp}$, and in particular, that the polytope defined by D is non-zero. Thus $P_\sigma(D) \neq 0$ and so α has vanishing intersection against a non-zero movable divisor. Lemma 6.6.9 shows that $\mathfrak{M}(\alpha) = 0$. □

Minkowski's theorem asserts the following. Suppose that $u_1, \ldots, u_s$ are unit vectors which span $\mathbb{R}^n$ and that $r_1, \ldots, r_s$ are positive real numbers. Then there exists a polytope P with unit normals $u_1, \ldots, u_s$ and with corresponding facet volumes $r_1, \ldots, r_s$ if and only if the u_i satisfy the balanced condition

$$r_1 u_1 + \ldots + r_s u_s = 0.$$

Moreover, the resulting polytope is unique up to translation. (See [Kla04] for a proof which is compatible with the results below.) If a vector u is a unit normal to a facet of P, we will use the notation $\mathrm{vol}(P^u)$ to denote the volume of the facet corresponding to u.

If α is positive on a spanning set of rays, then it canonically defines a polytope (up to translation) via Minkowski's theorem by choosing the vectors u_i to be the unit vectors in the directions v_i and assigning to each the constant

$$r_i = \frac{t_i |v_i|}{(n-1)!}.$$

Note that this is the natural choice of volume for the corresponding facet, as it accounts for :

– the discrepancy in length between u_i and v_i, and

– the factor $\frac{1}{(n-1)!}$ relating the volume of an $(n-1)$-simplex to the determinant of its edge vectors.

We denote the corresponding polytope in $M_{\mathbb{R}}$ defined by the theorem of Minkowski by P_α.

Theorem 6.8.2. *Suppose α is a movable curve class which is positive on a spanning set of rays and let P_α be the corresponding polytope. Then*

$$\mathfrak{M}(\alpha) = n!\,\mathrm{vol}(P_\alpha).$$

Furthermore, the big movable divisor L_α corresponding to the polytope P_α satisfies $\langle L_\alpha^{n-1} \rangle = \alpha$.

Proof. Let $L \in \mathrm{Mov}^1(X)$ be a big movable divisor class and denote the corresponding polytope by Q_L. We claim that the intersection number can be interpreted as a mixed volume :

$$L \cdot \alpha = n! V(P_\alpha^{n-1}, Q_L).$$

To see this, define for a compact convex set K the function $h_K(u) = \sup_{v \in K}\{v \cdot u\}$. Using [Kla04, Equation (5)]

$$\begin{aligned} V(P_\alpha^{n-1}, Q_L) &= \frac{1}{n} \sum_{u \text{ a facet of } P_\alpha + Q_L} h_{Q_L}(u)\,\mathrm{vol}(P_\alpha^u) \\ &= \frac{1}{n} \sum_{\text{rays } v_i} \left(\frac{a_i}{|v_i|}\right)\left(\frac{t_i |v_i|}{(n-1)!}\right) \\ &= \frac{1}{n!} \sum_{\text{rays } v_i} a_i t_i = \frac{1}{n!} L \cdot \alpha. \end{aligned}$$

Note that we actually have equality in the second line because L is big and movable. Recall that by the Brunn-Minkowski inequality

$$V(P_\alpha^{n-1}, Q_L) \geq \mathrm{vol}(P_\alpha)^{n-1/n}\,\mathrm{vol}(Q_L)^{1/n}$$

with equality only when P_α and Q_L are homothetic. Thus

$$\begin{aligned}\mathfrak{M}(\alpha) &= \inf_{L \text{ big movable class}} \left(\frac{L\cdot\alpha}{\mathrm{vol}(L)^{1/n}}\right)^{n/n-1} \\ &= \inf_{L \text{ big movable class}} \left(\frac{n!V(P_\alpha^{n-1}, Q_L)}{n!^{1/n}\,\mathrm{vol}(Q_L)^{1/n}}\right)^{n/n-1} \\ &\geq n!\,\mathrm{vol}(P_\alpha).\end{aligned}$$

Furthermore, the equality is achieved for divisors L whose polytope is homothetic to P_α, showing the computation of $\mathfrak{M}(\alpha)$. Furthermore, since the divisor L_α defined by the polytope computes $\mathfrak{M}(\alpha)$ we see that $\langle L_\alpha^{n-1}\rangle$ is proportional to α. By computing $\mathfrak{M}$ we deduce the equality :

$$\mathfrak{M}(\langle L_\alpha^{n-1}\rangle) = \mathrm{vol}(L) = n!\,\mathrm{vol}(P_\alpha) = \mathfrak{M}(\alpha).$$

□

6.8.3 Zariski decompositions

The work of the previous section shows :

Corollary 6.8.3. *Let α be a curve class in $\mathrm{Mov}_1(X)_{\mathfrak{M}}$. Then $\alpha \in \mathrm{CI}_1(X)$ if and only if the normal fan to the corresponding polytope P_α is refined by Σ. In this case we have*

$$\widehat{\mathrm{vol}}(\alpha) = n!\,\mathrm{vol}(P_\alpha).$$

Proof. By the uniqueness in Theorem 6.6.12, $\alpha \in \mathrm{CI}_1(X)$ if and only if the corresponding divisor L_α as in Theorem 6.8.2 is big and nef. □

The nef cone of divisors and pseudo-effective cone of curves on X can be computed algorithmically. Thus, for any face F of the nef cone, by considering the $(n-1)$-product and adding on any curve classes in the dual face, one can easily divide $\overline{\mathrm{Eff}}_1(X)$ into regions where the positive product is determined by a class on F. In practice this is a good way to compute the Zariski decomposition (and hence the volume) of curve classes on X.

In the other direction, suppose we start with a big curve class α. On a toric variety, every big and nef divisor is semi-ample (that is, the pullback of an ample divisor on a toric birational model). Thus, the Zariski decomposition is characterized by the existence of a birational toric morphism $\pi : X \to X'$ such that :

- the class $\pi_*\alpha \in N_1(X')$ coincides with A^{n-1} for some ample divisor A, and
- $\alpha - (\pi^*A)^{n-1}$ is pseudo-effective.

Thus one can compute the Zariski decomposition and volume for α by the following procedure.

1. For each toric birational morphism $\pi : X \to X'$, check whether $\pi_*\alpha$ is in the complete intersection cone. If so, there is a unique big and nef divisor $A_{X'}$ such that $A_{X'}^{n-1} = \pi_*\alpha$.
2. Check if $\alpha - (\pi^*A_{X'})^{n-1}$ is pseudo-effective.

The first step involves solving polynomial equations to deduce the equality of coefficients of numerical classes, but otherwise this procedure is completely algorithmic. Thus this procedure can be viewed as a solution to our isoperimetric problem. (Note that there may be no natural pullback from $\overline{\mathrm{Eff}}_1(X')$ to $\overline{\mathrm{Eff}}_1(X)$, and in particular, the calculation of $(\pi^*A_{X'})^{n-1}$ is not linear in $A_{X'}^{n-1}$.)

Example 6.8.4. Let X be the toric variety defined by a fan in $N = \mathbb{Z}^3$ on the rays

$$v_1 = (1,0,0) \qquad v_2 = (0,1,0) \qquad v_3 = (1,1,1)$$
$$v_4 = (-1,0,0) \qquad v_5 = (0,-1,0) \qquad v_6 = (0,0,-1)$$

with maximal cones

$$\langle v_1, v_2, v_3\rangle,\ \langle v_1, v_2, v_6\rangle,\ \langle v_1, v_3, v_5\rangle,\ \langle v_1, v_5, v_6\rangle,$$
$$\langle v_2, v_3, v_4\rangle,\ \langle v_2, v_4, v_6\rangle,\ \langle v_3, v_4, v_5\rangle,\ \langle v_4, v_5, v_6\rangle.$$

The Picard rank of X is 3. Letting D_i and C_{ij} be the divisors and curves corresponding to v_i and $\overline{v_i v_j}$ respectively, we have intersection product

	D_1	D_2	D_3
C_{12}	-1	-1	1
C_{13}	0	1	0
C_{23}	1	0	0

Standard toric computations show that :

$$\overline{\mathrm{Eff}}^1(X) = \langle D_1, D_2, D_3\rangle \qquad \mathrm{Nef}^1(X) = \langle D_1 + D_3, D_2 + D_3, D_3\rangle$$
$$\mathrm{Mov}^1(X) = \langle D_1 + D_2, D_1 + D_3, D_2 + D_3, D_3\rangle$$

and

$$\overline{\mathrm{Eff}}_1(X) = \langle C_{12}, C_{13}, C_{23}\rangle \qquad \mathrm{Nef}_1(X) = \langle C_{12} + C_{13} + C_{23}, C_{13}, C_{23}\rangle.$$

X admits a unique flip and has only one birational contraction corresponding to the face of $\mathrm{Nef}^1(X)$ generated by D_1+D_3 and D_2+D_3. Set $B_{a,b} = aD_1 + bD_2 + (a+b)D_3$. The complete intersection cone is given by taking the convex hull of the boundary classes

$$B_{a,b}^2 = T_{a,b} = 2abC_{12} + (a^2 + 2ab)C_{13} + (b^2 + 2ab)C_{23}$$

and the face of $\mathrm{Nef}_1(X)$ spanned by C_{13}, C_{23}.

For any big class α not in $\mathrm{CI}_1(X)$, the positive part can be computed on the unique toric birational contraction $\pi : X \to X'$ given by contracting C_{12}. In practice, the procedure above amounts to solving $\alpha - tC_{12} = T_{a,b}$ for some a, b, t. If $\alpha = xC_{12} + yC_{13} + zC_{23}$, this yields the quadratic equation $4(y - x + t)(z - x + t) = (x - t)^2$. Solving this for t tells us $\gamma = tC_{12}$, and the volume can then easily be computed.

6.9 Hyperkähler manifolds

Throughout this section X will denote a hyperkähler variety of dimension n (with $n = 2m$). We will continue to work in the projective setting. However, as explained in Section 6.2.4, Demailly's conjecture on transcendental Morse inequality is known for hyperkähler manifolds. Thus all of the material in the previous sections will hold in the Kähler setting for hyperkähler varieties with no qualifications, and all the results in this section can extended accordingly.

Let σ be a symplectic holomorphic form on X. For a real divisor class $D \in N^1(X)$ the Beauville-Bogomolov quadratic form is defined as

$$q(D) = D^2 \cdot \{(\sigma \wedge \bar{\sigma})\}^{n/2-1},$$

where we normalize the symplectic form σ such that

$$q(D)^{n/2} = D^n.$$

As proved in [Bou04, Section 4], the bilinear form q is compatible with the volume function and σ-decomposition for divisors in the following way :

1. The cone of movable divisors is q-dual to the pseudo-effective cone.
2. If D is a movable divisor then $\mathrm{vol}(D) = q(D, D)^{n/2} = D^n$.
3. For a pseudo-effective divisor D write $D = P_\sigma(D) + N_\sigma(D)$ for its σ-decomposition. Then $q(P_\sigma(D), N_\sigma(D)) = 0$, and if $N_\sigma(D) \neq 0$ then $q(N_\sigma(D), N_\sigma(D)) < 0$.

The bilinear form q induces an isomorphism $\psi : N^1(X) \to N_1(X)$ by sending a divisor class D to the curve class defining the linear function $q(D, -)$. We obtain an induced bilinear form q on $N_1(X)$ via the isomorphism ψ, so that for curve classes α, β

$$q(\alpha, \beta) = q(\psi^{-1}\alpha, \psi^{-1}\beta) = \psi^{-1}\alpha \cdot \beta.$$

In particular, two cones $\mathcal{C}, \mathcal{C}'$ in $N^1(X)$ are q-dual if and only if $\psi(\mathcal{C})$ is dual to $\mathcal{C}'$ under the intersection pairing (and similarly for cones of curves). In this section we verify that the bilinear form q on $N_1(X)$ is compatible with the volume and Zariski decomposition for curve classes in the same way as for divisors.

Remark 6.9.1. Since the signature of the Beauville-Bogomolov form is $(1, \dim N^1(X) - 1)$, one can use the Hodge inequality to analyze the Zariski decomposition as in Example 6.4.7. We will instead give a direct geometric argument to emphasize the ties with the divisor theory.

We first need the following proposition.

Proposition 6.9.2. *Let D be a big movable divisor class on X. Then $\mathfrak{M}(\psi(D)) = \mathrm{vol}(D)^{1/n-1}$ and*

$$\psi(D) = \frac{\langle D^{n-1}\rangle}{\mathrm{vol}(D)^{n-2/n}}.$$

In particular, the complete intersection cone coincides with the ψ-image of the nef cone of divisors and if A is a big and nef divisor then $\widehat{\mathrm{vol}}(\psi(A)) = \mathrm{vol}(A)^{1/n-1}$.

Proof. First note that $\psi(D)$ is contained in $\mathrm{Mov}_1(X)$. Indeed, since the movable cone of divisors is q-dual to the pseudo-effective cone of divisors by [Bou04, Proposition 4.4], the ψ-image of the movable cone of divisors is dual to the pseudo-effective cone of divisors.

For any big movable divisor L, the basic equality for bilinear forms shows that

$$L \cdot \psi(D) = q(L, D) = \frac{1}{2}(\mathrm{vol}(L + D)^{2/n} - \mathrm{vol}(L)^{2/n} - \mathrm{vol}(D)^{2/n}).$$

Proposition 6.6.7 shows that $\mathrm{vol}(L + D)^{1/n} \geq \mathrm{vol}(L)^{1/n} + \mathrm{vol}(D)^{1/n}$ with equality if and only if L and D are proportional. Squaring and rearranging, we see that

$$\frac{L \cdot \psi(D)}{\mathrm{vol}(L)^{1/n}} \geq \mathrm{vol}(D)^{1/n}$$

with equality if and only if L is proportional to D. Thus $\mathfrak{M}(\psi(D)) = \mathrm{vol}(D)^{1/n-1}$ and this quantity is computed by the movable and big divisor D. This implies that

$$\psi(D) = \frac{\langle D^{n-1}\rangle}{\mathrm{vol}(D)^{n-2/n}}$$

by Theorem 6.6.12. The final statements follow immediately. □

Theorem 6.9.3. *Let q denote the Beauville-Bogomolov form on $N_1(X)$. Then :*

1. *The complete intersection cone of curves is q-dual to the pseudo-effective cone of curves.*
2. *If α is a complete intersection curve class then $\widehat{\mathrm{vol}}(\alpha) = q(\alpha, \alpha)^{n/2(n-1)}$.*
3. *For a big class α write $\alpha = B^{n-1} + \gamma$ for its Zariski decomposition. Then $q(B^{n-1}, \gamma) = 0$ and if γ is non-zero then $q(\gamma, \gamma) < 0$.*

Proof. For (1), since the complete intersection cone coincides with $\psi(\mathrm{Nef}^1(X))$ it is q-dual to the dual cone of $\mathrm{Nef}^1(X)$. For (2), by Proposition 6.9.2 we have

$$\begin{aligned} q(\psi(A),\psi(A)) = q(A,A) &= \mathrm{vol}(A)^{2/n} \\ &= \widehat{\mathrm{vol}}(\psi(A))^{2(n-1)/n}. \end{aligned}$$

For (3), we have

$$q(B^{n-1},\gamma) = \psi^{-1}(B^{n-1})\cdot\gamma = \mathrm{vol}(B)^{n-2/n}B\cdot\gamma = 0.$$

For the final statement $q(\gamma,\gamma)<0$, note that

$$q(\alpha,\alpha) = q(B^{n-1},B^{n-1}) + q(\gamma,\gamma)$$

so it suffices to show that $q(\alpha,\alpha) < q(B^{n-1},B^{n-1})$. Set $D = \psi^{-1}\alpha$. The desired inequality is clear if $q(D,D)\leq 0$, so by [Huy99, Corollary 3.10 and Erratum Proposition 1] it suffices to restrict our attention to the case when D is big. (Note that the case when $-D$ is big can not occur, since $q(D,A) = A\cdot\alpha > 0$ for an ample divisor class A.) Let $D = P_\sigma(D) + N_\sigma(D)$ be the σ-decomposition of D. By [Bou04, Proposition 4.2] we have $q(N_\sigma(D),B)\geq 0$. Thus

$$\begin{aligned} \mathrm{vol}(B)^{2(n-1)/n} = q(B^{n-1},B^{n-1}) &= q(\alpha,B^{n-1}) \\ &= \mathrm{vol}(B)^{n-2/n}q(D,B) \geq \mathrm{vol}(B)^{n-2/n}q(P_\sigma(D),B). \end{aligned}$$

Arguing just as in the proof of Proposition 6.9.2, we see that

$$q(P_\sigma(D),B) \geq \mathrm{vol}(P_\sigma(D))^{1/n}\,\mathrm{vol}(B)^{1/n}$$

with equality if and only if $P_\sigma(D)$ and B are proportional. Combining the two previous equations we obtain

$$\mathrm{vol}(B)^{n-1/n} \geq \mathrm{vol}(P_\sigma(D))^{1/n}.$$

and equality is only possible if B and $P_\sigma(D)$ are proportional. Then we calculate :

$$\begin{aligned} q(\alpha,\alpha) &= q(D,D) \\ &\leq q(P_\sigma(D),P_\sigma(D)) \text{ by [Bou04, Theorem 4.5]} \\ &= \mathrm{vol}(P_\sigma(D))^{2/n} \\ &\leq \mathrm{vol}(B)^{2(n-1)/n} = q(B,B). \end{aligned}$$

If $P_\sigma(D)$ and B are not proportional, we obtain a strict inequality at the last step. If $P_\sigma(D)$ and B are proportional, then $N_\sigma(D)>0$ (since otherwise $D=B$ and α is a complete intersection class). Then by [Bou04, Theorem 4.5] we have a strict inequality $q(P_\sigma(D),P_\sigma(D)) > q(D,D)$ on the second line. In either case we conclude $q(\alpha,\alpha) < q(B,B)$ as desired.

□

Remark 6.9.4. Suppose that α is a nef curve class that is not in the complete intersection cone. Then $q(\alpha,\alpha) = \mathfrak{M}(\alpha)^{2(n-1)/n}$ and $q(B^{n-1},B^{n-1}) = \widehat{\mathrm{vol}}(\alpha)^{2(n-1)/n}$. Theorem 6.7.1 shows that $q(\alpha,\alpha) < q(B^{n-1},B^{n-1})$, giving another proof of the final statement of Theorem 6.9.3.(3) in this special case.

6.10 Comparison with mobility

In this section we compare the volume function with the mobility function. Recall from the introduction that we are trying to show :

Theorem 6.10.1. *Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then :*

1. $\widehat{\mathrm{vol}}(\alpha) \leq \mathrm{mob}(\alpha) \leq n!\widehat{\mathrm{vol}}(\alpha)$.

2. Assume Conjecture 6.1.12, then $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$.

The upper bound improves the related result in the previous chapter (see also [Xia15a, Theorem 3.2]).

Proof. (1) We first prove the upper bound. By continuity and homogeneity it suffices to prove the upper bound for a class α in the natural sublattice of integral classes $N_1(X)_{\mathbb{Z}}$. Suppose that $p : U \to W$ is a family of curves representing $m\alpha$ of maximal mobility count for a positive integer m. Suppose that a general member of p decomposes into irreducible components $\{C_i\}$; arguing as in [Leh13b, Corollary 4.10], we must have $\mathrm{mc}(p) = \sum_i \mathrm{mc}(U_i)$, where U_i represents the closure of the family of deformations of C_i. We also let β_i denote the numerical class of C_i.

Suppose that $\mathrm{mc}(U_i) > 1$. Then we may apply Proposition 6.11.1 with all $k_i = 1$ and $r = \mathrm{mc}(U_i)-1$ to deduce that

$$\widehat{\mathrm{vol}}(\beta_i) \geq \mathrm{mc}(U_i) - 1.$$

If $\mathrm{mc}(U_i) \leq 1$ then Proposition 6.11.1 does not apply but at least we still know that $\widehat{\mathrm{vol}}(\beta_i) \geq 0 \geq \mathrm{mc}(U_i) - 1$. Fix an ample Cartier divisor A, and note that the number of components C_i is at most $mA \cdot \alpha$. All told, we have

$$\begin{aligned}\widehat{\mathrm{vol}}(m\alpha) &\geq \sum_i \widehat{\mathrm{vol}}(\beta_i)\\ &\geq \sum_i (\mathrm{mc}(U_i) - 1)\\ &\geq \mathrm{mc}(m\alpha) - mA \cdot \alpha.\end{aligned}$$

Thus,

$$\begin{aligned}\widehat{\mathrm{vol}}(\alpha) &= \limsup_{m\to\infty} \frac{\widehat{\mathrm{vol}}(m\alpha)}{m^{n/n-1}}\\ &\geq \limsup_{m\to\infty} \frac{\mathrm{mc}(m\alpha) - mA\cdot\alpha}{m^{n/n-1}} = \frac{\mathrm{mob}(\alpha)}{n!}.\end{aligned}$$

The lower bound relies on the Zariski decomposition of curves in Theorem 6.5.4. By [Leh13b, Theorem 6.11 and Example 6.2] we have

$$B^n \leq \mathrm{mob}(B^{n-1})$$

for any nef divisor B. With Theorem 6.5.2, this implies

$$\widehat{\mathrm{vol}}(B^{n-1}) \leq \mathrm{mob}(B^{n-1}).$$

In general, for a big curve class α we have

$$\begin{aligned}\mathrm{mob}(\alpha) &\geq \sup_{B \text{ nef},\ \alpha \succeq B^{n-1}} \mathrm{mob}(B^{n-1})\\ &\geq \sup_{B \text{ nef},\ \alpha \succeq B^{n-1}} B^n\\ &= \widehat{\mathrm{vol}}(\alpha).\end{aligned}$$

where the last equality follows from Theorem 6.5.4. This finishes the proof of the first statement.

(2) To prove the second half of Theorem 6.10.1, we need a result of [FL13] :

Lemma 6.10.2 (see [FL13] Corollary 6.16)**.** *Let X be a smooth projective variety of dimension n and let α be a big curve class. Then there is a big movable curve class β satisfying $\beta \preceq \alpha$ such that*

$$\mathrm{mob}(\alpha) = \mathrm{mob}(\beta) = \mathrm{mob}(\phi^*\beta)$$

for any birational map $\phi : Y \to X$ from a smooth variety Y.

We now prove the statement via a sequence of claims.

Claim: Assume Conjecture 6.1.12. If β is a movable curve class with $\mathfrak{M}(\beta) > 0$, then for any $\epsilon > 0$ there is a birational map $\phi_\epsilon : Y_\epsilon \to X$ such that

$$\mathfrak{M}(\beta) - \epsilon \leq \mathrm{mob}(\phi_\epsilon^* \beta) \leq \mathfrak{M}(\beta) + \epsilon.$$

□

By Theorem 6.6.12, we may suppose that there is a big divisor L such that $\beta = \langle L^{n-1} \rangle$. Without loss of generality we may assume that L is effective. Fix an ample effective divisor G as in [FL13, Proposition 6.24] ; the proposition shows that for any sufficiently small ϵ there is a birational morphism $\phi_\epsilon : Y_\epsilon \to X$ and a big and nef divisor A_ϵ on Y_ϵ satisfying

$$A_\epsilon \leq P_\sigma(\phi_\epsilon^* L) \leq A_\epsilon + \epsilon \phi_\epsilon^* G.$$

Note that $\mathrm{vol}(A_\epsilon) \leq \mathrm{vol}(L) \leq \mathrm{vol}(A_\epsilon + \epsilon \phi_\epsilon^* G)$. Furthermore, we have

$$\mathrm{vol}(A_\epsilon + \epsilon \phi_\epsilon^* G) \leq \mathrm{vol}(\phi_{\epsilon *} A_\epsilon + \epsilon G) \leq \mathrm{vol}(L + \epsilon G).$$

Applying [FL13, Lemma 6.21] and the invariance of the positive product under passing to positive parts, we have

$$A_\epsilon^{n-1} \preceq \phi_\epsilon^* \beta \preceq (A_\epsilon + \epsilon \phi_\epsilon^* G)^{n-1}.$$

Applying Conjecture 6.1.12 (which is only stated for ample divisors but applies to big and nef divisors by continuity of mob), we find

$$\mathrm{vol}(A_\epsilon) = \mathrm{mob}(A_\epsilon^{n-1}) \leq \mathrm{mob}(\phi_\epsilon^* \beta) \leq \mathrm{mob}((A_\epsilon + \epsilon \phi_\epsilon^*(G))^{n-1}) = \mathrm{vol}(A_\epsilon + \epsilon \phi_\epsilon^* G).$$

As ϵ shrinks the two outer terms approach $\mathrm{vol}(L) = \mathfrak{M}(\beta)$.

Claim: Assume Conjecture 6.1.12. If a big movable curve class β satisfies $\mathrm{mob}(\beta) = \mathrm{mob}(\phi^* \beta)$ for every birational ϕ then we must have $\beta \in \mathrm{CI}_1(X)$. □

When $\mathfrak{M}(\beta) > 0$, by the previous claim we see from taking a limit that $\mathrm{mob}(\beta) = \mathfrak{M}(\beta)$. By Theorem 6.10.1.(1) and Theorem 6.7.1 we get

$$\widehat{\mathrm{vol}}(\beta) \leq \mathfrak{M}(\beta) \leq \widehat{\mathrm{vol}}(\beta)$$

and Theorem 6.7.1 implies the result. When $\mathfrak{M}(\beta) = 0$, fix a class ξ in the interior of the movable cone and consider $\beta + \delta\xi$ for $\delta > 0$. By the previous claim, for any $\epsilon > 0$ we can find a sufficiently small δ and a birational map $\phi_\epsilon : Y_\epsilon \to X$ such that $\mathrm{mob}(\phi_\epsilon^*(\beta + \delta\xi)) < \epsilon$. We also have $\mathrm{mob}(\phi_\epsilon^* \beta) \leq \mathrm{mob}(\phi_\epsilon^*(\beta + \delta\xi))$ since the pullback of the nef curve class $\delta\xi$ is pseudo-effective. By the assumption on the birational invariance of $\mathrm{mob}(\beta)$, we can take a limit to obtain $\mathrm{mob}(\beta) = 0$, a contradiction to the bigness of β.

To finish the proof, recall that Lemma 6.10.2 implies that the mobility of α must coincide with the mobility of a movable class β lying below α and satisfying $\mathrm{mob}(\pi^* \beta) = \mathrm{mob}(\beta)$ for any birational map π. Thus we have shown

$$\mathrm{mob}(\alpha) = \sup_{B \text{ nef},\ \alpha \succeq B^{n-1}} \mathrm{mob}(B^{n-1}).$$

By Conjecture 6.1.12 again, we obtain

$$\mathrm{mob}(\alpha) = \sup_{B \text{ nef},\ \alpha \succeq B^{n-1}} B^n.$$

But the right hand side agrees with $\widehat{\mathrm{vol}}(\alpha)$ by Theorem 6.5.4. This proves the equality $\mathrm{mob}(\alpha) = \widehat{\mathrm{vol}}(\alpha)$ under the Conjecture 6.1.12. □

Theorem 6.10.1 yields two interesting consequences :

– The theorem indicates (loosely speaking) that if the mobility count of complete intersection classes is optimized by complete intersection curves, then the mobility count of *any* curve class is optimized by complete intersection curves lying below the class. This result is very surprising : it indicates that the "positivity" of a curve class is coming from ample divisors in a strong sense.
– The theorem suggests that the Zariski decomposition constructed in [FL13] for curves is not optimal : instead of defining a positive part in the movable cone, if Conjecture 6.1.12 is true we should instead define a positive part in the complete intersection cone. It would be interesting to see an analogous improvement for higher dimension cycles.

Remark 6.10.3. We expect Theorem 6.10.1 to also hold over any algebraically closed field, but we have not thoroughly checked the results on asymptotic multiplier ideals used in the proof of [FL13, Proposition 6.24].

6.10.1 Weighted mobility

The weighted mobility of a class α is defined similarly to the mobility, but it gives a higher "weight" to singular points. This better reflects the intersection theory on the blow-up of the points and indicates the close connection between the weighted mobility and Seshadri constants. We first define the weighted mobility count of a class $\alpha \in N_1(X)_{\mathbb{Z}}$ (see [Leh13b, Definition 8.7]) :

$$\mathrm{wmc}(\alpha) = \sup_{\mu} \max \left\{ b \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{c} \text{there is an effective cycle of class } \mu\alpha \\ \text{through any } b \text{ points of } X \text{ with} \\ \text{multiplicity at least } \mu \text{ at each point} \end{array} \right\}.$$

The supremum is shown to exist in [Leh13b] – it is then clear that the supremum is achieved by some positive integer μ. We define the weighted mobility to be

$$\mathrm{wmob}(\alpha) = \limsup_{m\to\infty} \frac{\mathrm{wmc}(m\alpha)}{m^{\frac{n}{n-k}}}.$$

Note that we no longer need the correction factor of $n!$. [Leh13b] shows that the weighted mobility is continuous and homogeneous on $\overline{\mathrm{Eff}}_1(X)$ and is 0 precisely along the boundary.

Theorem 6.10.4. *Let X be a smooth projective variety of dimension n and let $\alpha \in \overline{\mathrm{Eff}}_1(X)$ be a pseudo-effective curve class. Then $\widehat{\mathrm{vol}}(\alpha) = \mathrm{wmob}(\alpha)$.*

The key advantage is that the analogue of Conjecture 6.1.12 is known for the weighted mobility : [Leh13b, Example 8.22] shows that for any big and nef divisor B we have $\mathrm{wmob}(B^{n-1}) = B^n$.

Proof. We first prove the inequality $\geq$. The argument is essentially identical to the upper bound in Theorem 6.10.1.(1) : by continuity and homogeneity it suffices to prove it for classes in $N_1(X)_{\mathbb{Z}}$. Choose a positive integer μ and a family of class $\mu m\alpha$ achieving $\mathrm{wmc}(m\alpha)$. By splitting up into components and applying Proposition 6.11.1 with equal weight μ at every point we see that for any component U_i with class β_i we have

$$\widehat{\mathrm{vol}}(\beta_i) \geq \mu^{n/n-1}(\mathrm{wmc}(U_i) - 1)$$

Arguing as in Theorem 6.10.1.(1), we see that for any fixed ample Cartier divisor A we have

$$\widehat{\mathrm{vol}}(m\mu\alpha) \geq \mu^{n/n-1}(\mathrm{wmc}(m\alpha) - mA\cdot\alpha).$$

Rescaling by μ and taking a limit proves the statement.

We next prove the inequality $\leq$. Again, the argument is identical to the lower bound in Theorem 6.10.1.(1). It is clear that the weighted mobility can only increase upon adding an effective class. Using continuity and homogeneity, the same is true for any pseudo-effective class. Thus we have

$$\begin{aligned}\mathrm{wmob}(\alpha) &\geq \sup_{B \text{ nef, } \alpha \succeq B^{n-1}} \mathrm{wmob}(B^{n-1}) \\ &= \sup_{B \text{ nef, } \alpha \succeq B^{n-1}} B^n \\ &= \widehat{\mathrm{vol}}(\alpha).\end{aligned}$$

where the second equality follows from [Leh13b, Example 8.22]. □

6.11 Applications to birational geometry

We end with a discussion of several connections between positivity of curves and other constructions in birational geometry. There is a large body of literature relating the positivity of a divisor at a point to its intersections against curves through that point. One can profitably reinterpret these relationships in terms of the volume of curve classes. A key result conceptually is :

Proposition 6.11.1. *Let X be a smooth projective variety of dimension n. Choose positive integers $\{k_i\}_{i=1}^r$. Suppose that $\alpha \in \mathrm{Mov}_1(X)$ is represented by a family of irreducible curves such that for any collection of general points $x_1, x_2, \ldots, x_r, y$ of X, there is a curve in our family which contains y and contains each x_i with multiplicity $\geq k_i$. Then*

$$\widehat{\mathrm{vol}}(\alpha)^{\frac{n-1}{n}} \geq \frac{\sum_i k_i}{r^{1/n}}.$$

This is just a rephrasing of well-known results in birational geometry ; see for example [Kol96, V.2.9 Proposition].

Proof. By continuity and rescaling invariance, it suffices to show that if L is a big and nef Cartier divisor class then

$$\left(\sum_{i=1}^r k_i\right) \frac{\mathrm{vol}(L)^{1/n}}{r^{1/n}} \leq L \cdot C.$$

A standard argument (see for example [Leh13b, Example 8.22]) shows that for any $\epsilon > 0$ and any general points $\{x_i\}_{i=1}^r$ of X there is a positive integer m and a Cartier divisor M numerically equivalent to mL and such that $\mathrm{mult}_{x_i} M \geq mr^{-1/n} \mathrm{vol}(L)^{1/n} - \epsilon$ for every i. By the assumption on the family of curves we may find an irreducible curve C with multiplicity $\geq k_i$ at each x_i that is not contained M. Then

$$m(L \cdot C) \geq \sum_{i=1}^r k_i \,\mathrm{mult}_{x_i} M \geq \left(\sum_{i=1}^r k_i\right)\left(\frac{m\,\mathrm{vol}(L)^{1/n}}{r^{1/n}} - \epsilon\right).$$

Divide by m and let ϵ go to 0 to conclude. □

Example 6.11.2. The most important special case is when α is the class of a family of irreducible curves such that for any two general points of X there is a curve in our family containing them. Proposition 6.11.1 then shows that $\widehat{\mathrm{vol}}(\alpha) \geq 1$.

6.11.1 Seshadri constants

Let X be a smooth projective variety of dimension n and let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Recall that for points $\{x_i\}_{i=1}^r$ on X the Seshadri constant of A along the $\{x_i\}$ is

$$\varepsilon(x_1, \ldots, x_r, A) := \inf_{C \ni x_i} \frac{A \cdot C}{\sum_i \mathrm{mult}_{x_i} C}.$$

where the infimum is taken over all reduced irreducible curves C containing at least one of the points x_i. An easy intersection calculation on the blow-up of X at the r points shows that

$$\varepsilon(x_1, \dots, x_r, A) \leq \frac{\operatorname{vol}(A)^{1/n}}{r^{1/n}}.$$

When the r points are very general, r is large, and A is sufficiently ample, one "expects" the two sides of the inequality to be close. This heuristic can fail badly, but it is interesting to analyze how close it is to being true. In particular, the Seshadri constant should only be very small compared to the volume in the presence of a "Seshadri-exceptional fibration" (see [EKL95], [HK03]). This motivates the following definition :

Definition 6.11.3. Let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Set $\varepsilon_r(A)$ to be the Seshadri constant of A along r points $\mathbf{x} := \{x_i\}$ of X. We define the Seshadri ratio of A to be

$$sr_{\mathbf{x}}(A) := \frac{r^{1/n}\varepsilon(x_1, \dots, x_r, A)}{\operatorname{vol}(A)^{1/n}}.$$

Note that the Seshadri ratio is at most 1, and that low values should only arise in special geometric situations. The principle established by [EKL95], [HK03] is that if the Seshadri ratio for A is small, then the curves which approximate the bound in the Seshadri constant can not "move too much."

In this section we revisit these known results on Seshadri constants from the perspective of the volume of curves. In particular we demonstrate how the Zariski decomposition can be used to bound the classes of curves C which give small values in the Seshadri computations above.

Proposition 6.11.4. *Let X be a smooth projective variety of dimension n and let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Fix $\delta > 0$ and fix r points $x_1, \dots, x_r$. Suppose that C is a curve containing at least one of the x_i and such that*

$$\varepsilon(x_1, \dots, x_r, A)(1+\delta) > \frac{A \cdot C}{\sum_i \operatorname{mult}_{x_i} C}.$$

Letting α denote the numerical class of C, we have

$$sr_{\mathbf{x}}(A)(1+\delta) \geq r^{1/n} \frac{\widehat{\operatorname{vol}}(\alpha)^{n-1/n}}{\sum_i \operatorname{mult}_{x_i} C}$$

In fact, this estimate is rather crude ; with better control on the relationship between A and α, one can do much better.

Proof. One simply multiplies both sides of the first inequality by $r^{1/n}/\operatorname{vol}(A)^{1/n}$ to deduce that

$$sr_{\mathbf{x}}(A)(1+\delta) \geq r^{1/n} \frac{A \cdot C}{\operatorname{vol}(A)^{1/n} \sum_i \operatorname{mult}_{x_i} C}$$

and then uses the obvious inequality $(A \cdot C)/\operatorname{vol}(A)^{1/n} \geq \widehat{\operatorname{vol}}(C)^{n-1/n}$. □

We can then bound the Seshadri ratio of A in terms of the Zariski decomposition of the curve.

Proposition 6.11.5. *Let X be a smooth projective variety of dimension n and let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Fix $\delta > 0$ and fix r distinct points $x_i \in X$. Suppose that C is a curve containing at least one of the x_i such that the class α of C is big and*

$$\varepsilon(x_1, \dots, x_r, A)(1+\delta) > \frac{A \cdot C}{\sum_i \operatorname{mult}_{x_i} C}.$$

Write $\alpha = B^{n-1} + \gamma$ for the Zariski decomposition. Then $sr_{\mathbf{x}}(A)(1+\delta) > sr_{\mathbf{x}}(B)$.

Proof. By Proposition 6.11.4 it suffices to show that

$$r^{1/n}\frac{\widehat{\mathrm{vol}}(\alpha)^{n-1/n}}{\sum_i \mathrm{mult}_{x_i} C} \geq sr_{\mathbf{x}}(B),$$

But this follows from the definition of Seshadri constants along with the fact that $B \cdot C = \widehat{\mathrm{vol}}(C)$. □

These results are of particular interest in the case when the points are very general, when it is easy to deduce the bigness of the class of C.

Certain geometric properties of Seshadri constants become very clear from this perspective. For example, following the notation of [Nag61] we say that a curve C on X is abnormal for a set of r points $\{x_i\}$ and a big and nef divisor A if C contains at least one x_i and

$$1 > \frac{r^{1/n}(A \cdot C)}{\mathrm{vol}(A)^{1/n}\sum_i \mathrm{mult}_{x_i} C}.$$

Corollary 6.11.6. *Let X be a smooth projective variety of dimension n and let A be a big and nef $\mathbb{R}$-Cartier divisor on X. Fix r very general points $x_1, \ldots, x_r$. Then no abnormal curve goes through a very general point of X aside from the x_i.*

Proof. Since the x_i are very general, any curve going through at least one more very general point deforms to cover the whole space, so its class is big and nef. Then combine Proposition 6.11.4 and Proposition 6.11.1 to deduce that if the Seshadri constant of the $\{x_i\}$ is computed by a curve through an additional very general point then $sr_{\mathbf{x}}(A) = 1$. □

6.11.2 Rationally connected varieties

Given a rationally connected variety X of dimension n, it is interesting to ask for the possible volumes of curve classes representing rational curves. In particular, one would like to know if one can find classes whose volumes satisfy a uniform upper bound depending only on the dimension. There are four natural options :

1. Consider all classes of rational curves.
2. Consider all classes of chains of rational curves which connect two general points.
3. Consider all classes of irreducible rational curves which connect two general points.
4. Consider all classes of very free rational curves.

Note that each criterion is more special than the previous ones. We call a class of the second kind an RCC class and a class of the fourth kind a VF class. Every one of the classes (2), (3), (4) has positive volume ; indeed, [BCE+02] shows that if two general points of X can be connected via a chain of curves of class α, then α is a big class.

On a Fano variety of Picard rank 1, the minimal volume of an RCC class is determined by the degree and the minimal degree of an RCC class against the ample generator (or equivalently, the degree, the index, and the length of an RCC class). The minimum volume is thus related to these well studied invariants.

In higher dimensions, the work of [KMM92] and [Cam92] shows that there are constants $C(n)$, $C'(n)$ such that any n-dimensional smooth Fano variety carries an RCC class satisfying $-K_X \cdot \alpha \leq C(n)$, and a VF class satisfying $-K_X \cdot \beta \leq C'(n)$. We then also obtain explicit bounds on the minimal volume of an RCC or VF class on X. It is interesting to ask what happens for arbitrary rationally connected varieties.

Example 6.11.7. We briefly review bounds on the volumes of such classes for smooth surfaces. Consider first the Hirzebruch surfaces $\mathbb{F}_e$. It is clear that on a Hirzebruch surface a curve class is RCC if and only if it is big, and one easily sees that the minimum volume for an RCC class is $\frac{1}{e}$. Thus there is no non-trivial universal lower bound for the minimum volume of an RCC class.

In terms of upper bounds, note that if $\pi : Y \to X$ is a birational map and α is an RCC class, then $\pi_* \alpha$ is an RCC class as well. Conversely, given any RCC class β on X, there is some preimage β' on Y which is also an RCC class. Thus by Proposition 6.5.16, we see that any rational surface carries an RCC class of volume no greater than that of an RCC class on a minimal surface. This shows that any smooth rational surface has an RCC class of volume at most 1.

On a surface any VF class is necessarily nef, so the universal lower bound on the volume is 1. In the other direction, consider again the Hirzebruch surface $\mathbb{F}_e$. Any VF class will have the form $aC_0 + bF$ where C_0 is the section of negative self-intersection and F is the class of a fiber. Note that the self intersection is $2ab - a^2 e$. For a VF class we clearly must have $a \geq 1$, so that $b \geq ea$ to ensure nefness. Thus the smallest possible volume of a VF class is e, and this is achieved by the class $C_0 + eF$. Note that there is no uniform upper bound on the minimum volume of a VF class.

As indicated in the previous example, it is most interesting to look for upper bounds on the minimum volume of an RCC class. Indeed, by taking products with projective spaces, one sees that in any dimension the only uniform lower bound for volumes of RCC classes is 0. Furthermore, there is no uniform upper bound for the minimum volume of a VF class. The crucial distinction is that VF classes are nef, while RCC classes need not be, so that a uniform bound on the volume of a VF class can only be expected for bounded families of varieties.

The following question gives a "birational" version of the well-known results of [KMM92].

Question 6.11.8. Let X be a smooth rationally connected variety of dimension n. Is there a bound $d(n)$, depending only on n, such that X admits an RCC class of volume at most $d(n)$?

It is also interesting to ask for optimal bounds on volumes. The first situation to consider are the "extremes" in the examples above. Note that the lower bound of the volume of a VF class is 1 by Proposition 6.11.1, so it is interesting to ask when the minimum is achieved.

Question 6.11.9. For which varieties X is the smallest volume of an RCC class equal to 1 ?

For which varieties X is the smallest volume of a VF class equal to 1 ?

6.12 Appendix A

6.12.1 Reverse Khovanskii-Teissier inequalities

An important step in the analysis of the Morse inequality is the "reverse" Khovanskii-Teissier inequality for big and nef divisors A, B, and a movable curve class β :

$$n(A \cdot B^{n-1})(B \cdot \beta) \geq B^n (A \cdot \beta).$$

We prove a more general statement on "reverse" Khovanskii-Teissier inequalities in the analytic setting. Some related work has appeared independently in the recent preprint [Pop15].

Theorem 6.12.1. *Let X be a compact Kähler manifold of dimension n. Let $\omega, \beta, \gamma \in \overline{\mathcal{K}}$ be three nef classes on X. Then we have*

$$(\beta^k \cdot \alpha^{n-k}) \cdot (\alpha^k \cdot \gamma^{n-k}) \geq \frac{k!(n-k)!}{n!} \alpha^n \cdot (\beta^k \cdot \gamma^{n-k}).$$

Proof. The proof depends on solving Monge-Ampère equations and the method of [Pop14]. Without loss of generality, we can assume γ is normalised such that $\beta^k \cdot \gamma^{n-k} = 1$. Then we need to show

$$(\beta^k \cdot \alpha^{n-k}) \cdot (\alpha^k \cdot \gamma^{n-k}) \geq \frac{k!(n-k)!}{n!} \alpha^n. \tag{6.1}$$

We first assume α, β, γ are all Kähler classes. We will use the same symbols to denote the Kähler metrics in corresponding Kähler classes. By the Calabi-Yau theorem [Yau78], we can solve the following Monge-Ampère equation :

$$(\alpha + i\partial\bar{\partial}\psi)^n = \left(\int \alpha^n\right) \beta^k \wedge \gamma^{n-k}. \tag{6.2}$$

Denote by α_ψ the Kähler metric $\alpha + i\partial\bar{\partial}\psi$. Then we have

$$\begin{aligned}(\beta^k \cdot \alpha^{n-k}) \cdot (\alpha^k \cdot \gamma^{n-k}) &= \int \beta^k \wedge \alpha_\psi^{n-k} \cdot \int \alpha_\psi^k \wedge \gamma^{n-k} \\ &= \int \frac{\beta^k \wedge \alpha_\psi^{n-k}}{\alpha_\psi^n} \alpha_\psi^n \cdot \int \frac{\alpha_\psi^k \wedge \gamma^{n-k}}{\alpha_\psi^n} \alpha_\psi^n \\ &\geq \left(\int \left(\frac{\beta^k \wedge \alpha_\psi^{n-k}}{\alpha_\psi^n} \cdot \frac{\alpha_\psi^k \wedge \gamma^{n-k}}{\alpha_\psi^n} \right)^{1/2} \alpha_\psi^n \right)^2.\end{aligned}$$

The last line follows because of the Cauchy-Schwarz inequality. We claim that the following pointwise inequality holds :

$$\frac{\beta^k \wedge \alpha_\psi^{n-k}}{\alpha_\psi^n} \cdot \frac{\alpha_\psi^k \wedge \gamma^{n-k}}{\beta^k \wedge \gamma^{n-k}} \geq \frac{k!(n-k)!}{n!}.$$

Then by (6.2) it is clear the above pointwise inequality implies the desired inequality (6.1). For any fixed point $p \in X$, we can choose some coordinates such that at the point p :

$$\alpha_\psi = i \sum_{j=1}^n dz^j \wedge d\bar{z}^j, \quad \beta = i \sum_{j=1}^n \mu_j dz^j \wedge d\bar{z}^j,$$

and

$$\gamma^{n-k} = i^{n-k} \sum_{|I|=|J|=n-k} \Gamma_{IJ} dz_I \wedge d\bar{z}_J.$$

Denote by μ_J the product $\mu_{j_1}...\mu_{j_k}$ with index $J = (j_1 < ... < j_k)$ and denote by J^c the complement index of J. Then it is easy to see at the point p we have

$$\frac{\beta^k \wedge \alpha_\psi^{n-k}}{\alpha_\psi^n} \cdot \frac{\alpha_\psi^k \wedge \gamma^{n-k}}{\beta^k \wedge \gamma^{n-k}} = \frac{k!(n-k)!}{n!} \frac{(\sum_J \mu_J)(\sum_K \Gamma_{KK})}{\sum_J \mu_J \Gamma_{J^c J^c}} \geq \frac{k!(n-k)!}{n!}.$$

This finishes the proof of the case when α, β, γ are all Kähler classes. If they are just nef classes, by taking limits, then we get the desired inequality. □

Remark 6.12.2. By [Xia15a, Section 2.1.1], for $k = 1$ we can always replace γ^{n-1} in Theorem 6.12.1 by an arbitrary movable class.

Remark 6.12.3. It would be interesting to find an algebraic approach to Theorem 6.12.1, thus generalizing it to projective varieties defined over arbitrary fields.

6.12.2 Towards the transcendental holomorphic Morse inequality

Recall that the (weak) transcendental holomorphic Morse inequality over compact Kähler manifolds conjectured by Demailly is stated as follows :

Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in \overline{\mathcal{K}}$ be two nef classes. Then we have $\mathrm{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta$. *In particular, if $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$ then there exists a Kähler current in* *the class $\alpha - \beta$.*

Indeed, the last statement has been proved in the recent work [Pop14] (see also [Xia13]). The missing part is how to bound the volume $\mathrm{vol}(\alpha - \beta)$ by $\alpha^n - n\alpha^{n-1} \cdot \beta$.

By [Xia15a, Theorem 2.1 and Remark 2.3] the volume for transcendental pseudo-effective $(1,1)$-classes is conjectured to be characterized as following :

$$\mathrm{vol}(\alpha) = \inf_{\gamma \in \mathcal{M}, \mathfrak{M}(\gamma)=1} (\alpha \cdot \gamma)^n \tag{6.3}$$

For the definition of $\mathfrak{M}$ in the Kähler setting, see [Xia15a, Definition 2.2]. If we denote the right hand side of (6.3) by $\overline{\mathrm{vol}}(\alpha)$, then we can prove the following :

Theorem 6.12.4. *Let X be a compact Kähler manifold of dimension n, and let $\alpha, \beta \in \overline{\mathcal{K}}$ be two nef classes. Then we have*

$$\overline{\mathrm{vol}}(\alpha - \beta)^{1/n} \, \mathrm{vol}(\alpha)^{n-1/n} \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

Proof. We only need to consider the case when $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$. And [Pop14] implies the class $\alpha - \beta$ is big. By the definition of $\overline{\mathrm{vol}}$, we have

$$\overline{\mathrm{vol}}(\alpha - \beta)^{1/n} = \inf_{\gamma \in \mathcal{M}, \mathfrak{M}(\gamma)=1} (\alpha - \beta) \cdot \gamma.$$

So we need to estimate $(\alpha - \beta) \cdot \gamma$ with $\mathfrak{M}(\gamma) = 1$:

$$\begin{aligned}
(\alpha - \beta) \cdot \gamma &= \alpha \cdot \gamma - \beta \cdot \gamma \\
&\geq \alpha \cdot \gamma - \frac{n(\alpha^{n-1} \cdot \beta) \cdot (\alpha \cdot \gamma)}{\alpha^n} \\
&= \frac{\alpha \cdot \gamma}{\alpha^n} (\alpha^n - n\alpha^{n-1} \cdot \beta) \\
&\geq \mathrm{vol}(\alpha)^{1-n/n} (\alpha^n - n\alpha^{n-1} \cdot \beta),
\end{aligned}$$

where the second line follows from Theorem 6.12.1 and Remark 6.12.2, and the last line follows the definition of $\mathfrak{M}$ and $\mathfrak{M}(\gamma) = 1$.

By the arbitrariness of γ we get

$$\overline{\mathrm{vol}}(\alpha - \beta)^{1/n} \, \mathrm{vol}(\alpha)^{n-1/n} \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

□

Remark 6.12.5. Without using the conjectured equality (6.3), it is observed independently by [Tos15] and [Pop15] that one can replace $\overline{\mathrm{vol}}$ by the volume function vol in Theorem 6.12.4.

6.13 Appendix B

6.13.1 Non-convexity of the complete intersection cone

We give an example explicitly verifying the non-convexity of $\mathrm{CI}_1(X)$. Undoubtedly there are simpler examples, but this is the first one we wrote down.

Example 6.13.1. [FS09] gives an example of a smooth toric threefold X such that every nef divisor is big. We show that for this toric variety $\mathrm{CI}_1(X)$ is not convex.

Let X be the toric variety defined by a fan in $N = \mathbb{Z}^3$ on the rays

$$\begin{array}{llll}
v_1 = (1,0,0) & v_2 = (0,1,0) & v_3 = (0,0,1) & v_4 = (-1,-1,-1) \\
v_5 = (1,-1,-2) & v_6 = (1,0,-1) & v_7 = (0,-1,-2) & v_8 = (0,0,-1)
\end{array}$$

with maximal cones

$$\langle v_1, v_2, v_3\rangle,\ \langle v_1, v_2, v_6\rangle,\ \langle v_1, v_3, v_4\rangle,\ \langle v_1, v_4, v_5\rangle,$$
$$\langle v_1, v_5, v_6\rangle,\ \langle v_2, v_3, v_4\rangle,\ \langle v_2, v_4, v_8\rangle,\ \langle v_2, v_5, v_6\rangle,$$
$$\langle v_2, v_5, v_8\rangle,\ \langle v_4, v_5, v_7\rangle,\ \langle v_4, v_7, v_8\rangle,\ \langle v_5, v_7, v_8\rangle.$$

Since X is the blow-up of $\mathbb{P}^3$ along 4 rays, it has Picard rank 5. Let D_i be the divisor corresponding to the ray v_i and C_{ij} denote the curve corresponding to the face generated by v_i and v_j. Standard toric computations show that the pseudo-effective cone of divisors is simplicial and is generated by D_1, D_5, D_6, D_7, D_8. The pseudo-effective cone of curves is also simplicial and is generated by $C_{14}, C_{16}, C_{25}, C_{47}, C_{48}$. From now on we will write divisor or curve classes as vectors in these (ordered) bases.

The intersection matrix is :

	D_1	D_5	D_6	D_7	D_8
C_{14}	-2	1	0	0	0
C_{16}	1	1	-2	0	0
C_{25}	0	-1	1	0	1
C_{47}	0	1	0	-2	1
C_{48}	0	0	0	1	-2

The nef cone of divisors is dual to the pseudo-effective cone of curves. Thus it is simplicial and has generators $A_1, \ldots, A_5$ determined by the columns of the inverse of the matrix above :

$$\begin{aligned}
A_1 &= (1, 3, 2, 2, 1)\\
A_2 &= (3, 6, 4, 4, 2)\\
A_3 &= (6, 12, 9, 8, 4)\\
A_4 &= (2, 4, 3, 2, 1)\\
A_5 &= (4, 8, 6, 5, 2)
\end{aligned}$$

A computation shows that for real numbers $x_1, \ldots, x_5$,

$$\begin{aligned}
\left(\sum_{i=1}^{5} x_i A_i\right)^2 =&(1, 3, 6, 2, 4)(x_1^2 + 6x_1x_2 + 12x_1x_3 + 4x_1x_4 + 8x_1x_5)+\\
&(9, 22, 45, 15, 30)x_2^2+\\
&(12, 30, 60, 20, 40)(x_2x_4 + 2x_2x_5 + 3x_2x_3 + 3x_3^2 + 2x_3x_4 + 4x_3x_5)+\\
&(4, 10, 20, 6, 13)x_4^2+\\
&(16, 40, 80, 26, 52)(x_4x_5 + x_5^2).
\end{aligned}$$

Note that the five vectors above form a basis of $N_1(X)$ and each one is proportional to one of the A_i^2.

It is clear from this explicit description that the cone is not convex. For example, the vector

$$v = (9, 22, 45, 15, 30) + (4, 10, 20, 6, 13)$$

can not be approximated by curves of the form H^2 for an ample divisor H. Indeed, if we have a sequence of ample divisors $H_j = \sum x_{i,j} A_i$ with $x_{i,j} > 0$ such that H_j^2 converges to v, then

$$\lim_{j\to\infty} x_{2,j} = 1 \qquad \text{and} \qquad \lim_{j\to\infty} x_{4,j} = 1.$$

But then the limit of the coefficient of $(12, 30, 60, 20, 40)$ is at least 1, a contradiction. Exactly the same argument shows that the closure of the set of all products of 2 (possibly different) ample divisors is not convex.

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