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Cohomologie des fibrés holomorphes et classes de Chern

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Cohomologie des fibrés holomorphes et classes de Chern

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Abstract

This thesis consists of two parts. In the first part, we study the cohomology of a compact Kähler manifold with values in a pseudo-effective line bundle. This part also describes various results concerning pseudo-effective vector bundles. Since several of our definitions do not require the use of a Kähler metric, the corresponding results also apply to general compact complex manifolds. The second part of the thesis concentrates on finding adequate definitions of Chern classes (or equivalently, of Chern characteristic classes) in Bott-Chern cohomology with rational coefficients. A related intersection theory is developed for that purpose in the context of integral Bott-Chern cohomology.

The organisation of the thesis is as follows. In Chapter 2, we improve the hard Lefschetz theorem obtained by Demailly, Peternell and Schneider, and discuss the optimality of the resulting statement. We show in particular that the holomorphic sections constructed in this result are in fact parallel with respect to the given positive singular metrics. A consequence of this property is the existence of naturally related holomorphics foliations.

In Chapter 3, we study the numerical dimension of a pseudo-effective line bundle over a compact Kähler manifold, and, in the framework of L^2 estimates, we obtain vanishing theorems analogous to those of Fedor Bogomolov and Junyan Cao, expressed in terms of numerical dimension.

In Chapter 4, we introduce the definition of nefness in higher codimension, a concept that interpolates between usual nefness and pseudo-effectivity. In this setting, we give a simplified proof of a result of Nakayama on the non-existence of Zariski decompositions in dimension at least 3. We also state a variant of the Bogomolov theorem and study the surjectivity of the Albanese map of a compact Kähler manifold when the anticanonical line bundle is pseudo-effective.

Chapter 5 discusses the concept of strongly pseudo-effective vector bundle or torsion-free sheaf, and proves the result that a strongly pseudo-effective reflexive sheaf with vanishing first Chern class over a compact Kähler manifold is in fact a numerically flat vector bundle.

In Chapter 6, following some ideas of Julien Grivaux, we construct an intersection theory for the integral Bott-Chern cohomology that had been defined in 2007 by Michel Schweitzer. A combination of these works allows us to define Chern classes and to obtain a Riemann-Roch-Grothendieck formula in rational Bott-Chern cohomology.

Résumé

Cette thèse comporte deux parties. Dans la première partie, nous étudions la cohomologie des variétés kähleriennes compactes à valeurs dans un fibré en droites pseudo-effectif, et également différents résultats concernant les fibrés vectoriels pseudo-effectifs. Comme certaines de nos définitions ne nécessitent pas l'existence de métriques kähleriennes, les résultats correspondents s'appliquent aussi aux variétés complexes compactes arbitraires. Dans la seconde partie, nous nous attachons à trouver une définition appropriée des classes de Chern (ou, de façon équivalente, des classes de Chern caractéristiques) pour la cohomologie de Bott-Chern à coefficients rationnels. Nous développons parallèlement une théorie de l'intersection dans le contexte de la cohomologie de Bott-Chern entière.

L'organisation de la thèse est la suivante. Dans le Chapitre 2, nous améliorons le théorème de Lefschetz difficile à valeurs dans un fibré en droites démontré par Demailly, Peternell et Schneider, et discutons l'optimalité de l'énoncé qui en découle. Nous montrons en particulier que les sections holomorphes construites dans ce résultat sont en fait parallèles par rapport à la métrique singulière donnée. Une conséquence de cette propriété est l'existence de feuilletages holomorphes naturellement reliés.

Dans le Chapitre 3, nous étudions la dimension numérique d'un fibré en droites pseudo-effectif sur une variété kählerienne compacte, et, dans le cadre des estimations L^2, nous obtenons des théorèmes d'annulation analogues à ceux de Fedor Bogomolov et de Junyan Cao, exprimés en termes de la dimension numérique.

Dans le Chapitre 4, nous introduisons la définition du concept de fibré en droites "nef en dimension supérieure", qui interpole entre la propriété nef usuelle et la pseudo-effectivité. Dans ce contexte, nous donnons une preuve simpliée d'un résultat de Nakayama sur la non-existence de décompositions de Zariski en dimension au moins 3. Nous énonçons aussi une variante du théorème d'annulation de Bogomolov et étudions la surjectivité du morphisme d'Albanese d'une variété kählerienne compacte dont le diviseur anticanonique est pseudo-effectif.

Le Chapitre 5 propose une discussion de la notion de fibré vectoriel ou de faisceau sans torsion pseudo-effectif (au sens fort). Nous montrons qu'un faisceau réflexif pseudo-effectif au sens fort sur une variété kählerienne compacte ayant une première classe de Chern triviale est en fait numériquement plat.

Dans le Chapitre 6, en nous inspirant d'idées de Julien Grivaux, nous construisons une théorie de l'intersection pour la cohomologie de Bott-Chern entière, qui avait été introduite en 2007 par Michel Schweitzer. Une combinaison de ces travaux nous permet de définir les classes de Chern et d'obtenir une formule de Riemann-Roch-Grothendieck en cohomologie de Bott-Chern rationnelle.

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Introduction (Français)

0.1. Un peu d'histoire

Un siècle après les travaux révolutionnaires de Riemann sur les surfaces de Riemann, les progrès généraux en géométrie différentielle et en analyse globale sur les variétés ont abouti à des avancées majeures dans la théorie des variétés algébriques et analytiques de dimension arbitraire. Dans ce contexte, l'un des résultats les plus fondamentaux obtenu dans les années 1950 est le théorème d'annulation de Kodaira pour les fibré en droites positifs, qui est une conséquence profonde de la technique de Bochner et de la théorie des formes harmoniques initiée par Hodge dans les années 1940. Cette approche a permis à Kodaira d'obtenir son fameux théorème de plongement, qui est une vaste généralisation du critère de Riemann caractérisant les variétés abéliennes.

Pour expliquer comment les techniques analytiques modernes sont mises en jeu, nous rappelons ici brièvement l'argument de Kodaira. Le théorème de plongement caractérise les variétés projectives comme suit.

Théorème 0.1.1. (Critère de Kodaira)
Soit X une variété kählerienne compacte. Alors X est projective si et seulement s'il existe une métrique kählerienne ω dont la classe de cohomologie est image d'une classe entière par le morphisme d'inclusion $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$.

D'une façon équivalente, une variété complexe compacte est projective si et seulement si dans l'espace $H^2(X,\mathbb{R})$, le cône de Kähler, c'est-à-dire le cône convexe ouvert engendré par les formes kähleriennes, contient un point rationnel (un élément de $H^2(X,\mathbb{Q})$).

Si une variété est projective, la restriction de la métrique de Fubini-Study est une forme kählerienne de classe entière dans le cône de Kähler. Le point essentiel est de montrer la réciproque. La méthode de Kodaira pour la démontrer est basée sur le théorème d'annulation suivant.

Théorème 0.1.2. (Théorème d'annulation de Kodaira)
Si L est un fibré en droites positif sur une variété complexe compacte X (c'est-à-dire, s'il existe une métrique lisse h sur L telle que $i\Theta(L,h) > 0$), alors pour $q \geqslant 1$,

$$H^q(X, K_X \otimes L) = 0.$$

L'argument utilisé par Kodaira pour déduire le théorème de plongement du théorème d'annulation est le suivant : on montre l'existence de suffisamment de sections pour plonger la variété dans l'espace projectif. Plus précisément, on considère l'application de Kodaira pour m suffisamment grand

$$X \to \mathbb{P}(H^0(X, L^{\otimes m}))$$
$$x \mapsto \{s \in H^0(X, L^{\otimes m}) | s(x) = 0\}.$$

On montre que pour m suffisamment grand, l'application de Kodaira donne le plongement. Pour donner une idée de la preuve, montrons que l'application de Kodaira est un morphisme pour m suffisamment grand, ce qui est équivalent à montrer que pour tout $x \in X$, la restriction $H^0(X, L^{\otimes m}) \to L^{\otimes m} \otimes \mathcal{O}_{X,x}/\mathfrak{m}_x$ est surjective, où l'on a noté $\mathfrak{m}_x$ l'idéal maximal de $\mathcal{O}_{X,x}$ en x. Considérons l'éclatement $\pi : \tilde{X} \to X$ de X en x, et désignons par E le diviseur exceptionnel. La suite exacte courte

$$0 \to \mathcal{O}(-E) \otimes \pi^* L^{\otimes m} \to \pi^* L^{\otimes m} \to \pi^* L^{\otimes m}|_E \to 0$$

induit la suite exacte longue

$$H^0(\tilde{X}, \pi^* L^{\otimes m}) \cong H^0(X, L^{\otimes m}) \to H^0(E, \pi^* L^{\otimes m}|_E) \cong L^{\otimes m} \otimes \mathcal{O}_{X,x}/\mathfrak{m}_x \to H^1(\tilde{X}, \mathcal{O}(-E) \otimes \pi^* L^{\otimes m}).$$

D'après le théorème d'annulation de Kodaira, on a

$$H^1(\tilde{X}, \mathcal{O}(-E) \otimes \pi^* L^{\otimes m}) = 0$$

pour m suffisamment grand. Ceci démontre la surjectivité. Ceci correspond à la preuve que $L^{\otimes m}$ est sans point base pour m suffisamment grand. On voit ainsi apparaître un de problèmes centraux de la géométrie complexe : construire des sections holomorphes vérifiant des propriétés supplémentaires particulières.

De nouveaux développements de la technique de Bochner, notamment entre les mains de Kohn, Andreotti-Vesentini et Hörmander, ont conduit dix ans après Kodaira à la théorie des estimations de L^2 pour l'opérateur de Cauchy-Riemann. Ces généralisations permettent non seulement d'améliorer ou de généraliser les théorèmes d'annulation, mais, et c'est peut-être plus important encore, fournissent des informations nature quantitative pour les solutions des équations du type $\overline{\partial} u = v$. Par exemple, on peut "forcer" les zéros d'une solution d'une équation $\overline{\partial} u = v$ par le choix d'un poids plurisousharmonique singulier.

Une façon de généraliser les résultats de Kodaira est ainsi d'étudier des théorèmes d'annulation dans le contexte des métriques singulières positives, par exemple dans la direction du théorème d'annulation de Demailly-Nadel. Rappelons quelques définitions élémentaires sur les métriques singulières positives.

DÉFINITION 0.1.1. (Courants positifs)
*D'après [**Lel57**], un courant Θ de bidimension (p,p) est dit (faiblement) positif si pour chaque choix de $(1,0)$-formes lisses $\alpha_1, \cdots, \alpha_p$ sur X, la distribution*

$$\Theta \wedge i\alpha_1 \wedge \overline{\alpha_1} \wedge \cdots \wedge i\alpha_p \wedge \overline{\alpha_p}$$

est une mesure positive.

Pour tout $(1,1)$-courant T et toute $(1,1)$-forme lisse α, nous disons $T \geqslant \alpha$ au sens des courants si $T-\alpha$ est un courant positif.

DÉFINITION 0.1.2. (Fonction plurisousharmonique = psh / quasi-psh)
Soit X une variété complexe (pas nécessairement compacte). On dit que φ est une fonction psh (resp. une fonction quasi-psh) sur X, si on a $i\partial\overline{\partial}\varphi \geqslant 0$ au sens des courants (resp. $i\partial\overline{\partial}\varphi \geqslant \alpha$), où α est une forme lisse sur X. On dit qu'une fonction quasi-psh φ a des singularités analytiques, si localement φ est de la forme

$$\varphi(z) = c\log\Big(\sum_i |g_i|^2\Big) + O(1)$$

où $c > 0$ et (g_i) sont des fonctions holomorphes locales, et où $O(1)$ signifie un terme localement borné.

Le singularité d'une métrique peut être appréhendée au moyen du faisceau d'idéaux multiplicateurs introduit par Nadel [**Nad89**]. En particulier, si la métrique est lisse, le faisceau d'idéaux multiplicateurs est trivial (c'est-à-dire, égal à $\mathcal{O}_X$).

DÉFINITION 0.1.3. (Faisceau d'idéaux multiplicateurs)
Soit φ une fonction quasi-psh. Le faisceau d'idéaux multiplicateurs $\mathcal{I}(\varphi)$ est défini par

$$\mathcal{I}(\varphi)_x = \left\{ f \in \mathcal{O}_{X,x} | \exists U_x, \int_{U_x} |f|^2 e^{-2\varphi} < \infty \right\}$$

où U_x désigne un voisinage ouvert de x dans X.

Pour une métrique singulière h (c'est-à-dire dont le poids local φ est dans L^1_{loc}), on définit localement le faisceau d'idéaux multiplicateurs $\mathcal{I}(h)$ comme le faisceau d'idéaux multiplicateurs de φ.

Le faisceau d'idéaux multiplicateurs intervient dans le théorème d'annulation fondamental suivant :

THÉORÈME 0.1.3. (Théorème d'annulation de Demailly-Nadel)([**Nad89**], [**Dem93**])
Soit (X,ω) une variété kählérienne faiblement pseudoconvexe, et E un fibré en droites holomorphe sur X muni d'une métrique hermitienne h singulière de poids φ. Supposons qu'il existe une fonction continue positive ε sur X telle que la courbure satisfasse l'inégalité

$$i\Theta(E,h) \geqslant \varepsilon\omega$$

au sens des courants (on dit que L est gros). Alors

$$H^q(X, \mathcal{O}(K_X \otimes E) \otimes \mathcal{I}(h)) = 0$$

pour tout $q \geqslant 1$.

C'est une situation géométrique fréquente que la courbure d'une métrique singulière puisse « dégénérer » dans certaines directions. Cela conduit au concept de dimension numérique, qui, grosso modo, compte le nombre de « directions positives » au point générique. Un problème ouvert important en géométrie complexe, qui va largement au delà des résultats de Kodaira, est la conjecture d'abondance. Celle-ci prédit la croissance de la dimension des sections pluricanoniques (c'est-à-dire les sections de mK_X où K_X est le fibré en droites canonique) par rapport à l'exposant m, asymptotiquement en termes de la dimension numérique. En particulier, dans cette conjecture, toute hypothèse de positivité stricte est abandonnée. Ceci amène à considérer plutôt les cônes fermés de classes positives et les classes se situant à la frontière. Deux cône positifs fermés importants interviennent ici : les cônes nef et pseudo-effectif (psef).

Pour plus de généralité, notamment lorsque les variétés considérées ne sont pas kähleriennes, nous travaillerons dans cette thèse avec des classes prises au sens de la cohomologie de Bott-Chern complexe. Rappelons que la cohomologie Bott-Chern complexe de X est définie par

$$H^{p,q}_{BC}(X,\mathbb{C}) = \{(p,q)\text{-formes } d\text{-fermées}\}/\{(p,q)\text{-formes } \partial\overline{\partial}\text{-exactes }\}.$$

DÉFINITION 0.1.4. (Fibré en droites psef)
Soit L un fibré en droites holomorphe sur une variété complexe compacte X. On dit que L est pseudo-effective (en abrégé, psef) si $c_1(L) \in H^{1,1}_{BC}(X,\mathbb{C})$ est la classe de cohomologie d'un courant positif fermé T, ou, de façon équivalente, si L peut être équipé d'une métrique hermitienne singulière h telle que $T = \frac{1}{2\pi}\Theta_{L,h} \geqslant 0$ au sens des courants.

Une classe de cohomologie $\alpha \in H^{1,1}_{BC}(X,\mathbb{C})$ est dite pseudo-effective (psef) si elle contient un courant positif.

Une façon usuelle de construire une métrique singulière est d'utiliser des sections globales du fibré en droites. Par exemple, on a la formule de Lelong-Poincaré suivante : soit $f \in H^0(X,\mathcal{O}_X)$ une fonction holomorphe non nulle, $Z_f = \sum m_j Z_j, m_j \in \mathbb{N}$, le diviseur zéro de f et $[Z_f] = \sum m_j [Z_j]$ le courant d'intégration associé au diviseur zéro. Alors

$$\frac{i}{\pi}\partial\overline{\partial}\log|f| = [Z_f].$$

Si f est une section globale non nulle à valeurs dans un fibré en droites L, la même formule donne un courant positif représentant la première classe de Chern $c_1(L)$.

Une autre méthode puissante permettant de construire des métriques singulières à la limite repose sur l'utilisation de l'équation de Monge-Ampère. On peut ainsi construire une suite de métriques dont la masse se concentre de plus en plus au voisinage d'un ensemble analytique donné, par une application du théorème de Calabi-Yau [**Yau78**] (le cas Kähler-Einstein avec $c_1(X) < 0$ étant dû à Aubin).

THÉORÈME 0.1.4. (Yau) *Soit (X,ω) une variété kählerienne compacte de dimension n. Alors pour toute forme volume lisse $f > 0$ satisfaisant $\int_X f = \int_X \omega^n$, il existe une métrique kählerienne $\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ telle que $\tilde{\omega}^n = f$.*

On peut citer [**Mou98**], [**DP03**], [**DP04**] parmi les travaux utilisant ce cercle d'idées. Une autre technique utile consiste à utiliser des métriques Hermite-Einstein pour les fibrés vectoriels stables E sur une variété Kählerienne compacte (X,ω). Rappelons qu'une métrique Hermite-Einstein est une métrique telle que $\Lambda i\Theta(E) = c\,\mathrm{Id}_E$ où Λ est l'adjoint de $\omega \wedge \bullet$, et où c est une constante. L'existence de telles métriques a été prouvée par [**Don85**], [**UY86**], [**BS94**].

Un autre cône important est le cône nef. La définition relative au cas non algébrique a été introduite dans [**DPS94**].

DÉFINITION 0.1.5. (fibré en droites nef)

Un fibré en droites L sur une variété complexe compacte X est dit nef si pour tout $\varepsilon > 0$, il existe une métrique hermitienne lisse h_ε sur L telle que $i\Theta_{L,h_\varepsilon} \geqslant -\varepsilon\omega$ où ω est une métrique hermitienne lisse.

Une classe de cohomologie $\alpha \in H^{1,1}_{BC}(X,\mathbb{C})$ est dite nef si pour toute $\varepsilon > 0$, il existe un représentant lisse $\alpha_\varepsilon \in \alpha$ tel que $\alpha_\varepsilon \geqslant -\varepsilon\omega$, où ω est une métrique hermitienne lisse.

De manière générale, il est intéressant d'étudier les cônes positifs associés à des variétés complexes compactes et de les relier à la géométrie de la variété. Leur importance se reflète déjà dans les diverses reformulations ou généralisations du théorème de plongement de Kodaira. Pour le cas non algébrique, un tel énoncé est donné dans [**DP04**] : une variété complexe compacte X contient un courant kählerien (à savoir un courant $T \in H^{1,1}(X,\mathbb{R})$ tel que $T \geqslant \omega$ pour une certaine forme hermitienne lisse ω), si et seulement si elle est biméromorphe à une variété kählerienne. Dans la situation algébrique, on a le théorème d'annulation de Kawamata-Viehweg.

THÉORÈME 0.1.5. (Théorème d'annulation de Kawamata-Viehweg)

Soit X une variété algébrique projective lisse et soit F un fibré en droites sur X tel que F possède un multiple mF s' écrivant sous la forme $mF = L + D$ où L est un fibré en droites nef, et D un diviseur effectif. Alors

$$H^q(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(m^{-1}D)) = 0$$

pour $q \geqslant n - \mathrm{nd}(L) + 1$.

Un cas particulier du théorème d'annulation de Kawamata-Viehweg est le suivant. Si F est un fibré en droites nef, alors

$$H^q(X, \mathcal{O}(K_X + F)) = 0$$

pour $q \geqslant n - \mathrm{nd}(F) + 1$. Dans le théorème, $\mathrm{nd}(L)$ désigne la dimension numérique du fibré en droites nef. Dans ce cas, elle est définie par

$$\mathrm{nd}(L) = \max\{p \in [0,n]; (c_1(L))^p \neq 0\}.$$

Un autre outil fondamental de la géométrie complexe est la formule de Riemann-Roch-Hirzebruch. Elle calcule entre autres la caractéristique d'Euler du produit tensoriel d'un fibré en droites en termes de nombres d'intersection mettant en jeu les classes de Chern du fibré en droites et du fibré tangent de la variété. Si un fibré en droites donné est supposé posséder une métrique de courbure strictement positive (par exemple, si le fibré en droites est nef et gros), le théorème d'annulation de Kawamata-Viehweg implique que les groupes de cohomologie de degré supérieur à valeurs dans les puissances tensorielles élevées du fibré en droites tordu par le fibré en droites canonique seront triviaux – après avoir pris le produit tensoriel avec le faisceau d'idéaux multiplicateurs ad hoc. Dans ce cas, la formule de Riemann-Roch-Hirzebruch prédit la croissance des sections globales du produit tensoriel d'un fibré en droites. En particulier, si le fibré en droites L est nef et gros, on a une croissance maximale des sections globales de $L^{\otimes m}$ par rapport à m, et des groupes de cohomologie supérieurs triviaux pour m suffisamment grand.

En géométrie complexe, les classes de Chern peuvent être définies et déclinées suivant différentes théories cohomologiques : cohomologie singulière, cohomologie de De Rham, cohomologie de Dolbeault, cohomologie de Deligne, cohomologie de Bott-Chern complexe, etc. D'après les travaux de Michel Schweitzer [**Sch07**], il existe une théorie cohomologique plus précise que toutes les théories précédemment citées, à savoir la cohomologie de Bott-Chern à coefficients entiers. On veut dire par là qu'il existe des morphismes naturels de la cohomologie de Bott-Chern entière vers toutes ces autres théories cohomologiques.

C'est donc une question naturelle de savoir si l'on peut généraliser la formule de Riemann-Roch-Hirzebruch et définir les classes de Chern pour un faisceau cohérent sur une variété complexe compacte en cohomologie de Bott-Chern rationnelle.

Une difficulté, mise en évidence par un résultat frappant de C. Voisin [**Voi02a**], réside dans le fait que sur une variété complexe compacte arbitraire (même supposée kählerienne), une résolution d'un faisceau cohérent par des fibrés vectoriels n'existe pas nécessairement. Autrement dit, sur une variété complexe compacte quelconque, le groupe de Grothendieck des fibrés vectoriels n'est pas isomorphe au groupe de Grothendieck des faisceaux cohérents, même si c'est le cas pour une variété projective. Il nous faudra cependant donner un sens à la classe de Chern des images directes des faisceaux cohérents dans la formule de Riemann-Roch-Grothendieck. Il s'ensuit que la définition des classes Chern des faisceaux cohérents sur des variétés complexes compactes est beaucoup plus intriquée que dans le cas algébrique. Les résultats que nous avons pu obtenir dans cette direction sont exposés dans la dernière partie de cette thèse.

0.2. Un résumé des principaux résultats

Le théorème d'annulation de Demailly-Nadel implique que dans le cadre des métriques singulières, la positivité d'un fibré en droites entraîne des contraintes importantes sur les groupes de cohomologie.

La majeure partie de cette thèse portera sur les conséquences de l'existence de métriques positives singulières sur les groupes de cohomologie des fibrés vectoriels ou la structure géométriques des variétés mises en jeu.

Dans la dernière partie de la thèse, nous discutons de la construction des classes de Chern et de l'énoncé de la formule de Riemann-Roch-Grothendieck en cohomologie de Bott-Chern rationnelle (telle que définie par Michel Schweitzer).

0.2.1. Théorème de Lefschetz difficile pour un fibré en droites psef.

D'après la formule de Riemann-Roch-Grothendieck et le théorème d'annulation de Kawamata-Viehweg, les sections globales des grandes puissances tensorielles d'un fibré en droites nef et gros ont une croissance asymptotique maximale (de l'ordre de l'exposant élevé à une puissance égale à la dimension complexe). Dans le cas algébrique, one peut obtenir ces résultats en prenant une intersection par un hyperplan générique pour faire une récurrence sur la dimension.

En général, et dans le cas semi-positif, en particulier lorsque (L, h) est un fibré en droites pseudo-effectif (psef) possédant un faisceau d'idéaux multiplicateurs $\mathcal{I}(h)$, les groupes de cohomologie de degrés supérieurs calculés sur une variété kählerienne compacte (X, ω) à valeurs dans $K_X \otimes L \otimes \mathcal{I}(h)$ ne sont pas nécessairement triviaux.

Cette situation est étudiée dans [**DPS01**], où Demailly, Peternell et Schneider construisent des pré-images dans $H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$ pour le morphisme de Lefschetz, c'est-à-dire le morphisme induit par $\omega^q \wedge \bullet$ à valeurs dans $H^q(X, K_X \otimes L \otimes \mathcal{I}(h))$. Lorsque le fibré en droites L est trivial équipé de la métrique triviale, ce résultat redonne le théorème classique de Lefschetz difficile – dans ce cas, comme il est bien connu, le morphisme de Lefschetz est un isomorphisme.

THÉORÈME 0.2.1. *([**DPS01**])*
Soit (L, h) un fibré en droites pseudo-effectif sur une variété kählerienne compacte (X, ω) de dimension n. Soit $\Theta_{L,h} \geqslant 0$ son courant de courbure et $\mathcal{I}(h)$ le faisceau d'idéaux multiplicateurs associé.
Alors, l'opérateur de produit extérieur $\omega^q \wedge \bullet$ induit un morphisme surjectif

$$\Phi^q_{\omega,h} : H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h)).$$

Le cas spécial où L est nef est dû à Takegoshi [**Tak97**]. Un cas encore plus spécial est lorsque L est semi-positif, c'est-à-dire que L possède une métrique lisse ayant une courbure semi-positive. Dans ce cas, le faisceau d'idéaux multiplicateurs $\mathcal{I}(h)$ coïncide avec $\mathcal{O}_X$ et on obtient la conséquence suivante déjà observée par Enoki [**Eno93**] et Mourougane [**Mou95**], à savoir que le morphisme $H^0(X, \Omega_X^{n-q} \otimes L) \to H^q(X, \Omega_X^n \otimes L)$ est surjectif.

La stratégie de la preuve est la suivante. On approche la métrique singulière par une suite de métriques lisses en dehors d'ensembles analytiques propres, de sorte que le faisceau d'idéaux multiplicateurs soit préservé. Au cours du processus, on perd de manière inévitable un peu de positivité de courbure. Comme observé dans [**Dem82**], on peut modifier la métrique de Kähler de façon à obtenir des métriques complètes sur les ouverts complémentaires de chacun des ensembles analytiques. Pour une classe de cohomologie de degré q donnée, on peut ainsi appliquer l'inégalité de Bochner (valable dans le cas kählerien complet) aux représentants harmoniques de cette classe par rapport aux métriques approchées du fibré et aux métriques de Kähler complètes construites précédemment. Ceci permet de trouver une suite de préimages via l'isomorphisme de Lefschetz ponctuel. Grâce aux estimations L^2 obtenues, les préimages ont une limite faible qui sera holomorphe, quitte à passer à une sous-suite bien choisie. La limite faible de cette sous-suite est la section holomorphe souhaitée dans $H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$.

Dans le cas semi-positif, les choses sont beaucoup plus faciles car il n'est pas besoin de prendre une suite d'approximation des métriques singulières.

Dans le cas classique $L = \mathcal{O}_X$, on peut observer que toute section $u \in H^0(X, \Omega_X^{n-q})$ satisfait la condition supplémentaire $du = d_{h_0} u = 0$. Ceci se voit facilement à l'aide de la formule de Stokes, qui implique

$$\int_X i du \wedge \overline{du} \wedge \omega^{q-1} = \int_X \{du, du\}_{h_0} \wedge \omega^{q-1} = 0,$$

où h_0 est la métrique lisse triviale sur $\mathcal{O}_X$.

La preuve du théorème de Lefschetz difficile donnée dans [**DPS01**] est obtenue en construisant les préimages comme limites de formes données par l'isomorphisme ponctuel de Lefschetz. On utilise ensuite une suite de représentants harmoniques d'une classe donnée dans $H^q(X, K_X \otimes L \otimes \mathcal{I}(h))$, par rapport aux métriques hermitiennes approximatives h_ε, encore singulières, mais lisses sur des ouverts de Zariski. Il est alors naturel de se demander si la limite est harmonique par rapport à la métrique singulière originale h.

Dans le cadre singulier, l'opérateur ∂_h est encore un opérateur densément défini, mais il est a priori non évident d'évaluer le domaine de l'adjoint hilbertien ∂_h^*. Néanmoins, cela a encore un sens de se demander si la limite est parallèle par rapport à la métrique singulière originale h.

Le calcul effectué ci-dessus correspond au cas d'un fibré trivial muni d'une métrique triviale. Notre premier résultat détaillé dans [**Wu18**] fournit une réponse affirmative à la question générale en étudiant des estimations supplémentaires dans le processus d'approximation de [**DPS01**].

THÉORÈME 0.2.2. *Toutes les sections holomorphes produites par le théorème de Lefschetz difficile à valeurs dans un fibré en droites psef sont parallèles par rapport à la connexion de Chern associée à la métrique hermitienne singulière donnée h sur L, dès lors que celle-ci possède un courant de courbure semi-positif.*

Le point essentiel de la preuve consiste à montrer que l'opérateur de dérivée covariante est toujours bien définis dans le cadre singulier, et se comporte bien dans le processus d'approximation.

Plus précisément, soit φ la fonction de poids locale de la métrique singulière. Alors la dérivée $\partial\varphi$ est une fonction L^q_{loc} pour tous $q < 2$ (mais pas nécessairement pour $q = 2$, comme c'est le cas par exemple pour $\varphi = \log|z|$ sur $\mathbb{C}$). Localement, la dérivée covariante d'une section u par rapport à la métrique singulière h peut s'écrire sous la forme

$$\partial_h u = \partial u + \partial\varphi \wedge u.$$

Si u est une section holomorphe (donc en particulier localement bornée), le deuxième terme est le produit d'une forme L^q_{loc} par une section L^∞_{loc}, ce terme est donc L^q_{loc} pour tout $q < 2$. Dans le processus d'approximation, nous prenons en fait une section à valeurs dans $L^2_{loc}(e^{-\varphi})$. Pour montrer que le second terme est au moins bien défini dans L^1_{loc} par rapport à la mesure de Lebesgue, il suffit également d'observer que $\partial\varphi \in L^2_{\mathrm{loc}}(e^{\varphi})$, ce qui est toujours le cas pour une fonction psh. Regardons à titre d'exemple le cas typique où $\varphi = \log|z|$ sur $\mathbb{C}$. Alors la section u doit s'annuler en 0 et il suffit d'observer que

$$\partial\varphi e^{\varphi} = \frac{dz}{z} \times |z|^2 = \bar{z} dz.$$

Dans le cas pseudo-effectif, le morphisme de Lefschetz n'est en général plus injectif comme dans le cas classique du théorème de Lefschetz difficile. Un contre-exemple évident peut être obtenu en prenant $L = mA$ où A est un diviseur ample, de sorte que $h^0(X, \Omega_X^{n-q} \otimes L) \sim Cm^n$ pour m assez grand, mais $h^q(X, \Omega_X^n \otimes L) = 0$ si $q > 0$. Néanmoins, nous allons montrer qu'il y a un isomorphisme entre l'espace des sections qui sont parallèles par rapport à la métrique singulière et le groupe de cohomologie de degré supérieur considéré.

THÉORÈME 0.2.3. *Soit (L, h) un fibré en droites pseudo-effectif sur une variété kählerienne compacte (X, ω). Alors, le produit extérieur avec la forme de Kähler induit un isomorphisme*

$$\Phi^q_{\omega,h} : H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \cap \mathrm{Ker}(\partial_h) \longrightarrow H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h)).$$

En particulier si la métrique est semi-positive, on a un isomorphisme

$$\Phi^q_{\omega,h} : H^0(X, \Omega_X^{n-q} \otimes L) \cap \mathrm{Ker}(\Delta_{\partial_h}) \longrightarrow H^q(X, \Omega_X^n \otimes L).$$

Cet énoncé entraîne en particulier la surjectivité stipulée par le théorème précédent. En application de ces résultats, nous montrons que chaque section holomorphe obtenue comme préimage définit en fait une feuilletage sur X.

THÉORÈME 0.2.4. *Supposons que* $v \in H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$, $q \geqslant 1$, *soit une section parallèle par rapport à la métrique singulière h. En particulier, une section construite comme préimage par le théorème de Lefschetz difficile est parallèle. Le produit intérieur par v donne un* $\mathcal{O}_X$*-morphisme (défini sur X tout entier)*

$$F_v : T_X \to \Omega_X^{n-q-1} \otimes L, \qquad \xi \mapsto \iota_\xi v.$$

Le noyau de F_v *définit un sous-faisceau cohérent intégrable de* $\mathcal{O}(T_X)$*, et donc un feuilletage holomorphe.*

Ici, nous entendons par feuilletage holomorphe un feuilletage éventuellement singulier, c'est-à-dire qu'il existe un ensemble analytique irréductible V de l'espace total T_X tel que pour tout $x \in X$, $V_x := V \cap T_X$ soit un espace vectoriel complexe et le faisceau de sections $\mathcal{O}(V) \subset \mathcal{O}(T_X)$ soit stable par crochets de Lie. Il est équivalent de dire que l'on a un sous faisceau cohérent $\mathcal{O}(V)$ qui est stable par crochets de Lie et saturé, c-à-d. $\mathcal{O}(T_X)/\mathcal{O}(V)$ est sans torsion.

Observons qu'en général une section de $H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$ n'induit pas nécessairement un feuilletage singulier sur X. En fait, notre définition du noyau de F_v définit une feuilletage si et seulement si

$$d_h v \wedge v = 0$$

qui est le cas quand la section mise en jeu est parallèle par rapport à la métrique singulière. Une question naturelle est de savoir si ce feuilletage est "algébrique", au sens où il induit un espace quotient avec une structure d'espace complexe.

Il existe des exemples concrets présentant ce phénomène qui ont été initialement donnés par Beauville [**Bea00**]; ils nous ont été indiqués par Andreas Höring. Un calcul complet se trouve dans la section 4 de notre travail [**Wu18**].

Une autre possibilité pour généraliser le théorème de Lefschetz difficile est de se demander si l'on peut remplacer le faisceau d'idéaux multiplicateurs par un faisceau d'idéaux plus grand, "moins singulier". Demailly, Peternell et Schneider ont montré dans [**DPS01**] qu'on ne peut pas en tout cas omettre le faisceau d'idéal, même lorsque le fibré L est supposé nef, et ils ont donné un contre exemple lorsque $L = -mK_X$ est un multiple du fibré en droites anticanonique. Cependant, il pourrait encore être possible dans certaines situations "d'améliorer" le faisceau d'idéaux multiplicateur, par exemple en le remplaçant par $\lim_{\delta \to 0_+} \mathcal{I}((1-\delta)\varphi)$ qui peut être vu comme une intersection infinie de faisceaux cohérents contenant $\mathcal{I}(\varphi)$. Même lorsque φ a des singularités analytiques, il peut arriver que l'on obtienne ainsi un faisceau d'idéaux strictement plus grand que $\mathcal{I}(\varphi)$, et même que la limite ne soit pas nécessairement un faisceau cohérent:

PROPOSITION 0.2.1. *Il existe un exemple de fonction psh* φ *telle que*

$$\lim_{\delta \to 0+} \mathcal{I}((1-\delta)\varphi) = \bigcap_{\delta > 0} \mathcal{I}((1-\delta)\varphi)$$

ne soit pas cohérent.

0.2.2. Théorème d'annulation dans L^2.

Les singularités d'une métrique se reflètent notamment dans leurs idéaux multiplicateurs associés. Une situation géométrique fréquente est que la courbure d'une métrique singulière "dégénère" dans certaines directions. Ce phénomène conduit au concept de dimension numérique, qui, en gros, mesure le nombre de "directions de courbure positives" en un point générique. Un problème ouvert important de la géométrie complexe est la conjecture dite d'abondance. Cette dernière peut être vue comme une vaste généralisation des résultats actuellement connus sur la dimension de Kodaira $\kappa(X) = \kappa(K_X)$, qui compte la croissance des sections pluricanoniques, c'est-à-dire les sections des multiples mK_X où K_X est le fibré en droites canonique. Par définition, pour tout fibré en droites L, la dimension de Kodaira-Iitaka est

$$\kappa(L) = \limsup_{m \to +\infty} \frac{\log h^0(X, mL)}{\log m}.$$

Un théorème bien connu de Siegel entraîne que $\kappa(L) \in \{-\infty, 0, 1, \ldots, n\}$ où $n = \dim X$, et qu'en dehors du cas $-\infty$, $\kappa(L)$ est le maximum des dimensions des images pluricanoniques $\Phi_{mL}(X) \subset \mathbb{P}(H^0(X, mK_X))$. La conjecture d'abondance prédit que le fibré canonique K_X atteint toujours sa croissance asymptotique maximale possible comme $m \to +\infty$, et que $\kappa(K_X)$ coïncide avec la dimension numérique (redéfinie plus loin).

Conjecture 0.2.1. (Conjecture d'abondance généralisée dans le cas kählérien, cf. [**BDPP13**]). *Pour une variété kählerienne compacte arbitraire X, la dimension de Kodaira coïncide avec la dimension numérique:*

$$\kappa(K_X) = \mathrm{nd}(c_1(K_X)).$$

Une version kählerienne de la définition de la dimension numérique est donnée dans [**Dem14**] ou [**Bou02a**].

Définition 0.2.1. (Dimension numérique)

Pour un fibré en droites psef L sur une variété kählerienne compacte (X, ω), la dimension numérique de L est définie comme

$$\mathrm{nd}(L) := \max\left\{p \in [0, n]; \exists c > 0, \forall \varepsilon > 0, \exists h_\varepsilon, i\Theta_{L,h_\varepsilon} \geqslant -\varepsilon\omega, \text{telle que} \int_{X \smallsetminus Z_\varepsilon} (i\Theta_{L,h_\varepsilon} + \varepsilon\omega)^p \wedge \omega^{n-p} \geqslant c\right\}.$$

Ici, les métriques h_ε sont supposées avoir des singularités analytiques, et on désigne par Z_ε l'ensemble singulier de la métrique.

Pour un fibré en droites nef, cette définition coincide avec la définition donnée dans la section précédente. Une définition équivalente peut être donnée en termes du produit (d'intersection) positif défini dans [**BEGZ10**]. Le produit positif est la classe de cohomologie réelle $\langle\alpha^p\rangle$ de bidegré (p, p) de la limite

$$\langle\alpha^p\rangle := \lim_{\delta\to 0} \ \{\langle T^p_{\min,\delta\omega}\rangle\},$$

où $T_{\min,\delta\omega}$ est le courant positif à singularité minimale contenu dans la classe $\alpha + \delta\{\omega\}$, et où $\langle T^p_{\min,\delta\omega}\rangle$ est le produit non pluripolaire.

Avec cette notion, la dimension numérique de α est définie comme

$$\mathrm{nd}(\alpha) := \max\{p | \langle\alpha^p\rangle \neq 0\}$$

qui est aussi égale à

$$\max\left\{p \mid \int_X \langle\alpha^p\rangle \wedge \omega^{n-p} > 0\right\}.$$

Une définition plus intuitive du produit positif est donnée dans [**BDPP13**], comme suit. Supposons que α soit une classe grosse sur une variété kählerienne compacte (X, ω) (c'est-à-dire que α contient un courant T tel que $T \geqslant C\omega$ pour une certaine constante $C > 0$). Pour déterminer la valeur du produit, il suffit de connaître son accouplement avec n'importe quelle forme test de bidegré $(n-p, n-p)$, et, en fait avec une famille dénombrable dense de formes dans l'espace des formes lisses.

Puisque toute forme u de bidegré $(n-p, n-p)$ peut s'écrire $u = C\omega^{n-p} - (C\omega^{n-p} - u)$ avec deux formes $C\omega^{n-p}$ et $C\omega^{n-p} - u$ fortement positives sur la variété compacte X si $C > 0$ est assez grand, il suffit de prendre en compte les accouplements avec une famille dénombrable dense de formes fortement positives.

Fixons une forme fermée fortement positive du type $(n-p, n-p)$ u sur X. On sélectionne des courants de Kähler $T \in \alpha$ avec singularités analytiques et une résolution logarithmique $\mu : \tilde{X} \to X$ telle que

$$\mu^* T = [E] + \beta$$

où $[E]$ est le courant associé à un $\mathbb{R}-$diviseur et β est une forme semi-positive. Nous considérons le courant image directe $\mu_*(\beta \wedge \ldots \wedge \beta)$. Étant donné deux courants $(1, 1)$ positifs fermés $T_1, T_2 \in \alpha$, nous pouvons écrire $T_j = \theta + i\partial\overline{\partial}\varphi_j$ $(j = 1, 2)$ pour certains forme lisse $\theta \in \alpha$. Définissons $T := \theta + i\partial\overline{\partial}\max(\varphi_1, \varphi_2)$. On obtient ainsi un courant à singularités analytiques moins singulier que les deux courants T_1, T_2. De cette façon, si on change le représentant T en un autre courant T', on peut toujours prendre une log-résolution simultanée telle que $\mu^* T' = [E'] + \beta'$, et supposer que $E' \leqslant E$. Par un calcul direct, on trouve

$$\int_{\tilde{X}} \beta' \wedge \ldots \wedge \beta' \wedge \mu^* u \geqslant \int_{\tilde{X}} \beta \wedge \ldots \wedge \beta \wedge \mu^* u.$$

On peut montrer que les courants positifs fermés $\mu_*(\beta \wedge \ldots \wedge \beta)$ sont uniformément bornés en masse. Pour chacune des intégrales associées à une famille dénombrable dense de formes u, le supremum de l'intégrale $\int_{\tilde{X}} \beta \ldots \wedge \beta \wedge \mu^* u$ est réalisé par une suite de courants $(\mu_m)_*(\beta_m \wedge \ldots \wedge \beta_m)$ obtenus comme images directes, pour une suite appropriée de modifications $\mu_m : \tilde{X}_m \to X$ et pour des formes β_m appropriées. En extrayant une sous-suite, on peut supposer que cette suite est faiblement convergente et on définit

$$\langle\!\langle\alpha^p\rangle\!\rangle := \lim_{m\to+\infty} \uparrow \{(\mu_m)_*(\beta_m \wedge \ldots \wedge \beta_m)\}.$$

Si α est seulement psef, on définit

$$\langle\!\langle\alpha^p\rangle\!\rangle := \lim_{\delta\downarrow 0} \downarrow \langle\!\langle(\alpha + \delta\{\omega\})^p\rangle\!\rangle.$$

On peut vérifier que

PROPOSITION 0.2.2. *Les deux produits positifs définis dans* [**BEGZ10**] *et* [**BDPP13**] *coïncident pour toute classe psef.*

DEFINITION 0.1. *Pour tout fibré en droites psef L sur une variété Kählerienne compacte, on pose*

$$\mathrm{nd}(L) = \mathrm{nd}(c_1(L)).$$

Dans le cas particulier d'une variété projective X, la définition précédente de la dimension numérique coïncide avec la définition algébrique suivante :

$$nd(L) = \sup_{A \text{ ample sur } X} \limsup_{m \to +\infty} \frac{\log h^0(X, mL + A)}{\log m}$$

(et on peut voir aisément qu'elle ne dépend elle aussi que de la classe numérique $c_1(L)$).

Dans nos articles [**Wu19a**] et [**Wu19b**], nous démontrons quelques théorèmes d'annulation L^2 s'appuyant sur la notion de dimension numérique d'un fibré en droites psef. Rappelons qu'un théorème d'annulation classique de Bogomolov dit que pour tout fibré L sur une variété projective lisse X on a

$$H^0(X, \Omega_X^p \otimes L^{-1}) = 0$$

pour $p < \kappa(L)$. Il est intéressant de savoir si on peut remplacer la dimension de Kodaira $\kappa(L)$ par la dimension numérique $\mathrm{nd}(L)$. Dans [**Dem02**], il est prouvé que pour tout fibré en droites pseudo-effiectif L sur une variété kählerienne compacte X, et toute section holomorphe non nulle $\theta \in H^0(X, \Omega^p \otimes L^{-1})$, où $0 \leqslant p \leqslant n = \dim X$, alors θ induit un feuilletage sous le même sens que dans le théorème 0.2.4. Le théorème d'annulation de Bogomolov interdit l'existence d'une telle section non nulle pour $p \geqslant \kappa(L)$. (D'après notre résultat, la même conclusion est vraie pour $p \geqslant \mathrm{nd}(L)$.)

L'article [**Mou98**] démontre la version suivante du théorème d'annulation de Bogomolov : si L est un fibré en droites nef sur une variété kählerienne compacte X, alors

$$H^0(X, \Omega_X^p \otimes L^{-1}) = 0$$

pour $p < \mathrm{nd}(L)$. Dans notre travail [**Wu19b**], nous obtenons une généralisation du cas nef au cas psef en affinant les estimations de Mourougane dans [**Mou98**]. Une preuve similaire avait été donnée dans [**Bou02a**] au moyen d'une version singulière de l'équation Monge-Ampère ; nous donnons ici une autre preuve qui ne nécessite que la résolution d'équations de Monge-Ampère "classiques ".

THÉORÈME 0.2.5. *Si L est un fibré en droites psef sur une variété kählerienne compacte X, alors*

$$H^0(X, \Omega_X^p \otimes L^{-1}) = 0$$

pour $p < \mathrm{nd}(L)$.

En nous inspirant des travaux de [**Cao14**], nous obtenons le théorème d'annulation suivant de type Kawamata-Viehweg dans [**Wu19a**]. La preuve est une modification celle donnée par Junyan Cao:

THÉORÈME 0.2.6. *Soit L est un fibré en droites psef sur une variété kählerienne compacte X de dimension n. Alors le morphisme induit en cohomologie par l'inclusion $K_X \otimes L \otimes \mathcal{I}(h_{\min}) \to K_X \otimes L$, soit*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_{\min})) \to H^q(X, K_X \otimes L),$$

s'annule pour tous $q \geqslant n - \mathrm{nd}(L) + 1$. La même conclusion est valable pour toute métrique singulière à courbure semi-positive au lieu de $h_{\min}$.

Le théorème de Junyan Cao est le suivant : soit (L, h) un fibré en droites psef sur une variété kählerienne compacte X de dimension n. Alors

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0$$

pour tous $q \geqslant n - \mathrm{nd}(L, h) + 1$.

Remarquons que le résultat de Junyan Cao s'exprime en termes de la dimension numérique de la métrique singulière considérée, plutôt que de celle du fibré en droites. En général, ces notions sont différentes, comme le montre l'exemple 1.7 dans [**DPS94**] : il existe un fibré en droites nef L sur une surface réglée $X \to C$ au dessus d'une courbe elliptique, de telle sorte qu'il existe une unique métrique singulière à courbure positive sur L, donc le courant de courbure est le courant d'intégration $[\widetilde{C}]$ sur une section de $X \to C$. Cette métrique est à singularités analytiques et sa courbure est nulle sur un ouvert de Zariski, donc la dimension numérique vaut 0. Mais la construction de [**DPS94**] montre que la dimension numérique du fibré en droites L est égale à 1.

Remarquons aussi qu'en général on ne peut pas espérer l'annulation du groupe de cohomologie à valeurs $\mathcal{I}(h_{\min})$, mais seulement l'annulation de l'image dans la cohomologie à valeurs dans L. En effet, d'après le même exemple que celui donné dans le dernier paragraphe, $h^2(X, K_X \otimes L \otimes \mathcal{I}(h_{\min})) = 1$ tandis que

$h^2(X, K_X \otimes L) = 0$. En fait, la situation envisagée ici est plus facile que celle étudiée par Junyan Cao, puisque nous ne gardons pas de faisceau d'idéaux multiplicateurs dans l'image.

Le dernier résultat que nous énonçons dans cette partie est un théorème d'annulation de type Kodaira-Nakano-Akizuki ([**Wu19c**]), exprimé en termes de lieux base augmentés.

THÉORÈME 0.2.7. *Soit X une variété projective de dimension n et L un fibré en droites holomorphes nef sur X. Alors on a*

$$H^p(X, \Omega_X^q \otimes L) = 0$$

pour tout $p + q > n + \max(\dim(B_+(L)), 0)$. Ici $B_+(L)$ désigne le lieu base augmenté (ou lieu non ample) de L. Lorsque $B_+(L) = \varnothing$, on pose par convention que la dimension est -1.

0.2.3. Fibré en droite nef en dimension supérieure.

Comme on l'a rappelé dans l'historique du début, la projectivité d'une variété compacte est caratérisée par l'existence d'une classe rationnelle dans le cône de Kähler. De manière générale, il est intéressant d'étudier les cônes positifs attachés aux variétés complexes compactes et de les relier à la géométrie de ces variétés. En géométrie algébrique classique ou complexe, l'accent est mis sur deux types de cônes positifs: les cônes nef et psef, qui sont définis comme étant les cônes convexes fermés engendrés par les classes nef et les classes psef, respectivement. Le cône nef est bien entendu contenu dans le cône psef.

Remarquons qu'en géométrie algébrique, les propriétés de dualité des cônes apparaissent dans de nombreux contextes, et que les cônes fermés sont souvent plus aisés à décrire que les cône ouverts.

Les travaux de Boucksom [**Bou02a**] définissent et étudient un cône défini comme étant le "cône nef modifié", pour une variété complexe compacte arbitraire. En utilisant ce concept, Boucksom a pu montrer l'existence d'une décomposition de Zariski divisorielle pour toute classe psef (c'est-à-dire toute classe de cohomologie contenant un courant positif). Le cône modifié se trouve être compris entre les cônes nef et psef.

En nous inspirant de la définition de Boucksom, nous introduisons dans [**Wu19d**], pour toute variété complexe compacte, un concept de cône nef en codimension arbitraire ; les cônes associés aux diverses codimensions possibles fournissent une interpolation entre les cônes positifs psef et nef.

DÉFINITION 0.2.2. (Multiplicités minimales) ([**Bou02a**])
La multiplicité minimale en un point $x \in X$ d'une classe pseudo-effective $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ est définie par

$$\nu(\alpha, x) := \sup_{\varepsilon > 0} \nu(T_{\min,\varepsilon}, x)$$

où $T_{\min,\varepsilon}$ est un représentant de la classe d'équivalence des courants $T \in \alpha$ à singularités minimales tels que $T \geqslant -\varepsilon\omega$, et où $\nu(T_{\min,\varepsilon}, x)$ désigne le nombre de Lelong de $T_{\min,\varepsilon}$ en x. Lorsque Z est un sous-ensemble analytique irréductible, on définit la multiplicité minimale générique de α le long de Z par

$$\nu(\alpha, Z) := \inf\{\nu(\alpha, x), x \in Z\}.$$

DÉFINITION 0.2.3. *Soit $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ une classe psef. Nous dirons que α est nef en codimension k, si pour tout sous-ensemble analytique irréductible $Z \subset X$ de codimension au plus égal à k, on a*

$$\nu(\alpha, Z) = 0.$$

Avec cette terminologie, le cône nef est le cône nef en codimension n, où n est la dimension complexe de la variété, tandis que le cône psef est le cône nef en codimension 0, et le cône nef modifié est le cône nef en codimension 1. Dans le même article, nous montrons au moyen d'exemples explicites que ces cônes sont en général différents.

Comme application, nous obtenons la généralisation suivante ([**Wu19d**]), du cas nef au cas psef, d'un résultat voisin énoncé dans [**DP03**].

THÉORÈME 0.2.8. *Soit (X, ω) une variété kählerienne compacte de dimension n et L un fibré en droites sur X qui est nef en codimension 1. Supposons que $\langle L^2 \rangle \neq 0$ où $\langle \bullet \rangle$ est le produit positif défini dans* [**Bou02a**]. *Supposons qu'il existe un diviseur entier effectif D tel que $c_1(L) = c_1(D)$. Alors*

$$H^q(X, K_X + L) = 0$$

pour $q \geqslant n - 1$.

La preuve du théorème repose sur une récurrence sur la dimension, en utilisant le théorème 0.2.6 du chapitre précédent. Une différence par rapport au cas nef étudié dans [**DP03**] réside dans le fait que le produit positif (ou nombre d'intersection mobile) n'est plus linéaire dans le cas psef. Cependant, sous la condition que L soit nef en codimension supérieure, nous avons l'estimation suivante.

PROPOSITION 0.2.3. *Soit α une classe nef en codimension p sur une variété kählerienne compacte (X, ω). Alors pour tout $k \leqslant p$ et toute $(n-k, n-k)$-forme Θ positive fermée, on a*

$$(\alpha^k, \Theta) \geqslant \langle \alpha^k, \Theta \rangle.$$

Grâce à cette inégalité, le calcul du nombre d'intersection effectué dans [**DP03**] se trouve être toujours valide, de même que les calculs de cohomologie qui en résultent. Remarquons que le courant à singularités minimales n'est pas toujours à singularités analytiques, comme cela a été observé par Matsumura [**Mat13**] pour la classe α construite par [**Nak04**], qui est grosse et nef en codimension 1 mais non en codimension 2. Une conséquence directe de l'observation de Matsumura est que l'hypothèse supplémentaire de notre version du théorème d'annulation de Kawamata-Viehweg énoncée ci-dessus, suivant laquelle le fibré en droites est numériquement équivalent à un diviseur entier effectif, est bien nécessaire.

Dans le cas nef étudié dans [**DP03**], il se trouve que les auteurs parviennent à déduire de l'hypothèse que le fibré en droites L est nef avec $(L^2) \neq 0$ que L est bien numériquement équivalent à un diviseur entier effectif D, de sorte qu'il existe une métrique singulière positive h sur L telle que $\mathcal{I}(h) = \mathcal{O}(-D)$. Cependant, pour un fibré en droites L sur une variété kählerienne compacte (X, ω), qui est gros et nef en codimension 1 mais non nef en codimension 2 et tel que $\langle L^2 \rangle \neq 0$, le courant $\frac{i}{2\pi}\Theta(L, h_{\min})$ n'est pas associé à un diviseur entier effectif.

Une autre conséquence est un exemple (probablement déjà connu) d'une variété projective X telle que $-K_X$ soit psef, pour laquelle le morphisme d'Albanese n'est pas surjectif. Il a été démontré dans [**Cao13**], [**Pau17**] (et [**Zha06**] pour le cas projectif) que le morphisme d'Albanese d'une variété kählerienne compacte avec $-K_X$ nef est toujours surjectif. Remplacer la propriété nef par la pseudo-effectivité dans l'étude du morphisme d'Albanese semble donc être un problème non trivial. Un résultat positif partiel est celui de notre article déjà cité, affirmant que le morphisme d'Albanese d'une variété compacte Kählerienne qui a un fibré en droites anticanoniques $-K_X$ psef et satisfaisant une condition d'intégralité est encore surjectif.

THÉORÈME 0.2.9. *Soit (X, ω) une variété kählerienne compacte de dimension n telle que $-K_X$ soit psef. Supposons qu'il existe une suite $\varepsilon_\nu > 0$ telle que $\lim_{\nu \to \infty} \varepsilon_\nu = 0$ et $\mathcal{I}(h_{\varepsilon_\nu}) = \mathcal{O}_X$ pour une suite de métriques h_{ε_ν} sur $-K_X$ à singularités analytiques et telles que $i\Theta(-K_X, h_{\varepsilon_\nu}) \geqslant -\varepsilon_\nu \omega$. Alors le morphisme d'Albanese α_X est surjectif à fibres connexes. Plus précisément, le morphisme d'Albanese est une submersion en dehors d'un ensemble analytique de codimension au moins égale à 2.*

Notons que lorsque $-K_X$ est nef, l'hypothèse du théorème ci-dessus est satisfaite. La stratégie de la preuve est analogue à celle de Junyan Cao dans [**Cao13**]. On considère la filtration de Harder-Narasimhan de T_X

$$0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = T_X.$$

Le point essentiel est de prouver que les pentes de $\mathcal{E}_{i+1}/\mathcal{E}_i$ sont semi-positives. Supposons pour simplifier que tous les $\mathcal{E}_{i+1}/\mathcal{E}_i$ soient des fibrés vectoriels. D'après [**UY86**], l'équation de Hermite-Einstein admet toujours une solution pour des fibrés vectoriels stables.

En consédérant le signe des pentes, on voit que la trace de la courbure de $\mathcal{E}_{i+1}/\mathcal{E}_i$ est semi-positive, ce qui permet de construire une métrique sur T_X dont la partie négative de la courbure de Ricci peut être prise arbitrairement petite. Grâce à la technique de Bochner, on vérifie que les sections non nulles de $H^0(X, \Omega_X^1)$ ne s'annulent en aucun point, ce qui conclut la surjectivité du morphisme d'Albanese.

L'idée pour prouver la semi-positivité des pentes est la suivante. Grâce à la condition de stabilité, il suffit de prouver que les pentes de $T_X/\mathcal{E}_i$ sont semi-positives. Grosso modo, on veut utiliser une équation de Kähler-Einstein pour construire une métrique de Kähler sur le fibré tangent d'un modèle biméromorphe ayant une borne inférieure de courbure de Ricci arbitrairement petite, et prendre la métrique quotient de celle-ci sur $T_X/\mathcal{E}_i$.

Le problème est que bien que l'on puisse résoudre une équation de Kähler-Einstein singulière grâce au travail de [**BEGZ10**], la métrique quotient n'a pas toujours de sens précis. Cependant, d'après les travaux de [**CGP13**] et [**GP16**], le potentiel a un comportement connu pour une équation de Monge-Ampère à singularité conique, à la fois le long du diviseur de singularités, et on sait aussi que la solution est lisse sur l'ouvert de Zariski complémentaire. Grâce à cette dernière solution, on peut obtenir une solution lisse dans le complémentaire du lieu singulier, qui induit donc une métrique lisse sur $T_X/\mathcal{E}_i$ sur cet ouvert de Zariski. D'après ce résultat de régularité de l'équation de Kähler-Einstein appliqué sur un modèle biméromorphe de la variété où tous les diviseurs deviennent simples à croisements normaux, on conclut que la masse de $c_1(T_X/\mathcal{E}_i)$ est bornée près du lieu singulier. D'après le théorème de Skoda-El Mir, le courant de courbure quasi positif s'étend à travers le lieu singulier avec lequel on estime la pente.

0.2.4. Faisceaux reflexifs fortement pseudo-effectifs.

Une question centrale de géométrie analytique est de classifier les variétés complexes vérifiant diverses conditions. En ce qui concerne la structure des variétés projectives ayant un fibré en droites anticanonique nef, un ingrédient clé utilisé par Junyan Cao [**Cao19**] pour la preuve de l'isotrivialité du morphisme d'Albanese est la trivialité numérique de certains fibrés vectoriels.

La notion de fibré vectoriel numériquement plat peut être définie de manière purement algébrique, mais sur une variété complexe quelconque on peut observer qu'un tel fibré vectoriel est soumis à de fortes contraintes métriques quant à sa courbure. Dans [**DPS94**], Demailly, Peternell et Schneider ont prouvé qu'un fibré vectoriel numériquement plat E sur une variété kählerienne compacte X admet une filtration par des fibrés vectoriels dont le gradué est somme directe de fibrés hermitiens plats. En ce sens, la platitude métrique est le correspendant analytique de la notion algébrique de platitude numérique.

Dans les travaux [**CCM19**] et [**HIM19**], les auteurs considèrent la question suivante. Si on a (dans un sens adéquat) un fibré vectoriel pseudo-effectif sur une variété projective ayant une première classe de Chern triviale, ce fibré vectoriel est-il numériquement plat? Puisqu'un fibré vectoriel E est numériquement plat si et seulement si E et $\det(E)^{-1}$ sont nefs (ou encore, si et seulement si E et E^* sont nefs), la question revient à se demander si un tel fibré vectoriel est en fait nef.

Intuitivement, une métrique singulière à courbure semi-positive sur le fibré vectoriel E devrait induire une métrique singulière à courbure semi-positive sur le déterminant $\det(E)$. Comme la première classe Chern de E (c'est-à-dire la classe Chern de $\det(E)$) est supposée triviale, une métrique à courbure semi-positive est nécessirement plate et elle ne peut donc posséder aucune singularité. Ceci implique que toute métrique singulière à courbure semi-positive sur E est nécessairement lisse. On s'attend à ce qu'une telle propriété ait lieu de manière générale pour une variété kählerienne compacte arbitraire, puisque les propriétés mises en jeu font encore sens dans cette situation. Nous montrerons que c'est bien le cas dans le chapitre 5 :

THÉORÈME 0.2.10. *Soit E un fibré vectoriel fortement pseudo-effectif tel que $c_1(E) = 0$. Alors E est un fibré vectoriel nef.*

On peut en fait s'attendre à un certain nombre de propriétés plus générales des fibrés fortement psef impliquant le résultat précédent comme cas particulier. Si E est fortement psef, la classe de cohomologie $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ est psef et contient une métrique pas trop singulière (la définition implique grosso modo que la projection du lieu singulier sur X est contenue dans un ensemble analytique de codimension au moins 1). Ceci entraîne que les puissances extérieures pas trop élevées de la classe Chern $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ sont positives, et donc que leur images directes sous la projection $\pi : \mathbb{P}(E) \to X$ le sont aussi. En particulier, on peut espérer que la deuxième classe de Segré $\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)))^{r+1}$ soit semi-positive (c'est-à-dire, qu'elle contienne un courant positif) où r est le rang de E. Rappelons que c'est aussi la classe $c_1(E)^2 - c_2(E)$. D'après l'inégalité de Bogomolov, lorsque $c_1(E) = 0$ et que E est semi-stable, l'intégrale de $c_2(E) \wedge \omega^{n-2}$ sur X est positive pour toute forme de Kähler ω sur X, où n est la dimension de X. En comparant les deux inégalités, on conclut que $c_2(E) = 0$, donc l'inégalité de Bogomolov sera en fait une égalité.

Remarquons que pour un faisceau reflexif $\mathcal{F}$, les classes de Chern peuvent être définies comme suit. Soit $\sigma : \widehat{X} \to X$ une modification telle que $\sigma^*\mathcal{F}/\mathrm{Tors}$ soit un fibré vectoriel. Alors pour tout $i = 1, 2$, $c_i(\mathcal{F}) := \sigma_* c_i(\sigma^*\mathcal{F})$ est indépendant du choix de la modification σ. Moralement, nous espérons que les mêmes calculs que ci-dessus s'appliquent en passant à un modèle birationnel, et en prenant des images directes, que l'égalité dans l'inégalité de Bogomolov soit atteinte.

Notons le résultat important suivant de [**BS94**] : pour un faisceau réflexif polystable $\mathcal{F}$ de rang r sur une variété kählerienne compacte (X, ω) de dimension n, on a l'inégalité de Bogomolov

$$\int_X (2rc_2(\mathcal{F}) - (r-1)c_1(\mathcal{F})^2) \wedge \omega^{n-2} \geqslant 0.$$

De plus, l'égalité a lieu si et seulement si $\mathcal{F}$ est localement libre (c'est-à-dire si $\mathcal{F}$ est un fibré vectoriel), et si sa métrique Hermite-Einstein donne une connexion projectivement plate. Les notions de faisceau réflexif nef ou psef sont définies ici comme suit.

DÉFINITION 0.2.4. *Un faisceau sans torsion $\mathcal{F}$ sur une variété complexe compacte (resp. compacte kählerienne) est dit nef* (*resp. fortement psef*) *s'il existe une modification $\sigma : \tilde{X} \to X$ telle que $\sigma^*\mathcal{F}$ modulo torsion soit un fibré vectoriel nef* (*resp. fortement psef*).

Comme conséquence de ce qui précède, il est naturel d'espérer le fait plus fort suivant : un faisceau réflexif fortement psef sur une variété kählerienne compacte (X, ω) ayant une première classe de Chern triviale est en fait un fibré vectoriel nef. Au chapitre 5, nous prouvons que c'est vraiment le cas. Une difficulté de l'approche précédente réside dans la fait qu'en général un produit extérieur de courants positifs n'est pas nécessairement bien défini. Pour contourner cette difficulté, nous commençons par prouver le résultat suivant.

THÉORÈME 0.2.11. *Soit $\mathcal{F}$ un faisceau réflexif nef sur une variété kählerienne compacte (X, ω) tel que $c_1(\mathcal{F}) = 0$. Alors $\mathcal{F}$ est un fibré vectoriel nef.*

En combinant maintenant les deux théorèmes ci-dessus, on parvient alors à l'énoncé suivant.

THÉORÈME 0.2.12. *Soit $\mathcal{F}$ un faisceau fortement psef réflexif sur une variété ählerienne compacte (X, ω) avec $c_1(\mathcal{F}) = 0$. Alors $\mathcal{F}$ est un fibré vectoriel nef.*

On observe que dans l'approche ci-dessus, tous les produits extérieurs sont bien définis sans restriction sur la codimension du lieu singulier de la métrique. En d'autres termes, pour un fibré vectoriel fortement psef E, on peut trouver un courant positif représentant la classe de cohomologie $c_1(E)$ (mais ce n'est pas nécessairement le cas pour $c_2(E)$). Dans le chapitre 5, nous donnons une définition d'un fibré vectoriel psef essentiellement équivalente à la version kählerienne proposée dans [**BDPP13**].

DÉFINITION 0.2.5. *Soit (X, ω) une variété kählerienne compacte et E un fibré vectoriel holomorphe sur X. On dit que E est fortement pseudo-effectif (en abrégé, fortement psef) si le fibré en droites $\mathcal{O}_{\mathbb{P}(E)}(1)$ est pseudo-effectif sur le projectivisé $\mathbb{P}(E)$ des hyperplans de E, et si pour tout $\varepsilon > 0$, on peut trouver une métrique singulière h_ε sur $\mathcal{O}_{\mathbb{P}(E)}(1)$ ayant une courbure $i\Theta(h_\varepsilon) \geqslant -\varepsilon\pi^*\omega$ (où $\pi : \mathbb{P}(E) \to X$ est la projection naturelle), à singularités analytiques, telle que la projection $\pi(\mathrm{Sing}(h_\varepsilon))$ du lieu singulier ne recouvre pas X tout entier.*

De manière équivalente, E est fortement psef si et seulement si le fibré en droites $\mathcal{O}_{\mathbb{P}(E)}(1)$ est pseudo-effectif sur la variété projectivisée $\mathbb{P}(E)$ des hyperplans de E, et si la projection $\pi(L_{\mathrm{nnef}}(\mathcal{O}_{\mathbb{P}(E)}(1)))$ du lieu non nef de $\mathcal{O}_{\mathbb{P}(E)}(1)$ ne recouvre pas X tout entier.

Rappelons qu'une métrique hermitienne sur $\mathcal{O}_{\mathbb{P}(E)}(1)$ correspond à une métrique de Finsler dans le sens suivant ([**Kob75**], [**Dem99**]).

DÉFINITION 0.2.6. *Une métrique de Finsler (définie positive) sur un fibré vectoriel holomorphe E est une fonction homogène complexe positive*

$$\xi \to \|\xi\|_x$$

définie sur chaque fibre E_x, c'est-à-dire telle que $\|\lambda\xi\|_x = |\lambda|\|\xi\|_x$ pour chaque $\lambda \in \mathbb{C}$, et telle que $\|\xi\|_x > 0$ pour $\xi \neq 0$.

On peut montrer que les métriques de Finsler sur un fibré vectoriel fortement psef E peuvent être approximées par des métriques induites par des métriques hermitiennes sur de grandes puissances symétriques $S^m E^*$.

PROPOSITION 0.2.4. *Soit $E \to X$ un fibré vectoriel et $p : S^m E^* \to X$ la projection naturelle. Les propriétés suivantes sont équivalentes:*

(1) E est fortement psef.

(2) Il existe une suite de fonctions quasi-psh $w_m(x, \xi) = \log(|\xi|_{h_m})$ à singularités analytiques, induites par des métriques hermitiennes h_m sur $S^m E^*$, telles que le lieu des singularités se projette dans un ensemble Zariski fermé propre $Z_m \subset X$, avec

$$i\partial\overline{\partial} w_m \geqslant -m\varepsilon_m p^*\omega$$

au sens des courants et $\lim \varepsilon_m = 0$.

(3) Il existe une suite de fonctions quasi-psh $w_m(x, \xi) = \log(|\xi|_{h_m})$ à singularités analytiques, induites par des métriques hermitiennes h_m sur $S^m E^*$, telles que le lieu des singularités se projette dans un ensemble Zariski fermé propre $Z_m \subset X$, avec

$$i\Theta_{S^m E^*, h_m} \leqslant m\varepsilon_m \omega \otimes \mathrm{Id}$$

sur $X \smallsetminus Z_m$ dans le sens de Griffiths et $\lim \varepsilon_m = 0$.

Grâce à cette condition équivalente, nous pouvons montrer que certaines opérations algébriques habituelles peuvent toujours être faites pour des fibrés vectoriels fortement psefs. Par exemple, la somme directe ou le produit tensoriel des fibrés vectoriels fortement psefs est toujours fortement psef.

Comme conséquence, on peut définir des formes de Segre (ou courants de Segre) associés, c'est-à-dire des courants positifs fermés de bidegré (k, k), obtenus par image directe des puissances extérieures du courant de courbure de $\mathcal{O}_{\mathbb{P}(E)}(1)$, sous une hypothèse sur la codimension de lieu singulier.

THÉORÈME 0.2.13. *Soit E un fibré vectoriel fortement psef de rang r sur une variété kählerenne compacte (X, ω). Soit h_ε une métrique singulière sur $(\mathcal{O}_{\mathbb{P}(E)}(1)$, ayant des singularités analytiques et telle que*

$$i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) \geqslant -\varepsilon\pi^*\omega,$$

la codimension de $\pi(\mathrm{Sing}(h_\varepsilon))$ dans X étant au moins égale à k. Alors, il existe un courant positif de bidegré (k, k) représentant la classe $\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \varepsilon\pi^*\{\omega\})^{r+k-1}$. En particulier, $\det(E)$ *est un fibré en droites psef.*

Une construction similaire a été faite dans [**LRRS18**].

À la fin du chapitre, en tant qu'application géométrique, nous classifions les surfaces kähleriennes compactes et les variétés de dimension 3 ayant un fibré tangent fortement psef et une première classe de Chern triviale. Par notre théorème principal, ce sont les mêmes que les surfaces kähleriennes compactes et les variétés de dimension 3 ayant un fibré tangent nef et une première classe de Chern triviale, qui ont été classées en particulier dans [**DPS94**]. Une conséquence est que le fibré tangent d'une surface K3 kählerienne ne peut pas être fortement psef. Ce résultat généralise ceux de [**DPS94**] et de [**Nak04**] dans le cas projectif. Plus généralement, les variétés symplectiques irréductibles ou de Calabi-Yau ont des fibrés tangents et cotangents qui ne sont pas fortement psefs. Dans le cas singulier et projectif, un résultat plus fort est prouvé dans le théorème 1.6 de [**HP19**], et dans le corollaire 6.5 de [**Dru18**], pour le cas de la dimension 3. En fait, pour le cas projectif, $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ ou $\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(1)$ n'est pas un fibré en droites psef sur une variété symplectique irréductible ou de Calabi-Yau X.

Nous généralisons également au cas compact kählerien les principaux résultats de [**LOY20**] sur les fibrés tordus par des $\mathbb{Q}$-diviseurs.

0.2.5. Théorie de l'intersection et classes de Chern en cohomologie de Bott-Chern.

Il est attendu que la formule de Riemann-Roch-Grothendieck soit vérifiée pour toutes les théories de cohomologie naturelles associées aux variétés algébriques ou analytiques. En particulier, une question intéressante est de savoir si la formule de Riemann-Roch-Grothendieck est vérifiée pour la cohomologie de Bott-Chern à coefficients rationnels. Pour donner un sens précis à la formule, nous devons définir les classes de Chern associées à l'image directe d'un fibré vectoriel (et même à toutes les images directes supérieures). Lorsqu'un morphisme entre deux variétés est propre, le théorème des images directes de Grauert énonce que ces images directes sont des faisceaux cohérents. En conséquence, il serait intéressant de pouvoir construire une théorie des classes de Chern en cohomologie de Bott-Chern entière (ou au moins rationnelle), pour des faisceaux cohérents arbitraires.

Lorsque la variété est projective, cela découle d'un travail inédit de Junyan Cao dans lequel il définit d'abord les classes Chern de fibrés vectoriels pour la cohomologie de Bott-Chern à coefficients entiers. Comme on l'a expliqué dans la section précédente, le cas général des faisceaux cohérents est beaucoup plus compliqué.

Pour traiter la situation similaire de la cohomologie de Deligne rationnelle, Julien Grivaux propose dans son travail [**Gri10**] une approche générale pour définir les classes caractéristiques de Chern dans une théorie axiomatique de la cohomologie rationnelle. Ceci se fait en spécifiant que la théorie de la cohomologie doit satisfaire certains axiomes de la théorie de l'intersection.

La ligne générale de la construction est la suivante. On «force» le théorème de Grothendieck-Riemann-Roch à être valable pour une immersion fermée d'hypersurfaces lisses. Ensuite, par dévissage, on peut déduire des axiomes de la théorie de l'intersection proposés par Grivaux que le théorème de Grothendieck-Riemann-Roch est valable pour toute immersion fermée. Puisque chaque morphisme projectif peut par définition être factorisé en la composition d'une projection et d'une immersion fermée, le théorème de Grothendieck-Riemann-Roch est valable pour tout morphisme projectif, comme observé par Grothendieck. (Bien sûr, nous utilisons également les axiomes de la théorie de l'intersection pour traiter le cas d'une projection.)

En particulier, en suivant l'approche de Grivaux, nous sommes en mesure de définir les classes de Chern comme des classes de cohomologie de Bott-Chern rationnelles. En principe, l'image réciproque d'une classe de cohomologie est induit par l'image réciproque d'une forme lisse, tandis que le poussé en avance de la classe de cohomologie est mieux vu en prenant des images directes de courants. La principale difficulté est alors de contrôler le comportement des classes de cohomologie sous la composition de l'image réciproque et d'une image directe.

Plus précisément, le complexe de Bott-Chern à coefficients entiers est quasi-isomorphe à différents types de complexes, à savoir le complexe formé par le faisceau localement constant $\mathbb{Z}$ complété par un complexe de formes différentielles lisses, soit comme $\mathbb{Z}$ remplacé par un complexe de courants construit à l'aide des courants localement intégraux. Pour définir l'image réciproque ou le poussé en avance dans l'hypercohomologie du complexe de Bott-Chern entier, nous sommes alors amenés à utiliser ces différents complexes quasi-isomorphes. Lorsqu'on traite de la composition de l'image réciproque et du poussé en avance de la cohomologie, il est commode de passer à la catégorie dérivée pour montrer que les morphismes sont bien définis et commutent dans la catégorie dérivée des complexe de groupes abéliens, puis de prendre l'hypercohomologie.

Il se trouve que l'image réciproque des courants n'est pas toujours bien définie en général, bien qu'elle le soit pour des courants satsifaisant des hypothèses spéciales adéquates. Par exemple, supposons que Y, Z soient deux cycles lisses se coupant transversalement le long de W. L'images réciproque du courant $[Z]$ sous l'immersion fermée de i_Y de Y dans X est bien défini comme étant égal à $[W]$. Nous devons montrer que via certains quasi-isomorphismes, ces types particuliers de morphismes entre représentants «spéciaux» conduisent à des morphismes de cohomologie bien définis.

La principale difficulté par rapport au cas de la cohomologie de Deligne entière est que la structure multiplicative de la cohomologie de Bott-Chern entière est beaucoup plus compliquée. Nous choisissons une définition de la multiplication telle que le morphisme naturel de la cohomologie de Bott-Chern entière vers la cohomologie de Bott-Chern complexe soit un morphisme d'anneau, et pas seulement un morphisme de groupe. On remarque pour cela que la cohomologie de Bott-Chern complexe peut être représentée par des formes lisses globales. Le produit extérieur des formes lisses passe en hypercohomologie, lorsqu'on effectue une multiplication de classes de cohomologie de Bott-Chern complexes.

THÉORÈME 0.2.14. *Soit $p : X \to S$ un morphisme projectif de variétés complexes compactes et $\mathcal{F}$ un faisceau cohérent sur X. Alors, nous avons la formule de Riemann-Roch-Grothendieck dans la cohomologie de Bott-Chern rationnelle et la cohomologie de Bott-Chern complexe*

$$ch(R^\bullet p_*\mathcal{F})Td(T_S) = p_*(\mathrm{Ch}(\mathcal{F})\mathrm{Td}(T_X))$$

où $R^\bullet p_*\mathcal{F} = \sum_i R^i p_*\mathcal{F}$.

THÉORÈME 0.2.15. *Si X est compacte et si K_0X est l'anneau de Grothendieck des faisceaux cohérents sur X, on peut définir un morphisme «caractère de Chern» $\mathrm{Ch} : K_0X \to \oplus_k H^{k,k}_{BC}(X, \mathbb{Q})$ tel que*
(1) le morphisme caractère de Chern est fonctoriel pour l'image réciproque par un morphisme holomorphe.
(2) le morphisme caractère de Chern est une extension du morphisme habituel défini pour les fibrés vectoriels.
(3) Le théorème de Riemann-Roch-Grothendieck est valable pour les morphismes projectifs entre variétés compactes complexes lisses.

Grâce à la dualité entre la cohomologie de Bott-Chern complexe et la cohomologie d'Aeppli, nous montrons également que la cohomologie de plus haut degré d'une variété complexe connexe compacte peut être calculée en cohomologie de Bott-Chern entière, contrairement à ce qui se passe pour la cohomologie de Deligne.

PROPOSITION 0.2.5. *Pour une variété complexe connexe compacte X, on a une suite exacte courte*

$$0 \to H^{2n-1}(X, \mathbb{C})/H^{2n-1}(X, \mathbb{Z}) \to H^{n,n}_{BC}(X, \mathbb{Z}) \to \mathbb{Z} \to 0.$$

CHAPTER 1

Introduction and elementary definitions

1.1. Introduction

The modern language of complex geometry relies for a large part on cohomology theory, e.g. in the context of coherent sheaves. One of the earliest general results is the Kodaira embedding theorem. The original proof by Kodaira is based on the so-called Kodaira vanishing theorem: under a strict positivity assumption for the Chern curvature of a given smooth hermitian line bundle, one shows the existence of sufficiently many sections to embed the manifold into a projective space. One way to generalize the work of Kodaira is to study vanishing theorems in the context of singular positive metrics, such as the Demailly-Nadel vanishing theorem (cf. [**Nad89**], [**Dem93**]).

THEOREM 1.1. *Let (X, ω_X) be a Kähler weakly pseudo-convex manifold with a Kähler metric ω_X and let L be a line bundle on X with a singular metric h. Assume that $i\Theta_h(L) \geqslant \varepsilon\omega_X$ in the sense of currents for some $\varepsilon > 0$. Then*

$$H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(h)) = 0$$

for all $q \geqslant 1$, where $\mathcal{I}(h) = \mathcal{I}(\varphi)$ is the multiplier ideal sheaf associated to φ for the local weight φ with $h = e^{-\varphi}$.

The Demailly-Nadel vanishing theorem reflects that in the singular metric setting, the positivity of a line bundle may have strong obstruction on the cohomology group. The major part of this thesis is concerned by the implications of the hypothesis of the existence of positively curved singular metrics on the geometric structure of a manifold, or on cohomology groups with values in a vector bundle. In the last part of the thesis, we discuss the construction of Chern classes and give the Riemann-Roch-Grothendieck formula in the rational Bott-Chern cohomology (defined by Michael Schweitzer).

1.1.1. Hard Lefschetz theorem for pseudoeffective line bundles.

A fundamental tool in complex geometry is the Riemann-Roch-Hirzebruch formula. It predicts the growth of the Euler number of the tensor product of a line bundle in terms of the intersection numbers of the Chern classes of the line bundle and the tangent bundle T_X. If a given line bundle is assumed to possess a metric of strictly positive curvature (for example, if the line bundle is nef and big), the Kawamata-Viehweg vanishing theorem states that the higher degree cohomology groups with values in high tensor powers of the line bundle twisted by the canonical bundle K_X (maybe after taking the tensor product with an ad hoc multiplier ideal sheaf) are trivial. In particular, asymptotically (which means we consider sufficient high tensor powers of the line bundle), the global sections have a maximal asymptotic growth (with exponent equal to the complex dimension). In the algebraic case, we can take a generic hyperplane intersection to perform the induction on dimension.

In general, in the semi-positive case, especially when (L, h) is a pseudoeffective (psef) line bundle with multiplier ideal sheaf $\mathcal{I}(h)$, the higher degree cohomology groups of a compact Kähler manifold (X, ω) with values in $K_X \otimes L \otimes \mathcal{I}(h)$ are not necessarily trivial. This situation is studied in [**DPS01**], where the authors construct a non-trivial preimage in $H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$ of the Lefschetz morphism, i.e. the morphism induced by $\omega^q \wedge \bullet$, for any non-trivial class in $H^q(X, K_X \otimes L \otimes \mathcal{I}(h))$. When the line bundle L is chosen to be the trivial line bundle equipped with the trivial metric, this result recovers the classical hard Lefschetz theorem. In this case, it is well-known that the Lefschetz morphism is in fact an isomorphism.

THEOREM 1.2. (see [**DPS01**]). *Let (L, h) be a pseudo-effective line bundle on a compact Kähler manifold (X, ω) of dimension n, $\sqrt{-1}\Theta_{L,h} \geqslant 0$ its curvature current and $\mathcal{I}(h)$ the associated multiplier ideal sheaf. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$\Phi^q_{\omega,h} : H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h)).$$

The special case when L is nef is due to Takegoshi [**Tak97**]. An even more special case is when L is semi-positive, i.e. L possesses a smooth metric with semi-positive curvature. In that case, the multiplier ideal sheaf $\mathcal{I}(h)$ coincides with $\mathcal{O}_X$ and we get the following consequence already observed by Enoki [**Eno93**] and Mourougane [**Mou95**].

The strategy of the proof is the following. We approximate the singular metric by a sequence of metrics which are smooth outside a proper analytic set and preserve the multiplier ideal sheaf. During this process, one inevitably loses some positivity of the curvature. As observed in [**Dem82**], one can modify the Kähler metric in such a way that the complement of any proper analytic set becomes complete with respect to the modified Kähler metric. For a fixed degree q cohomology class, one can apply a Bochner type inequality (which still works for a complete Kähler manifold) to the harmonic representatives of the approximated metrics and to the approximated Kähler metrics. In this manner, one finds a sequence of preimages via the pointwise Lefschetz isomorphism. By the L^2-estimates, the preimages have a holomorphic weak limit up to taking some subsequence. The weak limit is the desired section in $H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$.

In the smooth semi-positive case, the arguments are much easier since there is no need to take an approximating sequence for a given singular metric.

In the classical case $L = \mathcal{O}_X$, one can observe that any section $u \in H^0(X, \Omega_X^{n-q})$ satisfies the additional condition $du = d_{h_0} u = 0$. This is easily seen by the Stokes formula, which implies

$$\int_X i du \wedge \overline{du} \wedge \omega^{q-1} = \int_X \{du, du\}_{h_0} \wedge \omega^{q-1} = 0,$$

where h_0 is the trivial smooth metric on $\mathcal{O}_X$.

The proof of the hard Lefschetz theorem in [**DPS01**] is obtained by constructing preimages as limits of forms given by the pointwise Lefschetz isomorphism. One then deals with a sequence of harmonic representatives of a given class in $H^q(X, K_X \otimes L \otimes \mathcal{I}(h))$, with respect to the approximated hermitian metrics h_ε, which are still singular but smooth on a Zariski open set. It is natural to ask whether the limit is harmonic with respect to the original singular metric h.

In the singular setting, the operator ∂_h is still a densely defined operator, but it is a priori not evident to find explicitly the domain of definition of the Hilbert adjoint of ∂_h. However, it is still meaningful to consider whether the limit is parallel with respect to the original singular metric h; the above "classical" calculation corresponds to the case of a trivial bundle with its trivial metric. Our first result in [**Wu18**] confirms a positive answer to this question in the general case, by providing further estimates in the approximation process of [**DPS01**].

THEOREM 1.3. *All holomorphic sections produced by the bundle valued hard Lefschetz theorem are parallel with respect to the Chern connection associated with any given singular hermitian metric h on L, possessing a semi-positive curvature current.*

The main point of the proof is to show that the covariant derivative operator is still well-defined in the singular setting, and behaves well in the approximation process.

More precisely, let φ be the local weight function of the singular metric. Then the derivative $\partial\varphi$ is a L^q_{loc} function for all $q < 2$, but not necessarily $q = 2$. This is the case for example for $\varphi = \log|z|$ on $\mathbb{C}$. The covariant derivative of a section u with respect to the singular metric h locally can be written under the form

$$\partial_h u = \partial u + \partial\varphi \wedge u.$$

If u is a holomorphic (and in particular locally bounded) section, the second term is the product of a L^q_{loc} function with a section in L^∞_{loc}, hence is a L^q_{loc} for all $q < 2$. In the approximation process, we deal with sections with values in $L^2_{\text{loc}}(e^{-\varphi})$. To show that the second term is at least well defined in L^1_{loc} with respect to the Lebesgue measure, it is enough to observe that $\partial\varphi \in L^2_{\text{loc}}(e^\varphi)$. But this is always the case for a psh function. To give an idea of what is going on in a typical case, let us just look at $\varphi = \log|z|$ on $\mathbb{C}$. Then the section u has to vanish at 0 and it is enough to observe that

$$\partial\varphi e^\varphi = \frac{dz}{z} \times |z|^2 = \bar{z} dz.$$

In the pseudoeffective case, the Lefschetz morphism is in general no longer injective as in the classical hard Lefschetz theorem. An obvious counterexample can be obtained by taking $L = mA$ where A is an ample divisor, so that $h^0(X, \Omega_X^{n-q} \otimes L) \sim Cm^n$ for m large enough, but $h^q(X, \Omega_X^n \otimes L) = 0$ if $q > 0$. However, one can show that there is a linear isomorphism between the space of parallel sections with respect to the singular metric and the corresponding higher degree cohomology groups.

THEOREM 1.4. *Let (L, h) be a psef line bundle over a compact Kähler manifold (X, ω). Then the Lefschetz morphism obtained by taking the wedge product with a power of the Kähler form induces a linear isomorphism*

$$\Phi^q_{\omega,h} : H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \cap Ker(\partial_h) \longrightarrow H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h)).$$

In particular, when the metric is semi-positive, there is a linear isomorphism

$$\Phi^q_{\omega,h} : H^0(X, \Omega_X^{n-q} \otimes L) \cap Ker(\Delta_{\partial_h}) \longrightarrow H^q(X, \Omega_X^n \otimes L).$$

Observe that the surjectivity property is a consequence of the previous theorem.

As an application, we show that each preimage actually defines a foliation on the given Kähler manifold.

THEOREM 1.5. *Assume that* $v \in H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)), q \geqslant 1$ *is a parallel section with respect to the singular metric* h (*such as any section constructed by the hard Lefschetz theorem*). *The interior product with* v *gives an* $\mathcal{O}_X$*-morphism (which is well defined on the whole of* X*)*

$$F_v : T_X \to \Omega_X^{n-q-1} \otimes L$$

$$U \mapsto \iota_U v.$$

The kernel of F_v *defines an integrable saturated coherent subsheaf of* $\mathcal{O}(T_X)$, *and thus a* (*possibly singular*) *holomorphic foliation.*

Here the concept of (possibly singular) holomorphic foliation is defined as follows: assuming X to be connected, one means that there exists an irreducible analytic set V of the total space T_X such that for any $x \in X$, $V_x := V \cap T_{X,x}$ is a complex vector space and the sheaf of sections $\mathcal{O}(V) \subset \mathcal{O}(T_X)$ is closed under the Lie bracket. It is equivalent to take a coherent analytic subsheaf $\mathcal{O}(V) \subset \mathcal{O}(T_X)$ that is closed under Lie bracket and saturated, i.e. such that $\mathcal{O}(T_X)/\mathcal{O}(V)$ is torsion free.

Let us observe that in general a section in $H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$ does not necessarily induce a singular foliation on X. In fact, our definition for the kernel of F_v defines a foliation if and only if

$$d_h v \wedge v = 0$$

which is the case when the section is parallel with respect to the singular metric. Thus, to any element in $H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h))$ is associated in a canonical way of a holomorphic foliation defined by the section produced via the hard Lefschetz theorem. A natural question is whether this foliation is always "algebraic" in the sense that the leaf space has the structure of a complex space quotient.

This is however not the case: there are concrete examples initially given by Beauville [**Bea00**] exhibiting this phenomenon; they were indicated to us by Andreas Höring. A complete calculation can be found in Section 4 of our work [**Wu18**].

Another possibility to extend or improve the hard Lefschetz theorem would be to see whether one can replace the multiplier ideal sheaf by some bigger, "less singular", ideal sheaf. Demailly, Peternell and Schneider have already shown in [**DPS01**] that one cannot entirely omit the ideal sheaf, even when L is taken to be nef, and they gave a counterexample when $L = -mK_X$ is a multiple of the anticanonical bundle. However, it might still be possible in some cases to "improve" the ideal sheaf, for instance to replace it with the limit $\lim_{\delta \to 0_+} \mathcal{I}((1-\delta)\varphi)$, which is an infinite intersection containing $\mathcal{I}(\varphi)$. When φ has analytic singularities, it may happen that one gets a strictly larger sheaf, and in general the limit need not even be a coherent sheaf:

PROPOSITION 1.1.1. There exists an example of a psh function φ such that

$$\lim_{\delta \to 0+} \mathcal{I}((1-\delta)\varphi) := \bigcap_{\delta > 0} \mathcal{I}((1-\delta)\varphi)$$

is not coherent.

1.1.2. L^2 vanishing theorems.

The singularities of a metric are reflected in particular in their associated multiplier ideal sheaves. A frequent geometric situation is that the curvature of a singular metric can "degenerate" in some directions. This leads to the concept of numerical dimension, that, loosely speaking, counts the number of "positive directions of curvature" at a generic point. One important open problem in complex geometry is the Abundance conjecture. The latter can be seen as a very broad generalisation of the results known so far on the Kodaira dimension $\kappa(X) = \kappa(K_X)$, which counts the growth of pluricanonical sections, i.e. sections of mK_X where K_X is the canonical line bundle. By definition, for any line bundle L, the Kodaira-Iitaka dimension is

$$\kappa(L) = \limsup_{m \to +\infty} \frac{\log h^0(X, mL)}{\log m}.$$

It is a consequence of Siegel's well known theorem that $\kappa(L) \in \{-\infty, 0, 1, \ldots, n\}$ where $n = \dim X$, and that it is either $-\infty$ or the maximum of the dimensions of the pluricanonical images $\Phi_{mL}(X) \subset P(H^0(X, mK_X)^*)$. The Abundance conjecture predicts that the canonical bundle K_X always achieves its maximum possible asymptotic growth as $m \to +\infty$, and that $\kappa(K_X)$ coincides with the numerical dimension (redefined further below).

CONJECTURE 1.1.1. (Generalized abundance conjecture in the Kähler case, see [**BDPP13**])
For an arbitrary compact Kähler manifold X, the Kodaira dimension should be equal to the numerical dimension :

$$\kappa(K_X) = \mathrm{nd}(c_1(K_X)).$$

A Kähler version of the definition of numerical dimension can be found in [**Dem14**] or [**Bou02a**].

DEFINITION 1.6. *(numerical dimension)*
For L a psef line bundle on a compact Kähler manifold (X, ω), one defines

$$\mathrm{nd}(L) := \max\{p \in [0,n]; \exists c > 0, \forall \varepsilon > 0, \exists h_\varepsilon, i\Theta_{L,h_\varepsilon} \geqslant -\varepsilon\omega, \text{such that} \int_{X \setminus Z_\varepsilon} (i\Theta_{L,h_\varepsilon} + \varepsilon\omega)^p \wedge \omega^{n-p} \geqslant c\}.$$

Here the metrics h_ε are supposed to have analytic singularities and Z_ε is the singular set of the metric.

When the line bundle L is nef, a simpler definition can be given:

$$\mathrm{nd}(L) = \max\{p; c_1(L)^p \neq 0\}.$$

An equivalent definition can be given in terms of the positive product defined in [**BEGZ10**]. The positive product is the real (p,p) cohomology class $\langle \alpha^p \rangle$ of the limit

$$\langle \alpha^p \rangle := \lim_{\delta \to 0} \{\langle T^p_{\min,\delta\omega} \rangle\}$$

where $T_{\min,\delta\omega}$ is the positive current with minimal singularity in the class $\alpha + \delta\{\omega\}$ and $\langle T^p_{\min,\delta\omega} \rangle$ is the non-pluripolar product. With this notion, the numerical dimension of α is defined as

$$\mathrm{nd}(\alpha) := \max\{p | \langle \alpha^p \rangle \neq 0\}$$

which is also equal to $\max\{p | \int_X \langle \alpha^p \rangle \wedge \omega^{n-p} > 0\}$.

A more intuitive definition of positive product is defined in [**BDPP13**] as follows. Assume that α is a big class on a compact Kähler manifold (X, ω). To determine the product, it is enough to know the value of the product pairing with any $(n-p, n-p)$-form, in fact it is enough to know its value with a countable dense family of forms in the space of smooth forms. Since for any $(n-p, n-p)$-form u, $u = C\omega^{n-p} - (C\omega^{n-p} - u)$ and both $C\omega^{n-p}$ and $C\omega^{n-p} - u$ are strongly positive forms on the compact manifold X if $C > 0$ is big enough, it is enough to consider only a countable dense family of strongly positive forms.

Fix a smooth closed $(n-p, n-p)$ strongly-positive form u on X. We select Kähler currents $T \in \alpha$ with analytic singularities, and a log-resolution $\mu : \tilde{X} \to X$ such that

$$\mu^* T = [E] + \beta$$

where $[E]$ is the current associated to an effective $\mathbb{R}$-divisor and β is a semi-positive form. We consider the direct image current $\mu_*(\beta \wedge \ldots \wedge \beta)$. Given two closed positive $(1,1)$ currents $T_1, T_2 \in \alpha$, we write $T_j = \theta + i\partial\bar{\partial}\varphi_j$ $(j = 1, 2)$ for some smooth form $\theta \in \alpha$. Define $T := \theta + i\partial\bar{\partial}\max(\varphi_1, \varphi_2)$. We get a current with analytic singularities that is less singular than these two currents. In this way, if we change the representative T with another current T', we may always take a simultaneous log-resolution such that $\mu^* T' = [E'] + \beta'$, and we can always assume that $E' \leqslant E$. By a calculation, we find

$$\int_{\tilde{X}} \beta' \wedge \ldots \wedge \beta' \wedge \mu^* u \geqslant \int_{\tilde{X}} \beta \wedge \ldots \wedge \beta \wedge \mu^* u.$$

It can be shown that the closed positive current $\mu_*(\beta \wedge \ldots \wedge \beta)$ is uniformly bounded in mass. For each of the integrals associated with a countable dense family of forms u, the supremum of $\int_{\tilde{X}} \beta \wedge \ldots \wedge \beta \wedge \mu^* u$ is achieved by a sequence of currents $(\mu_m)_*(\beta_m \wedge \ldots \wedge \beta_m)$ obtained as direct images by a suitable sequence of modifications $\mu_m : \tilde{X}_m \to X$ and suitable β_m's. By extracting a subsequence, we can achieve that this sequence is weakly convergent and we set

$$\langle\langle \alpha^p \rangle\rangle := \lim_{m \to +\infty} \uparrow \{(\mu_m)_*(\beta_m \wedge \ldots \wedge \beta_m)\}$$

If α is only psef, we define

$$\langle\langle \alpha^p \rangle\rangle := \lim_{\delta \downarrow 0} \downarrow \langle\langle (\alpha + \delta\{\omega\})^p \rangle\rangle.$$

One can check:

PROPOSITION 1.1.2. The two positive products defined in [**BEGZ10**] and [**BDPP13**] coincide for every psef class.

DEFINITION 1.7. *For a psef line bundle L over a compact Kähler manifold, one defines*

$$\mathrm{nd}(L) = \mathrm{nd}(c_1(L)).$$

In the special case of a projective manifold X, the above numerical dimension can be seen to coincide with the following more algebraic definition:

$$\mathrm{nd}(L) = \sup_{A \text{ ample on } X} \limsup_{m\to+\infty} \frac{\log h^0(X, mL + A)}{\log m}$$

(and one can also easily see that this definition only depends on the numerical class $c_1(L)$).

In our papers [**Wu19a**] and [**Wu19b**], we prove some new L^2 vanishing theorems in terms of the numerical dimension of a psef line bundle.

The Bogomolov vanishing theorem [**Bog**] asserts that

$$H^0(X, \Omega_X^p \otimes L^{-1}) = 0$$

for $p < \kappa(L)$. It is interesting to ask whether we can replace the Kodaira dimension $\kappa(L)$ by the numerical dimension $\mathrm{nd}(L)$. In [**Dem02**], it is proven that for any pseudo-effective line bundle L on X a compact Kähler manifold, and any nonzero holomorphic section $\theta \in H^0(X, \Omega^p \otimes L^{-1})$, where $0 \leqslant p \leqslant n = \dim X$, then θ induces a foliation in the same terms as for Theorem 1.5. The Bogomolov vanishing theorem forbids the existence of such non zero section for $p \geqslant \kappa(L)$. (By our result, the same happens for $p \geqslant \mathrm{nd}(L)$.) In [**Mou98**], the following version of the Bogomolov vanishing theorem is stated: if L is a nef line bundle over a compact Kähler manifold X, then

$$H^0(X, \Omega_X^p \otimes L^{-1}) = 0$$

for $p < \mathrm{nd}(L)$. In our work [**Wu19b**], we get a generalization from the nef case to the psef case by refining Mourougane's estimates from [**Mou98**]. A similar proof had been given in [**Bou02a**] by using a singular Monge-Ampère equation. Here, we give another proof that only requires solving "classical" Monge-Ampère equations.

THEOREM 1.8. *Let L be a psef line bundle over a compact Kähler manifold X. Then*

$$H^0(X, \Omega_X^p \otimes L^{-1}) = 0$$

for $p < \mathrm{nd}(L)$.

Inspired by the work of Junyan Cao [**Cao14**], we get the following Kawamata-Viehweg type vanishing theorem in [**Wu19a**]. The proof follows closely Cao's proof:

THEOREM 1.9. *Let L be a pseudoeffective line bundle on a n-dimensional compact Kähler manifold X. Then the morphism induced by the inclusion $K_X \otimes L \otimes \mathcal{I}(h_{\min}) \to K_X \otimes L$*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_{\min})) \to H^q(X, K_X \otimes L)$$

vanishes for every $q \geqslant n - \mathrm{nd}(L) + 1$. The same holds for any positive singular metric h instead of $h_{\min}$.

The theorem of Junyan Cao is as follows: Let (L, h) be a pseudoeffective line bundle on a compact Kähler n-dimensional manifold X. Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0$$

for every $q \geqslant n - \mathrm{nd}(L, h) + 1$.

Let us observe that the result of Junyan Cao is expressed in terms of the numerical dimension of a singular metric $\mathrm{nd}(L, h)$, which is defined as the numerical dimension of the current $\Theta_{L,h}$, instead of the numerical dimension $\mathrm{nd}(L)$ of the line bundle L itself. In general, these notions are different. A typical example is the example 1.7 in [**DPS94**]. There exists a nef line bundle L over a ruled surface $X \to C$ over an elliptic curve C, possessing a unique positively curved singular metric. In fact, the current $\Theta_{L,h}$ associated with this unique singular metric h turns out to be the current of integration $[\tilde{C}]$ over a section of $X \to C$. Since this current is zero on a Zariski open set, the numerical dimension of the singular metric is easily seen to be 0. However, the construction of [**DPS94**] shows that the numerical dimension of the line bundle is 1.

Observe also that in general one cannot hope to obtain the vanishing of the cohomology groups with values in $\mathcal{I}(h_{\min})$ instead of simply obtaining a zero image into the cohomology with values in L. In fact by the same example of the last paragraph, $h^2(X, K_X \otimes L \otimes \mathcal{I}(h_{\min})) = 1$ while $h^2(X, K_X \otimes L) = 0$. In fact, the situation we consider is easier than the one studied by Junyan Cao since we do not keep the multiplier ideal sheaf.

Our last vanishing result is a Kodaira-Nakano-Akizuki type vanishing theorem ([**Wu19c**]), stated in term of augmented base loci.

THEOREM 1.10. *Let X be an n-dimensional projective manifold and L a nef holomorphic line bundle over X. Then we have*

$$H^p(X, \Omega_X^q \otimes L) = 0$$

for any $p+q > n+\max(\dim(B_+(L)), 0)$. Here $B_+(L)$ denotes the augmented base locus(or non-ample locus) of L. When $B_+(L) = \varnothing$, we define by convention that its dimension is -1.

1.1.3. Nefness in higher codimension.

One of the reformulation of the Kodaira embedding theorem is that a compact complex manifold is projective if and only if the Kähler cone, i.e. the convex cone spanned by Kähler forms in $H^2(X, \mathbb{R})$, contains a rational point (i.e., an element in $H^2(X, \mathbb{Q})$).

As a general matter of fact, it is obviously interesting to study positive cones attached to compact complex manifolds and to relate them with the geometry of the manifold. In classical algebraic or complex geometry, the emphasis is on two types of positive cones: the nef and psef cones, which are defined to be the closed convex cones spanned by nef classes and psef classes, respectively. The nef cone is of course contained in the psef cone.

The work of Boucksom [**Bou02a**] defines and studies the so-called modified nef cone, for an arbitrary compact complex manifold. Thanks to this definition, Boucksom was able to show the existence of a divisorial Zariski decomposition for any psef class (i.e., any cohomology class containing a positive current). The modified cone just sits between the nef and psef cones.

Inspired by Boucksom's definition, we introduce in [**Wu19d**], for any compact complex manifold, the concept of a nef cone in arbitrary codimension, which is an interpolation between the above positive cones.

DEFINITION 1.11. *(Minimal multiplicities) ([**Bou02a**])*
The minimal multiplicity at $x \in X$ of the pseudo-effective class $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ is defined as

$$\nu(\alpha, x) := \sup_{\varepsilon > 0} \nu(T_{\min,\varepsilon}, x)$$

where $T_{\min,\varepsilon}$ is the minimal element $T \in \alpha$ such that $T \geqslant -\varepsilon\omega$ and $\nu(T_{\min,\varepsilon}, x)$ is the Lelong number of $T_{\min,\varepsilon}$ at x. When Z is an irreducible analytic subset, we define the generic minimal multiplicity of α along Z as

$$\nu(\alpha, Z) := \inf\{\nu(\alpha, x), x \in Z\}.$$

DEFINITION 1.12. *Let $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ be a psef class. We say α is nef in codimension k, if for any irreducible analytic subset $Z \subset X$ of codimension at most equal to k, we have*

$$\nu(\alpha, Z) = 0.$$

With this terminology, the nef cone is the nef cone in codimension n, where n is the complex dimension of the manifold, while the psef cone is the nef cone in codimension 0, and the modified nef cone is the nef cone in codimension 1. In the same paper, we show that these cones are in general different, and construct explicit examples where these cones are different.

As an application, we obtain the following generalisation from the nef case to the psef case of a similar result stated in [**DP03**] (see [**Wu19d**]).

THEOREM 1.13. *Let (X, ω) be a compact Kähler manifold of dimension n and L a line bundle on X that is nef in codimension 1. Assume that $\langle L^2 \rangle \neq 0$ where $\langle \bullet \rangle$ is the positive product defined in* [**Bou02a**]. *Assume that there exists an effective integral divisor D such that $c_1(L) = c_1(D)$. Then*

$$H^q(X, K_X + L) = 0$$

for $q \geqslant n-1$.

The proof of the above theorem is an induction on dimension, using theorem 1.8 of the previous chapter. A difference compared with the nef case treated in [**DP03**] is that we need passing from an intersection number to a positive product (or movable intersection number), which is a non linear operation. Nevertheless, under a condition of nefness in higher codimension, we get the following estimate.

LEMMA 1.14. *Let α be a nef class in codimension p on a compact Kähler manifold (X, ω), then for any $k \leqslant p$ and Θ any positive closed $(n-k, n-k)-$form we have*

$$(\alpha^k, \Theta) \geqslant \langle \alpha^k, \Theta \rangle.$$

With this inequality, the intersection number calculation in [**DP03**] is still valid and thus the cohomology calculations can be recycled.

Observe that a current with minimal singularities need not have analytic singularities for every big class α that is nef in codimension 1 but not nef in codimension 2; such an example was given by [**Nak04**], and also observed by Matsumura [**Mat13**].

As a consequence of Matsumura's observation, the assumption of our Kawamata-Viehweg vanishing theorem that the line bundle is numerically equivalent to a effective integral divisor is actually required. In the nef case considered in [**DP03**], the authors deduce from their assumption that the line bundle L is nef with $(L^2) \neq 0$ that L is numerically equivalent to an effective integral divisor D, and that there exists a positive singular metric h on L such that $\mathcal{I}(h) = \mathcal{O}(-D)$.

However, for a big line bundle L that is nef in codimension 1 but not nef in codimension 2 over an arbitrary compact Kähler manifold (X, ω), we have $\langle L^2 \rangle \neq 0$ and $\frac{i}{2\pi}\Theta(L, h_{\min})$ need not be a current associated with an effective integral divisor.

Another by-product is a (probably already known) example of a projective manifold X with $-K_X$ psef, for which the Albanese morphism is not surjective. It was proven in [**Cao13**], [**Pau17**] (and [**Zha06**] for the projective case) that the Albanese morphism of a compact Kähler manifold with $-K_X$ nef is always surjective. Thus replacing nefness by pseudoeffectivity in the study of the Albanese morphism seems to be a non-trivial problem. In the same paper, we show that the Albanese morphism of a compact Kähler manifold which has its anticanonical line bundle $-K_X$ psef and satisfying some integrability condition is still surjective.

THEOREM 1.15. *Let (X, ω) be an n-dimensional compact Kähler manifold such that $-K_X$ is psef. Assume that there exists a sequence $\varepsilon_\nu > 0$ such that $\lim_{\nu\to\infty} \varepsilon_\nu = 0$ and $\mathcal{I}(h_{\varepsilon_\nu}) = \mathcal{O}_X$ for a sequence of singular metrics with analytic singularities h_{ε_ν} on $-K_X$ such that $i\Theta(-K_X, h_{\varepsilon_\nu}) \geqslant -\varepsilon_\nu\omega$. Then the Albanese morphism α_X is surjective with connected fibres. In fact, the Albanese map is a submersion outside an analytic set of codimension bigger than 2.*

Notice that when $-K_X$ is nef, the extra multiplier ideal sheaf assumption made in the above theorem is satisfied. The condition is also satisfied when there exists a singular positive metric h on $-K_X$ such that $\mathcal{I}(h) = \mathcal{O}_X$, in which case the surjectivity of the Albanese map is shown in [**BDPP13**] and [**Pau17**] (Remark 2.3, in the projective case).

The strategy of the proof follows closely the arguments of Junyan Cao in [**Cao13**]. We consider the Harder-Narasimhan filtration of T_X

$$0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = T_X.$$

The essential point is to prove that the slopes of $\mathcal{E}_{i+1}/\mathcal{E}_i$ is positive. Assume for simplification, that all the $\mathcal{E}_{i+1}/\mathcal{E}_i$ are vector bundles. By [**UY86**], the solution of Hermitian-Einstein equations for stable vector bundles always exists. By considering the sign of the slopes, the trace of the curvature is positive on each quotient $\mathcal{E}_{i+1}/\mathcal{E}_i$. By this property, we can construct a metric on T_X whose Ricci curvature has an arbitrarily small negative part. Then the Bochner formula shows any non zero section of $H^0(X, \Omega^1_X)$ does not vanish anywhere, and this implies the surjectivity of the Albanese morphism.

The idea to prove the positivity of the slopes is the following. By the stability condition, it is enough to prove that the slopes of $T_X/\mathcal{E}_i$ is positive. Grosso modo, we want to construct from a Kähler-Einstein equation a Kähler metric on T_X with arbitrary small Ricci curvature lower bound. Such a metric will induce a quotient metric on $T_X/\mathcal{E}_i$. The problem is that although we can solve a singular Kähler-Einstein equation by the work of [**BEGZ10**], the quotient metric need not have a precise meaning. However, by the work of [**CGP13**] and [**GP16**], we know the regularity and the behaviour of solutions for a Monge-Ampère equation with conic singularity along a divisor. In that case, the solution is known in particular to be smooth on a Zariski open set.

By taking the solution of a singular Monge-Ampère equation over some bimeromorphic model, we can obtain a solution that is smooth outside the singular set and that induces a smooth metric on $T_X/\mathcal{E}_i$ outside on that same singular set. We show by the regularity result for the Kähler-Einstein equation on a birational model of the manifold (on which all the divisors are simple normal crossing) that the mass of the curvature of the induced metric on the pull back of $\det(T_X/\mathcal{E}_i)$ with respect to the solution of the Kähler-Einstein equation is bounded near the singular set. By the Skoka-El Mir theorem, the quasi-positive curvature current extends across the singular set on the chosen bimeromorphic model. In this manner, we can obtain the required slope estimate for the extended current.

1.1.4. Pseudo-effective reflexive sheaves.

A central question of geometry is to obtain a classification of complex manifolds satisfying various natural positivity or negativity conditions. In order to elucidate the structure of a projective variety with nef anticanonical line bundle, a key ingredient is the proof by Junyan Cao [**Cao19**] of the isotriviality of the Albanese morphism, which is based in turn on the numerical flatness of some related vector bundles.

In fact, the numerical flatness property of a vector bundle is a completely algebraic concept that brings in analytic terms a strong obstruction for the curvature of any psef metric. In [**DPS94**], Demailly, Peternell and Schneider proved that a numerically flat bundle E on the compact Kähler manifold X admits a filtration by vector bundles whose graded pieces are hermitian flat. In some sense, numerical flatness is the algebraic counterpart of the concept of metric flatness.

In the work of [**CCM19**] and [**HIM19**], the authors consider the following question. If one has a pseudo-effective vector bundle over a projective manifold with vanishing first Chern class, is this vector bundle necessarily numerically flat? An easy reformulation of the definition is that a vector bundle E is numerically flat if and only if both E and $\det(E)^{-1}$ are nef. As a consequence, the above question amounts to ask whether the given pseudo-effective vector bundle with vanishing first Chern class is in fact nef.

Intuitively, a positive singular metric on the vector bundle E would induce a positive singular metric on the determinant $\det(E)$. But since the first Chern class of E (i.e. the Chern class of $\det(E)$) is trivial, one checks that it cannot support any singularity anywhere. Therefore the given positively curved singular metric has to be smooth.

From this point of view, the same should hold on an arbitrary compact Kähler manifold, and not only on a projective manifold, since all hypotheses and conclusions are independent of the projectivity condition and still make sense in the Kähler situation. In Chapter 5 ([**Wu20**]), we show that this is the case. Namely we prove that

Theorem 1.16. *Let E be a strongly psef vector bundle over a compact Kähler manifold (X, ω) with $c_1(E) = 0$. Then E is a nef vector bundle.*

In reality, one can expect something even stronger. Since E is strongly psef, the class $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ is psef. Intuitively, $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ contains a current that is not too singular (this means that the projection of the singular part onto X is contained in some analytic set of codimension at least 1). Thus the wedge power of the first Chern class to a not so high exponent is well defined and positive, and so is its direct image under $\pi : \mathbb{P}(E) \to X$. In particular, if r is the rank of E, one can hope that the second Segre class $\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)))^{r+1}$ is positive (in the sense that its cohomology class contains a positive current).

Remind that the second Segre class it is also the class $c_1(E)^2 - c_2(E)$. By the Bogomolov inequality on any Kähler n-fold, when $c_1(E) = 0$, the integration of $c_2(E) \wedge \omega^{n-2}$ on X is positive for every Kähler form ω on X. By comparing these two facts, one concludes that $c_2(E) = 0$ and that the Bogomolov inequality is in fact an equality.

We observe that for a reflexive sheaf $\mathcal{F}$, its Chern class can be defined as follows. Let σ be any modification such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a vector bundle. Then for any $i = 1, 2$, $c_i(\mathcal{F}) = \sigma_* c_i(\sigma^*\mathcal{F}/\mathrm{Tors})$ which is independent of the choice of modification σ. Morally, we hope that the same calculations hold on some birational model. By taking direct images, the equality in the Bogomolov inequality is attained.

On the other hand, we have the following important result of Bando-Siu [**BS94**]. For a poly-stable reflexive sheaf $\mathcal{F}$ of generic rank r over a compact n-dimensional Kähler manifold (X, ω), we have the following Bogomolov inequality:

$$\int_X (2rc_2(\mathcal{F}) - (r-1)c_1(\mathcal{F})^2) \wedge \omega^{n-2} \geqslant 0.$$

Moreover, the equality holds if and only if $\mathcal{F}$ is local free and its Hermitian-Einstein metric gives a projective flat connection.

We will define a nef (or strongly psef) torsion free coherent sheaf as follows.

Definition 1.17. *A torsion free coherent sheaf $\mathcal{F}$ over a compact complex manifold (resp. compact Kähler manifold) is called nef (resp. strongly psef) if there exists some modification $\sigma : \tilde{X} \to X$ such that $\sigma^*\mathcal{F}$ modulo torsion is a nef (resp. strongly psef) vector bundle.*

In conclusion, we hope the stronger fact that a strongly psef reflexive sheaf over a compact Kähler manifold (X, ω) with trivial first Chern class is in fact a nef vector bundle.

In Chapter 5, we prove that this is again actually the case. A difficulty of the above approach is that in general a wedge product of positive currents is not necessarily well defined. Instead of proving our contention directly, we first prove the following result.

Theorem 1.18. *Let $\mathcal{F}$ be a nef reflexive sheaf over a compact Kähler manifold (X, ω) with $c_1(\mathcal{F}) = 0$. Then $\mathcal{F}$ is a nef vector bundle.*

Now combining the above two theorems, we conclude

Theorem 1.19. *Let $\mathcal{F}$ be a strongly psef reflexive sheaf over a compact Kähler manifold (X, ω) with $c_1(\mathcal{F}) = 0$. Then $\mathcal{F}$ is a nef vector bundle.*

Observe that in the above approach, the wedge products involved are well defined without having to make a restriction on the codimension of the singular set of the metric. In other words, we can then find a positive current in $c_1(E)$ for any psef vector bundle E, but this will not be necessarily the case for $c_2(E)$.

In the chapter, we give a definition of strongly psef vector bundles in the Kähler situation that is essentially equivalent to the one proposed in [**BDPP13**].

DEFINITION 1.20. *Let (X,ω) be a compact Kähler manifold and E a holomorphic vector bundle on X. Then E is said to be strongly pseudo-effective (by short, strongly psef) if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective on the projectivized bundle $\mathbb{P}(E)$ of hyperplanes of E, and if the projection $\pi(\mathrm{Sing}(h_\varepsilon))$ of the singular part of some singular metric with analytic singularities on $(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)$ with curvature $i\Theta(h_\varepsilon) \geqslant -\varepsilon\pi^*\omega$ does not cover all of X for any $\varepsilon > 0$ where $\pi : \mathbb{P}(E) \to X$.*

Equivalently, E is strongly psef if and only if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective on the projectivized bundle $\mathbb{P}(E)$, and if the projection $\pi(L_{nnef}(\mathcal{O}_{\mathbb{P}(E)}(1)))$ of the non-nef locus of $\mathcal{O}_{\mathbb{P}(E)}(1)$ onto X does not cover all of X.

Remind that a Hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ corresponds to a Finsler metric in the following sense ([**Kob75**], [**Dem99**]).

DEFINITION 1.21. *A (positive definite) Finsler metric on a holomorphic vector bundle E is a positive complex homogeneous function*

$$\xi \to \|\xi\|_x$$

defined on each fibre E_x, that is, such that $\|\lambda\xi\|_x = |\lambda|\|\xi\|_x$ for each $\lambda \in \mathbb{C}$ and $\xi \in E_x$, and $\|\xi\|_x > 0$ for $\xi \neq 0$.

It is shown in [**Wu20**] that a Finsler metric with positive curvature current on a strongly psef vector bundle E can be approximated and induced in the limit by a sequence of Hermitian metrics on large symmetric powers $S^m E^*$.

PROPOSITION 1.1.3. *The following properties are equivalent:*

(1) E is strongly psef

(2) There exists a sequence of quasi-psh functions $w_m(x,\xi) = \log(|\xi|_{h_m})$ with analytic singularity induced from hermitian metrics h_m on $S^m E^*$ such that the singularity locus projects into a proper Zariski closed set Z_m in X, and

$$i\partial\bar{\partial} w_m \geqslant -m\varepsilon_m p^*\omega$$

in the sense of current with $\lim \varepsilon_m = 0$. Here $p : S^m E^* \to X$ is the projection.

(3) There exists a sequence of quasi-psh functions $w_m(x,\xi) = \log(|\xi|_{h_m})$ with analytic singularities induced from hermitian metrics h_m on $S^m E^*$ such that the singularity locus projects into a proper Zariski closed set Z_m of X, and

$$i\Theta_{S^m E^*, h_m} \leqslant m\varepsilon_m \omega \otimes Id$$

on $X \backslash Z_m$ in the sense of Griffiths with $\lim \varepsilon_m = 0$.

By the equivalence of the above conditions, one can show that the psef property is preserved by a number of usual algebraic operations. For example, a direct sum or tensor product of strongly psef vector bundles is still strongly psef.

As consequence, we can define Segre forms (or Segre currents) i.e. a $(k,k)-$ closed positive current defined as the direct image of the wedge product of a curvature current of $\mathcal{O}_{\mathbb{P}(E)}(1)$, under a suitable codimension condition on the singular locus.

THEOREM 1.22. *Let E be a strongly psef vector bundle of rank r over a compact Kähler manifold (X,ω). Let $(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)$ be singular metric with analytic singularities such that*

$$i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) \geqslant -\varepsilon\pi^*\omega$$

and the codimension of $\pi(\mathrm{Sing}(h_\varepsilon))$ is at least k in X. Then there exists a $(k,k)-$positive current in the class $\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \varepsilon\pi^*\{\omega\})^{r+k-1}$.

In particular, $\det(E)$ *is a psef line bundle.*

A similar construction has been done in [**LRRS18**].

At the end of the chapter, as a geometric application, we classify compact Kähler surfaces and 3-folds with strongly psef tangent bundles and with vanishing first Chern class. By our Main theorem, they are the same as compact Kähler surfaces or 3-folds with nef tangent bundles and with zero first Chern class, that were classified in [**DPS94**]. As a consequence, the tangent bundle of a Kähler K3 surface is not strongly psef. This generalises the work of [**DPS94**] and [**Nak04**] from the projective setting. More generally, an irreducible symplectic or Calabi-Yau manifold does not have strongly psef tangenet bundle or cotangent bundle. In the singular and projective setting, a stronger result has been proven in Theorem 1.6 of [**HP19**], and in the case of threefolds, in Corollary 6.5 of [**Dru18**]. (They prove that in this case $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not a psef line bundle where E is the tangent bundle or the cotangent bundle.)

In the compact Kähler setting, we also generalise our main results to the $\mathbb{Q}$-twisted case considered in [**LOY20**].

1.1.5. Intersection theory and Chern classes in Bott-Chern cohomology.

Important cohomology invariants of complex manifolds are provided by their Chern classes. In complex geometry, Chern classes can be defined in various cohomology theories: singular cohomology, De Rham cohomology, Dolbeault cohomology, Deligne cohomology, complex Bott-Chern cohomology, etc. By the work of [**Sch07**], there exists a more precise cohomology theory than all the above cohomology theories, namely integral Bott-Chern cohomology, in the sense that there exists a natural morphism from integral Bott-Chern cohomology into all other cohomology theories.

It is proven that the Riemann-Roch-Grothendieck formula is verified for all the above cohomology theories. A natural question is thus the Riemann-Roch-Grothendieck formula is verified for rational Bott-Chern cohomology. To give a precise meaning to the formula, we have to define the Chern classes of the direct image of a vector bundle (even all higher degree direct images). When the map between two manifolds is proper, by Grauert direct image theorem, the direct image along with higher degree direct images of a vector bundle is coherent. As a consequence, it would be interesting to be able to build a theory of Chern classes in the integral Bott-Chern cohomology for arbitrary coherent sheaves.

When the manifold is projective, this follows from an unpublished work of Junyan Cao in which he defines the Chern classes of vector bundles in the integral Bott-Chern cohomology. Since any coherent bundle can be resolved by a finite sequence of vector bundles on a projective manifold, we can as well define Chern classes for coherent sheaves via such resolutions. However, according to a striking result of Voisin [**Voi02a**], for an arbitrary compact complex manifold (even assumed to be Kähler), the resolution of a coherent sheaf by vector bundles does not necessarily exist. It follows that the definition of Chern classes of coherent sheaves on compact complex manifolds is much more involved.

To treat the similar situation for the rational Deligne cohomology, in the work [**Gri10**], Julien Grivaux proposes more generally an approach to define the Chern characteristic classes in a rational axiomatic cohomology theory. This has been done by specifying that the cohomology theory must satisfy some intersection theory axioms.

The general line of the construction is as follows. One "forces" the Grothendieck–Riemann–Roch theorem to be valid for a closed immersion of smooth hypersurfaces. Then by "devissage", one can derive from the intersection theory axioms that the Grothendieck–Riemann–Roch theorem is valid for any closed immersion. Since every projective morphism is by definition factorising into the composition of a projection and a closed immersion, the Grothendieck–Riemann–Roch theorem is valid for any projective morphism, as observed by Grothendieck. (Of course, we also use the axioms of the intersection theory to treat the projection case.)

In particular, following the approach of Grivaux, we are able in [**Wu19e**] to define Chern classes as rational Bott-Chern cohomology classes. In principle, the pull back of a cohomology class is induced by the pull-back of a smooth form, while the push-forward of cohomology class is better seen by pushing forward currents. The main difficulty is then to control the behaviour of cohomology classes under the composition of pull-back and push-forward. More precisely, the integral Bott-Chern complex is quasi-isomorphic to different types of complexes. To define pull-back or push-forward for the hypercohomology (the integral Bott-Chern cohomology), we have to use different quasi-isomorphic complexes. When we deal with the effect of taking pull-backs and push-forwards in cohomology, we pass to the derived category to show that the morphisms are still well-defined and that they commute in the derived category of complexes of abelian groups, after passing to hypercohomology.

In certain situations, the pull-back of currents can still exist, although it is not always well-defined in general. For example, let Y, Z be two smooth cycles intersecting transversally along W. The pull-back of the current $[Z]$ under the closed immersion i_Y is well defined as $[W]$. We have to show that via some quasi-isomorphisms, these special types of morphisms between special representatives lead to well defined cohomology morphisms.

The main difficulty compared to the integral Deligne case is that the multiplication structure of the integral Bott-Chern cohomology is much more complicated. We choose this multiplication definition such that the natural morphism from the integral Bott-Chern cohomology to the complex Bott-Chern cohomology is a ring morphism not only a group morphism. Remark that the complex Bott-Chern cohomology can be represented by global smooth forms. The wedge product of smooth forms pass to hypercohomology the multiplication of the complex Bott-Chern cohomology.

THEOREM 1.23. *Let $p : X \to S$ be a projective morphism of compact complex manifolds and $\mathcal{F}$ a coherent sheaf over X. Then we have the Riemann-Roch-Grothendieck formula in the rational and complex Bott-Chern cohomology*

$$\mathrm{ch}(R^\bullet p_* \mathcal{F})\,\mathrm{td}(T_S) = p_*(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(T_X))$$

where $R^\bullet p_*\mathcal{F} = \sum_i R^i p_*\mathcal{F}$.

THEOREM 1.24. *If* X *is compact and* K_0X *is the Grothendieck ring of coherent sheaves on* X*, one can define a Chern character morphism* $\mathrm{ch} : K_0X \to \bigoplus_k H^{k,k}_{BC}(X, \mathbb{Q})$ *such that*
(1) the Chern character morphism is functorial by pull back of holomorphic maps;
(2) the Chern character morphism is an extension of the usual Chern character morphism for vector bundles;
(3) the Riemann–Roch-Grothendieck theorem holds for projective morphisms between smooth complex compact manifolds.

Thanks to the duality between complex Bott-Chern cohomology and Aeppli cohomology, we also show that the top degree cohomology of a compact connected complex manifold can be calculated in integral Bott-Chern cohomology, unlike what happens for Deligne cohomology.

PROPOSITION 1.25. *For a compact connected complex manifold* X*, we have a short exact sequence*

$$0 \to H^{2n-1}(X, \mathbb{C})/H^{2n-1}(X, \mathbb{Z}) \to H^{n,n}_{BC}(X, \mathbb{Z}) \to \mathbb{Z} \to 0.$$

1.2. Elementary definitions and results

In this section, we recall some elementary definitions and fix the notations which will appear in all the thesis. In all the thesis, without specifying we assume the manifold to be compact complex. For more details, we refer to the books Analytic Methods in Algebraic Geometry [**Dem12a**] and Complex analytic and differential geometry [**Dem12b**].

We start by recalling the definition of positive currents and of plurisubharmonic / quasi-plurisubharmonic functions (psh / quasi-psh for brevity).

DEFINITION 1.26. (Positive currents)

*According to [**Lel57**], a current* Θ *of bidimension* (p, p) *is said to be (weakly) positive if for every choice of smooth* $(1, 0)-$ *forms* $\alpha_1, \cdots, \alpha_p$ *on* X*, the distribution*

$$\Theta \wedge i\alpha_1 \wedge \overline{\alpha_1} \wedge \cdots \wedge i\alpha_p \wedge \overline{\alpha_p}$$

is a positive measure.

For any $(1, 1)-$*current* T *and any smooth* $(1, 1)-$*form* α*, we say* $T \geqslant \alpha$ *in the sense of currents if* $T - \alpha$ *is a positive current.*

DEFINITION 1.27. (Psh / quasi-psh functions)

Let X *be a complex manifold (not necessary compact). We say that* φ *is a psh function (resp. a quasi-psh function) on* X*, if* $i\partial\overline{\partial}\varphi \geqslant 0$*,(resp.* $i\partial\overline{\partial}\varphi \geqslant \alpha$*) in the sense of currents where* α *is some smooth form on* X*.*

We say that a quasi-psh function φ *has analytic singularities, if locally* φ *is of the form*

$$\varphi(z) = c\log(\sum_i |g_i|^2) + O(1)$$

with $c > 0$ *and* (g_i) *some local holomorphic functions. Here* $O(1)$ *means a locally bounded term.*

An important example of closed positive current is the current associated to an effective cycle due to Lelong [**Lel57**]. Every closed analytic set $A \subset X$ of pure dimension p is associated a current of integration $[A]$ defined as follows:

$$\langle [A], \alpha \rangle = \int_{A_{reg}} \alpha, \alpha \in \mathcal{D}_{p,p}(X),$$

obtained by integrating over the regular points of A.

To show that the current $[A]$ is closed and to extend a current across an analytic set, we have the following fundamental theorem.

THEOREM 1.28. (Skoda [**Sko82**], El Mir [**EM84**], Sibony [**Sib85**])

Let E *be a closed complete pluripolar set in* X *(i.e. there is an open covering* (Ω_j) *of* X *and psh functions* u_j *on* Ω_j *with* $E \cap \Omega_j = u_j^{-1}(-\infty)$*), and let* Θ *be a closed positive current on* $X \backslash E$ *such that the coefficients* $\Theta_{I,J}$ *of* Θ *are measures with locally finite mass near* E*. Then the trivial extension* $\tilde{\Theta}$ *obtained by extending the measures* $\Theta_{I,J}$ *by 0 on* E *is still a closed positive current on* X*.*

Let us observe that Lelong's result asserting that $d[A] = 0$ for any (closed) analytic set A can be obtained by applying the Skoda-El Mir theorem to $\Theta = [A_{reg}]$ on $X \backslash A_{sing}$.

Another important property of closed positive currents is the following support theorem (see e.g. Demailly [**Dem12b**] Chap. III (2.10)). Recall that a support of a current is the complement of the maximal open set on which the restriction of the current is 0.

THEOREM 1.29. *Let Θ be a current of degree q on a real manifold M, such that both Θ and $d\Theta$ have measure coefficients (i.e. are normal currents). Suppose that the support Θ is contained in a real submanifold A with $codim_{\mathbb{R}} A > q$. Then $\Theta = 0$.*

Let A be a complex analytic subset of X a complex manifold with global irreducible components A_j of pure dimension p. Then any closed current $\Theta \in \mathcal{D}'^{p,p}(X)$ of order 0 with support in A is of the form

$$\Theta = \sum \lambda_j [A_j]$$

where $\lambda_j \in \mathbb{C}$. Moreover,if Θ is positive, then all coefficients λ_j are $\geqslant 0$.

An important application of the support theorem is the Lelong-Poincaré formula.

Let $f \in H^0(X, \mathcal{O}_X)$ be a non zero holomorphic function, $Z_f = \sum m_j Z_j, m_j \in \mathbb{N}$, the zero divisor of f and $[Z_f] = \sum m_j [Z_j]$ the current associated to the zero divisor. Then

$$\frac{i}{\pi}\partial\overline{\partial}\log|f| = [Z_f].$$

An important measure of singularity is the Lelong number introduced by Lelong [**Lel57**]. Let Θ be a closed positive current of bidimension (p,p) on a coordinate open set $\Omega \subset \mathbb{C}^n$. The Lelong number of Θ at a point $x \in \Omega$ is defined to be the limit

$$\nu(\Theta, x) = \lim_{r \to 0+} \nu(\Theta, x, r)$$

with $\nu(\Theta, x, r) = \frac{1}{r^{2p}} \int_{B(x,r)} \Theta(z) \wedge (\frac{i}{2\pi}\partial\overline{\partial}|z|^2)^p$.

A few basic properties of Lelong number are summarised below.

THEOREM 1.30.
*(1) ([**Lel57**]) For every positive current Θ, the ratio $\nu(\Theta, x, r)$ is a non-negative increasing function of r, in particular the limit $\nu(\Theta, x)$ as $r \to 0+$ always exists.*
*(2) ([**Lel57**]) If $\Theta = i\partial\overline{\partial}\varphi$ is the bidegree (1,1)-current associated with a psh function φ, then*

$$\nu(\Theta, x) = \nu(\varphi, x) = \sup\{\gamma > 0; \varphi(z) \leqslant \gamma\log|z - x| + O(1) \text{ at } x\}.$$

*(3) ([**Siu74**]) For every $c > 0$, the set $E_c(\Theta) := \{x \in X; \nu(\Theta, x) > c\}$ is a closed analytic subset of X of dimension at most p.*

A related notion is the concept of multiplier ideal sheaf.

DEFINITION 1.31. (Multiplier ideal sheaf). *Let φ be a quasi-psh function. The multiplier ideal sheaves $\mathcal{I}(\varphi)$ is defined as*

$$\mathcal{I}(\varphi)_x = \{f \in \mathcal{O}_{X,x} | \exists U_x, \int_{U_x} |f|^2 e^{-2\varphi} < \infty\}$$

where U_x is some open neighbourhood of x in X.

A basic property of the multiplier ideal sheaf due to [**Nad89**] is that it is always a coherent ideal sheaf.

Now we recall what are the main concepts of positive cones in complex geometry. In general, we work in the complex Bott-Chern cohomology, which is define as follows:

$$H^{p,q}_{BC}(X, \mathbb{C}) = \{d\text{-closed } (p,q)\text{-forms}\}/\{\partial\overline{\partial}\text{-exact } (p,q)\text{-forms}\}.$$

DEFINITION 1.32. (Psef line bundles)

Let L be a holomorphic line bundle on a compact complex manifold X. L is pseudo-effective (by short, psef) if $c_1(L) \in H^{1,1}_{BC}(X, \mathbb{C})$ is the cohomology class of some closed positive current T, i.e. if L can be equipped with a singular Hermitian metric h (which means the local weight function is L^1_{loc}) with $T = \frac{i}{2\pi}\Theta_{L,h} \geqslant 0$ as a current.

A cohomology class $\alpha \in H^{1,1}_{BC}(X, \mathbb{C})$ is said to be psef if it contains some positive current. A cohomology class $\alpha \in H^{k,k}_{BC}(X, \mathbb{C})$ for some $k \in \mathbb{N}$ is said to be positive if it contains some strongly positive current in the sense of Lelong. For a $(1,1)-$class, a class is psef if and only if it is positive.

Currents with minimal singularities in a given psef class are defined below.

DEFINITION 1.33. (See [**DPS01**]). *Let φ_1, φ_2 be two quasi-psh functions on X (i.e. $i\partial\overline{\partial}\varphi_i \geqslant -C\omega$ in the sense of currents for some $C \geqslant 0$). Then, φ_1 is said to be less singular than φ_2 (and we write $\varphi_1 \preceq \varphi_2$) if we have $\varphi_2 \leqslant \varphi_1 + C_1$ for some constant C_1. Let α be a psef class in $H^{1,1}_{BC}(X, \mathbb{R})$, and γ be a smooth real $(1,1)-$form. Let $T_1, T_2, \theta \in \alpha$ with θ smooth and $T_i = \theta + i\partial\overline{\partial}\varphi_i (i = 1, 2)$. The potential φ_i is well defined up to an additive constant since X is compact. We say that $T_1 \preceq T_2$ if $\varphi_1 \preceq \varphi_2$.*

A minimal element $T_{\min,\gamma}$ with respect to the pre-order relation $\preceq$ can be shown to exist by taking the upper semi-continuous regularization of all φ_i such that $\theta + i\partial\overline{\partial}\varphi_i \geqslant \gamma$ and $\sup_X \varphi_i = 0$.

Another important cone is the nef cone. The following definition has been introduced in [**DPS94**] in the non necessarily algebraic case.

Definition 1.34. (Nef line bundles)
A line bundle L on a compact complex manifold X is said to be nef if for every $\varepsilon > 0$, there is a smooth Hermitian metric h_ε on L such that $i\Theta_{L,h_\varepsilon} \geqslant -\varepsilon\omega$ where ω is some smooth Hermitian metric.

A cohomology class $\alpha \in H^{1,1}_{BC}(X, \mathbb{C})$ is said to be nef if for every $\varepsilon > 0$, there is a smooth element $\alpha_\varepsilon \in \alpha$ such that $\alpha_\varepsilon \geqslant -\varepsilon\omega$ where ω is some smooth Hermitian metric.

By definition, the nef cone is contained in the psef cone. A basic measure for a psef class to be nef is the non-nef locus introduced in [**Bou04**].

Definition 1.35. (Non-nef locus)
The non-nef locus of a pseudo-effective class $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ is defined by

$$E_{nn}(\alpha) := \bigcup_{\varepsilon>0} \bigcup_{c>0} E_c(T_{\min,-\varepsilon\omega})$$

where ω is any Hermitian metric.

The notion of nefness can be generalized to the vector bundle case (cf. [**DPS94**]).

Definition 1.36. *A vector bundle E is said to be numerically effective (nef) if the canonical bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E)$, the projective bundle of hyperplanes in the fibres of E.*

A holomorphic vector bundle E over X is said to be numerically flat if both E and E^* are nef (or equivalently if and only if E and $(\det E)^{-1}$ are nef).

Finally, we recall the following regularization theorem due to Demailly.

Definition 1.37. *A $(1,1)$-current T is said to be quasi-positive if $T \geqslant \alpha$ where α is a smooth form, in other words if T is positive modulo smooth forms. (In particular, according to the definitions, a function φ is quasi-psh ifff $i\partial\bar{\partial}\varphi$ is quasi-positive).*

Theorem 1.38. *Let T be a quasi-positive closed $(1,1)$-current on a compact complex manifold X of dimension n such that $T \geqslant \gamma$ for some continuous $(1,1)$-form γ. Then there exists a sequence of currents T_m whose local potentials have the form*

$$\frac{1}{m}\log(\sum_i |g_{i,m}|^2) + O(1)$$

with $O(1)$ a locally bounded term and $(g_{i,m})$ some local holomorphic functions, and a decreasing sequence $\varepsilon_m > 0$ converging to 0 such that

(1) T_m converges weakly to T;
(2) $\nu(T,x) - \frac{n}{m} \leqslant \nu(T_m, x) \leqslant \nu(T,x)$ for every $x \in X$;
(3) $T_m \geqslant \gamma - \varepsilon_m\omega$ in the sense of currents.

CHAPTER 2

On the hard Lefschetz theorem for pseudoeffective line bundles

Abstract. In this note, we obtain a number of results related to the hard Lefschetz theorem for pseudoeffective line bundles, due to Demailly, Peternell and Schneider. Our first result states that the holomorphic sections produced by the theorem are in fact parallel, when viewed as currents with respect to the singular Chern connection associated with the metric. Our proof is based on a control of the covariant derivative in the approximation process used in the construction of the section. Then we show that we have an isomorphism between such parallel sections and higher degree cohomology. As an application, we show that the closedness of such sections induces a linear subspace structure on the tangent bundle. Finally, we discuss some questions related to the optimality of the hard Lefschetz theorem.

2.1. Introduction

In this note, we establish a closedness and harmonicity result that complements the hard Lefschetz theorem for pseudoeffective line bundles proved in [**DPS01**]. By following the arguments of the above paper, we show that the sections provided by the proof are in fact parallel, when viewed as currents with respect to the singular Chern connection of the metric. The first difficulty is to define the covariant derivative for such singular metrics, since in general the wedge product of two currents is not always well-defined. Another difficulty is to control the covariant derivative in the approximation process employed in the original proof.

Let X be a compact Kähler n-dimensional manifold, equipped with a Kähler metric, i.e. a positive definite Hermitian $(1,1)$-form $\omega = \mathrm{i}\sum_{1\leqslant j,k\leqslant n}\omega_{jk}(z)\,dz_j\wedge d\overline{z}_k$ such that $d\omega=0$. By definition a holomorphic line bundle L on X is said to be pseudoeffective if there exists a singular hermitian metric h on L, given by $h(z)=e^{-\varphi(z)}$ with respect to a local trivialization $L_{|U}\simeq U\times\mathbb{C}$, such that the curvature form

$$\mathrm{i}\Theta_{L,h} := \mathrm{i}\partial\overline{\partial}\varphi$$

is (semi)positive in the sense of currents, i.e. φ is locally integrable and $i\Theta_{L,h}\geqslant 0$: in other words, the weight function φ is plurisubharmonic (psh) on the corresponding trivializing open set U. In this trivialization, if the metric is in fact smooth, the (1,0) part of the covariant derivative with respect to the associated Chern connection is given in the form:

$$\partial_h = \partial + \partial\varphi\wedge\bullet,$$

and the total connection is $d_h = \partial_h + \overline{\partial}$. An important fact is that ∂_h and d_h still make sense for an arbitrary singular metric h as above. Another basic concept relative to a singular metric is the notion of *multiplier ideal sheaf*, introduced in [**Nad90**].

Definition 2.1. *To any psh function φ on an open subset U of a complex manifold X, one associates the "multiplier ideal sheaf" $\mathcal{I}(\varphi)\subset\mathcal{O}_{X|U}$ of germs of holomorphic functions $f\in\mathcal{O}_{X,x}$, $x\in U$, such that $|f|^2e^{-\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x. We also define the global multiplier ideal sheaf $\mathcal{I}(h)\subset\mathcal{O}_X$ of a hermitian metric h on $L\in\mathrm{Pic}(X)$ to be equal to $\mathcal{I}(\varphi)$ on any open subset U where $L_{|U}$ is trivial and $h=e^{-\varphi}$. In such a definition, we may in fact assume $\mathrm{i}\Theta_{L,h}\geqslant -C\omega$, i.e. locally $\varphi = psh + C^\infty$, we say in that case that φ is* quasi-psh.

The interest of considering quasi-psh functions is that on a compact manifold global psh functions are constant, while the space of quasi-psh functions is infinite dimensional. Among them, functions with analytic singularity will be of special concern for us. With this notation, the following bundle valued generalization of the hard Lefschetz theorem has been established in [**DPS01**]. The proof uses the natural L^2-resolution of the sheaf $\Omega^n_X\otimes L\otimes\mathcal{I}(h)$.

Theorem 2.2. ([**DPS01**]) *Let (L,h) be a pseudo-effective line bundle on a compact Kähler manifold (X,ω) of dimension n, let $\Theta_{L,h}\geqslant 0$ be its curvature current and $\mathcal{I}(h)$ the associated multiplier ideal sheaf. Then, the wedge multiplication operator $\omega^q\wedge\bullet$ induces a surjective morphism*

$$\Phi^q_{\omega,h} : H^0(X,\Omega^{n-q}_X\otimes L\otimes\mathcal{I}(h)) \longrightarrow H^q(X,\Omega^n_X\otimes L\otimes\mathcal{I}(h)).$$

The special case when L is nef is due to Takegoshi [**Tak97**] (for the definition of nef in the analytic setting, cf. [DPS94]). An even more special case is when L is semi-positive, i.e. L possesses a smooth metric with semi-positive curvature. In that case, the multiplier ideal sheaf $\mathcal{I}(h)$ coincides with $\mathcal{O}_X$ and we get the following consequence already observed by Enoki [**Eno93**] and Mourougane [**Mou95**].

COROLLARY 2.3. Let (L, h) be a semi-positive line bundle on a compact Kähler manifold (X, ω) of dimension n. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism

$$\Phi^q_\omega : H^0(X, \Omega_X^{n-q} \otimes L) \longrightarrow H^q(X, \Omega_X^n \otimes L).$$

It should be observed that although all objects involved in Theorem 2.2 are algebraic when X is a projective manifold, there is no known algebraic proof of the statement; it is not even clear how to define algebraically $\mathcal{I}(h)$ in the case when $h = h_{\min}$ is a metric with minimal singularity. The classical hard Lefschetz theorem is the case when L is trivial or unitary flat; then L has a (real analytic) metric h of curvature equal to 0, whence $\mathcal{I}(h) = \mathcal{O}_X$.

In the pseudoeffective case, the Lefschetz morphism is in general no longer injective as in the classical hard Lefschetz theorem. An obvious counterexample can be obtained by taking $L = mA$ where A is an ample divisor, so that $h^0(X, \Omega_X^{n-q} \otimes L) \sim Cm^n$ for m big enough, but $h^q(X, \Omega_X^n \otimes L) = 0$ if $q > 0$. We will show that to have an isomorphism, we should change the left hand side by the parallel sections with respect to the singular metric.

Notice that the proof of the hard Lefschetz theorem is given by constructing directly a pre-image for any element in $H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h))$. This is done by taking weak limit of some subsequence in a bounded family of some Hilbert space. Since for a bounded family of some Hilbert space, there exists some subsequence with a weak limit in the Hilbert space. However, there is no trivial reason that the weak limit is unique. Thus viewing the proof of the hard Lefschetz theorem as construction of an inverse operator

$$H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)),$$

a priori, this operator is not necessarily linear. Thus it is a natural question to demand whether the inverse operator is linear. More general, does there exist a sublinear space of $H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$ such that the inverse operator is an isomorphism of linear spaces?

In the classical case $L = \mathcal{O}_X$, one can observe that any section $u \in H^0(X, \Omega_X^{n-q})$ satisfies the additional condition $du = d_{h_0} u = 0$. This is easily seen by Stokes formula, which implies

$$\int_X i du \wedge \overline{du} \wedge \omega^{q-1} = \int_X \{du, du\}_{h_0} \wedge \omega^{q-1} = 0,$$

where h_0 is the trivial smooth metric on $\mathcal{O}_X$; in that formula (as well as in the rest of this paper), given a hermitian metric h, we denote by $\{u, v\}_h$ the natural sesquilinear pairing

$$\mathcal{C}^\infty(M, \wedge^p T_X^* \otimes L) \times \mathcal{C}^\infty(M, \wedge^q T_X^* \otimes L) \to \mathcal{C}^\infty(M, \wedge^{p+q} T_X^*)$$

$$(u, v) \mapsto \{u, v\}_h$$

given by

$$\{u, v\}_h = \sum_{\lambda, \mu} i u_\lambda \wedge \bar{v}_\mu \langle e_\lambda, e_\mu \rangle_h$$

where $u = \sum u_\lambda \otimes e_\lambda$, $v = \sum v_\mu \otimes e_\mu$. Another proof relies on the observation that $\overline{\partial} u = \overline{\partial}^* u = 0$ (the second equality holds since u is of bidegree $(n-q, 0)$), whence $\Delta_{\overline{\partial}} u = 0 = \Delta_\partial u$ by the Kähler identities. As a consequence, we have $\partial u = \partial^* u = 0$, and so $du = 0$.

More generally, the proof of the hard Lefschetz theorem in [**DPS01**] is obtained by constructing pre-images as limits of forms given by the pointwise Lefschetz isomorphism. One then deals with a sequence of harmonic representatives of a given class in $H^q(X, K_X \otimes L \otimes \mathcal{I}(h))$, with respect to approximated, less singular, hermitian metrics h_ε. It is thus natural to wonder whether the holomorphic sections provided by Theorem 2.2 also satisfy some sort of closedness property in the case of arbitrary pseudoeffective line bundles. In fact, we are going to prove that these sections are parallel with respect to the (possibly singular) Chern connection associated with the metric h; the proof employs similar arguments, but with the additional difficulty that one has to deal with non smooth metrics.

THEOREM 2.4. *All holomorphic sections produced by Theorem 2.2 are parallel with respect to the Chern connection associated with the singular hermitian metric h on L.*

More precisely, as h can be singular, this means that in local coordinates, any such holomorphic section $s \in H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$ satisfies

$$\partial_h s = \partial s + \partial\varphi \wedge s = 0$$

in the sense of currents. Since $\overline{\partial}s = 0$, we conclude that $d_h s = \partial_h s + \overline{\partial}s = 0$. This property can be expressed by saying that the section s is parallel with respect to d_h.

Now, let us consider the harmonicity. Assume first that the metric is semi-positive (i.e. a smooth metric with positive Chern curvature). By computing $\overline{\partial}(\partial_h s) = 0$, we get $\overline{\partial}\partial\varphi \wedge s = 0$, hence

$$i\Theta_{L,h} \wedge s = 0.$$

As $\Delta_{\overline{\partial}} s = 0$ (s is a holomorphic section and $\overline{\partial}^* s = 0$ by a bidegree consideration), the Kodaira-Nakano identity implies

$$\Delta_{\overline{\partial}} s - \Delta_{\partial_h} s = [i\Theta_{L,h}, \Lambda]s = i\Theta_{L,h}\Lambda s - \Lambda i\Theta_{L,h} s = -\Lambda i\Theta_{L,h} s = 0,$$

by the fact that $\Lambda s = 0$. Therefore $\Delta_{\partial_h} s = 0$. Since the metric is smooth, this is equivalent to the fact that $\partial_h s = 0$ and $\partial_h^* s = 0$. If the metric is singular, we still have

$$i\Theta_{L,h} \wedge s = 0$$

by the same arguments. However, in the latter case, although the operator ∂_h is still a densely defined operator on $L^2(X, \Omega_X^{n-q} \otimes L, h)$ (cf. Remark 1), it is difficult to give an explicit expression of his Hilbert adjoint ∂_h^*. There may exist the boundary condition on the domain of ∂_h^* caused by integration by parts, while the singular part of a general positive singular metric could have very difficult topology. Thus it is difficult to discuss the Hilbert adjoint ∂_h^* in general. Nevertheless, the fact that the section is parallel with respect to the singular metric is sufficient to characterize the pre-image of the wedge multiplication operator in the hard Lefschetz theorem.

THEOREM 2.5. *Let (L, h) be a pseudo-effective line bundle on a compact Kähler manifold (X, ω) of dimension n, let $\Theta_{L,h} \geqslant 0$ be its curvature current and $\mathcal{I}(h)$ the associated multiplier ideal sheaf. Then, the wedge multiplication operator $\omega^q \cdot \bullet$ induces a linear isomorphism*

$$\Phi_\omega^q : H^0(X, \Omega_X^{n-q} \otimes L) \cap \mathrm{Ker}(\partial_h) \longrightarrow H^q(X, \Omega_X^n \otimes L).$$

In section 4, as a geometric application, we use the closedness property of the holomorphic sections produced by the hard Lefschetz theorem to derive the existence of a "singular foliation" of X (in fact a linear subspace structure of T_X).

THEOREM 2.6. *Assume that $v \in H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)), q \geqslant 1$ is a parallel section with respect to the singular metric h. In particular a section constructed by the hard Lefschetz theorem is such a section. The interior product with v gives an $\mathcal{O}_X$- morphism (which is well defined throughout X)*

$$F_v : T_X \to \Omega_X^{n-q-1} \otimes L$$

$$X \mapsto \iota_X v.$$

The kernel of F_v defines an integrable coherent subsheaf of $\mathcal{O}(T_X)$, i.e. a holomorphic foliation.

At the end of section 4, we show by a concrete example indicated to the author by Professor Andreas Höring that for a general pre-image, instead of the one constructed by the hard Lefschetz theorem, the above process does not necessarily induce a foliation. In fact, the kernel of F_v defined in the theorem 2.6 defines a foliation if and only if v is a parallel section.

Finally, in the last sections of this work, we discuss the optimality of the multiplier ideal sheaf $\mathcal{I}(h) = \mathcal{I}(\varphi)$ involved in the hard Lefschetz theorem. Demailly, Peternell and Schneider already showed in [**DPS01**] that one cannot omit the ideal sheaf even when L is taken to be nef, and gave a counterexample when $L = -K_X$ is the anticanonical bundle. However, it might still be possible in some cases to "improve" the ideal sheaf, for instance to replace it with $\lim_{\delta \to 0_+} \mathcal{I}((1-\delta)\varphi) \supset \mathcal{I}(\varphi)$. When φ has analytic singularities, it may happen that the inclusion be strict, but in general the limit need not even be a coherent sheaf (see section 5). The abundance conjecture and the nefness of $L = K_X$ would imply the semiampleness of L, so in that case, the ideal sheaf is definitely not needed. For the general case, this seems to be a difficult problem. Some discussions of these issues are conducted in section 6.

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2.2. Definition of the covariant derivative

In this section, we consider a pseudoeffective line bundle (L, h) on a Kähler (non necessarily compact) manifold (Y, ω) where $h(z) = e^{-\varphi(z)}$ with respect to a local trivialization $L_{|U} \simeq U \times \mathbb{C}$ and ω is smooth. We denote by $|\ \ | = |\ \ |_{\omega,h}$ the pointwise hermitian norm on $\Lambda^{p,q}T^\star_Y \otimes L$ associated with ω and h, and by $\|\ \ \| = \|\ \ \|_{\omega,h}$ the global L^2 norm

$$\|u\|^2 = \int_Y |u|^2 dV_\omega \qquad \text{where} \quad dV_\omega = \frac{\omega^n}{n!}.$$

Recall that since φ is a quasi-psh function on U, its derivative $d\varphi$ belongs to $L^p_{\text{loc}}(U)$ with respect to Lebesgue measure for every $p < 2$ (cf. e.g. Theorem 1.48 in [**GZ17**]). This regularity is optimal since on $\mathbb{C}$, the psh function $\log|z|$ has a derivative not in $L^2_{\text{loc}}(\mathbb{C})$. We fix a smooth reference metric h_0 on L (not necessarily semi-positive) from which we can view any other singular metric as given by $h = h_0 e^{-\varphi}$ where φ is a quasi-psh function defined on Y. In general, for $u \in L^2_{\text{loc}}(U, \Lambda^{p,q}T^\star_Y \otimes L, \omega, h_0)$, $\partial\varphi \wedge u$ is not a priori well defined as a form with coefficients in $L^1_{\text{loc}}(U, \Lambda^{p+1,q}T^\star_Y \otimes L, \omega, h_0)$ (with respect to the Lebesgue measure), at least if we make a naive use of the Cauchy-Schwarz inequality to get a current on U. (Note that in this case, $\partial\varphi \in L^1_{\text{loc}}(U, \Lambda^{p+1,q}T^\star_Y \otimes L, \omega, h_0)$ *is however* a current on U.)

We can overcome this problem in our proof, because in the construction of sections in the proof of the bundle valued hard Lefschetz theorem, this type of product can always be defined. In fact we always have additional assumptions on either u or φ, as we will see next, and this will be enough to prove our main theorem. At the end of this section, we prove that the wedge product $\partial\varphi \wedge u$ is closed with respect to the L^2 topology when φ is any psh function and u is in $L^2_{\text{loc}}(e^{-\varphi})$; this will be used in the following section.

In the sequel, we will make use two types of such wedge products. The first type is when u is holomorphic, so that the coefficients of u are in fact bounded on any compact set, hence in L^∞_{loc}, thus $\partial\varphi \wedge u$ has coefficients in

$$L^1_{\text{loc}}(U, \Lambda^{p,q}T^\star_Y \otimes L, \omega, h_0) \times L^\infty_{\text{loc}}(U, \Lambda^{1,0}T^\star_Y \otimes L, \omega, h_0) \subset L^1_{\text{loc}}(U, \Lambda^{p+1,q}T^\star_Y \otimes L, \omega, h_0).$$

Moreover, if φ_i a sequence of quasi-psh functions such that $\varphi_i \to \varphi$ in $L^1_{\text{loc}}(U, \omega, h_0)$, we have $\partial\varphi_i \to \partial\varphi$ in $L^1_{\text{loc}}(U, \Lambda^{1,0}T^\star_Y \otimes L, \omega, h_0)$ hence $\partial\varphi_i \wedge u \to \partial\varphi \wedge u$ in $L^1_{\text{loc}}(U, \Lambda^{p+1,q}T^\star_Y \otimes L, \omega, h_0)$, which implies in particular the weak convergence as currents (cf. e.g. theorem 1.48 in [**GZ17**]).

The second type is when φ is an arbitrary psh function, taken as a local weight function of h, and $u \in L^2_{\text{loc}}(U, \Lambda^{p,q}T^\star_Y \otimes L, \omega, h)$.

To understand what happens, we start by the case when φ has analytic singularities, although this consideration is not necessary for the proof of general case. Suppose that φ has analytic singularities along a simple normal crossing divisor, i.e. in some coordinates,

$$\varphi = c\log|z_1^{a_1}...z_n^{a_n}| + \mathrm{C}^\infty.$$

We only need to check the current is well defined near a point in $\mathrm{Sing}(h)$, a situation which happens only in case $c > 0$. When $u \in L^2_{\text{loc}}(U, \Lambda^{p,q}T^\star_Y \otimes L, \omega, h)$, we have to show that $\partial\varphi \wedge u$ is locally integrable with respect to the Lebesgue measure, and without loss of generality, we can suppose that the section is integrable on U, and not only on every compact in U, i.e.

$$\int_U |\partial\varphi \wedge u|_{\omega,h_0} dV_\omega < \infty.$$

It is true since

$$\leqslant C(\int_U |z_1^{a_1}...z_n^{a_n}|^c| \sum \frac{a_i}{2}\frac{dz_i}{z_i}|^2_{\omega,h_0} dV_\omega)^{\frac{1}{2}} (\int_U |u|^2_{\omega,h} dV_\omega)^{\frac{1}{2}}$$

$$\leqslant C(\int_U |z_1^{a_1}...z_n^{a_n}|^c \sum \frac{a_i^2}{4|z_i|^2} \prod i dz_i \wedge \overline{dz_i})^{\frac{1}{2}} (\int_U |u|^2_{\omega,h} dV_\omega)^{\frac{1}{2}}.$$

Since U is a local coordinate chart, we can suppose U to be a poly-disc $\prod D(0, r_j)$. The integrability of the first term in the integral is given by for any j such that $a_j > 0$,

$$\int_U |z_1^{a_1}...z_n^{a_n}|^c \frac{a_j^2}{4|z_j|^2} \prod i dz_i \wedge \overline{dz_i} \leqslant C_j \int_0^{r_j} r^{2a_jc-1} < \infty$$

since $ca_j - 1 > -1$. By assumption the second term in the integral is finite, so the product is finite.

If φ has analytic singularities, there exists a modification of $\mu : \tilde{Y} \to Y$ such that $\mu^*(\mathcal{I}(h))$ is an invertible sheaf associated to a simple normal crossing divisor, thanks to Hironaka's desingularization theorem [**Hir64**]. Since we consider only local integrability of functions up to modification (by definition, a modification is a biholomorphism outside of proper analytic sets) and since analytic subsets are of Lebesgue measure zero, singularities are irrelevant with respect to integration. Therefore, we can reduce the general case by using a modification that converts the singular sets involved into simple normal crossing divisors.

For the general case where φ is arbitrary psh function. It is enough to prove that $\int_K |e^{\frac{\varphi}{2}}\partial\varphi|^2_{\omega,h_0} dV_\omega$ is finite for any compact set $K \Subset U$. After shrinking U into a smaller relatively compact open subset, we can suppose that $\varphi \leqslant C$ for some $C > 0$, and also that there exists a non increasing sequence of smooth psh functions $\varphi_{\varepsilon_\nu}$ converging to φ in $L^1(U)$ as $\varepsilon_\nu \to 0$. The smooth psh function sequence can be obtained by taking a convolution with radially symmetric approximations of the Dirac measure. The upper bound is obtained by the maximum principle. The same is true for φ_{ε_1}. In particular, $e^\varphi \in L^1(U)$. We prove that $e^\varphi \in PSH(U)$. Up to a subsequence, $e^{\varphi_{\varepsilon_\nu}} \to e^\varphi$ almost everywhere. The functions are uniformly bounded. By the dominated convergence theorem, $e^{\varphi_{\varepsilon_\nu}} \to e^\varphi$ in $L^1(U)$. Since the space of the psh functions is closed in $L^1_{\mathrm{loc}}(U)$, $e^\varphi \in PSH(U)$. Hence

$$i\partial\overline{\partial}e^\varphi = e^\varphi(i\partial\overline{\partial}\varphi + i\partial\varphi \wedge \overline{\partial}\varphi) \geqslant 0$$

as a current. For any compact set $K \subset U$, the mass of $i\partial\varphi \wedge \overline{\partial}\varphi e^\varphi \wedge \omega^{n-1}$ on K is the mass of $i\partial\overline{\partial}(e^\varphi) \wedge \omega^{n-1}$ on K minus the mass of $i\partial\overline{\partial}\varphi e^\varphi \wedge \omega^{n-1}$ on K which is finite. This means $\int_K |e^{\frac{\varphi}{2}}\partial\varphi|^2_{\omega,h_0} dV_\omega$ is finite. And it is closed with respect to the L^2 topology in the sense that considering a sequence $u_j, u \in L^2_{\mathrm{loc}}(U, \Lambda^{p,q}T^\star_Y \otimes L, \omega, h)$ such that $u_j \to u$, we have by the inequality

$$\begin{aligned}\int_U |\partial\varphi \wedge u|_{\omega,h_0} dV_\omega &= \int_U |\partial\varphi e^{\frac{\varphi}{2}}|_{\omega,h_0} |u|_{\omega,h} dV_\omega \\ &\leqslant \Big(\int_U |e^{\frac{\varphi}{2}}\partial\varphi|^2_{\omega,h_0} dV_\omega\Big)^{\frac{1}{2}} \Big(\int_U |u|^2_{\omega,h} dV_\omega\Big)^{\frac{1}{2}}\end{aligned}$$

which shows that $\partial\varphi \wedge u_j \to \partial\varphi \wedge u$ in $L^1_{\mathrm{loc}}(U, \Lambda^{p+1,q}T^\star_Y \otimes L, \omega, h_0)$, in particular as currents.

We should mention that some similar discussion of the definition of covariant derivative with respect to a singular metric can also be found in [**Dem02**]. (The author thanks Professor A. Höring for mentioning the reference.)

Remark 2.7. We check here that the operator

$$\partial_h : L^2(X, \wedge^{n-q}T^*_X \otimes L, h) \to L^2(X, \wedge^{n-q+1}T^*_X \otimes L, h)$$

is a closed densely defined operator.

By a partition of unity argument, it is enough to check this on a local coordinate chart U. Assume that we have $h = e^{-\varphi}$ on U for some psh function φ. We claim that functions of the type $e^{(1/2+\varepsilon)\varphi}f$ with any $\varepsilon > 0$ and f smooth with compact support are in the domain of definition of ∂_h and are dense in $L^2(U, \wedge^{n-q}T^*_X \otimes L, h)$. In fact, we have

$$\partial_h(e^{(1/2+\varepsilon)\varphi}f) = (3/2+\varepsilon)\partial\varphi \wedge e^{(1/2+\varepsilon)\varphi}f + e^{(1/2+\varepsilon)\varphi}\partial f.$$

Without loss of generality, we can assume that φ is bounded from above. Since $f, \partial f$ are bounded and $|\partial\varphi|^2 e^{2\varepsilon\varphi} dV_\omega \leqslant \frac{1}{4\varepsilon^2} i\partial\overline{\partial}(e^{2\varepsilon\varphi}) \wedge \omega^{n-1}$ is integrable, we have $\int_U |\partial\varphi \wedge e^{(1/2+\varepsilon)\varphi}f|^2 e^{-\varphi} dV_\omega < \infty$ and $\int_U |e^{(1/2+\varepsilon)\varphi}\partial f|^2 e^{-\varphi} dV_\omega < \infty$. Thus $e^{(1/2+\varepsilon)\varphi}f$ is in the domain of definition.

To prove the density, it is equivalent to show that smooth functions with compact support are dense in $L^2(U, e^{2\varepsilon\varphi}dV)$ where dV is the Lebesgue measure. Notice that we have an isomorphism of topological linear space between $L^2(U, e^{2\varepsilon\varphi}dV)$ and $L^2(U, e^{-\varphi}dV)$ by sending f to $e^{(1/2+\varepsilon)\varphi}f$. Since $e^{2\varepsilon\varphi}$ is locally bounded, thus $e^{2\varepsilon\varphi}dV_\omega$ is a locally finite measure. Any real function $u \in L^2(U, e^{2\varepsilon\varphi}dV)$ can be approximated in norm by a bounded function $\tilde{u}_\nu = \max(\min(u,\nu), -\nu)$, and then $\tilde{u}_\nu$ can be approximated by smooth compactly supported functions u_ν by taking the product of $\tilde{u}_\nu$ with a cut-off function and taking a convolution by dominated convergence theorem.

By the last paragraphs before the remark, if $u_\nu \to u$ in $L^2(e^{-\varphi})$ topology, then $\partial_h u_\nu \to \partial_h u$ in the weak topology of currents. This shows that ∂_h is a closed operator by definition.

Assuming for the moment that theorem 2.4 is valid, we infer theorem 2.5. A consequence is that the inverse operator in the proof of the hard Lefschetz theorem is linear, a fact that is a priori non trivial.

Proof of theorem 2.5. By theorem 2.4, we know that the morphism is surjective. Since the morphism is the restriction of the wedge multiplication operator on some subspace, it is linear. Thus to show that it is a linear isomorphism, it is enough to show that it is injective.

Assume that $u \in H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h))$ such that $\partial_h u = 0$ and $u \wedge \omega^q \equiv 0$ in $H^q(X, K_X \otimes L \otimes \mathcal{I}(h))$. It means that there exists $v \in L^2(X, \wedge^{n,q-1} T_X^* \otimes L, h)$ such that

$$u \wedge \omega^q = \overline{\partial} v.$$

To prove that $u = 0$, it is equivalent to prove that $u \wedge \omega^q = 0$ by the pointwise Lefschetz isomorphism. To prove that $u \wedge \omega^q = 0$, it is enough to prove that $\| \overline{\partial} v \| = 0$.

We have that

$$\| \overline{\partial} v \|^2 = \int_X \langle \overline{\partial} v, u \wedge \omega^q \rangle dV_\omega = \int_X \{\overline{\partial} v, u\}.$$

On the other hand, we have that

$$\overline{\partial}\{v, u\} = \{\overline{\partial} v, u\} + (-1)^{n+q-1}\{v, \partial_h u\}$$

since v is a $(n, q-1)$ form. By the assumption that $\partial_h u = 0$, we get $\overline{\partial}\{v, u\} = \{\overline{\partial} v, u\}$. Since u is a $(n-q, 0)$ form and v is a $(n, q-1)$ form, by a degree consideration, we find $\partial\{v, u\} = 0$.

Observe that $\{v, u\}$ is a well defined current (in fact L^1_{loc} with respect to any smooth metric on L) since both v, u are L^2 with respect to the singular metric h.

Thus by Stokes theorem (for a statement of the result in terms of currents, cf. e.g. [**deR60**]), we obtain

$$\| \overline{\partial} v \|^2 = \int_X d\{v, u\} = 0.$$

□

2.3. Proof of theorem 2.4

This section follows closely [**DPS01**] with some additional estimates for the integral norms of the terms involved at each step. First, we reproduce the variant of the Bochner formula used in [**DPS01**].

PROPOSITION 2.8. *Let (Y, ω) be a* complete *Kähler manifold and (L, h) a smooth Hermitian line bundle such that the curvature current possesses a uniform lower bound $\Theta_{L,h} \geqslant -C\omega$. For every measurable $(n-q, 0)$-form v with L^2 coefficients and values in L such that $u = \omega^q \wedge v$ has differentials $\overline{\partial} u$, $\overline{\partial}^* u$ also in L^2, we have*

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}_h^* u\|^2 = \|\overline{\partial} v\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j \in J} \lambda_j\Big) |u_{IJ}|^2$$

(here, all differentials are computed in the sense of distributions) and where $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ are the curvature eigenvalues of $i\Theta_{L,h}$ expressed in an orthonormal frame $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ (at some fixed point $x_0 \in Y$), in such a way that

$$\omega_{x_0} = \mathrm{i} \sum_{1 \leqslant j \leqslant n} dz_j \wedge d\overline{z}_j, \qquad (i\Theta_{L,h})_{x_0} = dd^c \varphi_{x_0} = \mathrm{i} \sum_{1 \leqslant j \leqslant n} \lambda_j dz_j \wedge d\overline{z}_j.$$

Now, X denotes a compact Kähler manifold equipped with a Kähler metric ω, and (L, h) a pseudoeffective line bundle on X. To fix the ideas, we first indicate the proof in the much simpler case when (L, h) has a smooth metric h (so that $\mathcal{I}(h) = \mathcal{O}_X$), and then treat the general case (although it is not really used in the proof of the general case).

Let $\{\beta\} \in H^q(X, \Omega_X^n \otimes L)$ be an arbitrary cohomology class. By standard Hodge theory, $\{\beta\}$ can be represented by a smooth harmonic $(0, q)$-form β with values in $\Omega_X^n \otimes L$. We can also view β as a (n, q)-form with values in L. The pointwise Lefschetz isomorphism produces a unique $(n-q, 0)$-form α such that $\beta = \omega^q \wedge \alpha$. Proposition 2.8 then yields

$$\|\overline{\partial}\alpha\|^2 + \int_X \sum_{I,J} \Big(\sum_{j \in J} \lambda_j\Big) |\alpha_{IJ}|^2 = \|\overline{\partial}\beta\|^2 + \|\overline{\partial}_h^*\beta\|^2 = 0,$$

and the curvature eigenvalues λ_j are non-negative by our assumption. Hence $\overline{\partial}\alpha = 0$ and $\{\alpha\} \in H^0(X, \Omega_X^{n-q} \otimes L)$ is mapped to $\{\beta\}$ by $\Phi_{\omega,h}^q = \omega^q \wedge \bullet$.

In this case, the proof of the closedness property of sections involves the identity

$$\int_X \{\partial_h v, \partial_h v\}_h \wedge \omega^{q-1} = \int_X (\partial\{v, \partial_h v\}_h - (-1)^{\deg v}\{v, \overline{\partial}\partial_h v\}_h) \wedge \omega^{q-1}.$$

Using the holomorphicity of v, the fact that (X, ω) is Kähler and the Stokes formula, we get

$$\text{RHS} = (-1)^{\deg v+1} \int_X \{v, -\partial_h \overline{\partial} v + i\Theta_{L,h} v\}_h \wedge \omega^{q-1} = (-1)^{\deg v+1} \int_X \{v, i\Theta_{L,h} v\}_h \wedge \omega^{q-1}$$

$$= -\int_X i\Theta_{L,h} \wedge \{v, v\}_h \wedge \omega^{q-1} \leqslant 0.$$

In the above calculation, we have used the formula

$$\partial_h\overline{\partial}+\overline{\partial}\partial_h = i\Theta_{L,h}\wedge\bullet.$$

The last inequality uses the curvature assumption. Therefore we have

$$\int_X\{\partial_h v,\partial_h v\}_h\wedge\omega^{q-1}=0,$$

and this implies $\partial_h v=0$.

Let us return to the case of an arbitrary plurisubharmonic weight φ. We will need the following "equisingular" approximation of psh functions; here, equisingularity is to be understood in the sense that the multiplier ideal sheaves are preserved. A proof can be found in [**DPS01**] or [**Dem14**].

THEOREM 2.9. *Let $T=\alpha+dd^c\varphi$ be a closed $(1,1)$-current on a compact Hermitian manifold (X,ω), where α is a smooth closed $(1,1)$-form and φ a quasi-psh function. Let γ be a continuous real $(1,1)$-form such that $T\geqslant\gamma$. Then one can write $\varphi=\lim_{m\to+\infty}\widetilde{\varphi}_m$ where*

(a) *$\widetilde{\varphi}_m$ is smooth in the complement $X\backslash Z_m$ of an analytic set $Z_m\subset X$;*

(b) *$\{\widetilde{\varphi}_m\}$ is a non-increasing sequence, and $Z_m\subset Z_{m+1}$ for all m;*

(c) *$\int_X(e^{-\varphi}-e^{-\widetilde{\varphi}_m})dV_\omega$ is finite for every m and converges to 0 as $m\to+\infty$;*

(d) *("equisingularity") $\mathcal{I}(\widetilde{\varphi}_m)=\mathcal{I}(\varphi)$ for all m ;*

(e) *$T_m=\alpha+dd^c\widetilde{\varphi}_m$ satisfies $T_m\geqslant\gamma-\varepsilon_m\omega$, where $\lim_{m\to+\infty}\varepsilon_m=0$.*

Fix $\varepsilon=\varepsilon_\nu$ and let $h_\varepsilon=h_{\varepsilon_\nu}$ be an approximation of h, such that h_ε is smooth on $X\backslash Z_\varepsilon$ (Z_ε being an analytic subset of X), $\Theta_{L,h_\varepsilon}\geqslant-\varepsilon\omega$, $h_\varepsilon\leqslant h$ and $\mathcal{I}(h_\varepsilon)=\mathcal{I}(h)$. As above we fix a reference smooth metric h_0 on L. We denote by β the curvature form of h_0 and $h_\varepsilon=h_0e^{-\varphi_\varepsilon}$ (φ_ε is hence a global quasi-psh function on X). The existence of a such metric is guaranteed by Theorem 2.9. Now, we can find a family

$$\omega_{\varepsilon,\delta}=\omega+\delta(\mathrm{i}\partial\overline{\partial}\psi_\varepsilon+\omega),\qquad\delta>0$$

of *complete Kähler* metrics on $X\backslash Z_\varepsilon$, where ψ_ε is a quasi-psh function on X with analytic singularity with $\psi_\varepsilon=-\infty$ on Z_ε, ψ_ε smooth on $X\backslash Z_\varepsilon$ and $\mathrm{i}\partial\overline{\partial}\psi_\varepsilon+\omega\geqslant0$ (see e.g. [**Dem82**], Théorème 1.5). By construction, $\omega_{\varepsilon,\delta}\geqslant\omega$ and $\lim_{\delta\to0}\omega_{\varepsilon,\delta}=\omega$. We look at the L^2 Dolbeault complex $K^\bullet_{\varepsilon,\delta}$ of $(n,\bullet)$-forms on $X\backslash Z_\varepsilon$, where the L^2 norms are induced by $\omega_{\varepsilon,\delta}$ on differential forms and by h_ε on elements in L. Specifically

$$K^q_{\varepsilon,\delta}=\Big\{u{:}X\backslash Z_\varepsilon{\to}\Lambda^{n,q}T^*_X\otimes L;\int_{X\backslash Z_\varepsilon}(|u|^2_{\Lambda^{n,q}\omega_{\varepsilon,\delta}\otimes h_\varepsilon}+|\overline{\partial}u|^2_{\Lambda^{n,q+1}\omega_{\varepsilon,\delta}\otimes h_\varepsilon})dV_{\omega_{\varepsilon,\delta}}<\infty\Big\}.$$

Let $\mathcal{K}^q_{\varepsilon,\delta}$ be the corresponding sheaf of germs of locally L^2 sections on X (the local L^2 condition should hold on X, not only on $X\backslash Z_\varepsilon$!). Then, for all $\varepsilon>0$ and $\delta\geqslant0$, $(\mathcal{K}^q_{\varepsilon,\delta},\overline{\partial})$ is a resolution of the sheaf $\Omega^n_X\otimes L\otimes\mathcal{I}(h_\varepsilon)=\Omega^n_X\otimes L\otimes\mathcal{I}(h)$. This is because L^2 estimates hold locally on small Stein open sets, and the L^2 condition on $X\backslash Z_\varepsilon$ forces holomorphic sections to extend across Z_ε ([**Dem82**], Lemma 6.9).

Let $\{\beta\}\in H^q(X,\Omega^n_X\otimes L\otimes\mathcal{I}(h))$ be a cohomology class represented by a smooth form with values in $\Omega^n_X\otimes L\otimes\mathcal{I}(h)$. Then

$$\|\beta\|^2_{\varepsilon,\delta}\leqslant\|\beta\|^2=\int_X|\beta|^2_{\Lambda^{n,q}\omega\otimes h}dV_\omega<+\infty.$$

The reason is that $|\beta|^2_{\Lambda^{n,q}\omega\otimes h}dV_\omega$ decreases as ω increases, see e.g. [**Dem82**], Lemma 3.2. Now, β is a $\overline{\partial}$-closed form in the Hilbert space defined by $\omega_{\varepsilon,\delta}$ on $X\backslash Z_\varepsilon$ and for $\delta>0$, the Kähler metric is complete on $X\backslash Z_\varepsilon$, so there is a $\omega_{\varepsilon,\delta}$-harmonic form $u_{\varepsilon,\delta}$ in the same cohomology class as β, such that

$$\|u_{\varepsilon,\delta}\|_{\varepsilon,\delta}\leqslant\|\beta\|_{\varepsilon,\delta}.$$

Let $v_{\varepsilon,\delta}$ be the unique $(n-q,0)$-form such that $u_{\varepsilon,\delta}=v_{\varepsilon,\delta}\wedge\omega^q_{\varepsilon,\delta}$ ($v_{\varepsilon,\delta}$ exists by the pointwise Lefschetz isomorphism). Then

$$\|v_{\varepsilon,\delta}\|_{\varepsilon,\delta}=\|u_{\varepsilon,\delta}\|_{\varepsilon,\delta}\leqslant\|\beta\|_{\varepsilon,\delta}\leqslant\|\beta\|.$$

As $\sum_{j\in J}\lambda_j\geqslant-q\varepsilon$ by the assumption on Θ_{L,h_ε}, the Bochner formula for $X\backslash Z_\varepsilon$ yields

$$\|\overline{\partial}v_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\leqslant q\varepsilon\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\leqslant q\varepsilon\|\beta\|^2.$$

But since Z_ε is an analytic set, the integral can also be seen taken on X; In the following, we use it abusively. These uniform bounds imply that there are subsequences $u_{\varepsilon,\delta_\nu}$ and $v_{\varepsilon,\delta_\nu}$ with $\delta_\nu\to0$, possessing weak-L^2 limits $u_\varepsilon=\lim_{\nu\to+\infty}u_{\varepsilon,\delta_\nu}$ and $v_\varepsilon=\lim_{\nu\to+\infty}v_{\varepsilon,\delta_\nu}$. The limit $v_\varepsilon=\lim_{\nu\to+\infty}v_{\varepsilon,\delta_\nu}$ is with respect to $L^2(\omega)=L^2(\omega_{\varepsilon,0})$. To check this, notice that in bidegree $(n-q,0)$, the space $L^2(\omega)$ has the weakest topology of all spaces $L^2(\omega_{\varepsilon,\delta})$; indeed, an easy calculation made in [**Dem82**], Lemma 3.2 yields

$$|f|^2_{\Lambda^{n-q,0}\omega\otimes h}dV_\omega\leqslant|f|^2_{\Lambda^{n-q,0}\omega_{\varepsilon,\delta}\otimes h}dV_{\omega_{\varepsilon,\delta}}\qquad\text{if } f \text{ is of type } (n-q,0).$$

On the other hand, the limit $u_\varepsilon = \lim_{\nu\to+\infty} u_{\varepsilon,\delta_\nu}$ takes place in all spaces $L^2(\omega_{\varepsilon,\delta})$, $\delta > 0$, since the topology gets stronger and stronger as $\delta \downarrow 0$ [possibly not in $L^2(\omega)$, though, because in bidegree (n,q) the topology of $L^2(\omega)$ might be strictly stronger than that of all spaces $L^2(\omega_{\varepsilon,\delta})$]. For fixed $\delta > 0$, for any $\delta' < \delta$,we have

$$\|u_{\varepsilon,\delta'}\|_{\varepsilon,\delta} \leqslant \|u_{\varepsilon,\delta'}\|_{\varepsilon,\delta'} \leqslant \|\beta\|$$

$$\|u_\varepsilon\|_{\varepsilon,\delta} \leqslant \liminf_{\delta'\to 0}\|u_{\varepsilon,\delta'}\|_{\varepsilon,\delta} \leqslant \|\beta\|$$

By Lebesgue's monotone convergence theorem, u_ε is $L^2(\omega_{\varepsilon,\delta}\otimes h_\varepsilon)$ bounded. The above estimates yield

$$\|v_\varepsilon\|^2_{\varepsilon,0} = \int_X |v_\varepsilon|^2_{\Lambda^{n-q,0}\omega\otimes h_\varepsilon} dV_\omega \leqslant \|\beta\|^2,$$

$$\|\overline{\partial} v_\varepsilon\|^2_{\varepsilon,0} \leqslant q\varepsilon\|\beta\|^2_{\varepsilon,0} = q\varepsilon\|\beta\|^2,$$

$$u_\varepsilon = \omega^q \wedge v_\varepsilon \equiv \beta \qquad \text{in} \quad H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h_\varepsilon)).$$

The last equality can be checked via the De Rham-Weil isomorphism, by using the fact that the map $\alpha \mapsto \{\alpha\}$ from the cocycle space $Z^q(\mathcal{K}^\bullet_{\varepsilon,\delta})$ equipped with its L^2 topology, into $H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h))$ equipped with its finite vector space topology, is continuous.

For the closedness property, we want to control the $L^1_{\rm loc}$ norm of the covariant derivative with respect to the Lebesgue measure, which is well defined on X since the metric is smooth outside an analytic set and the section is locally L^2 with respect to the metric. For any smooth $(n-q,0)$-form v with compact support in $X\backslash Z_\varepsilon$, we can apply the Stokes formula to get

$$\int_X \{\partial_{h_\varepsilon} v, \partial_{h_\varepsilon} v\}_{h_\varepsilon} \wedge \omega^{q-1}_{\varepsilon,\delta} = (-1)^{\deg v+1}\int_X \{v, -\partial_{h_\varepsilon}\overline{\partial} v + i\Theta_{L,h_\varepsilon} v\}_{h_\varepsilon} \wedge \omega^{q-1}_{\varepsilon,\delta}$$

$$= \int_X (\overline{\partial}\{v, \overline{\partial} v\}_{h_\varepsilon} - \{\overline{\partial} v, \overline{\partial} v\}_{h_\varepsilon} - i\Theta_{L,h_\varepsilon} \wedge \{v,v\}_{h_\varepsilon}) \wedge \omega^{q-1}_{\varepsilon,\delta}$$

$$= \int_X (-\{\overline{\partial} v, \overline{\partial} v\}_{h_\varepsilon} - i\Theta_{L,h_\varepsilon} \wedge \{v,v\}_{h_\varepsilon}) \wedge \omega^{q-1}_{\varepsilon,\delta}.$$

We want to apply this identity to $v = v_{\delta,\varepsilon}$ that does not necessarily have compact support in $X\backslash Z_\varepsilon$. However, the metric $\omega_{\varepsilon,\delta}\otimes h_\varepsilon$ is smooth and complete on $X\backslash Z_\varepsilon$, and this will allow us to extend the identity to $v = v_{\varepsilon,\delta}$. In fact, there exists a sequence of smooth forms $v_{\varepsilon,\delta,\nu}$ with compact support on $X\backslash Z_\varepsilon$ obtained by truncating $v_{\varepsilon,\delta}$ and by taking the convolution with a regularizing kernel, in such a way that $v_{\varepsilon,\delta,\nu} \to v_{\varepsilon,\delta}$ in $L^2(\omega_{\varepsilon,\delta}\otimes h_\varepsilon)$ (and therefore in $L^2(\omega\otimes h_0)$ as well). For simplicity of notation, we put $\partial_\varepsilon = \partial_{h_\varepsilon}$ and denote by $\partial^*_{\varepsilon,\delta}$ its dual with respect to the metric $\omega_{\varepsilon,\delta}\otimes h_\varepsilon$ (the latter operator depends on δ, since the Hodge $*$ operator depends on the Kähler metric). By taking $v = v_{\varepsilon,\delta,\nu}$ in the above identity, neglecting the non positive term involving $\overline{\partial} v$ and using the curvature condition, we obtain

$$\|\partial_\varepsilon v_{\varepsilon,\delta,\nu}\|^2_{\varepsilon,\delta} \leqslant q\varepsilon\|v_{\varepsilon,\delta,\nu}\|^2_{\varepsilon,\delta}.$$

Let us put $C = e^{max_X(\varphi_{\varepsilon_1})}$ (we have $C < \infty$ as X is compact). Then by using $\omega_{\varepsilon,\delta} \geqslant \omega$, $h_\varepsilon \geqslant \frac{1}{C}h_0$, we get

$$\|\partial_\varepsilon v_{\varepsilon,\delta,\nu}\|^2_{L^2(\omega\otimes h_0)} \leqslant C\|\partial_\varepsilon v_{\varepsilon,\delta,\nu}\|^2_{\varepsilon,\delta},$$

By the Cauchy-Schwarz inequality and the fact that X is compact and that the metrics ω, h_0 are smooth, we find

$$\|\partial_\varepsilon v_{\varepsilon,\delta,\nu}\|_{L^1(\omega\otimes h_0)} \leqslant C'\|\partial_\varepsilon v_{\varepsilon,\delta,\nu}\|_{L^2(\omega\otimes h_0)},$$

Since the covariant derivative is a closed operator and $v_{\varepsilon,\delta,\nu} \to v_{\varepsilon,\delta}$, $v_{\varepsilon,\delta} \to v_\varepsilon$ in $L^2(\omega_{\varepsilon,0}\otimes h_\varepsilon)$, we conclude that

$$\|\partial_\varepsilon v_{\varepsilon,\delta}\|_{L^1(\omega\otimes h_0)} \leqslant C''\sqrt{q\varepsilon}\|\beta\|,$$

$$\|\partial_\varepsilon v_\varepsilon\|_{L^1(\omega\otimes h_0)} \leqslant C''\sqrt{q\varepsilon}\|\beta\|.$$

Again, by arguing in a fixed Hilbert space $L^2(h_{\varepsilon_0})$ (since $\omega_\varepsilon = \omega$, the notation $L^2(h_{\varepsilon_0})$ will be used for fixed $\varepsilon_0 > 0$), we find L^2 convergent subsequences $u_\varepsilon \to u$, $v_\varepsilon \to v$ as $\varepsilon \to 0$, and in this way get $\overline{\partial} v = 0$ and

$$\|v\|^2 \leqslant \|\beta\|^2,$$

$$u = \omega^q \wedge v \equiv \beta \qquad \text{in} \quad H^q(X, \Omega^n_X \otimes L \otimes \mathcal{I}(h)).$$

By closedness of the covariant derivative and by continuity of the injection $L^2(\omega\otimes h_0) \hookrightarrow L^1(\omega\otimes h_0)$ on the compact manifold X, we obtain

$$\|\partial_{\varepsilon_0} v\|^2_{L^1(\omega\otimes h_0)} \leqslant Cq\varepsilon_0\|\beta\|^2.$$

As $\varphi = \lim_{\varepsilon\to 0}\varphi_\varepsilon$ and $\partial\varphi = \lim_{\varepsilon\to 0}\partial\varphi_\varepsilon$ in $L^1_{\rm loc}(h_0)$, and as we haven proven that v is in fact holomorphic, by the continuity of the covariant derivative operator, we infer that $\partial\varphi\wedge v = \lim_{\varepsilon\to 0}\partial\varphi_\varepsilon \wedge v$ in the sense of distributions, and we have $\|\partial_h v\|^2_{L^1(\omega\otimes h_0)} = 0$, which means that $\partial_h v = 0$. The closedness property is proved along the same lines.

2.4. Foliation induced by sections

We show that the closedness property of the holomorphic section provided by the hard Lefschetz theorem induces a foliation on X. Here foliation means that there exists an irreducible analytic set V of the total space T_X such that for any $x \in X$, $V_x := V \cap T_X$ is a complex vector space and the section sheaf $\mathcal{O}(V) \subset \mathcal{O}(T_X)$ is closed under the Lie bracket. It is equivalent to say that $\mathcal{O}(V)$ is closed under Lie bracket and that $\mathcal{O}(T_X)/\mathcal{O}(V)$ is torsion free.

We consider $v \in H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)), q \geqslant 1$ a parallel section with respect to the singular metric h. In particular a section constructed by the hard Lefschetz theorem is such a section. The interior product with v gives an $\mathcal{O}_X$-morphism (which is well defined on the whole of X)

$$F_v : T_X \to \Omega_X^{n-q-1} \otimes L$$
$$X \mapsto \iota_X v.$$

First we observe that the kernel $\mathrm{Ker}(F_v)$ is coherent and locally free over a Zariski open set – this merely relies on the fact that v is holomorphic, and although the proof is purely formal, we repeat here the standard argument for the reader's convenience. For any $z \in X$, take an open neighbourhood V of z such that $L|_V$ is trivial and on this open set $v(z) = \sum_{|I|=n-q} v_I(z) dz_I$ where $v_I \in \Gamma(V, \mathcal{O}_X)$. Consider $\xi = \sum \xi_j(z) \frac{\partial}{\partial z_j}$ a local tangent vector field on V. For any multiindex I and any $j \in I$, we write it in the form $I = (j, I'_j)$. Then $\xi \in \mathrm{Ker}(F_v)$ if and only if $\sum_{j,I,|I|=n-q-1} \xi_j u_{(j,I)} dz_I = 0$, i.e. if and only if for any $I, |I| = n-q-1$, $\sum_j \xi_j(z) u_{(j,I)}(z) = 0$. This gives a local system of analytic equations defining $\mathrm{Ker}(F_v)$. In particular, we see that $\mathrm{Ker}(F_v)$ is locally free over the Zariski open set where the holomorphic linear system $\sum_j \xi_j(z) u_{(j,I)}(z) = 0$ $(|I| = n-q-1)$ achieves its generic rank.

Next, we show that the spaces of sections of $\mathrm{Ker}(F_v)$ are closed under Lie brackets; this uses of course the closedness property of v. Since the closedness under Lie brackets is a local property, we can take an open set U such that there exists a nowhere vanishing local generator s_L of the line bundle L on U, and we verify the closedness of the Lie bracket on U. On U, $v = u \otimes s_L$ for some $u \in H^0(U, \Omega_X^{n-q})$. Denote by X, Y two local tangent vector fields in $\mathrm{Ker} F_v \subset \mathcal{O}(T_X)$ defined on U. Observe that $d_h(u \otimes s_L)$ is only almost everywhere defined (instead of pointwise defined). The above equalities are calculated in the sense of currents. We have

$$\begin{aligned}
0 &= d_h(u \otimes s_L)(X, Y, \bullet) \\
&= (du \otimes s_L + (-1)^{\deg u} u \wedge d_h s_L)(X, Y, \bullet) \\
&= du(X, Y, \bullet) \otimes s_L + (-1)^{\deg u} u \wedge d_h s_L(X, Y, \bullet) \\
&= du(X, Y, \bullet) \otimes s_L + (-1)^{\deg u}[u(X, \bullet) d_h s_L(Y) - u(Y, \bullet) d_h s_L(X) + ...] \\
&= du(X, Y, \bullet) \otimes s_L
\end{aligned}$$

The above dots ... mean terms of the form $\pm u(X, Y, \bullet) d_h s_L(\bullet)$. The last equality uses of course the fact that $X, Y \in \mathrm{Ker} F_v$.

For any $X_0, ..., X_{n-q}$ tangent vector fields of U such that $X_0 = X, X_1 = Y$, we have

$$\begin{aligned}
0 = du(X_0, ..., X_{n-q}) &= \sum_{i=0}^{n-q} (-1)^i X_i [u(X_0, ..., \hat{X}_i, ..., X_k)] \\
+ \sum_{0 \leqslant i < j \leqslant n-q} &(-1)^{i+j} u([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{n-q}) \\
&= -u([X, Y], X_2, ..., X_{n-q}),
\end{aligned}$$

which means that $[X, Y] \in \mathrm{Ker}(F_v)$.

By the Frobenius theorem, the subsheaf $\mathrm{Ker}(F_v) \subset T_X$ defines a regular holomorphic foliation on a Zariski open set. Notice additionally that $\mathrm{Ker}(F_v)$ is saturated in T_X, i.e. $T_X/\mathrm{Ker}(F_v) \simeq \mathrm{Im}(F_v)$ is torsion free, as a subsheaf of the locally free sheaf $\Omega_X^{n-q-1} \otimes L$.

We can also reformulate our conclusions in the following form: denote by r the generic rank of $\mathrm{Ker}(F_v)$. Then, looking at F_v as a morphism of bundles rather than as a morphism of sheaves, we get a meromorphic morphism

$$X \dashrightarrow \mathrm{Gr}(T_X, r)$$
$$z \mapsto \mathrm{Ker}(F_{v,z})$$

where $\mathrm{Gr}(TX, r)$ is the Grassmannian bundle of r-dimensional subspaces of T_X, and the corresponding distribution of subspaces is integrable on the Zariski open set where the above map is holomorphic.

Let us observe that the foliation property only holds for the parallel sections. In general, a non trivial section $v \in H^0(X, \Omega_X^{n-q} \otimes L), q \geqslant 1$, does not necessarily induce a foliation. We give below a concrete example of the non-integrability of $\mathrm{Ker}(F_v)$ for such a section v, and thank Professor A. Höring for pointing out the example. It is interesting at this point to compare the situation with the following result proved in [**Dem02**]: if L is a psef line bundle over a compact Kähler manifold X and $0 \leqslant q \leqslant n = \dim X$, then for every non-zero holomorphic section $v \in H^0(X, \Omega^q_X \otimes L^{-1})$, the kernel $\mathrm{Ker}(F_v)$ automatically defines a foliation on X.

The example pointed out by A. Höring first appeared in the paper of Beauville [**Bea00**]. Let A be an abelian surface and $X = A \times \mathbb{P}^1$. Let (U, V) be a basis of $H^0(A, T_A)$, and let S, T be two vector fields on $\mathbb{P}^1$ which do not commute. For example, in the homogeneous coordinates $[w_1 : w_2]$ of $\mathbb{P}^1$, we can take

$$S = w_2 \frac{\partial}{\partial w_1}, \; T = w_1 \frac{\partial}{\partial w_2}.$$

Then the vector fields $U + S$ and $V + T$ span a rank 2 subbundle Σ of T_X. Since $U + S$, $V + T$ have no common root, $\Sigma \cong \mathcal{O}_X^{\oplus 2}$. In particular, Σ is not integrable, i.e. Σ is not closed under the Lie bracket of vector fields. Consider the short exact sequence of vector bundles

$$0 \to \Sigma \to T_X \to T_X/\Sigma \to 0.$$

We deduce that $T_X/\Sigma \cong -K_X$. The quotient map $T_X \to T_X/\Sigma \cong -K_X$ induces by duality a vector bundle morphism $K_X \to \Omega^1_X$. Thus we have a non trivial section $\eta_{S,T} \in H^0(X, \Omega^1_X \otimes (-K_X))$.

To use the hard Lefschetz theorem, we take the following smooth metric on $-K_X$. Denote by $\pi_1 : X \to A$, $\pi_2 : X \to \mathbb{P}^1$ the natural projections. $-K_X = \pi_2^* \mathcal{O}_{\mathbb{P}^1}(2)$. Thus $-K_X$ is a semiample divisor. By taking the smooth metric h induced by a basis of global sections $\pi_2^* H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ (or a base point free system of global sections), we get a smooth positive metric on $-K_X$. In particular, the multiplier ideal sheaf associated to this metric is trivial. Moreover, by construction, the metric is real analytic. In other words, we have a section $v \in H^0(X, \Omega^1_X \otimes (-K_X))$ such that $\mathrm{Ker}(F_v)$ is not integrable, while the metric is positive and real analytic.

Fix any Kähler metric ω on X. By the hard Lefschetz theorem, we have a surjective map

$$H^0(X, \Omega^1_X \otimes (-K_X)) \to H^2(X, \mathcal{O}_X).$$

The image $\omega^2 \wedge \eta_{S,T}$ has a pre-image $\eta_{S,T}$ which does not define a foliation on X with the above choice of S, T.

Next, we derive by an explicit calculation what is the pre-image given by the hard Lefschetz theorem, and show that this pre-image indeed defines a foliation on X. To simplify our exposition, we keep the same notation as above without assuming any longer that S, T do not commute. Fix ω_A a flat metric on A such that U, V form an orthonormal basis at each point. Fix $\omega_{\mathbb{P}^1}$ a Kähler metric on $\mathbb{P}^1$ induced by the Fubini-Study metric and fix $\omega = \pi_1^* \omega_A + \pi^* \omega_{\mathbb{P}^1}$ a Kähler metric on X. In particular, with this choice of metric, the induced metric $\wedge^3 \omega \otimes h$ on $K_X + (-K_X)$ is trivial.

We begin by showing that for any choice of S, T, the image $\omega^2 \wedge \eta_{S,T}$ is the same. To verify this claim, we use the following isomorphism of $\mathbb{C}$-vector spaces. Notice that $H^2(X, \mathcal{O}_X) \cong \pi_1^* H^2(A, \mathcal{O}_A) \cong \mathbb{C}$. Fix some $x \in \mathbb{P}^1$. Consider the morphism

$$\iota : H^2(X, \mathcal{O}_X) \to \mathbb{C}$$
$$\{u\} \mapsto \int_{A \times \{x\}} u \wedge iU^* \wedge V^*.$$

Here u is a $C^\infty_{(0,2)}(X)$ representative of $\{u\} \in H^2(X, \mathcal{O}_X)$. It is surjective since a generator of $H^2(X, \mathcal{O}_X)$ can be represented by $\pi_1^*(\overline{U}^* \wedge \overline{V}^*)$ whose image is equal to $\int_A \omega_A^2$. Since both sides are isomorphic to $\mathbb{C}$, we have an isomorphism.

For any $x \in \mathbb{P}^1$, let W be a local generator $T_{\mathbb{P}^1}$ with norm 1 with respect to $\omega_{\mathbb{P}^1}$. In particular, locally U, V, W form an orthonormal basis with respect to ω pointwise. Assume that locally $S = fW$ and $T = gW$. There exists a C^∞ splitting of the short exact sequence $0 \to \Sigma \to T_X \to T_X/\Sigma \to 0$ by $T_X \cong \Sigma \oplus T_X/\Sigma$ which is induced by ω. Locally, T_X is spanned by orthogonal basis $fU + gV - W$, $U + fW$ and $V + gW$. With this identification, η can be locally given by for any $\xi \in T_X$

$$\eta(\xi) = \frac{\langle \xi, fU + gV - W \rangle}{|fU + gV - W|^2} (fU + gV - W).$$

Thus η is given by

$$\left(\frac{f}{1 + f^2 + g^2} U^* + \frac{g}{1 + f^2 + g^2} V^* - \frac{1}{1 + f^2 + g^2} W^*\right) \otimes (fU + gV - W).$$

The anticanonical line bundle $-K_X$ is locally generated by

$$(fU + gV - W) \wedge (U + fW) \wedge (V + gW) = -(1 + f^2 + g^2) U \wedge V \wedge W.$$

In other words, the identification of $\Sigma^\perp \cong T_X/\Sigma \cong -K_X$ means the identification of $fU + gV - W$ with $-(1+f^2+g^2)U \wedge V \wedge W$. Thus $\omega^2 \wedge \eta$ seen as a $C^\infty_{(0,2)}$ form is given by

$$f\overline{V}^* \wedge \overline{W}^* + g\overline{U}^* \wedge \overline{W}^* + \overline{U}^* \wedge \overline{V}^*.$$

Using this expression, $\iota(\omega^2 \wedge \eta_{S,T})$ is the same for any S, T. Since ι is an isomorphism of vector spaces, $\omega^2 \wedge \eta_{S,T}$ is independent of the choice of S, T.

In the following, we show that the section constructed in the hard Lefschetz theorem for $\omega^2 \wedge \eta_{S,T}$ is $\eta_{S,T}$ associated with $S = T = 0$. We remark that since the metric is smooth, we can directly use the result of [**Eno93**] without employing the equisingular approximation of [**DPS01**]. In other words, the pre-image is given by the pointwise Lefschetz isomorphism of the harmonic representative of an element in $H^2(X, \mathcal{O}_X)$.

We claim that a generator of $H^2(X, \mathcal{O}_X)$ can be represented by the harmonic $(0,2)$-form $\overline{U}^* \wedge \overline{V}^*$. The reason is as follows. Since the metric is trivial on $\mathcal{O}_X$, the covariant derivative coincides with the exterior derivative. Since U, V are global parallel holomorphic sections, $dU^* = dV^* = 0$. This implies in particular that $\overline{\partial}(\overline{U}^* \wedge \overline{V}^*) = 0$. On the other hand, $\overline{U}^* \wedge \overline{V}^*$ is independent of the choice of coordinate on $\mathbb{P}^1$. To prove that $\overline{\partial}^*(\overline{U}^* \wedge \overline{V}^*) = 0$, it is enough to make a calculation in a normal coordinate chart centred at x. In other words, locally $\omega = iU^* \wedge \overline{U}^* + iV^* \wedge \overline{V}^* + iW^* \wedge \overline{W}^*$ with $dW(x) = 0$. (The existence of the normal coordinate chart is ensured by the assumption that ω is Kähler.) Since $\overline{\partial}^* = - * \partial *$, we have $\overline{\partial}^*(\overline{U}^* \wedge \overline{V}^*)(x) = 0$, as this form involves only the value $dW(x)$ at x. By the pointwise Lefschetz isomorphism, the pre-image of $\overline{U}^* \wedge \overline{V}^*$ in the hard Lefschetz theorem is given by $U \wedge V \in H^0(X, \Omega^1_X \otimes -K_X) \cong H^0(X, \wedge^2 T_X) \cong H^0(A, K_A)$. It defines a foliation of T_X generated by U, V, which has leaves $A \times \{x\}$ $(x \in \mathbb{P}^1)$.

2.5. Counterexample to coherence

In this section, we wonder whether it is possible to replace the multiplier ideal sheaf by its "lower semi-continuous regularization", i.e.

$$\mathcal{I}_-(\varphi) := \bigcap_{\delta > 0} \mathcal{I}((1-\delta)\varphi),$$

which could be thought of as some sort of limit $\lim_{\delta \to 0_+} \mathcal{I}((1-\delta)\varphi)$. A priori, as an infinite intersection of ideal sheaves, this lower semi-continuous regularization might not be coherent. It contains certainly $\mathcal{I}(\varphi)$ and can be different from it if 1 is a jumping coefficient of the multiplier ideal sheaf. In this section, we show by a counterexample that the above infinite intersection $\bigcap_{\delta>0} \mathcal{I}((1-\delta)\varphi)$ need not be coherent for arbitrary psh functions; hence some further conditions should be added to ensure coherence and possible applications to algebraic geometry, thanks to Serre's GAGA theorem [**Ser56**].

PROPOSITION 2.10. *Let B be the ball of radius $\frac{1}{2}$ centred at 0 in $\mathbb{C}^2$, and consider the plurisubharmonic function*

$$\varphi(z, w) = \log|z| + \sum_{k \geqslant 1} \varepsilon_k \log(|z| + |w - a_k|^{N_k})$$

where a_k is any sequence converging to 0 smaller than $\frac{1}{2}$ and $\varepsilon_k > 0$ and $N_k \in \mathbb{N}^*$ are suitable numbers (to be determined later). Then φ defines multiplier sheaves such that the intersection ideal $\bigcap_{\delta>0} \mathcal{I}((1-\delta)\varphi)$ is not coherent.

The potential used above is a modification of the one given in [**GL16**] (and was suggested to the author by Demailly). Assume that the a_k's are distinct and not equal to zero. We recall the following elementary calculation of [**Siu01**].

LEMMA 2.11. *Let a, b, and c be some positive numbers such that a and $c(1 - \frac{\lceil a \rceil - a}{b})$ are not integers and $\lceil a \rceil - a < b < 1$. Let $p_0 = \lceil a - 1 \rceil$ and $q_0 = \lfloor c(1 - \frac{\lceil a \rceil - a}{b}) \rfloor$. Then on $\mathbb{C}^2$, the multiplier ideal sheaf for the weight function*

$$a \log|z| + \log(|z|^b + |w|^c)$$

is generated by z^{p_0+1} and $z^{p_0} w^{q_0}$. Here $\lfloor \cdot \rfloor$ denotes the round-down and $\lceil \cdot \rceil$ denotes the round-up.

Using this lemma, we can calculate the multiplier ideal sheaf at $(0, a_k)$ since near $(0, a_k)$ the function is equisingular to $\log|z| + \varepsilon_k \log(|z| + |w - a_k|^{N_k})$. Using the trivial inequality

$$\frac{1}{2}(\alpha^\gamma + \beta^\gamma) \leqslant (\alpha + \beta)^\gamma \leqslant 2^\gamma(\alpha^\gamma + \beta^\gamma)$$

for α, β, γ non negative, one can easily reduce the required check to the lemma. In order to compute the multiplier ideal sheaf associated to $(1-\delta)\varphi$ at $(0, a_k)$, $0 < \delta < 1$, we apply the lemma to $a = 1 - \delta$, $b = 1 - \delta$ and $c = (1-\delta)N_k\varepsilon_k$. Once ε_k, N_k are fixed, the number $c(1 - \frac{\lceil a \rceil - a}{b})$ is an integer only for countably many values of δ, a situation that does not affect $\mathcal{I}_-(\varphi)$. When ε_k converge to 0 fast enough, φ well define a psh function on B. In particular, we can choose ε_k positive such that $\sum \varepsilon_k < \infty$. By this assumption, $\varphi \geqslant (1+\sum \varepsilon_k)\log|z|$. Hence it is not identically infinite. In particular, φ is the limit of a decreasing sequence of psh functions $\log|z| + \sum_{k_0 \geqslant k \geqslant 1} \varepsilon_k \log(|z| + |w-a_k|^{N_k})$. Hence it is a psh function on B for any choice of N_k.

Now fix $C > 1$ and choose N_k so that $N_k\varepsilon_k \geqslant C$ and $N_k\varepsilon_k$ is not an integer. Consider a given index k. For such a choice and δ small enough, $q_{k,\delta} = \lfloor N_k\varepsilon_k(1-2\delta)\rfloor \geqslant 1$. By the lemma, $\mathcal{I}((1-\delta)\varphi)$ is generated at $(0, a_k)$ by $z, (w-a_k)^{q_{k,\delta}}$. In particular, $(z, (w-a_k)^{\lfloor N_k\varepsilon_k \rfloor}) \subset (\mathcal{I}_-(\varphi), a_k)$. Now we prove that $\mathcal{I}_-(\varphi)$ is not coherent by contradiction. If $\mathcal{I}_-(\varphi)$ is coherent, since B is a Stein manifold, by Cartan theorem A for any $(0, a_k)$ the map $H^0(B, \mathcal{I}_-(\varphi)) \to \mathcal{I}_-(\varphi)_{(0,a_k)}$ is surjective. For any $f \in H^0(B, \mathcal{I}_-(\varphi))$, $f(0, a_k) = 0$ for any k. Since $(0, a_k)$ has a cluster point 0 on the complex line $\{z = 0\}$, we have $f|_{\{z=0\}} \equiv 0$. In other words, f can be divided by z. But $(w-a_k)^{\lfloor N_k\varepsilon_k \rfloor}$ should then be the restriction of such a function f, and this contradiction yields the proposition.

We check below that the coherence may however hold for psh functions that are not too badly behaved. By definition, it is enough to treat the case when 1 is actually a jumping value of the multiplier ideal sheaves $t \mapsto \mathcal{I}(t\varphi)$. First, we observe that when φ has analytic singularity, we have $\mathcal{I}_-(\varphi) = \mathcal{I}((1-\delta)\varphi)$ for $\delta > 0$ small enough, in particular, $\mathcal{I}_-(\varphi)$ is coherent. In fact, if φ has the form $\varphi = \sum \alpha_j \log|g_j|$ where $D_j = g_j^{-1}(0)$ are non-singular irreducible divisors with normal crossings, then $\mathcal{I}(\varphi)$ is the sheaf of functions f on open sets $U \subset X$ such that $\int_U |f|^2 \prod |g_j|^{-2\alpha_j} dV < \infty$. Since locally the g_j can be taken to be coordinate functions from a local coordinate system $(z_1, \ldots, z_n)$, the integrability condition is that f be divisible by $\prod g_j^{m_j}$ where $m_j > \lfloor \alpha_j \rfloor$. Hence $\mathcal{I}(\varphi) = \mathcal{O}(-\lfloor D \rfloor) = \mathcal{O}(-\sum \lfloor \alpha_j \rfloor D_j)$. Saying that 1 is a jumping coefficient in this case means that there exist some index subset J such that for any $j_0 \in J$ we have $\alpha_{j_0} = \lfloor \alpha_{j_0} \rfloor$. In this case for δ small enough we have that

$$\mathcal{I}((1-\delta)\varphi) = \mathcal{O}(-\sum_{j \in J} (\alpha_j + 1) D_j - \sum_{j \notin J} \lfloor \alpha_j \rfloor D_j)$$

and the conclusion follows. More generally, if φ has arbitrary analytic singularity, there exists a smooth modification $\nu : \tilde{X} \to X$ of X such that $\nu^*\mathcal{I}(\varphi)$ is an invertible sheaf $\mathcal{O}(-D)$ associated with a normal crossing divisor $D = \sum \lambda_j D_j$, where (D_j) are the components of the exceptional divisor of ν. Now, we have $K_{\tilde{X}} = \nu^* K_X + R$ where $R = \sum \rho_j D_j$ is the zero divisor of the Jacobian determinant of the blow-up map. By the direct image formula, we get

$$\mathcal{I}(\varphi) = \nu_*(\mathcal{O}(R) \otimes \mathcal{I}(\varphi \circ \nu)),$$

and the proof is reduced to the divisorial case.

Even more generally, for any psh function φ and any psh function ψ with zero Lelong numbers (i.e., for every x, $\nu(\psi, x) = 0$), we have $\mathcal{I}(\varphi) = \mathcal{I}(\varphi + \psi)$ (cf. Proposition 2.3 [**Kim15**]). By the above discussion we thus get $\mathcal{I}_-(\varphi + \psi) = \mathcal{I}((1-\delta)(\varphi + \psi))$ for $\delta > 0$ small if φ has analytic singularities.

In particular, when X is 1-dimensional, Siu's decomposition theorem [**Siu74**] can be used, to decompose $dd^c\varphi$ into the sum of a convergent series of Dirac masses and of a current with zero Lelong numbers; only the locally finite set of points where the Lelong number number is at least 1 plays a role; we then see that $\mathcal{I}_-(\varphi) = \mathcal{I}((1-\delta)\varphi)$ for δ small enough, hence $\mathcal{I}_-(\varphi)$ is coherent. More generally, the following variant of Nadel's proof on the coherence of multiplier ideal sheaf [**Nad90**] can be exploited.

LEMMA 2.12. *For any psh function φ on $\Omega \subset X$ such that $E_1(\varphi) := \{x; \nu(\varphi, x) \geqslant 1\}$ consists of isolated points, the sheaf $\mathcal{I}_-(\varphi)$ is a coherent sheaf of ideals over Ω.*

PROOF. We follow the proof of Nadel. Without loss of generality, we can assume that Ω is the unit ball. By the strong noetherian property of coherent sheaves, the family of sheaves generated by finite subsets of $H^2_-(\Omega, \varphi) := \{f \in \mathcal{O}_\Omega(\Omega); \int_\Omega |f|^2 e^{-2(1-\delta)\varphi} < \infty, \forall \delta \in]0,1[\}$ has a maximal element on each compact subset of Ω, hence $H^2_-(\Omega, \varphi)$ generates a coherent ideal sheaf $\mathcal{J}$ in $\mathcal{O}_\Omega$. By definition we have $\mathcal{J} \subset \mathcal{I}_-(\varphi)$. We will prove that in fact $\mathcal{J} = \mathcal{I}_-(\varphi)$, which shows in particular that $\mathcal{I}_-(\varphi)$ is coherent.

For the other direction, it is enough to prove that $\mathcal{J}_x + \mathcal{I}_-(\varphi)_x \cap m_x^{s+1} = \mathcal{I}_-(\varphi)_x$ for every integer s, by the Krull lemma. Let $f \in \mathcal{I}_-(\varphi)_x$ be defined in a neighbourhood V of x and let θ be a cut-off function with support in V such that $\theta = 1$ in some neighbourhood of x. We solve the $\overline{\partial}$ equation $\overline{\partial} u = \overline{\partial}(\theta f)$ by Hörmander's L^2 estimates ,with respect to the strictly psh weight

$$\tilde{\varphi}(z) := \varphi(z) + (n+s)\log|z-x| + |z|^2.$$

The integrability is ensured by the fact that $\overline{\partial}(\theta f)$ vanishes near x and the Skoda integrability theorem [**Sko72**]. We remark that the Lelong number outside a small open neighbourhood of 0 is strictly less than 1 pointwise by the assumption that $E_1(\varphi)$ is isolated at x.

Hence we get a solution u such that $\int_\Omega |u|^2 e^{-2\varphi}|z-x|^{-2(n+s)}d\lambda < \infty$, thus $F = \theta f - u$ is holomorphic. $F \in H^2_-(\Omega,\varphi)$ as a sum of a function in $L^2(\Omega,\varphi)$ and a function in $H^2_-(\Omega,\varphi)$. Moreover, $f_x - F_x = u_x \in \mathcal{I}_-(\varphi)_x \cap m_x^{s+1}$. This finishes the proof. □

2.6. On the optimality of multiplier ideal sheaves

We study here whether the ideal sheaves $\mathcal{I}(\varphi)$ involved in the hard Lefschetz theorem can be replaced by ideals $\mathcal{I}((1-\delta)\varphi) \supset \mathcal{I}(\varphi)$. In other words, if (L,h) is a pseudo-effective line bundle on a compact Kähler manifold (X,ω) of dimension n, $i\Theta_{L,h} \geqslant 0$ its curvature current and $\mathcal{I}(h)$ the associated multiplier ideal sheaf, we study whether for any $\delta \in [0,1]$ small enough the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism

$$\Phi^q_{\omega,h} : H^0(X,\Omega_X^{n-q}\otimes L\otimes \mathcal{I}((1-\delta)h)) \longrightarrow H^q(X,\Omega_X^n\otimes L\otimes\mathcal{I}((1-\delta)h)).$$

First, we recall the following special case of the hard Lefschetz theorem. Assume that L admits a smooth metric h_0 such that its curvature form α is semi-positive. Then, the wedge multiplication operator $\omega^q\wedge\bullet$ induces a surjective morphism for any $\delta\in[0,1]$

$$\Phi^q_{\omega,h} : H^0(X,\Omega_X^{n-q}\otimes L\otimes \mathcal{I}((1-\delta)h)) \longrightarrow H^q(X,\Omega_X^n\otimes L\otimes\mathcal{I}((1-\delta)h)).$$

The proof of this case just consists of applying the hard Lefschetz theorem to the Hermitian line bundle $(L, h_0^\delta h^{1-\delta})$. If the line bundle admits a positive singular metric h_0 such that the corresponding Lelong numbers are equal to 0 at every point, by Proposition 2.3 in [**Kim15**], for any $\delta\in[0,1]$, the metric $(L,h_0^\delta h^{1-\delta})$ has a multiplier ideal sheaf equal to $\mathcal{I}((1-\delta)h)$. Then the bundle valued hard Lefschetz theorem also implies the surjectivity property.

The condition that the line bundle admits a positive singular metric such that the Lelong number of this metric is pointwise 0 implies in particular by regularization (see e.g. Theorem 14.12 in [**Dem12a**]) that the line bundle is nef. However, the converse is false by example 1.7 in [**DPS94**], in which the only positive singular metric on the nef line bundle is the singular one induced by a section. An alternative example is given in [**Koi17**]: there, Koike considers the anticanonical line bundle $-K_X$ of the blow-up of $\mathbb{P}^2$ at 9 points, and shows that there exists some configuration of the nine points such that $-K_X$ is nef, while the singular metric with minimal singularities is induced by a section $s\in H^0(X,-K_X)\backslash\{0\}$. In particular, there exists no singular metric on $-K_X$ with curvature $\geqslant 0$, such that the Lelong number of the singular metric is equal to 0 at each point.

This condition is also non equivalent to the semipositivity of the line bundle, although it is obviously implied by semipositivity. A counter example for the converse direction is provided by [**BEGZ10**], example 5.4 and [**Kim07**], example 2.14. Take a non-trivial rank 2 extension V of the trivial line bundle by itself, over an elliptic curve C, and an ample line bundle A over C. Then consider $X = \mathbb{P}(V\oplus A)$ and the associated line bundle $\mathcal{O}(1)$. It is big and nef, and this is enough to conclude that it admits a semi-positive singular metric with Lelong numbers equal to 0. In fact, it is enough to argue for the semi-positive metric with minimal singularity. By the Kodaira lemma, there exists $m_0\in\mathbb{N}$ such that $\mathcal{O}(m_0) = \tilde{A}+E$ where $\tilde{A}$ is an ample line bundle over X and E is an effective line bundle over X. For any $m\geqslant m_0$, a metric on $\mathcal{O}(m)$ is induced by a smooth strictly positive metric on the ample line bundle $\tilde{A}+\mathcal{O}((m-m_0))$ and by a singular metric induced by a non zero section on the effective line bundle E. This metric itself induces a metric on $\mathcal{O}(1)$ which is by definition more singular than the metric with minimal singularity. It has pointwise Lelong numbers at most equal to $\frac{1}{m}$. Hence the metric with minimal singularity has Lelong numbers equal to 0 pointwise. However, $\mathcal{O}(1)$ cannot admit a smooth semi-positive metric: for this, note that X has a submanifold $Y\cong\mathbb{P}(V)$ given by the surjective bundle morphism $V\oplus A\to V$; a smooth semipositive metric on $\mathcal{O}(1)$ would induce a smooth semipositive metric on $\mathcal{O}_Y(1)$ by restriction, which is impossible by [**DPS94**].

As we have seen, the extension is possible if the minimal metric is not "too bad". This is also true in the purely exceptional case, as we will now see.

Let X be the blow up a point of some smooth complex manifold Y of dimension n. Denote by E the exceptional divisor. Let L be a semi-positive line bundle on X such that $L|_E$ is not trivial on E. Consider the line bundle $L+E$. Take h to be metric on $L+E$ induced by the canonical section of the effective divisor E, tensor product with the given semi-positive metric on L. We start by remarking that for any $\delta\in\,]0,1]$ we have $\mathcal{I}((1-\delta)h) = \mathcal{O}_X$. Hence the lower semi-continuous regularization of the multiplier ideal sheaf is trivial. We claim that the map

$$H^0(X,\Omega_X^{n-q}\otimes L\otimes E)\to H^q(X,K_X\otimes L\otimes E)$$

is surjective for every $q\geqslant 1$. First, by the hard Lefschetz theorem, we find that

$$H^0(X,\Omega_X^{n-q}\otimes L)\to H^q(X,K_X\otimes L)$$

is surjective for every $q \geqslant 1$. On the other hand, we have the following commutative diagram

$$\begin{array}{ccc} H^0(X, \Omega_X^{n-q} \otimes L) & \longrightarrow & H^0(X, \Omega_X^{n-q} \otimes L \otimes E) \\ \downarrow & & \downarrow \\ H^q(X, K_X \otimes L) & \longrightarrow & H^q(X, K_X \otimes L \otimes E). \end{array}$$

To show that the right arrow is surjective, it is enough to show that the bottom arrow is surjective. By Serre duality, this is equivalent to proving that

$$H^{n-q}(X, -L-E) \to H^{n-q}(X, -L)$$

is injective. By considering the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{O}_X(-L-E) \to \mathcal{O}_X(-L) \to \mathcal{O}(-L)|_E \to 0,$$

it is enough to show that for any $q \geqslant 1$

$$H^{n-q-1}(E, -L|_E) = 0.$$

Remind that $E \cong \mathbb{P}^{n-1}$. For any $q \in \mathbb{Z}$, for $0 < i < n-1$, we have that $H^i(\mathbb{P}^{n-1}, \mathcal{O}(q)) = 0$. Remind also that the Picard group of $\mathbb{P}^{n-1}$ is $\mathbb{Z}$. This finishes the case $q \leqslant n-2$, and the case $q = n-1$ also holds, since our assumptions $L \geqslant 0$ and $L|_E$ non trivial imply $H^0(E, -L|_E) = 0$. The same arguments also work for $L = \mathcal{O}_X$. We have an exact sequence

$$H^0(X, \mathcal{O}_X) \to H^0(E, \mathcal{O}_E) \to H^1(X, \mathcal{O}(-E)) \to H^1(X, \mathcal{O}_X).$$

The first morphism is an isomorphism – it is just a restriction morphism applied to constant functions – hence $H^1(X, \mathcal{O}(-E)) \to H^1(X, \mathcal{O}_X)$ is injective.

In general, as discussed in [**DPS96**], the minimal singular metric of a psef line bundle can still be very singular, and this fact might lead to a non coherent lower semi-continuous regularization of the multiplier ideal sheaf. It thus seems to be a difficult problem to improve the hard Lefschetz theorem by replacing the given multiplier ideal sheaf by its lower semi-continuous regularization, if at all possible.

CHAPTER 3

Numerical dimension and vanishing theorems

In the first part of this chapter, we compare different definitions of numerical dimension of a psef class or a psef line bundle. Although it is perhaps well-known for experts, we still give the complete proofs here. In the second part of this chapter, some L^2 vanishing theorems in terms of numerical dimension are given. The variant of Junyan Cao's vanishing will be also used in the next chapter to give a Kawamata-Viehweg type vanishing theorem without multiplier ideal sheaf.

3.1. Numerical dimension

We first recall the Kähler version of the definition of numerical dimension as stated in [**Dem14**]. For L a psef line bundle on a compact Kähler manifold (X, ω), we define

$$\mathrm{nd}(L) := \max\{p \in [0, n]; \exists c > 0, \forall \varepsilon > 0, \exists h_\varepsilon, i\Theta_{L,h_\varepsilon} \geqslant -\varepsilon\omega, \text{such that} \int_{X\setminus Z_\varepsilon} (i\Theta_{L,h_\varepsilon} + \varepsilon\omega)^p \wedge \omega^{n-p} \geqslant c\}.$$

Here the metrics h_ε are supposed to have analytic singularities and Z_ε is the singular set of the metric. Fix a family of metric h_ε as stated in the definition. For such metrics, for $p > \mathrm{nd}(L)$, by definition,

$$\lim_{\varepsilon\to 0} \int_{X\setminus Z_\varepsilon} (i\Theta_{L,h_\varepsilon} + \varepsilon\omega)^p \wedge \omega^{n-p} = 0.$$

If the line bundle L is nef, we can take h_ε to be smooth and $Z_\varepsilon = \varnothing$, (cf. the proof of point (i) in [**BDPP13**], or [**Bou02a**]) and we have for any p

$$\lim_{\varepsilon\to 0} \int_{X\setminus Z_\varepsilon} (i\Theta_{L,h_\varepsilon} + \varepsilon\omega)^p \wedge \omega^{n-p} = \lim_{\varepsilon\to 0} \int_X (c_1(L) + \varepsilon\omega)^p \wedge \omega^{n-p} = \int_X c_1(L)^p \wedge \omega^{n-p}.$$

The integral condition in the definition of the numerical dimension in the nef case means that $p = \mathrm{nd}(L)$ is the largest integer such that

$$\int_X c_1(L)^p \wedge \omega^{n-p} \neq 0.$$

Since for each p, $c_1(L)^p$ can be represented by a positive closed (p,p)-current, the triviality of the mass is equivalent to the triviality of the current. In other words,

$$\mathrm{nd}(L) = \max\{p; c_1(L)^p \neq 0\},$$

which corresponds to the definition of the numerical dimension for a nef line bundle.

In fact denoting $\alpha := c_1(L)$, the numerical dimension for the psef line bundle L is the numerical dimension of the class of α defined in [**BEGZ10**].

To see it, we need the definition of moving intersection product for any psef $(1,1)$-class α for any $1 \leqslant p \leqslant n$. We start by recalling the following definition.

Definition 3.1. (See [**DPS01**]). *Let φ_1, φ_2 be two quasi-psh functions on X (i.e. $i\partial\overline{\partial}\varphi_i \geqslant -C\omega$ in the sense of currents for some $C \geqslant 0$). Then, φ_1 is less singular than φ_2 (and write $\varphi_1 \preceq \varphi_2$) if we have $\varphi_2 \leqslant \varphi_1 + C_1$ for some constant C_1. Let α be a psef class in $H^{1,1}_{BC}(X, \mathbb{R})$ and γ be a smooth real $(1,1)$-form. Let $T_1, T_2, \theta \in \alpha$ with θ smooth and such that $T_i = \theta + i\partial\overline{\partial}\varphi_i (i = 1, 2)$. φ_i is well defined up to constant since X is compact. We say $T_1 \preceq T_2$ if and only if $\varphi_1 \preceq \varphi_2$.*

The minimal element $T_{min,\gamma}$ with the pre-order relation $\preceq$ exists by taking the upper semi-continuous envelope of all φ_i such that $\theta + \gamma + i\partial\overline{\partial}\varphi_i \geqslant 0$ and $\sup_X \varphi_i = 0$.

The positive product defined in [**BEGZ10**] is the real (p,p) cohomology class $\langle\alpha^p\rangle$ of the limit

$$\langle\alpha^p\rangle := \lim_{\delta\to 0}\{\langle T^p_{\min,\delta\omega}\rangle\}$$

where $T_{\min,\delta\omega}$ is the positive current with minimal singularity in the class $\alpha + \delta\{\omega\}$ and $\langle T^p_{\min,\delta\omega}\rangle$ is the non-pluripolar product. The numerical dimension of α is defined as

$$\mathrm{nd}(\alpha) := \max\{p | \langle\alpha^p\rangle \neq 0\}$$

which is also equal to $\max\{p|\int_X \langle \alpha^p \rangle \wedge \omega^{n-p} > 0\}$. The equivalence of two numerical dimensions given here is an adapted version of arguments in [**Tos**]. We will also need the definition of non-Kähler locus defined in [**Bou02b**].

DEFINITION 3.2. *Let α be a big class in $H^{1,1}(X,\mathbb{R})$. The non-Kähler locus is defined to be*

$$E_{nK}(\alpha) := \bigcap_{T\in\alpha} E_+(T)$$

where T ranges all Kähler currents in α and $E_+(T) := \bigcup_{c>0} E_c(T)$.

We will also need the following lemma in [**Bou02b**] which implies in particular that the non-Kähler locus is in fact an analytic set.

LEMMA 3.3. *Let α be a big class. There exists a Kähler current $\tilde{T}$ with analytic singularities such that $E_{nK}(\alpha) = E_+(\tilde{T})$.*

PROOF. By regularization, we have equivalently that $E_{nK}(\alpha) = \bigcap_{T\in\alpha} E_+(T)$ where T ranges all Kähler currents with analytic singularities. Since T has analytic singularities, $E_+(T)$ is a proper analytic set. By the strong Noether property, there exist $T_i (i \in I)$ finite Kähler currents with analytic singularities such that $E_{nK}(T) = \bigcap_{i\in I} E_+(T_i)$. Take a regularization $\tilde{T}$ of $\min_{i\in I} T_i$. Then we have

$$\nu(\tilde{T}, x) \leqslant \min_{i\in I} \nu(T_i, x)$$

for any $x \in X$. In particular, this implies that

$$E_+(\tilde{T}) \subset \bigcap_{i\in I} E_+(T_i).$$

Since $\tilde{T}$ itself is a Kähler current with analytic singularities, we get in fact an equality in the statement. □

We will need the following result stated in [**BEGZ10**, Prop. 1.16].

PROPOSITION 3.4. *For $j = 1, \cdots, p$, let T_j and T'_j be two closed positive $(1,1)$-currents with small unbounded locus (i.e. there exists a (locally) complete pluripolar closed subset A of X outside which the potential is locally bounded) in the same cohomology class, and assume also that T_j is less singular than T'_j. Then the cohomology classes of their non-pluripolar products satisfy $\{\langle T_1 \wedge \cdots \wedge T_p \rangle\} \geqslant \{\langle T'_1 \wedge \cdots \wedge T'_p \rangle\}$ in $H^{p,p}(X,\mathbb{R})$, where $\geqslant$ means that the difference is pseudo-effective, i.e. representable by a closed positive (p,p)-current.*

Now we are prepared to prove that

PROPOSITION 3.5. *For L a psef line bundle, we have that*

$$\mathrm{nd}(c_1(L)) = \mathrm{nd}(L).$$

PROOF. Let h_ε be a family of metric with analytic singularities as stated in the definition of $\mathrm{nd}(L)$. Denote $A_\varepsilon := E_{nK}(\alpha + \varepsilon\{\omega\})$. Since $T_{\min,\varepsilon\omega} \preceq i\Theta_{L,h_\varepsilon} + \varepsilon\omega$, we have by proposition 3.4 that for any $1 \leqslant p \leqslant n$

$$\int_{X\setminus Z_\varepsilon} (i\Theta_{L,h_\varepsilon} + \varepsilon\omega)^p \wedge \omega^{n-p} = \int_{X\setminus(Z_\varepsilon\cup A_\varepsilon)} \langle (i\Theta_{L,h_\varepsilon} + \varepsilon\omega)^p \rangle \wedge \omega^{n-p} \leqslant \int_{X\setminus(Z_\varepsilon\cup A_\varepsilon)} \langle T^p_{\min,\varepsilon\omega} \rangle \wedge \omega^{n-p}.$$

The right hand term is the same as $\int_X \langle T^p_{\min,\varepsilon\omega} \rangle \wedge \omega^{n-p}$, since the non-pluripolar product has no mass on any analytic set. It has limit equal to $\int_X \langle c_1(L)^p \rangle \wedge \omega^{n-p}$. In particular, this implies that $\mathrm{nd}(c_1(L)) \geqslant \mathrm{nd}(L)$. We remark that A_ε is an analytic set hence is a small unbounded locus.

For the other direction, we construct a family of metrics with analytic singularities with control of the Monge-Ampère mass from below. Denote $p := \mathrm{nd}(c_1(L))$. Since $\langle c_1(L)^p \rangle \neq 0$, for ε small enough, $\langle T^p_{\min,\varepsilon\omega} \rangle \neq 0$ and

$$\int_X \langle T^p_{\min,\varepsilon\omega} \rangle \wedge \omega^{n-p} \geqslant c$$

for some constant $c > 0$ uniform for ε small enough. Let $T_{\varepsilon,\delta}$ be a sequence of regularisation of $T_{\min,\varepsilon\omega}$ with analytic singularities such that

$$T_{\varepsilon,\delta} \geqslant -\delta\omega$$

and the potentials of $T_{\varepsilon,\delta}$ decrease to the potential of $T_{\min,\varepsilon\omega}$. Hence $T_{\min,\varepsilon\omega} + \varepsilon\omega$ and $T_{\varepsilon,\delta} + \varepsilon\omega$ are closed positive currents in the cohomology class $\alpha + 2\varepsilon\{\omega\}$ if $\delta \leqslant \varepsilon$. By lemma 3.3, $A_\varepsilon = E_+(T_\varepsilon)$ for some Kähler current with analytic singularities. Thus $T_{\min,\varepsilon\omega} \preceq T_\varepsilon$, whose potential is locally bounded outside A_ε, as the potential of T_ε is. So the potentials of $T_{\varepsilon,\delta}$ are also locally bounded outside A_ε. By weak continuity of the Bedford-Taylor Monge-Ampère operators with respect to decreasing sequences of functions, we have on $X\backslash A_\varepsilon$ that,

$$(T_{\varepsilon,\delta}+\varepsilon\omega)^l \to (T_{\min,\varepsilon\omega}+\varepsilon\omega)^l$$

for any l. By the Fatou lemma, we have that

$$\int_{X\backslash A_\varepsilon}(T_{\min,\varepsilon\omega}+\varepsilon\omega)^p\wedge\omega^{n-p}\leqslant \liminf_{\delta\to 0}\int_{X\backslash A_\varepsilon}(T_{\varepsilon,\delta}+\varepsilon\omega)^p\wedge\omega^{n-p}.$$

Take any sequence $\delta(\varepsilon)$ such that $\delta(\varepsilon)\leqslant\varepsilon$ and $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$. Let h_ε be the metric on L with analytic singularities such that $i\Theta_{L,h_\varepsilon}=T_{\varepsilon,\delta(\varepsilon)}-\varepsilon\omega$. The metric h_ε is uniquely defined up to a multiple. To normalise it, we can assume for example, that the maximum of the potentials on X equals to 0. Hence we have that

$$i\Theta_{L,h_\varepsilon}\geqslant -(\varepsilon+\delta(\varepsilon))\omega\geqslant -2\varepsilon\omega.$$

Then the sequence of metric satisfies the condition demanded in the definition of $\mathrm{nd}(L)$. □

REMARK 3.6. $T_{\min,\varepsilon'\omega}\preceq T_{\min,\varepsilon\omega}+(\varepsilon'-\varepsilon)\omega$ for any $\varepsilon\leqslant\varepsilon'$. Denote $T_{\min,\varepsilon\omega}=\theta+\varepsilon\omega+i\partial\overline{\partial}\varphi_{\min,\varepsilon\omega}$. We can arrange that

$$\varphi_{\min,0}\leqslant\varphi_{\min,\varepsilon\omega}\leqslant\varphi_{\min,\varepsilon'\omega}.$$

The Bergman kernel regularisation perserves the ordering of potentials (cf. [**Dem14**]), so we have

$$\varphi_{0,\delta}\leqslant\varphi_{\varepsilon,\delta}\leqslant\varphi_{\varepsilon',\delta}.$$

for any $\delta>0$. If $\delta(\varepsilon)$ is increasing with respect to ε, by the proof of the proposition, we can choose the metric h_ε to be decreasing with respect to ε. The limit of $\varphi_{\varepsilon,\delta(\varepsilon)}$ as $\varepsilon\to 0$ is equal to $\varphi_{\min,0}$ corresponding to the metric with minimal singularities on L.

REMARK 3.7. Similar to the definition of [**Dem14**] for the numerical dimension of a psef line bundle, we can define in a similar way the numerical dimension of a psef cohomology class. The above proof in fact shows that the two definitions of numerical dimension of a psef cohomology class coincide.

In the rest of the section, we show that the movable intersection of cohomology classes defined in [**BDPP13**] coincides with the positive product defined in [**BEGZ10**] which might be well-known for experts. In particular, using movable intersection instead of positive intersection, we can give a third equivalent definition of numerical dimension of a psef cohomology class.

To distinguish the notations, we will denote by $\langle,\rangle$ for the positive product and $\langle\langle,\rangle\rangle$ for the movable intersection. In other words, it shows that the numerical definition of the psef class α can either defined to be the largest number such that $\langle\alpha^p\rangle\neq 0$ or such that $\langle\langle\alpha^p\rangle\rangle\neq 0$.

We start by recalling the definition of the movable intersection given in Theorem 3.5 of [**BDPP13**]. Let (X,ω) be a compact Kähler manifold and α be a psef class on X. To simplify the notations, we only define $\langle\langle\alpha^p\rangle\rangle$ where the general case is similar. First assume that α is big. To know the value of the product pairing with any $(n-p,n-p)$-smooth form, it is enough to know its value with a countable dense family of smooth forms. Since for any $(n-p,n-p)$-smooth form u, $u=C\omega^{n-p}-(C\omega^{n-p}-u)$. For $C>0$ big enough, both $C\omega^{n-p}$ and $C\omega^{n-p}-u$ is strongly positive forms (since X is compact). Thus it is enough to consider only a countable dense family of strongly positive forms.

Fix a smooth closed $(n-p,n-p)$ strongly-positive form u on X. We select Kähler currents $T\in\alpha$ with analytic singularities, and a log-resolution $\mu:\tilde{X}\to X$ such that

$$\mu^*T=[E]+\beta$$

where $[E]$ is the current associated to a $\mathbb{R}$-divisor and β a semi-positive form. We consider the direct image current $\mu_*(\beta\wedge\ldots\wedge\beta)$. Given two closed positive $(1,1)$-currents $T_1,T_2\in\alpha$, we write $T_j=\theta+i\partial\overline{\partial}\varphi_i$ $(j=1,2)$ for some smooth form $\theta\in\alpha$. Define $T:=\theta+i\partial\overline{\partial}\max(\varphi_1,\varphi_2)$. We get a current with analytic singularities less singular than these two currents. By this way, if we change the representative T with another current T', we may always take a simultaneous log-resolution μ such that $\mu^*T'=[E']+\beta'$, and we can always assume that $E'\leqslant E$. By calculation, we find

$$\int_{\tilde{X}}\beta'\wedge\ldots\wedge\beta'\wedge\mu^*u\geqslant\int_{\tilde{X}}\beta\ldots\wedge\beta\wedge\mu^*u.$$

In fact, we have

$$\begin{aligned}\int_{\tilde{X}}\beta'\wedge\beta\wedge\ldots\wedge\beta\wedge\mu^*u&=\int_{\tilde{X}}(\beta+[E]-[E'])\wedge\beta\wedge\ldots\wedge\beta\wedge\mu^*u\\&\geqslant\int_{\tilde{X}}\beta\ldots\wedge\beta\wedge\mu^*u.\end{aligned}$$

A similar substitution applies to change all β' by β.

It can be shown that the closed positive currents $\mu_*(\beta \wedge \ldots \wedge \beta)$ are uniformly bounded in mass. In fact, for any Kähler metric ω in X, there exists a constant $C > 0$ such that $C\{\omega\} - \alpha$ is a Kähler class. In other words, there exists some Kähler form γ on X in the cohomology class $C\{\omega\} - \alpha$. By pulling back with μ, we find

$$C\mu^*\omega - ([E] + \beta) \equiv \mu^*\gamma,$$

hence $\beta \equiv C\mu^*\omega - ([E] + \mu^*\gamma)$ where $\equiv$ means in the same cohomology class. By performing again a substitution in the integrals, we find

$$\int_{\tilde{X}} \beta^k \wedge \mu^*\omega^{n-k} \leqslant C^k \int_{\tilde{X}} \mu^*\omega^n = C^k \int_X \omega^n.$$

For each of the integrals associated with a countable dense family of forms u, the supremum of $\int_{\tilde{X}} \beta \ldots \wedge \beta \wedge \mu^* u$ is achieved by a sequence of currents $(\mu_m)_*(\beta_m \wedge \ldots \wedge \beta_m)$ obtained as direct images by a suitable sequence of modifications $\mu_m : \tilde{X}_m \to X$ and suitable β_m's. By extracting a subsequence, we can achieve that this sequence is weakly convergent and we set

$$\langle\langle \alpha^p \rangle\rangle := \lim_{m \to +\infty} \uparrow \{(\mu_m)_*(\beta_m \wedge \ldots \wedge \beta_m)\}$$

In the general case when α is only psef, we define

$$\langle\langle \alpha^p \rangle\rangle := \lim_{\delta \downarrow 0} \downarrow \langle\langle (\alpha + \delta\{\omega\})^p \rangle\rangle.$$

We now prove that these two products coincide for psef classes. Since in the two cases, the products are the limit of the products of big classes in $H^{p,p}(X, \mathbb{R})$, without loss of generality, we can assume α to be big. We state it in the following lemma.

LEMMA 3.8. *For α a big class, for any $1 \leqslant p \leqslant n$, we have*

$$\langle \alpha^p \rangle = \langle\langle \alpha^p \rangle\rangle.$$

PROOF. It is enough to prove by duality that for any smooth closed $(n-p, n-p)$-strongly positive form u on X, we have

$$\int_X \langle \alpha^p \rangle \wedge u = \int_X \langle\langle \alpha^p \rangle\rangle \wedge u.$$

Denote by A the non-Kähler locus of α which is the pole of some Kähler current T in α with analytic singularities. Denote by $T_{\min} \in \alpha$, the current with minimal singularities in α. By definition, it is less singular than the Kähler current T. In particular, the potential of $T_{\min}$ is locally bounded outside A. Let T_ε be a regularisation of $T_{\min}$ such that $T_\varepsilon \geqslant -\varepsilon\omega$. Their potentials are locally bounded outside A as $T_{\min}$'s is. By weak continuity of the Bedford-Taylor Monge-Ampère operator along decreasing sequences we have on $X \backslash A$ for any $\delta > 0$ and $\varepsilon \leqslant \delta$

$$(T_{\min} + \delta\omega)^p \to (T_\varepsilon + \delta\omega)^p$$

as current. By Fatou lemma we have

$$\int_{X \backslash A} (T_{\min} + \delta\omega)^p \wedge u \leqslant \liminf_{\varepsilon \to 0} \int_{X \backslash A} (T_\varepsilon + \delta\omega)^p \wedge u.$$

Since the non-pluripolar product of currents has no mass along any analytic set, the left hand term has limit equal to $\int_X \langle \alpha^p \rangle \wedge u$. We remark that both $\langle \alpha^p \rangle$ and $\langle\langle \alpha^p \rangle\rangle$ depend continuously on α in the big cone. Since T_ε has analytic singularities, there exists a modification $\mu : \tilde{X} \to X$ such that

$$\mu^* T_\varepsilon = [E] + \beta$$

with E associated to a $\mathbb{R}$-divisor and μ a biholomorphism on $X \backslash A$. So we have

$$\int_{X \backslash A} (T_\varepsilon + \delta\omega)^p \wedge u = \int_{X \backslash A} \mu_*(\beta + \delta\mu^*\omega)^p \wedge u \leqslant \int_X \langle\langle (\alpha + \delta\omega)^p \rangle\rangle \wedge u.$$

We remark that $T_\varepsilon + \delta\omega$ is a Kähler current with analytic singularities for $\varepsilon < \delta$. When $\delta \to 0$, we have

$$\liminf_{\varepsilon \to 0} \int_{X \backslash A} (T_\varepsilon + \delta\omega)^p \wedge u \leqslant \int_X \langle\langle \alpha^p \rangle\rangle \wedge u.$$

For the other direction, for any $T \in \alpha$ a Kähler current with analytic singularities, there exists a modification $\mu : \tilde{X} \to X$ such that

$$\mu^* T = [E] + \beta$$

as above. By the definition of non Kähler locus, T is locally bounded outside A. The modification can be achieved by a composition of blow-ups with smooth centres in A, so μ is a biholomorphism on $X\backslash A$. So we have

$$\int_{X\backslash A} T^p \wedge u = \int_{X\backslash A} \mu_*(\beta^p) \wedge u = \int_{\tilde{X}} \beta^p \wedge \mu^* u \leqslant \int_{X\backslash A} \langle T^p_{\min} \rangle \wedge u = \int_X \langle T^p_{\min} \rangle \wedge u.$$

The inequality use proposition 1.16 in [**BEGZ10**] cited above and the fact that $T_{\min}$ is less singular than T. By taking supremum among all Kähler currents with analytic singularities in the cohomology class α, we have

$$\int_X \langle \alpha^p \rangle \wedge u \geqslant \int_X \langle\langle \alpha^p \rangle\rangle \wedge u.$$

□

3.2. Vanishing theorems

In this section, we generalise some L^2 vanishing theorems in terms of numerical dimension of a psef line bundle. At the end of this section, we give a variant of Nakano vanishing theorem. The relation of the Nakano vanishing theorem with the others is as follows. Recall the classical Kawamata-Viehweg vanishing theorem states that for a nef line bundle L over a projective manifold X, for any $q > n - \mathrm{nd}(L)$ we have that

$$H^q(X, K_X \otimes L) = 0.$$

It is natural to ask whether the canonical line bundle can be changed by Ω^p_X to get a Nakano type vanishing theorem without strict positivity curvature condition. By the example of Ramanujam, it is not always possible. The last section gives some laboratory discussions.

3.2.1. Bogomolov vanishing theorem.

Let L be a holomorphic line bundle over a compact Kähler manifold X, the Bogomolov vanishing theorem [**Bog**] asserts that

$$H^0(X, \Omega^p_X \otimes L^{-1}) = 0$$

for $p < \kappa(L)$.

In [**Mou98**], the following two versions of Bogomolov vanishing theorem are given.

THEOREM 3.9. *If L is a nef line bundle over a compact Kähler manifold X, then*

$$H^0(X, \Omega^p_X \otimes L^{-1}) = 0$$

for $p < \mathrm{nd}(L)$.

THEOREM 3.10. *If L is a psef line bundle over a compact Kähler manifold X, then*

$$H^0(X, \Omega^p_X \otimes L^{-1}) = 0$$

for $p < e(L)$, where $e(L)$ is the biggest natural number k such that there exists $T \in c_1(L)$ a positive $(1,1)$-current whose absolute part has rank k on a strictly positive Lebesgue measure set on X. (The absolute part exists and is unique by the Lebesgue decomposition theorem.)

In this note, we give the following improved version of the Bogomolov vanishing theorem, following the ideas of [**Mou98**].

THEOREM 3.11. *Let L be a psef line bundle over a compact Kähler manifold X. Then we have*

$$H^0(X, \Omega^p_X \otimes L^{-1}) = 0$$

if $p < \mathrm{nd}(L)$.

In this section, we prove a numerical dimension version of the Bogomolov vanishing theorem. Now, let us denote $l := \mathrm{nd}(L)$. Then we have

$$\lambda_\varepsilon := \frac{\int_{X\backslash Z_\varepsilon} (i\Theta_{L,h_\varepsilon} + \varepsilon\omega)^n}{\int_X \omega^n} \geqslant c\varepsilon^{n-l}.$$

The first step of the proof consists in the use of Yau's theorem [**Yau78**], so as to show that one can turn the above integral inequality into a pointwise lower bound, more precisely, the inequality $(*)$ given below. Up to a re-parametrisation of ε, we can assume that

$$i\Theta_{L,h_\varepsilon} + \varepsilon\omega \geqslant \frac{\varepsilon}{2}\omega.$$

Let $\nu_\varepsilon : X_\varepsilon \to X$ be a log resolution of the analytic singularities of h_ε. We then have

$$\nu_\varepsilon^*(i\Theta_{L,h_\varepsilon} + \varepsilon\omega) = [D_\varepsilon] + \beta_\varepsilon$$

where $\beta_\varepsilon \geqslant \frac{\varepsilon}{2}\nu_\varepsilon^*\omega \geqslant 0$ is a smooth positive closed $(1,1)$-form on X_ε. It is strictly positive on the complement $X_\varepsilon \backslash E$ of the exceptional divisor E (we denote its irreducible components as E_l). $[D_\varepsilon]$ is the closed positive current associated to a $\mathbb{R}$-divisor. By the theorem of Hironaka [**Hir64**] we can assume that the exceptional divisor is simple normal crossing divisors and the morphism is obtained as a composition of a sequence of blow up with smooth centres. In this situation, there exist arbitrary small numbers $\eta_l > 0$ such that the cohomological class of $\beta_\varepsilon - \sum \eta_l [E_l]$ is a Kähler class (which means that there exists a Kähler form in this class).

Hence we can find a quasi psh function $\hat{\theta}_\varepsilon$ on X_ε such that

$$\hat{\beta}_\varepsilon := \beta_\varepsilon - \sum \eta_l [E_l] + i\partial\overline{\partial}\hat{\theta}_\varepsilon$$

is a Kähler metric on X_ε. By taking η_l small enough, we can assume that

$$\int_{X_\varepsilon} (\hat{\beta}_\varepsilon)^n \geqslant \frac{1}{2}\int_X \beta_\varepsilon^n.$$

The assumption on the numerical dimension implies there exists $c > 0$ such that with $Z_\varepsilon := \nu_\varepsilon(E) \subset X$, we have

$$\int_{X_\varepsilon} \beta_\varepsilon^n = \int_{X\backslash Z_\varepsilon} (i\Theta_{L,h_\varepsilon} + \varepsilon\omega)^n$$

$$\geqslant \binom{n}{l}(\frac{\varepsilon}{2})^{n-l}\int_{X\backslash Z_\varepsilon} (i\Theta_{L,h_\varepsilon} + \frac{\varepsilon}{2}\omega)^l \wedge \omega^{n-l} \geqslant c\varepsilon^{n-l}\int_X \omega^n.$$

Hence we have

$$\int_{X_\varepsilon} (\hat{\beta}_\varepsilon)^n \geqslant \frac{c}{2}\varepsilon^{n-l}\int_X \omega^n.$$

By Yau's theorem [**Yau78**], there exists a quasi-psh potential $\hat{\tau}_\varepsilon$ on X_ε such that $\hat{\beta}_\varepsilon + i\partial\overline{\partial}\hat{\tau}_\varepsilon$ is a Kähler metric on X_ε with any prescribed volume form $\hat{f}$ such that $\int_{X_\varepsilon} \hat{f} = \int_{X_\varepsilon} (\hat{\beta}_\varepsilon)^n$. By the integral condition, we can choose a smooth volume form on X_ε such that

$$(*) \qquad \hat{f} > \frac{c}{3}\varepsilon^{n-l}\nu_\varepsilon^*\omega^n$$

everywhere on X_ε. Fix h a smooth metric on L and let φ_ε be the weight function of h_ε (i.e. $h_\varepsilon = he^{-2\varphi_\varepsilon}$). We impose the additional normalization condition that $\sup_{X_\varepsilon}(\nu_\varepsilon^*\varphi_\varepsilon + \hat{\theta}_\varepsilon + \hat{\tau}_\varepsilon) = 0$.

We now work again on X (e.g. by taking direct images to construct a sequence of singular metrics on X). Consider $\theta_\varepsilon := \nu_{\varepsilon *}\hat{\theta}_\varepsilon$ and $\tau_\varepsilon := \nu_{\varepsilon *}\hat{\tau}_\varepsilon \in L^1_{\mathrm{loc}}(X)$. Define $\Phi_\varepsilon := \varphi_\varepsilon + \theta_\varepsilon + \tau_\varepsilon$. This is a quasi psh potential on X since it satisfies the condition

$$\nu_\varepsilon^*(i\Theta_{L,h} + \varepsilon\omega + i\partial\overline{\partial}\Phi_\varepsilon) = [D_\varepsilon] + \sum \eta_l [E_l] + \hat{\beta}_\varepsilon + i\partial\overline{\partial}\hat{\tau}_\varepsilon \geqslant 0.$$

Define $\tilde{Z}_\varepsilon := \nu_\varepsilon(D_\varepsilon)$ which includes Z_ε since the support of the divisor D_ε includes all components of the exceptional divisor by Hironaka theorem [**Hir64**]. By construction, Φ_ε is smooth on $X\backslash\tilde{Z}_\varepsilon$. By the normalised condition we have that $\sup_X \Phi_\varepsilon = 0$. Since $i\Theta_{L,h} + \varepsilon\omega + i\partial\overline{\partial}\Phi_\varepsilon$ is a family of $(1,1)$-forms in a bounded family of cohomology classes, with the above normalisation, we have, up to taking a subsequence, that the family of quasi-psh potentials Φ_ε converges almost everywhere to $\Phi \in L^1(X)$ by weak compactness. It satisfies that

$$i\Theta_{L,h} + i\partial\overline{\partial}\Phi \geqslant 0.$$

We also have that

$$\nu_\varepsilon^* \mathbb{1}_{X\backslash\tilde{Z}_\varepsilon}(i\Theta_{L,h} + i\partial\overline{\partial}\Phi_\varepsilon + \varepsilon\omega)^n \geqslant \hat{\beta}_\varepsilon^{\,n} \geqslant \frac{c}{3}\varepsilon^{n-l}\nu_\varepsilon^*\omega^n.$$

In other words, on $X\backslash\tilde{Z}_\varepsilon$

$$(i\Theta_{L,h} + i\partial\overline{\partial}\Phi_\varepsilon + \varepsilon\omega)^n \geqslant \frac{c}{3}\varepsilon^{n-l}\omega^n.$$

To use the Bochner-Kodaira-Nakano inequality, we need to change the Kähler metric in such a way that $X\backslash\tilde{Z}_\varepsilon$ becomes a complete manifold. We define a family of Kähler metrics $\omega_{\varepsilon,\delta} := \omega + \delta(i\partial\overline{\partial}\psi_\varepsilon + \omega)$, for $\delta > 0$ which is complete metrics on $X\backslash\tilde{Z}_\varepsilon$, where ψ_ε is a quasi-psh function on X with $\psi_\varepsilon = -\infty$ on $\tilde{Z}_\varepsilon$, ψ_ε smooth on $X\backslash\tilde{Z}_\varepsilon$ and $i\partial\overline{\partial}\psi_\varepsilon + \omega \geqslant 0$ (see e.g. [**Dem82**], Théorème 1.5).

Here we choose ψ_ε more explicit for better control. Since we will use the Bochner-Kodaira-Nakano inequality on $X\backslash\tilde{Z}_\varepsilon$, to simplify the notations, we identify it with $X_\varepsilon\backslash\mathrm{Supp}(D_\varepsilon)$. We define

$$\psi_\varepsilon := -\sqrt{-\nu_\varepsilon^*\varphi_\varepsilon - C}$$

with $C \in \mathbb{R}$ such that $\sup_{X_\varepsilon} \nu_\varepsilon^*\varphi_\varepsilon + C = -1$. Now ψ_ε satisfies the condition of [**Dem82**], Théorème 1.5 following its calculation.

We want to prove that $e^{\Phi_\varepsilon} dV_{\omega_{\varepsilon,\delta}}$ is a current on X_ε. Since X_ε is compact, it has finite mass on X_ε. In particular, it has finite mass on $X_\varepsilon \backslash \mathrm{Supp}(D_\varepsilon)$. It is enough to prove that $e^{\Phi_\varepsilon}(i\partial\overline{\partial}\psi_\varepsilon)^p$ defines a current on X_ε for any $p > 0$. More precisely, we prove that $e^{\nu_\varepsilon^* \varphi_\varepsilon}(i\partial\overline{\partial}\psi_\varepsilon)^p$ defines a current on X_ε for any $p > 0$. Since

$$i\partial\overline{\partial}\psi_\varepsilon = \frac{i\partial\overline{\partial}\nu_\varepsilon^*\varphi}{2\psi_\varepsilon} + \frac{i\partial\nu_\varepsilon^*\varphi \wedge \overline{\partial}\nu_\varepsilon^*\varphi}{4\psi_\varepsilon},$$

it is equivalent to prove that $e^{\nu_\varepsilon^*\varphi}(i\partial\overline{\partial}\nu_\varepsilon^*\varphi)^p \wedge (i\partial\nu_\varepsilon^*\varphi \wedge \overline{\partial}\nu_\varepsilon^*\varphi)^q$ $(p, q \geqslant 0)$ defines a current. By anti-commutativity, we can assume q is either 0 or 1.

$$e^{\frac{\nu_\varepsilon^*\varphi_\varepsilon}{n}} i\partial\overline{\partial}\nu_\varepsilon^*\varphi_\varepsilon = e^{\frac{\nu_\varepsilon^*\varphi_\varepsilon}{n}}([D_\varepsilon] + \text{smooth terms}) = e^{\frac{\nu_\varepsilon^*\varphi_\varepsilon}{n}} \text{smooth terms}$$

since $\nu_\varepsilon^*\varphi_\varepsilon$ vanishes along D_ε. Thus it is smooth on X_ε vanishing along D_ε.

On the other hand, in local coordinates,

$$\nu_\varepsilon^*\varphi_\varepsilon = \sum \alpha_i \log(|z_i|^2)$$

with $\alpha_i > 0$. So

$$e^{\frac{\nu_\varepsilon^*\varphi_\varepsilon}{n}} i\partial\nu_\varepsilon^*\varphi_\varepsilon \wedge \overline{\partial}\nu_\varepsilon^*\varphi_\varepsilon = \prod |z_k|^{\frac{2\alpha_k}{n}} i \sum \frac{\alpha_i dz_i}{z_i} \wedge \sum \overline{\frac{\alpha_j dz_j}{z_j}}$$

has all coefficients in L^1_{loc}. (This is because $\alpha_k > 0$ for any k, although a priori, the derivative of a quasi-psh function is not necessarily in L^2_{loc}.) Hence the current is well defined as a wedge product of locally integrable functions and smooth forms.

In conclusion, we have that $\int_{X\backslash Z_\varepsilon} e^{\Phi_\varepsilon} dV_{\omega_{\varepsilon,\delta}}$ is finite and uniformly bounded for δ small enough.

Remark 3.12. Let us indicate an alternative argument in a more general situation, following a suggestion by Demailly. It is not necessary for our proof, but may be interesting for other uses. Let (X, ω) be a compact Hermitian manifold and D a SNC divisor in X. Let $u \in H^0(X, \mathcal{C}^0_{p,q,X} \otimes L)$ be a (p,q) continuous forms with value in some line bundle (L, h) endowed with some continuous metric h. In this remark, we construct a family of complete metrics ω_δ on $X\backslash D$ such that ω_δ decreasing to ω as $\delta \to 0$ and

$$\int_{X\backslash D} |u|_h^2 dV_{\omega_\delta} \leqslant C$$

where C is a universal constant independent of δ.

To begin with, we recall some facts about the local model: the Poincaré metric on the punctured disk. The Poincaré metric on $\mathbb{H} := \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ is given by $\frac{idz \wedge \overline{dz}}{|\operatorname{Im} z|^2}$. There exists an infinite cover from $\mathcal{H}$ to $\mathcal{D}^* = \{z \in \mathbb{C} ||z| < 1\}$ given by $z \mapsto e^{iz}$. The Poincaré metric on $\mathcal{H}$ is the pull back of the Poincaré metric on $\mathcal{D}^*$ given by

$$\frac{idz \wedge \overline{dz}}{|z|^2 \, |\log(|z|)|^2}.$$

Since the Poincaré metric on $\mathcal{H}$ is complete and the cover is locally diffeomorphism, the Poincaré metric on $\mathcal{D}^*$ is also geodesic complete. It is well known that the Poincaré metric is of volume finite near the origin:

$$\int_{0<|z|<\frac{1}{2}} \frac{idz \wedge \overline{dz}}{|z|^2 ||\log(|z|)|^2} = \int_0^{2\pi} d\theta \int_0^{\frac{1}{2}} \frac{dr}{r(\log(r))^2} = \frac{2\pi}{\log 2} < \infty.$$

Now we return to the construction of our metrics. Let U_α be a finite system of coordinate charts of X(X is compact) such that for any U_α such that $U_\alpha \cap D \neq \varnothing$, (we denote the set of all such indices as I) we have in this coordinate chart

$$U_\alpha \cap D = \{z_1 = \cdots = z_r = 0\},$$
$$U_\alpha \subset \{|z_i| < 1, \forall i\}.$$

This is possible since D is a SNC divisor. Let χ_α be a partition of unity adapted to this cover. Define the family of metric ω_δ on $X\backslash D$ as follows:

$$g_\alpha := \sum_{i=1}^{r} \frac{idz_i \wedge \overline{dz_i}}{|z_i|^2 \, |\log(|z_i|)|^2} + \sum_{i=r+1}^{n} idz_i \wedge \overline{dz_i}$$

$$\omega_\delta := \omega + \delta \sum_{\alpha \in I} \chi_\alpha g_\alpha.$$

The sum converges since we take finite sums. We have by construction $\omega_\delta \geqslant \omega$ decreasing to ω. By a similar calculation to the one made above, we have

$$\int_{X\backslash D} |u|_h^2 dV_{\omega_\delta} \leqslant C.$$

We remark that $|u|_h^2$ a priori depends on ω_δ. However in a local chart $U_\alpha(\alpha \in I)$, we can write

$$u = \sum_{J,K,|J|=p,|K|=q} u_{J,K} dz^J \wedge \overline{dz^K}$$

using J(resp. K) for multi index of length p (resp. q). Denote for $1 \leqslant i \leqslant n$ and a multi index I, $\delta_{iI} = 1$ if $i \in I$ and $\delta_{iI} = 0$ if $i \notin I$. Then we have

$$|u|^2_{g_\alpha,h} = \sum_{J,K} \prod_{i=1}^n (|z_i|\ |\log|z_i||)^{\delta_{iJ}} \prod_{i=1}^n (|z_i|\ |\log|z_i||)^{\delta_{iK}} |u_{JK}|^2 e^{-2\varphi_\alpha}$$

where φ_α is the weight function of h on U_α (i.e. $h = e^{-2\varphi_\alpha}$ on U_α). Hence $|u|^2_{g_\alpha,h}$ is uniformly bounded since $|z_i||\log|z_i||$ is bounded for $|z_i| < 1$ and all terms are continuous.

It remains to prove that ω_δ is complete. We prove it by contradiction. Let $\gamma(t)$ be a geodesic of ω_δ with natural parametrization for $\delta > 0$ whose maximal defining interval is $]t_0, t_1[$ with $t_1 < \infty$. By property of ordinary differential equation (the solution goes outside any compact subset), the adherent point(s) must be contained in D with respect to the background topology of X. Since X is compact, there exists a sequence $\gamma(t_\nu) \to x \in D$ with $t_\nu \to t_1$, $t_\nu < t_1$. Up to taking a subsequence we can assume that such a sequence is contained in some chart U_{α_0}. Then $|\gamma'(t)|_{g_{\alpha_0}} \leqslant \delta |\gamma'(t)|_{\omega_\delta} = \delta$ where the second equality is from the fact that $\gamma(t)$ be a geodesic of ω_δ. Hence $\gamma(t_\nu)$ is a Cauchy sequence with respect to g_α. Since the Poincaré type metric g_α is complete, the limit $x \in U_{\alpha_0} \backslash D$ exists, which gives a contradiction.

We recall the Bochner-Kodaira-Nakano inequality in the non compact case.

THEOREM 3.13. *Let h be a smooth hermitian metric on L over (X, ω) a complete Kähler manifold. We assume that the curvature possesses a uniform lower bound*

$$i\Theta_{L,h} \geqslant -C\omega.$$

Then for an arbitrary (p,q)-form $u \in C^\infty(X, \wedge^{p,q} T_X^* \otimes L)$ which is L^2 integrable, the following basic a priori inequality holds

$$\| \overline{\partial} u \|^2 + \| \overline{\partial}^* u \|^2 \geqslant \int_X \langle [i\Theta_{L,h}, \Lambda] u, u \rangle dV_\omega.$$

PROOF. For u with compact support, the inequality is just the classical one. When u is just L^2-integrable case, since (X, ω) is assumed to be complete, there exists a sequence of smooth forms u_ν with compact support in X (obtained for example by truncating u and taking the convolution with a regularizing kernel) such that $u_\nu \to u$ in L^2 and such that $\overline{\partial} u_\nu \to \overline{\partial} u, \overline{\partial}^* u_\nu \to \overline{\partial}^* u$ in L^2.

By our curvature assumption the term on the right is controlled by $C|u|^2$ which is L^2. We thus get the inequality by passing to the limit, using Lebesgue's dominated convergence theorem. □

We now return to the proof of the Bogomolov vanishing theorem.

Let u be a holomorphic p-form with value in L^{-1}. We take the metric induced from $(L, he^{-\Phi_\varepsilon})$. The Bochner-Kodaira-Nakano inequality on the complete manifold $(X \backslash \tilde{Z}_\varepsilon, \omega_{\varepsilon,\delta})$ gives

$$0 \geqslant \int_{X \backslash Z_\varepsilon} \langle [i\Theta_{L,h}, \Lambda] u, u \rangle e^{\Phi_\varepsilon} dV_{\omega_{\varepsilon,\delta}},$$

by using the degree condition and the fact that the form is holomorphic. We remark that the form is L^2-integrable by the above discussion and the fact that u has globally bounded coefficients on X (hence on $X \backslash \tilde{Z}_\varepsilon$).

Let us observe that by [**Dem82**] Lemma 3.2, $(p,0)$-forms get larger L^2 norms as the metric increases. In other words, in bidegree $(p,0)$, the space $L^2(\omega)$ has the weakest topology of all spaces $L^2(\omega_{\varepsilon,\delta})$. Indeed, an easy calculation made in the above lemma yields

$$|f|^2_{\wedge^{p,0}\omega \otimes h} dV_\omega \leqslant |f|^2_{\wedge^{p,0}\omega_{\varepsilon,\delta} \otimes h} dV_{\omega_{\varepsilon,\delta}}$$

if f is of type $(p,0)$. By Lebesgue's dominated convergence theorem, we have

$$0 \geqslant \int_{X \backslash Z_\varepsilon} \langle [i\Theta_{L,h}, \Lambda] u, u \rangle e^{\Phi_\varepsilon} dV_\omega$$

by taking $\delta \to 0$.

The rest part of the proof follows in general the proof of [**Mou98**].

Let $-\varepsilon \leqslant \lambda_1^\varepsilon \leqslant \cdots \leqslant \lambda_n^\varepsilon$ the eigenvalues of $i\Theta_{L,h_\varepsilon}$ with respect to ω on $X \backslash \tilde{Z}_\varepsilon$.

Then we have

$$\int_{X \backslash \tilde{Z}_\varepsilon} (\lambda_n^\varepsilon + \varepsilon) dV_\omega \leqslant \int_{X \backslash \tilde{Z}_\varepsilon} ((\lambda_1^\varepsilon + \varepsilon) + \cdots + (\lambda_n^\varepsilon + \varepsilon)) dV_\omega$$

$$\leqslant \int_{X\setminus\tilde{Z}_\varepsilon} (i\Theta_{L,h_\varepsilon} + \varepsilon\omega) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$\leqslant \int_X (i\Theta_{L,h_\varepsilon} + \varepsilon\omega) \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_X (c_1(L) + \varepsilon\omega) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

$$\leqslant \int_X (c_1(L) + \omega) \wedge \frac{\omega^{n-1}}{(n-1)!} =: A.$$

Let $\delta > 0$ such that

$$\nu := \frac{n-l}{n-l+1} + \delta\frac{l-1}{n-l+1} < 1.$$

Hence $V_\varepsilon := \{x \in X\setminus\tilde{Z}_\varepsilon | \lambda_n^\varepsilon + \varepsilon \geqslant A\varepsilon^{-\delta}\}$ has volume smaller that $\varepsilon^\delta \int_X \omega^n$.

On the other hand, by the Monge-Ampère equation, on $X\setminus\tilde{Z}_\varepsilon$ we have

$$\prod_{i=1}^n (\lambda_i^\varepsilon + \varepsilon) \geqslant \frac{c}{3}\varepsilon^{n-l}.$$

Hence on $X\setminus(V_\varepsilon \cup \tilde{Z}_\varepsilon)$ we have

$$\begin{aligned}\lambda_{n-l+1}^\varepsilon + \varepsilon &\geqslant ((\lambda_{n-l+1}^\varepsilon + \varepsilon)\cdots(\lambda_1^\varepsilon + \varepsilon))^{\frac{1}{n-l+1}}\\ &\geqslant c\varepsilon^{\frac{n-l}{n-l+1}}(\lambda_n^\varepsilon + \varepsilon)^{\frac{l-1}{n-l+1}}\\ &\geqslant c\varepsilon^\nu.\end{aligned}$$

Combining this with the Bochner-Kodaira-Nakano inequality, we find

$$0 \geqslant \int_{X\setminus\tilde{Z}_\varepsilon} (\lambda_1^\varepsilon + \cdots + \lambda_{n-l+1}^\varepsilon + \cdots + \lambda_{n-p}^\varepsilon)|u|^2_{L^{-1},h^{-1}} e^{\Phi_\varepsilon} dV_\omega$$

$$\geqslant \int_{X\setminus(\tilde{Z}_\varepsilon\cup V_\varepsilon)} (c\varepsilon^\nu - (n-p)\varepsilon)|u|^2_{L^{-1},h^{-1}} e^{\Phi_\varepsilon} dV_\omega + \int_{V_\varepsilon} -(n-p)\varepsilon|u|^2_{L^{-1},h^{-1}} e^{\Phi_\varepsilon} dV_\omega.$$

In other words,

$$\int_{X\setminus\tilde{Z}_\varepsilon} |u|^2_{L^{-1},h^{-1}} e^{\Phi_\varepsilon} dV_\omega \leqslant (1 + \frac{n-p}{c\varepsilon^\nu - (n-p)}) \int_{V_\varepsilon} |u|^2_{L^{-1},h^{-1}} e^{\Phi_\varepsilon} dV_\omega$$

$$\leqslant C\int_{V_\varepsilon} \omega^n \leqslant C\varepsilon^\nu,$$

where we use that Φ_ε is uniformly bounded from above. Since $\tilde{Z}_\varepsilon$ is of Lebesgue measure 0,

$$\int_{X\setminus\tilde{Z}_\varepsilon} |u|^2_{L^{-1},h^{-1}} e^{\Phi_\varepsilon} dV_\omega = \int_X |u|^2_{L^{-1},h^{-1}} e^{\Phi_\varepsilon} dV_\omega.$$

Again by Lebesgue's dominated convergence theorem (there is an upper bound by constant), we have

$$\int_X |u|^2_{L^{-1},h^{-1}} e^{\Phi} dV_\omega \leqslant 0$$

by taking $\varepsilon \to 0$. This implies that $u = 0$ and finishes the proof of the Bogomolov vanishing theorem.

Remark 3.14. In example 1.7 of [**DPS94**], we consider a nef line bundle $\mathcal{O}(1)$ over the projectivisation of the unique non-trivial rank 2 vector bundle as extension of two trivial line bundle over an elliptic curve. An explicit calculation shows that there exists a unique singular positive metric on $\mathcal{O}(1)$ whose curvature is the current associated to a smooth curve. Hence in this example $e(\mathcal{O}(1)) = 0$. But the numerical dimension is $\mathrm{nd}(\mathcal{O}(1)) = 1$ since the line bundle is non trivial and not big. In fact, $(\mathcal{O}(1))^2 = 0$.

Remark 3.15. Our Bogomolov vanishing theorem can be reformulated as follows:

The sheaf of holomorphic p-forms over X has no subsheaf of rank one associated to a psef line bundle of numerical dimension strictly larger than p.

According to the fundamental work of Campana [**Cam04**] [**Cam11**] on special manifolds, the above results suggest to give the following variant of Campana's definition.

Definition 3.16. *Let $L \subset \Omega_X^p$ be a saturated, coherent and rank one subsheaf. We call it a "numerical Bogomolov sheaf" of X if* $\mathrm{nd}(X, L) = p > 0$.

We say that X is "numerically special" if it has no Bogomolov sheaf. A compact complex analytic space is said to be "numerically special" if some (or any) of its resolutions is "numerically special".

REMARK 3.17. It is conjectured by Campana that specialness is equivalent to the numerical specialness defined here.

One possibility to address Campana's conjecture would be study the following statement of the Bogomolov vanishing theorem incorporating the numerical dimension instead of the Kodaira-Iitaka dimension:

For a numerical Bogomolov subsheaf, does there exist a fibration $f : X \to Y$ such that $L = f^*(K_Y)$ over the generic point of Y (i.e., L and $f^*(K_Y)$ have the same saturation in Ω_X^p) ?

In case the Kodaira dimension case is used, the existence of the fibration comes directly from the Kodaira-Iitaka morphism. However, in case one uses the numerical dimension instead, the existence of the fibration is not guaranteed, i.e. there are examples of non abundant numerical Bogomolov sheaves. One can take for instead X to be a Hilbert modular surface obtained as a smooth quotient $\mathbb{D} \times \mathbb{D}/\Gamma$ with an irreducible subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{D}) \times \mathrm{Aut}(\mathbb{D})$ (in such a way that no subgroup of finite index of Γ splits). It is equipped with two natural foliations $\mathcal{F}$, $\mathcal{G}$ coming from the two factors $\mathbb{D}$, and $T_X = \mathcal{F} \oplus \mathcal{G}$. Then one can check that $\mathcal{F}^*, \mathcal{G}^* \subset \Omega_X^1$ satisfy $\mathrm{nd}(\mathcal{F}^*) = \mathrm{nd}(\mathcal{G}^*) = 1$, but $\kappa(\mathcal{F}^*) = \kappa(\mathcal{G}^*) = -\infty$ (see e.g. [**Br03**]).

REMARK 3.18. It should be remarked the above Bogomolov vanishing theorem was first proven in [**Bou02a**]. The strategy of the both proofs is based on the nef case proven in [**Mou98**]. The difficulty is the control by Monge-Ampère equation in the pseudo-effective case. The difficulty is overcome in [**Bou02a**] by a singular version of Monge-Ampère equation, and we give here another proof that only requires solving "classical" Monge-Ampère equations.

3.2.2. Junyan Cao's vanishing theorem.

In [**Cao14**], Junyan Cao has proven the following Kawamata-Viehweg-Nadel type vanishing theorem.

THEOREM 3.19. *Let (L, h) be a pseudo-effective line bundle on a compact Kähler n-dimensional manifold X. Then*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0$$

for every $q \geqslant n - \mathrm{nd}(L, h) + 1$.

The numerical dimension $\mathrm{nd}(L, h)$ used in Cao's theorem is the numerical dimension of the closed positive $(1,1)$-current $i\Theta_{L,h}$ defined in his paper. Since we will not need this definition, we refer to his paper for further information. We just recall the remark on page 22 of [**Cao14**]. In the example 1.7 of [**DPS94**], they consider the nef line bundle $\mathcal{O}(1)$ over the projectivisation of a rank 2 vector bundle over the elliptic curve C which is the only non-trivial extension of $\mathcal{O}_C$. They prove that there exists a unique positive singular metric h on $\mathcal{O}(1)$. For this metric, $\mathrm{nd}(\mathcal{O}(1), h) = 0$. But the numerical dimension of $\mathcal{O}(1)$ is equal to 1. We recall that for a nef line bundle L the numerical dimension is defined as

$$\mathrm{nd}(L) := \max\{p; c_1(L)^p \neq 0\}.$$

We also remark that Cao's technique of proof actually yields the result for the upper semi-continuous regularization of multiplier ideal sheaf defined as

$$\mathcal{I}_+(h) := \lim_{\varepsilon \to 0} \mathcal{I}(h^{1+\varepsilon})$$

instead of $\mathcal{I}(h)$, but we can apply Guan-Zhou's Theorem [**GZ15c**] [**GZ14a**] [**GZ15a**] to see that the equality $\mathcal{I}_+(h) = \mathcal{I}(h)$ always holds. In particular, by the Noetherian property of ideal sheaves, we have

$$\mathcal{I}_+(h) = \mathcal{I}(h^{\lambda_0}) = \mathcal{I}(h)$$

for some $\lambda_0 > 1$. This fact will also be used in our result.

In this part, we prove the following version of Junyan Cao's vanishing theorem, following closely the ideas of Junyan Cao [**Cao14**] and the version that was a bit simplified in [**Dem14**].

THEOREM 3.20. *Let L be a pseudo-effective line bundle on a compact Kähler n-dimensional manifold X. Then the morphism induced by inclusion $K_X \otimes L \otimes \mathcal{I}(h_{\min}) \to K_X \otimes L$*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_{\min})) \to H^q(X, K_X \otimes L)$$

is 0 map for every $q \geqslant n - \mathrm{nd}(L) + 1$.

REMARK 3.21. In the example 1.7 of [**DPS94**], since the rank 2 vector bundle is the only non-trivial extension of $\mathcal{O}_C$, there exists a surjective morphism from this vector bundle to $\mathcal{O}_C$ which induces a closed immersion C into the ruled surface. The only positive metric on $\mathcal{O}(1)$ has curvature $[C]$ the current associated to C. On the other hand, $\mathcal{O}(1) = \mathcal{O}(C)$. So we have $H^2(X, K_X \otimes \mathcal{O}(1)) = H^0(X, \mathcal{O}(-1)) = H^0(X, \mathcal{O}(-C)) = 0$ and $H^2(X, K_X \otimes \mathcal{O}(1) \otimes \mathcal{I}(h_{\min})) = H^2(X, K_X \otimes \mathcal{O}(1) \otimes \mathcal{O}(-C)) = H^0(X, \mathcal{O}_X) = \mathbb{C}$. This shows that to get a numerical dimension version of theorem the best that we can hope for is that the morphism is 0 map instead of that $H^q(X, K_X \otimes L \otimes \mathcal{I}(h_{\min})) = 0$. We notice that in general one would expect the vanishing result

$$H^q(X, K_X \otimes L) = 0$$

for $q \geqslant n - \mathrm{nd}(L) + 1$, whenever L is a nef line bundle. Here the difficulty is to prove a general Kähler version, since the results follows easily from an inductive hyperplane section argument when X is projective (cf. eg. Corollary (6.26) of [**Dem12a**]).

By remark 3.6 in the previous section, we can even assume that h_ε as stated in the definition of the numerical dimension is increasing to $h_{\min}$ as $\varepsilon \to 0$. What we need here is that the weight functions φ_ε has limit $\varphi_{\min}$ and is pointwise at least equal to $\varphi_{\min}$ with a universal upper bound on X.

Before giving the proof of the vanishing theorem, we give the general lines of the ideas and compare it with Cao's theorem. The idea is using the L^2 resolution of the multiplier ideal sheaf and proving that every $\overline{\partial}$-closed $L^2(h_{\min})$ global section can be approximated by $\overline{\partial}$-exact $L^2(h_\infty)$ global sections with h_∞ some smooth reference metric on L. To prove it, we solve the $\overline{\partial}$-equation using a Bochner technique with error term (as in [**DP03**]), and we prove that the error term tends to 0.

For this propose, we need to estimate the curvature asymptotically by some special approximating hermitian metrics constructed by means of the Calabi-Yau theorem. Cao tried to prove that the error term tends to 0 in the topology induced by the L^2-norm, with respect to the given singular metric. In this way, he tried to keep the multiplier ideal sheaf unchanged when approximating the singular metric, by means of suitable "equisingular approximation". For our propose, we try to prove that the error term tends to 0 in the topology induced by L^2-norm with respect to some (hence any) smooth metric. It would be enough for us that the multiplier ideal sheaf of $h_{\min}$is included in the multiplier ideal sheaf of the approximating hermitian metric. In some sense, Cao's theorem is more precise in studying the singularity of the metric which somehow explains why his approach works for any singular metric while our approach applies only for the image of the natural inclusion.

We start the proof of the vanishing theorem by the following technical curvature and singularity estimate.

PROPOSITION 3.22. *Let $(L, h_{\min})$ be a pseudo-effective line bundle on a compact Kähler manifold (X, ω). Let us write $T_{\min} = \frac{i}{2\pi}\Theta_{L,h_{\min}} = \alpha + \frac{i}{2\pi}\partial\overline{\partial}\varphi_{\min}$ where α is the curvature of some smooth metric h_∞ on L and $\varphi_{\min}$ is a quasi-psh potential. Let $p = \mathrm{nd}(L)$ be the numerical dimension of L. Then, for every $\gamma \in]0, 1]$ and $\delta \in]0, 1]$, there exists a quasi-psh potential $\Phi_{\gamma,\delta}$ on X satisfying the following properties:*
(a) *$\Phi_{\gamma,\delta}$ is smooth in the complement $X \backslash Z_\delta$ of an analytic set $Z_\delta \subset X$.*
(b) *$\alpha + \delta\omega + \frac{i}{2\pi}\partial\overline{\partial}\Phi_{\gamma,\delta} \geqslant \frac{\delta}{2}(1 - \gamma)\omega$ on X.*
(c) *$(\alpha + \delta\omega + \frac{i}{2\pi}\partial\overline{\partial}\Phi_{\gamma,\delta})^n \geqslant a\,\gamma^n\delta^{n-p}\omega^n$ on $X \backslash Z_\delta$.*
(d) *$\sup_X \Phi_{1,\delta} = 0$, and for all $\gamma \in]0, 1]$ there are estimates $\Phi_{\gamma,\delta} \leqslant A$ and*

$$\exp\big(-\Phi_{\gamma,\delta}\big) \leqslant e^{-(1+b\delta)\varphi_{\min}} \exp\big(A - \gamma\Phi_{1,\delta}\big)$$

(e) *For γ_0, $\delta_0 > 0$ small, $\gamma \in]0, \gamma_0]$, $\delta \in]0, \delta_0]$, we have*

$$\mathcal{I}_+(\varphi_{\min}) = \mathcal{I}(\varphi_{\min}) \subset \mathcal{I}(\Phi_{\gamma,\delta}).$$

Here a, b, A, γ_0, δ_0 are suitable constants independent of γ, δ.

PROOF. Denote by ψ_ε the (non-increasing) sequence of weight functions as stated in the definition of numerical dimension. We have $\psi_\varepsilon \geqslant \varphi_{\min}$ for all $\varepsilon > 0$, the ψ_ε have analytic singularities and

$$\alpha + \frac{i}{2\pi}\partial\overline{\partial}\psi_\varepsilon \geqslant -\varepsilon\omega.$$

Then for $\varepsilon \leqslant \frac{\delta}{4}$, we have

$$\begin{aligned}\alpha + \delta\omega + \frac{i}{2\pi}\partial\overline{\partial}\big((1 + b\delta)\psi_\varepsilon\big) &\geqslant \alpha + \delta\omega - (1 + b\delta)(\alpha + \varepsilon\omega)\\ &\geqslant \delta\omega - (1 + b\delta)\varepsilon\omega - b\delta\alpha \geqslant \tfrac{\delta}{2}\omega\end{aligned}$$

for $b \in]0, \frac{1}{5}]$ small enough such that $\omega - b\alpha \geqslant 0$.

Let $\mu : \widehat{X} \to X$ be a log-resolution of ψ_ε, so that

$$\mu^*\big(\alpha + \delta\omega + \frac{i}{2\pi}\partial\overline{\partial}((1 + b\delta)\psi_\varepsilon)\big) = [D_\varepsilon] + \beta_\varepsilon$$

where $\beta_\varepsilon \geqslant \frac{\delta}{2}\mu^*\omega \geqslant 0$ is a smooth closed $(1, 1)$-form on $\widehat{X}$ that is strictly positive in the complement $\widehat{X}\backslash E$ of the exceptional divisor, and D_ε is an effective $\mathbb{R}$-divisor that includes all components E_ℓ of E. The map μ can be obtained by Hironaka [**Hir64**] as a composition of a sequence of blow-ups with smooth centres, and we can even achieve that D_ε and E are normal crossing divisors. For arbitrary small enough numbers

$\eta_\ell > 0$, $\beta_\varepsilon - \sum \eta_\ell [E_\ell]$ is a Kähler class on $\widehat{X}$. Hence we can find a quasi-psh potential $\widehat{\theta}_\varepsilon$ on $\widehat{X}$ such that $\widehat{\beta}_\varepsilon := \beta_\varepsilon - \sum \eta_\ell [E_\ell] + \frac{i}{2\pi}\partial\overline{\partial}\widehat{\theta}_\varepsilon$ is a Kähler metric on $\widehat{X}$. By taking the η_ℓ small enough, we may assume that

$$\int_{\widehat{X}} (\widehat{\beta}_\varepsilon)^n \geqslant \frac{1}{2}\int_{\widehat{X}} \beta_\varepsilon^n.$$

We will use Yau's theorem [**Yau78**] to construct a form in the cohomology class of $\widehat{\beta}_\varepsilon$ with better volume estimate. We have

$$\begin{aligned} \alpha + \delta\omega + \frac{i}{2\pi}\partial\overline{\partial}\big((1+b\delta)\psi_\varepsilon\big) &\geqslant \alpha + \varepsilon\omega + \frac{i}{2\pi}\partial\overline{\partial}\psi_\varepsilon + (\delta-\varepsilon)\omega - b\delta(\alpha+\varepsilon\omega) \\ &\geqslant (\alpha + \varepsilon\omega + \frac{i}{2\pi}\partial\overline{\partial}\psi_\varepsilon) + \frac{\delta}{2}\omega. \end{aligned}$$

The assumption on the numerical dimension of L implies the existence of a constant $c > 0$ such that, with $Z = \mu(E) \subset X$, we have

$$\begin{aligned} \int_{\widehat{X}} \beta_\varepsilon^n &= \int_{X\backslash Z} \big(\alpha + \delta\omega + \frac{i}{2\pi}\partial\overline{\partial}((1+b\delta)\psi_\varepsilon)\big)^n \\ &\geqslant \binom{n}{p}\Big(\frac{\delta}{2}\Big)^{n-p}\int_{X\backslash Z} \big(\alpha + \varepsilon\omega + \frac{i}{2\pi}\psi_\varepsilon\big)^p \wedge \omega^{n-p} \geqslant c\delta^{n-p}\int_X \omega^n. \end{aligned}$$

Therefore, we may assume

$$\int_{\widehat{X}} (\widehat{\beta}_\varepsilon)^n \geqslant \frac{c}{2}\,\delta^{n-p}\int_X \omega^n.$$

We take $\widehat{f}$ a volume form on $\widehat{X}$ such that $\widehat{f} > \frac{c}{3}\delta^{n-p}\mu^*\omega^n$ everywhere on $\widehat{X}$ and such that $\int_{\widehat{X}} \widehat{f} = \int_{\widehat{X}} \widehat{\beta}_\varepsilon^n$. By Yau's theorem [**Yau78**], there exists a quasi-psh potential $\widehat{\tau}_\varepsilon$ on $\widehat{X}$ such that $\widehat{\beta}_\varepsilon + \frac{i}{2\pi}\partial\overline{\partial}\widehat{\tau}_\varepsilon$ is a Kähler metric on $\widehat{X}$ with the prescribed volume form $\widehat{f} > 0$.

Now push our focus back to X. Set $\theta_\varepsilon = \mu_*\widehat{\theta}_\varepsilon$ and $\tau_\varepsilon = \mu_*\widehat{\tau}_\varepsilon \in L^1_{loc}(X)$. We define

$$\Phi_{\gamma,\delta} := (1+b\delta)\psi_\varepsilon + \gamma(\theta_\varepsilon + \tau_\varepsilon).$$

By construction it is smooth in the complement $X\backslash Z_\delta$ i.e. property (a). It satisfies

$$\begin{aligned} \mu^*\big(\alpha + \delta\omega + \frac{i}{2\pi}\partial\overline{\partial}((1+b\delta)\psi_\varepsilon + \gamma(\theta_\varepsilon+\tau_\varepsilon))\big) &= [D_\varepsilon] + (1-\gamma)\beta_\varepsilon + \gamma\Big(\sum_\ell \eta_\ell[E_\ell] + \widehat{\beta}_\varepsilon + \frac{i}{2\pi}\partial\overline{\partial}\widehat{\tau}_\varepsilon\Big) \\ &\geqslant (1-\gamma)\beta_\varepsilon \geqslant \frac{\delta}{2}(1-\gamma)\,\mu^*\omega \end{aligned}$$

since $\widehat{\beta}_\varepsilon + \frac{i}{2\pi}\partial\overline{\partial}\widehat{\tau}_\varepsilon$ is a Kähler metric on $\widehat{X}$. Thus the property (b) is satisfied. Putting $Z_\delta = \mu(|D_\varepsilon|) \supset \mu(E) = Z$, we have on $X\backslash Z_\delta$

$$\begin{aligned} \mu^*\big(\alpha + \delta\omega + \frac{i}{2\pi}\partial\overline{\partial}\Phi_{\gamma,\delta}\big)^n &\geqslant \big(\beta_\varepsilon + \gamma\frac{i}{2\pi}\partial\overline{\partial}(\widehat{\theta}_\varepsilon + \widehat{\tau}_\varepsilon)\big)^n \\ &\geqslant \gamma^n\,(\widehat{\beta}_\varepsilon + \frac{i}{2\pi}\partial\overline{\partial}\widehat{\tau}_\varepsilon)^n \geqslant \frac{c}{3}\,\gamma^n\delta^{n-p}\mu^*\omega^n. \end{aligned}$$

Since $\mu : \widehat{X}\backslash D_\varepsilon \to X\backslash Z_\delta$ is a biholomorphism, the condition (c) is satisfied if we set $a = \frac{c}{3}$.

We adjust constants in $\widehat{\theta}_\varepsilon + \widehat{\tau}_\varepsilon$ so that $\sup_X \Phi_{1,\delta} = 0$. Since $\varphi_{\min} \leqslant \psi_\varepsilon \leqslant \psi_{\varepsilon_0} \leqslant A_0 := \sup_X \psi_{\varepsilon_0}$ for $\varepsilon \leqslant \varepsilon_0$ and

$$\Phi_{\gamma,\delta} = (1+b\delta)\psi_\varepsilon + \gamma\big(\Phi_{1,\delta} - \psi_\varepsilon\big) \geqslant (1+b\delta)\varphi_{\min} + \gamma\Phi_{1,\delta} - \gamma A_0$$

and we have $\Phi_{\gamma,\delta} \leqslant (1-\gamma+b\delta)A_0$. Thus the property (d) is satisfied if we set $A := (1+b)A_0$.

We observe that $\Phi_{1,\delta}$ satisfies $\alpha + \omega + dd^c\Phi_{1,\delta} \geqslant 0$ and $\sup_X \Phi_{1,\delta} = 0$, hence $\Phi_{1,\delta}$ belongs to a compact family of quasi-psh functions. By theorem 2.50 a uniform version of Skoda's integrability theorem in [**GZ17**], there exists a uniform small constant $c_0 > 0$ such that $\int_X \exp(-c_0\Phi_{1,\delta})dV_\omega < +\infty$ for all $\delta \in]0,1]$. If $f \in \mathcal{O}_{X,x}$ is a germ of holomorphic function and U a small neighbourhood of x, the Hölder inequality combined with estimate (d) implies

$$\int_U |f|^2\exp(-\Phi_{\gamma,\delta})dV_\omega \leqslant e^A\Big(\int_U |f|^2 e^{-p(1+b\delta)\varphi_{\min}}dV_\omega\Big)^{\frac{1}{p}}\Big(\int_U |f|^2 e^{-q\gamma\Phi_{1,\delta}}dV_\omega\Big)^{\frac{1}{q}}.$$

Take $p \in]1,\lambda_0[$ (say $p = (1+\lambda_0)/2$), and take

$$\gamma \leqslant \gamma_0 := \frac{c_0}{q} = c_0\frac{\lambda_0-1}{\lambda_0+1} \quad \text{and} \quad \delta \leqslant \delta_0 \in]0,1] \text{ so small that } p(1+b\delta_0) \leqslant \lambda_0.$$

Then $f \in \mathcal{I}_+(\varphi_{\min}) = \mathcal{I}(\lambda_0\varphi_{\min})$ implies $f \in \mathcal{I}(\Phi_{\gamma,\delta})$ which proves the condition (e). □

The rest of the proof follows from the proof of [**Cao14**] (cf. also [**Dem14**], [**DP03**], [**Mou98**]). We will just give an outline of the proof for completeness.

Let $\{f\}$ be a cohomology class in the group $H^q(X, K_X \otimes L \otimes \mathcal{I}(h_{\min}))$, $q \geqslant n - \mathrm{nd}(L) + 1$. The sheaf $\mathcal{O}(K_X \otimes L) \otimes \mathcal{I}(h_{\min})$ can be resolved by the complex $(K^\bullet, \overline{\partial})$ where K^i is the sheaf of (n,i)-forms u such that both u and $\overline{\partial} u$ are locally L^2 with respect to the weight $\varphi_{\min}$. So $\{f\}$ can be represented by a (n,q)-form f such that both f and $\overline{\partial} f$ are L^2 with respect to the weight $\varphi_{\min}$, i.e. $\int_X |f|^2 \exp(-\varphi_{\min}) dV_\omega < +\infty$ and $\int_X |\overline{\partial} f|^2 \exp(-\varphi_{\min}) dV_\omega < +\infty$.

We can also equip L by the hermitian metric h_δ defined by the quasi-psh weight $\Phi_\delta = \Phi_{\gamma_0,\delta}$ obtained in Proposition 1, with $\delta \in]0, \delta_0]$. Since Φ_δ is smooth on $X \backslash Z_\delta$, the Bochner-Kodaira inequality shows that for every smooth (n,q)-form u with values in $K_X \otimes L$ that is compactly supported on $X \backslash Z_\delta$, we have

$$\|\overline{\partial} u\|_\delta^2 + \|\overline{\partial}^* u\|_\delta^2 \geqslant 2\pi \int_X (\lambda_{1,\delta} + \ldots + \lambda_{q,\delta} - q\delta)|u|^2 e^{-\Phi_\delta} dV_\omega,$$

where $\|u\|_\delta^2 := \int_X |u|^2_{\omega,h_\delta} dV_\omega = \int_X |u|^2_{\omega,h_\infty} e^{-\Phi_\delta} dV_\omega$. The condition (b) of Proposition 3.22 shows that

$$0 < \frac{\delta}{2}(1-\gamma_0) \leqslant \lambda_{1,\delta}(x) \leqslant \ldots \leqslant \lambda_{n,\delta}(x)$$

where $\lambda_{i,\delta}$ are at each point $x \in X$, the eigenvalues of $\alpha + \delta\omega + \frac{i}{2\pi}\partial\overline{\partial}\Phi_\delta$ with respect to the base Kähler metric ω. In other words, we have up to a multiple 2π

$$\|\overline{\partial} u\|_\delta^2 + \|\overline{\partial}^* u\|_\delta^2 + \delta\|u\|_\delta^2 \geqslant \int_X (\lambda_{1,\delta} + \ldots + \lambda_{q,\delta})|u|^2_{\omega,h_\infty} e^{-\Phi_\delta} dV_\omega.$$

By the proof of theorem 3.3 in [**DP03**], we have the following lemma:

LEMMA 3.23. *For every L^2 section of $\Lambda^{n,q} T_X^* \otimes L$ such that $\|f\|_\delta < +\infty$ and $\overline{\partial} f = 0$ in the sense of distributions, there exists a L^2 section $v = v_\delta$ of $\Lambda^{n,q-1} T_X^* \otimes L$ and a L^2 section $w = w_\delta$ of $\Lambda^{n,q} T_X^* \otimes L$ such that $f = \overline{\partial} v + w$ with*

$$\|v\|_\delta^2 + \frac{1}{\delta}\|w\|_\delta^2 \leqslant \int_X \frac{1}{\lambda_{1,\delta} + \ldots + \lambda_{q,\delta}} |f|^2 e^{-\Phi_\delta} dV_\omega.$$

By lemma 3.23 and condition (d) of proposition 3.22, the error term w satisfies the L^2 bound,

$$\int_X |w|^2_{\omega,h_\infty} e^{-A} dV_\omega \leqslant \int_X |w|^2_{\omega,h_\infty} e^{-\Phi_\delta} dV_\omega \leqslant \int_X \frac{\delta}{\lambda_{1,\delta} + \ldots + \lambda_{q,\delta}} |f|^2_{\omega,h_\infty} e^{-\Phi_\delta} dV_\omega.$$

We will show that the right hand term tends to 0 as $\delta \to 0$. To do it, we need to estimate the ratio function $\rho_\delta := \frac{\delta}{\lambda_{1,\delta} + \ldots + \lambda_{q,\delta}}$. The ratio function is first estimated in [**Mou98**].

By estimates (b,c) in Proposition 3.22, we have $\lambda_{j,\delta}(x) \geqslant \frac{\delta}{2}(1-\gamma_0)$ and $\lambda_{1,\delta}(x) \ldots \lambda_{n,\delta}(x) \geqslant a\gamma_0^n \delta^{n-p}$. Therefore we already find $\rho_\delta(x) \leqslant 2/q(1-\gamma_0)$. On the other hand, we have

$$\int_{X \backslash Z_\delta} \lambda_{n,\delta}(x) dV_\omega \leqslant \int_X (\alpha + \delta\omega + dd^c \Phi_\delta) \wedge \omega^{n-1} = \int_X (\alpha + \delta\omega) \wedge \omega^{n-1} \leqslant \mathrm{Const},$$

therefore the "bad set" $S_\varepsilon \subset X \backslash Z_\delta$ of points x where $\lambda_{n,\delta}(x) > \delta^{-\varepsilon}$ has a volume with respect to ω $\mathrm{Vol}(S_\varepsilon) \leqslant C\delta^\varepsilon$ converging to 0 as $\delta \to 0$. Outside of S_ε,

$$\lambda_{q,\delta}(x)^q \delta^{-\varepsilon(n-q)} \geqslant \lambda_{q,\delta}(x)^q \lambda_{n,\delta}(x)^{n-q} \geqslant a\gamma_0^n \delta^{n-p}.$$

Thus we have $\rho_\delta(x) \leqslant C\delta^{1-\frac{n-p+(n-q)\varepsilon}{q}}$. If we take $q \geqslant n - \mathrm{nd}(L) + 1$ and $\varepsilon > 0$ small enough, the exponent of δ in the final estimate is strictly positive. Thus there exists a subsequence (ρ_{δ_ℓ}), $\delta_\ell \to 0$, that tends almost everywhere to 0 on X.

Estimate (e) in Proposition 3.22 implies the Hölder inequality

$$\int_X \rho_\delta |f|^2_{\omega,h_\infty} \exp(-\Phi_\delta) dV_\omega \leqslant e^A \Big(\int_X \rho_\delta^p |f|^2_{\omega,h_\infty} e^{-p(1+b\delta)\varphi_{\min}} dV_\omega \Big)^{\frac{1}{p}} \Big(\int_X |f|^2_{\omega,h_\infty} e^{-q\gamma_0 \Phi_{1,\delta}} dV_\omega \Big)^{\frac{1}{q}}$$

for suitable $p, q > 1$ as in the proposition. $|f|^2_{\omega,h_\infty} \leqslant C$ for some constant $C > 0$ since X is compact. Taking $\delta \to 0$ yields that $w_\delta \to 0$ in $L^2(h_\infty)$ by Lebesgue dominating theorem.

$H^q(X, K_X \otimes L)$ is a finite dimensional Hausdorff vector space whose topology is induced by the L^2 Hilbert space topology on the space of forms. In particular the subspace of coboundaries is closed in the space of cocycles. Hence f is a coboundary which completes the proof.

For any singular positive metric h on L, by definition, h is more singular that $h_{\min}$ which implies that $\mathcal{I}(h) \subset \mathcal{I}(h_{\min})$. A direct corollary of theorem 3.19 is the following.

COROLLARY 3.24. Let (L,h) be a pseudo-effective line bundle on a compact Kähler n-dimensional manifold X. Then the morphism induced by inclusion $K_X \otimes L \otimes \mathcal{I}(h) \to K_X \otimes L$

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes L)$$

is 0 map for every $q \geqslant n - \mathrm{nd}(L) + 1$.

3.2.3. Nakano vanishing theorem.

In this part, we give the following generalized version of the Nakano vanishing theorem.

THEOREM 3.25. *Let X be a n-dimensional projective manifold and L a nef holomorphic line bundle over X. Then we have*

$$H^p(X, \Omega_X^q \otimes L) = 0$$

for any $p+q > n+\max(\dim(B_+(L)), 0)$. Here $B_+(L)$ denotes the augmented base locus(or non-ample locus) of L. When $B_+(L) = \varnothing$, we define by convention that its dimension is -1.

Here we recall the definition of $B_+(L)$. Given an ample line bundle A over X. the augmented base locus is defined by

$$B_+(L) := \cap_{m>0} \mathrm{Bs}(mL - A)$$

where Bs means the base locus of a line bundle.

We recall classically (cf. [**BBP13**]) that $B_+(L) = \varnothing$ if and only if L is ample and $B_+(L) \neq X$ if and only if L is big. Thus we have the Nakano vanishing theorem in the case that $B_+(L) = \varnothing$.

We notice that by the example of [**Ram**], we can not change the augmented base locus by the base locus. In his example, we take X the blow up of $\mathbb{P}^3$ at one point and L the pull back of $\mathcal{O}_{\mathbb{P}^3}(1)$ under the blow up. Thus L is a big and nef line bundle with $\cap_{m>0}\mathrm{Bs}(mL) = \varnothing$. But by calculation of cohomology class we can show that

$$H^2(X, \Omega_X^2 \otimes L) \neq 0.$$

We observe that in this example $B_+(L) = E$ where E is the exceptional divisor.

Now, we return to the proof of the theorem. We argue by induction on the dimension of $B_+(L)$ and apply of the Nakamaye theorem. First note that we can assume L big, otherwise $B_+(L) = X$ and the theorem is void.

Let $l := \dim(B_+(L))$. When $l = -1$, the theorem is true by the Nakano vanishing theorem. When $l \leqslant 0$, we show that in fact L is ample. In this case, there exists some $m > 0$ and $s_0, \cdots, s_k \in H^0(X, mL - A)$ such that

$$\mathrm{Bs}(s_0, \cdots, s_k) = \{x_0, \cdots, x_l\}.$$

These sections induce a singular metric h_0 on $mL - A$ with analytic singularity at the discrete points $\{x_0, \cdots, x_l\}$. Its curvature is a closed positive (1,1)-current which is smooth outside $\{x_0, \cdots, x_l\}$. By [**Dem92a**] Lemma 6.3 $mL - A$ is nef. Hence L is ample.

Now let $l > 0$ and suppose by induction that the theorem has be verified for $\dim(B_+(L)) \leqslant l - 1$. We recall the concepts involved in the theorem of Nakamaye on base loci [**Nak04**].

DEFINITION 3.26. *Given a nef and big divisor L on X, the null locus* $\mathrm{Null}(L)$ *of L is the union of all positive dimensional subvarieties $V \subset X$ with*

$$(L^{\dim V} \cdot V) = 0.$$

We observe that for any smooth divisor D of X and such a line bundle,

$$\mathrm{Null}(L|_D) \subset \mathrm{Null}(L).$$

THEOREM 3.27. (Nakamaye). *If L is an arbitrary nef and big divisor on X, then*

$$B_+(L) = \mathrm{Null}(L).$$

Fix A_2 a very ample divisor on X. By Bertini theorem with a general choice we can assume that $D \in |A_2|$ is smooth. Since A_2 is very ample we can assume that $D \cap B_+(L) \subsetneq B_+(L)$. More precisely, for a general choice of D, no $l-$dimensional component of $B_+(L)$ is contained in D. Since L is nef and big, we have by Nakayame theorem $\mathrm{Null}(L) = B_+(L)$. By the definition of $\mathrm{Null}(L)$ we have

$$(L^{n-1} \cdot D) > 0.$$

In other words, $L|_D$ is big. Using another time the Nakamaye theorem, we find that

$$B_+(L|_D) = \mathrm{Null}(L|_D) \subset \mathrm{Null}(L) \cap D \subsetneq B_+(L).$$

In particular, $\dim B_+(L|_D) \leqslant \dim B_+(L) - 1$.

Recall the following elementary lemma (3.24) in [**SS**].

LEMMA 3.28. *Let L be a holomorphic line bundle over X, let D be a smooth hyper-surface in X, and let $p, q \geqslant 0$ be fixed. If*

$$(a) H^p(X, \Omega_X^q \otimes [D] \otimes L) = 0,$$
$$(b) H^{p-1}(D, \Omega_D^{q-1} \otimes L|_D) = 0,$$
$$(c) H^{p-1}(D, \Omega_D^q \otimes ([D] \otimes L)|_D) = 0,$$

then we have

$$H^p(X, \Omega_X^q \otimes L) = 0.$$

Since $[D] \otimes L$ is ample (L is nef), the hypotheses (a) (c) of the lemma is verified by the Nakano vanishing theorem. Since

$$(p-1) + (q-1) > \dim D + l - 1,$$

the condition (b) is satisfied by the inductive hypothesis.

This finishes the proof.

REMARK 3.29. It would be interesting to know whether the theorem is still valid without assuming L to be nef. Here principally, we use the nef condition in two places: in the Nakamaye theorem and in the fact that the sum of an ample divisor and a nef divisor is ample.

Here, following some ideas of Demailly, we give the following more general version of the Nakano vanishing theorem.

THEOREM 3.30. *Let X be a n-dimensional projective manifold, L a holomorphic line bundle and A an ample line bundle over X. Assume for sufficient grand $m \in \mathbb{N}$ and general hyper-surfaces in the linear system $H_1, \cdots, H_k \in |mA|$, we have that $L|_{H_1 \cap \cdots \cap H_k}$ is ample. Then for $p + q > n$,we have*

$$H^q(X, \Omega_X^p \otimes L) = 0.$$

PROOF. By duality, it is equivalent to show that for $p + q < n - k$,we have

$$H^q(X, \Omega_X^p \otimes L^{-1}) = 0.$$

Since the hyper-surface H_i is supposed to be general, we can assume that any intersection of type $H_1 \cap \cdots \cap H_l$ is smooth for any l and of dimension $n - l$ for any $l \leqslant k$.

For m big enough such that $mA + L$ is ample, hence by Nakano vanishing theorem we have the vanishing $p + q < n - k$

$$H^q(X, \Omega_X^p \otimes L^{-1} \otimes \mathcal{O}(-H_1)) = 0.$$

From the short exact sequence

$$0 \to \Omega_X^p \otimes L^{-1} \otimes \mathcal{O}(-H_1) \to \Omega_X^p \otimes L^{-1} \to (\Omega_X^p \otimes L^{-1})|_{H_1} \to 0$$

we know that to prove the desired vanishing it is enough to show that for $p + q < n - k$

$$H^q(X, (\Omega_X^p \otimes L^{-1})|_{H_1}) = 0.$$

From the short exact sequence

$$0 \to T_{H_1} \to T_X|_{H_1} \to \mathcal{O}(H_1)|_{H_1} \to 0$$

we have the exact sequence (using the fact that $\mathcal{O}(H_1)$ is of rank one)

$$0 \to \mathcal{O}(-H_1)|_{H_1} \otimes \Omega_{H_1}^{p-1} \to \Omega_X^p|_{H_1} \to \Omega_{H_1}^p \to 0.$$

We take the tensor product with $L^{-1}|_{H_1}$ and the long exact sequence associated to the coreesponding short exact sequence. By the Nakano vanishing theorem, we find

$$H^i(H_1, \Omega_{H_1}^j \otimes (L^{-1} \otimes \mathcal{O}(-H_1))|_{H_1}) = 0$$

for any $i + j < n - 1$. It is enough to prove that

$$H^q(H_1, (\Omega_{H_1}^p \otimes L^{-1}|_{H_1}) = 0$$

for $p + q < n - k$.

We continue this process and change X with H_1, then H_1 with $H_1 \cap H_2$ etc. Taking from the beginning m so big that $mA + L$ is ample, we get for every l that $mA + L|_{H_1 \cap \cdots \cap H_l}$ is ample on $H_1 \cap \cdots \cap H_l$. Hence in each step, we can use the Nakano vanishing theorem. Finally, we are reduced to proving that

$$H^q(H_1 \cap \cdots \cap H_k, \Omega_{H_1 \cap \cdots \cap H_k}^p \otimes L^{-1}|_{H_1 \cap \cdots \cap H_k}) = 0$$

for $p + q < n - k$. But this is true by the Nakano vanishing theorem and our assumption. □

REMARK 3.31. By the proof of the theorem, it is enough to take m so large that $mA + L$ is ample, and $H_i \in |mA|$ so that $H_1 \cap \cdots \cap H_l$ is smooth and of dimension $n - l$ for any $l \leqslant k$, and $L|_{H_1 \cap \cdots \cap H_k}$ is ample.

As pointed out by A. Höring, it is interesting to compare this result to the following theorem 2 of [**Kur13**]: Let X be a smooth projective variety, L a divisor, A a very ample divisor on X. If $L|_{E_1\cap\cdots\cap E_k}$ is big and nef for a general choice of $E_1, \cdots, E_k$, then $H^i(X, \mathcal{O}_X(K_X+L)) = 0$ for $i > k$.

REMARK 3.32. Our first theorem is a special case of this general version. Since L is nef, it is nef on the complete intersection of the hyper-surfaces $H_1, \cdots, H_l$ where $l := \dim(B_+(L))$. On the other hand, for such general hyper-surfaces, we can assume that the intersection $B_+(L) \cap H_1 \cap \cdots \cap H_l$ is finite points. By the definition of stable base locus, $L|_{H_1\cap\cdots\cap H_l}$ is ample outside these finite points. Hence in fact, $L|_{H_1\cap\cdots\cap H_l}$ is ample.

The k-ampleness condition defined by Sommese [**Som**] is also a sufficient condition for the condition stated in Theorem 3.30. We start by recalling the definition.

DEFINITION 3.33. *A holomorphic line bundle L on a compact complex manifold X is said to be k-ample $(0 \leqslant k \leqslant n-1)$ if there exists a positive integer N such that NL spans at each point of X and the Kodaira morphism associated to NL has at most k-dimensional fibres.*

Changing N in the definition by a possible large multiple of N we can assume that the Kodaira morphism associated to NL is the Iitaka fibration. Denote $\Phi : X \to Z$ the fibration where Z is a projective variety. Denote $A_{z,j}$ $(z \in Z, j \in \mathbb{N})$ the irreducible components of the fibre of z (i.e. $\Phi^{-1}(z)$). By a general choice of H_1, we can assume that for any z, j the hyper-surface H_1 intersecting $A_{z,j}$ defines a divisor of $A_{z,j}$ by the lemma stated below. Similarly, with a general choice of $H_1, \cdots, H_k$ we can assume that for any z, j $H_1 \cap \cdots \cap H_k \cap A_{z,j}$ is a finite set, by the assumption that $\dim A_{z,j} \leqslant k$. In other words, the restriction of the Kodaira morphism

$$\Phi : H_1 \cap \cdots \cap H_k \to Z$$

is a finite morphism. Since $L|_{H_1\cap\cdots\cap H_k}$ is pull back of $\mathcal{O}(1)$ via Φ, $L|_{H_1\cap\cdots\cap H_k}$ is ample on $H_1 \cap \cdots \cap H_k$. (Recall that the pull back of an ample line bundle under a finite morphism is ample.)

LEMMA 3.34. *Let $\Phi : X \to Z$ be the fibration such that all the fibers have dimension $\leqslant k$. Assume X is projective. Then there exists $H \subset X$ a general very ample divisor such that the restriction $\Phi_H : H \to Z$ of Φ on H has all fibers of dimension $\leqslant (k-1)$.*

PROOF. Denote $A_{z,j}$ $(z \in Z, j \in \mathbb{N})$ the irreducible components of the fibre of z (i.e. $\Phi^{-1}(z)$). It is equivalent to demand the restriction to each $A_{z,j}$ of the defining section σ of H is non trivial. Let A be an ample divisor on X. Denote $V_{z,j}$ the linear subspace of $H^0(X, mA)$ such that $\sigma|_{A_{z,j}} \equiv 0$. We want to choose σ such that $\sigma \in H^0(X, mA) \smallsetminus \bigcup_{z,j} V_{z,j}$. Notice that the family $A_{z,j}$ parametrized by z, j forms a bounded family in the Hilbert scheme of X. A sufficient condition to find σ as above is that for m large enough

$$\dim Z + \dim V_{z,j} < h^0(X, mA).$$

Without loss of generality, we can assume that A is very ample on X. Hence, by boundedness, we have for m large enough independent of z, j a surjective restriction morphism

$$H^0(X, mA) \to H^0(A_{z,j}, mA).$$

As $V_{z,j}$ is the kernel of this morphism, it is enough to take m so large that

$$\dim Z < h^0(A_{z,j}, mA).$$

For $A_{z,j}$ with positive dimension, the regular part of $A_{z,j}$ is a smooth submanifold of X. Since A is very ample, it generates 1-jets of the regular part of $A_{z,j}$ at any point. Hence $H^0(A_{z,j}, NA)$ generates any m-fold symmetric product of 1-jets of $A_{z,j}$ at some regular point. In other words,

$$h^0(A_{z,j}, mA) > \binom{m}{\dim A_{z,j}} \geqslant m.$$

□

CHAPTER 4

Considerations on nefness in higher codimension

ABSTRACT. In this note, following the fundamental work of Boucksom we construct the nef cone of a compact complex manifold in higher codimension and give explicit examples where these cones are different. In the third section, we give two versions of Kawamata-Viehweg vanishing theorems in terms of nefness in higher codimension and numerical dimensions. We also show by examples the optimality of the divisoral Zariski decomposition given in [**Bou04**]. In the last section, we discuss the surjectivity of the Albanese morphism for a compact Kähler manifold with $-K_X$ psef and some additional assumptions on the regularity of approximated metrics.

4.1. Nefness in higher codimension

We first recall some technical preliminaries introduced in [**Bou04**]. Throughout this paper, X is assumed to be a compact complex manifold equipped with some reference Hermitian metric ω (i.e. a smooth positive definite $(1,1)$-form); we usually take ω to be Kähler if X possesses such metrics. The Bott-Chern cohomology group $H^{1,1}_{BC}(X,\mathbb{R})$ is the space of d-closed smooth (1,1)-forms modulo $i\partial\overline{\partial}$-exact ones. By the quasi-isomorphism induced by the inclusion of smooth forms into currents, $H^{1,1}_{BC}(X,\mathbb{R})$ can also be seen as the space of d-closed $(1,1)$-currents modulo $i\partial\overline{\partial}$-exact ones. A cohomology class $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ is said to be pseudo-effective iff it contains a positive current; α is nef iff, for each $\varepsilon > 0$, α contains a smooth form α_ε such that $\alpha_\varepsilon \geqslant -\varepsilon\omega$; α is big iff it contains a Kähler current, i.e. a closed $(1,1)$-current T such that $T \geqslant \varepsilon\omega$ for $\varepsilon > 0$ small enough.

DEFINITION 4.1. ([**DPS01**]) *Let φ_1, φ_2 be two quasi-psh functions on X (i.e. $i\partial\overline{\partial}\varphi_i \geqslant -C\omega$ in the sense of currents for some $C \geqslant 0$). The function φ_1 is said to be less singular than φ_2 (one then writes $\varphi_1 \preceq \varphi_2$) if $\varphi_2 \leqslant \varphi_1 + C_1$ for some constant C_1. Let α be a fixed psef class in $H^{1,1}_{BC}(X,\mathbb{R})$. Given $T_1, T_2, \theta \in \alpha$ with θ smooth, and $T_i = \theta + i\partial\overline{\partial}\varphi_i$ with φ_i quasi-psh $(i = 1, 2)$, we write $T_1 \preceq T_2$ iff $\varphi_1 \preceq \varphi_2$ (notice that for any choice of θ, the potentials φ_i are defined up to smooth bounded functions, since X is compact). If γ is a smooth real $(1,1)$-form on X, the collection of all potentials φ such that $\theta + i\partial\overline{\partial}\varphi \geqslant \gamma$ admits a minimal element $T_{min,\gamma}$ for the pre-order relation $\preceq$, constructed as the semi-continuous upper envelope of the subfamily of potentials $\varphi \leqslant 0$ in the collection.*

DEFINITION 4.2. (Minimal multiplicities). *The minimal multiplicity at $x \in X$ of the pseudo-effective class $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ is defined as*

$$\nu(\alpha, x) := \sup_{\varepsilon>0} \nu(T_{\min,\varepsilon}, x)$$

where $T_{\min,\varepsilon}$ is the minimal element $T_{\min,-\varepsilon\omega}$ in the above definition and $\nu(T_{\min,\varepsilon}, x)$ is the Lelong number of $T_{\min,\varepsilon}$ at x. When Z is an irreducible analytic subset, we define the generic minimal multiplicity of α along Z as

$$\nu(\alpha, Z) := \inf\{\nu(\alpha, x), x \in Z\}.$$

When Z is positive dimensional, there exists for each $\ell \in \mathbb{N}^*$ a countable union of proper analytic subsets of Z denoted by $Z_\ell = \bigcup_p Z_{\ell,p}$ such that $\nu(T_{\min,\frac{1}{\ell}}, Z) := \inf_{x\in Z} \nu(T_{\min,\frac{1}{\ell}}, x) = \nu(T_{\min,\frac{1}{\ell}}, x)$ for $x \in Z \smallsetminus Z_\ell$. By construction, when $\varepsilon_1 < \varepsilon_2$, $T_{\min,\varepsilon_1} \succeq T_{\min,\varepsilon_2}$. Hence for a very general point $x \in Z \smallsetminus \bigcup_{\ell\in\mathbb{N}^*} Z_\ell$,

$$\nu(\alpha, Z) \leqslant \nu(\alpha, x) = \sup_{\ell} \nu(T_{\min,\frac{1}{\ell}}, Z).$$

On the other hand, for any $y \in Z$,

$$\sup_{\ell} \nu(T_{\min,\frac{1}{\ell}}, Z) \leqslant \sup_{\ell} \nu(T_{\min,\frac{1}{\ell}}, y) = \nu(\alpha, y).$$

In conclusion, $\nu(\alpha, Z) = \nu(\alpha, x)$ for a very general point $x \in Z \smallsetminus \bigcup_{\ell\in\mathbb{N}^*} Z_\ell$ and $\nu(\alpha, Z) = \sup_\varepsilon \nu(T_{\min,\varepsilon}, Z)$.

Now we can define the concept of nefness in higher codimension implicitly used in [**Bou04**]. It is the generalisation of the concept of "modified nefness" to the higher codimensional case.

DEFINITION 4.3. *Let $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ be a psef class. We say that α is nef in codimension k, if for every irreducible analytic subset $Z \subset X$ of codimension at most equal to k, we have*

$$\nu(\alpha, Z) = 0.$$

We denote by $\mathcal{N}_k$ the cone generated by nef classes in codimension k. By Proposition 3.2 in [**Bou04**], a psef class α is nef iff for any $x \in X$, $\nu(\alpha, x) = 0$. By our definition, psef is equivalent to nef in codimension 0, and nef is equivalent to nef in codimension $n := \dim_{\mathbb{C}} X$. In this way, we get a bunch of positive cones on X, satisfying the inclusion relations

$$\mathcal{N} = \mathcal{N}_n \subset \cdots \subset \mathcal{N}_1 \subset \mathcal{N}_0 = \mathcal{E}.$$

By a proof similar to those of propositions 3.5, 3.6 in [**Bou04**], we get

PROPOSITION 4.1.1. (1) For every $x \in X$ and every irreducible analytic subset Z, the map $\mathcal{E} \to \mathbb{R}^+$ defined on the cone $\mathcal{E}$ of psef classes by $\alpha \mapsto \nu(\alpha, Z)$ is convex and homogeneous. It is continuous on the interior $\mathcal{E}^\circ$, and lower semi-continuous on the whole of $\mathcal{E}$.

(2) If $T_{\min} \in \alpha$ is a positive current with minimal singularities, we have $\nu(\alpha, Z) \leqslant \nu(T_{\min}, Z)$.

(3) If α is moreover big, we have $\nu(\alpha, Z) = \nu(T_{\min}, Z)$.

The following lemma is a direct application of the proposition.

LEMMA 4.4. *Let $Y \subset X$ be a smooth submanifold of X and $\pi : \tilde{X} \to X$ be the blow-up of X along Y. We denote by E the exceptional divisor. If $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ is a big class, we have*

$$\nu(\alpha, Y) = \nu(\pi^* \alpha, E).$$

For Z any irreducible analytic set not included in Y, we denote by $\tilde{Z}$ the strict transform of Z. Then

$$\nu(\alpha, Z) = \nu(\pi^* \alpha, \tilde{Z}).$$

For W any irreducible analytic set in Y, we have

$$\nu(\alpha, W) = \nu(\pi^* \alpha, \mathbb{P}(N_{Y/X}|_W)).$$

PROOF. Since α is big, we know that by taking a suitable regularisation, there exists a Kähler current $T \in \alpha$ with analytic singularities. The pull back $\pi^* T$ of this current is a smooth Kähler current on some dense open set U where π is a biholomorphism. Hence the volume of $\pi^* \alpha$ defined as $\int_{T \in \pi^* \alpha, T \geqslant 0,} T^n_{ac}$ (ac means the absolute part of the current) is larger than the mass of $\pi^* T$ on U which is strictly positive. By [**Bou02a**] $\pi^* \alpha$ is thus big.

By the proposition, we have

$$\nu(\alpha, Y) = \inf_{T \in \alpha} \nu(T, Y), \quad \nu(\pi^* \alpha, E) = \inf_{S \in \pi^* \alpha} \nu(S, E).$$

On the other hand, the push forward and pull back operators acting on positive $(1,1)$ currents induce bijections between positive currents in the class α and positive currents in the class $\pi^* \alpha$. Let $\theta \in \alpha$ be a smooth form such that $T = \theta + i\partial\overline{\partial}\varphi$. We recall that for any irreducible analytic set W with local generators $(g_1, \cdots, g_r)$ near a regular point $w \in W$, the generic Lelong number along W is the largest γ such that $\varphi \leqslant \gamma \log(\sum |g_i|^2) + O(1)$ near w. Since $\pi^*(g_1, \cdots, g_r) \cdot \mathcal{O}_{\tilde{X}} = \mathcal{I}_E$, we have $\nu(T, Y) = \nu(\pi^* T, E)$. In particular, this implies that

$$\nu(\alpha, Y) = \nu(\pi^* \alpha, E).$$

For W any irreducible analytic set in the centre Y, since the exceptional divisor is isomorphic to $\mathbb{P}(N_{Y/X})$, the preimage of W under the blow-up is isomorphic to $\mathbb{P}(N_{Y/X}|_W)$. In suitable local coordinates $(z_1, \cdots, z_n)$ on X and $(w_1, \cdots, w_n)$ on $\tilde{X}$, the blow-up map is given by

$$\pi(w_1, \cdots, w_n) = (w_1, w_1 w_2, \cdots, w_1 w_s; w_{s+1}, \cdots, w_n).$$

In these coordinates, the centre Y is given by the zero variety $V(z_{s+1}, \cdots, z_n)$. Assume that in this chart, $W = V(z_{s+1}, \cdots, z_n; f_1, \cdots, f_r)$ where f_i is a function of $z_1, \cdots, z_s$ (as we can assume without loss of generality). Then $\pi^*(\mathcal{I}_W) \cdot \mathcal{O}_{\tilde{X}} = (w_1, f_1(w_1, \cdots, w_s), \cdots, f_r(w_1, \cdots, w_s)) = \mathcal{I}_{\mathbb{P}(N_{Y/X}|_W)}$. In particular, this implies that

$$\nu(\alpha, W) = \nu(\pi^* \alpha, \mathbb{P}(N_{Y/X}|_W)).$$

For the second statement, we just observe that the generic Lelong number along Z (resp. $\tilde{Z}$) is equal to the Lelong number at some very general point. Since Z is not contained in Y we can assume without loss of generality that the very general point is not in Y (resp. E). Since the Lelong number is a coordinate invariant local property, for such very general point $x \in \tilde{Z}$ near which π is a local biholomorphism and any $T \in \alpha, T \geqslant 0$, $\nu(T, Z) = \nu(T, \pi(x)) = \nu(\pi^* T, x) = \nu(\pi^* T, \tilde{Z})$. Hence we have

$$\nu(\alpha, Z) = \nu(\pi^* \alpha, \tilde{Z}).$$

□

As a corollary, we find

COROLLARY 4.5. *Let $\mu : \tilde{X} \to X$ be a composition of finitely many blow-up with smooth centres in X. If $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ is a big class on X such that $\mu^*\alpha$ is nef in codimension k, then α is a nef class in codimension k.*

PROOF. Without loss of generality, we can reduce ourselves to the case where μ is a blow-up of smooth centre Y in X. By Lemma 4.4, the generic minimal multiplicity of α along any irreducible analytic set of X of codimension at most equal to k is equal to the generic minimal multiplicity of $\mu^*\alpha$ along certain irreducible analytic set of $\tilde{X}$ of codimension at most equal to k. So by the definition of nefness in codimension k, the fact $\mu^*\alpha$ is nef in codimension k implies that α is nef in codimension k. □

REMARK 4.6. Let X be a compact complex manifold X whose big cone is non empty. Recall that by Proposition 2.3 of [**Bou04**], a class α is modified Kähler (i.e. α is in the interior of nef cone in codimension 1) iff there exists a modification $\mu : \tilde{X} \to X$ and a Kähler class $\tilde{\alpha}$ on $\tilde{X}$ such that $\alpha = \mu_*\tilde{\alpha}$. As a consequence, for $\mu : \tilde{X} \to X$ a modification between compact Kähler manifolds and $\tilde{\alpha} \in H^{1,1}_{BC}(\tilde{X}, \mathbb{R})$ a big and nef class on $\tilde{X}$ in codimension k, it is false in general that $\mu_*\tilde{\alpha}$ is a nef class in codimension k.

To give an equivalent definition of nefness in higher codimension, we will need the following definition.

DEFINITION 4.7. (Non-nef locus)
The non-nef locus of a pseudo-effective class $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ is defined by

$$E_{nn}(\alpha) := \{x \in X, \nu(\alpha, x) > 0\}.$$

PROPOSITION 4.1.2. A psef class α is nef in codimension k iff for any $\varepsilon > 0$, any $c > 0$, the codimension of any irreducible component of $E_c(T_{\min,\varepsilon})$ is larger than $k+1$.

PROOF. By the definition of non-nef locus, we have

$$E_{nn}(\alpha) = \bigcup_{\varepsilon>0} \bigcup_{c>0} E_c(T_{\min,\varepsilon}) = \bigcup_{m\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}^*} E_{\frac{1}{n}}(T_{\min,\frac{1}{m}}).$$

We know by Siu's theorem [**Siu74**] that $E_{\frac{1}{n}}(T_{\min,\frac{1}{m}})$ is an analytic set. Hence the non-nef locus is a countable union of irreducible analytic set. If for any $\varepsilon > 0$, any $c > 0$, the codimension of any irreducible component of $E_c(T_{\min,\varepsilon})$ is larger than $k+1$, then for any irreducible analytic set Z of codimension k, $E_{nn}(\alpha) \cap Z$ is strictly contained in Z. Hence $\nu(\alpha, Z) = 0$.

On the other direction, assume there exists an irreducible component Z of $E_{\frac{1}{n}}(T_{\min,\frac{1}{m}})$ has codimension at most equal to k. On each point x of this irreducible component, $\nu(\alpha, x) \geqslant \nu(T_{\min,\frac{1}{m}}, x) \geqslant \frac{1}{n}$. In particular, $\nu(\alpha, Z) \geqslant \frac{1}{n}$, which contradicts the fact that α is nef in codimension k. □

REMARK 4.8. If the manifold X is projective, it is enough to test the minimal multiplicity along irreducible analytic subsets of codimension k to prove that the class is nef in codimension k. The argument is as follows:

For any irreducible analytic set Z of codimension strictly smaller than k, for any $z \in Z$, since X is projective, there exists some hypersurfaces H_i such that $z \in H_i$ and the irreducible component of $Z \cap \bigcap_i H_i$ containing z has codimension k. In other words, Z is covered by the irreducible analytic subsets of codimension exactly k. By assumption, the generic minimal multiplicity along any of these irreducible analytic subsets is 0. This implies that the generic minimal multiplicity along Z at most equal to the generic minimal multiplicity along any these irreducible analytic set is 0.

REMARK 4.9. In the general setting of compact complex manifolds, it is important to test the generic minimal multiplicity along any analytic set of codimension at most equal to k, instead of any analytic set of codimension k, to obtain the inclusion of the various positive cones. The problem is that there may exist too few analytic subsets in an arbitrary compact complex manifold.

A typical example can be taken as follows. For example let X_1 be a compact manifold such that the nef cone is strictly contained in the psef cone (for example we can take the projectivisation of an unstable rank two vector bundle over a curve of genus larger than 2, whose cones are explicited calculated on page 70 [**Laz04**]) and X_2 be a very general torus such that the only analytic sets in X_2 are either union of points or X_2. Let β be a psef but not nef class on X_1. Let $X := X_1 \times X_2$ with natural projections π_1, π_2 and $\alpha := \pi_1^*\beta$. Assume that $\dim(X_1) < \dim(X_2)$. Fix ω_1, ω_2 two reference Hermitian metrics on X_1, X_2.

Now α is a psef but not nef class on X. The only analytic subsets of codimension $\dim(X_1)$ is the fibre of π_2. α has generic minimal multiplicity 0 along any fibre of π_2. The reason is as follows: The minimal current in α larger than $-\varepsilon(\pi_1^*\omega_1 + \pi^*\omega_2)$ denoting $\min\{T \in \alpha, T \geqslant -\varepsilon(\pi_1^*\omega_1 + \pi^*\omega_2)\}$ is less singular than the pull back of the minimal current in β larger than $-\varepsilon\omega_1$ denoting $\min\{S \in \beta, S \geqslant -\varepsilon\omega_1\}$ and the restriction of these minimal currents on the fibre of π_2 is trivial. In other words, the generic Lelong number of $\min\{T \in \alpha, T \geqslant -\varepsilon(\pi_1^*\omega_1 + \pi^*\omega_2)\}$ along the fibres is smaller than the generic Lelong number of the pull back of $\min\{S \in \beta, S \geqslant -\varepsilon\omega_1\}$ which is 0. Hence it is itself 0.

On the other hand, for any positive integers m, n, take Z a positive dimensional irreducible component of $E_{\frac{1}{n}}(T_{\min,\frac{1}{m}})$ in the non-nef locus of β. The existence of such an irreducible component will be shown in the lemma 4.11, which implies that α has to be nef in codimension at most equal to $n-2$. Now $Z \times X_2$ is an irreducible analytic set of codimension strictly smaller than $\dim(X_1)$. But the generic minimal multiplicity along $Z \times X_2$ is larger than $\frac{1}{n}$. In particular this shows that α is not nef in codimension $\dim(X_1) - \dim(Z)$.

REMARK 4.10. Let us mention that our definition of nefness in codimension 1 is equivalent to the definition of modified nefness. By definition, a psef class is modified nef iff its generic minimal multiplicity is 0 along any prime divisor. To prove the equivalence, it is enough to show that for any psef class α on X we automatically have

$$\nu(\alpha, X) = 0.$$

It is because that $\nu(\alpha, X) \leqslant \nu(T_{\min}, X)$ where the latter is 0. We notice that by Siu's decomposition theorem [**Siu74**], the set $E_{c>0}(T_{\min}) = \bigcup_{n\in\mathbb{N}^*} E_{\frac{1}{n}}(T_{\min})$ is countable union of proper analytic sets.

By this observation, we can also say that the "nef in codimension 0" cone is exactly the psef cone.

In analogy to the case of surfaces for which the nef cone coincides with the modified nef cone, the nef cone in codimension $n-1$ coincides with the nef cone.

LEMMA 4.11. *Let α be a psef class, then α is nef in codimension $n-1$ iff α is nef.*

PROOF. If α is nef, by inclusion of different positive cones, it is nef in codimension $n-1$. On the other direction, we will need the following proposition 3.4 in [**Bou04**] which is a reformulation of a result of Păun [**Paun98**].

A pseudo-effective class α is nef iff $\alpha|_Y$ is pseudo-effective for every irreducible analytic subset $Y \subset E_{nn}(\alpha)$.

Given a class α that is nef in codimension $n-1$, proposition 4.1.2 implies that for any $\varepsilon > 0$ and any $c > 0$ the analytic set $E_c(T_{\min,\varepsilon})$ is a finite set. Therefore, the non-nef locus which is a countable union of finite sets has at most countably many points. In particular, this implies that the restriction of α on any $Y \subset E_{nn}(\alpha)$ is 0, hence psef. By the above proposition, α is nef. □

REMARK 4.12. Recall that a line bundle L over a projective manifold is nef iff its intersection number with any curve satisfies $(L \cdot C) \geqslant 0$. By the important work of [**BDPP13**], a class is psef iff its pairing with any movable curve is positive. Here a curve C is said to be movable if $C = C_{t_0}$ is a member of an analytic family $(C_t)_{t\in S}$ such that $\bigcup_{t\in S} C_t = X$ and, as such, C is a reduced irreducible 1-cycle. Remark also that nef is equivalent to nef in codimension $n-1$ and psef is equivalent to nef in codimension 0.

Then it is natural to conjecture that a class over a projective manifold is nef in codimension k if and only if its pairing with any movable curve in codimension k is positive. Here a curve C is said to be movable in codimension k if $C = C_{t_0}$ is a member of an analytic family $(C_t)_{t\in S}$ such that $\bigcup_{t\in S} C_t$ is an analytic subset of X of codimension k and, as such, C is a reduced irreducible 1-cycle.

REMARK 4.13. Inspired by the result of Păun, it seems to be natural to conjecture that a psef class $\{T\}$ with T a positive current on X is nef in codimension k if and only if that for any irreducible component of codimension at most k in $\bigcup_{c>0} E_c(T)$ $\{T\}|_Z$ is nef in codimension $k - \operatorname{codim}(Z, X)$. When $k = n$, this is exactly the result of Păun. When $k = 0$, it is trivial. The "only if" part is quite similar. The restriction of the potentials of $T_{\min,\varepsilon}$ on any irreducible analytic set of codimension at most k decreases to a potential on the submanifold. If we fix the maximum of the potentials on X to be 0, they form a compact family. The limit potential would be quasi-psh and thus the restriction of the class on the analytic set is psef. The "if" part is of course true if the manifold is a Kähler surface by Păun's result.

The "if" part is also true for the case $k = 1$ if the manifold is hyperkähler. By lemma 4.9 [**Bou04**] (see also [**Huy03**]) a psef class α on a hyperkähler manifold is modified nef if and only if for any prime divisor D one has $q(\alpha, D) \geqslant 0$. Here, we let σ be a symplectic holomorphic form on X, and define

$$q(\alpha, \beta) := \int_X \alpha \wedge \beta \wedge (\sigma \wedge \overline{\sigma})^{\frac{n}{2}-1}$$

to be the Beauville–Bogomolov quadratic form for any $(1,1)$-classes α, β. For a psef $(1,1)$-class α such that $\alpha|D$ is psef for any prime divisor D, we have

$$q(\alpha, \{[D]\}) = \int_X \alpha \wedge \{[D]\} \wedge (\sigma \wedge \overline{\sigma})^{\frac{n}{2}-1} = \int_D \alpha \wedge (\sigma \wedge \overline{\sigma})^{\frac{n}{2}-1} \geqslant 0.$$

Thus α is nef in codimension 1.

A natural idea to attack this question in general consists in trying to extend the current on this subvariety Z to X. If this is possible, the current with minimal singularity would have a potential larger than that of the extended current. In particular, the current with minimal singularity would have generic Lelong number 0 along Z.

In this direction, Collins and Tosatti proved the following results in [**CT15**] and [**CT14**], which we now recall.

THEOREM 4.14. (Theorem 3.2 in [**CT14**]). *Let X be a compact Fujiki manifold and α a closed smooth real (1,1)-form on X with $\{\alpha\}$ nef and $\int_X \alpha^n > 0$. Let $E = V \cup \bigcup_{i=1}^{I} Y_i$ be an analytic subvariety of X, with V, Y_i its irreducible components, and V a positive dimensional compact complex submanifold of X. Let $R = \alpha + i\partial\overline{\partial}F$ be a Kähler current in the class $\{\alpha\}$ on X with analytic singularities precisely along E and let $T = \alpha|_V + i\partial\overline{\partial}\varphi$ be a Kähler current in the class $\{\alpha|_V\}$ on V with analytic singularities. Then there exists a Kähler current $\tilde{T} = \alpha + i\partial\overline{\partial}\Phi$ in the class $\{\alpha\}$ on X with $\tilde{T}|_V$ smooth in a neighbourhood of the very general point of V.*

THEOREM 4.15. (Theorem 1.1 in [**CT15**]). *Let (X, ω) be a compact Kähler manifold and let $V \subset X$ be a positive-dimensional compact complex submanifold. Let T be a Kähler current with analytic singularities along V in the Kähler class $\{\omega|_V\}$. Then there exists a Kähler current $\tilde{T}$ on X in the class $\{\omega\}$ with $T = \tilde{T}|_V$.*

Using their results, in a given Kähler class, one can extend Kähler currents with analytic singularities defined in a smooth subvariety. If the class is just nef and big on the Kähler manifold, one can only show the existence of a Kähler current whose potential is not identically infinity along the submanifold. Following example 5.4 in [**BEGZ10**], one can show that in a nef and big class on a Kähler manifold X, one cannot always extend a positive current along a submanifold into a positive current on X. In their example, the positive current on the submanifold can even be chosen to be smooth. More precisely there exists C, a submanifold of a certain compact Kähler manifold X, $\{\alpha\}$ a nef and big class on X with a smooth representative α and $\varphi \in L^1_{\mathrm{loc}}(C)$ with $\alpha|_C + i\partial\overline{\partial}\varphi \geqslant 0$, such that there does not exist a $\psi \in L^1_{\mathrm{loc}}(X)$ satisfying $\alpha + i\partial\overline{\partial}\psi \geqslant 0$ and $\psi|_C = \varphi$.

Let us start the construction of the example. Let C be an elliptic curve and let A be an ample divisor on C. Let V be the rank 2 vector bundle over C the unique non-trivial extension of $\mathcal{O}_C$. Define $X := \mathbb{P}(V \oplus A)$ and $\{\alpha\} := c_1(\mathcal{O}_X(1))$ with smooth representative α. Then $\mathcal{O}_X(1)$ is a big and nef line bundle over X. The quotient map $V \oplus A \to \mathcal{O}_C$ induces a closed immersion $C \to X$. In particular, we have $\mathcal{O}_X(1)|_C = \mathcal{O}_C$. Since $c_1(\mathcal{O}_X(1)|_C) = 0$, there exists a smooth function φ on C such that $\alpha|_C + i\partial\overline{\partial}\varphi = 0$. We prove by contradiction that there does not exist $\psi \in L^1_{\mathrm{loc}}(X)$ such that $\alpha + i\partial\overline{\partial}\psi \geqslant 0$ and $\psi|_C = \varphi$. The quotient map $V \oplus A \to V$ induces a closed immersion $\mathbb{P}(V) \to X$. On the contrary, we would have $\alpha|_{\mathbb{P}(V)} + i\partial\overline{\partial}\psi|_{\mathbb{P}(V)} \geqslant 0$ in the class $c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$. By the calculation made in example 1.7 of [**DPS94**], we know that

$$\alpha|_{\mathbb{P}(V)} + i\partial\overline{\partial}\psi|_{\mathbb{P}(V)} = [C]$$

where $[C]$ is the current associated with C. In particular, this shows that $\psi|_C \equiv -\infty$, a contradiction.

In other words, theorem 1.1 of [**CT15**] cannot be strengthened to obtain an extension of an arbitrary closed positive current in a class that is merely nef and big. Similarly, one cannot drop the Kähler current condition in the theorem of [**CT14**].

Let us return to our previous question. To get an analogue of Păun's result, the above discussion shows that we need to generalise theorem 3.2 of [**CT14**] to the class of a big class that is nef in codimension k by adding a small Kähler form to the class and by using the semi-continuity of the generic minimal multiplicity. Unfortunately, we do not know how to do it at this point.

4.2. Kawamata-Viehweg vanishing theorem

We first give a "numerical dimension version" of the Kawamata-Viehweg vanishing theorem in the projective case. In the following, we study various properties of nef classes in higher codimension. Then we end the section by a numerical version of the Kawamata-Viehweg vanishing theorem in the Kähler case.

To start with, we need the relation between movable intersection defined in [**BDPP13**], [**Bou02b**] and intersection number.

LEMMA 4.16. *Let α be a nef class in codimension p on a compact Kähler manifold (X, ω). Then for any $k \leqslant p$ and Θ any positive closed $(n-k, n-k)$-form we have*

$$(\alpha^k, \Theta) \geqslant \langle \alpha^k, \Theta \rangle.$$

Here we use the definition of movable intersection defined in [**Bou02b**] and [**BDPP13**]. The movable intersection number $\langle\alpha^k,\Theta\rangle$ in [**Bou02b**] is defined as the limit for $\varepsilon>0$ converging to 0 of the quantity:

$$\sup_{T_i}\int_{X\smallsetminus F}(T_1+\varepsilon\omega)\wedge\cdots(T_k+\varepsilon\omega)\wedge\Theta$$

where T_i ranges all closed current with analytic singularities in the class α such that $T_i\geqslant-\varepsilon\omega$ and F is the union of all singular part of T_i. (In [**Bou02b**], the movable intersection number is defined for any closed positive current Θ. In the following, we will take Θ to be ω^{n-k}. Thus we consider only the case when Θ is a positive closed form.)

The proof of the boundedness of the quantity is a consequence of regularisation and the theory of Monge-Ampère operator. In the general case, we approximate the current T_i decreasingly by the smooth forms by [**Dem82**] with a uniform lower bound $-C\omega$ depending on (X,ω) and $\{T_i\}$. Now on $X\smallsetminus F$ the current $(T_1+C\omega)\wedge\cdots(T_k+C\omega)\wedge\Theta$ is the limit of corresponding terms changing T_i by its smooth approximation, using the continuity of Monge-Ampère operator with respect to decreasing sequence. But the integral on $X\smallsetminus F$ obtained for the smooth approximation is bounded by its integral on X, which is the intersection number of cohomology classes $\{T_i+C\omega\}$ and $\{\Theta\}$.

Proof. Our observation is that with better regularity on the cohomology class α, we can define directly the Monge-Ampère operator on X. So comparing to the general case, we can skip the approximation process and get rid of the dependence of C which only depends on (X,ω) and α but not explicitly.

We recall the following theorem (4.6) on the Monge-Ampère operators in chapter 3 of [**Dem12b**].

Let $u_1,\cdots,u_q$ be quasi-plurisubharmonic functions on X and T be a closed positive current of bidimension (p,p). The currents $u_1 i\partial\overline{\partial}u_2\wedge\cdots\wedge i\partial\overline{\partial}u_q\wedge T$ and $i\partial\overline{\partial}u_1\wedge i\partial\overline{\partial}u_2\wedge\cdots\wedge i\partial\overline{\partial}u_q\wedge T$ are well defined and have locally finite mass in X as soon as $q\leqslant p$ and

$$H_{2p-2m+1}(L(u_{j_1})\cap\cdots\cap L(u_{j_m})\cap\mathrm{Supp}(T))=0$$

for all choices of indices $j_1<\cdots<j_m$ in $\{1,\cdots,q\}$.

Here $H_{2p-2m+1}$ means the $(2p-2m+1)$-dimensional Hausdorff content of the subset of X seen as a metric space induced by the Kähler metric. The unbounded locus $L(u)$ is defined to be the set of points $x\in X$ such that u is unbounded in every neighbourhood of x. When u has analytic singularities, it is the singular part of u (i.e. $\{u=-\infty\}$).

Now return to the proof of the lemma. By definition $T_{i,\min,-\varepsilon\omega}$ is less singular than T_i. Since for any $c>0$, $E_c(T_{i,\min,-\varepsilon\omega})$ has codimension larger than $p+1$, the singular set of T_i which has analytic singularities is also of codimension larger than $p+1$. By the theorem (4.6) cited above, the current $(T_1+\varepsilon\omega)\wedge\cdots(T_k+\varepsilon\omega)\wedge\Theta$ is well-defined on X. Thus we have

$$\begin{aligned}\int_{X\smallsetminus F}(T_1+\varepsilon\omega)\wedge\cdots(T_k+\varepsilon\omega)\wedge\Theta&\leqslant\int_X(T_1+\varepsilon\omega)\wedge\cdots(T_k+\varepsilon\omega)\wedge\Theta\\&=(\alpha+\varepsilon\{\omega\})\cdots\cdot(\alpha+\varepsilon\{\omega\})\cdot\{\Theta\}.\end{aligned}$$

Taking $\varepsilon\to0$, we get $(\alpha^k,\Theta)\geqslant\langle\alpha^k,\Theta\rangle$. □

We can now give in the projective case the following version of the Kawamata-Viehweg theorem in terms of nefness in higher codimension. The simple proof given below has been suggested to us by Demailly.

Theorem 4.17. *Let X be a projective manifold and L a nef line bundle in codimension $p-1$. If $\langle c_1(L)^p\rangle\neq0$, then for any $q\geqslant n-p+1$ we have*

$$H^q(X,K_X\otimes L)=0.$$

Proof. The proof is an induction on the dimension of X. Let A be an ample divisor on X and $\omega\in c_1(A)$ be a Kähler form. Let $Y\in|kA|$ be a generic smooth hypersurface. With the choice of k big enough, we can assume that $H^q(X,L^{-1}\otimes\mathcal{O}(-Y))=0$ for any $q<n$ by Kodaira vanishing theorem. By Serre duality, the statement of the theorem is equivalent to prove that for any $q\leqslant p-1$ we have

$$H^q(X,L^{-1})=0.$$

Consider the long exact sequence associated to the short exact sequence

$$0\to L^{-1}\otimes\mathcal{O}(-Y)\to L^{-1}\to L^{-1}|_Y\to0.$$

It turns out that it is enough to prove that $H^q(Y,L^{-1})=0$ for any $q\leqslant p-1$.

We check that conditions are preserved under the intersection with a generic hypersurface. Since α is nef in codimension $p-1$, we find that any irreducible component of

$$E_{nn}(\alpha)=\bigcup_{m\in\mathbb{N}^*}\bigcup_{n\in\mathbb{N}^*}E_{\frac{1}{n}}(T_{\min,\frac{1}{m}}).$$

has codimension larger than p. By regularisation of $T_{\min,\frac{1}{m}}$, there exists currents T_m with analytic singularities in α larger that $-\frac{2}{m}\omega$. Any irreducible component of the singular set of these currents have codimension larger than p. For generic Y the restriction of these currents on Y is well defined for any m. Since the inclusion of analytic sets is a Zariski closed condition, for generic Y we can also assume that the singular set of T_m is not contained in Y for any m.

On the other hand, in the class $\alpha|_Y$, the current with minimal singularities that admits a lower bound $-\frac{2}{m}\omega|_Y$ is certainly less singular than $T_m|_Y$. The upper-level set of the Lelong number of these minimal currents is included in the singular set of $T_m|_Y$, so it has codimension larger than p. This means that $\alpha|_Y$ is nef in codimension $p-1$.

The condition $\langle \alpha^p \rangle \neq 0$ implies that

$$\int_X \langle \alpha^p \rangle \wedge \omega^{n-p} > 0.$$

In other words, there exist a sequence of currents with analytic singularities $T_m \in \alpha$ such that $T_m \geqslant -\frac{1}{m}\omega$ and

$$\int_{X \smallsetminus F_m} (T_m + \frac{1}{m}\omega)^p \wedge \omega^{n-p} > c$$

for some $c > 0$ independent of m where F_m is the singular set of T_m.

With a generic choice of Y, we can still assume that the restriction of T_m is a current with analytic singularities. They satisfy the conditions $T_m|_Y \geqslant -\frac{1}{m}\omega|_Y$ and

$$\int_{Y \smallsetminus F_m} (T_m|_Y + \frac{1}{m}\omega|_Y)^p \wedge \omega^{n-p-1} > \frac{c}{k}.$$

In other words, $\langle \alpha|_Y^p \rangle \neq 0$.

By induction on the dimension, we are reduced to proving the case where X has dimension p and L is nef in codimension $p-1$, in which case L is (plainly) nef by lemma 4.11. The condition of the movable intersection reduces to $\langle c_1(L)^p \rangle \neq 0$. By lemma 4.14, this implies that $(L^p) > 0$. In particular, L is a nef and big line bundle. Now the vanishing of cohomology classes follows from the classical Kawamata-Viehweg theorem. □

As pointed out to us by A. Höring, this can also be proven using the result of [**Kur13**].

Remark 4.18. When $p = n$, the above theorem is the classical Kawamata-Viehweg vanishing theorem for a nef and big line bundle. We notice that $\langle c_1(L)^n \rangle = \mathrm{Vol}(L)$ by theorem 3.5 of [**BDPP13**]. When $p = 1$, the theorem states that if L is a psef line bundle with $\langle c_1(L) \rangle \neq 0$, then $H^n(X, K_X \otimes L) = 0$. This case is trivial by the following easy lemma. The first interesting case is when L is nef in codimension 1 and $\langle c_1(L)^2 \rangle \neq 0$. In the following example, we show that we can not weaken the condition to the case that L is only psef and $\langle c_1(L)^2 \rangle \neq 0$. On the other hand, by the divisorial Zariski decomposition, we can write any psef line bundle numerically as a sum of a nef class in codimension 1 and of an effective class. This shows that in some sense, this kind of theorem is the best we can hope for.

Now we begin our example. Let V be the unique non-trivial rank 2 extension of $\mathcal{O}_C$ over an elliptic curve C. Let X be the blow-up of a point of $\mathbb{P}(V) \times \mathbb{P}^1$ and L be the pull back of $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$. $\mathcal{O}_{\mathbb{P}(V)}(1)$ is a nef line bundle. We also notice that $c_1(\mathcal{O}_{\mathbb{P}(V)}(1))^2 = 0$ and $c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) \neq 0$. So L is a nef line bundle over X and $\mathrm{nd}(L) = 2$. By the above theorem we have that $H^2(X, K_X + L) = 0$. Let E be the exceptional divisor of the blow-up. The short exact sequence

$$0 \to K_X + L \to K_X + L + E \to K_X + L + E|_E \to 0$$

induces the long exact sequence

$$H^2(X, K_X + L) \to H^2(X, K_X + L + E) \to H^2(E, K_X + L + E|_E) = H^0(E, -L) \to H^3(X, K_X + L).$$

By Serre duality and the following lemma, $H^3(X, K_X + L) = H^0(X, -L) = 0$. Since $L|_E = \mathcal{O}_E$, $H^0(E, -L) \cong \mathbb{C}$. Thus we have that

$$H^2(X, K_X + L + E) \cong \mathbb{C}.$$

Now $L + E$ is a psef line bundle over X and $\mathrm{nd}(L+E) \geqslant 2$ but $H^2(X, K_X + L + E) \neq 0$. The reason of the numerical dimension is as follows. By the super-additivity of movable intersection, we have that

$$\langle (L+E)^2 \rangle \geqslant \langle L^2 \rangle + \langle E^2 \rangle + 2\langle L \cdot E \rangle \geqslant \langle L^2 \rangle.$$

Lemma 4.19. *Let (L, h) be a non-trivial (i.e. $L \neq \mathcal{O}_X$) psef line bundle over a compact complex manifold X. Then we have*

$$H^0(X, L^{-1}) = 0.$$

PROOF. We argue by contradiction. Let s be a non-zero section in $H^0(X, L^{-1})$. Consider the function $\log|s|^2_{L^{-1},h^{-1}}$. Let φ be the local weight of h such that $h = e^{-\varphi}$ locally. Thus the above function can be locally written as $\log|s|^2 + \varphi$. In particular, it is a psh function on X. Since X is compact, the only psh functions are the constant functions. On the other hand,

$$i\partial\overline{\partial}\log|s|^2_{L^{-1},h^{-1}} = [s = 0] + i\Theta_{L,h} = 0$$

where $[s = 0]$ is the current associated to the (possible trivial) divisor $s = 0$ and $i\Theta_{L,h}$ is the curvature of (L, h). Since both $[s = 0]$ and $i\Theta_{L,h}$ are positive currents, they are 0. In particular, s never vanishes on X which contradicts the fact that L is a non-trivial line bundle. □

A classical result on nef line bundles is the following. Let $A + B$ be a nef line bundle over a compact manifold X where A, B are effective $\mathbb{R}$-divisors without intersection. Then A, B are both nef divisors. In the case of nefness in lower codimension, we have the following generalised version.

LEMMA 4.20. *Let $A+B$ be a line bundle that is nef in codimension k over a compact manifold X (by this, we mean that $c_1(A + B)$ is nef in codimension k), where A, B are effective $\mathbb{R}$-divisors without intersection. Then the divisors A, B are both nef in codimension k.*

More generally, let $\alpha + c_1(E)$ be a class that is nef in codimension k over a compact manifold X, where E is an effective $\mathbb{R}$-divisor and $E_{nn}(\alpha) \cap E = \varnothing$. Then α is nef in codimension k.

PROOF. Fix $\alpha_0 \in \alpha, \beta_0 \in c_1(E)$ two smooth representatives. By assumption, for any $\varepsilon > 0$, there exists a quasi-psh function φ_ε on X with analytic singularities such that

$$\alpha_0 + i\partial\overline{\partial}\varphi_\varepsilon \geqslant -\varepsilon\omega$$

where ω is some Hermitian metric on X (not necessarily Kähler). (For example, we can take a regularisation of the minimal potential $\varphi_{\min,-\frac{\varepsilon}{2}}$.) We can assume that the singular set of φ_ε has empty intersection with V_E. Here V_E is some small tubular neighbourhood of E.

Let ψ_ε be a family of quasi-psh functions on X with analytic singularities such that

$$\alpha_0 + \beta_0 + i\partial\overline{\partial}\psi_\varepsilon \geqslant -\varepsilon\omega.$$

We can assume that the singular set of ψ_ε has codimension at least $k + 1$.

Let φ_E be a quasi-psh function on X such that $\beta_0 + i\partial\overline{\partial}\varphi_E = [E]$ where $[E]$ is the current associated to E. But definition, the pole of φ_E is exactly the support of E. In particular we have that $\psi_\varepsilon - \varphi_E$ is a well-defined quasi-psh function outside E such that

$$\alpha_0 + i\partial\overline{\partial}(\psi_\varepsilon - \varphi_E) \geqslant -\varepsilon\omega$$

on $X \smallsetminus E$.

Now we glue the potentials to get a quasi-psh function Φ_ε with analytic singularities on X, such that

$$\alpha_0 + i\partial\overline{\partial}\Phi_\varepsilon \geqslant -\varepsilon\omega.$$

We also demand that the singular set of Φ_ε be included in the singular set of ψ_ε. This will finish the proof of the lemma.

On $X \smallsetminus V_E$ we define $\Phi_\varepsilon = \max(\psi_\varepsilon - \varphi_E, \varphi_\varepsilon + C_\varepsilon)$ where C_ε is a constant which will be determined latter. In particular, on $X \smallsetminus V_E$ we have

$$\alpha_0 + i\partial\overline{\partial}\Phi_\varepsilon \geqslant -\varepsilon\omega.$$

On V_E, we define $\Phi_\varepsilon = \varphi_\varepsilon + C_\varepsilon$. On $X \smallsetminus V_E$, φ_E is bounded and ψ_ε is bounded from above. Near the boundary of V_E, φ_ε is also bounded since the singular set of φ_ε has empty intersection with V_E. Thus for C_ε large enough near the boundary of V_E $\psi_\varepsilon - \varphi_E < \varphi_\varepsilon + C_\varepsilon$. In particular, Φ_ε is a global well defined quasi-psh function such that $\alpha_0 + i\partial\overline{\partial}\Phi_\varepsilon \geqslant -\varepsilon\omega$. The singular set of Φ_ε in $X \smallsetminus V_E$ is included in the singular set of ψ_ε. On V_E, Φ_ε is smooth. This finishes our construction. □

REMARK 4.21. The condition that the intersection is empty is necessary for the lemma. Otherwise, we have the following counter-example.

The construction uses Cutkosky's construction detailed in the next section. Let Y be a projective manifold such that $\mathcal{N}_Y = \mathcal{E}_Y$. Let $\beta \in H^{1,1}(Y, \mathbb{R})$ be a non psef class. Let A_1, A_2 be very ample divisors on Y. Define

$$t_0 := \min\{t | \beta + tc_1(A_1) \text{ nef}\}.$$

We can assume that $\beta + t_0c_1(A_2)$ is nef. Define $X := \mathbb{P}(A_1 \oplus A_2)$ and denote by $\pi : \mathbb{P}(A_1 \oplus A_2) \to Y$ the natural projection. By proposition 4.3.1 below, $\pi^*\beta + t_0c_1(\mathcal{O}(1))$ and $c_1(\mathcal{O}(1))$ are nef. Notice that $\mathcal{O}(1)$ is an effective divisor since $H^0(X, \mathcal{O}(1)) = H^0(Y, A_1 \oplus A_2) \neq 0$.

By proposition 4.3.2, for any $t < t_0$, $\nu(\pi^*\beta + tc_1(\mathcal{O}(1)), \mathbb{P}(A_2)) > 0$ and $E_{nn}(\pi^*\beta + tc_1(\mathcal{O}(1)) = \mathbb{P}(A_2)$. This shows that for any $t < t_0$, $\pi^*\beta + tc_1(\mathcal{O}(1))$ is not nef in codimension 1. In other words, the nef class $\pi^*\beta + t_0 c_1(\mathcal{O}(1))$ is a sum of not nef in codimension 1 class $\pi^*\beta + tc_1(\mathcal{O}(1))$ and an effective divisor $(t_0 - t)\mathcal{O}(1)$. Let $(s_1, s_2) \in H^0(Y, A_1) \oplus H^0(Y, A_2) = H^0(X, \mathcal{O}(1))$ be a non-trivial section. Then we have

$$V(s_1, s_2) = \{(x, \xi^*) | \xi^* \in (A_1 \oplus A_2)^*, \ \xi^*(s_1, s_2) = 0\}.$$

Identify $\mathbb{P}(A_2)$ as Y,then we have $V(s_1, s_2) \cap \mathbb{P}(A_2) = V(s_2) \neq \varnothing$. Similar calculation shows that for any $E \in |\mathcal{O}(m)|$ for any $m \in N^*$

$$E_{nn}(\pi^*\beta + tc_1(\mathcal{O}(1)) \cap E \neq \varnothing.$$

Following the ideas of [**DP03**], we get the following Kähler version of the Kawamata-Viehweg vanishing theorem.

THEOREM 4.22. *Let (X, ω) be a compact Kähler manifold of dimension n and L a nef in codimension 1 line bundle on X. Assume that $\langle L^2 \rangle \neq 0$. Assume that there exists an effective $\mathbb{N}$-divisor D such that $c_1(L) = c_1(D)$. Then*

$$H^q(X, K_X + L) = 0$$

for $q \geqslant n - 1$.

PROOF. In the case $q = n$, we have $H^n(X, K_X + L) = H^0(X, -L)^*$ by Serre duality. For L psef, $-L$ has no section unless L is trivial by lemma 4.17. Since $\langle L^2 \rangle \neq 0$, L is not trivial. Therefore the only interesting case is $q = n - 1$. We divide the proof into two cases.

Case 1: We assume that $L = D$. Since the canonical section of D induces a positive singular metric on (L, h) with multiplier ideal sheaf $\mathcal{I}(h) \subset \mathcal{O}(-D)$. In fact we have equality outside an analytic set whose all irreducible components have codimension larger than 2. Write $D = \sum_i n_i D_i$ where $n_i \geqslant 0$ and D_i are the irreducible components of D. Define

$$Y = (D_{\text{red}})_{\text{Sing}} = \bigcup_{i \neq j} (D_i \cap D_j) \cup \bigcup_i D_{i,\text{Sing}}$$

where $D_{i,\text{Sing}}$ means the singular part of D_i. It is easy to see that we have an equality outside Y and that each irreducible components of Y is of codimension larger than 2.

In particular, the short exact sequence

$$0 \to \mathcal{I}(h) \to \mathcal{O}(-D) \to \mathcal{O}(-D)/\mathcal{I}(h) \to 0$$

induces that

$$H^{n-1}(X, K_X + L \otimes \mathcal{I}(h)) \to H^{n-1}(X, K_X + L - D) \to H^{n-1}(X, \mathcal{O}(-D)/\mathcal{I}(h)) = 0$$

since the support of $\mathcal{O}(-D)/\mathcal{I}(h)$ is included in Y.

Denote by $h_{\min}$ the minimal metric on L where we have a natural inclusion of $\mathcal{I}(h) \subset \mathcal{I}(h_{\min})$. Thus we have the following commuting diagram

$$\begin{array}{ccc} H^{n-1}(X, K_X + L \otimes \mathcal{I}(h)) & \longrightarrow & H^{n-1}(X, K_X + L \otimes \mathcal{I}(h_{\min})) \\ \downarrow & & \downarrow \\ H^{n-1}(X, K_X + L - D) & \longrightarrow & H^{n-1}(X, K_X + L). \end{array}$$

By Theorem 1.9 proved in chapter 3 and the condition that $\mathrm{nd}(L) \geqslant 2$, we know that the morphism

$$H^{n-1}(X, K_X + L \otimes \mathcal{I}(h_{\min})) \to H^{n-1}(X, K_X + L)$$

is the 0 map. Since the left vertical arrow is surjective in the above diagram, we conclude that the morphism

$$H^{n-1}(X, K_X + L - D) \to H^{n-1}(X, K_X + L)$$

is also the 0 map. Thus the short exact sequence

$$0 \to K_X + L - D \to K_X + L \to K_X + L|_D = K_D \to 0$$

gives in cohomology

$$H^{n-1}(X, K_X + L - D) \to H^{n-1}(X, K_X + L) \to H^{n-1}(D, K_D) \simeq H^0(D, \mathcal{O}_D) \to H^n(X, K_X + L - D) \to 0.$$

On the other hand, $H^n(X, K_X + L - D) \simeq H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$. Therefore we need only show that

$$h^0(D, \mathcal{O}_D) = 1.$$

More precisely, D is a effective Cartier divisor in the manifold X. Therefore D is a (possibly non reduced) Gorenstein variety. In this case the dualizing sheaf K_D is given by adjunction as $(K_X + D)|_D$. Moreover Serre duality holds in the same form as in the smooth case.

To calculate the dimension of global sections of D, first we show that D is connected. In fact, otherwise we would have $D = A + B$ with A and B effective non-trivial divisors such that $A \cap B = \varnothing$. In particular we have $(A \cdot B \cdot \omega^{n-2}) = 0$. But A and B are necessarily nef in codimension 1 by lemma 4.20.

We recall the Hodge Index Theorem on a compact Kähler manifold (X, ω) as theorem 6.33 and 6.34 in [**Voi02a**]. By the Hard Lefschetz theorem, we have

$$H^2(X, \mathbb{C}) = \mathbb{C}\{\omega\} \oplus H^2(X, \mathbb{C})_{\text{prim}}$$

where $H^2(X, \mathbb{C})_{\text{prim}}$ means primitive classes. The intersection form $(\alpha, \beta) \mapsto (\alpha \cdot \beta \cdot \omega^{n-2})$ has signature $(1, h^{1,1}(X) - 1)$ on $H^{1,1}(X)$ since $H^2(X, \mathbb{C})_{\text{prim}}$ is orthogonal to ω and the intersection form is negative definite on $H^2(X, \mathbb{C})_{\text{prim}}$.

On the other hand, by lemma 4.14, we have that

$$(A \cdot A \cdot \omega^{n-2}) \geqslant \langle A \cdot A \cdot \omega^{n-2} \rangle \geqslant 0$$

and similar inequality for B. We also notice that

$$(L \cdot L \cdot \omega^{n-2}) \geqslant \langle L \cdot L \cdot \omega^{n-2} \rangle > 0.$$

Since the intersection form (unlike the movable intersection) is bilinear, we have either $(A \cdot A \cdot \omega^{n-2}) > 0$ or $(B \cdot B \cdot \omega^{n-2}) > 0$. Without loss of generality, assume that $(A \cdot A \cdot \omega^{n-2}) > 0$. Thus $B \in A^{\perp}$ and $(B \cdot B \cdot \omega^{n-2}) \geqslant 0$. The Hodge index theorem implies that $B = 0$ which is a contradiction to our assumption. Hence D is connected, and if $h^0(D, \mathcal{O}_D) \geqslant 2$, then $\mathcal{O}_D$ contains a nilpotent section $t \neq 0$. In other words, the pull back of t under the natural morphism $D_{\text{red}} \to D$ is 0 but lies as a non trivial section in $H^0(D_{\text{red}}, \mathcal{O}(-\sum_{j \in I} \mu_j D_j))$ for some $1 \leqslant \mu_j \leqslant n_j$ for all j. Let

$$J := \{j \in J | \frac{n_j}{\mu_j} \text{ maximal}\}$$

and let $c = \frac{n_j}{\mu_j}$ be the maximal value. Notice that $\operatorname{div}(t)|_{D_i} = -\sum_{j \in I} \mu_j D_j|_{D_i}$ is effective (possibly 0) for all i. We claim that it is impossible that $c = \frac{n_j}{\mu_j}$ for all $j \in I$. Otherwise, $L|_{D_i} = c \sum \mu_j D_j|_{D_i}$ is psef. (L is nef in codimension 1, so its restriction to any prime divisor is psef.) Its dual is effective, hence $L|_{D_i} \equiv 0$ for all i. This implies that $(L \cdot L \cdot \omega^{n-2}) = 0$, contradiction.

Thus we find some j such that

$$c > \frac{n_j}{\mu_j}.$$

By connectedness of D we can choose $i_0 \in J$ in such a way that there exists $j_1 \in I \smallsetminus J$ with $D_{i_0} \cap D_{j_1} \neq \varnothing$. Now

$$\sum_{j \in I} (n_j - c\mu_j) D_j|_{D_{i_0}}$$

is pseudo-effective as a sum of a psef and an effective line bundle (this has nothing to do with the choice of i_0). Since the sum, taken over I, is the same as the sum taken over $I \smallsetminus \{i_0\}$, we conclude that

$$\sum_{j \neq i_0} (n_j - c\mu_j) D_j|_{D_{i_0}}$$

is pseudo-effective, too. Now all $n_j - c\mu_j \leqslant 0$ and $n_{j_1} - c\mu_{j_1} < 0$ with $D_{j_1} \cap D_{i_0} \neq \varnothing$, hence the dual of

$$\sum_{j \neq i_0} (n_j - c\mu_j) D_j | D_{i_0}$$

is effective and non-zero, a contradiction. This finishes the proof of case 1.

Case 2: general case. We can write

$$L = D + L_0$$

where $L_0^m \in \operatorname{Pic}^0(X)$ (The exponent m is there because there might be torsion in $H^2(X, \mathbb{Z})$; we take m to kill the denominator of the torsion part). We may in fact assume that $m = 1$; otherwise we pass to a finite étale cover $\tilde{X}$ of X and argue there (the vanishing on $\tilde{X}$ clearly implies the vanishing on X by Leray spectral sequence). In other words, we write L as a sum of D and a flat line bundle (L_0, h_0). Here h_0 is the flat metric. Thus there exists a bijection between singular positive metrics on L and those on D, via the tensor product by h_0. In particular, the minimal metric on L is the minimal metric on D, tensored by h_0.

The short exact sequence used above is modified into

$$0 \to K_X + L - D \to K_X + L \to (K_X + L)|_D = (K_D + L_0)|_D \to 0.$$

Taking cohomology as before and using a similar discussion, the arguments come down to proving

$$H^0(D, -L_0|_D) = 0$$

since $H^n(X, K_X + L - D) \simeq H^0(X, -L_0) = 0$.

The argument on the connectedness of D still works since the arguments only involve the first Chern class, and since L_0 has no contribution in the first Chern class. If $-L_0|_D \neq 0$, then we see as above that $-L_0|_D$ cannot have a nilpotent section. Since L_0 is flat, adding a multiple of L_0 does not change the pseudo-effectiveness. By adding a suitable such multiple, the arguments on the non-existence of nilpotent section are still valid.

So if $H^0(D, -L_0|_D) = 0$ fails, then $-L_0|_D$ has a section s such that $s|_{D_{\mathrm{red}}}$ has no zeroes. In other words $-L_0|_{D_{\mathrm{red}}}$ is trivial. But then $-L_0|_D$ is trivial, since the nowhere vanishing section of $H^0(X, -L_0 \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}_{D_{\mathrm{red}}})$ is mapped to a nowhere vanishing section in $H^0(X, -L_0 \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}_D)$ by passing to the quotient.

Now let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map with image Y. Then $L_0 = \alpha^*(L_0')$ with some line bundle L_0' on $\mathrm{Alb}(X)$. (We observe that $\mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(\mathrm{Alb}(X))$.) Notice that L_0' is a non trivial line bundle with $c_1(L_0') = 0$. Since $L_0|_D$ is trivial and L_0 is non trivial, we conclude that $\alpha(D) \neq Y$. We claim that $\alpha(D)$ is contained in some proper subtorus B of $\mathrm{Alb}(X)$.

The reason is as follows. Let $\nu : \tilde{X} \to X$ be a modification such that $\nu^*(D)$ is a SNC divisor. Denote by E_j the irreducible components of $\nu^*(D)$. Define $S \subset \prod_i \mathrm{Pic}^0(E_i)$ the connected component containing $(\nu^* L_0|_{E_i})$ of

$$\{(L_i) \in \prod_i \mathrm{Pic}^0(E_i) | L_i|_{E_i \cap E_j} = L_j|_{E_i \cap E_j}\}.$$

By proposition 1.5 of [**BL92**], S is a subtorus since S is a translation of the kernel of

$$\prod_i \mathrm{Pic}^0(E_i) \to \prod_{i,j,i\neq j} \mathrm{Pic}^0(E_i \cap E_j)$$

$$(L_i) \mapsto (L_i|_{E_i \cap E_j} - L_j|_{E_i \cap E_j}).$$

Notice that $\mathrm{Pic}^0(E_i)$ is a torus by Hodge theory since E_i is smooth. The natural group morphism of $\mathrm{Pic}^0(X) \to S$ given by $L \mapsto (\nu^* L|_{E_i})$ induces by duality the following commuting diagram

$$\begin{array}{ccc} \prod_i \mathrm{Pic}^0(E_i)^* & \longrightarrow & S^* \\ & \searrow & \downarrow \\ & & (\mathrm{Pic}^0(X))^* \cong \mathrm{Alb}(X). \end{array}$$

Since $L_0 \in S$ is non trivial, the image of S^* as a complex torus is a proper subtorus in $\mathrm{Alb}(X)$. We denote its image as B. (Let us observe that by proposition 1.5 of [**BL92**], the image of a homomorphism of complex tori is a subtorus.)

Consider the induced map

$$\beta : X \to \mathrm{Alb}(X)/B$$

and denote its image by Z. (Z can be singular!) The image $\beta(D)$ is a point p by construction. Let U be a Stein neighbourhood of p in Z(or some coordinate chart of p). Denote by m_p the maximal ideal of p in Z. In particular, for any $k \in \mathbb{N}^*$, m_p^k is globally generated on U (by Cartan theorem A).

Let $D = \sum_i n_i D_i$ and define $n_{\max} := \max(n_i)$. Then we have the inclusion $\beta^* H^0(U, m_p^{n_{\max}}) \subset H^0(D, \mathcal{O}(-n_{\max} D_{\mathrm{red}})|_D) \subset H^0(D, \mathcal{O}(-D)|_D)$ where the second inclusion uses the fact that $n_{\max} D_{\mathrm{red}} - D$ is an effective divisor in X. In particular, for any i, $H^0(D_i, \mathcal{O}(-D)|_{D_i}) \neq 0$. On the other hand, $\mathcal{O}(D)|_{D_i}$ is psef since D is nef in codimension 1. (Observe that nefness is a numerical property. Since $c_1(L_0) = 0$, D is nef in codimension 1 as L is.) By lemma 4.17, $D|_{D_i}$ is trivial.

Thus we have for any i

$$(D \cdot D_i \cdot \omega^{n-2}) = \int_{D_i} c_1(D|_{D_i}) \wedge \omega^{n-2} = 0.$$

This implies that $(L^2 \cdot \omega^{n-2}) = (D^2 \cdot \omega^{n-2}) = 0$. On the other hand, since L is nef in codimension 1, $(L^2 \cdot \omega^{n-2}) \geqslant \langle L^2 \cdot \omega^{n-2} \rangle$. But this is a contradiction with our assumption. □

REMARK 4.23. If D is a smooth reduced divisor, we can also argue as follows at the end of case 2. We observe that L_0 is a non-trivial element in a translate of the kernel of $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(D)$. On the other hand, we have

$$H^{n-1}(X, K_X + D) = H^1(X, -D) = 0 \to H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$$

since by case 1, $H^{n-1}(X, K_X + D) = 0$. However, $H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$ is the tangent map of $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(D)$. By proposition 1.5 of [**BL92**], the kernel is discrete. Moreover, the connected component containing the zero point of the kernel is of finite index in the kernel. In particular, L_0 is a torsion element. This yields a contradiction.

4.3. Examples and counter-examples

In this section, we first give for each $k \in \mathbb{N}^*$ an example of a psef class α_k on some manifold X_k, such that α_k is nef in codimension k but not nef in codimension $k+1$. This shows in particular that the inclusion of the various types of nef cones can be strict.

For the convenience of the reader, we recall Cutkosky's construction described in [**Bou04**], as well as all needed material for our use.

Let $\mathcal{E}$ be a vector bundle of rank r over a manifold Y and L be a line bundle over Y. Since there exists a surjective bundle morphism given by projection $\mathcal{E} \oplus L \to \mathcal{E}$, we can view $D := \mathbb{P}(\mathcal{E})$ as a closed submanifold of $\mathbb{P}(\mathcal{E} \oplus L)$. Note that the restriction of $\mathcal{O}_{\mathbb{P}(\mathcal{E}\oplus L)}(1)$ on $\mathbb{P}(\mathcal{E})$ is the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. We notice that the canonical line bundle of the projectivization of a vector bundle $\mathbb{P}(\mathcal{E})$ is given by

$$K_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-(r+1)) + \pi^*(K_Y + \det\mathcal{E})$$

where $\pi : \mathbb{P}(\mathcal{E}) \to Y$ is the projection. From the short exact sequence

$$0 \to T_{\mathbb{P}(\mathcal{E})} \to T_{\mathbb{P}(\mathcal{E}\oplus L)}|_{\mathbb{P}(\mathcal{E})} \to N_{\mathbb{P}(\mathcal{E})/\mathbb{P}(\mathcal{E}\oplus L)} = \mathcal{O}(D)|_{\mathbb{P}(\mathcal{E})} \to 0$$

we have

$$K_{\mathbb{P}(\mathcal{E}\oplus L)}|_{\mathbb{P}(\mathcal{E})} = K_{\mathbb{P}(\mathcal{E})} \otimes \mathcal{O}(-D)|_{\mathbb{P}(\mathcal{E})}.$$

Using the formula for the canonical line bundle, we have

$$\mathcal{O}(1)|_{\mathbb{P}(\mathcal{E})} = (\mathcal{O}(D) \otimes \pi^* L)|_{\mathbb{P}(\mathcal{E})}.$$

We observe that by the Leray-Hirsh theorem for Bott-Chern cohomology,

$$H^{1,1}_{BC}(\mathbb{P}(\mathcal{E}\oplus L), \mathbb{R}) = \mathbb{R}c_1(\mathcal{O}(1)) \oplus \pi^* H^{1,1}_{BC}(Y, \mathbb{R}).$$

In particular, this implies that the inclusion $i : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}\oplus L)$ induces an isomorphism on $H^{1,1}_{BC}$. Hence we find that on $\mathbb{P}(\mathcal{E}\oplus L)$

$$c_1(\mathcal{O}(1)) = c_1(\mathcal{O}(D)) + \pi^* c_1(L).$$

Now let Y be a compact complex manifold of dimension m and $L_0, \cdots, L_r$ the line bundles over Y. We define

$$X := \mathbb{P}(L_0 \oplus \cdots \oplus L_r).$$

We denote $H := \mathcal{O}(1)$ the tautological line bundle over the projectivization and $h := c_1(H)$. For any i, the projection $L_0 \oplus \cdots L_r \to L_0 \oplus \cdots \hat{L_i} \oplus \cdots \oplus L_r$ induces inclusions of hypersurfaces

$$D_i := \mathbb{P}(L_0 \oplus \cdots \oplus \hat{L_i} \oplus \cdots \oplus L_r).$$

By the above discussion

$$d_i + l_i = h$$

where $d_i := c_i(\mathcal{O}(D_i))$ and $l_i := c_1(L_i)$. In fact, by calculating the transition function, we can show that $\mathcal{O}(1)$ is linear equivalent to $L_i + D_i$. But the relation of Chern classes is enough for our use here.

We have the following explicit description of nef cone and psef cone in this case. We denote by $\mathcal{C}$ the cone generated by the l_i.

PROPOSITION 4.3.1. Let $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ be a class that is decomposed as $\alpha = \pi^*\beta + \lambda h$. Then
(1) α is nef iff $\lambda \geqslant 0$ and $\beta + \lambda\mathcal{C}$ is contained in $\mathcal{N}_Y$.
(2) α is psef iff $\lambda \geqslant 0$ and $(\beta + \lambda\mathcal{C}) \cap \mathcal{E}_Y \neq \emptyset$.

PROOF. We notice that if α contains a positive current $T = \theta + i\partial\bar{\partial}\varphi$ with θ smooth, then the pluripolar set $P(\varphi) = \{\varphi = -\infty\}$ is of Lebesgue measure 0. Hence, by the Fubini theorem, the set

$$\{y \in Y, \pi^{-1}(y) \subset P(\varphi)\}$$

is of Lebesgue measure 0. For y outside the measure 0 set, $\alpha|_{\pi^{-1}(z)}$ is the class of $T|_{\pi^{-1}(z)}$. It is also equal to the class of $\lambda c_1(\mathcal{O}_{\mathbb{P}^r}(1))$, and this implies that $\lambda \geqslant 0$. We always assume in the following that $\lambda \geqslant 0$.

(1) If α is nef, the restriction of α to $\mathbb{P}(L_i)$ for any i is also nef where $\mathbb{P}(L_i)$ is biholomorphic to Y given by π. Note that $\alpha|_{\mathbb{P}(L_i)} = \lambda l_i + \beta$ is nef as a restriction of nef class. So $\beta + \lambda\mathcal{C}$ is contained in $\mathcal{N}_Y$.

On the other hand, $\alpha = \pi^*\beta + h = \pi^*(\beta + \lambda l_i) + \lambda d_i$ for any i where $\beta + \lambda l_i$ is nef by assumption. Hence the non-nef locus of α is contained in D_i. Since the intersection of all D_i is empty, we conclude that α is nef.

(2) Let $t_i \in [0,1]$ such that $\sum t_i = 1$ and $\beta + \sum_{i=0}^r t_i l_i \in \mathcal{E}_Y$. Hence $h = \sum t_i h = \sum t_i \pi^* l_i + \sum t_i d_i$ and $\alpha = \pi^*(\beta + \lambda \sum t_i l_i) + \lambda \sum t_i d_i$. d_i is psef since it contains the positive current associated to D_i. As a sum of psef classes, α is psef.

For the other direction, we argue by induction. When $r = 0$, $X = Y$ and $\alpha = \beta + \lambda l_0$. By the assumption that α is psef, we have

$$\alpha \in (\beta + \lambda\mathcal{C}) \cap \mathcal{E}_Y.$$

Continue the induction on r. Let T be a closed positive current in α. we have that $\alpha - \nu(T, D_0)d_0$ is psef containing the current $T - \nu(T, D_0)[D_0]$. And $(\alpha - \nu(T, D_0)d_0)|_{D_0}$ is psef since the restriction of the current $T - \nu(T, D_0)[D_0]$ on D_0 is well defined. Now D_0 is the projectivisation of a vector bundle of rank r over Y. As a cohomology class

$$\alpha - \nu(T, D_0)d_0 = \pi^*(\beta + \lambda l_0) + (\lambda - \nu(T, D_0))d_0$$

Restrict α on some fibre of π as above. We have that $\lambda \geqslant \nu(T, D_0)$. By induction, we see that the psef class $(\alpha - \nu(T, D_0)d_0)|_{D_0}$, which is also equal to $\pi^*(\beta + \nu(T, D_0)l_0) + (\lambda - \nu(T, D_0))h$, satisfies

$$(\beta + \nu(T, D_0)l_0 + (\lambda - \nu(T, D_0))\mathcal{C}_0) \cap \mathcal{E}_Y \neq \varnothing$$

where $\mathcal{C}_0$ is the cone generated by $l_1, \cdots, l_r$. In other words,

$$(\beta + \lambda\mathcal{C}) \cap \mathcal{E}_Y \neq \varnothing.$$

□

We will also need the following explicit calculation of the generic minimal multiplicity in this example. From now on, we choose Y such that the nef cone $\mathcal{N}_Y$ and the psef cone $\mathcal{E}_Y$ coincide (for example we can take Y to be a Riemann surface).

We denote I a subset of $\{0, \cdots, r\}$ with complement J. We denote $V_I := \bigcap_{i\in I} D_i = \mathbb{P}(\oplus_{j\in J} L_j)$ and $\mathcal{C}_I$ the convex envelope of $l_i (i \in I)$.

We observe that the non-nef locus of any psef class is contained in the union of D_i. The reason is as follows: since $\alpha = \pi^*\beta + \lambda h$ is psef, by proposition 4.3.1 we know that there exist $t_i \in [0, 1]$ with $\sum t_i = 1$ such that $\beta + \lambda(\sum t_i l_i) \in \mathcal{E}_Y = \mathcal{N}_Y$. Hence

$$\alpha = \pi^*(\beta + \lambda(\sum t_i l_i)) + \lambda(\sum t_i d_i)$$

is a sum of nef divisor and effective divisor. (Since α is psef, $\lambda \geqslant 0$.) So the non-nef locus of α is contained in the union of D_i.

PROPOSITION 4.3.2. *Let α be a big class such that $\alpha = \pi^*\beta + \lambda h$. The generic minimal multiplicity of α along V_I is equal to*

$$\nu(\alpha, V_I) = \min\{t \geqslant 0, (\beta + tC_I + (\lambda - t)C_J) \cap N_Y \neq \varnothing\}.$$

More precisely, we have $\nu(\alpha, V_I) = \nu(\alpha, x)$ for any $x \in V_I \setminus \bigcup_{j\in J} D_j$.

PROOF. Let $\mu : X_I \to X$ the blow-up of X along V_I with exceptional divisor E_I. Hence we have $E_I = \mathbb{P}(N^*_{V_I/X})$ with $N^*_{V_I/X} = \oplus_{i\in I}\mathcal{O}_{V_i}(-D_i)$. By lemma 4.4, we get

$$\nu(\alpha, V_I) = \nu(\mu^*\alpha, E_I).$$

Denote by H_I the tautological line bundle over E_I where we have $\mathcal{O}_{E_I}(-E_I) = H_I$.

For $t \geqslant 0$, the restriction of $\mu^*\alpha - tc_1(\mathcal{O}(E_I))$ to E_I is psef is hence equivalent to that $\mu^*\alpha + tc_1(H_I)$ is psef. By proposition 4.3.1, the latter is equivalent to the fact that $\alpha + t\mathcal{C}(\pi^*l_i - h) = \alpha - th + t\pi^*\mathcal{C}(l_i)$ intersects $\mathcal{E}_{V_I}$ where $\mathcal{C}(l_i)$ is the convex envelop of l_i $(i \in I)$. Note also that

$$\alpha - th + t\pi^*\mathcal{C}(l_i) = \pi^*(\beta + t\mathcal{C}(l_i)) + (\lambda - t)h$$

where we denote by the same notation π to be the projection from V_I to Y and h to be the first Chern class of the tautological line bundle over V_I. By proposition 4.3.1, it is psef if and only if $\beta + t\mathcal{C}_I + (\lambda - t)C_J$ intersects the psef cone $\mathcal{E}_Y$.

Since the class $\mu^*\alpha - \nu(\alpha, V_I)c_1(\mathcal{O}(E_I))$ has positive current $\mu^*T_{\min} - \nu(T_{\min}, V_I)[E_I]$ whose restriction to E_I is well defined by Siu's decomposition theorem. By the last paragraph we have

$$\nu(\alpha, V_I) = \nu(T_{\min}, V_I) \geqslant \min\{t \geqslant 0, (\beta + tC_I + (\lambda - t)C_J) \cap N_Y \neq \varnothing\}.$$

On the other direction, let $\gamma := \beta + t\sum_{i\in I} a_i l_i + (\lambda - t)\sum_{j\in J} b_j l_j$ be a psef (equivalently nef by assumption) class on Y with $\sum a_i = \sum b_j = 1$. Hence $\alpha = \pi^*\gamma + t\sum a_i d_i + (\lambda - t)\sum b_j d_j$. For $x \in V_I \setminus \bigcup_{j\in J} D_j$,

$$\nu(\alpha, x) \leqslant t\sum a_i\nu([D_i], x) + (\lambda - t)\sum b_j\nu([D_j], x) \leqslant t\sum a_i = t.$$

In particular, this shows that

$$\nu(\alpha, V_I) \leqslant \min\{t \geqslant 0, (\beta + tC_I + (\lambda - t)C_J) \cap N_Y \neq \varnothing\}.$$

By the proof, the equality is attained for $x \in V_I \setminus \bigcup_{j\in J} D_j$. □

We notice that if we use the algebraic analogue in the projective case as in [**Nak04**], we can weaken the assumption to the case that α is just a psef class.

In particular, the proposition 4.3.2 shows that $\cup D_i$ is stratified by the sets $V_I \smallsetminus \bigcup_{j\in J} D_j$ with respect to the generic minimal multiplicity.

Now we are prepared to give our construction. Let Y as above be a projective manifold such that the nef cone coincides with the psef cone. Define $X_k = \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(A_1) \oplus \cdots \oplus \mathcal{O}_Y(A_{k+1}))$ where A_i are the ample line bundles over Y. Let $\beta \in H^{1,1}_{BC}(Y,\mathbb{R})$ be a not-nef class. Denote H be the tautological line bundle over X_k and denote h its first Chern class. Define $\alpha = \pi^*\beta + h$. We assume that:

For any i, $\beta + c_1(A_i)$ is nef and big.

As above, $\mathbb{P}(\mathcal{O}_Y) \simeq Y$ is a closed submanifold of X_k of codimension $k+1$ via the projection of $\mathcal{O}_Y \oplus \mathcal{O}_Y(A_1) \oplus \cdots \oplus \mathcal{O}_Y(A_{k+1}) \to \mathcal{O}_Y$. α is psef but not nef on X_k by proposition 4.3.1. In fact, if α is nef, its restriction to the submanifold Y (i.e. β) will be nef. For any subset $I \neq \{1, \cdots r\}$ (taking $L_0 := \mathcal{O}_Y$), by proposition 4.3.2, $\nu(\alpha, V_I) = 0$ since $\beta + \sum_{j\in J} c_1(A_j)$ is nef which means we can take $t = 0$ on the right hand of the equation. By proposition 4.3.2, $\nu(\alpha, x)$ is constant on $\mathbb{P}(\mathcal{O}_Y)$. The non-nef locus can not be empty otherwise α would be nef. But non-nef locus have to be contained in $\mathbb{P}(\mathcal{O}_Y)$. Hence the constant cannot be zero.

In conclusion, we have $\nu(\alpha, \mathbb{P}(\mathcal{O}_Y)) > 0$, which in particular shows that α is not nef in codimension $k+1$. On the other hand, the non-nef locus is also $\mathbb{P}(\mathcal{O}(Y))$ which in particular shows that α is nef in codimension k.

With the explicit calculation of generic minimal multiplicity, we discuss the optimality of the divisorial Zariski decomposition. Take $k = 1$ in the above construction. Take β to be the first Chern class of some line bundle. Hence by the above calculation α is nef in codimension 1 but not nef in codimension 2. Its non-nef locus is $\mathbb{P}(\mathcal{O}_Y)$. For α, there doesn't exist a unique decomposition of this psef class $\alpha = c_1(L)$ into a nef in codimension 2 $\mathbb{R}$-divisor P and an effective $\mathbb{R}$-divisor N such that the canonical inclusion $H^0(\lfloor kP \rfloor) \to H^0(kL)$ is an isomorphism for each $k > 0$. Here the round-down of an $\mathbb{R}$-divisor is defined coefficient-wise. On the contrary, this decomposition will also be the divisorial Zariski decomposition. But α is nef in codmension 1, the uniqueness of the divisorial Zariski decomposition shows that the nef in codimension 2 part have to be α itself. This is a contradiction. In particular, when Y is a Riemann surface, it gives an example in dimension 3 where the classical Zariski decomposition does not exist (although it is always possible in dimension 2).

Given a psef class α on some compact manifold X, in general there does not always exist a composition of finite blow-up(s) of smooth centres $\mu : \tilde{X} \to X$ such that the nef in codimension 1 part of $\mu^*\alpha$ is in fact nef. This example is first shown in [**Nak04**].

Let α be a big class on a compact Kähler manifold X. Assume that there exists no finite composition of blow-up(s) with smooth centres. such that the the nef in codimension 1 part of $\mu^*\alpha$ is in fact nef. For example, we can take the pull back of the classed constructed by Nakayama on X by $p : X \times T \to X$ where T a complex torus. We have following lemma to conclude that in fact there exists no modification such that the the nef in codimension 1 part of $\mu^*\alpha$ is in fact nef. In general, a modification is not necessarily a composition of blow-up(s) with smooth centres. However, by Hironaka's results, any modification is dominated by a finite composition of blow-up(s) with smooth centres. In other words, for $\nu : \tilde{X} \to X$ a modification, there exists a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & \tilde{X} \\ & \searrow^{g} & \downarrow{\scriptstyle \nu} \\ & & X \end{array}$$

where g is a finite composition of blow-up(s) with smooth centres and f is holomorphic. To prove that there exists no modification such that the nef in codimension 1 part of the pull back of some cohomology class is nef by the above argument, we have to prove that if $Z(\nu^*\alpha)$ is nef, $Z(g^*\alpha)$ is also nef. This is done by the following proposition. It shows in particular that in the above example, if $Z(\nu^*\alpha)$ is nef, $Z(g^*\alpha) = f^*Z(\nu^*\alpha)$ is also nef.

Notice that the initial argument of Nakayama already proves the non-existence of Zariski decomposition for any modification.

Proposition 4.3.3. (1) Let $f : Y \to X$ be a holomorphic map between two compact complex manifolds and α be a psef class on X. Assume that $Z(\alpha)$ is nef. Then $f^*N(\alpha) \geqslant N(f^*\alpha)$ where the inequality relation $\geqslant$ means the difference is a psef class.

(2) Let $f : Y \to X$ be a modification between two compact complex manifolds and α a big class on X. Then $N(f^*\alpha) \geqslant f^*N(\alpha)$.

Proof. (1) By the convexity of minimal multiplicity along the subvarieties,

$$N(f^*\alpha) \leqslant N(f^*N(\alpha)) + N(f^*Z(\alpha)).$$

Since $Z(\alpha)$ is nef, $f^*Z(\alpha)$ is also nef, and thus $N(f^*Z(\alpha)) = 0$. The conclusion follows observing that $N(f^*N(\alpha)) \leqslant f^*N(\alpha)$.

(2) We claim that for any positive current $T \in f^*\alpha$, there exists a positive current $S \in \alpha$ such that $T = f^*S$. It is proven in Propositon 1.2.7 [**Bou04**] in more general setting. For the convenience of the reader, we give a proof in this special case.

Fix a smooth representative α_∞ in α. There exists a quasi-psh function φ such that $T = f^*\alpha_\infty + i\partial\overline{\partial}\varphi$. Let U be a open set of X such that $\alpha_\infty = i\partial\overline{\partial}v$ on U. The function $v \circ f + \varphi$ is psh on $f^{-1}(U)$. All the fibres are compact and connected (the limit of general connected fibre, the points, is still connected), thus $v \circ f + \varphi$ is constant along the fibres. Thus there exists a function ψ on U such that $\varphi = \psi \circ f$. Since φ is L^1_{loc} and f is biholomorphic on a dense Zariski open set, ψ is also L^1_{loc}. It is easy to see that ψ is independent of the choice of v and is defined on X. Define $S = \alpha_\infty + i\partial\overline{\partial}\psi$ and we have $T = f^*S$.

In particular, the minimal current in $f^*\alpha$ is the pull back of the minimal current in α $T_{\min}$. Thus

$$N(f^*\alpha) = \{\sum \nu(f^*T_{\min}, E)[E]\} \geqslant \{\sum_{codim(f(E))=1} \nu(f^*T_{\min}, E)[E]\}$$

$$= \{\sum_{codim(f(E))=1} \nu(T_{\min}, f(E))[E]\} = f^*N(\alpha)$$

where the sum is taken over all irreducible hypersurfaces of Y. □

Let us point out that a current with minimal singularities does not necessarily have analytic singularities for such a big class α that is nef in codimension 1 but not nef in codimension 2; this has been observed by Matsumura [**Mat13**]. The reason is as follows. In such a situation, there exists a modification $\nu : \tilde{X} \to X$ such that the pull back of α has a minimal current of the form $\beta + [E]$ where β is a semi-positive smooth form and $[E]$ is the current associated to an effective divisor supported in the exceptional divisor. In particular, the sum $\{\beta\} + \{[E]\}$ as cohomology class gives the divisorial Zariski decomposition of the class $\nu^*\alpha$. Remind that for a big class, the Zariski projection of α is given by

$$\alpha - \sum_D \nu(T_{\min}, D)\{[D]\}$$

where D runs over all the irreducible divisors on X and $T_{\min}$ is the current with minimal singularity in the class α (cf. Proposition 3.6 of [**Bou04**]). On the other hand, the push forward ν_* and pull-pack ν^* induces isomorphism between $\nu^*\alpha_\infty$-psh functions on $\tilde{X}$ and α_∞-psh functions on X where α_∞ is a smooth element in α. In particular, the pull back of the minimal current of α is the minimal current in $\nu^*\alpha$ which is also a big class. Hence $\nu^*\alpha$ admits a divisorial Zariski decomposition where the Zariski projection is semi-positive (hence nef). This contradicts the last paragraph.

Remark 4.24. As a direct consequence of Matsumura's observation, it can be shown by an example that the strategy of proof of the Kawamata-Viehweg vanishing theorem used in [**DP03**] fails in the setting of theorem 4.22. In the nef case considered in [**DP03**], let h be any positive singular metric on L. Let $\frac{i}{2\pi}\Theta(L,h) = \sum_j \lambda_j D_j + G$ be the Siu's decomposition of the curvature current, where $\lambda_j \geqslant 0$, D_j are irreducible divisors, and G is a positive current such that G has Lelong numbers in codimension $\geqslant 2$. Define $D = \sum_j [\lambda_j] D_j$, which is an integral effective divisor. As in the beginning of the proof of theorem 4.22, $H^{n-1}(X, K_X \otimes L) \neq 0$ is equivalent to $H^0(X, (D-L)|_D) \neq 0$. To prove the vanishing theorem by contradiction, Demailly and Peternell made the following first reduction, based on the non-vanishing assumption $H^0(X, (D-L)|_D) \neq 0$ and the hypothesis that the line bundle L is nef with $(L^2) \neq 0$; namely, they showed that the curvature of h on L is the current of integration associated with an effective integral divisor, so that, in particular, L is numerically equivalent to an effective integral divisor.

Here we show that for a big line bundle L which is nef in codimension 1 but not nef in codimension 2 over a compact Kähler manifold (X, ω), the positive intersection product $\langle L^2 \rangle \neq 0$ and $\frac{i}{2\pi}\Theta(L,h)$ is not a current associated to an effective integral divisor for any singular metric h. In particular, the above situation occurs by nakayama's example, and the strategy of [**DP03**] no longer works. (Up to taking some multiple of L, since L is big, it can be represented by an effective divisor. By theorem 4.22, we still have vanishing cohomology groups for some multiple of L.)

By the observation of Matsumura, the curvature current of the minimal metric cannot even be a current associated to a real divisor. Since L is big, $\langle L^n \rangle = \mathrm{Vol}(L) \neq 0$. By the Teissier-Hovanskii inequalities, we get

$$\langle L^2 \cdot \omega^{n-2} \rangle = \langle L^2 \rangle \cdot \omega^{n-2} \geqslant \mathrm{Vol}(L)^{2/n} \mathrm{Vol}(\omega)^{(n-2)/n} > 0.$$

This shows in particular that $\langle L^2 \rangle \neq 0$.

REMARK 4.25. Let us observe that this kind of construction can also be used to give an example of manifold with psef anticanonical line bundle, for which the Albanese morphism is not surjective.

According to the knowledge of the author, this kind of question has been first proposed in [**DPS93**] where the authors ask whether the Albanese map of a compact Kähler manifold is surjective under the assumption that the anticanonical line bundle is nef. The statement has been proven first by Păun [**Pau17**] using the positivity of direct image and then by Junyan Cao [**Cao13**] via a different and simpler method. In case the manifold is projective and the anticanonical divisor is nef, this had been proven earlier by Qi Zhang [**Zha06**].

Let us use the same notation as above. Take Y to be a complex curve of genus larger than 2. By a classical result, the Albanese map of Y is the embedding of the curve into its Jacobian variety $Jac(Y)$. In particular, the Albanese map is not surjective. Fix A an ample divisor on Y. Define $E = A^{\otimes p} \oplus A^{\otimes -q}$ where $p, q \in \mathbb{N}$ will be determined latter. Denote $X = \mathbb{P}(E)$ with $\pi : X \to Y$.

We claim that the Albanese morphism of X is the composition of the natural projection π and the Albanese morphism of Y. The reason is as follows: (cf. Proposition 3.12 in [**DPS94**])

Since the fibres of π are $\mathbb{P}^1$ which is connected and since π is differentially a locally trivial fibre bundle, we have $R^0\pi_*\mathbb{R}_X = \mathbb{R}_Y$, while $R^1\pi_*\mathbb{R}_X = 0$. We remark that $H^1(\mathbb{P}^1, \mathbb{R}) = 0$. The Leray spectral sequence of the constant sheaf $\mathbb{R}_X$ over X satisfies

$$E_2^{s,t} = H^s(Y, R^t\pi_*\mathbb{R}_X), E_r^{s,t} \Rightarrow H^{s+t}(X, \mathbb{R}).$$

Since $R^1\pi_*\mathbb{R}_X = 0$, the Leray spectral sequence always degenerates in E_2. (In fact, by [**Bl56**], the Leray spectral sequence always degenerates in E_2 for Kähler fibrations.) Hence we have

$$H^1(X, \mathbb{R}_X) \cong H^1(Y, \mathbb{R}_Y).$$

Since Y and X is compact Kähler, we have by Hodge decomposition theorem that

$$H^0(X, \Omega^1_X) \cong H^0(Y, \Omega^1_Y).$$

Since $\pi^* : H^0(Y, \Omega^1_Y) \cong H^0(X, \Omega^1_X)$ is an injective morphism, it induces an isomorphism. Passing to the quotient, it induces an isomorphism $\pi^* : \mathrm{Alb}(X) \cong \mathrm{Alb}(Y)$. The claim is proven by the universality of the Albanese morphism:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ \mathrm{Alb}(X) & \xrightarrow{\pi^*} & \mathrm{Alb}(Y). \end{array}$$

We also claim that for well chosen p, q, the anticanonical line bundle $-K_X$ is big but not nef in codimension 1. In particular, this shows that there exists the compact Kähler manifold X such that $-K_X$ is psef but the Albanese morphism is not surjective. Recall that

$$K_X = \pi^*(K_Y \otimes \det E) \otimes \mathcal{O}_X(-2).$$

In particular for $q \gg p$, $-(K_Y \otimes \det E) = (q-p)A - K_Y$ is ample. On the other hand, $\mathcal{O}_X(1)$ is big since one of the component in the direct sum bundle E is big. Thus $-K_X$ is big for $q >> p$. On the other hand, the surjective morphism $E \to A^{\otimes p}$ induces the closed immersion $\mathbb{P}(A^{\otimes p}) \cong Y \to X$. We have that $-K_X|_{\mathbb{P}(A^{\otimes p})} = -K_Y - pA$. For p big enough, we can assume that $-K_Y - pA$ is not psef. As consequence, $-K_X$ is not nef in codimension 1.

In fact, we can calculate the generic minimal multiplicity as

$$\nu(c_1(-K_X), \mathbb{P}(A^{\otimes p})) = \min\{t, -K_Y + (q-p)A + tpA - (2-t)qA \text{ is nef}\}.$$

Since K_Y is ample, we know that the generic minimal multiplicity along $\mathbb{P}(A^{\otimes p})$ is strictly larger than 1. In particular, consider any singular metric h_ε on $-K_X$ such that its curvature satisfies $i\Theta(-K_X, h_\varepsilon) \geqslant -\varepsilon\omega$ where ω is some Kähler form on X. Then the multiplier ideal sheaf is not trivial. Near a point of $\mathbb{P}(A^{\otimes p})$, choose some local coordinate such that $\mathbb{P}(A^{\otimes p}) = \{z_1 = 0\}$. By Siu's decomposition, the local weight of h_ε is more singular than $\log(|z_1|^2)$. This implies that $\mathcal{I}(h_\varepsilon) \subset \mathcal{I}_{\mathbb{P}(A^{\otimes p})}$ where $\mathcal{I}_{\mathbb{P}(A^{\otimes p})}$ is the ideal sheaf associated to $\mathbb{P}(A^{\otimes p})$.

Therefore, some additional condition is certainly needed to ensure the surjectivity of Albanese morphism. In the next section, we will show that if there exist approximated singular metrics such that the associated multiplier ideal sheaves are trivial, then the Albanese morphism is surjective.

REMARK 4.26. Using Nakayama's algebraic definition of minimal multiplicities [**Nak04**], Lemma 4.4 holds for a psef class on a projective manifold. Our arguments based on the non existence of Zariski decomposition over a birational model obtained as composition of blow-up(s) of smooth centres also work for the example of John Lesieutre [**Les12**]. Consider the blow up of $\mathbb{P}^3$ at 9 points in very general position. There exist a class α that is nef in codimension 1 and a curve C such that $(\alpha, C) < 0$ constructed in [**Les12**]. In particular, α is not nef. Of course, we can construct a family of similar classes by considering $\alpha + \varepsilon c_1(A)$ with $\varepsilon > 0$ and A an ample divisor. For ε small enough, the intersection number is still strictly negative.

4.4. Surjectivity of the Albanese map

In this section, we discuss the surjectivity of the Albanese morphism of a compact Kähler manifold X, under the assumption that $-K_X$ psef and some additional integrability condition for its singular metrics.

We will need the following existence and regularity results of [**CGP13**] and [**GP16**] for solutions of singular Monge-Ampère equations.

THEOREM 4.27. (Main theorem in [**CGP13**] and theorem A in [**GP16**])
Let X be a n-dimensional compact Kähler manifold, and let $D = \sum_i a_i D_i$ be an effective $\mathbb{R}$-divisor with simple normal crossing support, such that for all $1 \leqslant i \leqslant r$, the coefficients satisfy $0 < a_i < 1$. Let ω be a Kähler metric on X, let dV be a smooth volume form, and let $\varepsilon > 0$. Then the weak solution of the Monge-Ampère equation

$$(\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi)^n = e^{\varepsilon\varphi}\frac{dV}{\prod |s_i|^{2a_i}}$$

exists and has conic singularities along D, with regularity $C^{2,\alpha,\beta}$ for any $1 > \alpha > 0$ and any angles $\beta = (1-a_1, \cdots, 1-a_r)$. Here s_i is the canonical section of $\mathcal{O}(D_i)$ and $|s_i|^2$ is the norm of s_i with respect to some smooth metric.

We notice that since the solution is bounded, the Monge-Ampère operator is well-defined in the sense of currents by Bedford-Taylor [**BT82**]. The operator coincides with the positive product defined in [**BEGZ10**]. By the theorem, in particular, the weak solution is a bounded ω-psh function which is smooth on $X \setminus \bigcup_i D_i$. We also find that $\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi$ has coefficients in L^1_{loc} by the above regularity result.

We now recall the definition of a singular metric on a vector bundle according to [**Paun16**].

DEFINITION 4.28. *A singular Hermitian metric h on E is given locally by a measurable possibly unbounded map with values in the set of semi-positive Hermitian matrices, such that $0 < \det h < \infty$ almost everywhere.*

By definition, a solution in the above theorem defines a singular metric on T_X. In particular, the solution also induces a singular metric on any quotient bundle of T_X. We observe that by the Monge-Ampère equation, the Ricci curvature of the singular metric is well defined as a current. However, one can notice that the curvature tensor of T_X is not necessarily well-defined as a current with values in semi-positive, possibly unbounded Hermitian matrices.

In fact, the work of [**Gue13**] and [**CGP13**] gives the following weak estimate for the following type of Monge-Ampère equation.

THEOREM 4.29. *Let X be a n-dimensional compact Kähler manifold, and let $D = \sum_i a_i D_i$, $E = \sum_j b_j E_j$ be two effective $\mathbb{R}$-divisors with simple normal crossing support, such that for all $1 \leqslant i \leqslant r$, $0 < a_i < 1$. Assume that D and E have no common irreducible component. Let ω be a Kähler metric on X, dV a smooth volume form, and let $\varepsilon > 0$. Then the weak solution of the Monge-Ampère equation*

$$\langle(\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi)^n\rangle = e^{\varepsilon\varphi}\frac{\prod |t_j|^{2b_j} dV}{\prod |s_i|^{2a_i}}$$

exists which is smooth on $X \setminus (D \cup E)$ and has upper bound by a metric with conic singularity along D. Here $\langle \bullet \rangle$ is the positive intersection product defined in [**BEGZ10**]. *Here s_i(resp. t_j) is the canonical section of $\mathcal{O}(D_i)$ (resp. $\mathcal{O}(E_j)$) and $|s_i|^2$ (resp. $|t_j|^2$) is the norm of s_i (resp. t_j) with respect to some smooth metric.*

We observe that the existence of a solution is proved in [**BEGZ10**]. As a consequence of their theorem, there exists $C > 0$ such that the solution has on $X \setminus (D \cup E)$ an upper bound

$$\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi \leqslant \frac{C\omega}{\prod_i |s_i|^{2a_i}}.$$

By the Monge-Ampère equation, we find on $X \setminus (D \cup E)$ a lower bound

$$\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi \geqslant e^{\varepsilon\varphi}\frac{\prod |t_j|^{2b_j}\omega}{\prod |s_i|^{2a_i}}(\frac{C}{\prod_i |s_i|^{2a_i}})^{-(n-1)}.$$

Notice that since the solution is smooth on $X \setminus (D \cup E)$, the above inequalities are satisfied pointwise. By the result of [**BEGZ10**], $|\varphi|$ is uniformly bounded on X. In particular, we have

$$\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi \geqslant \frac{C\prod|t_j|^{2b_j}\omega}{\prod|s_i|^{2a_i}}\left(\frac{C}{\prod_i|s_i|^{2a_i}}\right)^{-(n-1)}.$$

In conclusion outside $D \cup E$, the solution $\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi$ viewed as a Hermitian form over T_X with respect to ω has positive eigenvalues bounded from above by $\frac{C}{\prod_i|s_i|^{2a_i}}$ and bounded from below by $\frac{C\prod|t_j|^{2b_j}}{\prod|s_i|^{2a_i}}\left(\frac{C}{\prod_i|s_i|^{2a_i}}\right)^{-(n-1)}$.

Let us observe that for the singular metric on the determinant line bundle of the quotient bundle Q given by a short exact sequence of vector bundles

$$0 \to S \to T_X \to Q \to 0,$$

the curvature form is well-defined as a current. We detail the argument below.

Suppose that we are in the situation of Theorem 4.27, with the same notation as above. Since the metric is smooth outside $D \cup E$, we only need to study the neighbourhood of $D \cup E$. By a C^∞ splitting of the exact sequence we can view Q as a subbundle of T_X. $\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi$ thus induces a Hermitian form over Q which we will denote by $\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi|_Q$. By the minimax principle, for the induced Hermitian form on Q, the eigenvalues are bounded from above by $\frac{C}{\prod_i|s_i|^{2a_i}}$ and bounded from below by $\frac{C\prod|t_j|^{2b_j}}{\prod|s_i|^{2a_i}}\left(\frac{C}{\prod_i|s_i|^{2a_i}}\right)^{-(n-1)}$. To prove that the curvature of $\det(Q)$ is well-defined as a current (not necessarily positive), it is enough to prove that $\log(\det(\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi|_Q)) \in L^1_{\text{loc}}$. $\det(\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi|_Q)$ is the product of all eigenvalues of the Hermitian form $\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi|_Q$. Thus we get for the potentials the estimate

$$|\log(\det(\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi|_Q))| \leqslant \sum_i C_i\log|s_i|^2 + \sum_j C_j\log|t_j|^2 + C$$

for some $C_i > 0$, $C_j > 0$ and $C > 0$. In the following, we will refer to this type of control as potentials possessing at most logarithmic poles along $D \cup E$. Notice also that for any i $\log|z_i|$ is locally integrable with respect to the euclidean metric. In particular, the curvature of the induced metric on $\det(Q)$ is well defined as a current, since it is the $i\partial\overline{\partial}$ of some L^1_{loc} function.

Let U be a neighbourhood of some point in $D \cup E$ as above and let $\pi : \tilde{U} \to U$ be some ramified cover which can be written in local coordinate as

$$(z_1, z_2, \cdots, z_n) \mapsto (z_1^{p_1}, z_2^{p_2}, \cdots, z_n^{p_n})$$

for some $(p_1, \cdots, p_n) \in (\mathbb{N}^*)^n$. Notice that the pull back under π of the potential of our curvature current, namely $\pi^*\log(\det(\omega + \frac{i}{2\pi}\partial\overline{\partial}\varphi|_Q))$, is still L^1_{loc} with at most logarithmic poles along $D \cup E$.

In the following, instead of solving a Monge-Ampère type equation on X, we will solve a Monge-Ampère type equation on some bimeromorphic model of $\sigma : \tilde{X} \to X$. The bimeromorphic model is obtained by the work of Hironaka. We can thus assume that the modification σ is obtained as a finite composition of blows-up of smooth submanifold. Let us first study the case of blow-up of smooth submanifold $\pi : \tilde{Y} \to Y$.

The differential $d\pi$ induces a bundle morphism over $\tilde{Y}$ $T_{\tilde{Y}} \to \pi^*T_Y$. Assume we have biholomorphism between $\tilde{Y} \setminus E$ and $Y \setminus S$ where E is the exceptional divisor and S is the smooth submanifold to be blown-up. Over $\tilde{Y} \setminus E$, $d\pi$ is a pointwise linear isomorphism. Let us estimate the variation of the norm of the pointwise isomorphism. It will be enough for us to study the behaviour near the exceptional divisor. Otherwise the norm will be locally bounded by a constant.

LEMMA 4.30. *Let $\pi : \tilde{Y} \to Y$ be the blow-up of a smooth submanifold S. Let p be a point in the exceptional divisor. Choose coordinate of $\tilde{Y}$ and Y such that in local coordinates near p π is given by*

$$\pi(w_1, \cdots, w_n) = (w_1w_s, \cdots, w_{s-1}w_s, w_s, w_{s+1}, \cdots, w_n).$$

Then the norm of $d\pi$ and $(d\pi)^{-1}$ with respect to fixed smooth metric on $T_{\tilde{Y}}$ and π^*T_Y has estimate

$$\log\|d\pi(w_1, \cdots, w_n)\| \leqslant C_1\log|w_s|^2 + C_2$$

$$\log\|(d\pi)^{-1}(w_1, \cdots, w_n)\| \leqslant C_1\log|w_s|^2 + C_2$$

for some $C_1, C_2 > 0$.

PROOF. The differential of π at $(w_1, \cdots, w_n)$ is given by the matrix

$$\begin{bmatrix} w_s & 0 & \dots & 0 & w_1 & 0 \\ 0 & w_s & \dots & 0 & w_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & w_s & w_{s-1} & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & \mathrm{Id}_{n-s} \end{bmatrix}$$

where Id_{n-s} is the identity matrix of rank $n-s$.

The norm of $\|d\pi\|$ at $(w_1, \cdots, w_n)$ is bounded from above by the largest eigenvalue of the matrix $d\pi^\dagger d\pi$. While the norm $\|(d\pi)^{-1}\|$ at $(w_1, \cdots, w_n)$ is bounded from above by the inverse of the smallest eigenvalue. The product $d\pi^\dagger d\pi$ can be calculated in this local coordinate chart as

$$\begin{bmatrix} |w_s|^2 & 0 & \dots & 0 & w_1\bar{w}_s & 0 \\ 0 & |w_s|^2 & \dots & 0 & w_2\bar{w}_s & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & |w_s|^2 & w_{s-1}\bar{w}_s & 0 \\ w_s\bar{w}_1 & w_s\bar{w}_2 & \dots & w_s\bar{w}_{s-1} & 1+\sum_{j<s}|w_j|^2 & 0 \\ 0 & 0 & \dots & 0 & 0 & \mathrm{Id}_{n-s} \end{bmatrix}.$$

The eigenvalues are the roots of the polynomial $\det(d\pi^\dagger d\pi - \lambda \mathrm{Id}_n)$, which is $(1-\lambda)^{n-s}$ times

$$(1+\sum_{j<s}|w_j|^2-\lambda)(|w_s|^2-\lambda)^{s-1} - w_{s-1}\bar{w}_s w_s \bar{w}_{s-1}(|w_s|^2-\lambda)^{s-2}$$

$$-w_{s-2}\bar{w}_s w_s \bar{w}_{s-2}(|w_s|^2-\lambda)^{s-2} - \cdots$$

by developing the s-th column. The polynomial can be simplified as

$$(1+\sum_{j<s}|w_j|^2-\lambda)(|w_s|^2-\lambda)^{s-1} - (\sum_{j<s}|w_j|^2)|w_s|^2(|w_s|^2-\lambda)^{s-2}.$$

The product of the eigenvalues is

$$\det(d\pi^\dagger d\pi) = |\det(d\pi)|^2 = |w_s|^{2(s-1)}$$

while the sum of the eigenvalues is

$$n-s+|w_s|^{2(s-1)} - (s-2)|w_s|^{2(s-2)}(\sum_{j<s}|w_j|^2) + (s-1)(\sum_{j<s}|w_j|^2+1)|w_s|^{2(s-2)}.$$

In other words, the sum of the eigenvalues is $|w_s|^{2(s-2)}(s-1+\sum_{j\leqslant s}|w_j|^2)+n-s$.

Since $d\pi^\dagger d\pi$ is positive and Hermitian, all the eigenvalues are real and positive. In particular its largest eigenvalue is controlled from above by $|w_s|^{2(s-2)}(s-1+\sum_{j\leqslant s}|w_j|^2)+n-s$ and its smallest eigenvalue is controlled from below by $\det(d\pi^\dagger d\pi)\big(|w_s|^{2(s-2)}(s-1+\sum_{j\leqslant s}|w_j|^2)+n-s\big)^{-(n-1)}$. This implies the estimate of the norms $\|d\pi\|$ and $\|(d\pi)^{-1}\|$. □

PROPOSITION 4.4.1. Let $\sigma : \tilde{X} \to X$ be a finite composition of blows-up of smooth submanifolds. Denote by E the exceptional divisor. We have an estimate for the norm of $d\pi$ with respect to a fixed smooth metric on $T_{\tilde{Y}}$ and $\pi^* T_Y$ that reads

$$\log\|d\sigma\| \leqslant C_1 \log|s_E|^2 + C_2$$

where $C_1, C_2 > 0$ and s_E is the canonical section of the exceptional divisor. We also have a similar estimate for $(d\sigma)^{-1}$.

PROOF. Let $\sigma = \pi_d \circ \cdots \circ \pi_1$ where π_i are blows-up of smooth submanifolds. Since $d\sigma = d\pi_d \circ \cdots \circ d\pi_1$, we find

$$\|d\sigma\| \leqslant \|d\pi_d\| \cdot \cdots \cdot \|d\pi_1\|.$$

On the other hand, for each π_i, by the above lemma, the norm of $d\pi_i$ has upper bound with logarithmic pole along the exceptional divisor of this blow up. This singularity is independent of the choice of coordinate. The pull back of logarithmic pole along a divisor D under a modification is still logarithmic with pole supported in the exceptional divisor of the modification union the strict transform of D. This concludes the estimate of the upper bound of $\|d\sigma\|$. The estimate for $(d\sigma)^{-1}$ is similar. □

We will also need the following topological lemma. The definition of the first Chern class of a coherent sheaf $\mathcal{F}$ over a connected complex manifold can be found for example in section 6, Chap. V of [**Kob75**]. We define

$$c_1(\mathcal{F}) := c_1((\Lambda^r \mathcal{F})^{**})$$

where r is generic rank of $\mathcal{F}$.

LEMMA 4.31. *Let $\mathcal{F}$ be a torsion free sheaf over a compact complex manifold X. Let $\sigma : \tilde{X} \to X$ be a modification of X such that there exists a SNC divisor E in $\tilde{X}$ such that*

$$\sigma : \tilde{X} \smallsetminus E \to X \smallsetminus \pi(E)$$

is biholomorphism with E a SNC divisor and the codimension of $\pi(E)$ at least 2 and $\sigma^*\mathcal{F}/\mathrm{Tors}$ is locally free. Then we have

$$c_1(\mathcal{F}) = \sigma_*(c_1(\sigma^*\mathcal{F}/\mathrm{Tors})).$$

PROOF. First observe that such a modification always exists by the fundamental work of [**Ros68**], [**GR70**], [**Rie71**] (cf. eg. Theorem 3.5 of [**Ros68**]).

Without loss of generality we can assume that the dimension of X is at least 2. Otherwise, $\mathcal{F}$ is locally free and the result is straightforward. By Poincaré duality, it is equivalent to prove for any cohomology class α one has

$$\int_X c_1(\mathcal{F}) \wedge \alpha = \int_{\tilde{X}} (c_1(\sigma^*\mathcal{F}/\mathrm{Tors})) \wedge \sigma^*\alpha.$$

Recall that $\sigma^* \operatorname{ch}(\mathcal{F}) = \sum_i (-1)^i \operatorname{ch}(L^i\sigma^*\mathcal{F})$ where $L^i\sigma^*$ is the i-th left exact functor of σ^* (cf. eg. [**BS56**]). Without loss of generality, we can assume that $\mathcal{F}$ is locally free over $X \smallsetminus \pi(E)$. In particular, $L^i\sigma^*\mathcal{F}$ for any $i > 0$ is supported in the exceptional divisor. On the other hand, the torsion part of $\sigma^*\mathcal{F}$ is also supported in the exceptional divisor. Recall that for a torsion sheaf, its first Chern class is an effective divisor supported in the support of the sheaf. Thus we have

$$\int_{\tilde{X}} (c_1(\sigma^*\mathcal{F}/\mathrm{Tors})) \wedge \sigma^*\alpha = \int_{\tilde{X}} \sigma^*(c_1(\mathcal{F})) \wedge \sigma^*\alpha$$

since for any irreducible component of the exceptional divisor E_i, $\sigma^*\alpha|_{E_i} = 0$. This implies that

$$c_1(\mathcal{F}) = \sigma_*(c_1(\sigma^*\mathcal{F}/\mathrm{Tors})).$$

□

To prove the surjectivity of the Albanese map, we start by an analgue of the main result of [**Cao13**]. Now (X, ω) be a n-dimensional compact Kähler manifold such that $-K_X$ is psef. Notice that without loss of generality, we can assume that $n \geqslant 2$. Otherwise, $-K_X$ psef implies that $-K_X$ is nef in which case we know the surjectivity. By regularisation of the minimal metric larger than $-\varepsilon_\nu\omega$, for any ε_ν, there exists a current $T_{\varepsilon_\nu} = \mathrm{Ric}(\omega) + i\partial\bar{\partial} f_{\varepsilon_\nu} \in c_1(X)$ with analytic singularities such that $T_{\varepsilon_\nu} \geqslant -2\varepsilon_\nu\omega$. Let $\tilde{X}$ be a modification of X $\pi : \tilde{X} \to X$ such that $\pi^* T_{\varepsilon_\nu} = \beta_{\varepsilon_\nu} + [F_{\varepsilon_\nu}]$ where F_{ε_ν} is a simple normal crossing $\mathbb{R}$- divisor. We denote $[F_{\varepsilon_\nu}] = \sum_i b_i[D_i]$. We can also assume that the exceptional divisor is a SNC divisor.

Classically, we have

$$-K_{\tilde{X}} = \pi^*(-K_X) - cD$$

where $cD = \sum_i c_i D_i$ with $c_i \geqslant 0$. The condition that the singular metric $h_{\varepsilon_\nu} := \det(\omega)e^{-f_{\varepsilon_\nu}}$ has multiplier ideal sheaf $\mathcal{I}(h_{\varepsilon_\nu}) = \mathcal{O}_X$ means that $c_i - b_i < 1$ for any i. We will denote the irreducible components in D contained in the exceptional divisor as E_i. With this abuse of notation,

$$-K_{\tilde{X}} = \pi^*(-K_X) - cE.$$

THEOREM 4.32. *Let (X, ω) be a n-dimensional compact Kähler manifold such that $-K_X$ is psef. Assume that there exists a sequence $\varepsilon_\nu > 0$ such that $\lim_{\nu\to\infty} \varepsilon_\nu = 0$ and $\mathcal{I}(h_{\varepsilon_\nu}) = \mathcal{O}_X$ with the notation explained above. Let*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = T_X$$

be a filtration of torsion-free subsheaves such that $\mathcal{E}_{i+1}/\mathcal{E}_i$ is an ω-stable torsion-free subsheaf of $T_X/\mathcal{E}_i$ of maximal slope. Then for any i, the slope of $\mathcal{E}_{i+1}/\mathcal{E}_i$ with respect to ω^{n-1}, namely

$$\mu(\mathcal{E}_{i+1}/\mathcal{E}_i) := \int_X c_1(\mathcal{E}_{i+1}/\mathcal{E}_i) \wedge \omega^{n-1},$$

is positive.

Proof. We first consider a simple case.

Case 1: assume that the filtration is regular, i.e., that all $\mathcal{E}_i$, $\mathcal{E}_{i+1}/\mathcal{E}_i$ are vector bundles. By the stability condition, to prove the theorem, it is sufficient to prove that for any i

$$\int_X c_1(T_X/\mathcal{E}_i) \wedge \omega^{n-1} \geqslant 0.$$

The key step is the existence of positive closed $(1,1)$- current in a Kähler class on some birational model of X which is smooth outside a SNC divisor and whose Ricci curvature can be taken "arbitrary small" outside the divisor.

With the same notations as in the discussion before the theorem, for $\delta > 0$ sufficient small, $\pi^*\omega - \delta\{E\}$ is a Kähler class on $\tilde{X}$. We want to construct a positive closed current in the class $\pi^*\omega - \delta\{E\}$ with Ricci curvature lower bound using the theorems in [**GP16**] and [**CGP13**].

To get the lower bound, we want to solve the following Kähler-Einstein type of equation

$$\mathrm{Ric}(\omega_{\delta,\varphi}) = -\varepsilon_\nu \omega_{\delta,\varphi} + \varepsilon_\nu \omega_\delta + \pi^* T_{\varepsilon_\nu} - c[E]$$

where $\omega_{\delta,\varphi} := \omega_\delta + i\partial\overline{\partial}\varphi$ is the unknown in the class $\pi^*\omega - \delta\{E\}$ and ω_δ is a smooth Kähler representative. Notice that both sides belong to the class $c_1(-K_{\tilde{X}})$.

In order to solve the Kähler-Einstein type of equation, we thus solve the following Monge-Ampère equation. Let γ_{ε_ν} be a smooth representative of the class $\{F_{\varepsilon_\nu} - cE\}$ which is induced from the curvature forms of some smooth metrics $(\mathcal{O}(D_i), h_i)$. By the $\partial\overline{\partial}$-lemma, there exists $f_{\varepsilon_\nu} \in C^\infty(\tilde{X})$ such that $\beta_{\varepsilon_\nu} + \gamma_{\varepsilon_\nu} = \mathrm{Ric}(\omega_\delta) + \frac{i}{2\pi}\partial\overline{\partial} f_{\varepsilon_\nu,\delta}$. The Monge-Ampère equation equivalent to the Kähler-Einstein type of equation can be written as

$$\omega_{\delta,\varphi}^n = \frac{\omega_\delta^n e^{\varepsilon_\nu \varphi - f_{\varepsilon_\nu,\delta}}}{|s_i|_{h_i}^{2(c_i-b_i)}}.$$

By the assumption, we have

$$c_i < b_i + 1$$

which is exactly the integrability condition in the Theorem 4.27. Thus by theorem 4.27, the solutions exist and are smooth outside the support of D. In particular, we have the Kähler-Einstein type of equation pointwise outside D and

$$\mathrm{Ric}(\omega_{\delta,\varphi}) \geqslant -\varepsilon_\nu \omega_{\delta,\varphi}$$

in the sense of current. By lemma 2.7 in [**Cao13**] (which works even on non compact manifold since it is a local calculation), we have on $\tilde{X} \smallsetminus D$ that

$$i\Theta(T_{\tilde{X}}, \omega_{\delta,\varphi}) \wedge \omega_{\delta,\varphi}^{n-1}/\omega_{\delta,\varphi}^n \geqslant -\varepsilon_\nu \mathrm{Id}_{T_{\tilde{X}}}$$

pointwise.

The singular metric on $T_{\tilde{X}}$ from the solution of the Monge-Ampère equation induces a singular metric on π^*T_X by $d\pi : T_{\tilde{X}} \to \pi^*T_X$. Taking the quotient metric, it induces a singular metric on $\pi^*T_X/\mathcal{E}_i$ (we also denote it by $\omega_{\delta,\varphi}$). We get on $\tilde{X} \smallsetminus D$

$$i\Theta(\pi^*(T_X/\mathcal{E}_i), \omega_{\delta,\varphi}) \wedge \omega_{\delta,\varphi}^{n-1}/\omega_{\delta,\varphi}^n \geqslant -\varepsilon_\nu \mathrm{Id}_{\pi^*(T_X/\mathcal{E}_i)}$$

pointwise. In particular, we have that $(i\Theta(\pi^*\det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi}) + \varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi}) \wedge \omega_{\delta,\varphi}^{n-1}$ defines a closed positive (n,n)-current on $X \smallsetminus \pi(D) \cong \tilde{X} \smallsetminus D$.

Let us show that $i\Theta(\pi^*\det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})$ has L^1_{loc} weight. In fact the local weight has at most logarithmic pole along the divisor D. Notice that $i\Theta(\pi^*\det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})$ is in the first Chern class of $\pi^*c_1(T_X/\mathcal{E}_i)$.

Locally over $\tilde{X} \smallsetminus D$, identify the metric $\omega_{\delta,\varphi}$ as a Hermitian matrix $H_{\delta,\varphi}$. The induced metric on $\pi^*(T_X)$ can be identified as the Hermitian matrix over $\tilde{X} \smallsetminus D$

$$[(d\pi)^{-1}]^\dagger H_{\delta,\varphi}(d\pi)^{-1}.$$

By the minimax principle, the induced metric on $\pi^*(T_X/\mathcal{E}_i)$, as a Hermitian form, has eigenvalues that are controlled both from above and from below by the eigenvalues of the above matrix. More precisely, the maximal eigenvalue of the induced metric on $\pi^*(T_X/\mathcal{E}_i)$ over $\tilde{X} \smallsetminus D$ is bounded from above by the maximal eigenvalue of $H_{\delta,\varphi}$ times $\|(d\pi)^{-1}\|^2$. By the discussion after Theorem 4.27 and Proposition 4.4.1, the logarithm of the maximal eigenvalue of induced metric on $\pi^*(T_X/\mathcal{E}_i)$ has at most logarithmic pole along the divisor D. Similarly, the inverse of the minimal eigenvalue of induced metric on $\pi^*(T_X/\mathcal{E}_i)$ is bounded from above by the inverse of the minimal eigenvalue of $H_{\delta,\varphi}$ times $\|(d\pi)\|^2$. The absolute value of the logarithm of the minimal eigenvalue of induced metric on $\pi^*(T_X/\mathcal{E}_i)$ has also at most logarithmic pole along the divisor D.

The induced metric on $\pi^* \det(T_X/\mathcal{E}_i)$ has thus weight controlled both from above and from below by functions with at most logarithmic pole along the divisor D. In particular, the local weight is locally integrable.

We claim that $(i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi}) + \varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi}) \wedge \omega_{\delta,\varphi}^{n-1}$ extends by zero to be a closed positive (n,n)-current on $\tilde{X}$. Moreover, $(i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})) \wedge \omega_{\delta,\varphi}^{n-1}$ well defines a (n,n)-current on $\tilde{X}$ and it has zero mass along D.

To prove the claim, we start by showing that $(i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})) \wedge \omega_{\delta,\varphi}^{n-1}$ can be defined as a (n,n)-current on $\tilde{X}$. By a partition of unity, it is enough to show it in a finite open cover of $\tilde{X}$ such that D is the zero set of the coordinate functions in these charts. Let $p \in \mathbb{N}$ be a natural number large enough such that $p(c_i - b_i) > 1$ for any i such that $c_i - b_i > 0$. Let ψ be the local potential of $i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})$ on U a local chart. Assume that $F_{\varepsilon_\nu} - cE = E_1 - E_2$ with E_1, E_2 two effective divisors without common irreducible components. Assume that $E_1 = \sum_{i=1}^r a_i[z_i = 0]$. Let $p : \tilde{U} \to U$ be a local finite ramified cover given by

$$p(z_1, \cdots, z_r, z_{r+1}, \cdots, z_n) = (z_1^p, \cdots, z_r^p, z_{r+1}, \cdots, z_n).$$

By the discussion before Lemma 4.28, $p^*\psi$ is still L^1_{loc} since ψ can possess at most logarithmic pole along the divisor $E_1 \cup E_2$. On the other hand, $p^*\omega_{\delta,\varphi}$ is bounded from above by

$$C \sum_{i=1}^r id(z_i^{pa_i}) \wedge \overline{d(z_i^{pa_i})} + C \sum_{i=r+1}^n id(z_i) \wedge \overline{d(z_i)}$$

by the upper bound with conic singularity given by the theorem 4.27. Thus $p^*\psi \wedge p^*\omega_{\delta,\varphi}^{n-1}$ is well defined as current on $\tilde{U}$ with L^1_{loc} coefficients. We define $(i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})) \wedge \omega_{\delta,\varphi}^{n-1}$ on U to be

$$\frac{1}{p^r} p_*(i\partial\overline{\partial}(p^*\psi \wedge p^*\omega_{\delta,\varphi}^{n-1})).$$

This current coincides with the usual definition on $\tilde{X} \setminus D$.

Next, we show that $(i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})) \wedge \omega_{\delta,\varphi}^{n-1}$ defined above has zero mass along D. Let $\theta(z) \in C_c^\infty(\mathbb{C}^n)$ with compact support in U. Then $\theta_\varepsilon := \theta(\varepsilon z_1, \cdots, \varepsilon z_r, z_{r+1}, \cdots, z_n)$ is supported in a tubular neighbourhood of D of diameter ε in the coordinate chart. Then it remains to prove that for $\varepsilon > 0$ small enough the pair of the current $(i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})) \wedge \omega_{\delta,\varphi}^{n-1}$ with θ_ε is finite and has limit 0 as $\varepsilon \to 0$. We have

$$\int_U \theta_\varepsilon (i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})) \wedge \omega_{\delta,\varphi}^{n-1} = \frac{1}{p^r} \int_{\tilde{U}} \pi^*((i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi})) \wedge \omega_{\delta,\varphi}^{n-1})$$
$$= \frac{1}{p^r} \int_{\tilde{U}} \pi^*(\psi \wedge \omega_{\delta,\varphi}^{n-1} \wedge i\partial\overline{\partial}\theta_\varepsilon).$$

Here, $|i\partial\overline{\partial}\theta_\varepsilon|$ is bounded from above, with some constant $C > 0$, by

$$C \sum_{i=1}^r \frac{1}{\varepsilon^2} id(z_i) \wedge \overline{d(z_i)} + C \sum_{i=r+1}^n id(z_i) \wedge \overline{d(z_i)}.$$

On the other hand, $\int_{\tilde{U}} \pi^*(\psi \wedge \omega_{\delta,\varphi}^{n-1} \wedge i\partial\overline{\partial}\theta_\varepsilon)$ is bounded from above for some $r_0 > 0$ independent of ε by

$$C \prod_{i=1}^r \int_0^\varepsilon \frac{1}{\varepsilon^2} \log|z_i| |z_i|^{2pa_i - 2} idz_i \wedge \overline{dz_i} \prod_{i=r+1}^n \int_0^{r_0} idz_i \wedge \overline{dz_i}.$$

The upper bound is uniformly bounded and has limit 0 as $\varepsilon \to 0$ since

$$\int_0^\varepsilon \frac{1}{\varepsilon^2} \log(r) r^{2pa_i - 1} dr = \frac{1}{2pa_i \varepsilon^2} \log(\varepsilon) \varepsilon^{2pa_i} - \frac{1}{4p^2 a_i^2 \varepsilon^2} \varepsilon^{2pa_i}.$$

Notice that by the choice of p, for any i, $pa_i > 1$.

In conclusion, $(i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi}) + \varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi}) \wedge \omega_{\delta,\varphi}^{n-1}$ is a closed (n,n)- current with 0 mass along D. $(i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi}) + \varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi}) \wedge \omega_{\delta,\varphi}^{n-1}$ is a closed positive current on $\tilde{X} \setminus D$. By Skoda-El Mir theorem, it extends by 0 across E to be a positive closed current on $\tilde{X}$ which is $(i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi}) + \varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi}) \wedge \omega_{\delta,\varphi}^{n-1}$ defined above. The mass on $\tilde{X}$ is equal to the mass on $\tilde{X} \setminus E$. This can be seen from the following decomposition:

$$\int_U (i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi}) + \varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi}) \wedge \omega_{\delta,\varphi}^{n-1}$$
$$= \int_U \theta_\varepsilon (i\Theta(\pi^* \det(T_X/\mathcal{E}_i), \omega_{\delta,\varphi}) + \varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi}) \wedge \omega_{\delta,\varphi}^{n-1}$$

$$+\int_U (1-\theta_\varepsilon)(i\Theta(\pi^*\det(T_X/\mathcal{E}_i),\omega_{\delta,\varphi})+\varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi})\wedge\omega_{\delta,\varphi}^{n-1}.$$

By the dominated convergence theorem, the limit of the second term is

$$\int_{U\smallsetminus E}(i\Theta(\pi^*\det(T_X/\mathcal{E}_i),\omega_{\delta,\varphi})+\varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi})\wedge\omega_{\delta,\varphi}^{n-1}.$$

The limit of the first term is 0 by above discussion.

Now $(i\Theta(\pi^*\det(T_X/\mathcal{E}_i),\omega_{\delta,\varphi})+\varepsilon_\nu \mathrm{rank}(\pi^*(T_X/\mathcal{E}_i))\omega_{\delta,\varphi})\wedge\omega_{\delta,\varphi}^{n-1}$ is a positive closed (n,n)- current on $\tilde{X}$. It belongs to the cohomology class

$$(\pi^*c_1(T_X/\mathcal{E}_i)+\varepsilon_\nu \mathrm{rank}(T_X/\mathcal{E}_i)\pi^*\{\omega\}-\varepsilon_\nu \mathrm{rank}(T_X/\mathcal{E}_i)\delta\{E\})\wedge(\pi^*\{\omega\}-\delta\{E\})^{n-1}.$$

In particular, we have

$$(\pi^*c_1(T_X/\mathcal{E}_i)+\varepsilon_\nu \mathrm{rank}(T_X/\mathcal{E}_i)\pi^*\{\omega\}-\varepsilon_\nu \mathrm{rank}(T_X/\mathcal{E}_i)\delta\{E\})\wedge(\pi^*\{\omega\}-\delta\{E\})^{n-1}\geqslant 0.$$

If we let δ tend to 0, we find

$$(\pi^*c_1(T_X/\mathcal{E}_i)+\varepsilon_\nu \mathrm{rank}(T_X/\mathcal{E}_i)\pi^*\{\omega\})\wedge(\pi^*\{\omega\})^{n-1}\geqslant 0$$

which is also equal to

$$(c_1(T_X/\mathcal{E}_i)+\varepsilon_\nu \mathrm{rank}(T_X/\mathcal{E}_i)\{\omega\})\wedge(\{\omega\})^{n-1}.$$

By taking $\nu\to\infty$, one achieves the proof of case 1.

Case 2: general case.

To prove the theorem in the case when the filtration is given by subsheaves whose quotient sheaves are torsion free, we follow the arguments given in case 1.

In this situation, we first take a finite composition of blows-up of smooth submanifolds $\sigma:\tilde{X}\to X$ such that $\sigma^*(T_X/\mathcal{E}_i)/\mathrm{Tors}$ is a vector bundle over $\tilde{X}$. Then we take a further finite composition of blows-up of smooth submanifolds π to reduce the analytic singularity of h_{ε_ν} to the simple normal case. The proof given in case 1 changing X by $\tilde{X}$ and $T_X/\mathcal{E}_i$ by $\sigma^*(T_X/\mathcal{E}_i)/\mathrm{Tors}$ shows that

$$(c_1(\pi^*(\sigma^*(T_X/\mathcal{E}_i)/\mathrm{Tors}))+\varepsilon_\nu \mathrm{rank}(T_X/\mathcal{E}_i)\pi^*\sigma^*\{\omega\}-\varepsilon_\nu \mathrm{rank}(T_X/\mathcal{E}_i)\delta\{E\})\wedge(\pi^*\sigma^*\{\omega\}-\delta\{E\})^{n-1}\geqslant 0.$$

Notice that the metric is always well-defined on a Zariski open set and that its curvature defines a current in the first Chern class. The wedge product of the currents extends across the exceptional divisor over the bimeromorphic model for the same reasons. Letting δ tend 0 implies that

$$(c_1(\sigma^*(T_X/\mathcal{E}_i)/\mathrm{Tors})+\varepsilon_\nu \mathrm{rank}(T_X/\mathcal{E}_i)\sigma^*\{\omega\})\wedge(\sigma^*\{\omega\})^{n-1}\geqslant 0.$$

Notice that π depends on ν, however σ is independent of ν. Letting ν tend to infinity and using Lemma 4.29 concludes the proof.

□

Now the arguments of Proposition 5.1 of [**Cao13**] give the following corollary.

COROLLARY 4.33. *Let (X,ω) be a n-dimensional compact Kähler manifold such that $-K_X$ is psef. Assume that there exists a sequence $\varepsilon_\nu>0$ such that $\lim_{\nu\to\infty}\varepsilon_\nu=0$ and $\mathcal{I}(h_{\varepsilon_\nu})=\mathcal{O}_X$ for a sequence of singular metrics with analytic singularities h_{ε_ν} on $-K_X$ such that $i\Theta(-K_X,h_{\varepsilon_\nu})\geqslant-\varepsilon_\nu\omega$ Then the Albanese morphism α_X is surjective with connected fibres. In fact, the Albanese map is submersion outside an analytic set of codimension larger than 2.*

PROOF. The proof in [**Cao13**] only uses the fact that the slopes with respect to ω^{n-1} of the sheaves obtained as graded pieces of the Harder-Narasimhan filtration are positive. Hence using theorem 4.30, the result is a direct consequence of his arguments. For the convenience of the readers, we just give here the proof of the fact that the fibres of the Albanese map are connected. We follow the arguments in the Proposition 3.9 of [**DPS94**].

Let $X\to Y\to \mathrm{Alb}(X)$ be the Stein decomposition of the Albanese map with $Y=\mathrm{Spec}\,\alpha_{X*}\mathcal{O}_X$. Since X is smooth, Y is normal. We claim that the map $f:Y\to\mathrm{Alb}(X)$ is étale. The reason is as follows. By the arguments in [**Cao13**], there exists Z an analytic subset in $\mathrm{Alb}(X)$ with codimension at least 2 such that $X\smallsetminus\alpha_X^{-1}(Z)\to\mathrm{Alb}(X)\smallsetminus Z$ is submersion (thus a fibration). Thus $Y\smallsetminus f^{-1}(Z)\to\mathrm{Alb}(X)\smallsetminus Z$ is étale. We denote by F the fibre of the fibration $f|_{Y\smallsetminus f^{-1}(Z)}$ which is finite. By the long exact sequence associated to a fibration, we have

$$\pi_1(F)\to\pi_1(Y\smallsetminus f^{-1}(Z))\to\pi_1(\mathrm{Alb}(X)\smallsetminus Z)\to\pi_0(F)$$

where $\pi_1(F)=0$ and $\pi_0(F)$ is finite. In particular, $\pi_1(Y\smallsetminus f^{-1}(Z))$ is a free Abelian group of rank $2q:=2\mathrm{dim}_{\mathbb{C}}\mathrm{Alb}(X)$. Notice that by the codimension condition, we have $\pi_1(\mathrm{Alb}(X)\smallsetminus Z)\cong\pi_1(\mathrm{Alb}(X))$. $\mathrm{Alb}(X)$ is isomorphic to the quotient of the universal cover $\mathbb{C}^q$ of $\mathrm{Alb}(X)$ under the group action $\pi_1(\mathrm{Alb}(X))$. Define T to be the quotient of $\mathbb{C}^q$ under the group action $\pi_1(Y\smallsetminus f^{-1}(Z))$ with the natural cover $p:T\to\mathrm{Alb}(X)$.

By the homotopy lifting property, there exists a map $g : Y \smallsetminus f^{-1}(Z) \to T$ such that $p \circ g = f|_{Y \smallsetminus f^{-1}(Z)}$. Remark that g is holomorphic since it is given by the composition of f with the holomorphic local inverse of p. Since $Y \smallsetminus f^{-1}(Z) \to \mathrm{Alb}(X) \smallsetminus Z$ is finite, $f^{-1}(Z)$ is of codimension at least 2. Since Y is normal, g extends to a morphism $g : Y \to T$. Now g is a generically injective morphism between Y and T. Since T is smooth, the inverse map of $T \smallsetminus p^{-1}(Z) \to Y$ also extends across $p^{-1}(Z)$ which gives the inverse morphism of g. In conclusion g is a biholomorphism between T and Y which proves that f is étale.

In particular, Y is a finite étale cover of the torus $\mathrm{Alb}(X)$, so Y itself is a torus. By the universality of the Albanese morphism, there exists a morphism $h : \mathrm{Alb}(X) \to Y$ such that the morphism $X \to Y$ factorises through h. Since the morphisms $X \to Y$ and α_X are surjective, we have $h \circ f = id_Y$ and $f \circ h = id_{\mathrm{Alb}(X)}$. Thus f is a biholomorphism and the Albanese morphism has connected fibres. □

Notice that the assumption in the theorem 4.30 is satisfied when $-K_X$ is nef. In this case, all metrics are smooth and we do not need to take the blow up. Thus the above theorem can be seen as a generalisation of the result of [**Cao13**].

We remark that when $-K_X$ is psef and there exists a singular metric h on $-K_X$ such that $\mathcal{I}(h) = \mathcal{O}_X$, the surjectivity of the Albanese map is a direct consequence (Proposition 2.7.1 in [**DPS01**]) of the line bundle valued hard Lefschetz theorem in [**DPS01**]. For the convenience of the reader, we briefly recall the proof.

LEMMA 4.34. *Let (X, ω) be a compact Kähler manifold such that $-K_X$ is psef. Assume that there exists a singular metric h on $-K_X$ such that $\mathcal{I}(h) = \mathcal{O}_X$. Then the Albanese map is surjective.*

PROOF. By the hard Lefschetz theorem (main theorem in [**DPS01**]), we know that the morphism induced by taking the wedge product with ω

$$H^0(X, \Omega_X^{n-1} \otimes -K_X) \cong H^0(X, T_X) \to H^1(X, \mathcal{O}_X)$$

is surjective. Moreover, by the Hodge decomposition theorem, we have $H^1(X, \mathcal{O}_X) = \overline{H^0(X, \Omega_X^1)}$. For any $u \in H^0(X, \Omega_X^1)$, there exists a holomorphic vector field $\xi \in H^0(X, T_X)$ such that the image of ξ under the morphism induced by wedge product with ω is $\overline{u}$.

In particular, the inner product $i_\xi(u) \in H^0(X, \mathcal{O}_X)$ is a global holomorphic function. Thus $i_\xi(u)$ is constant. On the other hand $i_\xi(u) = |u|^2_\omega$ pointwise. Thus if $u \neq 0$, there exists some point x such that $|u|^2_\omega(x) \neq 0$. In other words, $i_\xi(u) \neq 0$. This implies that for any x, $u(x) \neq 0$, which implies in its turn that the Albanese morphism is surjective. □

The arguments of [**Cao13**] combined with theorem 4.30 also give the following affirmation of a conjecture of Mumford. The general conjecture of Mumford states that a projective or compact Kähler manifold X is rationally connected if and only if $H^0(X, (T_X^*)^{\otimes m}) = 0$ for any $m \geqslant 1$.

COROLLARY 4.35. *Let (X, ω) be a n-dimensional compact Kähler manifold such that $-K_X$ is psef. Assume that there exists a sequence $\varepsilon_\nu > 0$ such that $\lim_{\nu \to \infty} \varepsilon_\nu = 0$ and $\mathcal{I}(h_{\varepsilon_\nu}) = \mathcal{O}_X$ for a sequence of singular metrics with analytic singularities h_{ε_ν} on $-K_X$ such that $i\Theta(-K_X, h_{\varepsilon_\nu}) \geqslant -\varepsilon_\nu \omega$. Then the following properties are equivalent:*

(1) X is projective and rationally connected.

(2)$H^0(X, (T_X^*)^{\otimes m}) = 0$ for any $m \geqslant 1$.

(3) For every $m \geqslant 1$ and every finite étale cover $\tilde{X}$ of X, one has $H^0(\tilde{X}, \Omega_{\tilde{X}}^m) = 0$.

CHAPTER 5

Pseudo-effective and numerically flat reflexive sheaves

Abstract. In this note, we discuss the concept of strongly pseudoeffective vector bundle and also introduce strongly pseudoeffective torsion-free sheaves over compact Kähler manifolds. We show that a strongly pseudoeffective reflexive sheaf over a compact Kähler manifold with vanishing first Chern class is in fact a numerically flat vector bundle. A proof is obtained through a natural construction of positive currents representing the Segre classes of strongly pseudoeffective vector bundles.

5.1. Introduction

The concept of numerical flatness introduced in [**DPS94**] proved itself to be instrumental in the study and classification theory of compact Kähler manifolds with nef anticanonical bundles. It has been studied by many authors and in many works, cf. [**Cao18**], [**Cao19**], [**CH17**], [**CH19**], [**CCM19**], [**CP17**], [**HIM19**], [**HPS16**], [**Wang19**] among others.

Recall that a holomorphic vector bundle E is called numerically flat if both E and E^* are nef (equivalently if E and $(\det E)^{-1}$ are nef). In fact, the condition of being numerically flat yields strong restrictions for the curvature of the corresponding vector bundle. Actually, in [**DPS94**], Demailly, Peternell and Schneider proved that a numerically flat bundle E on a compact Kähler manifold X admits a filtration by vector bundles whose graded pieces are Hermitian flat. In some sense, numerical flatness is the algebraic analogue of metric flatness.

In [**CCM19**] and [**HIM19**], the authors consider the following question. If a strongly pseudo-effective vector bundle over a projective manifold has a vanishing first Chern class, is this vector bundle numerically flat? Since a vector bundle E is numerically flat if and only if E and $\det(E)^{-1}$ are nef, the question amounts to ask whether the vector bundle is in fact nef.

Intuitively, a positive singular metric on the vector bundle E would induce a positive singular metric on the determinant $\det(E)$. But since the first Chern class of E (i.e. the Chern class of $\det(E)$) is trivial, any metric with (semi)positive curvature must be flat and thus cannot possess any singularity. This implies that the given positive singular metric on E has to be smooth as well.

From this point of view, the same property should hold on an arbitrary compact Kähler manifold, and not just on projective manifolds, since all properties under consideration are independent of the projectivity condition. One of the goals of this work is to confirm this philosophy. Namely, we prove the following

Main Theorem. *Let E be a strongly psef vector bundle over a compact Kähler manifold (X, ω) with $c_1(E) = 0$. Then E is a nef vector bundle.*

The main technical tool is the construction of Segre currents. More precisely, we define a Segre (k, k)-closed positive current as the direct image of the wedge product of the curvature current of $\mathcal{O}_{\mathbb{P}(E)}(1)$, as soon as we have an appropriate codimension condition on the singular locus of the metric.

Main technical lemma. *Let E be a strongly psef vector bundle of rank r over a compact Kähler manifold (X, ω). Let $(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)$ be singular metric with analytic singularities such that*

$$i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) \geqslant -\varepsilon\pi^*\omega$$

and the codimension of $\pi(\mathrm{Sing}(h_\varepsilon))$ is at least k in X. Then there exists a (k, k)-positive current in the class $\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \varepsilon\pi^*\{\omega\})^{r+k-1}$.

The strategy of the proof of the Main theorem is as follows. We show that the Lelong numbers of the corresponding Segre current control the Lelong numbers of the weight functions of the singular metrics prescribed in the definition of a strongly pseudoeffective vector bundle. Then, we observe that the Lelong numbers of Segre currents must tend to 0 in the limit, as the unique (semi)positive current in $c_1(E)$ is the zero current. Thus the Lelong numbers of the weight functions uniformally tend to 0 as the Lelong numbers of the Segre currents. By Demailly's regularisation theorem, the weight functions of the metrics can be regularised, thus the vector bundle is actually nef.

In fact, we can expect an even stronger property. Since E is strongly psef, the class $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ is psef. Intuitively, $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ contains a not too singular current (in the sense that the projection of the singular part onto X is contained in some analytic subset of codimension at least 1). Thus the wedge powers of appropriate exponents of this current in the first Chern class are defined and positive, as well as their direct images under $\pi : \mathbb{P}(E) \to X$. In particular, if r is the rank of E, we can hope that the second Segre class $\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)))^{r+1}$ is positive (by this, we mean that its cohomology class contains a positive current)

Remind that the second Segre class is equal to $c_1(E)^2 - c_2(E)$. By the Bogomolov inequality if E is semistable, when $c_1(E) = 0$, the integration of $c_2(E) \wedge \omega^{n-2}$ on X is positive where ω is a Kähler form on X and n is the dimension of X. Comparing these two facts, one knows that $c_2(E) = 0$ and the Bogomolov inequality is in fact an equality.

For a reflexive sheaf $\mathcal{F}$, the Chern classes can be defined as follows. Let σ be any modification such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a vector bundle. The existence of such modification is provided by the fundamental work of [**Ros68**], [**GR70**] and [**Rie71**]. Then for $i = 1, 2$, $c_i(\mathcal{F}) = \sigma_* c_i(\sigma^*\mathcal{F}/\mathrm{Tors})$ which is independent of the choice of modification σ. The rough idea is that the above calculations should hold on some birational model of X, and by taking direct images, the equality in the Bogomolov inequality is also attained on X.

On the other hand, we have the following important result of [**BS94**]. For a polystable reflexive sheaf $\mathcal{F}$ of generic rank r over a compact n-dimensional Kähler manifold (X, ω), we have the Bogomolov inequality

$$\int_X (2rc_2(\mathcal{F}) - (r-1)c_1(\mathcal{F})^2) \wedge \omega^{n-2} \geqslant 0.$$

Moreover, the equality holds if and only if $\mathcal{F}$ is locally free and its Hermitian-Einstein metric yields a projectively flat connection.

In order to study the positivity of torsion free coherent sheaves, it is useful to define in full generality the nef (or strongly psef) property for such sheaves.

Definition. *A torsion free coherent sheaf $\mathcal{F}$ over a compact complex manifold is called nef (resp. strongly psef) if there exists some modification $\sigma : \tilde{X} \to X$ such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a nef (resp. strongly psef) vector bundle.*

The above considerations, combined with the result of [**BS94**], let us hope the stronger fact that over every compact Kähler manifold (X, ω), a strongly psef reflexive sheaf with trivial first Chern class is in fact a nef vector bundle. In section 5, we prove that this is actually the case. A difficulty of the above approach is that in general a wedge product of positive currents is not necessarily well defined. Instead of proceeding directly, we first prove the following result.

Lemma. *Let $\mathcal{F}$ be a nef reflexive sheaf over a compact Kähler manifold (X, ω) with $c_1(\mathcal{F}) = 0$. Then $\mathcal{F}$ is a nef vector bundle.*

Now combining the main theorem, we can conclude that

Corollary. *Let $\mathcal{F}$ be a strongly psef reflexive sheaf over a compact Kähler manifold (X, ω) with $c_1(\mathcal{F}) = 0$. Then $\mathcal{F}$ is a nef vector bundle.*

Note that in the above approach we have to take wedge products that are well defined without imposing any restriction on the codimension of singular part of the metric. In this situation, for a strongly psef vector bundle E, we can find a positive current in $c_1(E)$ but not necessarily in $c_2(E)$.

At the end of the paper, as a geometric application, we classify compact Kähler surfaces and 3-folds with strongly psef tangent bundles and with vanishing first Chern class. By our Main theorem, they are the same as compact Kähler surfaces or 3-folds with nef tangent bundles and with zero first Chern class, that were classified in [**DPS94**]. As a consequence, the tangent bundle of a Kähler K3 surface is not strongly psef. This generalise the work of [**DPS94**] and [**Nak04**] in the projective setting. More generally, an irreducible symplectic, or Calabi-Yau manifold does not possess a strongly psef tangent bundle or cotangent bundle. In the singular and projective setting, the "strongly psef" version is proven in Theorem 1.6 of [**HP19**] and Corollary 6.5 [**Dru18**] for threefolds. (They even prove in this case that the bundle is not weakly psef, i.e. that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not a psef line bundle whenever E is the tangent or cotangent bundle.)

We also generalise the main results to the $\mathbb{Q}-$twisted case analogous to the result of [**LOY20**] in the compact Kähler setting.

The organisation of this paper is as follows. In section 2, the concept of strongly psef vector bundles is discussed. We give a definition of strongly psef vector bundle of the Kähler version essentially equivalent to the one proposed in [**BDPP13**]. By this equivalent condition, we can show that some usual algebraic operations can still be taken for strongly psef vector bundles. For example, the direct sum or tensor product of strongly psef vector bundles is still strongly psef. In section 3, we investigate the concept of nef/strongly psef torsion free coherent sheaves and algebraic operations of these sheaves. Then we show that a numerically flat reflexive sheaf on an arbitrary compact Kähler manifold is in fact a vector bundle. This result can also be generalised to strongly pseudoeffective (strongly psef) reflexive sheaves $\mathcal{F}$ such that $c_1(\det \mathcal{F}) = 0$ in section 5. In section 4, we make a digression to introduce the definition of Segre forms (or Segre currents), as a tool to treat the strongly psef case. It should be observed that a similar construction has been done in [**LRRS18**].

In this note, all manifolds are supposed to be compact without any explicit mention.

5.2. Strongly pseudoeffective vector bundles

The following definition of a strongly psef vector bundle is a reformulation of the definition of [**BDPP13**] (Definition 7.1).

DEFINITION 5.1. *Let (X,ω) be a compact Kähler manifold and E a holomorphic vector bundle on X. Then E is said to be strongly pseudo-effective (strongly psef for short) if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective on the projectivized bundle $\mathbb{P}(E)$ of hyperplanes of E, i.e. if for every $\varepsilon > 0$ there exists a singular metric h_ε with analytic singularities on $\mathcal{O}_{\mathbb{P}(E)}(1)$ and a curvature current $i\Theta(h_\varepsilon) \geqslant -\varepsilon\pi^*\omega$, and if the projection $\pi(\mathrm{Sing}(h_\varepsilon))$ of the singular set of h_ε is not equal to X.*

One can observe that in [**BDPP13**] the definition is expressed rather in terms of the non-nef locus.

DEFINITION 5.2. *([**DPS01**]) Let φ_1, φ_2 be two quasi-psh functions on X (i.e. $i\partial\overline{\partial}\varphi_i \geqslant -C\omega$ in the sense of currents for some $C \geqslant 0$). Then, φ_1 is said to be less singular than φ_2 (we write $\varphi_1 \preceq \varphi_2$) if we have $\varphi_2 \leqslant \varphi_1 + C_1$ for some constant C_1. Let α be a psef class in $H^{1,1}_{BC}(X,\mathbb{R})$ and γ be a smooth real $(1,1)$-form. Let $T_1, T_2, \theta \in \alpha$ with θ smooth and $T_i = \theta + i\partial\overline{\partial}\varphi_i$ $(i = 1, 2)$, the potential φ_i being defined up to a constant since X is compact. We say that $T_1 \preceq T_2$, resp. singularity equivalent $T_1 \sim T_2$, if $\varphi_1 \preceq \varphi_2$, resp. if $\varphi_1 \preceq \varphi_2$ and $\varphi_2 \preceq \varphi_1$.*

A minimal element $T_{\min,\gamma}$ with respect to the pre-order relation $\preceq$ always exists. Such an element can be obtained by taking the upper semi-continuous upper envelope of all φ_i such that $\theta + i\partial\overline{\partial}\varphi_i \geqslant \gamma$ and $\sup_X \varphi_i = 0$. It is unique up to equivalence of singularities.

DEFINITION 5.3. *(Non-nef locus)*

The non-nef locus of a pseudo-effective class $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ is defined to be

$$E_{\mathrm{nn}}(\alpha) := \bigcup_{\varepsilon>0} \bigcup_{c>0} E_c(T_{\min,-\varepsilon\omega})$$

where ω is any Hermitian metric.

Let us observe that we can replace $\pi^*\omega$ by any smooth Kähler form $\tilde{\omega}$ on $\mathbb{P}(E)$ in the definition of a strongly psef vector bundle. The reason is as follows. On the one hand, $\pi^*\omega \leqslant C\tilde{\omega}$ for some $C > 0$ since X is compact. Thus, $i\Theta(h_\varepsilon) \geqslant -\varepsilon\pi^*\omega$ implies that $i\Theta(h_\varepsilon) \geqslant -C\varepsilon\tilde{\omega}$. On the other hand, since $\mathcal{O}_{\mathbb{P}(E)}(1)$ is relatively π-ample, we have $\varepsilon_0 i\Theta_{h_0}(\mathcal{O}_{\mathbb{P}(E)}(1)) + \pi^*\omega \geqslant \varepsilon_1\tilde{\omega}$ for any given smooth Hermitian metric h_0 on E, if $0 < \varepsilon_1 \ll \varepsilon_0 \ll 1$ are small enough. Assuming that there exists a singular metric h_ε on $\mathcal{O}_{\mathbb{P}(E)}(1)$ such that $i\Theta_{h_\varepsilon}(\mathcal{O}_{\mathbb{P}(E)}(1)) \geqslant -\varepsilon\tilde{\omega}$, we infer that the metric $h'_\varepsilon = h_0^{\varepsilon/\varepsilon_1} h_\varepsilon^{1-\varepsilon/\varepsilon_1}$ has a curvature lower bound

$$i\Theta_{h'_\varepsilon}(\mathcal{O}_{\mathbb{P}(E)}(1)) \geqslant \frac{\varepsilon}{\varepsilon_1}(\varepsilon_1\tilde{\omega} - \pi^*\omega) - \Big(1 - \frac{\varepsilon}{\varepsilon_1}\Big)\varepsilon\tilde{\omega} \geqslant -\frac{\varepsilon}{\varepsilon_1}\pi^*\omega.$$

In [**BDPP13**], a holomorphic vector bundle E was defined to be strongly pseudo-effective if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective on the projectivized bundle $\mathbb{P}(E)$ of hyperplanes of E, and if the projection $\pi(E_{\mathrm{nn}}(\mathcal{O}_{\mathbb{P}(E)}(1)))$ of the non-nef locus of $\mathcal{O}_{\mathbb{P}(E)}(1)$ onto X does not cover all of X. By definition,

$$E_{\mathrm{nn}}(c_1(\mathcal{O}_{\mathbb{P}(E)}(1))) \subset \bigcup_{\varepsilon>0} \mathrm{Sing}(T_{\min,-\varepsilon\tilde{\omega}}) \subset \bigcup_{\varepsilon>0} \mathrm{Sing}(h_\varepsilon).$$

Hence a strongly psef vector bundle defined in Definition 5.1 is strongly psef under the definition of [**BDPP13**]. On the other hand, by the regularization theorem, we can construct from $T_{\min,-\varepsilon\tilde{\omega}}$ a metric $h_{2\varepsilon}$ on $\mathcal{O}_{\mathbb{P}(E)}(1)$ with $i\Theta(h_{2\varepsilon}) \geqslant -2\varepsilon\tilde{\omega}$. By definition, $\mathrm{Sing}(h_{2\varepsilon}) \subset \bigcup_{c>0} E_c(T_{\min,-2\varepsilon\tilde{\omega}})$ thus it does not project onto X. Hence our definition is equivalent to the definition of [**BDPP13**].

We remark that the definition of strongly psef vector bundle we used is stronger than the widely used weak definition. A vector bundle E is called psef in the weak sense if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a psef line bundle over $\mathbb{P}(E)$. Of course, our definition of strongly psef vector bundle coincide with the widely used weak definition in the case of line bundle. However, this weak definition is too weak to give a classification even if we pose some strong topological obstruction like with vanishing first Chern class. For example, if X is a projective manifold and A is an ample line bundle over X, for any $p \neq 0$, $A^p \oplus (A^p)^*$ is a psef vector bundle in the weak sense with vanishing first Chern class. Intuitively, a psef vector bundle can have negative curvature in some direction which is not enough for our propose to construct some positive current in the first Chern class of the determinant bundle.

It should be noticed that pseudo-effectiveness in the weak sense is a Zariski closed condition while strong pseudo-effectiveness is not Zariski closed. More precisely, let $p : \mathfrak{X} \to \Delta$ be a proper holomorphic submersion which defines a family of compact Kähler manifolds over the unit disc Δ and E be a holomorphic vector bundle over $\mathfrak{X}$. Then the set $t \in \Delta$ such that the restriction $E|_{X_t}$ is a psef vector bundle over X_t in the weak sense is a Zariski closed set where $X_t := p^{-1}(t)$. A complete proof can be found e.g. in the appendix of [**AH19**] by Simone Diverio. However, the same does not hold for strong pseudo-effectiveness. For example, we can take the following example indicated to the author by Jean-Pierre Demailly.

EXAMPLE 5.4. (Theorem 2.2.5 [**OSS80**])

Let $x_1, \cdots, x_m$ be the points of the projective plane $\mathbb{P}^2$. There is a holomorphic rank 2 bundle E over $\mathbb{P}^2$ whose restriction to any line L, on which exactly a points of the set $\{x_1, \cdots, x_m\}$ lie, splits in the form

$$E|_L = \mathcal{O}_L(a) \oplus \mathcal{O}_L(-a).$$

The generic splitting type of this bundle is $(0, 0)$.

The construction of the vector bundle is as follows. Let $\sigma : Y \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ over $\{x_1, \cdots, x_m\}$ with exceptional divisor $C = \sum_{i=1}^m C_i$. Let E' be a rank two vector bundle over Y such that it satisfies the extension

$$0 \to \mathcal{O}_Y(C) \to E' \to \mathcal{O}_Y(-C) \to 0$$

and its restriction to each C_i satisfies the Euler sequence

$$0 \to \mathcal{O}_{C_i}(-1) \to E'|_{C_i} \cong \mathcal{O}_{C_i}^{\oplus 2} \to \mathcal{O}_{C_i}(1) \to 0.$$

It can be proved that E' is the pull back of some vector bundle E over $\mathbb{P}^2$. We have the short exact sequence

$$0 \to \mathcal{O}_{\tilde{L}}(a) \to E'|_{\tilde{L}} \to \mathcal{O}_{\tilde{L}}(-a) \to 0.$$

where a is the number of $\{x_1, \cdots, x_m\}$ which lie in L. The short exact sequence splits since $H^1(\tilde{L}, \mathcal{O}_{\tilde{L}}(2a)) = 0$. The blow up induces a biholomorphism between the strict transform of a line $\tilde{L}$ to L which gives the conclusion.

Thus we can construct a family of vector bundles whose restriction to some special fibers is not strongly psef although the restriction to the general fiber is strongly psef (in fact trivial). The lines in the projective plane form a family of $\mathbb{P}^1$ over the Grassmannian $Gr(2, 3)$. The total space $\mathfrak{X}$ is a closed submanifold of $\mathbb{P}^2 \times Gr(2, 3)$. Consider the vector bundle which is the restriction over $\mathfrak{X}$ of the pull back of the previous constructed bundle under $p_1 : \mathbb{P}^2 \times Gr(2, 3) \to \mathbb{P}^2$.

A related definitions in the projective case is also widely used in the literature, which is weak positivity in the sense of Nakayama (cf. eg. [**Nak04**] Definition 3.20). A torsion free coherent sheaf $\mathcal{F}$ is weakly positive at $x \in X$ a projective manifold if, for any $a \in \mathbb{N}^*$ and for any ample line bundle A on X, there exists $b \in \mathbb{N}^*$ such that $(\mathrm{Sym}^{ab}\mathcal{F})^{\vee\vee} \otimes A^b$ is globally generated at x, where $(\mathrm{Sym}^{ab}\mathcal{F})^{\vee\vee}$ is the double dual of ab-th symmetric power of $\mathcal{F}$. A torsion free coherent sheaf is called weak positive in the sense of Nakayama if it is weak positive at some point. It is proven in Proposition 7.2 [**BDPP13**] that for a vector bundle E over a projective manifold X, E is psef in our strong sense if and only if E is weak positive in the sense of Nakayama.

Now we give still another equivalent definition of a strongly psef vector bundle. The argument is analogous to the one of [**Dem92a**, theorem 4.1] in the singular setting. Intuitively, being strongly psef is equivalent to the existence of “algebraic" approximation currents. Here “algebraic" means that the approximation can be obtained from the sections of higher degree tensor product of the vector bundle. (Of course the sections are local since the global sections on X does not necessarily exist.) We construct approximating metrics by use of a Bergman kernel technique and use a Hörmander type L^2 estimate to get the required curvature estimates. For the convenience of the reader, we recall the basic L^2 estimate that we need.

LEMMA 5.5. (Corollary 5.3 in [**Dem12a**])

Let (X, ω) be a Kähler manifold, dim $X = n$. Assume that X is weakly pseudo-convex (in particular it is the case for any compact Kähler manifold). Let F be a holomorphic line bundle equipped with a degenerate metric whose local weights are denoted $\varphi \in L^1_{loc}$, i.e. $H = e^{-\varphi}$. Suppose that

$$i\Theta_{F,h} = \frac{i}{\pi}\partial\overline{\partial}\varphi \geqslant \varepsilon\omega$$

in the sense of currents for some $\varepsilon > 0$. Then for any form $g \in L^2(X, \Lambda^{n,q}T_X^* \otimes F)$ satisfying $\overline{\partial}g = 0$, there exists $f \in L^2(X, \Lambda^{n-1,q}T_X^* \otimes F)$ such that $\overline{\partial}f = g$ and

$$\int_X |f|^2 e^{-\varphi} dV_\omega \leqslant \frac{1}{q\varepsilon}\int_X |g|^2 e^{-\varphi} dV_\omega.$$

We will also need the following lemma stated by Demailly to glue the local weights into a global one, via a partition of unity.

LEMMA 5.6. (Lemma 13.11 in [**Dem12a**])

Let $U'_j \subset\subset U''_j$ be locally finite open coverings of a (not necessarily compact) complex manifold X by relatively compact open sets, and let θ_j be smooth non-negative functions with support in U''_j, such that $\theta_j \leqslant 1$ on U''_j and $\theta_j = 1$ on U'_j. Let $A_j \geqslant 0$ be such that

$$i(\theta_j \partial\overline{\partial}\theta_j - \partial\theta_j \wedge \overline{\partial}\theta_j) \geqslant -A_j \omega$$

on $U''_j \setminus U'_j$ for some positive (1,1)-form ω. Finally, let w_j be almost psh functions on U_j with the property that $i\partial\overline{\partial} w_j \geqslant \gamma$ for some real(1,1)-form γ on M, and let C_j be constants such that

$$w_j(x) \leqslant C_j + \sup_{k \neq j, x \in U'_k} w_k(x)$$

on $U''_j \setminus U'_j$.

Then the function $w := \log(\sum \theta_j^2 e^{w_j})$ is almost psh and satisfies

$$i\partial\overline{\partial} w \geqslant \gamma - 2(\sum_j \mathbb{1}_{U''_j \setminus U'_j} A_j e^{C_j})\omega.$$

PROPOSITION 5.2.1. *The following properties are equivalent:*

(1) E is strongly psef

(2) There exists a sequence of quasi-psh functions $w_m(x,\xi) = \log(|\xi|_{h_m})$ with analytic singularities induced from Hermitian metrics h_m on $S^m E^*$ such that the singularity locus projects into a proper Zariski closed set Z_m, and

$$i\partial\overline{\partial} w_m \geqslant -m\varepsilon_m p^* \omega$$

in the sense of currents with $\lim \varepsilon_m = 0$. Here $p : S^m E^* \to X$ is the projection.

(3) There exists a sequence of quasi-psh functions $w_m(x,\xi) = \log(|\xi|_{h_m})$ with analytic singularities induced from Hermitian metrics h_m on $S^m E^*$, such that the singularity locus projects into a proper Zariski closed set Z_m, and

$$i\Theta_{S^m E^*, h_m} \leqslant m\varepsilon_m \omega \otimes \mathrm{Id}$$

on $X \setminus Z_m$ in the sense of Griffiths with $\lim \varepsilon_m = 0$.

PROOF. Note that when a metric over F a vector bundle over X is smooth near a point x, we have the following equivalence (cf. Lemma 4.4 in [**Dem92a**]): for any real $(1,1)$ form γ near x, over a neighbourhood U near x

1. $i\Theta(F) \geqslant \gamma \otimes \mathrm{Id}_F$ in the sense of Griffiths;
2. $-i\Theta(F^*) \geqslant \gamma \otimes \mathrm{Id}_F$ in the sense of Griffiths;
3. $\frac{i}{2\pi}\partial\overline{\partial}\log|\xi|^2 \geqslant p^*\gamma,\ \ \xi \in F^*$, where $\log|\xi|^2$ is seen as a function on $p^{-1}(U)$ and $p : F^* \to X$ is the projection.

In particular, (2) implies (3) by this observation.

The more substantial part of the proof consists of showing that (1) implies (2). The proof follows closely the proof of theorem 4.1 in [**Dem92a**].

It is enough to show that for any $\varepsilon > 0$, there exists a sequence of quasi-psh functions $w_m(x,\xi) = \log(|\xi|_{h_m})$ with analytic singularities induced from Hermitian metrics h_m on $S^m E^*$, such that the singularity locus projects into a proper Zariski closed set Z_m, and

$$i\partial\overline{\partial} w_m \geqslant -m\varepsilon p^* \omega$$

in the sense of currents. Here $p : S^m E^* \to X$ is the projection.

We construct the metrics on the symmetric powers of vector bundles, starting from a singular metric h_ε on $\mathcal{O}_{\mathbb{P}(E)}(1)$ given in the definition of strongly psef vector bundle. Namely, we start with a singular metric such that the singularity locus projects into a proper Zariski closed set Z, and

$$\frac{i}{2\pi}\Theta_{\mathcal{O}_{\mathbb{P}(E)}(1)} \geqslant -\varepsilon \pi^* \omega.$$

Since X is compact, we can select a finite covering (W_ν) of X with open coordinate charts. For any $\delta > 0$, we take in each W_ν a maximal family of points with (coordinate) distance to the boundary $> 3\delta$ and mutual distance $> \delta/2$. In this way, we get for any $\delta > 0$ small enough a finite covering of X by open balls U'_j of radius δ (actually every point is even at distance $\leqslant \delta/2$ of one of the centres, otherwise the family of points would not be maximal), such that the concentric ball U_j of radius 2δ is relatively compact in the corresponding chart W_ν.

Let $\tau_j : U_j \to B(a_j, 2\delta)$ be the isomorphism given by the coordinates of W_ν. Let $\varepsilon(\delta)$ be a modulus of continuity for $\gamma := -\varepsilon\omega$ on the sets U_j, such that $\lim_{\delta\to 0}\varepsilon(\delta) = 0$ and $\omega_x - \omega_{x'} \leqslant \varepsilon(\delta)\omega_x$ for all $x, x' \in U_j$. We denote by γ_j the (1,1)-form with constant coefficients on $B(a_j, 2\delta)$ such that $\tau_j^*\gamma_j$ coincides with $\gamma - \varepsilon(\delta)\omega$ at $\tau_j^{-1}(a_j)$. Then we have

$$0 \leqslant \gamma - \tau_j^*\gamma_j \leqslant 2\varepsilon(\delta)\omega \tag{1}$$

on U_j' for $\delta > 0$ small enough. Let $\tilde{v}_j(z_j)$ be the associated quadratic function such that $\gamma_j = \frac{i}{\pi}\partial\overline{\partial}\tilde{v}_j$.

Now, we consider the Hilbert space $\mathcal{H}_j(m)$ of holomorphic sections $f \in H^0(\pi^{-1}(U_j), \mathcal{O}_{\mathbb{P}(E)}(m))$ with the L^2 norm

$$\|f\|_j^2 := \int_{\pi^{-1}(U_j)} |f|^2 e^{2m\tilde{v}_j(z_j)} dV,$$

where dV is a volume element on $\mathbb{P}(E)$ (fixed once for all) and $|f|^2$ is the pointwise norm on $\mathcal{O}_{\mathbb{P}(E)}(m)$ induced by the given (singular) Hermitian metric h_ε on $\mathcal{O}_{\mathbb{P}(E)}(1)$. It can be viewed as a metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$, twisted by the local weight $\tilde{v}_j$. Thus the corresponding curvature form is

$$\frac{i}{2\pi}\Theta_{\mathcal{O}_{\mathbb{P}(E)}(1)} - \frac{i}{\pi}\partial\overline{\partial}\tilde{v}_j \geqslant \pi^*(\gamma - \tau_j^*\gamma_j) \geqslant 0$$

by (1). Let $U_j' \Subset U_j'' \Subset U_j$ be concentric balls such that (U_j') still cover X and let θ_j be smooth functions with support in U_j'', such that $0 \leqslant \theta_j \leqslant 1$ on U_j'' and $\theta_j = 1$ on U_j'.

We define a Bergman kernel type metric on $S^m E^*$ as follows: for all $x \in X$ and $\xi \in S^m E_x^*$ we set

$$\|\xi\|_{(m)}^2 := \sum_j \theta_j^2(x)\exp(2m\tilde{v}_j(z_j) + \sqrt{m}(r_j'^2 - |z_j|^2))\sum_l |\sigma_{j,l}(x)\cdot\xi|^2, \tag{2}$$

where r_j' is the radius of U_j' and $(\sigma_{j,l})_{l\geqslant 1}$ is an orthonormal basis of $\mathcal{H}_j(m)$. The local sections $\sigma_{j,l}$ can be viewed as sections in $H^0(U_j, S^m E)$, and here $\sigma_{j,l}(x)\cdot\xi$ is computed via the natural pairing between $S^m E$ and $S^m E^*$. The metric is Hermitian since it is a sum of square of linear forms in $S^m E^*$. Since the metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ can be singular, the Hermitian metric can also be degenerate. It is degenerate at a point x if $\sigma_{j,l}(x) = 0$ for all j, l.

However, the infinite sum $\sum_l |\sigma_{j,l}(x)\cdot\xi|^2$ is smooth. In fact, the sum converges locally uniformly above every compact subset of U_j. This sum is the square of evaluation linear form

$$f \mapsto f(x)\cdot\xi$$

which is continuous on $\mathcal{H}_j(m)$. The reason is as follows. Given σ an element of $H^0(U_j, S^m E)$. It can be identified as an element of $H^0(\pi^{-1}(U_j), \mathcal{O}_{\mathbb{P}(E)}(m)) \cong H^0(U_j, S^m E)$ by considering the quotient of $\pi^*\sigma \in H^0(\pi^{-1}(U_j), \pi^* S^m E)$ under the tautological map $\pi^* S^m E \to \mathcal{O}_{\mathbb{P}(E)}(m)$. On the other hand, $\xi \in S^m E_x^*$ can be pulled back to $\mathbb{P}(E)$ as an element of $\mathcal{O}_{\mathbb{P}(E)}(-m)_{x,[\xi]} \subset \pi^* S^m E^*_{x,[\xi]}$. The natural pairing between $S^m E^*$ and $S^m E$ of $f(x)$ and ξ is equal to the natural pairing between $\mathcal{O}_{\mathbb{P}(E)}(-m)_{x,[\xi]}$ and $\mathcal{O}_{\mathbb{P}(E)}(m)_{x,[\xi]}$ under the above identification. In particular,

$$|f(x)\cdot\xi| \leqslant |f|(x,[\xi])|\xi|(x,[\xi])$$

Here we identify $\mathcal{O}_{\mathbb{P}(E)}(m)_{x,[\xi]}$ as $\mathbb{C}$ under any local trivialization near $(x,[\xi])$. The supremum of $|f|(x,[\xi])$ for $f \in \mathcal{H}_j(m)$, $\|f\| \leqslant 1$ is by definition the norm of the continuous linear function $f \mapsto f(x)$ under the chosen local trivialization near $(x,[\xi])$. (Remark that in the trivialization, by mean value inequality, the value of the holomoprhic function at the center of a ball is bounded from above by the L^2 norm of the function on the ball which is bounded from above by the L^2 norm of the section on $\mathbb{P}(E)$ with the singular weight.) Thus $f \mapsto f(x)\cdot\xi$ is a continuous linear function. The square of its norm is $\sum_l |\sigma_{j,l}(x)\cdot\xi|^2$ since $\sigma_{j,l}(x)\cdot\xi$ is the l-th coordinate in the orthonormal basis $\sigma_{j,l}$ of $\mathcal{H}_j(m)$. By Montel's theorem, $\sum_{l,k}\sigma_{j,l}(x)\cdot\xi\,\overline{\sigma_{j,l}(w)\cdot\eta}$ is a holomorphic function for $(x, w, \xi, \eta) \in U_j \times \overline{U_j} \times E \times \overline{E}$. Thus its restriction $\sum_l |\sigma_{j,l}(x)\cdot\xi|^2$ to the diagonal $U_j \times E$ is a real analytic function.

As a consequence, the metric $\|\cdot\|_{(m)}$ is a smooth metric, except for the fact that it might degenerate at some points. To show that this metric has analytic singularities and obtain the curvature estimate, we use lemma 5.6 for $w(x,\xi) := \log\|\xi\|_{(m)}^2$ and

$$w_j(x,\xi) = 2m\tilde{v}_j(z^j) + \sqrt{m}(r'^2 - |z^j|^2) + \log\sum_l |\sigma_{j,l}(x)\cdot\xi|^2 \tag{3}$$

on the total space $S^m E^*$ covered by $p^{-1}(U_j')$ where $p : S^m E^* \to X$ is the projection.

To proceed further, we need the following lemma 5.7 to compare the behaviour of w_j on different open sets. As a consequence of lemma 5.7, the functions $w_j(x,\xi)$ satisfy $w_j(x,\xi) \leqslant w_k(x,\xi)$ for any $x \in (U_j'' \setminus U_j') \cap U_k'$ for m large enough. (Remark that $r_j'^2 - |z^j|^2 \leqslant 0$ and $r_k'^2 - |z^k|^2 > 0$ for such x.) The choice of m depends on the value $r_k'^2 - |z^k|^2 > 0$. But the function on $U_j'' \smallsetminus U_j'$, $\sup_{k\neq j, x\in U_k'} |a_k - x|$ has a uniform strictly positive lower bound since $U_j'' \smallsetminus U_j'$ is compact. Thus there exists m_0 such that for $m \geqslant m_0$ we have

$$w_j(x) \leqslant \sup_{k\neq j, x\in U_k'} w_k(x)$$

on $U_j'' \smallsetminus U_j'$. We have a curvature estimate

$$\frac{i}{2\pi}\partial\overline{\partial} w_j \geqslant mp^*v_j - \sqrt{m}\frac{i}{2\pi}\partial\overline{\partial}|z^j|^2 \geqslant mp^*(\gamma - 3\varepsilon\omega)$$

in the sense of currents, since $\gamma_j \geqslant \gamma - 2\varepsilon\omega$ for $m \geqslant m_0' \geqslant m_0$ large enough (independent of x). Then lemma 5.6 implies that

$$\frac{i}{2\pi}\partial\overline{\partial} w \geqslant mp^*(\gamma - 3\varepsilon\omega) - p^*\left(2\sum_j \mathbb{1}_{U_j''\smallsetminus U_j'} A_j\omega\right).$$

The right side hand is bigger than $mp^*(\gamma - 4\varepsilon\omega)$ for $m \geqslant m_0'' \geqslant m_0'$.

We observe that the metric has analytic singularities. By the following lemma 5.7, there exist constants $C_{j,k}, C_{j,k}'$ such that

$$w_j - C_{j,k}' \leqslant w_k \leqslant w_j + C_{j,k}.$$

Note that w_j can be $-\infty$ at some point. Thus we have

$$\log\left(\sum_j \theta_j^2 e^{C_{j,k}'} e^{w_k}\right) \leqslant w = \log\left(\sum_j \theta_j^2 e^{w_j}\right) \leqslant \log\left(\sum_j \theta_j^2 e^{C_{j,k}} e^{w_k}\right).$$

Without loss of generality, we can assume that θ_j is a partition of unity, and in particular that $\sum_j \theta_j^2$ is strictly positive on any relative compact set. Thus $w = w_k + O(1)$ which implies w has analytic singularities along with w_k.

Now we show that (2) implies (1). The sequence of metrics in (2) induces a sequence of Hermitian metrics on $\mathcal{O}(1)$ over $\mathbb{P}(S^mE)$. Observe that we have the following commutative diagram given by the Veronese embedding

$$\begin{array}{ccc} \mathcal{O}(m) & \longrightarrow & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ \mathbb{P}(E) & \xrightarrow{i} & \mathbb{P}(S^mE). \end{array}$$

Since the metric is smooth over the pre-image of a dense Zariski open set of X. The restriction of singular metrics is well defined and still has analytic singularities. Define a sequence of metrics on $\mathcal{O}_{\mathbb{P}(E)}(1)$ induced from the restricted metrics. This sequence of metrics is the one required in the definition of a strongly psef vector bundle.

The arguments needed to show that (3) implies (2) are similar. By the observation made at the beginning of the proposition, the inequality holds on a dense Zariski open set V where the metric is smooth. The Skoda-El Mir extension theorem implies that $\mathbb{1}_V i\partial\overline{\partial} w_m \geqslant -m\varepsilon_m p^*\omega$. Since w_m has analytic singularities, the current $i\partial\overline{\partial} w_m$ is normal, and by the support theorem $\mathbb{1}_{S^mE^*\smallsetminus V} i\partial\overline{\partial} w_m$ is a sum of closed positive currents obtained by integration on analytic sets with positive coefficients. Thus the same inequality holds for $i\partial\overline{\partial} w_m = \mathbb{1}_V i\partial\overline{\partial} w_m + \mathbb{1}_{S^mE^*\smallsetminus V} i\partial\overline{\partial} w_m$. □

LEMMA 5.7. *There exist constants $C_{j,k}$ independent of m such that the almost psh functions*

$$\tilde{w}_j(x,\xi) := 2mv_j(z_j) + \log\sum_l |\sigma_{j,l}(x)\cdot\xi|^2, (x,\xi) \in p^{-1}(U_j'') \subset S^mE^*$$

satisfy on $p^{-1}(U_j'' \cap U_k'')$ a bound

$$\tilde{w}_j \leqslant \tilde{w}_k + (2n+2)\log m + C_{j,k}.$$

PROOF. By construction $E|_{U_j} \cong U_j \times \mathbb{C}^r$ is trivial over U_j. Define a Hermitian metric h_∞ on $E|_{U_j}$ with strict positive curvature by taking

$$|\xi|^2 := \sum_\lambda |\xi_\lambda|^2 e^{-\sum_j |z^j|^2}.$$

The associated curvature form on $(\mathcal{O}_{\mathbb{P}(E|_{U_j})}(1), h_\infty)$ is strictly positive and thus defines a Kähler metric ω_j on $\pi^{-1}(U_j)$. In fact, $\Theta_E = \omega_{\mathrm{eucl}} \otimes \mathrm{Id}_E$ where ω_{eucl} is the standard (flat) Hermitian metric on U_j. By a standard formula (cf. formula (15.15) in Chap V of [**Dem12b**]), the curvature of $(\mathcal{O}_{\mathbb{P}(E|_{U_j})}(1), h_\infty)$ is equal to the direct sum of the Euclidean metric of U_j and of the Fubini-Study metric of $\mathbb{P}^{r-1}$. In particular, the Ricci curvature of ω_j is non-negative. Define $\tau(z) := n\log|z^j - z^j(x)|$ depending only on the base variables and possessing a logarithmic pole at x. This is a psh function on a neighbourhood of $\pi^{-1}(U_j)$. Define a singular metric on $\mathcal{O}_{\mathbb{P}(E|_{U_j})}(m)$ as follows. Twist the metric $h_\varepsilon^{\otimes(m-1)} \otimes h_\infty$ by $(m-1)\tilde{v}_j(z^j) + \tau(z^j)$. The resulting curvature form on $\mathcal{O}_{\mathbb{P}(E|_{U_j})}(m)$ is given by

$$(m-1)\Big(\frac{i}{2\pi}\Theta_{\mathcal{O}_{\mathbb{P}(E)}(1)}(h_\varepsilon) - \frac{i}{\pi}\partial\overline{\partial}\tilde{v}_j\Big) + \omega_j + \frac{i}{\pi}\partial\overline{\partial}\tau \geqslant \omega_j$$

by (1). We consider the Hilbert space $F_j^{0,q}(m)$ of $(0,q)$-forms $(q = 0,1)$ f on $\pi^{-1}(U_j)$ with values in $\mathcal{O}_{\mathbb{P}(E)}(m)$, equipped with the L^2 norm $\|f\|_{j,q}^2 = \int_{\pi^{-1}(U_j)} |f|_j^2 dV_j$, where $dV_j = \omega_j^{n+r-1}/(n+r-1)!$ and where the pointwise norm $|f|_j$ is induced by ω_j and of the metric defined above on $\mathcal{O}_{\mathbb{P}(E)}(m)$.

Now, we apply Hörmander's L^2 estimates for the bundle $-K_X + \mathcal{O}_{\mathbb{P}(E)}(m)$ and an arbitrary $(0,1)$ form g in $F_j^{0,1}(m)$ with $\overline{\partial} g = 0$, (i.e. a $\overline{\partial}$-closed L^2 $(n,1)$-form valued in $-K_X + \mathcal{O}_{\mathbb{P}(E)}(m)$). We conclude that there exists a $(0,0)$-form in $F_j^{0,0}(m)$ such that $\overline{\partial} f = g$ and $\|f\|_{j,0} \leqslant \|g\|_{j,1}$. (Note that $\mathrm{Ric}(\omega_j) \geqslant 0$.)

It remains to choose a suitable section g to prove the inequality. Fix a point $x \in U_j'' \cap U_k''$ and $\xi \in S^m E_x^*$. There exists $h \in \mathcal{H}_k(m)$ with $\|h\|_k = 1$ such that

$$|h(x)\cdot\xi|^2 = \sum_l |\sigma_{k,l}(x)\cdot\xi|^2.$$

If the right rank side is 0, we can take h to be any element in the orthonormal basis. Otherwise, the linear functional $f \mapsto f(x)\cdot\xi$ is a non zero functional whose kernel defines a closed hypersurface in $\mathcal{H}_k(m)$. Thus there exists $h \in \mathcal{H}_k(m)$ with $\|h\|_k = 1$ which is orthogonal to the kernel. It is easy to see that such a point h is a maximum of the function $\mathcal{H}_k(m) \smallsetminus 0 \to \mathbb{R}$:

$$v \mapsto \frac{|v\cdot\xi|}{\|v\|^2},$$

and hence we have the equality. Let χ be a cut-off function with support in the (coordinate) ball $B(x,1/m)$, equal to 1 on $B(x,1/2m)$ and with $|\partial\chi| \leqslant m$. For $m \geqslant m_0$ large enough (independent of $x \in U_j'' \cap U_k''$) we have $B(x,1/m) \subset U_j \cap U_k$. We consider the solution of the equation $\overline{\partial} f = h\overline{\partial}(\chi\circ\pi)$ on $\pi^{-1}(U_j)$. We then get a holomorphic section

$$h' := h(\chi\circ\pi) - f \in H^0(\pi^{-1}(U_j), \mathcal{O}_{\mathbb{P}(E)}(m)).$$

The section h' coincide with h over $\pi^{-1}(x)$, since the Lelong number of the local weight at a point in $\pi^{-1}(x)$ is at least that of the local weight of τ which is n. The fact that the section f is in L^2 implies that it has to vanish along $\pi^{-1}(x)$. On the other hand, we have

$$\begin{aligned}
\|h\overline{\partial}(\chi\circ\pi)\|_{j,1}^2 &\leqslant m^2 \int_{\pi^{-1}(B(x,1/m)\smallsetminus B(x,1/2m))} \frac{|h|^2_{h_\varepsilon^{\otimes(m-1)}\otimes h_\infty} e^{2(m-1)\tilde{v}_j(z^j)}}{|z^j - z^j(x)|^{2n}} dV_j \\
&\leqslant Cm^{2n+2} \int_{\pi^{-1}(B(x,1/m)\smallsetminus B(x,1/2m))} |h|^2_{h_\varepsilon^{\otimes(m-1)}\otimes h_\infty} e^{2(m-1)\tilde{v}_j(z^j)} dV_j \\
&\leqslant Cm^{2n+2} \int_{\pi^{-1}(B(x,1/m))} |h|^2_{h_\varepsilon^{\otimes(m-1)}\otimes h_\infty} e^{2(m-1)\tilde{v}_j(z^j)} dV_j \\
&\leqslant Cm^{2n+2} \int_{\pi^{-1}(B(x,1/m))} |h|^2_{h_\varepsilon^{\otimes m}} e^{2(m-1)\tilde{v}_j(z^j)} dV_j \\
&\leqslant Cm^{2n+2} e^{2m(\tilde{v}_j(z^j(x)) - \tilde{v}_k(z^k(x)))} \int_{\pi^{-1}(B(x,1/m))} |h|^2_{h_\varepsilon^{\otimes m}} e^{2m\tilde{v}_k(z^k(z^k))} dV_k \\
&\leqslant Cm^{2n+2} e^{2m(\tilde{v}_j(z^j(x)) - \tilde{v}_k(z^k(x)))} \|h\|_k^2
\end{aligned}$$

All the constants are independent of x and m. For the fourth inequality we use the fact that $h_\varepsilon \geqslant Ch_\infty$ for some C on $\mathbb{P}(E|_{U_j})$, since h_ε has analytic singularities, h_∞ is smooth and the U_j's are relatively compact. For the fifth inequality, we use the fact that the oscillation of $\tilde{v}_j$ and $\tilde{v}_k$ on $B(x,1/m)$ is $O(1/m)$. By Hörmander's L^2 estimates we obtain

$$\|f\|_{j,0}^2 \leqslant Cm^{2n+2} e^{2m(\tilde{v}_j(z^j(x)) - \tilde{v}_k(z^k(x)))} \|h\|_k^2.$$

Since $\tau \leqslant 0$ and $h_\varepsilon \geqslant Ch_\infty$, we have for some C

$$\|f\|_j^2 \leqslant C\|f\|_{j,0}^2.$$

The norm $\|h(\chi\circ\pi)\|_j$ satisfies a similar estimate

$$\|h(\chi\circ\pi)\|_j \leqslant Cm^2 e^{2m(\tilde{v}_j(z^j(x)) - \tilde{v}_k(z^k(x)))} \|h\|_k^2$$

where C comes from the change of volume form from dV_j to dV_k and the oscillation of $\tilde{v}_j$ and $\tilde{v}_k$ on $B(x,1/m)$. Thus we have

$$\|h'\|_j \leqslant Cm^{2n+2} e^{2m(\tilde{v}_j(z^j(x)) - \tilde{v}_k(z^k(x)))},$$

$$\sum_l |\sigma_{j,l}(x)\cdot\xi|^2 \geqslant C^{-1}m^{-2n-2}e^{-2m(\tilde v_j(z^j(x))-\tilde v_k(z^k(x)))}|h'(x)\cdot\xi|^2$$
$$\geqslant C^{-1}m^{-2n-2}e^{-2m(\tilde v_j(z^j(x))-\tilde v_k(z^k(x)))}\sum_l |\sigma_{k,l}(x)\cdot\xi|^2$$

since $h'(x)=h(x)$ and $\sum_l|\sigma_{k,l}(x)\cdot\xi|^2=|h(x)\cdot\xi|^2$. By taking logarithms, we infer the desired inequality. □

REMARK 5.8. We have formulated the proposition in terms of E^* instead of E for the following reason. According to [**BP08**] and section 16 of [**HPS16**], the dual metric of a singular metric of vector bundle is always pointwise well defined. However the dual metric is not necessarily continuous if the original metric is continuous. Let us consider a case where the metric has analytic singularities. Assume that $\log|\xi|_h$ has analytic singularities as a function on the total space V for some vector bundle (V,h) and is the form of $\log\sum|f_i(x)\cdot\xi|^2+\psi(x)$ with f_i are holomorphic vector bundle sections and ψ is bounded. This is for instance the case for the approximating metrics used in Proposition 5.2.1. The function $\log|\xi^*|_{h*}$ on the total space V^* is the difference of two real analytic functions modulo bounded terms, on the dense Zariski open set where the metric is smooth. At points where the metric is smooth, we have $\log|\xi|_h^2=\log(\xi^\dagger H(x)\xi)$ for some Hermitian matrix $H(x)$ where $\dagger$ means the Hermitian transpose. Thus one has $\log|\xi^*|_{h*}=\log(\xi^{*\dagger}(H^{-1}(x))\xi^*)$ which can be calculated from the determinant and the adjugate matrix of $H(x)$. Each component of the adjoint matrix and of the determinant is the product of a bounded function times a real analytic series in the z^j's (coordinates of x) and in ξ. Near the singular locus of the metric h, both functions can tend to infinity for fixed ξ^*. These facts would result in more difficulties to be dealt with.

Here is a concrete example taken from Raufi [**Rau15**]. Let E be the trivial rank 2 vector bundle over $\mathbb{C}$ where the metric at $z\in\mathbb{C}$ is represented by the matrix

$$H:=\begin{pmatrix}1+|z|^2 & z\\ \overline{z} & |z|^2\end{pmatrix}.$$

On $\mathbb{C}^*$, the dual metric can be represented by the matrix

$$(H^{-1})^\dagger=\frac{1}{|z|^4}\begin{pmatrix}|z|^2 & -z\\ -\overline{z} & 1+|z|^2\end{pmatrix}.$$

Thus $\log|\xi^*|_{h*}=\log(|z\xi_2^*|^2+|\bar z\xi_1^*+\xi_2^*|^2)-\log|z|^4$. At $\xi^*=(1,0)$, $\log|\xi^*|_{h*}$ is a difference of two functions both tending to infinity when z tends to 0.

REMARK 5.9. We can also interpret the inequality

$$i\Theta_{S^mE*,h_m}\leqslant m\varepsilon_m\omega\otimes\mathrm{Id}$$

in the sense of currents as follows: for any non-trivial local section s of S^mE^*, $m\varepsilon_m\omega+i\partial\overline\partial\log|s|^2_{h_m}$ is a positive current. The local section can be seen as a map i from an open subset of X to the total space S^mE^*. If we pull back the current (2) to U via i, we see that $m\varepsilon_m\omega+i\partial\overline\partial\log|s|^2_{h_m}$ is a positive current. Here $|s|_{h_m}$ is not identically zero since it is non vanishing outside of the zero locus of s and of singular locus of h_m.

Further discussions of these points can be found in [**Paun16**]. The above proposition also answers partially to a question proposed in remark 2.11 of [**Paun16**]. Given a singular Finsler metric with analytic singularities on a vector bundle, one can produce singular Hermitian metrics on high order symmetric tensor products of the given vector bundle, with arbitrary small loss of positivity.

As a direct consequence of the approximation statement, we have the following corollary.

COROLLARY 5.10. *If E is a strongly psef vector bundle of rank r over a compact Kähler manifold (X,ω), then* $\det(E)$ *is a psef line bundle.*

PROOF. On $X\smallsetminus Z_m$, the curvature inequality

$$i\Theta_{S^mE*,h_m}\leqslant m\varepsilon_m\omega\otimes\mathrm{Id}$$

implies that $i\Theta_{\det S^mE,\det h_m^*}\geqslant-\mathrm{rank}(S^mE)m\varepsilon_m\omega$. On the other hand

$$\det S^mE=(\det E)^{\otimes\frac{m\mathrm{rank}(S^mE)}{r}}.$$

Therefore, the induced metric on $\det(E)$ satisfies on $X\smallsetminus Z_m$ the curvature inequality

$$i\Theta_{\det(E)}\geqslant-r\varepsilon_m\omega.$$

Let us point out that the metric h_m is smooth on X (although it might vanish at some points). The induced metric on $-\det(E)$ is locally bounded. In other words, the local weight of the dual metric on $\det(E)$ is locally bounded from above. By the Riemann extension theorem, the curvature inequality holds in the sense of currents throughout X, and not only on $X\smallsetminus Z_m$. By weak compactness, up to taking some subsequence, we get in the limit a closed positive current belonging to the class $c_1(\det E)$. This shows that $\det(E)$ is psef. □

Another direct application of the approximation is the following corollary.

COROLLARY 5.11. *Let E be a vector bundle over a compact Kähler manifold (X, ω). The following properties are equivalent.*
(1) E is strongly psef.
(2) For any $m \in \mathbb{N}^*$, $S^m E$ is strongly psef.
(3) There exists $m \in \mathbb{N}^*$ such that $S^m E$ is strongly psef.

PROOF. (2) implies (3) trivially. (3) implies (1) as in the proof of (2) implying (1) in Proposition 5.2.1. (1) implies (2) is a direct consequence of Proposition 5.2.1. All symmetric products $S^{mp}E$ of E ($p \in N^*$) are quotients of symmetric products of $S^p(S^m E)$. On the other hand, the induced metric on the quotient bundle of a vector bundle will satisfy similar curvature condition as the original metric as in point (1) the following corollary. □

As a consequence, one can also define "$\mathbb{Q}-$twisted" strongly psef vector bundles as follows.

DEFINITION 5.12. *Let (X, ω) be a compact Kähler manifold and E a holomorphic vector bundle on X and D be a $\mathbb{Q}-$line bundle. Then $E\langle D\rangle$ is said to be $\mathbb{Q}-$twisted strongly pseudo-effective ($\mathbb{Q}-$strongly psef for short) if $S^m E \otimes \mathcal{O}_X(mD)$ is strongly psef for some (hence any by Corollary 5.11) $m > 0$ such that $\mathcal{O}_X(mD)$ is a line bundle.*

As in [**DPS94**], one can derive some natural algebraic properties of strongly psef vector bundles.

COROLLARY 5.13 (Algebraic properties of strongly psef vector bundles).
(1) A quotient bundle of a strongly psef vector bundle is strongly psef.
(2) A direct summand of strongly psef vector bundles is strongly psef.
(3) A direct sum of strongly psef vector bundles is strongly psef.
(4) A tensor product (or Schur functor of positive weight) of strongly psef vector bundles is strongly psef.

PROOF. One can obtain lower bounds of the curvature through calculations very similar to those of [**DPS94**]. We first show that the induced singular metric has analytic singularities.

Assume E to be strongly psef. The surjective bundle morphism $E \to Q$ induces a closed immersion of $\mathbb{P}(Q)$ into $\mathbb{P}(E)$, and the restriction of $\mathcal{O}_{\mathbb{P}(E)}(1)$ to $\mathbb{P}(Q)$ is $\mathcal{O}_{\mathbb{P}(Q)}(1)$. The singular metrics on $\mathcal{O}_{\mathbb{P}(E)}(1)$ prescribed in the definition of a strongly psef vector bundle induce by restriction singular metrics with analytic singularities on $\mathcal{O}_{\mathbb{P}(Q)}(1)$. If we observe that all metrics involved are smooth over inverse images of non-empty Zariski open sets, we infer that the restricted metrics are not identically infinite. This concludes the proof of (1).

(1) implies (2) since a direct summand can be seen as a quotient bundle. Now, let E, F be two strongly psef vector bundles. The Hermitian metrics on $\mathcal{O}_{\mathbb{P}(E)}(1)$ and $\mathcal{O}_{\mathbb{P}(F)}(1)$ correspond to Finsler metrics on E^* and F^* denoted by h_E, h_F. Then $h_E + h_F$ defines a Finsler metric with analytic singularities on $E^* \oplus F^*$. It corresponds to a Hermitian metric on $\mathcal{O}_{\mathbb{P}(E \oplus F)}(1)$, and the properties required in the definition can easily be checked for $h_E + h_F$ if they are satisfied for h_E and h_F. This concludes the proof of (3).

By Corollary 5.11 and (3), $S^2(E \oplus F)$ is strongly psef as soon as E, F are. Since

$$S^2(E \oplus F) \cong S^2 E \oplus (E \otimes F) \otimes S^2 F,$$

we infer by (2) that $E \otimes F$ is strongly psef. Finally, the fact that a Schur tensor power is a direct summand of a tensor product implies (4). □

COROLLARY 5.14. *Let*

$$0 \to S \to E \to Q \to 0$$

be an exact sequence of holomorphic vector bundles. If E and $(\det(Q))^{-1}$ are strongly psef, then S is strongly psef.

PROOF. We have $S = \Lambda^{s-1} S^* \otimes \det S$ where s is the rank of S. By dualizing and taking the $s-1$ exterior product, we get a surjective bundle morphism

$$\Lambda^{s-1} E^* \to \Lambda^{s-1} S^* = S \otimes (\det S)^{-1}.$$

On the other hand, we have $\det E \cong \det S \otimes \det Q$, thus we have a surjective bundle morphism

$$\Lambda^{r-s-1} E \otimes (\det Q)^{-1} \to S$$

where r is the rank of E by tensoring $\det E$. By (4) of Corollary 5.13, $\Lambda^{r-s-1} E \otimes (\det Q)^{-1}$ is strongly psef. By (1) of Corollary 5.13, S is strongly psef. □

5.3. Reflexive sheaves

In this section, we show that a numerically flat reflexive sheaf on a compact Kähler manifold is in fact a vector bundle. We need the following topological lemmata.

LEMMA 5.15. *Let X be an arbitrary complex manifold (non necessarily compact) and E be a vector bundle on X. Let X_0 be a Zariski open set in X with $codim(X \smallsetminus X_0) \geqslant 3$. Then the morphism induced by the restriction morphism $H^1(X, E) \to H^1(X_0, E)$ is surjective.*

PROOF. We start by proving that

$$H^1(\mathbb{C}^3 \smallsetminus \{(0,0,0)\}, \mathcal{O}_{\mathbb{C}^3 \smallsetminus \{(0,0,0)\}}) = 0.$$

It is done by direct calculation. Cover $\mathbb{C}^3 \smallsetminus \{(0,0,0)\}$ by three Stein open sets isomorphic to $\mathbb{C}^* \times \mathbb{C}^2$, say $U_i = \{z_i \neq 0\}$, with coordinates (z_0, z_1, z_2). A 1-cochain can be identified with a triple of convergent power series (f_{01}, f_{02}, f_{12}) with f_{12} (say) of type

$$\sum_{(\alpha,\beta,\gamma) \in \mathbb{Z}^2 \times \mathbb{N}} c_{\alpha\beta\gamma} z_0^\alpha z_1^\beta z_2^\gamma$$

over $\mathbb{C}^{*2} \times \mathbb{C}$ (the intersection of two Stein open sets). Similarly, f_{02} is a sum over $(\alpha, \beta, \gamma) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{Z}$ and f_{01} is a sum over $(\alpha, \beta, \gamma) \in \mathbb{N} \times \mathbb{Z}^2$.

The condition that (f_{01}, f_{02}, f_{12}) is closed means that $f_{01} - f_{02} + f_{12} = 0$ on the intersection of the three Stein open sets $U_0 \cap U_1 \cap U_2$, biholomorphic to $\mathbb{C}^{*3}$. We can write f_{01} as a sum of three convergent power series g_{01}^0, g_{01}^1, g_{01} such that g_{01} has only positive power terms, g_{01}^0 has only negative power terms in z_0 and g_{01}^1 has only negative power terms in z_1. Similarly, we decompose f_{02}, f_{12}. Now the closeness condition is equivalent to

$$g_{01} - g_{02} + g_{12} = 0, \quad g_{01}^0 = g_{02}^0, \quad g_{12}^2 = g_{02}^2, \quad g_{01}^1 + g_{12}^1 = 0.$$

We define a 0-cochain in such a way that its differential is (f_{01}, f_{02}, f_{12}). On U_0, resp. U_1, U_2, we take the convergent power series $g_{01} + g_{01}^0$, resp. g_{12}^1, $-g_{12} - g_{02}^2$. This implies that every 1-cocycle is exact, hence

$$H^1(\mathbb{C}^3 \smallsetminus \{(0,0,0)\}, \mathcal{O}_{\mathbb{C}^3 \smallsetminus \{(0,0,0)\}}) = 0.$$

Now, on every polydisc D in $\mathbb{C}^3$, a holomorphic function is uniquely determined by its Taylor expansion at origin, and the same calculation shows that

$$H^1(D \smallsetminus \{(0,0,0)\}, \mathcal{O}_{D \smallsetminus \{(0,0,0)\}}) = 0.$$

By a similar calculation, we can show that for any polydisc D of dimension at least 3,

$$H^1(D \smallsetminus \{0\}, \mathcal{O}_{D \smallsetminus \{0\}}) = 0.$$

By the Künneth formula, for $B' \times (B'' \smallsetminus \{0\})$ where B', B'' are polydiscs with dimension of B'' at least 3, we have $H^1(B' \times (B'' \smallsetminus \{0\}), \mathcal{O}_{B' \times (B'' \smallsetminus \{0\})}) = 0$.

We now return to the general case. By the standard lemma below ensuring the existence of stratifications of analytic sets, we can reduce ourselves to the situation where $X \smallsetminus X_0$ is a closed manifold.

Cover X by the Stein open sets U_α and $B_\beta := B'_\beta \times B''_\beta$ such that X_0 is covered by U_α and $B'_\beta \times (B''_\beta \smallsetminus \{0\})$ where B'_β, B''_β are polydiscs with dimension of B''_β at least 3. Assume that E is trivial on U_α and B_β. Cover $B'_\beta \times (B''_\beta \smallsetminus \{0\})$ by B_β^γ $(1 \leqslant \gamma \leqslant \dim B''_\beta)$ such that each B_β^γ is isomorphic to a polydisc minus a hyperplane defined as zero set of one coordinate. Since U_α, B_β^γ are Stein, the cohomology on X_0 can be calculated as the Čech cohomology with respect to this open covering of X_0, which we denote by $\mathcal{V}$. We also denote by $\mathcal{U}$ the open covering of X consisting of the sets U_α, B_β. Any element s of $H^1(X_0, E)$ can be represented by a family of sections

$$(s_{\alpha_1,\alpha_2}, s_{\alpha\beta}^\gamma, s_\beta^{\gamma_1,\gamma_2}, s_{\beta_1,\beta_2}^{\gamma_1,\gamma_2}) \in$$
$$\prod \Gamma(U_{\alpha_1} \cap U_{\alpha_2}, E) \times \prod \Gamma(U_\alpha \cap B_\beta^\gamma, E) \times \prod \Gamma(B_\beta^{\gamma_1} \cap B_\beta^{\gamma_2}, E) \times \prod \Gamma(B_{\beta_1}^{\gamma_1} \cap B_{\beta_2}^{\gamma_2}, E).$$

Since $H^1(B'_\beta \times B''_\beta, E) = 0$ by the previous case, there exists

$$(s_\beta^\gamma) \in \prod \Gamma(B_\beta^\gamma, E)$$

such that for any β fixed

$$s_\beta^{\gamma_1,\gamma_2} = (-1)^{\gamma_1+1} s_\beta^{\gamma_1} + (-1)^{\gamma_2+1} s_\beta^{\gamma_2}.$$

Define a 0-cochain

$$(s_\beta^\gamma, 0) \in \prod \Gamma(B_\beta^\gamma, E) \times \prod \Gamma(U_\alpha, E).$$

Then we have $(s_{\alpha_1,\alpha_2}, s^{\gamma}_{\alpha\beta}, s^{\gamma_1,\gamma_2}_{\beta}, s^{\gamma_1,\gamma_2}_{\beta_1,\beta_2}) + \delta(-s^{\gamma}_{\beta}, 0)$ as another representative of the same cohomology class on X_0. The components in $\Gamma(B^{\gamma_1}_{\beta} \cap B^{\gamma_2}_{\beta}, E)$ are 0 by construction. Thus we can assume that the components in $\Gamma(B^{\gamma_1}_{\beta} \cap B^{\gamma_2}_{\beta}, E)$ are 0 from the beginning.

Since the representative is closed, the components in $\Gamma(B^{\gamma}_{\beta} \cap U_\alpha, E)$ glue to a section $s_{\alpha,\beta} \in \Gamma(B_\beta \smallsetminus (B'_\beta \times \{0\}) \cap U_\alpha, E)$ when γ varies. By the Hartogs theorem, this section extends across the submanifold $B'_\beta \times \{0\}$, as its codimension is at least 3. The components in $\Gamma(B^{\gamma_1}_{\beta_1} \cap B^{\gamma_2}_{\beta_2}, E)$ can be glued into a section of $\Gamma(B^{\gamma_1}_{\beta_1} \cap B_{\beta_2}, E)$ when γ_2 varies, and into a section of $\Gamma(B_{\beta_1} \cap B^{\gamma_2}_{\beta_2}, E)$ when γ_1 varies. By the unique continuation theorem for holomorphic functions, in fact they define a holomorphic section s_{β_1,β_2} of E on $B_{\beta_1} \cap B_{\beta_2}$.

We claim that after performing this glueing, the sections

$$(s_{\alpha_1,\alpha_2}, s_{\alpha,\beta}, s_{\beta_1,\beta_2}) \in \prod \Gamma(U_{\alpha_1} \cap U_{\alpha_2}, E) \times \prod \Gamma(U_\alpha \cap B_\beta, E) \times \prod \Gamma(B_{\beta_1} \cap B_{\beta_2}, E)$$

define a 1-cocycle of X with respect to the open covering U_α, B_β, and that its class in $H^1(X_0, E)$ is exactly s.

The reason is as follows. The image of $(s_{\alpha_1,\alpha_2}, s_{\alpha,\beta}, s_{\beta_1,\beta_2})$ from $H^1(\mathcal{U}, E)$ to $H^1(\mathcal{U} \cap X_0, E)$ is just the restriction of sections. The covering $\mathcal{V}$ is a refinement of $\mathcal{U} \cap X_0$ given by the inclusion of open sets: $U_\alpha \subset U_\alpha$, $B^{\gamma}_{\beta} \subset B_\beta$. The image under this refinement of open sets is precisely s. □

LEMMA 5.16 (Stratification of analytic sets, see e.g. Proposition 5.6 in Chap. II of [**Dem12b**]). *Let $Z \subset X$ be an analytic subset of dimension n. Then Z admits a stratification $\varnothing = Z_{n+1} \subset \cdots \subset Z_0 = Z$ by closed analytic sets Z_k of dimension $n_k > n_{k+1}$ such that $Z_k \smallsetminus Z_{k+1}$ is a closed complex submanifold of dimension n_k of $X \smallsetminus Z_{k+1}$.*

Let us point out that the result is false if the codimension is equal to 2. For example, the group $H^1(\mathbb{C}^2 \smallsetminus \{(0,0)\}, \mathcal{O}_{\mathbb{C}^2 \smallsetminus \{(0,0)\}})$ is infinite dimensional, while $H^1(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}) = 0$ by Cartan's theorem B.

LEMMA 5.17. (analogue of lemma 11.13 in [**Voi02a**])

Let X be a complex manifold (not necessary compact) and Y be a closed submanifold of codimension at least $r+1$. Then the restriction map

$$H^l(X, \mathbb{R}) \to H^l(X \smallsetminus Y, \mathbb{R})$$

is an isomorphism for $l \leqslant 2r$.

PROOF. We have the long exact sequence of relative cohomology

$$\cdots H^l(X, X \smallsetminus Y, \mathbb{R}) \to H^l(X, \mathbb{R}) \to H^l(X \smallsetminus Y, \mathbb{R}) \to H^{l+1}(X, X \smallsetminus Y, \mathbb{R}) \cdots .$$

On the other hand, we have by the excision lemma that for U a tubular neighborhood of Y

$$H^l(X, X \smallsetminus Y, \mathbb{R}) \cong H^l(U, U \smallsetminus Y, \mathbb{R}).$$

By Thom isomorphism theorem, we have

$$H^{l-2r}(Y, \mathbb{R}) \cong H^l(U, U \smallsetminus Y, \mathbb{R}).$$

We remark that X as a complex manifold is orientable, so does U. Hence the Thom class in coefficient $\mathbb{Z}$ exists by Theorem 4.D.10. in [**Hat02**]. The natural inclusion $\mathbb{Z} \to \mathbb{R}$ sends the Thom class in coefficient $\mathbb{Z}$ to the Thom class in coefficient $\mathbb{R}$. Thus we have the Thom isomorphism by the Corollary 4.D.9 in [**Hat02**]. It follows that for $j < \operatorname{codim} Y$, $H^j(X, X \smallsetminus Y, \mathbb{R}) = 0$. This finishes the proof of the lemma using the exact sequence. □

LEMMA 5.18. *Let X be a complex manifold (not necessary compact) and Y be a closed analytic subset of codimension at least $r+1$. Then the restriction map*

$$H^l(X, \mathbb{R}) \to H^l(X \smallsetminus Y, \mathbb{R})$$

is an isomorphism for $l \leqslant 2r$.

PROOF. It is a direct consequence of lemmata 5.16 and 5.17. □

We recall briefly the construction of Chern classes of a coherent sheaf $\mathcal{F}$ in the de Rham cohomology. We refer to [**GR58**] for more details. If X is connected complex compact manifold (or more generally a Zariski open set U of in X), by [**Voi02a**], $\mathcal{F}$ does not necessarily admit a resolution by holomorphic vector bundles. On the other hand, a real analytic coherent sheaf possesses a resolution by real analytic vector bundles. Let

$$0 \to E^{2n} \to \cdots E^0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\mathbb{R}-\mathrm{an}} \to 0$$

be a resolution of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\mathbb{R}-\mathrm{an}}$ by real analytic vector bundles E^i where $\mathcal{O}_X^{\mathbb{R}-\mathrm{an}}$ is sheaf of real analytic function on X and n is the complex dimension of X. Define the total Chern class of $\mathcal{F}$ by

$$c_\bullet(\mathcal{F}) := \prod_i c_\bullet(E^i)^{(-1)^i}.$$

By restriction on U, same formula defines $c_\bullet(\mathcal{F}|_U)$. It can be check that this is independent the choice of resolution.

LEMMA 5.19. *Let $\mathcal{F}$ be a coherent torsion sheaf over a compact complex manifold X (not necessarily Kähler) of dimension n. Assume that $\mathcal{F}$ is supported in a SNC divisor $E = \cup_i E_i$ where E_i are the irreducible components. Let α be a smooth closed form over X such that $\alpha|_{E_i} = 0$ for any i. Then for any $i < n$,*

$$\int_X c_i(\mathcal{F}) \wedge \alpha^{n-i} = 0.$$

More generally we have for any $i < n$ and any cohomology class β of X,

$$\int_X \mathrm{ch}(\mathcal{F}) \wedge \beta \wedge \alpha^{n-i} = 0.$$

PROOF. Denote for any divisor D (not necessarily irreducible) $G_D(X)$ the Grothendieck group of coherent sheaves over X supported in D. We have exact sequence

$$\oplus_i G_{E_i}(X) \to G_E(X) \to 0.$$

Let $(\mathcal{F}_i) \in \oplus_i G_{D_i}(X)$ be a preimage of $\mathcal{F}$. Then we have by construction of Chern character class (cf. [**Gri10**]),

$$\mathrm{ch}(\mathcal{F}) = \sum_i i_{E_i *}(\mathrm{ch}(\mathcal{F}_i)\,\mathrm{td}(N_{E_i/X})^{-1})$$

where i_{E_i} is the closed immersion and $\mathrm{td}(N_{E_i/X})$ is the Todd class of the normal bundle of E_i. For any cohomology class β on X,

$$\int_X \mathrm{ch}(\mathcal{F}) \wedge \beta \wedge \alpha^{n-i} = \sum_i \int_{E_i} \mathrm{ch}(\mathcal{F}_i)\,\mathrm{td}(N_{E_i/X})^{-1} \wedge i_{E_i}^* \beta \wedge i_{E_i}^* \alpha^{n-i} = 0$$

since $i_{E_i}^* \alpha = 0$. □

As an application of this lemma, we have the following result.

LEMMA 5.20. *Let $\mathcal{F}$ be a reflexive sheaf over a compact complex manifold X. Let $\sigma : \tilde{X} \to X$ be a modification of X such that there exists a SNC divisor E in $\tilde{X}$ such that*

$$\sigma : \tilde{X} \smallsetminus E \to X \smallsetminus \pi(E)$$

is biholomorphism with E a SNC divisor and the codimension of $\pi(E)$ at least 3 and $\sigma^*\mathcal{F}/\mathrm{Tors}$ is locally free. Then we have that for $i = 1, 2$

$$c_i(\mathcal{F}) = \sigma_*(c_i(\sigma^*\mathcal{F}/\mathrm{Tors})).$$

PROOF. First observe that such a modification always exists by the fundamental work of [**Ros68**], [**GR70**], [**Rie71**].

Without loss of generality we can assume that the dimension of X is at least 3. Otherwise, $\mathcal{F}$ is locally free and the result is direct. By Poincaré duality, it is the same to prove that for $i = 1, 2$ and any cohomology class α we have that

$$\int_X c_i(\mathcal{F}) \wedge \alpha = \int_{\tilde{X}} (c_i(\sigma^*\mathcal{F}/\mathrm{Tors})) \wedge \sigma^*\alpha.$$

Recall that $\sigma^* \mathrm{ch}(\mathcal{F}) = \sum_i (-1)^i \mathrm{ch}(L^i\sigma^*\mathcal{F})$ where $L^i\sigma^*$ is the i-th left derived functor of σ^*. Without loss of generality, we can assume that $\mathcal{F}$ is locally free over $X \smallsetminus \pi(E)$. In particular, $L^i\sigma^*\mathcal{F}$ for any $i > 0$ is supported in the exceptional divisor. On the other hand, the torsion part of σ^* is also supported in the exceptional divisor. By the above lemma, we have that

$$\int_{\tilde{X}} (c_i(\sigma^*\mathcal{F}/\mathrm{Tors})) \wedge \sigma^*\alpha = \int_{\tilde{X}} \sigma^*(c_i(\mathcal{F})) \wedge \sigma^*\alpha$$

which concludes the proof. □

We now introduce the definition of nef and strongly psef torsion-free sheaves.

DEFINITION 5.21 (Nef/ Strongly psef torsion-free sheaf).
Assume that $\mathcal{F}$ is a torsion free sheaf over a compact complex manifold X. We say that $\mathcal{F}$ is nef (resp. strongly psef) if there exists some modification $\pi : \tilde{X} \to X$ such that $\pi^*\mathcal{F}/\mathrm{Tors}$ is a nef (resp. strongly psef) vector bundle where Tors *means the torsion part.*

Notice that for any further modification $\pi' : \tilde{X}' \to \tilde{X}$, $\pi'^*(\pi^*\mathcal{F}/\mathrm{Tors}) = (\pi \circ \pi')^*\mathcal{F}/\mathrm{Tors}$ (in particular, further pull back is still a nef or strongly psef vector bundle). In fact, for any morphism π', π, there exist natural surjective morphisms

$$(\pi \circ \pi')^*\mathcal{F} = \pi'^*\pi^*\mathcal{F} \to \pi'^*(\pi^*\mathcal{F}/\mathrm{Tors}) \to (\pi'^*(\pi^*\mathcal{F}/\mathrm{Tors}))/\mathrm{Tors}$$

which induces a surjection $(\pi' \circ \pi)^*\mathcal{F}/\mathrm{Tors} \to (\pi'^*(\pi^*\mathcal{F}/\mathrm{Tors}))/\mathrm{Tors}$. It is generic isomorphism on which $\mathcal{F}$ is locally free. Thus the kernel of the induced morphism is a torsion sheaf. Since $(\pi' \circ \pi)^*\mathcal{F}/\mathrm{Tors}$ is torsion free, the morphism is also injective.

More generally, we show in the next remark that the definition is independent of the choice of the pull back.

REMARK 5.22. By the work of [**Ros68**], [**GR70**], [**Rie71**], for any torsion-free sheaf $\mathcal{F}$ over a compact complex manifold, there exists a modification $\pi : \tilde{X} \to X$ such that $\pi^*\mathcal{F}/\mathrm{Tors}$ is a locally free sheaf (i.e. a vector bundle). In the above definition, we say that $\mathcal{F}$ is nef or strongly psef if $\pi^*\mathcal{F}/\mathrm{Tors}$ is nef or strongly psef.

Let us recall here theorem 1.B.1 of [**Paun98**]. Let $f : Y \to X$ be a surjective holomorphic map between compact complex manifolds. Let α be a cohomology class in the Bott-Chern cohomology class $H^{1,1}_{BC}(X, \mathbb{C})$. Then α is nef if and only if $f^*\alpha$ is nef.

For the vector bundle case, a modification $\sigma : \tilde{X} \to X$ induces a surjection $\tilde{\sigma} : \mathbb{P}(\sigma^*E) \to \mathbb{P}(E)$ where E is a vector bundle over X. The pull back of $\mathcal{O}_{\mathbb{P}(E)}(1)$ under $\tilde{\sigma}$ is $\mathcal{O}_{\mathbb{P}(\sigma^*E)}(1)$. Thus σ^*E is nef if and only if $\mathcal{O}_{\mathbb{P}(\sigma^*E)}(1)$ is nef which is equivalent to say that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef, i.e. E is nef.

Thus in the above definition, it is same to say that $\mathcal{F}$ is nef if and only if for *every* modification $\sigma : \tilde{X} \to X$ such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a vector bundle, $\sigma^*\mathcal{F}/\mathrm{Tors}$ is nef.

Similarly, let $f : Y \to X$ be a surjective holomorphic map between compact complex manifolds. Let α be a cohomology class in the Bott-Chern cohomology class $H^{1,1}_{BC}(X, \mathbb{C})$. Then α is psef if and only if $f^*\alpha$ is psef. The pull back of a strongly psef vector bundle E under a modification σ is psef if and only if E itself is psef. Once a smooth metric has been fixed on E, the singular metrics on $\mathcal{O}_{\mathbb{P}(\sigma^*E)}(1)$ (resp. on $\mathcal{O}_{\mathbb{P}(E)}(1)$) are identified with quasi-psh functions. Let us observe that the push forward of a psh function with analytic singularities under a proper modification is still a psh function with analytic singularities. The singular set of the pushed forward weight on $\mathcal{O}_{\mathbb{P}(E)}(1)$ is the image of the singular set of the weight function on $\mathcal{O}_{\mathbb{P}(\sigma^*E)}(1)$.

More precisely, denote by $\tilde{\pi} : \mathbb{P}(\sigma^*E) \to \tilde{X}$ and $\pi : \mathbb{P}(E) \to X$ the projections. We have $\pi \circ \tilde{\sigma} = \sigma \circ \tilde{\pi}$. For a simple blow-up with a smooth irreducible centre, the opposite of the cohomology class of the exceptional divisor has a smooth representative that is positive along the fibers of the projectivised normal bundle. From this, it is easy to see that exists a smooth form ω_E on $\tilde{X}$ such that $\sigma^*\omega_X + \omega_E$ is a Kähler form on $\tilde{X}$, and $\{\omega_E\} = -\{[E]\}$ for a suitable combination $E = \sum \delta_j E_j$, $\delta_j \in \mathbb{R}_{>0}$ of the irreducible components E_j of the exceptional divisor. Notice that $\{\sigma_*\omega_E\}$ is the zero cohomology class. Denote by φ a quasi psh function on $\tilde{X}$ such that

$$\omega_E = -[E] + i\partial\overline{\partial}\varphi.$$

Assume that σ^*E is strongly psef and let us use a reference metric σ^*h_∞ induced by a smooth metric h_∞ on E. Then there exist quasi-psh functions ψ_ε with analytic singularities such that

$$i\Theta(\mathcal{O}_{\mathbb{P}(\sigma^*E)}(1), \sigma^*h_\infty e^{-\psi_\varepsilon}) \geqslant -\varepsilon\tilde{\pi}^*(\sigma^*\omega_X + \omega_E),$$

and $\sigma^*h_\infty e^{-\psi_\varepsilon - \tilde{\pi}^*\varphi}$ are singular metrics with analytic singularities on $\mathcal{O}_{\mathbb{P}(\sigma^*E)}(1)$. By taking the push-forward of the quasi-psh functions $\psi_\varepsilon + \varepsilon\tilde{\pi}^*\varphi$ under the modification $\tilde{\sigma}$, we get singular metrics $h_\varepsilon := h_\infty e^{-\tilde{\sigma}_*(\psi_\varepsilon + \varepsilon\tilde{\pi}^*\varphi)}$ on $\mathcal{O}_{\mathbb{P}(E)}(1)$ possessing analytic singularities and satisfying the condition

$$i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) \geqslant -\varepsilon\pi^*\omega_X.$$

In the above definition, it is thus the same to say that $\mathcal{F}$ is strongly psef if and only if for *every* modification $\sigma : \tilde{X} \to X$ such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a vector bundle, $\sigma^*\mathcal{F}/\mathrm{Tors}$ is strongly psef.

In fact, following the arguments in [**Paun98**] and [**DPS94**], we can prove a more general result.

THEOREM 5.23. *Let $f : Y \to X$ be a surjective holomorphic map between compact Kähler manifolds. Let E be a vector bundle over X. Then f^*E is strongly psef if and only if E is strongly psef.*

PROOF. It is easy to see that E is strongly psef implies that f^*E is strongly psef. To prove the inverse direction, we use the Hironaka flattening theorem which shows the existence of a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\pi_2} & \tilde{X} \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ Y & \xrightarrow{f} & X \end{array}$$

where Z is a compact Kähler complex space, π_2 a flat morphism (i.e. with equidimensional fibres) and σ a composition of blow-ups of smooth centres. In the previous remark, we prove that the pull back of a vector bundle under a blow-up of smooth center is strongly psef if and only if it is itself strongly psef. The result will follow if we prove that the pull back of a vector bundle under a flat morphism is strongly psef if and only if it is itself strongly psef. Intuitively, we would want to take the quasi-psh weight at any point to be the supremum of the quasi-psh weight on the pre-image of that point. But this operation does not necessarily give the desired lower bound of curvature. In order to overcome this difficulty, we use a modified version of the argument given in [**DPS94**] proposition 1.8, as follows. □

PROPOSITION 5.3.1. *Let $f : Y \to X$ be a surjective holomorphic map with equidimensional fibres where X is a compact Kähler manifold and Y is a compact Kähler complex space. Let E be a vector bundle over X. Then f^*E is strongly psef if and only if E is strongly psef.*

PROOF. The proof is essentially the same as for Théorème 1.B.1 in [**Paun98**] and Proposition 1.8 in [**DPS94**]. We just outline the arguments with the necessary modifications.

We denote by the same symbol f the induced map $\mathbb{P}(f^*E) \to \mathbb{P}(E)$. Let α be the curvature form in the cohomology class $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ induced by some smooth metric on E. Let ψ_ε be quasi psh functions with analytic singularities on $\mathbb{P}(f^*E)$ such that

$$f^*\alpha + \frac{i}{2\pi}\partial\overline{\partial}\psi_\varepsilon \geqslant -\varepsilon\omega', \quad \varepsilon > 0,$$

for some Kähler form ω' on $\mathbb{P}(f^*E)$. The existence follows from the definition of a strongly psef vector bundle (the definition of a strongly psef vector bundle is still valid for a compact Kähler complex space).

Denote by p the dimension of fibres. For every $y \in \mathbb{P}(f^*E)$ there exist local holomorphic functions $w_1, \cdots, w_p$ in a neighbourhood U of y such that $z \mapsto (f(z), w_1(z), \cdots, w_p(z))$ is a proper finite morphism from U to a neighbourhood of $\{f(y)\} \times \{0\}$ in $\mathbb{P}(E) \times \mathbb{C}^p$. Thus there exist local coordinates centered at $f(y)$ on $\mathbb{P}(E)$ such that

$$|F(z) - F(y)|^2 + \sum_{1 \leqslant j \leqslant p} |w_j(z)|^2 > 0$$

on $\overline{\partial} U$, where $F = (F_1, \cdots, F_n)$ denote the local coordinate components of f.

Since $\mathbb{P}(f^*E)$ is compact, we can cover $\mathbb{P}(f^*E)$ by finitely many such sets U_k centered at $y_k \in \mathbb{P}(f^*E)$, and find corresponding holomorphic functions $(w_1^{(k)}, \cdots, w_p^{(k)})$ on $\overline{U}_k$, as well as components $F^{(k)}$. Each U_k can be supposed to be embedded as a closed analytic set of some open set in $\mathbb{C}^{N_k}$ with coordinates $(w_1^{(k)}, \cdots, w_p^{(k)}, \cdots, w_{N_k}^{(k)})$ (i.e., we complete $(w_1^{(k)}, \cdots, w_p^{(k)})$ into a local coordinate system of $\mathbb{C}^{N_k}$). By construction,

$$2\delta_k := \inf_{\partial U_k} |F^{(k)}(z) - F^{(k)}(y_k)|^2 + \sum_{1 \leqslant j \leqslant p} |w_j^{(k)}(z)|^2 > 0.$$

We can even suppose that the open sets

$$V_k := \{z \in U_k; |F^{(k)}(z) - F^{(k)}(y_k)|^2 + \sum_{1 \leqslant j \leqslant p} |w_j^{(k)}(z)|^2 < \delta_k\}$$

cover $\mathbb{P}(f^*E)$. Define for $z \in \overline{U_k}$

$$\lambda_\varepsilon^{(k)}(z) := \varepsilon^3 \sum_{1 \leqslant i \leqslant N_k} |w_i^{(k)}|^2 - \varepsilon^2(|F^{(k)}(z) - F^{(k)}(y_k)|^2 + \sum_{1 \leqslant j \leqslant p} |w_j^{(k)}(z)|^2 - \delta_k),$$

and for $x \in \mathbb{P}(E)$,

$$\varphi_\varepsilon := \sup_{y \in f^{-1}(x) \cap \overline{U}_k} (\psi_{\varepsilon^4}(y) + \lambda_\varepsilon^{(k)}(y))$$

where the supremum is also taken with respect to k. The curvature condition is checked in the same way as in [**Paun98**] and [**DPS94**].

Let us observe that by using a regularization, one can assume that the quasi psh weight ψ_ε is continuous (i.e. locally the weight is of the form $c \log \sum |g_j|^2 + f$ where f is continuous, and not just bounded).

By choosing ε small enough, we get φ_ε continuous with values in $[-\infty, \infty[$. In fact, for ε small enough, $\lambda_\varepsilon^{(k)}$ is strictly negative on the boundary of U_k and positive on V_k. Thus the function $\Psi_\varepsilon(y) := \psi_{\varepsilon^4}(y) + \sup_{y \in \overline{U}_k} \lambda_\varepsilon^{(k)}(y)$ is continuous on Y. Since $\varphi_\varepsilon(x) = \sup_{y \in f^{-1}(x)} \Psi_\varepsilon(y)$, φ_ε is continuous on X.

We now turn ourselves to the proof that φ_ε has analytic singularities. Observe that φ_ε has the same singularities as the function $\sup_{y \in f^{-1}(x)} \psi_{\varepsilon^4}(y)$ on X, since the functions $\lambda_\varepsilon^{(k)}$ are bounded. We claim the following more general fact: let $f : Y \to X$ be a proper morphism between complex spaces, and φ_ε be a quasi psh function with analytic singularities on Y, then the function

$$f_*\varphi(x) := \sup_{y \in f^{-1}(x)} \varphi(y)$$

has analytic singularities on X. Here "φ is a quasi-psh function over a complex space" means that φ can be locally extended as a quasi-psh function to any open set of $\mathbb{C}^N$ in which Y can be embedded as a closed analytic set; that φ has analytic singularities means for every $y \in Y$, there exists an open set on which $\varphi = c\log(\sum |g_i|^2) + f$, with holomorphic functions g_i and a bounded function f.

By Hironaka, there exists a modification $\sigma : \tilde{Y} \to Y$ such that $\tilde{Y}$ is smooth. By considering $f \circ \sigma$ and $\varphi \circ \sigma$, we are reduced to the case where Y is smooth. For every $x \in X$, we can cover $f^{-1}(x)$ by finite open sets U_k such that the restriction of φ to each open set is of the form $c\log \sum |g_i^{(k)}|^2 + O(1)$, where g_i are holomorphic functions on this open set and $O(1)$ is a bounded term. There exists an open neighbourhood V of x such that $f^{-1}(V) \subset \cup_k U_k$. For every $x \in V$,

$$f_*\varphi(x) = \sup_k \sup_{y \in f^{-1}(x) \cap U_k} \varphi(y).$$

Since a finite supremum of quasi-psh functions with analytic singularities still has analytic singularities, it is enough to show that $\sup_{y \in f^{-1}(x) \cap U_k} \varphi(y)$ has analytic singularities for every k. Since we take a finite supremum, the bounded terms will remain bounded after taking the supremum, therefore we are only concerned with the logarithmic term in what follows.

Let J_k be the maximal germ of ideal sheaf at x such that $f^* J_k|_{U_k} \subset (g_i^{(k)})$ with respect to the inclusion relation. (Here one may have to shrink the open set U_k, i.e. the inclusion is to be understood in the sense of germs at any point of $f^{-1}(x)$.) Then the ideal $(g_i^{(k)})$ is generated by finitely many holomorphic functions that are either of the form $f^* h_\alpha^{(k)}$ for some holomorphic function germ at x, or of the form $f_\beta^{(k)}$ for some holomorphic function on U_k. We claim that the zero set $V(f_\beta^{(k)})$ is not of the form $f^{-1}(f(V(f_\beta^{(k)})))$. Otherwise, by Hilbert's Nullstensatz, $f_\beta^{(k)}$ is contained in the germ of pull back of the prime ideal sheaf vanishing on $f(V(f_\beta^{(k)}))$, contradicting the maximality of J_k. Therefore

$$\log(\sum_\alpha |g_\alpha^{(k)}|^2)(x) = \sup_{y \in f^{-1}(x) \cap U_k} \log(\sum_\alpha |f^* h_\alpha^{(k)}|^2 + \sum_\beta |f_\beta^{(k)}|^2),$$

which also has analytic singularities. □

REMARK 5.24. Observe that when the manifold X is projective, there exists the subtlety in the definition of strongly psef torsion free sheaf. Recall that a torsion free sheaf $\mathcal{F}$ over a projective manifold X with an ample line bundle A is called weakly positive in the sense of Nakayama (cf. [**Nak04**]) if for any $a \in \mathbb{N}$, there exists $b \in \mathbb{N}^*$ such that $(\mathrm{S}^{ab}\mathcal{F})^{\vee\vee} \otimes A^b$ is globally generated at some point (hence generically globally generated). Our definition of strongly psef torsion free sheaf implies that it is the weak positive torsion free sheaf in the sense of Nakayama, but not inversely in general.

First we show that if $\mathcal{F}$ is a strongly psef torsion free sheaf, then it is weakly positive in the sense of Nakayama. Let $\sigma : \tilde{X} \to X$ be a composition of blow-ups of the smooth centers such that the pull back of torsion free coherent sheaf $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a strongly psef vector bundle. Let A be an ample line bundle over X. For b large enough, $\sigma^* A^b - E$ is an ample line bundle over $\tilde{X}$ where E is the exceptional divisor. $\mathrm{S}^{ab}(\sigma^*\mathcal{F}/\mathrm{Tors}) \otimes \sigma^* A^b \otimes \mathcal{O}(-E)$ is generically globally generated over $\tilde{X}$ by possible larger b and by changing $\mathcal{O}(-E)$ by its multiple. It is from the assumption that $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a strongly psef vector bundle over $\tilde{X}$. By tensoring the canonical section of the line bundle $\mathcal{O}(E)$, $\mathrm{S}^{ab}(\sigma^*\mathcal{F}/\mathrm{Tors}) \otimes \sigma^* A^b$ is generically globally generated over $\tilde{X}$. Thus the same holds for $(\mathrm{Sym}^{ab}\mathcal{F})^{\vee\vee} \otimes A^b$ over X by the natural isomorphism $(\mathrm{Sym}^{ab}\mathcal{F})^{\vee\vee} \otimes A^b \to [\sigma_*(\mathrm{Sym}^{ab}(\sigma^*\mathcal{F}/\mathrm{Tors}) \otimes \sigma^* A^b)]^{\vee\vee}$.

To indicate the subtlety, we use the same notations as above. For the inverse direction, we hope to show that $(\mathrm{S}^{ab}(\sigma^*\mathcal{F}/\mathrm{Tors}))^{\vee\vee} \otimes \sigma^* A^b$ is generically globally generated over $\tilde{X}$ for large b from the fact that $(\mathrm{S}^{ab}\mathcal{F})^{\vee\vee} \otimes A^b$ is generically globally generated over X for large b. Let S be the analytic set of codimension at least 2 in X such that $\sigma : \tilde{X} \setminus E \to X \setminus S$ is biholomorphic. But the global sections

$$H^0(X, (\mathrm{S}^{ab}\mathcal{F})^{\vee\vee} \otimes A^b) \cong H^0(X \setminus S, (\mathrm{S}^{ab}\mathcal{F})^{\vee\vee} \otimes A^b) \cong H^0(\tilde{X} \setminus E, \mathrm{S}^{ab}(\sigma^*\mathcal{F}/\mathrm{Tors}) \otimes \sigma^* A^b)$$

do not necessarily extend over $\tilde{X}$ even seen as a section of $\mathrm{S}^{ab}(\sigma^*\mathcal{F}/\mathrm{Tors}) \otimes \sigma^* A^b \otimes \tilde{A}$ where $\tilde{A}$ is an arbitrary ample line bundle over $\tilde{X}$. The reason is that the sections may have essentially singularity along E.

A typical example is the following. Let S be an analytic set of codimension at least two over a projective manifold X and let $\mathcal{I}_S$ be the ideal sheaf associated to S. The bidual of $\mathcal{I}_S$ is $\mathcal{O}_X$ as well as all the symmetric power. Let U be any open set of X. We have that for any m and any vector bundle E

$$H^0(U,(S^m\mathcal{I}_S)^{\vee\vee}\otimes E)\cong H^0(U\smallsetminus S,(S^m\mathcal{I}_S)^{\vee\vee}\otimes E)=H^0(U\smallsetminus S,E)\cong H^0(U,E)$$

where the last equality follows from the Hartogs theorem. As a consequence, for any ample divisor A and any $a\in\mathbb{N}^*$, $(S^{ab}\mathcal{I}_S)^{\vee\vee}\otimes A^b$ is globally generated for b large enough. In particular, $\mathcal{I}_S$ is weakly positive in the sense of Nakayama.

However, observe that $\mathcal{I}_S$ has some "negativity" along S, even if it is weakly positive in the sense of Nakayama. It can be seen as follows. Let $\sigma:\tilde{X}\to X$ be a composition of blow-ups with smooth centers such that $\sigma^*\mathcal{I}_S/\mathrm{Tors}=\mathcal{O}_{\tilde{X}}(-E)$ where E is an effective divisor supported in the exceptional divisor. By the definition of strongly psef torsion free sheaf, if $\mathcal{I}_S$ is strongly psef, $\sigma^*\mathcal{I}_S/\mathrm{Tors}$ should be a psef vector bundle since it is a line bundle. But it is not the case which means that $\mathcal{I}_S$ is not strongly psef. In other words, our definition of strong psef torsion free sheaf is reasonable which forbids the above kind of negativity which will appear in some birational models.

Like the bundle case, the strongly psef torsion-free sheaf is stable under the usual algebraic operations, with the consideration of taking torsion free part.

Example 5.25. (The pull back of a torsion free sheaf is not necessarily torsion free)

According to the knowledge of the author, this example can be found in [**GR70**]. Let X be the blow up of the origin of $\mathbb{C}^2$ with $\pi:X\to\mathbb{C}^2$. Let (x,y) be the coordinate of $\mathbb{C}^2$. The maximal ideal at the origin can be resolved by the Koszul complex

$$\mathcal{O}_{\mathbb{C}^2}\xrightarrow{(-y,x)}\mathcal{O}_{\mathbb{C}^2}^{\oplus 2}\to\mathfrak{m}_0\to 0$$

where the second arrow sends (f,g) to $xf+yg$. The pull back is right exact which induces the exact sequence

$$\mathcal{O}_X\xrightarrow{(-v,uv)}\mathcal{O}_X^{\oplus 2}\to\pi^*\mathfrak{m}_0=\mathfrak{m}_0\otimes_{\mathcal{O}_{\mathbb{C}^2}}\mathcal{O}_X\to 0$$

where in local coordinates $\pi(u,v)=(uv,v)$ and the the second arrow sends (f,g) to $f\otimes x+g\otimes y$. We denote the second arrow as ε. We claim that $\varepsilon(-1,u)$ is not a zero element in $\pi^*\mathfrak{m}_0$. Otherwise, $(-1,u)$ is in the kernel of ε which by exactitude is of the form $(-vf,uvf)$ for some f. Contradiction.

On the other hand $(-1,u)$ is torsion in $\pi^*\mathfrak{m}_0$ since $v\varepsilon(-1,u)=-v\otimes x+vu\otimes y=(-v\pi^*x+vu\pi^*y)\otimes 1=0$. Consider the composition $\mathcal{O}_X^{\oplus 2}/\mathrm{Ker}(\varepsilon)\to\pi^*\mathfrak{m}_0\to\mathcal{O}_X$. The image is $\mathcal{I}_E$ where $\mathcal{I}_E$ is the ideal sheaf associated to the exceptional divisor E. The first morphism is in fact isomorphism. In local coordinates the composition sends (f,g) to $uvf+vg$. The kernel is $\mathcal{O}_X(-1,u)$ which is torsion. We have isomorphism between $\mathcal{O}_X^{\oplus 2}/\mathrm{Ker}(\varepsilon)$ modulo this kernel and $\mathcal{I}_E$. Thus the image under ε (i.e. $\mathcal{O}_X\varepsilon(-1,u)$) gives all the torsion elements. In other words, $\pi^*\mathfrak{m}_0/\mathrm{Tors}\cong\mathcal{O}_X(-E)$.

In fact, the morphism $u:\mathcal{O}_{\mathbb{C}^2}\to\mathcal{O}_{\mathbb{C}^2}^{\oplus 2}$ induces a meromorphic map from $\mathbb{C}^2$ to $\mathrm{Gr}(1,2)$ which sends z to the image of $u(z)$. The meromorphic map induces a holomorphic map from the blow up of the origin to $\mathrm{Gr}(1,2)$ which resolves the indeterminacy set of the meromorphic morphism. The total space of $\mathcal{O}_{\mathbb{P}^2}(-1)$ is also the blow-up of $\mathbb{C}^2$ at the origin with the natural projection $\tau:X\to\mathbb{P}^2$. The pull back of the tautological line bundle over $\mathrm{Gr}(1,2)$ admits exact sequence

$$\mathcal{O}_X^{\oplus 2}=\tau^*\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}\to\tau^*\mathcal{O}_{\mathbb{P}^2}(1)\to 0.$$

The image of the kernel of ε in $\tau^*\mathcal{O}_{\mathbb{P}^2}(1)$ is 0. Thus we have factorisation $\mathcal{O}_X^{\oplus 2}/\mathrm{Ker}(\varepsilon)\cong\pi^*\mathfrak{m}_0\to\tau^*\mathcal{O}_{\mathbb{P}^2}(1)\to 0$. The kernel of the factorisation is supported in the exceptional divisor which is thus torsion. In conclusion, we have isomorphism $\pi^*\mathfrak{m}_0/\mathrm{Tors}\cong\tau^*\mathcal{O}_{\mathbb{P}^2}(1)$. This shows in this special case how to find a modification such that the pull back of a torsion free sheaf is locally free modulo torsion. This construction was generalised in [**Ros68**], [**GR70**], [**Rie71**]. (We recall it briefly in Lemma 5.30.)

Example 5.26. (The symmetric and wedge power of torsion free sheaves are not necessarily torsion free)

Consider the maximal ideal sheaf $\mathfrak{m}_0$ in $X=\mathbb{C}^2$. The wedge power $\Lambda^2\mathfrak{m}_0$ is supported at the origin, however $z_1\wedge z_2$ is a non zero element of germ of $\Lambda^2\mathfrak{m}_0$ at the origin.

For the symmetric powers, let us first recall the following important theorem in [**Mic64**] (cf. also theorem 3 of [**LaB14**]). Let A be a domain, M be a finitely generated A-module. Then $\oplus_{i\geqslant 0}S^iM$ is a domain if and only if S^iM is torsion free, for all $i\geqslant 0$. To give a concrete example, consider a surjection from a holomorphic vector bundle E to a torsion free sheaf $\mathcal{F}$ over a compact manifold X. Then $\mathbb{P}(\mathcal{F})$ is a closed analytic set of $\mathbb{P}(E)$. If $\mathbb{P}(\mathcal{F})$ is not irreducible, by the above theorem there exists $i>0$ such that $S^i\mathcal{F}$ is not torsion free.

We summarise the algebraic properties of strongly psef torsion free sheaf in the following propositions.

PROPOSITION 5.3.2. *Let $\mathcal{F}$ be a torsion free sheaf over a compact Kähler manifold (X,ω). The following properties are equivalent.*

(1) $\mathcal{F}$ is strongly psef.

(2) For any $m \in \mathbb{N}^*$, $S^m\mathcal{F}$ modulo its torsion part is strongly psef.

(3) There exists $m \in \mathbb{N}^*$ such that $S^m\mathcal{F}$ modulo its torsion part is strongly psef.

PROOF. (1) implies (2) as follows. Let σ be a modification of X such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ and $\sigma^*(S^m\mathcal{F}/\mathrm{Tors})/\mathrm{Tors}$ are vector bundles where Tors means the torsion part. We have a surjection

$$\sigma^*S^m\mathcal{F} \cong S^m\sigma^*\mathcal{F} \to \sigma^*(S^m\mathcal{F}/\mathrm{Tors}).$$

It induces a surjection

$$S^m(\sigma^*\mathcal{F}/\mathrm{Tors}) \to \sigma^*(S^m\mathcal{F}/\mathrm{Tors})/\mathrm{Tors}.$$

This is justified as follows. Recall that there exists an exact sequence

$$\mathrm{Tors} \otimes S^{m-1}\sigma^*\mathcal{F} \to S^m(\sigma^*\mathcal{F}) \to S^m(\sigma^*\mathcal{F}/\mathrm{Tors}) \to 0$$

induced by

$$0 \to \mathrm{Tors} \to \sigma^*\mathcal{F} \to \sigma^*/\mathrm{Tors} \to 0.$$

The image of $\mathrm{Tors} \otimes S^{m-1}\sigma^*\mathcal{F}$ consists of torsion elements, and induces the morphism $S^m(\sigma^*\mathcal{F}/\mathrm{Tors}) \to \sigma^*(S^m\mathcal{F}/\mathrm{Tors})/\mathrm{Tors}$. Thus Corollary 5.11 and (1) of Corollary 5.13 implies that $\sigma^*(S^m\mathcal{F}/\mathrm{Tors})/\mathrm{Tors}$ is a strongly psef vector bundle.

We finally check that (3) implies (1). With the same notation, the above surjection is in fact an isomorphism since both sides have the same rank. Thus Corollary 5.11 implies that $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a strongly psef vector bundle. □

DEFINITION 5.27. *Let (X,ω) be a compact Kähler manifold, $\mathcal{F}$ be a torsion free sheaf on X and D be a $\mathbb{Q}-$Cartier divisor. Then $\mathcal{F}\langle D\rangle$ is said to be $\mathbb{Q}-$twisted strongly pseudo-effective ($\mathbb{Q}-$twisted strongly psef for short) if $S^m\mathcal{F}/\mathrm{Tors} \otimes \mathcal{O}_X(mD)$ is strongly psef for some (hence any by Proposition 5.3.2) $m > 0$ such that $\mathcal{O}_X(mD)$ is a line bundle.*

PROPOSITION 5.3.3.

(1) A torsion free quotient sheaf of a strongly psef torsion free sheaf is strongly psef.

(2) A direct summand of strongly psef torsion free sheaf is strongly psef.

(3) A direct sum of strongly psef torsion free sheaves is strongly psef.

(4) A tensor product (or Schur functor of positive weight) modulo its torsion part of strongly psef torsion free sheaves is strongly psef.

PROOF. Let $\mathcal{F} \to \mathcal{Q}$ be a surjective morphism of torsion free sheaves with $\mathcal{F}$ strongly psef over X. Let $\sigma : \tilde{X} \to X$ be a modification such that σ^*F/Tors,$\sigma^*\mathcal{Q}/\mathrm{Tors}$ are vector bundles. By assumption σ^*F/Tors is a strongly psef vector bundle. σ^* is right exact which induces surjection $\sigma^*F/\mathrm{Tors} \to \sigma^*\mathcal{Q}/\mathrm{Tors}$ passing to quotient. Thus $\sigma^*\mathcal{Q}/\mathrm{Tors}$ is a quotient bundle of $\sigma^*\mathcal{F}/\mathrm{Tors}$ Using Proposition 5.2.1 we can conclude that $\sigma^*\mathcal{Q}/\mathrm{Tors}$ is strongly psef. The other conclusions are similar and can be obtained in a formal manner. □

A natural operation for torsion free sheaves consists of taking the bidual. The relationships between a torsion free sheaf and its bidual will be stated in the next propositions. The following example indicates some of the occurring phenomena.

EXAMPLE 5.28. Let D be a smooth effective divisor over a compact Kähler manifold X with canonical section s_D. We have generic surjective sheaf morphism

$$\alpha : \mathcal{O}_X^{\oplus 2} \to \mathcal{O}_X(D) \oplus \mathcal{O}_X(2D)$$

induced by global section (s_D, s_D^2). Then $\det(\alpha) \cong \mathcal{O}_X(3D)$ has a global section s_D^3. The division by this global section induces a bimeromorphic map between the total spaces of $(\mathcal{O}_X(D) \oplus \mathcal{O}_X(2D))^*$ and $\mathcal{O}_X^{\oplus 2} \otimes \det(\alpha)^*$. Since $\mathcal{O}_X^{\oplus 2}$ is strongly psef, there exists a global (quasi-)psh function on the total space of its dual. Pairing with s_D^3 induces a global (quasi-)psh function on the total space of $\mathcal{O}_X^{\oplus 2} \otimes \det(\alpha)^*$ which induces a (quasi-)psh function on the total space of $(\mathcal{O}_X(D) \oplus \mathcal{O}_X(2D))^*$ outside a smooth divisor. We claim that this (quasi-)psh function extends across the divisor by boundedness from above. In particular, $(\mathcal{O}_X(D) \oplus \mathcal{O}_X(2D))$ is strongly psef.

For example, locally consider the psh function on the total space of $\mathcal{O}_X^{\oplus 2} \otimes \det(\alpha)^*$

$$\varphi(z,\xi_1,\xi_2) := \log(|z|^6(|\xi_1|^2 + |\xi_2|^2))$$

where $\alpha(z,\xi_1,\xi_2) = (z, z\xi_1, z^2\xi_2)$. The induced psh function outside the divisor $\{z = 0\}$ is given by

$$\varphi(z,\xi_1,\xi_2) := \log(|z|^6(|\xi_1/z|^2 + |\xi_2/z^2|^2)),$$

which is bounded from above near the divisor and can thus be extended across the divisor.

PROPOSITION 5.3.4. *Let $\mathcal{E}, \mathcal{F}$ be two torsion free sheaves over a compact Kähler manifold (X, ω). Let $\alpha : \mathcal{E} \to \mathcal{F}$ be a morphism of sheaves which is an isomorphism over a Zariski open set $X \smallsetminus A$. Assume that $\mathcal{E}$ is strongly psef. Then $\mathcal{F}$ is strongly psef.*

PROOF. Let σ be a modification of X such that $\sigma^*\mathcal{E}/\mathrm{Tors}$ and $\sigma^*\mathcal{F}/\mathrm{Tors}$ are locally free and $\sigma^*\mathcal{E}/\mathrm{Tors}$ is strongly psef. We can assume that $\sigma^*\alpha$ is an isomorphism outside a divisor E. Then $\det(\sigma^*\alpha)$ is an effective divisor supported in E. Division by this global section induces bimeromorphic map between the total spaces of $(\sigma^*\mathcal{F}/\mathrm{Tors})^*$ and $(\sigma^*\mathcal{E}/\mathrm{Tors})^* \otimes \det(\sigma^*\alpha)^*$. By Proposition 5.2.1, the fact that $\sigma^*\mathcal{E}/\mathrm{Tors}$ is strongly psef implies the existence of quasi-psh functions with analytic singularities on the total space of the symmetric powers of $(\sigma^*\mathcal{E}/\mathrm{Tors})^*$. Pairing with the canonical section of $\det(\sigma^*\alpha)$ induces global (quasi-)psh functions on the total space of $S^m(\sigma^*\mathcal{E}/\mathrm{Tors} \otimes \det(\alpha))^*$. We denote these quasi-psh functions by w_m. The functions w_m induce quasi-psh function on the total space of $(S^m\sigma^*\mathcal{F}/\mathrm{Tors})^*$ outside the divisor E. We claim that these quasi-psh functions extend across all the irreducible components of the divisor E by boundedness from above. In particular, by Proposition 5.2.1, $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a strongly psef vector bundle.

The claim is proven by a local coordinate calculation. In local coordinate $\sigma^*\alpha(z, \xi) = (z, A(z)\xi)$ where $A(z)$ a matrix of holomorphic functions. Locally

$$w_m(z, \xi) = \log(\sum_j |B_j(z)\xi|^2) + O(1) + \log(|\det(A(z))|^2)$$

where $B_j(z)$ are matrices of holomorphic functions. The induced quasi-psh functions outside the divisor E over $(S^m\sigma^*\mathcal{F}/\mathrm{Tors})^*$ are of the form

$$\tilde{w}_m(z, \xi) = \log(\sum_j |B_j(z)A^{-1}(z)\xi|^2) + O(1) + \log(|\det(A(z))|^2).$$

Since the inverse is given by the co-adjoint of the matrix divided by its determinant, $\tilde{w}_m$ is locally bounded from above near the divisor. □

The inverse direction is in general false. To get a counter-example, we consider an inclusion $\mathcal{I}_A \to \mathcal{O}_X$ where A is an analytic set of codimension at least 2. Then $\mathcal{I}_A$ is not strongly psef, while $\mathcal{O}_X$ is. However, the inclusion is an isomorphism over $X \smallsetminus A$.

PROPOSITION 5.3.5. *Let*

$$0 \to \mathcal{S} \to \mathcal{F} \to \mathcal{Q} \to 0$$

be an exact sequence of torsion free sheaves. If $\mathcal{F}, (\det(\mathcal{Q}))^{-1}$ are strongly psef and $\mathcal{S}$ is reflexive, then $\mathcal{S}$ is strongly psef.

PROOF. We have $S = \Lambda^{s-1}S^* \otimes \det S$ where s is the rank of S outside an analytic set A of codimension at least 2. Assume that all three sheaves are locally free outside A. We have a surjective bundle morphism over $X \smallsetminus A$

$$\Lambda^{r-s-1}\mathcal{F}/\mathrm{Tors} \otimes (\det \mathcal{Q})^{-1} \to S$$

where r is the rank of $\mathcal{F}$. Since $\mathcal{S}$ is reflexive (hence normal), the morphism extends as a morphism of sheaves over X. By (4) of Proposition 5.3.3, $\Lambda^{r-s-1}\mathcal{F}/\mathrm{Tors} \otimes (\det \mathcal{Q})^{-1}$ is strongly psef. By (1) of Proposition 5.3.3, the image of this sheaf morphism is strongly psef. Since the image ans $\mathcal{S}$ are isomorphism over $X \smallsetminus A$, by Proposition 5.3.4, S is strongly psef. □

PROPOSITION 5.3.6. *Let $\mathcal{F}$ be a strongly psef torsion free sheaf of rank r. Then $\det(\mathcal{F})$ is a psef line bundle.*

PROOF. By (4) of Proposition 5.3.3, $\Lambda^r\mathcal{F}/\mathrm{Tors}$ is strongly psef. Since $\Lambda^r\mathcal{F}/\mathrm{Tors}$ and $\det(\mathcal{F})$ is generic isomorphism, by Proposition 5.3.4, $\det(\mathcal{F})$ is a psef line bundle. □

PROPOSITION 5.3.7. *Let $\mathcal{F}$ be a strongly psef torsion free sheaf with $c_1(\mathcal{F}) = 0$. Then $\mathcal{F}^*$ is a strongly psef reflexive sheaf.*

PROOF. The fact that $\mathcal{F}^*$ is reflexive is purely algebraic. Outside an analytic set of codimension at least 2, $\mathcal{F}$ is locally free. Over this open set, we have an isomorphism

$$\Lambda^{r-1}\mathcal{F}/\mathrm{Tors} \otimes (\det(\mathcal{F}))^{-1} \to \mathcal{F}^*.$$

Since $\mathcal{F}^*$ is reflexive, this morphism extends across the analytic set. By (4) of Proposition 5.3.3, the left hand term is strongly psef. Thus the image is strongly psef. Moreover, the fact that we have a generic isomorphism implies that $\mathcal{F}^*$ is strongly psef. □

LEMMA 5.29. *Let $\mathcal{F}$ be a strongly psef torsion free sheaf with $c_1(\mathcal{F}) = 0$ over X. Let $\sigma : \tilde{X} \to X$ be a modification such that both $\sigma^*\mathcal{F}/\mathrm{Tors}$ and $\sigma^*\mathcal{F}^*/\mathrm{Tors}$ are locally free. Then $c_1(\sigma^*\mathcal{F}/\mathrm{Tors}) = 0$.*

PROOF. There exists a natural morphism

$$\sigma^*\mathcal{F}^*/\mathrm{Tors} \to (\sigma^*\mathcal{F}/\mathrm{Tors})^*$$

which is a generic isomorphism. Note that $(\sigma^*\mathcal{F}/\mathrm{Tors})^* \cong (\sigma^*\mathcal{F})^*$ by Corollary 4.9 Chap. V [**Kob75**]. The above morphism is induced by $\sigma^*\mathcal{F}^* \to (\sigma^*\mathcal{F}/\mathrm{Tors})^* \cong (\sigma^*\mathcal{F})^*$ under which the torsion part is in the kernel since $(\sigma^*\mathcal{F})^*$ is torsion free. By proposition 5.3.4, $(\sigma^*\mathcal{F}/\mathrm{Tors})^*$ is strongly psef. In other words, both $\sigma^*\mathcal{F}/\mathrm{Tors}$ and its dual are strongly psef vector bundle which infers that its first Chern class is 0. □

We can now prove the main result of this section assuming the main theorem (whose proof is independent of the main result of this section). For the convenience of readers, we recall here the construction of reduction of torsion free sheaf to the vector bundle case modulo torsion. For a complete proof, we recommend the paper of [**Ros68**].

LEMMA 5.30. *Let $\mathcal{F}$ be a torsion free sheaf of generic rank r over X a complex manifold. There exists some modification $\sigma : \tilde{X} \to X$ such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ is locally free. Then for every $i = 1, 2$, the Chern class $c_i(\mathcal{F})$ is well defined in the Bott-Chern cohomology group $H^{i,i}_{BC}(X, \mathbb{C})$.*

If X is compact Kähler and $\mathcal{F}$ is a reflexive sheaf, these two Chern classes can be represented by normal currents (in fact differences of two closed positive currents).

PROOF. Cover X by Stein open sets U_α. On each U_α, there exists an exact sequence

$$\mathcal{O}_{U_\alpha}^{\oplus M_\alpha} \to \mathcal{O}_{U_\alpha}^{\oplus N_\alpha} \to \mathcal{F}|_{U_\alpha} \to 0$$

which induces a meromorphic map

$$f_\alpha : U_\alpha \dashrightarrow \mathrm{Gr}(r, N_\alpha).$$

The maps $\mathcal{O}_{U_\alpha}^{\oplus M_\alpha} \to \mathcal{O}_{U_\alpha}^{\oplus N_\alpha}$ are locally given as holomorphic matrices $A_\alpha(z)$ which are of constant rank over Zariski open sets, and f_α sends z to the image of $A_\alpha(z)$. Let $\hat{U}_\alpha$ be the graph of this map $\hat{f}_\alpha : \hat{U}_\alpha \to \mathrm{Gr}(r, N_\alpha)$ be the corresponding morphism (given by the second projection of the graph). The $\hat{U}_\alpha$ glue into a complex space $\hat{X}$ sitting over X, and by Hironaka, we can find a modification $\sigma : \tilde{X} \to \hat{X} \to X$ such that $\tilde{X}$ is smooth and $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a vector bundle (the pull-back to $\hat{X}$ comes locally from the tautological quotient bundle Q_α of $\mathrm{Gr}(r, N_\alpha)$ generically, hence is already a vector bundle generically). It can be shown that the surjection $\sigma^*\mathcal{F} \to Q_\alpha$ which is in fact generic isomorphism. This infers in particular that the kernel is torsion and isomorphism $\sigma^*\mathcal{F}/\mathrm{Tors} \to Q_\alpha$. We equip Q_α with a smooth metric (e.g. the standard one coming from a Hermitian structure on $\mathbb{C}^{N_\alpha}$) and use a partition of unity to endow $\sigma^*\mathcal{F}/\mathrm{Tors}$ with a smooth metric h. Then the Chern forms $c_i(\sigma^*\mathcal{F}/\mathrm{Tors}, h)$ associated with the curvature tensor represent the Chern classes $c_i(\sigma^*\mathcal{F}/\mathrm{Tors})$ in Bott-Chern cohomology on $\tilde{X}$. We define the Chern classes $c_i(\mathcal{F}/\mathrm{Tors})$ in Bott-Chern cohomology on X to be the direct images $\sigma_* c_i(\sigma^*\mathcal{F}, h)$ for $i = 1, 2$ as in Lemma 5.20. (Notice that in lemma 5.20, we work with the de Rham cohomology. By the work of [**Gri10**] and the result in Chapter 6, the same formula holds in the complex Bott-Chern cohomology.) It is well known that these classes are independent of the choice of the metric h.

Assume now that X is a compact Kähler manifold. Then $\tilde{X}$ is also a compact Kähler manifold. Let ω be a smooth Kähler form on $\tilde{X}$. Then for C large enough, $c_i(\sigma^*\mathcal{F}/\mathrm{Tors}, h)$ can be written as difference of two positive forms $c_i(\sigma^*\mathcal{F}/\mathrm{Tors}, h) + C\omega^i$ and $C\omega^i$. The second statement holds by taking direct images of these positive forms. □

PROPOSITION 5.3.8. *Let $\mathcal{F}$ be a nef reflexive sheaf over a compact Kähler manifold (X, ω) with $c_1(\mathcal{F}) = 0$. Then $\mathcal{F}$ is a nef vector bundle.*

PROOF. The proof is analogous to those of [**CCM19**] and [**HIM19**]. The essential point is the following result of [**BS94**]: *for a polystable reflexive sheaf $\mathcal{F}$ of rank r over a compact n-dimensional Kähler manifold (X, ω), one has the Bogomolov inequality*

$$\int_X (2rc_2(\mathcal{F}) - (r-1)c_1(\mathcal{F})^2) \wedge \omega^{n-2} \geqslant 0,$$

and the equality holds if and only if $\mathcal{F}$ is locally free and its Hermitian-Einstein metric gives a projective flat connection.

The proof is obtained by an induction on the rank of $\mathcal{F}$. The general strategy of the induction is the same as in [**HIM19**]. For the convenience of the reader, we outline here the arguments with the necessary modifications. In the rank one case, reflexive sheaves are locally free, hence line bundles, and the conclusion is immediate. Let us observe however that the reflexivity condition is necessary even in that case; for example, the ideal sheaf associated with an analytic set of codimension at least 2 is of generic rank one, torsion free, but not locally free.

In the higher rank case, we consider the Harder-Narasimhan filtration of $\mathcal{F}$ with respect to ω, say

$$\mathcal{F}_0 = 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_m := \mathcal{F}$$

where $\mathcal{F}_i/\mathcal{F}_{i-1}$ is ω-stable for every i and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$, and where $\mu_j = \mu_\omega(\mathcal{F}_j/\mathcal{F}_{j-1})$ is the slope of $\mathcal{F}_j/\mathcal{F}_{j-1}$ with respect to ω. Now, consider the coherent subsheaf $\mathcal{S} = \mathcal{F}_{m-1}$. Notice that by construction $\mathcal{S}$ can be chosen to be reflexive by taking the double dual if necessary, as this preserves the rank, first Chern class and slope. Then we get a short exact sequence

$$0 \to \mathcal{S} \to \mathcal{F} \to \mathcal{Q} \to 0,$$

and $\mathcal{Q}$ is a torsion-free coherent sheaf. Pick a modification σ such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ and $\sigma^*(\mathcal{Q})/\mathrm{Tors}$ are vector bundles, with $\sigma^*\mathcal{F}/\mathrm{Tors}$ being nef. The pull back functor is right exact, so we have surjective bundle morphism $\sigma^*\mathcal{F}/\mathrm{Tors} \to \sigma^*(\mathcal{Q})/\mathrm{Tors}$. Thus $\sigma^*(\mathcal{Q})/\mathrm{Tors}$ is a nef vector bundle. By definition, we conclude that $\mathcal{Q}$ is nef.

In particular, its first Chern class $c_1(\mathcal{Q})$ is pseudo-effective by Proposition 5.3.6. On the other hand, we have

$$0 = c_1(\mathcal{F}) = c_1(\mathcal{S}) + c_1(\mathcal{Q})$$

by the assumption. Thus

$$\int_X c_1(\mathcal{Q}) \wedge \omega^{n-1} = -\int_X c_1(\mathcal{S}) \wedge \omega^{n-1} \leqslant 0,$$

and $c_1(\mathcal{Q}) = c_1(\mathcal{S}) = 0$.

Let X_0 be the largest open set on which $\mathcal{F}$ is locally free. We claim that $\mathcal{S}$ is a vector subbundle of $\mathcal{F}$ on X_0, and that the morphism $\mathcal{S} \to \mathcal{F}$ is a bundle morphism on X_0; for this, we apply corollary 1.20 of [**DPS94**] and prove that $\det(\mathcal{Q}^*) \to \Lambda^p\mathcal{F}^*$ is an injective bundle morphism on X_0, where p is the rank of $\mathcal{Q}$. This corresponds to a global section $\tau \in H^0(X, (\Lambda^p\mathcal{F}^*)^{**} \otimes \det(\mathcal{Q}^{**}))$ since $X \smallsetminus X_0$ is of codimension at least 3.

There exists a modification $\sigma : \tilde{X} \to X$ such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ and $\sigma^*\mathcal{F}^*/\mathrm{Tors}$ are vector bundles. We can assume that σ is obtained as a composition of smooth centres in $X \smallsetminus X_0$. We can view $\sigma^*\tau$ as an element in $H^0(\tilde{X}, \sigma^*[(\Lambda^p\mathcal{F}^*)^{**} \otimes \det(\mathcal{Q}^{**})])$ as well as an element in $H^0(\tilde{X}, \Lambda^p(\sigma^*\mathcal{F}/\mathrm{Tors})^* \otimes \sigma^*\det(\mathcal{Q}^{**}))$ under natural morphism

$$\sigma^*[(\Lambda^p\mathcal{F}^*)^{**}] \to [\sigma^*(\Lambda^p\mathcal{F}^*)]^{**} \to \Lambda^p(\sigma^*\mathcal{F}/\mathrm{Tors})^*.$$

More precisely, the natural morphism $\sigma^*\mathcal{F}^* \to (\sigma^*\mathcal{F})^*$ induces

$$\sigma^*(\Lambda^p\mathcal{F}^*) = \Lambda^p\sigma^*(\mathcal{F}^*) \to \Lambda^p\sigma^*(\mathcal{F})^* = \Lambda^p(\sigma^*\mathcal{F}/\mathrm{Tors})^*.$$

By taking the bidual, we obtain the second morphism.

Let us observe that $\Lambda^p\sigma^*\mathcal{F}/\mathrm{Tors}$ is nef, and also that $\det(Q^*)$ is nef since $c_1(\mathcal{Q}) = 0$. Thus $\sigma^*\tau$ cannot vanish at any point of $\tilde{X}$ by Prop. 1.16 of [**DPS94**]. Thus τ does not vanish on X_0. This concludes the proof of the claim. In particular, $\mathcal{Q}$ is a vector bundle over X_0.

Let s be the rank of $\mathcal{S}$, which must be strictly smaller than the rank r of $\mathcal{F}$. We consider the surjective bundle morphism

$$\Lambda^{r-s+1}\mathcal{F} \otimes \det\ \mathcal{Q}^* \to \mathcal{S}$$

on X_0. Since $\mathcal{F}$ is nef and $\det \mathcal{Q}^*$ is numerical trivial, we infer that $\mathcal{S}$ is a strongly psef reflexive sheaf by Proposition 5.3.5. Thus over some bimeromorphic model of X, the pull back of $\mathcal{S}$ is a strongly psef vector bundle modulo torsion with vanishing first Chern class by Lemma 5.29. By Theorem 5.48, over the bimeromorphic model, the vector bundle is in fact nef. By the induction hypothesis, $\mathcal{S}$ is in fact a nef vector bundle over X.

$\mathcal{Q}$ is a priori not necessarily a reflexive sheaf, but the double dual $\mathcal{Q}^{**}$ is. To conclude that $\mathcal{Q}^{**}$ is in fact a vector bundle by the result of Bando-Siu recalled at the beginning, it is enough to prove that $c_2(\mathcal{Q}^{**}) = 0$. Since $\mathcal{Q}$ is locally free on X_0 and the codimension of $X \smallsetminus X_0$ is at least 3, $\mathcal{Q}$ coincides with $\mathcal{Q}^{**}$ on X_0. Let i be the inclusion $X_0 \to X$. Since the restriction map $i^* : H^4(X, \mathbb{R}) \to H^4(X_0, \mathbb{R})$ is an isomorphism by Lemma 5.18 and

$$i^*c_2(\mathcal{Q}) = c_2(\mathcal{Q}|_{X_0}) = c_2(\mathcal{Q}^{**}|_{X_0}) = i^*c_2(\mathcal{Q}^{**}),$$

we infer that $c_2(\mathcal{Q}) = c_2(\mathcal{Q}^{**})$. Let $\pi : \mathbb{P}(\sigma^*\mathcal{Q}/\mathrm{Tors}) \to X$ be the projectivization of the nef vector bundle $\sigma^*\mathcal{Q}/\mathrm{Tors}$, viewed as a quotient of the nef vector bundle $\sigma^*\mathcal{F}/\mathrm{Tors}$. By the definition of Segre classes, we have

$$\pi_*(c_1(\mathcal{O}_{\mathbb{P}(\sigma^*\mathcal{Q}/\mathrm{Tors})}(1))^{r-s+1}) = s_2(\sigma^*\mathcal{Q}/\mathrm{Tors}) = c_1^2(\sigma^*\mathcal{Q}/\mathrm{Tors}) - c_2(\sigma^*\mathcal{Q}/\mathrm{Tors}).$$

In particular,

$$\int_{\tilde{X}} s_2(\sigma^*\mathcal{Q}/\mathrm{Tors}) \wedge \tilde{\omega}^{n-2} = \int_{\mathbb{P}(\sigma^*\mathcal{Q}/\mathrm{Tors})} c_1(\mathcal{O}_{\mathbb{P}(\sigma^*\mathcal{Q}/\mathrm{Tors})}(1))^{r-s+1} \wedge \tilde{\omega}^{n-2} \geqslant 0,$$

as $\sigma^*\mathcal{Q}/\mathrm{Tors}$ is a nef vector bundle and thus $s_2(\sigma^*\mathcal{Q}/\mathrm{Tors}) = -c_2(\sigma^*\mathcal{Q}/\mathrm{Tors})$ is a positive class, containing a closed positive $(2,2)$-current. Here $\tilde{\omega}$ is any Kähler form on $\tilde{X}$. Since $c_1(\sigma^*\mathcal{Q}/\mathrm{Tors}) = 0$ by Lemma 5.29, we deduce that

$$\int_{\tilde{X}} c_2(\sigma^*\mathcal{Q}/\mathrm{Tors}) \wedge \tilde{\omega}^{n-2} \leqslant 0.$$

The inequality is valid for any Kähler form on $\tilde{X}$. In particular, we can take a sequence of Kähler metrics on $\tilde{X}$ converging to $\pi^*\omega$, and this implies

$$\int_X c_2(\mathcal{Q}) \wedge \omega^{n-2} = \int_{\tilde{X}} c_2(\sigma^*\mathcal{Q}/\mathrm{Tors}) \wedge \pi^*\omega^{n-2} \leqslant 0.$$

Notice that the first equality is by Lemma 5.20. The Bogomolov inequality shows that

$$\int_X c_2(\mathcal{Q}^{**}) \wedge \omega^{n-2} = 0.$$

$\mathcal{Q}^{**}$ is thus in fact a vector bundle by the result of Bando-Siu.

The extension class obtained from the exact sequence on X_0 can be extended to the extension class (defined on X) of $\mathcal{S}$ and $\mathcal{Q}^{**}$ by lemma 5.15. The extended class by construction determines a vector bundle whose restriction to X_0 is isomorphic to $\mathcal{F}$. Since $\mathcal{F}$ is a reflexive sheaf, in fact we have an isomorphism on X. This proves that $\mathcal{F}$ is in fact a vector bundle. By remark 5.22, it is a nef vector bundle. □

REMARK 5.31. It has been observed by Demailly, that the previous proposition can be derived from Theorem 1.18 of [**DPS94**] (cf. also [**Deng16**]). Let us recall the statement of this theorem. *Let E be a numerically flat vector bundle over a compact Kähler manifold (X,ω). Then there exists a filtration of E*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_p = E$$

by vector subbundles such that the quotients E_k/E_{k-1} are hermitian flat, i.e. given by unitary representations $\pi_1(X) \to U(r_k)$.

Since $\mathcal{F}$ is a nef reflexive sheaf with $c_1(\mathcal{F}) = 0$, there exists a modification such that $\sigma : \tilde{X} \to X$ such that $\sigma^*\mathcal{F}/\mathrm{Tors}$ is a nef vector bundle with vanishing first Chern class by lemma 5.29. By the above theorem, there exists a filtration of $\sigma^*\mathcal{F}/\mathrm{Tors}$

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_p = \sigma^*\mathcal{F}/\mathrm{Tors}$$

by vector bundles over $\tilde{X}$ such that $\tilde{E}_k/\tilde{E}_{k-1}$ are hermitian flat.

We claim that $\tilde{E}_k/\tilde{E}_{k-1} = \sigma^*(E_k/E_{k-1})$ for some vector bundle E_k/E_{k-1} over X for each k. (For the moment, E_k/E_{k-1} is just a notion, not the quotient of two vector bundles over X. But it is the case which is proven in the next paragraph.) The reason is as follows. $\sigma_* : \pi_1(\tilde{X}) \to \pi_1(X)$ is an isomorphism since we can assume that σ is composition of a sequence of blows-up of smooth centres and as a CW complex a blow-up of smooth center changes skeleton of (real) codimension at least 2 which preserves the fundamental group. Thus we have unitary representations $\pi_1(X) \to U(r_k)$ which proves the claim.

Let A be the analytic set such that $\mathcal{F}$ is locally free over $X \smallsetminus A$. Since $\mathcal{F}$ is reflexive, A is of codimension at least 3 in X. Without loss of generality, we can assume that σ induces an isomorphism between $\sigma^{-1}(X \smallsetminus A)$ and $X \smallsetminus A$. Thus we have extension of vector bundles over $X \smallsetminus A$

$$0 \to E_{k-1}|_{X\smallsetminus A} \to E_k|_{X\smallsetminus A} \to E_{k-1}/E_{k-1}|_{X\smallsetminus A} \to 0$$

where E_k are a priori vector bundles defined over $X \smallsetminus A$. By lemma 5.15, the extensions extend across A. Thus there exist vector bundles E_k over X which are the extensions of E_{k-1} and E_k/E_{k-1}.

By construction, we have isomorphism $\mathcal{F}|_{X\smallsetminus A} \cong E_p|_{X\smallsetminus A}$. Since $\mathcal{F}$ is reflexive, we have isomorphism $\mathcal{F} \cong E_p$ over X. In particular, $\mathcal{F}$ is a vector bundle.

REMARK 5.32. In the proof, we have shown that $c_2(\mathcal{F}) = c_2(\mathcal{F}^{**}) \in H^4(X,\mathbb{R})$ from the fact that $\mathcal{F} = \mathcal{F}^{**}$ outside an analytic set of codimension at least 3. In fact, the equality also holds in Bott-Chern cohomology, and the latter equality induces the previous one by the natural morphism from the Bott-Chern cohomology to the de Rham cohomology.

The proof is an easy consequence of the following diagram, using the same notation as in the proof.

$$\begin{array}{ccc} H^{2,2}_{BC}(X,\mathbb{C}) & \longrightarrow & H^{2,2}_{BC}(X \smallsetminus A,\mathbb{C}) \\ \downarrow & & \downarrow \\ H^4(X,\mathbb{C}) & \xrightarrow{\cong} & H^4(X \smallsetminus A,\mathbb{C}). \end{array}$$

By the Hodge decomposition theorem, the left vertical arrow is an injection, and this implies that the map $H^{2,2}_{BC}(X,\mathbb{C}) \to H^{2,2}_{BC}(X \smallsetminus A,\mathbb{C})$ is also injective.

REMARK 5.33. The difficulty to extend the above proof to the case where $\mathcal{F}$ is a strongly psef reflexive sheaf is to prove that

$$\int_{\mathbb{P}(\sigma^*\mathcal{F}/\mathrm{Tors})} c_1(\mathcal{O}_{\mathbb{P}(\sigma^*\mathcal{F}/\mathrm{Tors})}(1))^{r+1} \wedge \tilde{\omega}^{n-2} \geqslant 0$$

on some bimeromorphic model of X. In the nef case, with small loss of positivity, the cohomology class can be represented by smooth forms. Thus the above inequality is trivial when taking the small loss tending to 0. In the strongly psef case, the cohomology class can be represented by a current with analytic singularities only at the expense of some loss of positivity. However a wedge product of arbitrary currents is not always well defined. In the next section, we make a "digression" and discuss what we call Segre currents to investigate the strongly psef case.

5.4. Segre forms

In this section, we are interested in the following problem. Assume that E is a holomorphic vector bundle of rank r over a compact Kähler manifold (X,ω). Can one find a (k,k)-closed positive current in the Segre class $s_k(E) := \pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{k+r-1})$? We have to point out that a similar construction is made in [**LRRS18**], based on Demailly's improvement ([**Dem92a**]) of the Bedford-Taylor theory ([**BT82**]) of Monge-Ampère operators. The authors define the corresponding current as a limit of smooth forms induced from local smooth regularizations of the metric given in [**Rau15**]. Compared to theirs, our construction has the advantage that we define the relevant current as a limit of currents defined by Monge-Ampère operators without necessarily employing a regularizing sequence. In that way, we are still in a position to estimate the Lelong number of the limiting Segre current in terms by the Lelong number of the approximating sequence of weights. On the other hand, in the case of [**LRRS18**], the approximation is given by smooth forms, hence the Lelong number of the approximation forms is identically zero, and one does not a priori obtain any information on the Lelong number of the limiting current. The Lelong number estimate will be necessary in the next section.

In particular, starting from a singular metric with analytic singularities on $\mathcal{O}_{\mathbb{P}(E)}(1)$, the construction yields a singular metric on $\det(E)$ which is unique up to a constant and, as a consequence, the curvature of the induced metric of $\det(E)$ is uniquely determined by the curvature of the metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$.

To start with, we state some results of pluripotential theory. Some of this material is not essentially needed in the construction, but it provides intuition for a few arguments. The following statement is an improvement by Demailly of the Bedford-Taylor theory ([**BT82**]) of Monge-Ampère operators.

LEMMA 5.34 (Proposition 10.2 [**Dem93**]).
Let ψ be a plurisubharmonic function on a (non necessarily compact) complex manifold X such that ψ is locally bounded on $X \smallsetminus A$, where A is an analytic subset of X of codimension $\geqslant p+1$ at each point. Let θ be a closed positive current of bidimension (p,p).

Then $\theta \wedge i\partial\overline{\partial}\psi$ can be defined in such a way that $\theta \wedge i\partial\overline{\partial}\psi = \lim_{\nu\to\infty} \theta \wedge i\partial\overline{\partial}\psi_\nu$ in the weak topology of currents, for any decreasing sequence $(\psi_\nu)_{\nu\geqslant 1}$ of plurisubharmonic functions converging to ψ. Moreover, at every point $x \in X$ we have

$$\nu\Big(\theta \wedge \frac{i}{\pi}\partial\overline{\partial}\psi, x\Big) \geqslant \nu(\theta,x)\nu(\psi,x).$$

PROPOSITION 5.4.1. *Let T be a (k,k)-closed positive current in the cohomology class α, over a compact Kähler manifold (X,ω). Let U be a coordinate open set of X such that on U,*

$$C^{-1}\omega \leqslant \frac{i}{2\pi}\partial\overline{\partial}|z|^2 \leqslant C\omega.$$

Then for any $r_0 > 0$ and for any $x \in U$ with $d(x,\partial U) \geqslant r_0$ with respect to the Euclidean metric in the coordinate chart, we have for $r \leqslant r_0$

$$\frac{1}{r^{2n-2k}}\int_{B(x,r)} T \wedge \omega^{n-k} \leqslant \frac{C^{2n-2k}}{r_0^{2n-2k}}(\alpha\cdot\{\omega\}^{n-r}).$$

Here $(\alpha\cdot\{\omega\}^{n-r})$ is the intersection product of cohomology classes.

PROOF. It is enough to prove that

$$\frac{1}{r^{2n-2k}}\int_{B(x,r)} T \wedge \Big(\frac{i}{2\pi}\partial\overline{\partial}|z|^2\Big)^{n-k} \leqslant \frac{C^{n-k}}{r_0^{2n-2k}}(\alpha\cdot\{\omega\}^{n-r}).$$

By a basic observation of Lelong in [**Lel68**], the left hand term is a increasing function with respect to r. Thus we have

$$\frac{1}{r^{2n-2k}}\int_{B(x,r)} T\wedge\left(\frac{i}{2\pi}\partial\bar{\partial}|z|^2\right)^{n-k} \leqslant \frac{1}{r_0^{2n-2k}}\int_{B(x,r_0)} T\wedge\left(\frac{i}{2\pi}\partial\bar{\partial}|z|^2\right)^{n-k}.$$

However, the right hand term is at most

$$\frac{1}{r_0^{2n-2k}}\int_{B(x,r_0)} T\wedge (C\omega)^{n-k} \leqslant \frac{C^{2n-2k}}{r_0^{2n-2k}}(\alpha\cdot\{\omega\}^{n-r})$$

since T is a positive current. □

We will need the following standard local parametrization theorem for analytic sets.

LEMMA 5.35 (local parametrization theorem, cf. e.g. Theorem 4.19, Chap. II [**Dem12b**]). *Let $\mathcal{I}$ be an ideal in $\mathcal{O}_n$, let $A=V(\mathcal{I})$ and A_j be the irreducible components of A whose dimension is equal to the dimension of A. For every j and $d=d_j=\dim A_j$, there exists a generic choice of coordinates*

$$(z',z'')=(z_1,\cdots,z_d;z_{d+1},\cdots,z_n)\in\Delta'\times\Delta''$$

such that the restriction of the canonical projection to the first component $\pi_j: A_j\cap(\Delta'\times\Delta'')\to\Delta'$ is a finite and proper ramified cover, which moreover yields an étale cover $A_j\cap\pi^{-1}(\Delta'\smallsetminus S)\to\Delta'\smallsetminus S$, where S is an analytic subset in Δ'.

LEMMA 5.36. *Let A be a compact analytic subset of a complex manifold M. Assume that $\dim_{\mathbb{C}}A=d$ and $\dim_{\mathbb{C}}M=n$. Let (W_ν) be relatively compact coordinate charts which form a finite open covering of A. Without loss of generality, assume that W_ν is taken to be relatively compact in some larger coordinate chart, and is the coordinate chart provided by the local parametrization theorem. Then there exists $C>0$ such that for $r>0$ small enough, the open neighbourhood $\bigcup_\nu\{x\in W_\nu, d(A,x)<r\}$ of A can be covered by at most $\frac{C}{r^{2d}}$ balls of radius r. Here the distance is calculated by the coordinate distance in each coordinate chart.*

PROOF. It is enough to prove this for each W_ν. We verify that the volume of the open set $\{x\in W_\nu, d(A,x)<r\}$ has an upper bound Cr^{2n-2d} for r small enough. We take in each local tubular neighbourhood a maximal family of points with mutual coordinate distance $\geqslant r$. For r small enough, every point is at distance $\leqslant r$ to at least one of the centres, otherwise the family of points would not be maximal. In particular, balls of radius $2r$ centered at these points cover the tubular neighbourhood. On the other hand, balls of radius $r/2$ centered at these points are disjoint. Therefore, the number of such balls N_ν satisfies the relation

$$c_nN_\nu\left(\frac{r}{2}\right)^{2n}\leqslant \mathrm{Vol}(\{x\in W_\nu, d(A,x)<r\})\leqslant Cr^{2n-2d}.$$

Here c_n is the volume of the unit ball in $\mathbb{C}^n$. The lemma follows from the inequality.

The proof of the volume estimate for the tubular neighbourhood is obtained by induction on the dimension of the analytic set A. When $d=0$, i.e. when A consists of a finite set, the estimate is trivial. Assume that we have already proven the result for all analytic sets of dimension $d\leqslant\dim_{\mathbb{C}}(A)-1$. Then, we use the local parametrization theorem and the fact that $A\cap\pi^{-1}(S)$ is a proper analytic set of $A\cap W_\nu$. By the induction hypothesis, we have

$$\mathrm{Vol}(\{x\in W_\nu, d(A\cap\pi^{-1}(S),x)<r\})\leqslant Cr^{2n-2d+2},$$

and a similar estimate holds for the open set of points with distance $<r$ to the irreducible components of A of dimension $\leqslant d-1$. On the other hand, $A\cap\pi^{-1}(\Delta'\smallsetminus S)$ is contained in the union of $A_j+\sum_{i=d+1}^n\mathbb{D}(0,r)e_i$ where e_i is the standard basis of $\mathbb{C}^n$ and $\mathbb{D}(0,r)$ is the disc in $\mathbb{C}$ centered at 0 of radius r. Here A_j are the irreducible components of dimension d of A intersecting $\pi^{-1}(\Delta'\smallsetminus S)$. Each open set $A_j+\sum_{i=d+1}^n\mathbb{D}(0,r)e_i$ has volume equal to $c(n,d)\mathrm{Vol}(A_j)r^{2n-2d}$ where $c(n,d)$ is the volume of the unit disc in $\mathbb{C}^{n-d}$. This is because that π induces a biholomorphism between $A_j+\sum_{i=d+1}^n\mathbb{D}(0,r)e_i$ and $\Delta'\times 0+\sum_{i=d+1}^n\mathbb{D}(0,r)e_i$ which preserves the Lebesgue volume form. On the other hand the tubular neighbourhood of A $\{d(x,A)<r\}$ is included in the union of the union of $A_j+\sum_{i=d+1}^n\mathbb{D}(0,r)e_i$, the open set of points whose distance to the dimension $\leqslant d-1$ irreducible components of $A<r$ and $\{x\in W_\nu, d(A\cap\pi^{-1}(S),x)<r\}$ from which the estimate follows. □

PROPOSITION 5.4.2. *Let T be a (k,k)-closed positive current in the cohomology class α, over a compact Kähler manifold (X,ω). Let A be an analytic subset of X of dimension d. There exists a sequence of open neighbourhoods $U(r)$ of A (independent of T) such that $\bigcap_{r>0}U(r)=A$ and the volume of $U(r)$ is at most Cr^{2n-2d}, with a constant C independent of T. Moreover there exists C' independent of T such that*

$$\int_{U(r)} T\wedge\omega^{n-k}\leqslant C'r^{2n-2k-2d}.$$

Here C' depends on α, (X,ω) and A.

PROOF. This is a direct consequence of Proposition 5.4.1 and Lemma 5.36. □

Remark that in particular, if A is codimension at least $k+1$, the contribution of mass of T on $U(r)$ vanishes asymptotically as $r \to 0$, and the above Proposition holds uniformly for all positive currents T in the cohomology class α. The codimension condition is optimal since that the mass of the current $[A]$ associated with a k-dimensional analytic set A does not vanish in the limit.

Now we return to the construction of positive currents in the Segre classes. Observe that a codimension condition is needed to ensure the existence of such closed positive currents; this is shown by the following easy example.

EXAMPLE 5.37. Let X be the blow up of $\mathbb{P}^2$ at some point and let D be the exceptional divisor. Consider the vector bundle $E := \mathcal{O}(D)^{\oplus r}$ of rank $r \geqslant 2$ over X. Corollary 5.13 shows that E is a strongly psef vector bundle as a direct sum of strongly psef line bundles.

An equivalent definition of total Segre class (i.e. $\sum_k s_k(E)$) is the inverse of the total Chern class. Remark that for any vector bundles E, F, the total Chern class satisfies the axiom $c(E \oplus F) = c(E)c(F)$. Thus the same relation holds for the total Segre class since the cohomological ring is commutative. In particular, $s(E) = s(\mathcal{O}(D))^r$ with $s_2(E) = \binom{r}{2}(c_1(\mathcal{O}(D))^2) = -\binom{r}{2}$. Thus there exists no closed positive current in the class $s_2(E)$.

For the convenience of the reader, we recall the definition of a Finsler metric on a vector bundle, as introduced in [**Kob75**] (cf. also [**Dem99**]).

DEFINITION 5.38. *A (positive definite) Finsler metric on a holomorphic vector bundle E is a positive complex homogeneous function*

$$\xi \to \|\xi\|_x$$

defined on each fibre E_x, that is, such that $\|\lambda\xi\|_x = |\lambda|\|\xi\|_x$ for each $\lambda \in \mathbb{C}$ and $\xi \in E_x$, and $\|\xi\|_x > 0$ for $\xi \neq 0$.

We say that the metric is smooth if it is smooth outside of the zero section on the total space of E. Observe that a Finsler metric on a line bundle L is the same as a Hermitian metric on L. A Finsler metric on E^* can also be viewed as a Hermitian metric h^* on the line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ (as the total space of $\mathcal{O}_{\mathbb{P}(E)}(-1)$ coincides with the blow-up of E^* along the zero section). In particular, $\mathcal{O}_{\mathbb{P}(E)}(1)$ carries a smooth Hermitian metric of positive Chern curvature form if and only if E carries a smooth Finsler metric whose logarithmic indicatrix defined by

$$\chi(x,\xi) := \log\|\xi\|_x$$

is plurisubharmonic on the total space. Let us observe that the logarithmic indicatrix has a pole along the zero section and can be extended as a global psh function on the total space, even though it is a priori psh only outside of the zero section.

Assume that we have a smooth Hermitian metric on (E,h) rather than just a Finsler metric on E, and let us consider the corresponding Hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$. We have the following calculation, which can be seen as a direct consequence of intersection theory, and is still valid on the level of forms without passing to cohomology classes: for every $k \in \mathbb{N}$

$$\pi_* \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1),h)\right)^{r+k} = s_k(E,h).$$

Note that the Segre classes can be written in terms of Chern classes and the Chern classes can be represented by the Chern forms derived from the curvature tensor. For our application, we only detail the calculation for the case $k=1$ that we need. For the general case, we refer for example to the papers [**Div16**], [**Gul12**] and [**Mou04**]. The author thanks Simone Diverio for the references.

LEMMA 5.39. *Let E be a holomorphic vector bundle of rank r on a (non necessarily compact) complex manifold X. Let π be the canonical projection $\mathbb{P}(E) \to X$. Assume that E is endowed with a smooth Hermitian metric h, and consider the induced metrics on $\mathcal{O}_{\mathbb{P}(E)}(1)$ and $\det(E)$ (which we still denote by h). Then*

$$\pi_* \left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1),h)\right)^{r} = \frac{i}{2\pi}\Theta(\det(E),\det(h))$$

where Θ means the curvature tensor.

PROOF. To start with, we recall formula (15.15) of Chap. V in [**Dem12b**], expressing the curvature of $\mathcal{O}(1)$ for the projectivisation of a vector bundle. Let (e_λ) be a normal coordinate frame of E at $x_0 \in X$ and let

$$i\Theta(E)_{x_0} = \sum c_{jk\lambda\nu} i dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\nu$$

be the curvature tensor of E. At any point $a \in \mathbb{P}(E)$ represented by a vector $\sum_\lambda a_\lambda e_\lambda^* \in E_{x_0}^*$ of norm 1, the curvature of $\mathcal{O}_{\mathbb{P}(E)}(1)$ is

$$\Theta(\mathcal{O}_{\mathbb{P}(E)}(1))_a = \sum c_{jk\nu\lambda} a_\lambda \bar{a}_\nu dz_j \wedge d\bar{z}_k + \sum_{1 \leqslant \lambda \leqslant r-1} d\xi_\lambda \wedge d\bar{\xi}_\lambda,$$

where (ξ_λ) are the coordinates near a on $\mathbb{P}(E)$, induced by unitary coordinates of the hyperplane $a^\perp \subset E_{x_0}^*$. In other words, if $\mathbb{P}(E|_U)$ is locally isomorphic to $U \times \mathbb{P}^{r-1}$ with coordinates $(z, [\xi])$, we have a canonical projection $pr_2 : \mathbb{P}(E) \to \mathbb{P}^{r-1}$ and the curvature at $(z, [\xi])$ is given by

$$\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1))(z, [\xi]) = -\frac{\langle \frac{i}{2\pi}\Theta_{E^*}\xi, \xi\rangle_h}{\langle \xi, \xi\rangle_h} + pr_2^*\omega_{FS}$$

where ω_{FS} is the Fubini-Study metric on $\mathbb{P}^{r-1}$. Therefore we have

$$\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h)\right)^r = -r\int_{\mathbb{P}^{r-1}} \frac{\langle \frac{i}{2\pi}\Theta_{E^*}\xi, \xi\rangle_h}{\langle \xi, \xi\rangle_h} \wedge \omega_{FS}^{r-1}.$$

Observe that $\mathbb{P}^{r-1} \cong S^{2r-1}/S^1$ by the Hopf fibration. The Fubini-Study metric is the metric induced on the quotient $\mathbb{P}^{r-1}$ by the restriction of the standard Euclidean metric to the unit hypersphere. We denote by $d\sigma$ the volume form of the standard Euclidean metric restricted to that sphere. Then we have

$$\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h)\right)^r = -r\int_{S^{r-1}} \langle \frac{i}{2\pi}\Theta_{E^*}\xi, \xi\rangle_h \wedge d\sigma.$$

Note that for a Hermitian form $Q(\xi, \xi) = \sum \lambda_i |\xi_i|^2$ we have

$$\int_{S^{r-1}} Q(\xi, \xi) d\sigma(\xi) = \frac{1}{r}\mathrm{tr}(Q) = \frac{1}{r}\sum \lambda_i,$$

since $\int_{S^{r-1}} |\xi_i|^2 d\sigma(\xi) = \frac{1}{r}$ by symmetry. Thus we get

$$\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h)\right)^r = -\mathrm{tr}_\xi \langle \frac{i}{2\pi}\Theta_{E^*}\xi, \xi\rangle_h = \frac{i}{2\pi}\Theta(\det(E), h).$$

□

As a direct consequence of the above formula, if h is a smooth semi-positive metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$, the induced metric on $\det(E)$ is also semi-positive. This is the positive form what we want. More generally, the forms $\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h)\right)^{r+k} = s_k(E, h)$ are smooth positive currents in the $k-$th Segre class. Hence if h is a smooth semi-positive metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$, we can find positive forms in the Segre classes, which we will call Segre forms (or Segre currents) in the sequel.

In the case where the metric is singular, the construction is more complicated. The difficulty is that Monge-Ampère operators are not always well-defined for arbitrary closed positive currents.

In general, for a strongly psef vector bundle, in order to get a singular metric with analytic singularities, we have to allow a bounded negative part. Accordingly, we have to work in a more general setting. Let E be a vector bundle of rank r on a compact Kähler manifold (X, ω), and let T be a closed positive $(1,1)$-current on $\mathbb{P}(E)$, in the cohomology class of a fixed closed smooth form α. Notice that that the restriction of the cohomology class $\{\alpha\}$ is constant on any fibre of $\pi : \mathbb{P}(E) \to X$. A typical case is $\{\alpha\} = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + C\pi^*\omega$ for some $C \geqslant 0$. Write

$$T = \alpha + i\partial\bar{\partial}\varphi.$$

Assume that φ is smooth over $\mathbb{P}(E) \smallsetminus A$ where A is an analytic set in $\mathbb{P}(E)$ such that $A = \pi^{-1}(\pi(A))$ and $\pi(A)$ is of codimension at least k in X. We wish to define a current $\pi_* T^{r-1+k}$. A priori, this Monge-Ampère operator is not well defined by just invoking the codimension condition, since the exponent $r-1+k$ is larger than the codimension k. This problem can be overcome by defining the desired current as a weak limit of a sequence of less singular currents, in such a way that the limit is still unique.

Let ψ be a quasi-psh function on $\mathbb{P}(E)$ that is smooth outside an analytic set A' such that A' is of dimension at most $n-k-1$. In other words, the codimension of A' in $\mathbb{P}(E)$ is at least $k+r$. This implies that the codimension of $\pi(A')$ in X is at least $k+1$. Then the Monge-Ampère operator $(\alpha + i\partial\bar{\partial}\log(e^\varphi + \delta e^\psi))^{r-1+k}$ is well defined for every $\delta > 0$, as a consequence of Demailly's techniques [**Dem92a**]. Thus, by a weak compactness argument, the sequence of currents

$$\pi_*(\alpha + i\partial\bar{\partial}\log(e^\varphi + \delta_\nu e^\psi))^{r-1+k}$$

which all belong to the cohomology class $\pi_*\alpha^{r-1+k}$, has a weak limit as $\delta_\nu \to 0$ for some subsequence. Observe that if we take $\psi = 0$, for any $\delta > 0$, the function $\log(e^\varphi + \delta)$ is a bounded quasi-psh function. In that case the wedge product

$$\pi_*(\alpha + i\partial\overline{\partial}\log(e^\varphi + \delta))^{r-1+k}$$

is already well defined as a current by the work of [**BT82**]. However, we want the flexibility of choosing a non constant potential ψ in order to get quasi-psh functions with isolated singularities that can be used to get Lelong number estimates. Note that since all currents involved are closed, the limit current is still closed.

Now, we show that the limit is uniquely defined. The intuition is as follows. As we have observed at the end of Proposition 5.4.2, the family of currents indexed by δ has a contribution of mass 0 along the singular part of $\pi(A')$, and we can therefore guess that the limit should be independent of the choice of ψ. (Nevertheless, without passing to the limit, each current may still have a positive Lelong number at some point of $\pi(A')$.)

LEMMA 5.40. *The limit current is independent of the choice of the smooth representative α, as well as of the choice of ψ.*

PROOF. Fix a sequence δ_ν tending to 0 such that the weak limit corresponding to α and $\psi = 0$ exists. Up to taking a subsequence which preserves the weak limit, we can assume in the following that the same sequence δ_ν gives a weak limit for different choice of α and ψ. We will prove that the weak limits are the same, although a priori they might be different.

Let $\tilde{\alpha}$, α be two representatives in the same cohomology class. Then there exists a smooth function f on $\mathbb{P}(E)$ such that

$$\tilde{\alpha} = \alpha + i\partial\overline{\partial} f.$$

Let $\tilde{\varphi}$ be the quasi-psh function such that $T = \tilde{\alpha} + i\partial\overline{\partial}\tilde{\varphi}$. Without loss of generality, we can assume that $\tilde{\varphi} = \varphi - f$. Thus we have

$$\pi_*(\tilde{\alpha} + i\partial\overline{\partial}\log(e^{\tilde{\varphi}} + \delta_\nu e^\psi))^{r-1+k} = \pi_*(\alpha + i\partial\overline{\partial}\log(e^\varphi + \delta_\nu e^{\psi+f}))^{r-1+k}.$$

Thus to prove that the limit is independent of the choice of α, it is enough to prove that the limit is independent of ψ, and this is what the proof will be devoted to from now on. On the regular part $X \smallsetminus (\pi(A) \cup \pi(A'))$, the limit current is equal to

$$\pi_*(\alpha + i\partial\overline{\partial}\varphi)^{r-1+k}$$

by the continuity of Monge-Ampère operators with respect to bounded decreasing sequences and the fact that the currents are smooth on the pre-image of $X \smallsetminus (\pi(A) \cup \pi(A'))$. Thus the limit currents corresponding to different choices of ψ coincide on the regular part. Now, consider a Kähler form $\tilde{\omega}$ on $\mathbb{P}(E)$ satisfying the conditions

$$\alpha \geqslant -\tilde{\omega}/2, \quad i\partial\overline{\partial}\psi \geqslant -\tilde{\omega}/2.$$

We can assume that the restriction of $\tilde{\omega}$ over all the fibres $\mathbb{P}^{r-1}$ is a fixed cohomology class. For example, we can take

$$\tilde{\omega} = C\pi^*\omega + c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h_\infty)$$

for some $C \gg 0$ and for a smooth metric h_∞ on $\mathcal{O}_{\mathbb{P}(E)}(1)$ induced by a Hermitian metric on E. For any $\delta > 0$ we have

$$(*)\qquad \begin{aligned} &\alpha + i\partial\overline{\partial}\log(e^\varphi + \delta e^\psi) \\ &\geqslant \alpha + \frac{e^\varphi}{e^\varphi + \delta e^\psi}(i\partial\overline{\partial}\varphi) + \frac{\delta e^\psi}{e^\varphi + \delta e^\psi}(i\partial\overline{\partial}\psi) + \frac{\delta e^{\varphi+\psi}}{(e^\varphi + \delta e^\psi)^2} i\partial(\psi - \varphi) \wedge \overline{\partial}(\psi - \varphi) \geqslant -\tilde{\omega}\end{aligned}$$

in the sense of currents, and the lower bound is independent of δ.

By adding and subtracting $\tilde{\omega}$ and using the Newton binomial formula, we see that the current $(\alpha + i\partial\overline{\partial}\log(e^\varphi + \delta e^\psi))^{r+k-1}$ can be written as a difference of two closed positive currents equal to summations of terms

$$(\alpha + i\partial\overline{\partial}\log(e^\varphi + \delta e^\psi) + \tilde{\omega})^i \wedge \tilde{\omega}^j$$

with $i + j = r + k - 1$. Since the direct image functor transforms closed positive currents into closed positive currents, $\pi_*(\alpha + i\partial\overline{\partial}\log(e^\varphi + \delta e^\psi))^{r+k-1}$ can also be written as a difference. If we compute the limit as δ tends to 0 (up to taking some convergent subsequence), the limit current will be a difference of two closed positive currents, in particular, $\lim_{\nu\to\infty} \pi_*(\alpha + i\partial\overline{\partial}\log(e^\varphi + \delta_\nu e^\psi))^{r+k-1}$ is a normal current.

Denote by T_1, T_2 the limit currents obtained with different choices of ψ, namely ψ_1 and ψ_2. Assume that A' is the union of the singular loci of ψ_1 and ψ_2. By assumption, $\pi(A')$ is of codimension at least $k+1$ in X. Then $T_1 - T_2$ is a normal (k,k)-current supported in $\pi(A) \cup \pi(A')$. If the codimension of $\pi(A)$ in X

is at least $k+1$, standard support theorems imply that $T_1 = T_2$. If the codimension of $\pi(A)$ in X is k, the support theorem yields

$$T_1 - T_2 = \sum_\nu c_\nu [Z_\nu]$$

where Z_ν are the codimension k irreducible components of $\pi(A)$ and $c_\nu \in \mathbb{R}$, and there exists no components of $\pi(A')$ as its codimension is higher. We now check that the limit current is independent of the choice of ψ by a Lelong number calculation, i.e. by showing that $c_\nu = 0$.

For any $x \in Z_{\nu_0,reg} \smallsetminus (\bigcup_{\nu\neq\nu_0} Z_\nu \cup \pi(A'))$, there exists a coordinate chart V such that $x = 0$, $V \Subset X \smallsetminus \pi(A')$, and $Z_{\nu_0} = \{z_1 = \cdots = z_k = 0\}$ locally. Take a cut-off function θ supported in V and define

$$T_{1,\delta} = \alpha + i\partial\overline{\partial}\log(e^\varphi + \delta e^{\psi_1}),$$

$$T_{2,\delta} = \alpha + i\partial\overline{\partial}\log(e^\varphi + \delta e^{\psi_2}).$$

It is enough to prove that

$$\lim_{\delta\to 0}\int_X \left(\pi_* T_{1,\delta}^{k+r-1} - \pi_* T_{2,\delta}^{k+r-1}\right) \wedge \theta\omega^{n-k} = 0$$

which will imply that

$$\int_X (T_1 - T_2) \wedge \theta\omega^{n-k} = 0.$$

By a direct calculation, we have that

$$\begin{aligned} T_{1,\delta}^{k+r-1} - T_{2,\delta}^{k+r-1} &= \left(\sum_{j=0}^{k+r-1} T_{1,\delta}^j \wedge T_{2,\delta}^{r+k-1-j}\right) \wedge (T_{1,\delta} - T_{2,\delta}) \\ &= \left(\sum_{j=0}^{k+r-1} T_{1,\delta}^j \wedge T_{2,\delta}^{r+k-1-j}\right) \wedge i\partial\overline{\partial}\log\left(\frac{e^\varphi + \delta e^{\psi_1}}{e^\varphi + \delta e^{\psi_2}}\right). \end{aligned}$$

An integration by parts gives

$$\int_X \left(\pi_* T_{1,\delta}^{k+r-1} - \pi_* T_{2,\delta}^{k+r-1}\right) \wedge \theta\omega^{n-k} = \int_{\mathbb{P}(E)} i\partial\overline{\partial}\theta \wedge \omega^{n-k} \wedge \left(\sum_{j=0}^{r+k-1} T_{1,\delta}^j \wedge T_{2,\delta}^{r+k-1-j}\right)\log\left(\frac{e^\varphi + \delta e^{\psi_1}}{e^\varphi + \delta e^{\psi_2}}\right).$$

Define

$$F_\delta := \log\left(\frac{e^\varphi + \delta e^{\psi_1}}{e^\varphi + \delta e^{\psi_2}}\right),$$

which is a uniformly bounded function on V since $\bar{V}$ is outside of the image of the singular locus of ψ_1, ψ_2 under π. Note also that the bound is independent of δ. Moreover, F_δ tends to 0 almost everywhere as $\delta \to 0$. The convergence is locally uniform outside of the pole set A of φ.

Define $Z_\eta := \{z \in V, d(z, \pi(A)) \leqslant \eta\}$ with respect to the Kähler metric ω. The volume of Z_η with respect to ω tends to 0 as $\eta \to 0$ by the assumption that $V \cap \pi(A)$ is a smooth submanifold in V. Now we separate the estimate in different terms

$$\begin{aligned} \int_X \left(\pi_* T_{1,\delta}^{k+r-1} - \pi_* T_{2,\delta}^{k+r-1}\right) \wedge \theta\omega^{n-k} &= \int_{\pi^{-1}(Z_\eta)} i\partial\overline{\partial}\theta \wedge \omega^{n-k} \wedge \left(\sum_{j=0}^{r+k-1} T_{1,\delta}^j \wedge T_{2,\delta}^{r+k-1-j}\right) F_\delta \\ &+ \int_{\pi^{-1}(V\smallsetminus Z_\eta)} i\partial\overline{\partial}\theta \wedge \omega^{n-k} \wedge \left(\sum_{j=0}^{r+k-1} T_{1,\delta}^j \wedge T_{2,\delta}^{r+k-1-j}\right) F_\delta, \end{aligned}$$

and we use the Fubini theorem to perform a double integration with respect to the base direction $V \smallsetminus Z_\eta$ (resp. Z_η) and the fibration direction $\mathbb{P}^{r-1}$, for V sufficiently small. The first term in the integration is bounded by

$$C\omega^{n-k+1} \wedge \left(\sum_{j=0}^{r+k-1} (T_{1,\delta} + \tilde{\omega})^j \wedge (T_{2,\delta} + \tilde{\omega})^{r+k-1-j}\right)$$

with C independent of δ since F_δ is uniformly bounded on $\bar{V}$ and $i\partial\overline{\partial}\theta$ is bounded by $C\omega$ for C large enough.

The currents $T_{1,\delta}$ and $T_{2,\delta}$ are not smooth on Z_η, thus some attention has to be paid to apply the Fubini theorem. Let $U(\eta)$ (resp. $U'(\eta)$) be the open neighbourhoods of A (resp. A') in $\mathbb{P}(E)$ given by Proposition 5.4.2. Note that $T_{1,\delta}$ and $T_{2,\delta}$ are smooth near the boundary of $U(\eta) \cup U'(\eta)$. Without loss of generality, we can assume that $\pi^{-1}(Z_\eta)$ is contained in $U(\eta) \smallsetminus U'(\eta)$. Take smooth currents $\tilde{T}_{i,\delta}$ on $U(\eta) \cup U'(\eta)$

cohomologous to $T_{i,\delta}$, which coincide with $T_{i,\delta}$ ($i = 1, 2$) near the boundary of $U(\eta) \cup U'(\eta)$. By Stokes' theorem,

$$\int_{U(\eta)\cup U'(\eta)} \omega^{n-k+1} \wedge \left(\sum_{j=0}^{r+k-1} (T_{1,\delta} + \tilde{\omega})^j \wedge (T_{2,\delta} + \tilde{\omega})^{r+k-1-j} \right)$$
$$= \int_{U(\eta)\cup U'(\eta)} \omega^{n-k+1} \wedge \left(\sum_{j=0}^{r+k-1} (\tilde{T}_{1,\delta} + \tilde{\omega})^j \wedge (\tilde{T}_{2,\delta} + \tilde{\omega})^{r+k-1-j} \right).$$

Therefore we can apply the Fubini theorem in the right hand side since all terms are smooth. The integral on $\pi^{-1}(Z_\eta)$ is bounded from above by the integral of the same term on $U(\eta) \cup U'(\eta)$ by the inclusion relation $\pi^{-1}(Z_\eta) \subset U(\eta) \cup U'(\eta)$.

We first perform the integration along the fibres $\mathbb{P}^{r-1}$. The integration of $\sum_{j=0}^{r+k-1} (\tilde{T}_{1,\delta} + \tilde{\omega})^j \wedge (\tilde{T}_{2,\delta} + \tilde{\omega})^{r+k-1-j}$ along the fibre direction is a cohomological constant since we assume that the restriction of cohomology class of α along each fibres is a fixed cohomology class on $\mathbb{P}^{r-1}$. Thus the integral on $U(\eta) \cup U'(\eta)$ is bounded from above by $C \int_{U(\eta)\cup U'(\eta)} \omega^n$, for some C independent of δ. Observe that the constant is the same as the supremum of $|F_\delta|$ on $\overline{V}$ (independent of δ), since for η small enough $\overline{V} \cap U'(\eta) = \varnothing$.

The second term appearing in the integral is bounded by

$$\sup_{\pi^{-1}(X \smallsetminus Z_\eta)} |F_\delta| \sup_X |i\partial\overline{\partial}\theta|_\omega \omega^{n-k+1} \wedge \left(\sum_{j=0}^{r+k-1} (T_{1,\delta} + \tilde{\omega})^j \wedge (T_{2,\delta} + \tilde{\omega})^{r+k-1-j} \right).$$

On $V \smallsetminus Z_\eta$, the currents $T_{1,\delta}$ and $T_{2,\delta}$ are smooth, thus the Fubini theorem applies. We first integrate along $\mathbb{P}^{r-1}$. The integration of $\sum_{j=0}^{r+k-1} ((T_{1,\delta} + \tilde{\omega})^j \wedge (T_{2,\delta} + \tilde{\omega})^{r+k-1-j})$ along the fibre direction is a cohomological constant as above. Thus the second term obtained after integrating is bounded from above by $C \sup_{\pi^{-1}(X \smallsetminus Z_\eta)} |F_\delta|$, for some C independent of δ.

For every $\varepsilon' > 0$, there exist η such that $C \int_{U(\eta)\cap U'(\eta)} \omega^n < \frac{\varepsilon'}{2}$. There also exists δ_0 such that $C \sup_{X \smallsetminus Z_\eta} |F_\delta| < \frac{\varepsilon'}{2}$ for every $\delta \leqslant \delta_0$. Thus the two parts of estimate (integration on $U(\eta) \cup U'(\eta)$ and on $\pi^{-1}(V \smallsetminus Z_\eta)$) are both bounded from above by $\frac{\varepsilon'}{2}$ for $\delta \leqslant \delta_0$. This concludes the proof that the limit current is independent the choice of ψ.

□

In what follows we show that the weak limit is also independent of the subsequence δ_ν if the weight function φ has analytic singularities. It seems that the independence of the weak limit does not hold in general if we only require that φ is smooth outside an analytic set of sufficient high codimension. However some special cases can be easily checked.

EXAMPLE 5.41. Assume that there exists some $C_2 \geqslant C_1 > 0$ such that

$$C_1 \delta'_\nu \leqslant \delta_\nu \leqslant C_2 \delta'_\nu$$

up to taking some subsequence but with the same limit currents. Then the function

$$\log\left(\frac{e^\varphi + \delta_\nu e^\psi}{e^\varphi + \delta'_\nu e^\psi} \right)$$

is uniform bounded on $\mathbb{P}(E)$ (independently of ν). It is locally uniformly convergent to 0 on $\pi^{-1}(X \smallsetminus Z_\eta)$. The same arguments as above can be used to achieve the proof.

Another easy case is when the projection of the singular part of φ is of codimension at least $k+1$. In this case, different choices of subsequence δ_ν will have the same closed positive limit outside an analytic set of codimension at least $k+1$. By standard support theorems, they have to coincide over X.

The case of potentials with analytic singularities comes from the following observation of Demailly.

PROPOSITION 5.4.3. *Let φ be a quasi-psh function with analytic singularities over on a (connected) complex n-dimensional manifold X, and $u \in C^\infty(X)$. Then for any exponent p ($1 \leqslant p \leqslant n$), the asymptotic limit of Monge-Ampère operator $\lim_{\delta\to 0}(i\partial\overline{\partial}\log(e^\varphi + \delta e^u))^p$ is always well defined as a current (but not necessarily positive, even when $i\partial\overline{\partial}\varphi \geqslant 0$, and the limit may depend on u).*

PROOF. By writing $\log(e^\varphi + \delta e^u) = \log(e^{\varphi-u} + \delta) + u$ and using a binomial expansion, it is sufficient to consider the case $u = 0$, after replacing φ with $\varphi - u$. Let us now consider the divisorial case, i.e., assume that $X = \mathbb{C}^n$ and that φ is of the form $\varphi = \log|f|^2 + \psi$ for some holomorphic function $f = \prod_{i=1}^m z_i^{m_i} \in \mathcal{O}(X)$ and $\psi \in C^\infty(X)$. We can define $h = e^\psi$ a smooth Hermitian metric on $L := \mathcal{O}_X$. We denote by ∇_h the associated Chern connection.

Then, for every $\delta > 0$, we have $i\partial\overline{\partial}\log(e^\varphi + \delta) = i\partial\overline{\partial}\log(|f|_h^2 + \delta)$ which converge to $i\partial\overline{\partial}\varphi$ as $\delta \to 0+$. We will define the Monge-Ampère operator $(i\partial\overline{\partial}\varphi)^p$ as the limit of $\big(i\partial\overline{\partial}\log(|f|_h^2 + \delta)\big)^p$ as $\delta \to 0+$. For every $\delta > 0$, we have

$$i\partial\overline{\partial}\log(|f|_h^2 + \delta) = i\partial\frac{\langle f, \nabla_h f\rangle}{|f|_h^2 + \delta} = \frac{i\langle \nabla_h f, \nabla_h f\rangle}{|f|_h^2 + \delta} - i\frac{\langle \nabla_h f, f\rangle}{|f|_h^2 + \delta} \wedge \frac{\langle f, \nabla_h f\rangle}{|f|_h^2 + \delta} + i\frac{\langle f, \nabla_h^{0,1}\nabla_h^{1,0} f\rangle}{|f|_h^2 + \delta}$$

$$= \frac{\delta}{(|f|_h^2 + \delta)^2}\, i\langle \nabla_h f, \nabla_h f\rangle - \frac{|f|_h^2}{|f|_h^2 + \delta}\, i\,\Theta_{L,h}.$$

Now, $i\langle \nabla_h f, \nabla_h f\rangle$ is a $(1,1)$-form of rank 1. In particular, its wedge powers of exponents > 1 are equal to 0. If we raise to power p, the Newton binomial formula implies

$$\Big(\frac{i}{2\pi}\partial\overline{\partial}\log(|f|_h^2 + \delta)\Big)^p = \frac{p\delta}{(|f|_h^2 + \delta)^2}\Big(\frac{|f|_h^2}{|f|_h^2 + \delta}\Big)^{p-1}\frac{i}{2\pi}\langle \nabla_h f, \nabla_h f\rangle \wedge \Big(-\frac{i}{2\pi}\Theta_{L,h}\Big)^{p-1}$$

$$+\Big(\frac{|f|_h^2}{|f|_h^2 + \delta}\Big)^p\Big(-\frac{i}{2\pi}\Theta_{L,h}\Big)^p.$$

The last term converges almost everywhere to $(-\frac{i}{2\pi}\Theta_{L,h})^p$, thus it converges weakly to the same limit by the bounded convergence theorem as $\delta \to 0+$. We claim that

$$(*) \qquad \frac{p\delta\,|f|_h^{2p-2}}{(|f|_h^2 + \delta)^{p+1}}\,\frac{i}{2\pi}\langle \nabla_h f, \nabla_h f\rangle \to [Z_f]$$

weakly, where $[Z_f]$ is the current of integration on the zero divisor of f. Terms that depend on h in $\nabla_h f$ are equal to $f\partial\varphi$, and they can be seen to yield zero limits, using the Cauchy-Schwarz formula and the fact that

$$\frac{p\delta\,|f|_h^{2p-2}}{(|f|_h^2 + \delta)^{p+1}} \cdot |f|_h^2 \leqslant p$$

converges to zero almost everywhere. In fact the limit (if it exists) is a positive current as a limit of positive currents. It will also be closed, since

$$\overline{\partial}\Big(\frac{p\delta\,|f|_h^{2p-2}}{(|f|_h^2 + \delta)^{p+1}}\,\frac{i}{2\pi}\langle \nabla_h f, \nabla_h f\rangle\Big) = \frac{p\delta\,|f|_h^{2p-2}}{(|f|_h^2 + \delta)^{p+1}}\,\frac{1}{2\pi}\langle f, \nabla_h f\rangle \wedge \Theta_{L,h}$$

and we can again apply a Cauchy-Schwarz argument to see that the right hand side converges to 0. A priori the limit current (if it exists) should be supported on $|Z_f|$. However, at any regular point of Z_f we can find local holomorphic coordinates in which $f(z) = z_1^m$, where m is the multiplicity of the irreducible component. An easy calculation yields

$$(**) \qquad \int_{z_1\in\mathbb{C}} \frac{p\delta\,|z_1^m|^{2p-2}}{(|z_1^m|^2 + \delta)^{p+1}}\,\frac{i dz_1^m \wedge d\overline{z_1^m}}{2\pi} = m.$$

Equality $(**)$ can be checked e.g. by putting $w = z_1^m$, using polar coordinates $w = re^{i\theta}$ and making a change of variables $t = \frac{r^2}{r^2+\delta}$. More generally, if $f(z) = \prod z_i^{m_i}$, we have to consider the integration

$$\int_{\{|z_i|\leqslant 1\}} \frac{p\delta\,|\prod_{i=1}^m z_i^{m_i}|^{2p-2}}{(|\prod_{i=1}^m z_i^{m_i}|^2 + \delta)^{p+1}}\,\frac{i d(\prod_{i=1}^m z_i^{m_i}) \wedge d(\overline{\prod_{i=1}^m z_i^{m_i}})}{(2\pi)^n} \wedge \omega_{\mathrm{eucl}}^{n-1}$$

where ω_{eucl} is the standard $(1,1)$-form associated with the euclidean metric on $\mathbb{C}^n$. It is bounded by sums of integrals of the type

$$\int_{\{0<|z_i|\leqslant 1, 2\leqslant i\leqslant n\}} \frac{p\delta\,||\prod_{i=2}^m z_i^{m_i}|z_1^{m_1}|^{2p-2}}{(||\prod_{i=2}^m z_i^{m_i}|z_1^{m_1}|^2 + \delta)^{p+1}}\,\frac{i|\prod_{i=2}^m z_i^{m_i}|d(z_1^{m_1}) \wedge |\prod_{i=2}^m z_i^{m_i}|d\overline{z_1^{m_1}}}{(2\pi)^n} \wedge \omega_{\mathrm{eucl}}^{n-1}.$$

The integral is finite by the Fubini theorem and a calculation similar to $(**)$, putting e.g. $w = |\prod_{i=2}^m z_i^{m_i}|z_1^{m_1}$. In particular, up to taking a subsequence, the limit in formula $(*)$ exists as $\delta \to 0+$. By the support theorem any limit current is associated to a divisor supported in $|Z_f|$. To show that the weak limit is unique, it is sufficient to check formula $(*)$ at a regular point of $|Z_f|$ and to show that the coefficient is unique. This actually follows from equality $(**)$.

As a consequence of the above calculations, we find

$$\Big(\frac{i}{2\pi}\partial\overline{\partial}\log(|f|_h^2 + \delta)\Big)^p \to (-1)^{p-1}[Z_f] \wedge \Big(\frac{i}{2\pi}\Theta_{L,h}\Big)^{p-1} + (-1)^p\Big(\frac{i}{2\pi}\Theta_{L,h}\Big)^p.$$

For the general case, we apply Hironaka's theorem. There exists a certain modification $\sigma : \tilde{X} \to X$ of X such that $\sigma^*\varphi$ is locally of the form considered in the previous case, where f has a simple normal crossing divisor. Thus the limit

$$\lim_{\delta \to 0+} \left(i\partial\overline{\partial}\log(e^{\varphi}+\delta)\right)^p = \sigma_*\Big(\lim_{\delta \to 0+} \left(i\partial\overline{\partial}\log(e^{\sigma^*\varphi}+\delta)\right)^p\Big)$$

exists by the weak continuity of the direct image operator σ_*. By the filtering property of modifications, one can also see that the above limit is independent of the choice of the modification σ. □

It follows directly from the proposition that the limit current is independent of the subsequence δ_ν if the weight function φ has analytic singularities. It is communicated has communicated to the us by Richard Lärkäng that a similar calculation has been done in [**ABW19**] and [**Bl19**]. The advantage of the construction made in lemma 5.40 is that under the assumption that the weight function is smooth outside of an analytic set of sufficient high codimension, one can show that the limit current is positive. This is shown in theorem 5.43 below.

EXAMPLE 5.42. We describe below a special case of the previous construction. Let E be a strongly psef vector bundle over a compact Kähler manifold (X, ω). Let h_∞ be an arbitrary metric on E. Since $\mathcal{O}_{\mathbb{P}(E)}(1)$ is relatively ample with respect to the projection $\pi : \mathbb{P}(E) \to X$, there exists $C > 0$ big enough such that

$$i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\infty) + C\pi^*\omega > 0.$$

We take the above form as a smooth representative in the class $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + C\pi^*\{\omega\}$. By definition of a strongly psef vector bundle, there exists a singular metric h_ε with analytic singularities on $\mathcal{O}_{\mathbb{P}(E)}(1)$ such that

$$i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) \geqslant -\varepsilon\pi^*\omega.$$

By the above construction, $\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) + C\pi^*\omega\right)^r$ is well defined for ε small enough by taking that $\psi = 0$. In the construction, all currents are positive currents. In particular, $\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) + C\pi^*\omega\right)^r$ is a closed positive current on X for ε small enough. On the other hand,

$$\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) + C\pi^*\omega\right)^r = \pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)\right)^r$$

$$+ r\pi_*\left(C\pi^*\omega \wedge (\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon))^{r-1}\right) + \cdots .$$

In the $\cdots$ summation, there are terms of the form

$$\pi_*\left(\pi^*\omega^i \wedge (\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon))^{r-i}\right)$$

for $i \geqslant 2$. By the projection formula, we have

$$\pi_*\left(\pi^*\omega^i \wedge (\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon))^{r-i}\right) = \pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)\right)^{r-i} \wedge \omega^i.$$

By a degree consideration, for $i \geqslant 2$, the right hand side is 0 and for $i = 1$ it is equal to ω. In conclusion,

$$\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) + C\pi^*\omega\right)^r = \pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)\right)^r + Cr\omega \geqslant 0$$

in the sense of currents. In particular, $\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)\right)^r$ is a quasi-positive current (i.e. a current bounded below by a smooth form), belonging to the cohomology class $c_1(\det(E))$ by lemma 5.39.

More generally, we have the following Segre current construction.

THEOREM 5.43. *(Main technical lemma) Let E be a vector bundle of rank r over a compact Kähler manifold (X, ω), and let T be a closed positive $(1,1)$-current on $\mathbb{P}(E)$, belonging to the same cohomology class as a smooth form α. Write*

$$T = \alpha + i\partial\overline{\partial}\varphi.$$

Assume that φ is smooth over $\mathbb{P}(E) \setminus A$, where $\pi : \mathbb{P}(E) \to X$ is the projection and A is an analytic set in $\mathbb{P}(E)$ such that $A = \pi^{-1}(\pi(A))$ and $\pi(A)$ is of codimension at least k in X. Then there exists a (k,k)-positive current in the class $\pi_*\{\alpha\}^{r+k-1}$.

PROOF. The desired current $\pi_*(T^{r+k-1})$ has been constructed, and its uniqueness has been shown in the previous lemma. It remains to show that $\pi_*(T^{r+k-1})$ is positive. It is enough to prove this near an arbitrary point $x \in X$, since positivity is a local property. There exists a smooth function ψ on $\mathbb{P}(E)$ such that

$$\alpha + i\partial\overline{\partial}\psi \geqslant 0$$

on an open neighbourhood U of x. Thus over U, for every $\delta > 0$, we have

$$\alpha + i\partial\overline{\partial}\log(e^{\varphi} + \delta e^{\psi}) \geqslant 0$$

using $(*)$ in the previous lemma. Therefore, over U again, we see that

$$\pi_* T^{r+k-1} = \lim_{\delta\to 0} \pi_*(\alpha + i\partial\overline{\partial}\log(e^{\varphi} + \delta e^{\psi}))^{r+k-1}$$

is positive as a limit of positive currents. Let us note that the restriction of the cohomology class $\{\alpha\}$ is constant on the fibres of π – this property being automatically true for any smooth proper morphism. ☐

REMARK 5.44. In fact, the above construction would work for any submersion $\pi : X \to Y$ of relative dimension $r-1$ and any psef cohomology class $\{\alpha\} \in H^{1,1}(X, \mathbb{R})$, when X, Y are compact Kähler manifolds. The construction works for currents with analytic singularities of an adequate codimension, and in this way, one gets gives a closed positive current in the direct image of wedge powers of $\{\alpha\}$.

In the special case of Segre currents, we get

COROLLARY 5.45. *Let E be a strongly psef vector bundle of rank r over a compact Kähler manifold (X, ω). Let $(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon)$ be a singular metric with analytic singularities such that*

$$i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\varepsilon) \geqslant -\varepsilon\pi^*\omega$$

and the codimension of $\pi(\mathrm{Sing}(h_\varepsilon))$ is at least k in X. Then there exists a (k,k)-positive current in the cohomology class $\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \varepsilon\pi^*\{\omega\})^{r+k-1}$. In particular, $\det(E)$ is a psef line bundle.

PROOF. The first part is a direct consequence of theorem 5.43. The second part is consequence of the fact that when $k = 1$ one has

$$\pi_*(c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \varepsilon\pi^*\{\omega\})^r = c_1(\det(E)) + \varepsilon\omega.$$

☐

REMARK 5.46. Let h be a smooth metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ (not necessarily coming from a Hermitian metric on E). We can define an induced singular metric on $\det(E)$ in the following non canonical way. Fix an arbitrary smooth Hermitian metric h_∞ on $\mathbb{P}(E)$. Then there exists $\psi \in C^\infty(\mathbb{P}(E))$ such that $h = h_\infty e^{-\psi}$. Therefore we have

$$\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h) - \frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\infty) = \frac{i}{2\pi}\partial\overline{\partial}\psi.$$

Define a metric on $\det(E)$ by $\det(h_\infty)e^{-\varphi}$ with

$$\varphi := \pi_*\left(\psi \sum_{j=0}^{r-1} \Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h)\Big)^j \wedge \Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\infty)\Big)^{r-1-j}\right).$$

We have that

$$\frac{i}{2\pi}\partial\overline{\partial}\varphi = \pi_*\left(\Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h)\Big)^r - \Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\infty)\Big)^r\right).$$

In other words,

$$\frac{i}{2\pi}\Theta(\det(E), \det(h_\infty)e^{-\varphi}) = \pi_*\Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h)\Big)^r.$$

If h comes from a Hermitian metric of E, we get precisely the same curvature formula as in lemma 5.39.

REMARK 5.47. The definition in the previous remark is non canonical in the sense that it depends on the choice of the reference metric h_∞. This can be seen as follows. In analogy with the Monge-Ampère functional, we consider the functional

$$M_{h_\infty} : C^\infty(\mathbb{P}(E)) \to C^\infty(X)$$

$$\psi \mapsto \pi_*\left(\psi \sum_{j=0}^{r-1} \Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\infty) + \frac{i}{2\pi}\partial\overline{\partial}\psi\Big)^j \wedge \Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\infty)\Big)^{r-1-j}\right).$$

Let ψ_t be a smooth path in $C^\infty(\mathbb{P}(E))$. We compute the Fréchet differential

$$\frac{dM_{h_\infty}(\psi_t)}{dt} = \pi_*\left(\dot{\psi}_t \sum_{j=0}^{r-1} \Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\infty) + \frac{i}{2\pi}\partial\overline{\partial}\psi_t\Big)^j \wedge \Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1), h_\infty)\Big)^{r-1-j}\right) +$$

$$\pi_*\left(\psi_t\sum_{j=0}^{r-1} j\frac{i}{2\pi}\partial\overline{\partial}\dot{\psi}_t\wedge\Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1),h_\infty)+\frac{i}{2\pi}\partial\overline{\partial}\psi_t\Big)^{j-1}\wedge\Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1),h_\infty)\Big)^{r-1-j}\right)$$

which, by an integration by parts, is equal to

$$\pi_*\left(\dot{\psi}_t\Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1),h_\infty)\Big)^{r-1}\right).$$

Now let h_∞, $\tilde{h}_\infty$ be two smooth metrics on E and denote the induced metrics on $\mathcal{O}_{\mathbb{P}(E)}(1)$ by the same notation. Let ψ_t be a smooth path connecting h_∞ and $\tilde{h}_\infty$. For example we can take ψ_t such that $h_\infty e^{-\psi_t} = h_\infty^t\tilde{h}_\infty^{1-t}$. As a consequence of the calculation of Fréchet differential, our functional satisfies for any $\varphi\in C^\infty(\mathbb{P}(E))$ the cocycle relation

$$M_{h_\infty}(\varphi+\psi_1)=M_{\tilde{h}_\infty}(\varphi)+M_{h_\infty}(\psi_1).$$

Let us note that $M_{h_\infty}(\varphi+\psi_1)$ (resp. $M_{\tilde{h}_\infty}(\varphi)$) is the weight function of the induced metric on $\det(E)$ with respect to the reference metric h_∞ (resp. $\tilde{h}_\infty$), associated with the weight function $\varphi+\psi_1$ (resp. φ) on $\mathbb{P}(E)$. In particular, they correspond to metrics on $\det(E)$ that are induced by the same metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$, but with different reference metrics $\tilde{h}_\infty$ and h_∞. Since $i\partial\overline{\partial}M_{h_\infty}(\varphi)$ is independent of the choice of the reference metric h_∞, we have $i\partial\overline{\partial}M_{h_\infty}(\psi_1)\equiv 0$, and this means that $M_{h_\infty}(\psi_1)$ is a constant. Therefore the metric defined in the previous remark is uniquely defined up to a constant.

5.5. Strongly pseudoeffective and numerically trivial bundles

In this section, we use the Lelong number estimate to show that a strongly psef vector bundle with trivial first Chern class is in fact numerically flat. In particular, this implies that a strongly psef reflexive sheaf with trivial first Chern class is in fact a numerically flat vector bundle. As an application of the previous section, we get the following result.

THEOREM 5.48. *(Main theorem) Let E be a strongly psef vector bundle on a compact Kähler manifold (X,ω), such that $c_1(E)=0$. Then E is a nef (and thus numerically flat) vector bundle.*

PROOF. We show through Lelong number estimates and regularization, that the vector bundle E is in fact nef. Let h_ε be a singular metric with analytic singularities on $\mathcal{O}_{\mathbb{P}(E)}(1)$, such that

$$i\Theta(\mathcal{O}_{\mathbb{P}(E)}(1),h_\varepsilon)\geqslant -\varepsilon\pi^*\omega.$$

Let us write $h_\varepsilon = h_\infty e^{-\varphi_\varepsilon}$ with respect to some smooth reference metric h_∞ on $\mathcal{O}_{\mathbb{P}(E)}(1)$. Define

$$T_\varepsilon := \pi_*\Big(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)(1)},h_\infty)+\frac{i}{2\pi}\partial\overline{\partial}\varphi_\varepsilon\Big)^r$$

by means of Theorem 5.43. We have $T_\varepsilon\geqslant -\varepsilon\omega$. More precisely, we are going to prove the Lelong number estimate

$$\nu(T_\varepsilon,z)\geqslant\left(\sup_{w,\pi(w)=z}\nu(\varphi_\varepsilon,w)\right)^r.$$

The proof of this estimate is similar to the proof of theorem 10.2 of [**Dem93**]. For the convenience of the reader, we briefly outline the proof here. Fix $w_0\in\pi^{-1}(x)$ and $\gamma=\nu(\varphi_\varepsilon,w_0)$. The inequality is trivial when $\gamma=0$. Otherwise, for any $\varepsilon'<\gamma$, let us define

$$\psi := (\gamma-\varepsilon')\theta(w)\log|w-w_0|$$

where w is the coordinate near w_0 and θ is a cut off function near w_0. By lemma 5.40, we have

$$T_\varepsilon=\lim_{\delta\to 0}\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)(1)},h_\infty)+\frac{i}{2\pi}\partial\overline{\partial}\log(e^{\varphi_\varepsilon}+\delta e^{\psi})\right)^r$$

in the sense of currents. For every η so small that $\{|z|\leqslant\eta\}$ is contained in a coordinate chart with $\pi(w_0)=0$, we have

$$\int_{|z|\leqslant\eta}T_\varepsilon\wedge\Big(\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2\Big)^{n-1}\geqslant$$

$$\limsup_{\delta\to 0}\int_{|z|\leqslant\eta}\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)(1)},h_\infty)+\frac{i}{2\pi}\partial\overline{\partial}\log(e^{\varphi_\varepsilon}+\delta e^{\psi})\right)^r\wedge\Big(\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2\Big)^{n-1}$$

by the semi continuity of Monge-Ampère operators with respect to decreasing sequences. By construction, we have $\varphi_\varepsilon(w)\leqslant\gamma\log|w-w_0|+C$ near w_0, so $\frac{i}{2\pi}\partial\overline{\partial}\log(e^{\varphi_\varepsilon}+\delta e^{\psi})$ coincides with $\frac{i}{2\pi}\partial\overline{\partial}\psi$ on a small ball $B(w_0,\eta_\delta)\subset\pi^{-1}(B(0,\eta))$. Thus we have

$$\int_{|z|\leqslant\eta}\pi_*\left(\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)(1)},h_\infty)+\frac{i}{2\pi}\partial\overline{\partial}\log(e^{\varphi_\varepsilon}+\delta e^{\psi})\right)^r\wedge(\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{n-1}$$

$$\geqslant \int_{|w-w_0|\leqslant \eta_\delta} (\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)(1)}, h_\infty) + \frac{i}{2\pi}\partial\overline{\partial}\log(e^{\varphi_\varepsilon} + \delta e^{\psi}))^r \wedge (\frac{i}{2\pi}\partial\overline{\partial}\log|z|^2)^{n-1} \geqslant (\gamma - \varepsilon')^r.$$

Taking $\eta \to 0$ and $\varepsilon' \to 0$ gives the Lelong number estimate.

We have proven in Corollary 5.45 that $T_\varepsilon \geqslant -\varepsilon\omega$, and $T_\varepsilon + \varepsilon\omega$ is in the class $c_1(\det(E)) + \varepsilon\{\omega\}$. By weak compactness, there exists a convergent subsequence T_{ε_ν} with limit T in the class $c_1(\det(E))$. Since $T \geqslant 0$ and $c_1(\det(E)) = 0$, the only possibility is that $T = 0$.

Now, we recall the following version of the regularization theorem given in [**Dem82**]: *let $T = \theta + i\partial\overline{\partial}\varphi$ be a closed $(1,1)$-current, where θ is a smooth form. Suppose that a smooth $(1,1)$-form γ is given such that $T \geqslant \gamma$. Then there exists a decreasing sequence of smooth functions φ_k converging to φ such that, if we set $T_k := \theta + i\partial\overline{\partial}\varphi_k$, we have*

(1) *$T_k \to T$ weakly,*
(2) *$T_k \geqslant \gamma - C\lambda_k\omega$, where $C > 0$ is a constant depending on (X, ω) only, and λ_k is a decreasing sequence of continuous functions such that $\lambda_k(x) \to \nu(T, x)$ for all $x \in X$.*

By Corollary 5.51 below, we get

$$\lim_{\varepsilon\to 0} \sup_X \nu(T_\varepsilon, x) = 0,$$

thus

$$\lim_{\varepsilon\to 0} \sup_{\mathbb{P}(E)} \nu(\varphi_\varepsilon, w) = 0$$

thanks to the above Lelong number estimate. By the regularization theorem just recalled, there exists $\tilde{\varphi}_\varepsilon \in C^\infty(\mathbb{P}(E))$ such that

$$\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)(1)}, h_\infty) + \frac{i}{2\pi}\partial\overline{\partial}\tilde{\varphi}_\varepsilon \geqslant -2\varepsilon\tilde{\omega}$$

where $\tilde{\omega}$ is some Kähler form on $\mathbb{P}(E)$. In other words, the line bundle $\mathcal{O}_{\mathbb{P}(E)(1)}$ is nef. □

LEMMA 5.49. *Let X be a compact complex manifold. Let T_δ $(\delta > 0)$ be a sequence of closed positive (k,k)-currents. Assume that $T_\delta \to 0$ weakly as $\delta \to 0$. Then*

$$\lim_{\delta\to 0} \sup_X \nu(T_\delta, x) = 0.$$

PROOF. Since X is compact, we can cover X by finite coordinate open charts $V_i (\subset U_i \subset \tilde{U}_i)$ such that V_i is relatively compact in U_i and U_i is relatively compact in $\tilde{U}_i$. Thus we reduce the proof to the case of coordinate chart V_i.

Let ρ_i be cut off functions supported in $\tilde{U}_i$ such that $\rho_i \equiv 1$ on U_i and $0 \leqslant \rho_i \leqslant 1$. Since $T_\delta \to 0$ weakly, there exists a uniform $C > 0$ such that

$$\int_{U_i} T_\delta \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}|z|^2\right)^{n-k} \leqslant \int_{\tilde{U}_i} T_\delta \wedge \rho_i \left(\frac{i}{2\pi}\partial\overline{\partial}|z|^2\right)^{n-k} \leqslant C.$$

Define for $x \in \overline{V_i}$ and for small r

$$\nu(T_\delta, x, r) := r^{-2(n-k)} \int_{|z-x|<r} T_\delta \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}|z|^2\right)^{n-k}.$$

Then $\nu(T_\delta, x, r)$ is an increasing function with respect to r and we have that

$$\nu(T_\delta, x) = \lim_{r\to 0} \nu(T_\delta, x, r).$$

For small $r > 0$ such that $2r < d(V_i, \partial U_i)$, there exists a cut-off function θ_x supported in $B(x, 2r)$ such that $\theta_x \equiv 1$ on $B(x, r)$ and $0 \leqslant \theta_x \leqslant 1$. Then we have

$$\nu(T_\delta, x, r) \leqslant r^{-2(n-k)} \int_{U_i} T_\delta \wedge \theta_x \left(\frac{i}{2\pi}\partial\overline{\partial}|z|^2\right)^{n-k}.$$

Since θ_x can be obtained by translation of the same function, $(\theta_x)_{x\in\overline{V_i}}$ for small r is a compact family with respect to C^∞ topology. Thus for fixed small r, for every $x, y \in \overline{V_i}$,

$$r^{-2(n-k)} \int_{U_i} T_\delta \wedge (\theta_x - \theta_y) \left(\frac{i}{2\pi}\partial\overline{\partial}|z|^2\right)^{n-1} \leqslant Cr^{-2(n-k)} \parallel \theta_x - \theta_y \parallel_{L^\infty(U_i)}.$$

Thus $r^{-2(n-k)} \int_{U_i} T_\delta \wedge \theta_x \left(\frac{i}{2\pi}\partial\overline{\partial}|z|^2\right)^{n-k}$ tends to 0 as $\delta \to 0$ uniformly with respect to $x \in \overline{V_i}$.

In particular, $\nu(T_\delta, x, r)$ tends to 0 as $\delta \to 0$ uniformly with respect to $x \in \overline{V_i}$, hence the same property holds for $\nu(T_\delta, x)$. □

REMARK 5.50. For a family of $(1,1)$-closed positive currents, the proof is much simpler, using the observation of Proposition 5.4.1.

Let γ be a Gauduchon metric over X (i.e. a smooth metric such that $i\partial\overline{\partial}(\gamma^{n-1})=0$). With the same notation as in the proof, we have for r_0 small enough

$$\nu(T_\delta, x, r) \geqslant \nu(T_\delta, x, r_0) \leqslant \frac{C}{r_0^{2n-2}} \int_X T_\delta \wedge \gamma^{n-1}.$$

Since the right-hand side term (which is cohomological) tends to 0 along with δ, the Lelong number tends to zero locally uniformly. Since X is compact, the convergence is uniform.

COROLLARY 5.51. *Let (X,ω) be a compact Kähler manifold. Let T_δ $(\delta > 0)$ be a sequence of closed $(1,1)$-currents such that*

$$T_\delta \geqslant -\delta\omega$$

in the sense of currents. Assume that $T_\delta \to 0$ weakly as $\delta \to 0$. Then

$$\lim_{\delta\to 0} \sup_X \nu(T_\delta, x) = 0.$$

PROOF. This is a direct consequence of the previous lemma if we consider $T_\delta + \delta\omega$ instead of T_δ. □

Now we can easily conclude our result.

COROLLARY 5.52. *Let $\mathcal{F}$ be a strongly psef reflexive sheaf over a compact Kähler manifold (X,ω) with $c_1(\mathcal{F})=0$. Then $\mathcal{F}$ is a nef (and numerically flat) vector bundle.*

PROOF. By our assumption, there exists a modification such that the pull back of $\mathcal{F}$ modulo torsion is a strongly psef vector bundle with vanishing first Chern class by lemma 5.29. By theorem 5.48, this vector bundle is in fact nef. Thus by Proposition 5.3.8, we conclude the corollary. □

As a geometric application, we obtain the following generalisation of Theorem 7.7 in [**BDPP13**].

COROLLARY 5.53. *Let X be a (non necessarily projective) K3-surface or a Calabi-Yau 3-fold. Then the tangent bundle T_X is not strongly psef. In other words, for a compact Kähler surface or 3-fold if $c_1(X)=0$ and T_X is strongly psef, then a finite étale cover of X is a torus.*

PROOF. Assume X is a compact Kähler surface such that $c_1(X)=0$ and T_X is strongly psef. Then by Theorem 5.48, T_X is in fact numerically flat. In particular, the second Chern class of X is 0. By classification of compact surface with nef tangent bundle (Theorem 6.1 and 6.2) in [**DPS94**], a finite étale cover of X must be a torus. Remind that the difference between the projective case and the compact complex case is whether the torus is abelian or Kodaira surface or Hopf surface. The later two surfaces are nevertheless non Kähler.

Then proof of the dimension 3 case is similar. Instead of the Theorem 6.1 and 6.2, we use the classification of compact 3-folds with nef tangent bundle (Theorem 7.1 and 7.2) in [**DPS94**]. □

In fact, we can show the following more general fact. A stronger result in the projective singular setting can be found in Theorem 1.6 of [**HP19**] (Instead of proving "non strong psefness", they prove "non weak psefness".)

COROLLARY 5.54. *For a compact Kähler manifold if $c_1(X)=0$ and T_X is strongly psef, then a finite étale cover of X is a torus. In particular, an irreducible symplectic, or Calabi-Yau manifold does not have strongly psef tangenet bundle or cotangent bundle.*

PROOF. By the Beauville-Bogomolov theorem, up to a finite étale cover $\pi : \tilde{X} \to X$, $\tilde{X}$ is a product of $\prod T_i \times \prod S_j \times \prod Y_k$ where T_i are complex tori, S_j are Calabi-Yau manifolds and Y_k are irreducible symplectic manifolds. Since the tangent bundle of $\tilde{X}$ is numerical flat under the assumption and by Theorem 5.48, the tangent bundle of all the components in the direct sum is numerical flat. In particular, all the components have vanishing second Chern class by Corollary 1.19 of [**DPS94**]. (A stronger result in the projective and singular setting can be found in Theorem 1.8 of [**HP19**].) By representation theory, the tangent bundle of the Calabi-Yau or irreducible symplectic components is stable. Thus we have the equality case in the Bogomolov inequality which implies that the tangent bundle of the Calabi-Yau or irreducible symplectic components is projectively flat. Since the first Chern class of the Calabi-Yau or irreducible symplectic components vanishes, the tangent bundle is in fact unitary flat. In particular, the restricted holonomy groups of the Calabi-Yau or irreducible symplectic components are trivial. In other words, only the complex tori components appear in the decomposition. □

Inspired by the work of [**LOY20**], we can slightly generalise Corollary 5.52 in the following form.

LEMMA 5.55. *(analogue of Lemma 4.5 [**LOY20**]) Let (X,ω) be a compact Kähler manifold of dimension $n>2$, and let $\mathcal{F}$ be a reflexive coherent sheaf of rank $r>2$ on X. Then, for any positive integer $m>2$, we have*

$$c_2(S^{[m]}\mathcal{F}) = Ac_1^2(\mathcal{F}) + Bc_2(\mathcal{F}),$$

where A and B are non-zero rational numbers depending only on m and r, and satisfy the relation

$$A + \frac{r-1}{r}B - \frac{(R-1)Rm^2}{2r^2} = 0$$

where $R = \binom{r+m-1}{r}$ is the rank of $S^{[m]}\mathcal{F}$.

PROOF. The proof is almost identical to Lemma 4.5 in [**LOY20**]. The only difference is the abandonment of the use of the auxiliary ample line bundle. For this reason, we only sketch the proof. We have trivially the form of the equality over the open set where the sheaf is locally free. By Lemma 5.18, the same equality should hold on X. By splitting principle, it is enough to prove the formula for $\mathcal{F} = \oplus^r L$ where L is a hermitian (complex) line bundle (not necessarily holomorphic).

In this case, $\mathcal{F}$ is a polystable and projectively flat vector bundle, thus we have the equality case in the Bogomolov-Lübke inequality,

$$(c_2(\mathcal{F}) - \frac{r-1}{r}c_1(\mathcal{F})^2)\cdot\omega^{n-2} = 0.$$

Develop $c_2(S^m\mathcal{F}\otimes L^{*\otimes m}) = 0$ in terms of $c_1(L)$. Combining with the above equality, we have

$$(A + \frac{r-1}{r}B - \frac{(R-1)Rm^2}{2r^2})c_1(L)^2\cdot\omega^{n-2} = 0.$$

It suffices to show that there exists a hermitian (complex) line bundle such that $c_1(L)^q\cdot\omega^{n-q}\neq 0$ for any q. Recall that Théorème 4.3 of [**Lae02**] proved using Kronecker lemma that for any closed real $(1,1)-$form α on a compact complex manifold, for infinite k, $k\alpha$ can be approximated in C^∞ norm by the curvature of some hermitian (complex) line bundle L_k with respect to some hermitian connection. In particular, for such k large enough, $c_1(L_k)^q\cdot\omega^{n-q}\neq 0$ for any q.

By choosing $\mathcal{F}$ as some combination of L_k, L_k^* and $\mathcal{O}_X$, it can be shown that A, B are non-zero. □

For the convenience of the reader, we give here the proof of the compact Kähler version of proposition 4.6 Chap. IV of [**Nak04**].

PROPOSITION 5.5.1. *Let (X,ω) be a compact Kähler manifold and $\mathcal{F}$ be an ω-semi-stable reflexive sheaf with*

$$(c_2(\mathcal{F}) - \frac{r-1}{r}c_1(\mathcal{F})^2)\cdot\omega^{n-2} = 0.$$

Then $\mathcal{F}$ is locally free.

PROOF. We shall prove by induction on the rank of $\mathcal{F}$. If $\mathcal{F}$ is polystable, it is direct consequence of corollary 3 of [**BS94**]. We may assume $\mathcal{F}$ is not polystable. Then there is an exact sequence

$$0\to\mathcal{S}\to\mathcal{F}\to\mathcal{Q}\to 0,$$

where $\mathcal{S}$ and $\mathcal{Q}$ are non-zero torsion-free sheaves satisfying the relation of slope $\mu(\mathcal{S}) = \mu(\mathcal{F}) = \mu(\mathcal{Q})$. The sheaves $\mathcal{S}$ and $\mathcal{Q}^{**}$ are semi-stable.

Recall the formula (II.9) Chap. II [**Nak04**]

$$\hat{\Delta}_2(\mathcal{F}) = \hat{\Delta}_2(\mathcal{S}) + \hat{\Delta}_2(\mathcal{Q}) - \frac{\text{rank }\mathcal{S}\cdot\text{rank }\mathcal{Q}}{\text{rank }\mathcal{F}}(\mu(\mathcal{S})-\mu(\mathcal{Q}))^2$$

where $\hat{\Delta}_2(\mathcal{F}) := (c_2(\mathcal{F}) - \frac{r-1}{2r}c_1(\mathcal{F})^2)\cdot\omega^{n-2}$. The Bogomolov inequality gives $\hat{\Delta}_2(\mathcal{S})\geqslant 0$ and $\hat{\Delta}_2(\mathcal{Q}^{**})\geqslant 0$. On the other hand, $c_2(\mathcal{Q}^{**}/\mathcal{Q})$ is represented by an effective cycle supported in the support of the torsion sheaf $\mathcal{Q}^{**}/\mathcal{Q}$. Thus we have that

$$\hat{\Delta}_2(\mathcal{Q}^{**}) = \hat{\Delta}_2(\mathcal{Q}) = \hat{\Delta}_2(\mathcal{S}) = 0.$$

By induction, $\mathcal{S}$ and $\mathcal{Q}^{**}$ are locally free which by lemma 5.15 defines an extension of vector bundles over X. Since $\mathcal{F}$ coincides with a vector bundle outside an analytic set of codimension at least 3, $\mathcal{F}$ is locally free. □

As consequence of the lemma and the proposition, we have the following generalisation of Corollary 5.52.

Corollary 5.56. (analogue of Theorem 1.6 [**LOY20**])
Let (X,ω) be a compact Kähler manifold of dimension n, and let $\mathcal{F}$ be a reflexive coherent sheaf on X. Assume there exists a line bundle L and $m>0$ such that $S^{[m]}\mathcal{F}\otimes L$ is strongly psef with $c_1(S^{[m]}\mathcal{F}\otimes L)=0$. Then $\mathcal{F}$ is a vector bundle such that $\mathcal{F}\langle -\frac{1}{m}L\rangle$ is a $\mathbb{Q}-$twisted nef vector bundle.

In particular, let E be a vector bundle of rank r such that $E\langle -\frac{1}{r}\det(E)\rangle$ is $\mathbb{Q}-$twisted strongly psef vector bundle, then $E\langle -\frac{1}{r}\det(E)\rangle$ is $\mathbb{Q}-$twisted nef vector bundle.

Proof. By corollary 5.52, $S^{[m]}\mathcal{F}\otimes L$ is a numerically flat vector bundle. In particular, $c_2(S^{[m]}\mathcal{F}\otimes L)=0$ and $S^{[m]}\mathcal{F}\otimes L$ is semistable. In fact, $S^{[m]}\mathcal{F}\otimes L$ admits a filtration of vector bundles

$$0=\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_p=S^{[m]}\mathcal{F}\otimes L$$

such that for each i, $\mathcal{E}_i/\mathcal{E}_{i-1}$ is flat and polystable. For any subsheaf $\mathcal{S}$ of $S^{[m]}\mathcal{F}\otimes L$, let $i_0:=\max\{i,\mathcal{E}_i\subset\mathcal{S}\}$. Then if $\mathcal{S}=\mathcal{E}_{i_0}$, $\mu(\mathcal{S})=0$. Otherwise $\mathcal{F}/\mathcal{E}_{i_0}$ is a non zero subsheaf of $\mathcal{E}_{i_0+1}/\mathcal{E}_{i_0}$, thus $\mu(\mathcal{S})=\mu(\mathcal{S}/\mathcal{E}_{i_0})+\mu(\mathcal{E}_{i_0})\leqslant\mu(\mathcal{E}_{i_0+1}/\mathcal{E}_{i_0})=0$.

By the above lemma, direct calculations yield

$$(c_2(\mathcal{F})-\frac{r-1}{2r}c_1(\mathcal{F})^2)\cdot\omega^{n-2}=0.$$

We claim that $\mathcal{F}$ is also semistable. In fact, for any torsion free quotient sheaf $\mathcal{Q}$ of $\mathcal{F}$, we have generic surjective morphism

$$\alpha: S^{[m]}\mathcal{F}\otimes L\to S^{[m]}\mathcal{Q}\otimes L.$$

The image of α coincide with $S^{[m]}\mathcal{Q}\otimes L$ outside an analytic set of codimension at least 2, thus these two sheaves have the same slope. The inequality $\mu(S^{[m]}\mathcal{F}\otimes L)\leqslant\mu(S^{[m]}\mathcal{Q}\otimes L)$ implies that $\mu(\mathcal{F})\leqslant\mu(\mathcal{Q})$. In fact, $S^{[m]}\mathcal{F}$ and $\mathcal{F}$ are locally free outside a closed analytic set A of codimension at least 2. Since $H^2(X,\mathbb{C})\cong H^2(X\smallsetminus A,\mathcal{F})$,

$$c_1(S^{[m]}\mathcal{F})=\frac{1}{r}\binom{m+r-1}{m}c_1(\mathcal{F})$$

from the corresponding formula by restriction on $X\smallsetminus A$ on which the coherent sheaves are locally free. Here r is the rank of $\mathcal{F}$. We have of course similar formula for $\mathcal{Q}$.

For the general case, it is a direct consequence of the above proposition. Thus we can prove the following equivalent conclusion. $\mathcal{F}$ is locally free and there is a filtration of vector subbundles

$$0=\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_p=\mathcal{F}$$

such that $\mathcal{F}_i/\mathcal{F}_{i+1}$ are projectively flat vector bundles and $\mu(\mathcal{F}_i/\mathcal{F}_{i+1})=\mu(\mathcal{F})$ for any i. □

As pointed out to us by A. Höring, Corollary 5.54 can be established in the following easier way. As above, one shows that a compact Kähler manifold (X,ω) with strongly psef tangent bundle or cotangent bundle and $c_1(X)=0$ is a finite étale quotient of a complex torus. By our main theorem, T_X is a numerically flat vector bundle. In particular, it is well-known that T_X is ω-semi-stable and that $c_2(X)=0$. (This is the special case considered at the beginning of the proof of Corollary 5.56.) Thus we have the equality case in the Bogomolov inequality, and therefore the tangent bundle T_X is projectively flat. Since $c_1(X)=0$, T_X is flat, which, by the Bieberbach theorem, implies that X is a torus, up to a finite étale cover.

CHAPTER 6

Intersection theory and Chern classes in Bott-Chern cohomology

Abstract. In this article, we study the axiomatic approach of Grivaux in [**Gri10**] for rational Bott-Chern cohomology, and use it in particular to define Chern classes of coherent sheaves in rational Bott-Chern cohomology. This method also allows us to derive a Riemann-Roch-Grothendieck formula for a projective morphism between smooth complex compact manifolds. The appendix presents a calculation of integral Bott-Chern cohomology in top degree for a connected compact manifold.

In the general case of complex spaces, the Poincaré and Dolbeault-Grothendieck lemmas are not valid in general. For this reason, and to simplify the exposition, we only consider non singular complex spaces in the sequel, and let X denote throughout a complex manifold.

6.1. Introduction

Chern classes and Chern characteristic classes are very important topological invariants of complex vector bundles. In order to better reflect the complex structure of manifolds, we refine Chern classes and Chern characteristic classes, and define them in rational Bott-Chern cohomology. This is done by introducing suitable complexes of sheaves of holomorphic and anti-holomorphic forms. There exists a canonical morphism from the complex of rational Bott-Chern cohomology into the locally constant sheaf $\mathbb{Q}$, seen as a complex with a single term located in degree 0. Under this morphism, the image of Chern classes and Chern characteristic classes in rational Bott-Chern cohomology are the usual ones defined in singular cohomology.

In the fundamental article [**Gri10**], Grivaux showed that for suitable rational cohomology theories of compact complex manifolds, one can construct Chern characteristic classes of arbitrary coherent sheaves, and in particular of torsion sheaves, by induction on the dimension. This can be done provided one has a reasonable intersection theory, and provided Chern classes can be defined for vector bundles. One important argument consists of ensuring the validity of the Riemann-Roch-Grothendieck formula for closed immersions of smooth hypersurfaces.

We begin by recalling some background for this type of problems. For any complex manifold X, we denote by K^0X the Grothendieck group of vector bundles on X. For a vector bundle E, we denote by $[E]$ the class represented by E. By definition, K^0X is the quotient of the free abelian group on the set of isomorphism classes of vector bundles, modulo the relations

$$[E] = [E'] + [E'']$$

for all exact sequences $0 \to E' \to E \to E'' \to 0$. It can be endowed with a ring structure by taking tensor products of vector bundles.

In a similar way, we denote by K_0X the Grothendieck group of coherent sheaves on X, simply by replacing vector bundles in the definition of K^0X by coherent sheaves, and one has a natural morphism $K^0X \to K_0X$ by viewing vector bundles as coherent sheaves. This morphism is an isomorphism in the projective case. However, by the fundamental work of Voisin [**Voi02a**], K^0X can be strictly smaller than K_0X when X is a compact Kähler manifold. This phenomenon is caused by the lack of global resolutions of coherent sheaves by locally free sheaves.

Over $\mathbb{Q}$, Chern characteristic class can be seen through the $\mathbb{Q}$-linear morphism

$$\mathrm{ch} : K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to A(X),$$

where $A(X)$ means the cohomology ring in the cohomology theory under consideration. A priori, on arbitrary compact complex manifolds, it is not trivial that this morphism can extended into a morphism from $K_0(X)\otimes_{\mathbb{Z}} \mathbb{Q}$. Grivaux showed that this is possible once the cohomology theory satisfies suitable axioms of intersection theory. The aim of this note is to develop a similar intersection theory for integral (or rational) Bott-Chern cohomology.

Such theories have been considered in the work [**Sch07**] of M. Schweitzer, and have also been developed in a more recent unpublished work of Junyan Cao. They are more precise than Deligne cohomology or than complex Bott-Chern cohomology, in the sense that there always exist natural morphisms from the integral (or rational) Bott-Chern cohomology into the other ones. We use here Grivaux's axiomatic approach to construct Chern classes in rational Bott-Chern cohomology, for coherent sheaves on arbitrary compact complex manifolds.

In fact, it would be interesting to give a construction of Chern classes of coherent sheaves in the integral Bott-Chern cohomology rather than the rational one, but substantial difficulties remain. Let $\mathcal{F}$ be a coherent sheaf on a smooth hypersurface D of X. We denote by $i : D \to X$ the inclusion. One of the main difficulties is to express the total Chern class $c(i_*\mathcal{F})$ in function of $i_*c_\bullet(\mathcal{F})$ and $i_*c_\bullet(N_{D/X})$, where $N_{D/X}$ is the normal bundle of D in X. There exists a formulation of the Riemann-Roch-Grothendieck formula that does not involve denominators, but it does not seem to be easily applicable since Chern classes of coherent sheaves, unlike in the vector bundle case, may involve data in higher degrees than the generic rank.

Using the methods developped in this note combined with the work of [**Gri10**], we give as an application a more algebraic proof of the following theorem of Bismut [**Bis11**], [**Bis13**] under the additional assumption that the morphism is projective. However, we do not need the condition that the sheaf and all of its direct images are locally free, nor the condition that the morphism is a submersion.

THEOREM 6.1. *Let $p : X \to S$ a projective morphism of compact complex manifolds and $\mathcal{F}$ be a coherent sheaf over X. Then we have the Riemann-Roch-Grothendieck formula in the rational and complex Bott-Chern cohomology*

$$\mathrm{ch}(R^\bullet p_*\mathcal{F})\mathrm{Td}(T_S) = p_*(\mathrm{ch}(\mathcal{F})\mathrm{Td}(T_X))$$

where $R^\bullet p_*\mathcal{F} = \sum_i R^i p_*\mathcal{F}$.

The rational case is a direct consequence of the work of [**Gri10**], which uses classical arguments of Serre to reduce the prooof to the fact that the Riemann-Roch-Grothendieck formula holds for a closed immersion. It is proven by a construction of Chern characteristic classes (or equivalently of Chern classes in the rational coefficient case), using the prescribed axioms of intersection theory. The complex case can be derived by the natural morphism from the rational Bott-Chern cohomology to the complex Bott-Chern cohomology.

For the convenience of the reader, we summarize here the axioms needed in the axiomatic cohomology theory developped in [**Gri10**]. We assume that for any compact complex manifold X we can associate to X a graded commutative cohomology ring $A(X)$ which is also a $\mathbb{Q}(\subset A^0(X))$-algebra.

Axiom A (Chern classes for vector bundles)

(1) For each holomorphic map $f : X \to Y$, there exists a functorial pull-back morphism $f^* : A(Y) \to A(X)$ which is compatible with the products and the gradings.

(2) One has a group morphism $c_1 : \mathrm{Pic}(X) \to A^1(X)$ which is compatible with pull-backs.

(3) (Splitting principle) If E is a holomorphic vector bundle of rank r on X, then $A(\mathbb{P}(E))$ is a free graded module over $A(X)$ with basis $1, c_1(\mathcal{O}_E(1)), \cdots, (c_1(\mathcal{O}_E(1))^{r-1}$.

(4) (homotopy principle) For every t in $\mathbb{P}^1$, let i_t be the inclusion $X \times \{t\} \hookrightarrow X \times \mathbb{P}^1$. Then the induced pull-back morphism $i_t^* : A(X \times \mathbb{P}^1) \to A(X) \cong A(X \times \{t\})$ is independent of t.

(5) (Whitney formula) Let $0 \to E \to F \to G \to 0$ be an exact sequence of vector bundles, then one has $c(F) = c(E) \cdot c(F)$ and $\mathrm{ch}(F) = \mathrm{ch}(E) + \mathrm{ch}(G)$ where $c(E)$ means the total Chern class of E and $\mathrm{ch}(E)$ means the Chern characteristic class of E.

The construction of the pull-back will be given in the second section and the other parts are important results of Junyan Cao which will be given the fourth section.

Axiom B (Intersection theory)

If $f : X \to Y$ is a proper holomorphic map of relative dimension d, there is a functorial Gysin morphism $f_* : A^\bullet(X) \to A^{\bullet-d}(Y)$ satisfying the following properties:

(1) (Projection formula) For any $x \in A(X)$ and any $y \in A(Y)$ one has $f_*(x \cdot f^*y) = f_*(x) \cdot y$.

(2) Consider the following commutative diagram with p, q the projections on the first factors

$$\begin{array}{ccc} Y \times Z & \xrightarrow{i_{Y\times Z}} & X \times Z \\ \downarrow p & & \downarrow q \\ Y & \xrightarrow{i_Y} & X \end{array}$$

Assume Z to be compact and i_Y proper. Then one has $q^* i_{Y*} = i_{Y\times Z*} p^*$.

(3) Let $f : X \to Y$ be a surjective proper map between compact manifolds, and let D be a smooth divisor of Y. We denote $f^*D = m_1\tilde{D}_1 + \cdots + m_N\tilde{D}_N$ with $\tilde{D}_i$ simply normal crossing. Let $\tilde{f}_i : \tilde{D}_i \to D$ $(1 \leqslant i \leqslant N)$ be the restriction of f to $\tilde{D}_i$. Then one has

$$f^* i_{D*} = \sum_{i=1}^N m_i\, i_{\tilde{D}_i *} \tilde{f}_i^*.$$

(4)Consider the commutative diagram, where Y and Z are compact and intersect transversally with $W =$

$Y \cap Z$:

$$\begin{array}{ccc} W & \xrightarrow{i_{W/Y}} & Y \\ {\scriptstyle i_{W/Z}}\downarrow & & \downarrow{\scriptstyle i_Y} \\ Z & \xrightarrow[i_Z]{} & X. \end{array}$$

Then one has $i_Y^* i_{Z*} = i_{W/Y*} i_{W/Z}^*$.
(5)(Excess formula) If Y is a smooth hypersurface of a compact complex manifold X, then for any cohomological class α we have

$$i_Y^* i_{Y*}\alpha = \alpha \cdot c_1(N_{Y/X}).$$

(6) The Hirzebruch–Riemann–Roch theorem holds for $(\mathbb{P}^n, \mathcal{O}(i))$ $(\forall i)$.
(7) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and $Y \subset X$ be a closed complex submanifold of complex codimension $r \geqslant 2$. Suppose that $p : \tilde{X} \to X$ is the blow-up of X along Y. We denote by E the exception divisor and $i : Y \to X$, $j : E \to \tilde{X}$ the inclusions, and $q : E \to Y$ the restriction of p on E. Then p^* is injective and there is an isomorphism induced by j^*

$$j^* : A^\bullet(\tilde{X})/p^* A^\bullet(X) \cong A^\bullet(E)/q^* A^\bullet(Y).$$

In other words, a class $\alpha \in A^\bullet(\tilde{X})$ is in the image of p^* if and only if the class $j^*\alpha$ is in the image of q^*.

The verification of axiom B will constitute the main substance of the fifth and sixth sections. In principle, pull-backs can be induced by taking the pull-back of smooth forms, and push-forwards can be induced by taking the push-forward of currents under proper morphisms. The proof of the first two axioms is then reduced to considering the natural pairing between smooth forms and currents. The third and fourth axioms are more complicated, since they demand taking pull-backs of currents. As in the case of Deligne cohomology, we first reduce the situation to the case of cycle classes. Then we reduce cycle classes to integral Bott-Chern (or Deligne) cohomology by means of Bloch cycle classes, which can be represented by holomorphic forms. Checking the remaining axioms is more standard. This will be done in the sixth section.

In conclusion, it can be shown that the cohomology ring $\oplus_k H_{BC}^{k,k}(X,\mathbb{Q})$ satisfies axiom A, B. In fact, the cohomology ring $\oplus_k H_{BC}^{k,k}(X,\mathbb{Z})$ satisfies axiom A, B except the sixth one of list B which demands rational coefficients to define Chern characteristic classes and the Todd class. As a consequence, by the work of [**Gri10**], for the rational Bott-Chern cohomology we get the following result.

THEOREM 6.2. *If X is compact and K_0X is the Grothendieck ring of coherent sheaves on X , one can define a Chern character morphism* $\mathrm{ch} : K_0X \to \oplus_k H_{BC}^{k,k}(X,\mathbb{Q})$ *such that*
(1) the Chern character morphism is functorial by pull-backs of holomorphic maps.
(2) the Chern character morphism is an extension of the usual Chern character morphism for locally free sheaves given in axiom A.
(3) The Riemann–Roch-Grothendieck theorem holds for projective morphisms between smooth complex compact manifolds.

The organisation of the paper is the following. Section two recalls basic definitions and introduces pull back and push forward morphisms. Section three introduces a ring structure on the integral Bott-Chern cohomology, in such a way that it is compatible with the ring structure of the complex Bott-Chern cohomology via the canonical map. Section four gives the construction of Chern classes associated with a vector bundle and verifies the list of axioms A. Section five introduces cycle classes in integral Bott-Chern cohomology and verifies the intersection theory part of axioms B. Section six studies the transformation of Chern classes under blow ups. This completes the verification of axioms B. At the end, we present an appendix in which we calculate the integral Bott-Chern cohomology of a connected compact manifold in top degree. The analogous result for integral Deligne cohomology do not seem to be as direct.

6.2. Definition of integral Bott-Chern cohomology classes

In this section, we recall the basic definitions associated with integral Bott-Chern cohomology. A reference for this part is [**Sch07**]. Notice that changing $\mathbb{Z}(p)$ by $\mathbb{C}$ in the integral Bott-Chern complex gives a quasi-isomorphic complex which defines the complex Bott-Chern cohomology. Hence one gets a canonical map from the integral Bott-Chern cohomology to the complex Bott-Chern cohomology. Next, we define pull backs and push forwards in integral Bott-Chern cohomology. We verify the axioms without involving the ring structure of the integral Bott-Chern cohomology (namely Axiom B (2), part of (7)).

DEFINITION 6.3. *The integral Bott-Chern cohomology group is defined as the hypercohomology group*

$$H_{BC}^{p,q}(X,\mathbb{Z}) = \mathbb{H}^{p+q}(X, \mathcal{B}_{p,q,\mathbb{Z}}^*)$$

of the integral Bott-Chern complex

$$\mathcal{B}^\bullet_{p,q,\mathbb{Z}} : \mathbb{Z}(p) \xrightarrow{\Delta} \mathcal{O} \oplus \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega^1} \to \cdots \to \Omega^{p-1} \oplus \overline{\Omega^{p-1}} \to \overline{\Omega^p} \to \cdots \to \overline{\Omega^{q-1}} \to 0$$

where $\mathbb{Z}(p) = (2\pi i)^p\mathbb{Z}$ at 0 degree and Δ is multiplication by 1 for the first component and multiplication by -1 for the second component. We call rational (or complex) Bott-Chern cohomology the hypercohomology of the complex obtained by changing $\mathbb{Z}(p)$ respectively into $\mathbb{Q}, \mathbb{C}$.

Notice that the choice of the sign in Δ is to ensure that the natural map from the integral Bott-Chern cohomology to the complex Bott-Chern cohomology is a ring morphism. This will be discussed in Section 3. The choice of $\mathbb{Z}(p)$ instead of $\mathbb{Z}(q)$ is more or less artificial, but since the Chern class always lies in $H^{p,p}_{BC}(X, \mathbb{Z})$ for some p, this choice poses no problem.

We begin by the definition of pull-backs of cohomology classes. Let $f : X \to Y$ be a holomorphic map, it induces a natural morphism of complexes of abelian group on any open set U of Y, $\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}(U) \xrightarrow{f^*} \mathcal{B}^\bullet_{p,q,\mathbb{Z},X}(f^{-1}(U))$ which induces the cohomological class morphism $H^{p,q}_{BC}(Y, \mathbb{Z}) \xrightarrow{f^*} H^{p,q}_{BC}(X, \mathbb{Z})$. More precisely, the pull-back of forms induces a morphism of complexes $f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y} \xrightarrow{f^*} \mathcal{B}^\bullet_{p,q,\mathbb{Z},X}$ on X which induces a cohomological morphism $\mathbb{H}^\bullet(X, f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}) \to \mathbb{H}^\bullet(X, \mathcal{B}^\bullet_{p,q,\mathbb{Z},X})$. On the other hand, there exists a natural morphism $\mathbb{H}^\bullet(Y, \mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}) \to \mathbb{H}^\bullet(X, f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y})$ since the pre-image of any open covering of Y gives an open covering of X. The composition of two morphisms gives the pull back morphism $H^{p,q}_{BC}(Y, \mathbb{Z}) \xrightarrow{f^*} H^{p,q}_{BC}(X, \mathbb{Z})$. The second morphism can be interpreted more formally as follows. There exists a natural morphism $\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y} \to Rf_*f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}$. Taking $R\Gamma(Y, -)$ on both sides gives $\mathbb{H}^\bullet(Y, \mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}) \to \mathbb{H}^\bullet(X, f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y})$.

For a proper holomorphic map $f : X \to Y$ of relative dimension d, we next construct a functorial Gysin morphism $f_* : H^{p,q}_{BC}(X, \mathbb{Z}) \to H^{p-d,q-d}_{BC}(Y, \mathbb{Z})$. The construction is a modification of the similar construction for Deligne cohomological class given in [**ZZ84**]. The condition of properness is necessary even if we just consider cycle classes, since the image of an analytic set is not necessarily an analytic set when the properness condition is omitted.

Let $K^\bullet$ be a complex of sheaves on the space X. One denotes by $\{F^pK^\bullet\}$ the stupid filtration which does not preserve the cohomology at degree p i.e. if $q \geqslant p$, $F^pK^q = K^q$, otherwise $F^pK^q = 0$. For the corresponding quotient complex, we denote it as $\sigma_pK^\bullet = K^\bullet/F^pK^\bullet$. We denote by $\Omega^\bullet$ the complex of sheaves of holomorphic forms on X. Let $i : \mathbb{Z}(p) \to \sigma_p\Omega^\bullet \oplus \sigma_q\overline{\Omega^\bullet}$ be the complex map defined by the diagonal map sending $\mathbb{Z}(p)$ into $\mathcal{O}_X \oplus \overline{\mathcal{O}_X}$ at degree 0 with a sign -1 at the second component and viewing $\mathbb{Z}(p)$ as a complex centred at degree 0. With the above notations, the integral Bott-Chern complex is the mapping cone of i which we denote as $\mathrm{Cone}^\bullet(i)[-1]$. The idea to define the push-forward of the cohomology class is to choose compatible resolutions of the complexes $\mathbb{Z}(p), \sigma_p\Omega^\bullet \oplus \sigma_q\overline{\Omega^\bullet}$ such that the both complexes are formed by some kind of currents for which the push-forward is well-defined.

For the convenience of the readers, we recall here some basic definitions and properties concerning currents and geometric measure theory. We will use them to define a resolution of $\mathbb{Z}(p)$. For more details and proofs, we refer to the article of [**Kin71**].

DEFINITION 6.4. *Let A, B be two metric spaces and $f : A \to B$ be a map. We say that f is Lipschitz if there exists $C > 0$ such that for any $a, b \in A$, we have*

$$d(f(a), f(b)) \leqslant Cd(a, b).$$

We now recall the definition of the mass of a current.

DEFINITION 6.5. *For any continuous form on a Riemannian manifold N and any $x \in N$, we define a function*

$$\|u\|(x) = \sup\{|u(\lambda)| \ : \ \lambda \text{ is a decomposable } r\text{-vector at } x \text{ with } |\lambda|_x \leqslant 1\}.$$

For any set $K \subset N$, the comass of u on K is

$$\nu_K(u) = \sup\{\|u\|(x), x \in K\}.$$

The mass of a current T is

$$M(T) = \{|T(u)| : u \in A^r_c(N), \nu_N(u) \leqslant 1\}$$

where $A^r_c(N)$ is the space of the smooth r-forms with compact support.

Let $U \subset \mathbb{R}^s$ be an open set of euclidean space and N a Riemannian manifold. Let P be a current defined by a finite sum of oriented linear simplices and $f : U \to N$ a Lipschitz map. We can approximate f by $f_i : U \to N$ which is $\mathcal{C}^1$ and we define f_*P to be the limit of $f_{i*}P$ in the sense of currents. Using this construction, one can define rectifiable currents.

DEFINITION 6.6. *For any compact set $K \Subset N$, one defines the space $R_{r,K}(N)$ of rectifiable r-currents in K as follows: $T \in R_{r,K}(N)$ if and only if $T \in \mathcal{E}'_r(N)$ (the dual space of smooth forms) and for any $\varepsilon > 0$, there exists $U \subset \mathbb{R}^s$ for some s, $f : U \to N$ a Lipschitz map and P a current defined by finite sum of oriented linear simplices such that*

$$M(T - f_*(P)) < \varepsilon.$$

One defines the space $R_r(N)$ of rectifiable r-currents by $R_r(N) := \bigcup_{K \Subset N} R_{r,K}(N)$ and the space $R_r^{\mathrm{loc}}(N)$ of locally rectifiable r-currents by

$$R_r^{\mathrm{loc}}(N) = \{T \in \mathcal{D}'_r(N) | \forall x \in N, \exists T_x \in R_r(N) \text{ s.t. } x \in N \smallsetminus \operatorname{supp}(T - T_x)\}.$$

Now, one defines locally integral currents.

DEFINITION 6.7. *The space of locally integral currents is defined by*

$$\mathcal{I}_r^{\mathrm{loc}}(N) := \{T \in R_r^{\mathrm{loc}}(N) | dT \in R_r^{\mathrm{loc}}(N)\}.$$

We have the following version of the Federer support theorem.

THEOREM 6.8. *Let $i : M \to N$ be an embedding of the submanifold M into N. One has*

$$i_*\mathcal{I}_r^{\mathrm{loc}}(M) = \{T \in \mathcal{I}_r^{\mathrm{loc}}(N) | \operatorname{supp}(T) \subset M\}.$$

As a corollary, the sheaf of locally integral currents is a soft sheaf, and is in particular acyclic.

COROLLARY 6.9. Let N be any Riemannian manifold. Then the sheaf of locally integral currents $\mathcal{I}_\bullet^{\mathrm{loc}}$ is a soft sheaf.

PROOF. Let F be a closed set of N with respect to the metric topology. Let $s \in \mathcal{I}_r^{\mathrm{loc}}(F) = \varinjlim_{F \subset U} \mathcal{I}_r^{\mathrm{loc}}(U)$ be a section on F. By definition, there exists s_U a section defined on U an open set of N such that $s_U|_F = s$. Consider $i : U \to N$ the inclusion. The Federer support theorem gives a section $\tilde{s} \in \mathcal{I}_r^{\mathrm{loc}}(N)$ such that $i_* s_U = \tilde{s}$. Hence $\tilde{s}$ extends s, and this proves that the sheaf of locally integral currents is soft. □

Notice that for any smooth morphism $f : M \to N$, f_* maps locally integral currents to locally integral currents even without the properness condition on f. To see that the complex of locally integral currents gives a resolution of the locally constant sheaf $\mathbb{Z}$, we need the fact that for $T \in \mathcal{I}_m^{\mathrm{loc}}(\mathbb{R}^n)$ such that $dT = 0$ there exists a $S \in \mathcal{I}_{m+1}^{\mathrm{loc}}(\mathbb{R}^n)$ such that $dS = T$ (cf. **[Fed96]** 4.2.10 as a consequence of the deformation theorem) and the following proposition in **[Kin71]** proposition 2.1.9 for the case of top degree.

THEOREM 6.10. *Let M be a Riemannian manifold of dimension n. If $T \in \mathcal{D}'^0(M)$ such that $dT = 0$ then T is the current defined by locally constant functions. If $T \in \mathcal{I}_n^{\mathrm{loc}}(M)$ then this function is integral valued.*

We now return to the construction of the push forward for hypercohomology. We denote by ${\mathcal{D}'_X}^{p,q}$ the sheaf of currents of type (p,q) on X. For each p, $({\mathcal{D}'_X}^{p,\bullet}, \bar{\partial})$ is a fine resolution of Ω_X^p. By taking the conjugation, $({\mathcal{D}'_X}^{\bullet,q}, \partial)$ is a fine resolution of $\overline{\Omega_X^q}$. The conjugate of differential forms induces the conjugate of currents. In particular, $\sigma_{p,\bullet}{\mathcal{D}'_X}^{\bullet,\bullet}$ (resp. $\sigma_{\bullet,q}{\mathcal{D}'_X}^{\bullet,\bullet}$) is a Cartan-Eilenberg resolution of $\sigma_p \Omega_X^\bullet$ (resp. $\sigma_q \overline{\Omega_X^\bullet}$). Taking the total complex of the double complex, we deduce that $\sigma_p {\mathcal{D}'_X}^\bullet$ is a resolution of $\sigma_p \Omega_X^\bullet$. Here, we use an abuse of notation, and actually mean that we take direct sums of spaces of currents of bidegree (k,l) with $k \leqslant p$. Similarly, $\sigma_q {\mathcal{D}'_X}^\bullet$ is a resolution of $\sigma_q \overline{\Omega_X^\bullet}$. By taking complex coefficients, locally integral currents extend into a complex of $\mathbb{C}$-vector spaces of currents instead of $\mathbb{Z}$-modules.

Let $\mathcal{I}_X^i$ be the complex valued extended sheaf of locally integral currents of real codimension i on X, as defined above. The complex $\mathcal{I}_X^\bullet$ is a soft resolution of $\mathbb{Z}$. The integral Bott-Chern complex is quasi-isomorphic to the following complex obtained by composing the natural inclusion of forms into currents:

$$\mathbb{Z}(p) \xrightarrow{\Delta} \sigma_p {\mathcal{D}'_X}^\bullet \oplus \sigma_q {\mathcal{D}'_X}^\bullet.$$

This morphism of complexes factorises into

$$\mathbb{Z}(p) \to \mathcal{I}_X^\bullet \xrightarrow{\Delta} \sigma_p {\mathcal{D}'_X}^\bullet \oplus \sigma_q {\mathcal{D}'_X}^\bullet.$$

The morphism of complexes Δ factorises itself into the composition of two maps : the first is the diagonal map with positive sign on the first component and negative sign on the second component with image in ${\mathcal{D}'_X}^\bullet \oplus {\mathcal{D}'_X}^\bullet$; the second map is the decomposition of locally integral currents into their components of adequate bidegrees.

Since the first inclusion is a quasi-isomorphism in the derived category in $D(\mathrm{Sh}(X))$, the integral Bott-Chern complex is quasi-isomorphic to $\mathrm{Cone}^\bullet(\Delta)[-1] : \mathcal{I}_X^\bullet \xrightarrow{\Delta} \sigma_p {\mathcal{D}'_X}^\bullet \oplus \sigma_q {\mathcal{D}'_X}^\bullet$. Note that the push-forward of currents and of the locally integral currents are both well-defined for a proper morphism. We also remark that the rule $df_* = f_* d$ holds for currents. Hence there exists a natural morphism of complexes on Y

$$f_* \mathcal{I}_X^\bullet \to \mathcal{I}_Y^{\bullet - d}, f_*(\sigma_p {\mathcal{D}'_X}^\bullet \oplus \sigma_q {\mathcal{D}'_X}^\bullet) \to \sigma_{p-d} {\mathcal{D}'_Y}^\bullet \oplus \sigma_{q-d} {\mathcal{D}'_Y}^\bullet$$

which, as will be explained below, induces a cohomological group morphism

$$f_* : H^{p,q}_{BC}(X,\mathbb{Z}) \to H^{p-d,q-d}_{BC}(Y,\mathbb{Z}).$$

Here, to define the push-forward for cohomology classes, it is enough to define it for global section representatives; in fact, the complex $\mathcal{I}^\bullet_X$ is soft, which means any section over any closed subset can be extended to a global section; a soft sheaf is in particular acyclic, thus the complex $\sigma_p\mathcal{D}'^{\bullet}_X \oplus \sigma_q\mathcal{D}'^{\bullet}_X$ is acyclic. The hypercohomology of the integral Bott-Chern complex is just the cohomology of the global sections of the mapping cone Δ. Now we define the push-forward of a cohomology class as the push-forward of any of the global currents representing the cohomology class. By construction, the pull-back and push-forward both satisfy the functoriality property.

Notice that the use of a resolution of the locally constant sheaf $\mathbb{Z}(p)$ seems to be necessary since a priori we have only natural morphism in inverse direction $\mathbb{H}^\bullet(Y, f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) \to \mathbb{H}^\bullet(X,\mathcal{B}^\bullet_{p,q,\mathbb{Z},X})$. The trace morphism $\mathrm{tr} : f_*\mathbb{Z}_X \to \mathbb{Z}_Y$ and the push forward of currents induces a morphism $\mathbb{H}^\bullet(Y, f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) \to \mathbb{H}^\bullet(Y,\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y})$ if X, Y have the same dimension. It seems to be not easy to induces from these two morphisms a morphism $\mathbb{H}^\bullet(X,\mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) \to \mathbb{H}^\bullet(Y,\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y})$. If we take the quasi-isomorphic acyclic resolution involving the locally integral currents, the hypercohomology of $\mathbb{H}^\bullet(X, B^\bullet_{p,q,\mathbb{Z},X})$ is represented by global sections. Then the restriction of the global section on the open sets induces a morphism $\mathbb{H}^\bullet(X,\mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) \to \mathbb{H}^\bullet(Y, f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X})$ in the desired direction. In this case, we have the following factorisation

$$\begin{array}{ccc} \mathbb{H}^\bullet(X,\mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) & \longrightarrow & \mathbb{H}^\bullet(Y, f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) \\ & \searrow^{f_*} & \downarrow \\ & & \mathbb{H}^\bullet(Y,\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}[-2d]) \end{array}$$

where d is the relative complex dimension. The vertical arrow is the morphism induced by pushing forward currents, under the assumption that f is proper.

Commutativity can be checked directly. Let T be the global section representing a cohomology class in $\mathbb{H}^\bullet(X,\mathcal{B}^\bullet_{p,q,\mathbb{Z},X})$. Let $(V_i)_i$ be an open Stein covering of Y such that the hypercohomology class on Y can be calculated by the hypercohomology associated with the open cover. We denote by $\{T_i\}$ the image of T in $\mathbb{H}^\bullet(Y, f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X})$ by restriction on V_i. More precisely, T_i is the restriction of T on $f^{-1}(V_i)$. Its image in $\mathbb{H}^\bullet(Y,\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}[-2d])$ is $\{f_*T_i\}$, and those sections glue into a global section f_*T.

The definition of the push-forward of cohomology classes can also be interpreted more formally as follows. In order to distinguish the different morphisms of complexes, we denote by Δ_X the map on X involving $\mathbb{Z}(p)$ and $\tilde{\Delta}_X$ the map on X involving locally integral currents. The complex $\mathrm{Cone}(\tilde{\Delta}_X)$ involving locally integral currents is a soft complex. Since f is proper, $f_*\mathrm{Cone}(\tilde{\Delta}_X)$ is a soft complex which means $f_*\mathrm{Cone}(\tilde{\Delta}_X) = Rf_*\mathrm{Cone}(\Delta_X)$ in $D(\mathrm{Sh}(Y))$. We denote by a_X (resp. a_Y) the morphism from X (resp. Y) to a point. The push forward of currents induces a morphism of complexes in $C(\mathrm{Sh}(Y))$: $f_*\mathrm{Cone}(\tilde{\Delta}_X) \to \mathrm{Cone}(\tilde{\Delta}_Y)[-2d]$. In other words, we have by composition a morphism in the derived category

$$Rf_*(\mathrm{Cone}(\Delta_X)) \to \mathrm{Cone}(\Delta_Y)[-2d].$$

Taking $R\Gamma(Y,-) = Ra_{Y*}$ on both sides, and using the fact that $R(a_Y \circ f)_* = Ra_{X*} = Ra_{Y*} \circ Rf_*$ (since f_* transforms soft complexes into soft complexes), we get $f_* : H^{p,q}_{BC}(X,\mathbb{Z}) \to H^{p-d,q-d}_{BC}(Y,\mathbb{Z})$ after taking cohomology.

In the following, once we want to view the push forward of the cohomology groups as a morphism in the cohomology level induced by a morphism of complexes, we use the above interpretation (for example, in the proof of the projection formula).

In the case where f is analytic fibration, in the sense that f is a proper surjective morphism and all fibres are connected, we can additionally define a morphism from the push forward of the locally constant sheaf $\mathbb{Z}_X$ to the locally constant sheaf $\mathbb{Z}_Y$, e.g. a morphism $f_*\mathbb{Z}_X \to \mathbb{Z}_Y$. Any modification f such as a composition of blows-up with smooth centers is an example of an analytic fibration in the above setting. We now use this morphism to prove that any modification p yields an injective morphism p^* between the corresponding integral Bott-Chern cohomology groups.

In this case, for any connected open set $V \subset Y$, we have $f_*\mathbb{Z}_X(V) = Z_X(f^{-1}(V))$ where $f^{-1}(V)$ is a connected open set, so it is enough to define the morphism $f_*\mathbb{Z}_X \to \mathbb{Z}_Y$ by asserting that it associates the constant function 1 on $f^{-1}(V)$ to the constant function 1 on V. In preparation for the next steps, we need the following lemma.

LEMMA 6.11. *For any algebraic fibration $f : X \to Y$, there is a commutative diagram*

$$\begin{array}{ccc} f_*\mathbb{Z}_X & \longrightarrow & f_*\mathcal{I}_X^0 \\ \downarrow & & \downarrow \\ \mathbb{Z}_Y & \longrightarrow & \mathcal{I}_Y^0. \end{array}$$

PROOF. This is directly verified on any connected open set $V \subset Y$. The map $\mathbb{Z}_X(f^{-1}(V)) \to \mathcal{I}_X^0(f^{-1}(V))$ is given by associating the constant function 1 to the integral current $[f^{-1}(V)]$ associated with $f^{-1}(V)$. The image of the constant function 1 under $\mathbb{Z}_X(f^{-1}(V)) \to \mathbb{Z}_Y(V)$ is the constant function 1 on V. The image of the constant function 1 under $\mathbb{Z}_Y(V) \to \mathcal{I}_Y^0(V)$ is the integral current $[V]$ associated with V which is also the image of $[f^{-1}(V)]$ under $f_*\mathcal{I}_X^0(V) \to \mathcal{I}_Y^0(V)$. □

Using an identification of the push forward of currents on X as currents on Y, we get the following commutative diagram

$$\begin{array}{ccc} f_*\mathrm{Cone}(\Delta_X) & \longrightarrow & f_*\mathrm{Cone}(\tilde{\Delta}_X) \\ \downarrow & & \downarrow \\ \mathrm{Cone}(\Delta_Y)[-2d] & \longrightarrow & \mathrm{Cone}(\tilde{\Delta}_Y)[-2d] \end{array}$$

with the above notations. Taking Ra_{Y*} and cohomology to the commutative diagram gives

$$\begin{array}{ccc} \mathbb{H}^\bullet(Y, f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) & \longrightarrow & \mathbb{H}^\bullet(X, \mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) \\ \downarrow & & \downarrow f_* \\ \mathbb{H}^\bullet(Y, \mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}[-2d]) & \xrightarrow{\mathrm{id}} & \mathbb{H}^\bullet(Y, \mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}[-2d]). \end{array}$$

In the case of a modification, one can prove that f^* is injective. This can be seen via the following

LEMMA 6.12. *For any modification $f : X \to Y$, one has*

$$f_* f^* = \mathrm{id} : H^{\bullet,\bullet}_{BC}(Y, \mathbb{Z}) \to H^{\bullet,\bullet}_{BC}(Y, \mathbb{Z}).$$

PROOF. Using the above commutative diagram, it is enough to show that for any open set $V \subset Y$ and any sheaf in the integral Bott-Chern complex one has the identity $f_*f^* = \mathrm{id}$, so that the identity will hold for any hypercocycle representing an integral Bott-Chern cohomology class.

Let A be an analytic set of X, Z be an analytic set of Y such that the map $f|_{X \smallsetminus A} : X \smallsetminus A \to Y \smallsetminus Z$ is biholomorphic. For any smooth form ω defined on V, we have $f_*f^*\omega = \omega$. In fact, for any smooth form $\tilde{\omega}$ with compact support in V, we can write

$$\begin{aligned} \langle f_*f^*\omega, \tilde{\omega}\rangle = \langle f^*\omega, f^*\tilde{\omega}\rangle &= \int_{f^{-1}V} f^*\omega \wedge f^*\tilde{\omega} = \int_{f^{-1}V \smallsetminus A} f^*\omega \wedge f^*\tilde{\omega} \\ &= \int_{V \smallsetminus Z} \omega \wedge \tilde{\omega} = \int_V \omega \wedge \tilde{\omega} = \langle \omega, \tilde{\omega}\rangle. \end{aligned}$$

Here, the third and fourth equality hold since the integral of a smooth form on an analytic set of lower dimension is 0 (such a set being of Lebesgue measure 0 in the relevant dimension).

For the locally constant sheaf $\mathbb{Z}$, since the analytic fibration has connected fibres, a straightforward argument yields $f_*f^* = \mathrm{id}$.

In conclusion the composition of sheaf morphisms: $\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y} \to f_*f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}$ (given by the canonical map), $f_*f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y} \to f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X}$ (induced by pull-back of smooth forms) and $f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X} \to \mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}$ (induced by push-forward of currents) is the identity map. Notice that a priori, the image complex of the last morphism should be the quasi-isomorphic complex involving currents instead of smooth forms. However, in the case of a modification, the push forward of a pull-back of a smooth form is still a smooth form. In particular, the composition of sheaf morphisms

$$\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y} \to f_*f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y} \to f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X} \to \mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}$$

is the identity map. This shows that the canonical map $\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y} \to f_*f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}$ is an isomorphism.

Thus we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{H}^\bullet(Y, \mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}) = \mathbb{H}^\bullet(Y, f_*f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}) & \longrightarrow & \mathbb{H}^\bullet(Y, f_*\mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) \\ \downarrow & & \downarrow \\ \mathbb{H}^\bullet(X, f^*\mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}) & \longrightarrow & \mathbb{H}^\bullet(X, \mathcal{B}^\bullet_{p,q,\mathbb{Z},X}). \end{array}$$

The vertical arrows are the canonical maps and the horizontal maps are given by pull-back of smooth forms. Notice that the composition of

$$\mathbb{H}^\bullet(Y, \mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}) \cong \mathbb{H}^\bullet(Y, f_* f^* \mathcal{B}^\bullet_{p,q,\mathbb{Z},Y}) \to \mathbb{H}^\bullet(Y, f_* \mathcal{B}^\bullet_{p,q,\mathbb{Z},X}) \to \mathbb{H}^\bullet(X, \mathcal{B}^\bullet_{p,q,\mathbb{Z},X})$$

is exactly the pull-back of cohomology classes. A comparison of this diagram with the diagram given before the lemma concludes the proof. □

The above observation is in particular useful to define the Chern class of a coherent sheaf on a complex manifold, using the following fundamental lemma (cf. [**GR70**], [**Hir64**], [**Rie71**], [**Ros68**]).

LEMMA 6.13. *Let X be a complex compact manifold and $\mathcal{F}$ be a coherent analytic sheaf on X. There exists a bimeromorphic morphism $\sigma : X' \to X$, which is a finite composition of blow-ups with smooth centres, such that $\sigma^*\mathcal{F}$ is locally free modulo torsion.*

Using the same notations as in the lemma, we recall briefly the strategy proposed by Grivaux [**Gri10**] to define the Chern classes of arbitrary coherent sheaf $\mathcal{F}$ on X. We force the equality

$$\sigma^* \operatorname{ch}(\mathcal{F}) := \sum_i (-1)^i \operatorname{ch}(L^i \sigma^* \mathcal{F})$$

to be always verified where $L^i\sigma^*$ is the i-th left derived functor of σ^*. On the other hand, we also force the equality

$$\operatorname{ch}(\sigma^*\mathcal{F}) = \operatorname{ch}(\sigma^*\mathcal{F}/\mathrm{Tors}) + \operatorname{ch}(\mathrm{Tors})$$

to be always verified where Tors is the torsion part. We first define the Chern classes for all the torsion sheaves $L^i\sigma^*\mathcal{F}(i \geqslant 1)$ and Tors as well as the Chern classes of the vector bundle $\sigma^*\mathcal{F}/\mathrm{Tors}$. Since σ^* is injective, we can thus define $\operatorname{ch}(\mathcal{F})$ to be the unique element such that these two equalities are verified.

Since the support of a torsion sheaf is a proper analytic subset, we can perform an induction on the dimension of the manifold to define Chern classes of a torsion sheaf. Intuitively, using an appropriate version of the Riemann-Roch-Grothendieck formula, one can construct Chern classes of a torsion sheaf over X' as a direct image under a closed immersion of a certain polynomial in the Chern classes of a positive rank sheaf over the support in X', and the normal bundle of that support.

The difficulty in defining Chern classes of an arbitrary coherent sheaf comes from the case where the coherent sheaf is torsion, especially since the support of a torsion sheaf may be an analytic subset with singularities, and not necessarily a submanifold. In order to make the construction, the results of [**Gri10**] will be applied thoroughly.

By the fact that the pull back of a current is always well defined in the case of a submersion, one gets the following proposition.

PROPOSITION 6.2.1. *Consider the commutative diagram below, where p, q are the projections on the first factors*

$$\begin{array}{ccc} Y \times Z & \xrightarrow{i_{Y\times Z}} & X \times Z \\ \downarrow p & & \downarrow q \\ Y & \xrightarrow{i_Y} & X. \end{array}$$

Assume Z to be compact and i_Y proper. Then one has $q^* i_{Y*} = i_{Y\times Z*} p^*$.

PROOF. The point is that the pull-back of a current is well defined and commutes with the exterior differential for a submersion, which is the case here. For any connected open set $V \subset Y$, we have the following commutative diagram

$$\begin{array}{ccc} Z_Y(V) & \xrightarrow{p^*} & Z_{Y\times Z}(p^{-1}(V)) \ . \\ \downarrow & & \downarrow \\ \mathcal{I}^\bullet_Y(V) & \xrightarrow{p^*} & \mathcal{I}^\bullet_{Y\times Z}(p^{-1}(V)) \end{array}$$

The vertical arrow is given by associating the constant 1 to the integral current associated with $[V]$ (resp. $[p^{-1}(V)]$).

Passing to hypercohomology, inclusion of forms and constants $\mathcal{B}^\bullet$ into currents and locally integral currents $\tilde{\mathcal{B}}^\bullet$ induces isomorphism on hypercohomology, so the morphisms of integral Bott-Chern cohomology groups induced by pulling back forms and pulling back currents are the same. In other words the commutative diagram

$$\begin{array}{ccc} p^*\mathcal{B}^\bullet_Y & \longrightarrow & p^*\tilde{\mathcal{B}}^\bullet_Y \\ \downarrow & & \downarrow \\ \mathcal{B}^\bullet_{Y\times Z} & \longrightarrow & \tilde{\mathcal{B}}^\bullet_{Y\times Z} \end{array}$$

induces in hypercohomology the commutative diagram

$$\begin{array}{ccc} H^*(Y,\mathcal{B}^\bullet_Y) & \xrightarrow{\cong} & H^*(Y,\tilde{\mathcal{B}}^\bullet_Y) \\ \downarrow & & \downarrow \\ H^*(Y\times Z,p^*\mathcal{B}^\bullet_Y) & \longrightarrow & H^*(Y\times Z,p^*\tilde{\mathcal{B}}^\bullet_Y) \\ \downarrow & & \downarrow \\ H^*(Y\times Z,\mathcal{B}^\bullet_{Y\times Z}) & \xrightarrow{\cong} & H^*(Y\times Z,\tilde{\mathcal{B}}^\bullet_{Y\times Z}). \end{array}$$

Here the terms containing a tilde indicate complexes involving currents, and the terms without tilde indicate complexes involving locally constant sheaves or forms.

To prove the equality at the level of hypercohomology, it is thus enough to prove the equality at the level of complexes with terms involving currents. In particular, we just take global representative and verify the equality. The proof is reduced to checking that for any current T defined on Y, one has

$$q^* i_{Y*}T = i_{Y\times Z*}p^*T.$$

By duality, this is equivalent to the fact that for any smooth form ω with compact support in $X\times Z$, one has

$$i_Y^* q_*\omega = p_* i_{Y\times Z}^*\omega.$$

This is indeed trivial, if we observe that p_* and q_* are just integration along the second factor. The integrals are finite by the assumption that Z is compact. □

The directions of arrows can also be reversed; this is exactly Axiom B (2). For complex Bott-Chern cohomology, the formula is valid, since the cohomology class can be represented by global smooth forms and since the push forward of global forms under the projection is just the integration over the second component, which commutes with the restriction on the corresponding (smooth) submanifold.

LEMMA 6.14. *Consider the commutative diagram below, where p,q are the projections onto the first factors*

$$\begin{array}{ccc} Y\times Z & \xhookrightarrow{i_{Y\times Z}} & X\times Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xhookrightarrow{i_Y} & X. \end{array}$$

Assume Z to be compact. Then one has in complex Bott-Chern cohomology an equality $i_Y^*q_* = p_*i_{Y\times Z}^*$.

To prove the case of integral coefficients, we need a relative version of pull back and push forward for cohomology classes. To do this, we recall some definitions of derived categories. For a more complete description, we refer to [**KS02**]. We start with the definition of a relative soft sheaf.

DEFINITION 6.15. *Let $f: X\to Y$ be a continuous proper morphism between topological spaces and F be a sheaf of abelian groups on X. Then we say that F is f-soft if for any $y\in Y$, $F|_{f^{-1}(y)}$ is soft.*

In general, to define Rf_* (or some right derived functor), one can take any f_*-injective resolution (or any relative injective resolution). In particular, we do not need to take an injective resolution (which is the key point of Axiom B (2)). We verify that a f-soft resolution gives a f_*-injective resolution.

DEFINITION 6.16. (Definition 1.8.2 in [**KS02**]) *Let $F:\mathcal{C}\to\mathcal{C}'$ be an additive functor between abelian categories. A full additive subcategory $\mathcal{S}$ of $\mathcal{C}$ is called injective with respect to F if*

(1) for any $X\in Ob(\mathcal{C})$ there exists $X'\in Ob(\mathcal{S})$ and an exact sequence $0\to X\to X'$.
(2) For any exact sequence $0\to X'\to X\to X''\to 0$ in $\mathcal{C}$, if $X',X\in Ob(\mathcal{S})$ then $X''\in Ob(\mathcal{S})$.
(3) For any exact sequence $0\to X'\to X\to X''\to 0$ in $\mathcal{C}$, if $X',X,X''\in Ob(\mathcal{S})$ then we have exact sequence

$$0\to F(X')\to F(X)\to F(X'')\to 0.$$

LEMMA 6.17. *The subcategory formed by f-soft modules in $C(\mathrm{Sh}(X))$ is injective with respect to f_* for f proper.*

PROOF. It is a variant version of Proposition 2.5.10 in [**KS02**]. We give the proof in the relative case.

Since any soft module is f-soft by definition and the subcategory formed by soft modules has enough injective element i.e. it satisfies condition 1, the subcategory formed by f-soft modules in $C(\mathrm{Sh}(X))$ also satisfies condition 1. Notice that since f is proper, for any $y \in Y$, $f^{-1}(y)$ is compact hence closed.

Condition 2 is a direct consequence of exercice II.10 in [**KS02**]. It says that for any exact sequence of $\mathbb{Z}_X$ modules $0 \to F' \to F \to F'' \to 0$ with F, F' f-soft and for any $y \in Y$, the hypothesis that $0 \to F'|_{f^{-1}(y)} \to F|_{f^{-1}(y)} \to F''|_{f^{-1}(y)} \to 0$ is exact implies that $F''|_{f^{-1}(y)}$ is soft. In particular, F'' is f-soft.

Now, we prove condition 3, i.e. that if $0 \to F' \to F \to F'' \to 0$ is an exact sequence of f-soft module, then there is an exact sequence

$$0 \to f_*F' \to f_*F \to f_*F'' \to 0.$$

Let $y \in Y$, we want to check that for any $s'' \in \Gamma(f^{-1}(y), F'')$ there exists $s \in \Gamma(f^{-1}(y), F)$ whose image is s''. Notice that since f is proper the functors f_* and $f_!$ are the same. By the base change theorem (proposition 2.5.2 in [**KS02**]), we have

$$(f_*F)_y \cong \Gamma(f^{-1}(y), F|_{f^{-1}(y)}).$$

Let K_i be a finite covering of $f^{-1}(y)$ by compact subsets such that there exists $s_i \in \Gamma(K_i, F)$ whose image is $s''|_{K_i}$. This is possible from the assumption that $F \in F''$ is surjective and the fact that $f^{-1}(y)$ is compact. Let us argue by the induction on the index of the covering to adjust the s_i's such that s_i's glue to a global section. For $n \geqslant 2$, on $(\bigcup_{i \leqslant n-1} K_i) \cap K_n$, we have s_1' the glued section constructed by induction and $s_2 \in \Gamma(K_n, F)$. Hence $s_1' - s_2 \in \Gamma((\bigcup_{i \leqslant n-1} K_i) \cap K_n, F')$ which extends to $s' \in \Gamma(f^{-1}(y), F')$ since F' is f-soft. Replacing s_2 by $s_2 + s'$ we may assume that

$$s_1'|_{(\bigcup_{i \leqslant n-1} K_i) \cap K_n} = s_2|_{(\bigcup_{i \leqslant n-1} K_i) \cap K_n}.$$

Therefore after finite times induction, there exists $s \in \Gamma(f^{-1}(y), F)$ such that $s|_{K_i} = s_i$.

(Notice that condition 2 can be deduced from condition 3 by the following commutative diagram. Let K be a closed subset of $f^{-1}(y)$. We have

$$\begin{array}{ccc} \Gamma(f^{-1}(y), F) & \longrightarrow & \Gamma(f^{-1}(y), F'') \\ \downarrow & & \downarrow \\ \Gamma(K, F) & \longrightarrow & \Gamma(K, F''). \end{array}$$

The fact that the bottom and left arrow are surjective implies that the right arrow is surjective.) ☐

We also need the following lemma (Lemma 3.1.2) in [**KS02**].

LEMMA 6.18. *Let $f : X \to Y$ be a continuous map of locally compact spaces and K be a flat and f-soft $\mathbb{Z}_X$ module. For any sheaf G on X, $G \otimes_{\mathbb{Z}_X} K$ is f-soft.*

This lemma entails the following useful corollary.

COROLLARY 6.19. *Let X, Z be two complex manifolds with Z compact. Let $F^\bullet$ be a flat complex (of sheaves of abelian groups) over X and $G^\bullet$ be a soft and flat complex over Z. Then $F^\bullet \boxtimes G^\bullet$ is flat and q-soft with respect to $q : X \times Z \to X$.*

PROOF. The flatness part is from the fact that for abelian groups flatness is equivalent to torsion-freeness. For any $x \in X$ we have $F^\bullet \boxtimes G^\bullet|_{\{x\} \times Z} = F_x^\bullet \otimes_{\mathbb{Z}_Z} G^\bullet$ which, by the lemma, is q-soft. ☐

Now, we are prepared for the proof of Axiom B (2).

PROPOSITION 6.2.2. *Consider the following commutative diagram where p, q are the projections on the first factors*

$$\begin{array}{ccc} Y \times Z & \xhookrightarrow{i_{Y \times Z}} & X \times Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xhookrightarrow{i_Y} & X \end{array}$$

Assume Z to be compact. Then one has in integral Bott-Chern cohomology an equality $i_Y^* q_* = p_* i_{Y \times Z}^*$.

PROOF. The idea is to use a resolution on $X \times Z$ formed by pulling back a resolution involving smooth forms on X, and tensoring with the pull back of a resolution involving currents on Z. This gives a q-soft resolution, and an explicit method to calculate Rq_*, via corollary 6.19.

Let $\mathcal{U}$ be an open covering of X formed by geodesic balls with small enough radius such that any finite intersection of such balls are diffeomorphic to euclidean ball. Therefore, the total complex of the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X)$ gives a resolution of $\mathbb{Z}_X$ by the Leray theorem. It is a flat complex on X since all terms are torsion free. Also, $\mathcal{I}_Z^\bullet$ is a flat and soft resolution of $\mathbb{Z}_Z$ on Z.

By the corollary, $\check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \boxtimes \mathcal{I}_Z^\bullet$ is a q-soft resolution of $\mathbb{Z}_{X\times Z} = \mathbb{Z}_X \boxtimes \mathbb{Z}_Z$ on $X \times Z$.

Now we perform a similar construction for the sheaves of smooth forms. The sheaves $C_\infty^{\bullet;\bullet}$ on $X \times Z$ can be viewed as flat $\mathbb{Z}_X$ modules and $\mathbb{Z}_Z$ modules. Thus we have

$$C_\infty^{\bullet;\bullet} \cong C_\infty^{\bullet;\bullet} \otimes_{\mathbb{Z}_X} \check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \cong C_\infty^{\bullet;\bullet} \otimes^L_{\mathbb{Z}_X} \mathbb{Z}_X.$$

Similarly we have

$$C_\infty^{\bullet;\bullet} \cong C_\infty^{\bullet;\bullet} \otimes_{\mathbb{Z}_Z} \mathcal{I}_Z^\bullet \cong C_\infty^{\bullet;\bullet} \otimes^L_{\mathbb{Z}_Z} \mathbb{Z}_Z.$$

Therefore, the integral Bott-Chern complex on $X \times Z$ in the derived category is quasi-isomorphic to

$$\mathcal{B}^\bullet_{\mathbb{Z},X\times Z} \cong \mathrm{Cone}(\check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \boxtimes \mathcal{I}_Z^\bullet \to \sigma_{p,\bullet} C_\infty^{\bullet;\bullet} \otimes_{\mathbb{Z}_X} \check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \otimes_{\mathbb{Z}_Z} \mathcal{I}_Z^\bullet \oplus \sigma_{\bullet,q} C_\infty^{\bullet;\bullet} \otimes_{\mathbb{Z}_X} \check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \otimes_{\mathbb{Z}_Z} \mathcal{I}_Z^\bullet)[-1]$$

with the natural inclusion morphism which is q-soft. Notice that the sheaves of smooth forms on $X \times Z$ are also q-soft. In particular, we have

$$Rq_*(\mathcal{B}^\bullet_{\mathbb{Z},X\times Z}) \cong q_*(\mathrm{Cone}(\check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \boxtimes \mathcal{I}_Z^\bullet \to \sigma_{p,\bullet} C_\infty^{\bullet;\bullet} \oplus \sigma_{\bullet,q} C_\infty^{\bullet;\bullet})[-1])$$

where $C_\infty^{\bullet;\bullet}$ means in fact $C_\infty^{\bullet;\bullet} \otimes_{\mathbb{Z}_X} \check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \otimes_{\mathbb{Z}_Z} \mathcal{I}_Z^\bullet$. In the following of the proof, we always use this simplified notation. We have morphisms in the derived category $D(\mathrm{Sh}(X))$

$$\check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \xleftarrow{\sim} \mathbb{Z}_X \xrightarrow{\sim} \mathcal{I}_X^\bullet.$$

We also have a morphism $q_* \mathrm{pr}_2^* \mathcal{I}_Z^\bullet \to q_* \mathcal{I}_{X\times Z}^\bullet \to \mathcal{I}_X^\bullet$. It induces a morphism $q_*(\check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \boxtimes \mathcal{I}_Z^\bullet) \to \check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X)$. We have commutative diagrams

$$\begin{array}{ccc} q_*(\mathbb{Z}_X^\bullet \boxtimes \mathcal{I}_Z^\bullet) & \longrightarrow & q_*(\check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \boxtimes \mathcal{I}_Z^\bullet) \\ \downarrow & & \downarrow \\ \mathbb{Z}_X^\bullet & \xrightarrow{\sim} & \check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X), \end{array} \qquad \begin{array}{ccc} q_*(\mathbb{Z}_X^\bullet \boxtimes \mathcal{I}_Z^\bullet) & \longrightarrow & q_*(\mathcal{I}_{X\times Z}^\bullet) \\ \downarrow & & \downarrow \\ \mathbb{Z}_X^\bullet & \xrightarrow{\sim} & \mathcal{I}_X^\bullet. \end{array}$$

On the other hand, since q is a submersion, we have a canonical morphism $q_*(C_\infty^{\bullet;\bullet}) \to C_\infty^{\bullet-n,\bullet-n}$ where $n = \dim_{\mathbb{C}} Z$. Thus we get a morphism

$$\begin{aligned} q_*(\mathrm{Cone}(\check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \boxtimes \mathcal{I}_Z^\bullet \to \sigma_{p,\bullet} C_\infty^{\bullet;\bullet} \oplus \sigma_{\bullet,q} C_\infty^{\bullet;\bullet})[-1]) \to \mathrm{Cone}(&\check{C}^{\bullet-2n}(\mathcal{U}, \mathbb{Z}_X) \\ &\to \sigma_{p-n,\bullet-n} C_\infty^{\bullet;\bullet} \oplus \sigma_{\bullet-n,q-n} C_\infty^{\bullet;\bullet})[-1]. \end{aligned}$$

Passing to hypercohomology, this morphism induces the push forward of integral Bott-Chern cohomology by q. The above commutative diagrams show that the push forward of cohomology classes defined in this way coincides with the previous one. This yields two ways of defining the same map $Rq_* \mathbb{Z}_{X\times Z} \to \mathbb{Z}_X$.

Since this resolution is flat, we can also use it to define the pull back of cohomology classes. More precisely, one can define the pull-back of cohomology class for a projection as follows. Since $i_{Y\times Z} = (i_Y, \mathrm{id}_Z)$, one has

$$\begin{aligned} i^*_{Y\times Z} : \mathrm{Cone}(\check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \boxtimes \mathcal{I}_Z^\bullet \to \sigma_{p,\bullet} C_\infty^{\bullet;\bullet} \oplus \sigma_{\bullet,q} C_\infty^{\bullet;\bullet})[-1] \to \mathrm{Cone}(&\check{C}^\bullet(\mathcal{U} \cap Y, \mathbb{Z}_Y) \boxtimes \mathcal{I}_Z^\bullet \\ &\to \sigma_{p,\bullet} C_\infty^{\bullet;\bullet} \oplus \sigma_{\bullet,q} C_\infty^{\bullet;\bullet})[-1] \end{aligned}$$

induced by pulling back forms and pulling back currents. Here id_Z is a submersion, so the pull back of currents is well defined (and is in fact the identity!). Passing to hypercohomology, we get another way of defining $i^*_{Y\times Z}$ for integral Bott-Chern cohomology. We next check that these two definitions coincide. The inclusion $\mathbb{Z}_Z \to \mathcal{I}_X^\bullet$ induces a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_{X\times Z} = \mathbb{Z}_X \boxtimes \mathbb{Z}_Z & \xrightarrow{i^*_{Y\times Z}} & \mathbb{Z}_{Y\times Z} = \mathbb{Z}_Y \boxtimes \mathbb{Z}_Z \\ \downarrow & & \downarrow \\ \check{C}^\bullet(\mathcal{U}, \mathbb{Z}_X) \boxtimes \mathcal{I}_Z^\bullet & \xrightarrow{i^*_{Y\times Z}} & \check{C}^\bullet(\mathcal{U} \cap Y, \mathbb{Z}_Y) \boxtimes \mathcal{I}_Z^\bullet. \end{array}$$

This commutative diagram implies that the two definitions of pull back coincide.

Similar arguments show that the pull back by i_Y^* and the push forward by p_* can be defined using the corresponding resolutions. Since the resolution is relative soft with respect to p or q, the hypercohomology can be represented by global sections. The sections are formed by currents and forms on the open set of

$U \times Z$ or $(U \cap Y) \times Z$ for some open set U of X which is some intersection of the open sets in the cover $\mathcal{U}$. The equality asserted in the proposition is satisfied for such forms and currents. This concludes the proof. $\square$

6.3. Multiplication of the Bott-Chern cohomology ring

In this section, we discuss a natural ring structure of the integral Bott-Chern cohomology and we verify the projection formula (Axiom B(1)). Some calculation of this part is borrowed from an unpublished work of Junyan Cao.

The complex Bott-Chern cohomology is represented by global differential forms. The exterior product of forms induces the multiplication of cohomology classes. To define a multiplication of integral Bott-Chern cohomology which preserves the ring structure under the canonical map from the integral Bott-Chern cohomology to the complex Bott-Chern cohomology, we start by defining a modified version of multiplication of Deligne cohomology. Recall that the integral Deligne complex $\mathcal{D}(p)^\bullet$ is the complex in $C(\mathrm{Sh}(X))$

$$\mathbb{Z}(p) \to \mathcal{O} \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0.$$

The integral Deligne complex admits a multiplication structure as follows.

$$\cup : \mathcal{D}(p)^\bullet \otimes_{\mathbb{Z}_X} \mathcal{D}(q)^\bullet \to \mathcal{D}(p+q)^\bullet$$

$$x \cup y = \begin{cases} x \cdot y, \text{if } \deg(x) = 0 \\ x \wedge dy, \text{if } \deg(x) > 0 \text{ and } \deg(y) = p \\ 0, \text{ otherwise.} \end{cases}$$

$\cup$ is a morphism of $\mathbb{Z}_X$-module sheaf complexes by a direct verification. A modified version of multiplication is given in the following definition.

Definition 6.20. *For the integral Deligne complex, we define*

$$\cup : \mathcal{D}(p)^\bullet \otimes_{\mathbb{Z}_X} \mathcal{D}(q)^\bullet \to \mathcal{D}(p+q)^\bullet$$

$$x \cup y = \begin{cases} x \cdot y, \textit{if } \deg(y) = 0 \\ (-1)^p dx \wedge y, \textit{if } \deg(y) > 0 \textit{ and } \deg(x) = p \\ 0, \textit{ otherwise.} \end{cases}$$

We verify that $\cup$ yields a well defined morphism of complexes, namely that

$$d(x \cup y) = dx \cup y + (-1)^{\deg(x)} x \cup dy.$$

Notice that for $x \in \mathcal{D}(p)^i$ with $i > 0$, x is a $(i-1, 0)$-form, and not a $(i, 0)$-form. This is frequently used in the following calculations.
If $\deg(y) = 0, \deg(x) < p$, $d(x \cup y) = d(yx) = ydx$, $dx \cup y + (-1)^{\deg(x)} x \cup dy = ydx + 0 = ydx$.
If $\deg(y) = 0, \deg(x) = p$, $d(x \cup y) = d(yx) = ydx$, $dx \cup y + (-1)^{\deg(x)} x \cup dy = 0 \cup y + dx \wedge y = ydx$.
If $\deg(y) > 0, \deg(x) = p$, $d(x \cup y) = d((-1)^p dx \wedge y) = dx \wedge dy$, $dx \cup y + (-1)^{\deg(x)} x \cup dy = 0 \cup y + dx \wedge dy = dx \wedge dy$. ($x$ is a $(p-1, 0)$-form here.)
If $\deg(y) = 0, \deg(x) = p-1$, $d(x \cup y) = 0$, $dx \cup y + (-1)^{\deg(x)} x \cup dy = (-1)^p ddx \wedge y + 0 = 0$.
In the other cases, both sides are 0.

Remark 6.21. For the definition of multiplication in the integral Bott-Chern complex, we need a modified Deligne complex where we change the signs. To be more precise, we consider the complex

$$\mathbb{Z}(p) \xrightarrow{-1} \mathcal{O} \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0.$$

In this case, we define the multiplication as follows:

$$x \cup y = \begin{cases} x \cdot y, \text{if } \deg(y) = 0 \\ (-1)^{p-1} dx \wedge y, \text{if } \deg(y) > 0 \text{ and } \deg(x) = p \\ 0, \text{ otherwise.} \end{cases}$$

The verification is similar. In the second case $x \cup dy = (-1)^{p-1} dx \wedge dy = (-1)^p ydx$ since $dy = -y$ in this case. The third and fourth cases consist of changing just a sign on both sides of the equations.

Proposition 6.3.1. *The multiplication is associative and homotopy graded-commutative. Thus, it induces a structure of an anti-commutative ring with unit on the integral Deligne cohomology.*

PROOF. Considering $\alpha \in \mathcal{D}(p)^\bullet$, $\tilde{\alpha} \in \mathcal{D}(p')^\bullet$ and $\deg(\alpha) = i$, $\deg(\tilde{\alpha}) = j$, we prove the formula

$$(-1)^{ij}\alpha \cup \tilde{\alpha} = \tilde{\alpha} \cup \alpha + (dH + Hd)(\tilde{\alpha} \otimes \alpha).$$

Here d is the differential of $\mathcal{D}(p)^\bullet \otimes \mathcal{D}(p')^\bullet$, and $d(\alpha \otimes \beta)$ is defined by $d(\alpha \otimes \beta) = d\,\alpha \otimes \beta + (-1)^{\deg(\alpha)}\alpha \otimes d\,\beta$. The modified homotopy operator H is defined by: $H(\tilde{\alpha} \otimes \alpha) = (-1)^{j-1}\tilde{\alpha} \wedge \alpha$, if $i \neq 0, j \neq 0$. Otherwise, $H(\tilde{\alpha} \otimes \alpha) = 0$. We prove it by a direct verification, case by case:

(1) $i = j = 0$: $(-1)^{ij}\alpha \cup \tilde{\alpha} = \alpha\tilde{\alpha}$, $\tilde{\alpha} \cup \alpha = \alpha\tilde{\alpha}$, $(dH + Hd)(\tilde{\alpha} \otimes \alpha) = 0$.

(2) $i = 0, j > 0$: $\alpha \cup \tilde{\alpha} = 0$, $\tilde{\alpha} \cup \alpha = \tilde{\alpha} \wedge \alpha$, $(dH + Hd)(\tilde{\alpha} \otimes \alpha) = d0 + H(d\tilde{\alpha} \otimes \alpha + (-1)^j\tilde{\alpha} \otimes d\alpha) = (-1)^{j+j-1}\tilde{\alpha} \wedge d\alpha = -\tilde{\alpha} \wedge \alpha$. Here we use $d\alpha = \alpha$, if $\deg\alpha = 0$.

(3) $j = 0, i > 0$: $\alpha \cup \tilde{\alpha} = \tilde{\alpha} \wedge \alpha$, $\tilde{\alpha} \cup \alpha = 0$, $(dH + Hd)(\tilde{\alpha} \otimes \alpha) = H(d\tilde{\alpha} \otimes \alpha + \tilde{\alpha} \otimes d\alpha) = d\tilde{\alpha} \wedge \alpha + 0 = \tilde{\alpha} \wedge \alpha$.

(4) $p' > j > 0, p > i > 0$:

$$\alpha \cup \tilde{\alpha} = \tilde{\alpha} \cup \alpha = 0,$$

$$(dH + Hd)(\tilde{\alpha} \otimes \alpha) = d((-1)^{j-1}\tilde{\alpha} \wedge \alpha) + H((-1)^j\tilde{\alpha} \otimes d\alpha + d\tilde{\alpha} \otimes \alpha)$$

$$= (-1)^{j-1}d\tilde{\alpha} \wedge \alpha + (-1)^{j-1+j-1}\tilde{\alpha} \wedge d\alpha + (-1)^{j+1-1}d\tilde{\alpha} \wedge \alpha + (-1)^{j+j-1}\tilde{\alpha} \wedge d\alpha = 0.$$

The second line uses the fact that $d\tilde{\alpha}$ is a $(j, 0)$-form.

(5) $j = p', p > i > 0$:

$$\alpha \cup \tilde{\alpha} = 0, \tilde{\alpha} \cup \alpha = (-1)^{p'}d\tilde{\alpha} \wedge \alpha,$$

$$(dH + Hd)(\tilde{\alpha} \otimes \alpha) = d((-1)^{j-1}\tilde{\alpha} \wedge \alpha) + H(d\tilde{\alpha} \otimes \alpha + (-1)^j\tilde{\alpha} \otimes d\alpha)$$

$$= (-1)^{j-1}d\tilde{\alpha} \wedge \alpha + \tilde{\alpha} \wedge d\alpha + (-1)^{j+j-1}\tilde{\alpha} \wedge d\alpha = (-1)^{p'-1}d\tilde{\alpha} \wedge \alpha.$$

The second line uses the fact that $d\tilde{\alpha}$ is a $(j, 0)$-form and that $d\tilde{\alpha} = 0$ in $\mathcal{D}(p')^\bullet$.

(6) $i = p, p' > j > 0$: the verification is similar to the previous case.

(7) $i = p, j = p'$:

$$\alpha \cup \tilde{\alpha} = (-1)^p d\alpha \wedge \tilde{\alpha}, \tilde{\alpha} \cup \alpha = (-1)^{p'}d\tilde{\alpha} \wedge \alpha,$$

$$(dH + Hd)(\tilde{\alpha} \otimes \alpha) = d((-1)^{j-1}\tilde{\alpha} \wedge \alpha) + H(0) = (-1)^{j-1}d\tilde{\alpha} \wedge \alpha + (-1)^{j-1+j-1}\tilde{\alpha} \wedge d\alpha.$$

For the equality, it remains to see that

$$(-1)^i d\alpha \wedge \tilde{\alpha} = (-1)^{ij}\tilde{\alpha} \wedge d\alpha.$$

This is true since $i = i(j - 1) + ij \bmod 2$.

The associativity is also checked by a direct calculation. Let $x \in \mathcal{D}(p)^\bullet, y \in \mathcal{D}(p')^\bullet, z \in \mathcal{D}(p'')^\bullet$. Then

(1) if $\deg z = \deg y = 0$, $x \cup (y \cup z) = (x \cup y) \cup z = xyz$.

(2) if $\deg z = 0$, $\deg y > 0$, $\deg x = p$, $x \cup (y \cup z) = (x \cup y) \cup z = (-1)^p dx \wedge yz$.

(3) if $\deg z > 0$, $\deg y = p'$, $\deg x = p$, $x \cup (y \cup z) = x \cup ((-1)^{p'}dy \wedge z) = (-1)^{p+p'}dx \wedge dy \wedge z = (x \cup y) \cup z = ((-1)^p dx \wedge y) \cup z = (-1)^{p+p+p'}d(dx \wedge y) \wedge z = (-1)^{p+p'}dx \wedge dy \wedge z$ since dx is a $(p, 0)$ form.

(4) otherwise, the product is 0.

□

REMARK 6.22. Similarly, for the integral Bott-Chern cohomology, the modified Deligne complex admits a homotopy operator defined by: $H(\tilde{\alpha} \otimes \alpha) = (-1)^j\tilde{\alpha} \wedge \alpha$, if $i \neq 0, j \neq 0$. Otherwise, $H(\tilde{\alpha} \otimes \alpha) = 0$. We also have the equality:

$$(-1)^{ij}\alpha \cup \tilde{\alpha} = \tilde{\alpha} \cup \alpha + (dH + Hd)(\tilde{\alpha} \otimes \alpha).$$

We prove it again by a direct calculation case by case:

(1) $i = j = 0$: $(-1)^{ij}\alpha \cup \tilde{\alpha} = \alpha\tilde{\alpha}$, $\tilde{\alpha} \cup \alpha = \alpha\tilde{\alpha}$, $(dH + Hd)(\tilde{\alpha} \otimes \alpha) = 0$.

(2) $i = 0, j > 0$: $\alpha \cup \tilde{\alpha} = 0$, $\tilde{\alpha} \cup \alpha = \tilde{\alpha} \wedge \alpha$, $(dH + Hd)(\tilde{\alpha} \otimes \alpha) = d0 + H(d\tilde{\alpha} \otimes \alpha + (-1)^j\tilde{\alpha} \otimes d\alpha) = (-1)^{j+j}\tilde{\alpha} \wedge d\alpha = -\tilde{\alpha} \wedge \alpha$. Here we use $d\alpha = -\alpha$, if $\deg\alpha = 0$.

(3) $j = 0, i > 0$: $\alpha \cup \tilde{\alpha} = \tilde{\alpha} \wedge \alpha$, $\tilde{\alpha} \cup \alpha = 0$, $(dH + Hd)(\tilde{\alpha} \otimes \alpha) = H(d\tilde{\alpha} \otimes \alpha + \tilde{\alpha} \otimes d\alpha) = -d\tilde{\alpha} \wedge \alpha + 0 = \tilde{\alpha} \wedge \alpha$. The last equality uses $d\alpha = -\alpha$, if $\deg\alpha = 0$.

(4) $p' > j > 0, p > i > 0$:

$$\alpha \cup \tilde{\alpha} = \tilde{\alpha} \cup \alpha = 0,$$

$$(dH + Hd)(\tilde{\alpha} \otimes \alpha) = d((-1)^j\tilde{\alpha} \wedge \alpha) + H((-1)^j\tilde{\alpha} \otimes d\alpha + d\tilde{\alpha} \otimes \alpha)$$

$$= (-1)^j d\tilde{\alpha} \wedge \alpha + (-1)^{j+j-1}\tilde{\alpha} \wedge d\alpha + (-1)^{j-1}d\tilde{\alpha} \wedge \alpha + (-1)^{j+j}\tilde{\alpha} \wedge d\alpha = 0$$

The second line uses the fact that $d\tilde{\alpha}$ is $(j, 0)$-form.

(5) $j = p', p > i > 0$:
$$\alpha \cup \tilde{\alpha} = 0, \tilde{\alpha} \cup \alpha = (-1)^{p'-1} d\tilde{\alpha} \wedge \alpha,$$
$$(dH + Hd)(\tilde{\alpha} \otimes \alpha) = d((-1)^j \tilde{\alpha} \wedge \alpha) + H(d\tilde{\alpha} \otimes \alpha + (-1)^j \tilde{\alpha} \otimes d\alpha)$$
$$= (-1)^j d\tilde{\alpha} \wedge \alpha - \tilde{\alpha} \wedge d\alpha + (-1)^{j+j} \tilde{\alpha} \wedge d\alpha = (-1)^{p'} d\tilde{\alpha} \wedge \alpha.$$
The second line uses the fact that $d\tilde{\alpha}$ is $(j,0)$-form and that $d\tilde{\alpha} = 0$ in $\mathcal{D}(p')^\bullet$.

(6) $i = p, p' > j > 0$: the verification is similar to the previous case.

(7) $i = p, j = p'$:
$$\alpha \cup \tilde{\alpha} = (-1)^{p-1} d\alpha \wedge \tilde{\alpha}, \tilde{\alpha} \cup \alpha = (-1)^{p'-1} d\tilde{\alpha} \wedge \alpha,$$
$$(dH + Hd)(\tilde{\alpha} \otimes \alpha) = d((-1)^j \tilde{\alpha} \wedge \alpha) + H(0) = (-1)^{p'} d\tilde{\alpha} \wedge \alpha + (-1)^{p'+p'-1} \tilde{\alpha} \wedge d\alpha.$$
For the equality, it remains to see that
$$(-1)^{i-1} d\alpha \wedge \tilde{\alpha} = (-1)^{ij-1} \tilde{\alpha} \wedge d\alpha.$$
This is true as above since $i = i(j-1) + ij \bmod 2$.

Once we have defined a morphism from a tensor product of two complexes to another complex. It naturally induces a product on the hypercohomology class. For self-containedness, we recall the construction.

DEFINITION 6.23. *Consider two complexes of sheaves $\mathcal{A}^\bullet, \mathcal{B}^\bullet$, such that there exists a multiplication denoted by $\cup\colon \mathcal{A}^\bullet \otimes_{\mathbb{Z}} \mathcal{B}^\bullet \to \mathcal{C}^\bullet$, $\alpha \otimes \beta \mapsto \alpha \cup \beta$ satisfying the relation $d(\alpha \cup \beta) = (d\alpha) \cup \beta + (-1)^{\deg(\alpha)} \alpha \cup d\beta$. Then one can define a product between $\mathbb{H}^\bullet(\mathcal{A}^\bullet)$ and $\mathbb{H}^\bullet(\mathcal{B}^\bullet)$ as follows: let $\beta \in \check{C}^k(\mathcal{A}^l)$ and $\tilde{\beta} \in \check{C}^{k'}(\mathcal{B}^{l'})$ (where $\check{C}$ means Čech hypercocycle). One defines a Čech hypercocycle $\beta \cdot \tilde{\beta} \in \check{C}^{k+k'}(\mathcal{C}^{l+l'})$ by*
$$(\beta \cdot \tilde{\beta})_{j_0 \ldots j_{k+k'}} := (-1)^{k \cdot l'} \beta_{j_0 \ldots j_k} \cup \tilde{\beta}_{j_k \ldots j_{k+k'}}.$$

We next check the derivation relation:
$$\check{\delta}(\beta \cdot \tilde{\beta}) = (\check{\delta}\beta) \cdot \tilde{\beta} + (-1)^{k+l} \beta \cdot (\check{\delta}\tilde{\beta})$$
where $\check{\delta}\beta = (-1)^l \delta\beta + d\beta$, δ is the Čech differential. By definition we have
$$\check{\delta}(\beta \cdot \tilde{\beta}) = (-1)^{k \cdot l'} \check{\delta}(\beta \cup \tilde{\beta}) = (-1)^{k \cdot l'}((-1)^{l+l'} \delta(\beta \cup \tilde{\beta}) + d(\beta \cup \tilde{\beta}))$$
$$= (-1)^{k \cdot l'}(-1)^{l+l'}(\delta\beta \cup \tilde{\beta} + (-1)^k \beta \cup \delta\tilde{\beta}) + (-1)^{k \cdot l'}(d\beta \cup \tilde{\beta} + (-1)^l \beta \cup d\tilde{\beta}).$$
$$(\check{\delta}\beta) \cdot \tilde{\beta} + (-1)^{k+l} \beta \cdot (\check{\delta}\tilde{\beta}) = (-1)^{(k+1)l'}(-1)^l (\delta\beta) \cup \tilde{\beta} + (-1)^{k \cdot l'} d\beta \cup \tilde{\beta}$$
$$+ (-1)^{k \cdot l'}(-1)^{k+l} \beta \cup (-1)^{l'} \delta\tilde{\beta} + (-1)^{k(l'+1)}(-1)^{k+l} \beta \cup d\tilde{\beta}.$$
The multiplicative structure on the integral Deligne complex induces a multiplicative structure on the integral Bott-Chern complex as follows. We denote $\epsilon_{\mathcal{D}}$ the canonical morphism of complexes from the integral Bott-Chern complex $\mathcal{B}^\bullet_{p,q,\mathbb{Z}}$ to the integral Deligne complex $\mathcal{D}(p)^\bullet$. We denote $\overline{\epsilon_{\mathcal{D}}}$ the canonical morphism of complexes from the integral Bott-Chern complex $\mathcal{B}^\bullet_{p,q,\mathbb{Z}}$ to the modified conjugated integral Deligne complex $\overline{\mathcal{D}(q)^\bullet} := 0 \to \mathbb{Z}(q) \xrightarrow{-1} \overline{\mathcal{O}_X} \to \cdots \to \overline{\Omega^{q-1}_X} \to 0$ with a multiplication of $(2\pi i)^{q-p}$ at degree 0. The modified multiplication of modified integral Deligne complex in the remark 6.22 induces a multiplication of modified conjugated integral Deligne complex. These two canonical maps induce a multiplicative structure on the integral Bott-Chern complex as follows. Let y', y'' be two elements of $\mathcal{D}(p)^i, \overline{\mathcal{D}(q)^i}$ over the same open set for some i. If $i = 0$, there exists a unique element x of $\mathcal{B}^0_{p,q,\mathbb{Z}}$ such that $\epsilon_{\mathcal{D}}(x) = y'$ and $\overline{\epsilon_{\mathcal{D}}}(x) = y'$ if and only if they satisfy $y'' = (2\pi i)^{q-p} y'$. The existence of the unique element is trivial for all positive degree. Hence we can define the multiplication $x \cup x'$ of two elements x, x' of $\mathcal{B}^i_{p,q,\mathbb{Z}}$ and $\mathcal{B}^j_{p',q',\mathbb{Z}}$ respectively just to be the unique element such that $\epsilon_{\mathcal{D}}(x \cup x') = x \cup x'$ and $\overline{\epsilon_{\mathcal{D}}}(x \cup x') = x \cup x'$ with the cup product of Deligne complex and the modified cup product of modified conjugated Deligne complex respectively. At degree 0, the multiplication is just the multiplication of the two integer at degree 0 up to a constant satisfying the compatible condition. Therefore the multiplication of the integral Bott-Chern complex is well-defined. In conclusion, the cup product of the complex is given explicitly by the following definition.

DEFINITION 6.24. *Let $w, \tilde{w}$ be two elements of the complex $\mathcal{B}^\bullet_{p,q} \otimes_{\mathbb{Z}} \mathcal{B}^\bullet_{p',q'}$, and let us use the following diagrams to denote the elements $w, \tilde{w}$ of mixed degrees*
$$w = \left(c, \begin{matrix} u^{0,0}, \ldots, u^{p-1,0} \\ v^{0,0}, \ldots\ldots, v^{0,q-1} \end{matrix} \right), \qquad \tilde{w} = \left(\tilde{c}, \begin{matrix} \tilde{u}^{0,0}, \ldots\ldots\ldots, \tilde{u}^{p'-1,0} \\ \tilde{v}^{0,0}, \ldots, \tilde{v}^{0,q'-1} \end{matrix} \right).$$
For instance, at degree 0, we denote w by c, at degree 1, we denote w by $(u^{0,0}, v^{0,0})$ etc. With the same notation, the cup product $w \cup \tilde{w}$ is represented by the diagram
$$\left(c \wedge \tilde{c}, \begin{matrix} c \wedge \tilde{u}^{0,0}, \ldots\ldots\ldots, c \wedge \tilde{u}^{p'-1,0}, u^{0,0} \wedge \partial\tilde{u}^{p'-1,0}, \ldots, u^{p-1,0} \wedge \partial\tilde{u}^{p'-1,0} \\ v^{0,0} \wedge \tilde{c}, \ldots, v^{0,q-1} \wedge \tilde{c}, (-1)^{q-1}\overline{\partial} v^{0,q-1} \wedge \tilde{v}^{0,0}, \ldots\ldots, (-1)^{q-1}\overline{\partial} v^{0,q-1} \wedge \tilde{v}^{0,q'-1} \end{matrix} \right).$$

The cup product of integral Bott-Chern cohomology is given explicitly by the following diagram.

DEFINITION 6.25. *Let $w, \tilde{w}$ be two representatives of hypercocycles of the complex $\mathcal{B}^{\bullet}_{p,q} \otimes_{\mathbb{Z}} \mathcal{B}^{\bullet}_{p',q'}$, and let us use the following diagrams to denote the elements $w, \tilde{w}$*

$$w = \left(c, \begin{matrix} u^{0,0}, \ldots, u^{p-1,0} \\ v^{0,0}, \ldots\ldots, v^{0,q-1} \end{matrix} \right), \qquad \tilde{w} = \left(\tilde{c}, \begin{matrix} \tilde{u}^{0,0}, \ldots\ldots\ldots, \tilde{u}^{p'-1,0} \\ \tilde{v}^{0,0}, \ldots, \tilde{v}^{0,q'-1} \end{matrix} \right).$$

For instance, at degree 0, we denote by c an element in $\check{C}^{p+q}(\mathcal{B}^0_{p,q})$, at degree 1, we denote by $(u^{0,0}, v^{0,0})$ an element in $\check{C}^{p+q-1}(\mathcal{B}^1_{p,q})$ etc. With the same notation, the cup product $w \cup \tilde{w}$ is represented by the diagram

$$\left(c \wedge \tilde{c}, \begin{matrix} \epsilon^{0,*} c \wedge \tilde{u}^{0,0}, \ldots\ldots\ldots, \epsilon^{p'-1,*} c \wedge \tilde{u}^{p'-1,0}, \epsilon^{p',*} u^{0,0} \wedge \partial \tilde{u}^{p'-1,0}, \ldots, \epsilon^{p+p'-1,*} u^{p-1,0} \wedge \partial \tilde{u}^{p'-1,0} \\ \epsilon^{*,0} v^{0,0} \wedge \tilde{c}, \ldots, \epsilon^{*,q-1} v^{0,q-1} \wedge \tilde{c}, \epsilon^{*,q} \overline{\partial} v^{0,q-1} \wedge \tilde{v}^{0,0}, \ldots\ldots, \epsilon^{*,q+q'-1} \overline{\partial} v^{0,q-1} \wedge \tilde{v}^{0,q'-1} \end{matrix} \right).$$

The signs $\epsilon^{R,*}$,$\epsilon^{*,S}$ are given as follows:

$$\epsilon^{R,*} = \begin{cases} (-1)^{(p+q)(R+1)}, \mathit{if} R \leqslant p' - 1 \\ (-1)^{p'(R+p+q)}, \mathit{if} R \geqslant p' \end{cases}$$

$$\epsilon^{*,S} = \begin{cases} 1, \mathit{if} S \leqslant q - 1 \\ (-1)^{pS+(p+1)(q+1)}, \mathit{if} S \geqslant q \end{cases}.$$

Notice that this cup product is just the cup product defined in [**Sch07**]. Let us also notice that there exists a more obvious natural product structure on the complex Bott-Chern cohomology induced by the wedge product of forms. The signs in the cup product defined in [**Sch07**] are exactly taken in such a way that the two products coincide under the natural morphism. The natural inclusion of the integral Bott-Chern complex into the complex Bott-Chern complex induces a ring morphism in hypercohomology. The morphism of complexes $\epsilon_{\mathcal{D}}$ also induces a ring morphism in hypercohomology.

PROPOSITION 6.3.2. *The multiplication is anti-commutative. Thus, it induces a structure of an anti-commutative ring with unit on the integral Bott-Chern cohomology.*

PROOF. As for Deligne cohomology, there is a natural homotopy operator. We identify the degree 0 sheaf in the integral Bott-Chern class $\mathbb{Z}(p)$ with a subsheaf of $\mathbb{Z}(p) \oplus \mathbb{Z}(q)$ via the map $1 \mapsto (1, (2\pi i)^{q-p})$. In this way, we can include the integral Bott-Chern complex into the direct sum of the integral Deligne cohomology and the (modified) conjugate integral Deligne complex. We define $H : \mathcal{B}^{\bullet}_{p,q,\mathbb{Z}} \otimes_{\mathbb{Z}_X} \mathcal{B}^{\bullet}_{p',q',\mathbb{Z}} \to \mathcal{B}^{\bullet}_{p+p',q+q',\mathbb{Z}}$ by the formula for any element $\varphi^i = (a^i, b^i) \in \mathcal{B}^i_{p,q,\mathbb{Z}}$, $\psi^j = (a'^j, b'^j) \in \mathcal{B}^j_{p',q',\mathbb{Z}}$,

$$H(\varphi^i \otimes \psi^j) := \begin{cases} ((-1)^i a^i \wedge b^j, (-1)^j a'^i \wedge b'^j), \text{if } i \neq 0, j \neq 0 \\ 0, \text{ otherwise.} \end{cases}$$

This is well defined since at degree 0, the homotopy operator is 0 map. We have checked that

$$(-1)^{ij} \psi^j \cup \varphi^i = \varphi^i \cup \psi^j + (dH + Hd)(\varphi^i \otimes \psi^j).$$

Therefore, passing to hypercohomology, we have defined an anti-commutative ring structure on the integral Bott-Chern cohomology. For reference, the formulas for the homotopy operator of the integral Deligne complex can be found in [**EV88**]. □

We write $\varphi \cdot \psi$ for the multiplication of cohomology classes. There exists also another description of cup product following [**EV88**] by introducing the Deligne-Beilinson complex. In this way, the projection formula can be expressed more formally.

We start by recalling the Deligne-Beilinson complex in [**EV88**]. The advantage of the Deligne-Beilinson complex is that the multiplication is either 0 or weight product of two forms. When changing the complex involving forms by the complex involving currents, it becomes clearer what the sign should be.

DEFINITION 6.26. *The Deligne-Beilinson complex is*

$$A(p)^{\bullet} = \text{Cone}(\mathbb{Z}(p) \oplus F^p \Omega^{\bullet}_X \xrightarrow{\epsilon - \ell} \Omega^{\bullet}_X)[-1]$$

where ϵ, ℓ are the natural maps and $F^p \Omega^{\bullet}_X$ the stupid filtration.

By the following easy lemma in [**EV88**], we know that the Deligne-Beilinson complex is quasi-isomorphic to the Deligne complex.

LEMMA 6.27. *Let $u_1 : A^{\bullet}_1 \to B^{\bullet}$ and $u_2 : A^{\bullet}_2 \to B^{\bullet}$ be two morphisms of complexes and $C^{\bullet} = \text{Cone}(A^{\bullet}_1 \oplus A^{\bullet}_2 \xrightarrow{u_1 - u_2} B^{\bullet})[-1]$. Then*

$$C^{\bullet} = \text{Cone}(A^{\bullet}_1 \xrightarrow{u_1} \text{Cone}(A^{\bullet}_2 \xrightarrow{-u_2} B^{\bullet}))[-1].$$

PROOF. Both complexes are equal to $A_1^\bullet \oplus A_2^\bullet \oplus B^\bullet[-1]$ with the differential

$$(a_1, a_2, b) \mapsto -(-da_1, -da_2, u_1(a_1) - u_2(a_2) + db).$$

□

A quasi-isomorphism $\alpha : \mathcal{D}(p)^\bullet \to A(p)^\bullet$ can be given by

$$\begin{array}{ccccccccccc}
\mathbb{Z}(p) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \cdots & \longrightarrow & \Omega^{p-2} & \longrightarrow & \Omega^{p-1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & & & \downarrow{\scriptstyle \alpha_{p-1}} & & \downarrow{\scriptstyle \alpha_p} & & \downarrow \\
\mathbb{Z}(p) & \xrightarrow{-\epsilon} & \mathcal{O}_X & \xrightarrow{-\delta_1} & \cdots & \longrightarrow & \Omega^{p-2} & \xrightarrow{-\delta_{p-1}} & \Omega^p \oplus \Omega^{p-1} & \xrightarrow{-\delta_p} & \Omega^{p+1} \oplus \Omega^p \cdots
\end{array}$$

with $\alpha_p(\omega) = (-1)^p(d\omega, \omega)$ and $\alpha_i(\omega) = (-1)^i\omega$. The symbol δ denotes the differential of the mapping cone, where in particular

$$\delta_{p-1}(\eta) = (0, d\eta), \delta_p(\psi, \eta) = (-d\psi, -\psi + d\eta).$$

The mapping cone has a negative sign, by the convention that for a complex $A^\bullet$, the complex $A^\bullet[d]$ has a differential in degree n defined by $(-1)^d d^{n-d}$. The cup product of the Deligne-Beilinson complex is defined as follows. We set

$$\cup_0 : A(p)^\bullet \otimes_{\mathbb{Z}_X} A(q)^\bullet \to A(p+q)^\bullet$$

$$x \cup_0 y = \begin{cases} x \cdot y, \text{if } x \in \mathbb{Z}(p), y \in \mathbb{Z}(q) \\ x \cdot y, \text{if } x \in \mathbb{Z}(p), y \in \Omega^\bullet \\ x \wedge y, \text{if } x \in F^p\Omega^\bullet, y \in F^q\Omega^\bullet \\ x \wedge y, \text{if } x \in \Omega^\bullet, y \in F^q\Omega^\bullet \\ 0, \text{ otherwise.} \end{cases}$$

A direct verification shows that the diagram

$$\begin{array}{ccc}
\mathcal{D}(p)^\bullet \otimes_{\mathbb{Z}_X} \mathcal{D}(q)^\bullet & \xrightarrow{\cup} & \mathcal{D}(p+q)^\bullet \\
\downarrow{\scriptstyle \alpha\otimes\alpha} & & \downarrow{\scriptstyle \alpha} \\
A(p)^\bullet \otimes_{\mathbb{Z}_X} A(q)^\bullet & \xrightarrow{\cup_0} & A(p+q)^\bullet
\end{array}$$

is commmutative and that $\cup_0$ is a morphism of complexes (cf. [**EV88**]). Since α is a quasi-isomorphism, we have a ring isomorphism at the level of hypercohomology.

For the analogue in the Bott-Chern case, we start by the modified cup product of Deligne-Beilinson complex. In this case, the cup product is defined as follows. We set

$$\cup_0 : A(p)^\bullet \otimes_{\mathbb{Z}_X} A(q)^\bullet \to A(p+q)^\bullet$$

$$x \cup_0 y = \begin{cases} x \cdot y, \text{if } x \in \mathbb{Z}(p), y \in \mathbb{Z}(q) \\ x \cdot y, \text{if } x \in \Omega^\bullet, y \in \mathbb{Z}(p) \\ x \wedge y, \text{if } x \in F^p\Omega^\bullet, y \in F^q\Omega^\bullet \\ (-1)^{\deg(x)} x \wedge y, \text{if } x \in F^p\Omega^\bullet, y \in \Omega^\bullet \\ 0, \text{ otherwise.} \end{cases}$$

The product can be described by the following table

	a_q	f_q	ω_q
a_p	$a_p \cdot a_q$	0	0
f_p	0	$f_p \wedge f_q$	$(-1)^{\deg(f_p)} f_p \wedge \omega_q$
ω_p	$\omega_p \cdot a_q$	0	0

representing elements of

	$\mathbb{Z}(q)$	$F^q\Omega^\bullet$	$\Omega^\bullet$
$\mathbb{Z}(p)$	$\mathbb{Z}(p+q)$	0	0
$F^p\Omega^\bullet$	0	$F^{p+q}\Omega^\bullet$	$\Omega^\bullet$
$\Omega^\bullet$	$\Omega^\bullet$	0	0

.

We verify that the cup product $\cup_0$ is a morphism of complexes, i.e. that

$$d(x \cup_0 y) = dx \cup_0 y + (-1)^{\deg(x)} x \cup_0 dy.$$

Both sides of the equation can be represented by the following table

	a_q	f_q	ω_q
a_p	$a_p \cdot a_q$	0	0
f_p	0	$(-df_p \wedge f_q - (-1)^{\deg(f_p)} f_p \wedge df_q, -f_p \wedge f_q)$	$(-1)^{\deg(f_p)} df_p \wedge \omega_q + f_p \wedge d\omega_q$
ω_p	$d\omega_p \cdot a_q$	0	0

The second line is calculated as follows:

$$(-df_p, -f_p) \cup_0 a_q + (-1)^{\deg(f_p)} f_p \cup_0 a_q = -f_p \wedge a_q + (-1)^{2\deg(f_p)} f_p \wedge a_q = 0,$$

$$(-df_p, -f_p) \cup_0 f_q + (-1)^{\deg(f_p)} f_p \cup_0 (-df_q, -f_q) = (-df_p \wedge f_q + (-1)^{\deg(f_p)} f_p \wedge (-f_q), (-1)^{2\deg(f_p)} f_p \wedge (-f_q)),$$

$$(-df_p, -f_p) \cup_0 \omega_q + (-1)^{\deg(f_p)} f_p \cup_0 d\omega_q = (-1)^{\deg(f_p)+1} (-df_p) \wedge \omega_q + (-1)^{2\deg(f_p)} f_p \wedge d\omega_q.$$

We now verify that the map from Deligne complex to Delinge-Beilinson complex is also commutative under the modified cup product.
If $\deg(y) = 0, \deg(x) < p$, $\alpha(x \cup y) = \alpha(xy) = (-1)^{\deg(x)} xy$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} x \cup_0 y = (-1)^{\deg(x)} xy$.
If $\deg(y) = 0, \deg(x) = p$, $\alpha(x \cup y) = \alpha(xy) = (-1)^{\deg(x)} xy$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} (dx, x) \cup_0 y = (-1)^{\deg(x)} xy$.
If $0 < \deg(y) < q, \deg(x) = p$, $\alpha(x \cup y) = \alpha((-1)^p dx \wedge y) = (-1)^{\deg(x)+\deg(y)+p} dx \wedge y$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} (dx, x) \cup_0 (-1)^{\deg(y)} y = (-1)^{\deg(x)+\deg(y)+p} dx \wedge y$.
If $\deg(y) = q, \deg(x) = p$, $\alpha(x \cup y) = \alpha((-1)^p dx \wedge y) = (-1)^{\deg(x)+\deg(y)+p} (d(dx \wedge y), dx \wedge y) = (-1)^{\deg(x)+\deg(y)} (dx \wedge dy, (-1)^p dx \wedge y)$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} (dx, x) \cup_0 (-1)^{\deg(y)} (dy, y) = (-1)^{\deg(x)+\deg(y)} (dx \wedge dy, (-1)^p dx \wedge y)$.
If $0 < \deg(y) < q, \deg(x) < p$, $\alpha(x \cup y) = \alpha(0) = 0$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} x \cup_0 (-1)^{\deg(y)} y = 0$. So in this case, we also have a ring isomorphism of Deligne cohomology and Deligne-Beilinson cohomology for the modified cup product.

The modified Deligne complex is quasi isomorphic to the following modified Deligne-Beilinson complex

$$A(p)^\bullet = \mathrm{Cone}(\mathbb{Z}(p) \oplus F^p \Omega_X^\bullet \xrightarrow{-\epsilon - \ell} \Omega_X^\bullet)[-1]$$

where ϵ, ℓ are the natural maps. We define the morphism α by the same formula. We change the definition of $\cup_0$ by the following table to give a modified cup product

	a_q	f_q	ω_q
a_p	$a_p \cdot a_q$	0	0
f_p	0	$-f_p \wedge f_q$	$(-1)^{\deg(f_p)-1} f_p \wedge \omega_q$
ω_p	$\omega_p \cdot a_q$	0	0

The verification that this is a morphism of complexes can be represented by the table

	a_q	f_q	ω_q
a_p	$-a_p \cdot a_q$	0	0
f_p	0	$(df_p \wedge f_q + (-1)^{\deg(f_p)} f_p \wedge df_q, f_p \wedge f_q)$	$(-1)^{\deg(f_p)-1} df_p \wedge \omega_q - f_p \wedge d\omega_q$
ω_p	$d\omega_p \cdot a_q$	0	0

The second line is calculated as follows:

$$(-df_p, -f_p) \cup_0 a_q + (-1)^{\deg(f_p)} f_p \cup_0 (-a_q) = -f_p \wedge a_q + (-1)^{2\deg(f_p)-1} f_p \wedge (-a_q) = 0,$$

$$(-df_p, -f_p) \cup_0 f_q + (-1)^{\deg(f_p)} f_p \cup_0 (-df_q, -f_q) = (df_p \wedge f_q - (-1)^{\deg(f_p)} f_p \wedge (-df_q), (-1)^{2\deg(f_p)-1} f_p \wedge (-f_q)),$$

$$(-df_p, -f_p) \cup_0 \omega_q + (-1)^{\deg(f_p)} f_p \cup_0 d\omega_q = (-1)^{\deg(f_p)} (-df_p) \wedge \omega_q + (-1)^{2\deg(f_p)-1} f_p \wedge d\omega_q.$$

We verify that the map from modified Deligne complex to the modified Delinge-Beilinson complex is also commutative under the modified cup product.
If $\deg(y) = 0, \deg(x) < p$, $\alpha(x \cup y) = \alpha(xy) = (-1)^{\deg(x)} xy$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} x \cup_0 y = (-1)^{\deg(x)} xy$.
If $\deg(y) = 0, \deg(x) = p$, $\alpha(x \cup y) = \alpha(xy) = (-1)^{\deg(x)} xy$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} (dx, x) \cup_0 y = (-1)^{\deg(x)} xy$.
If $0 < \deg(y) < q, \deg(x) = p$, $\alpha(x \cup y) = \alpha((-1)^{p-1} dx \wedge y) = (-1)^{\deg(x)+\deg(y)+p-1} dx \wedge y$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} (dx, x) \cup_0 (-1)^{\deg(y)} y = (-1)^{\deg(x)+\deg(y)+p-1} dx \wedge y$.
If $\deg(y) = q, \deg(x) = p$, $\alpha(x \cup y) = \alpha((-1)^{p-1} dx \wedge y) = (-1)^{\deg(x)+\deg(y)+p-1} (d(dx \wedge y), dx \wedge y) = (-1)^{\deg(x)+\deg(y)-1} (dx \wedge dy, (-1)^p dx \wedge y)$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} (dx, x) \cup_0 (-1)^{\deg(y)} (dy, y) = (-1)^{\deg(x)+\deg(y)-1} (dx \wedge dy, (-1)^p dx \wedge y)$.
If $0 < \deg(y) < q, \deg(x) < p$, $\alpha(x \cup y) = \alpha(0) = 0$, $\alpha(x) \cup_0 \alpha(y) = (-1)^{\deg(x)} x \cup_0 (-1)^{\deg(y)} y = 0$. Hence passing to hypercohomology, we have a ring isomorphism for the modified Deligne cohomology and the modified Deligne-Beilinson cohomology.

As above, the cup product of the Deligne-Beilinson complex and the modified cup product of the conjugate modified Deligne-Beilinson complex induce a cup product on the integral Bott-Chern complex. Indeed, the latter is quasi-isomorphic to

$$\mathcal{B}^\bullet_{p,q,\mathbb{Z}} = \mathrm{Cone}(\mathbb{Z}(p) \oplus F^p\Omega^\bullet_X \oplus F^q\overline{\Omega^\bullet_X} \xrightarrow{(\epsilon,-(2\pi i)^{q-p}\epsilon)+(-\ell,-\bar{\ell})} \Omega^\bullet_X \oplus \overline{\Omega^\bullet_X})[-1]$$

where ϵ is the natural map $\mathbb{Z}(p) \to \Omega^\bullet_X$ and $\ell, \bar{\ell}$ are the natural maps $F^p\Omega^\bullet_X \to \Omega^\bullet_X$ and $F^p\overline{\Omega^\bullet_X} \to \overline{\Omega^\bullet_X}$. With this quasi-isomorphism it becomes easier to check the projection formula.

Proposition 6.3.3. (Projection formula) *For a proper morphism f, one has*

$$(1)\ f^*\varphi \cdot f^*\psi = f^*(\varphi \cdot \psi)$$

$$(2)\ f_*(\varphi \cdot f^*\psi) = f_*\varphi \cdot \psi.$$

Proof. For the first equality, we can in fact check that on the level of complexes

$$f^*\varphi \cup f^*\psi = f^*(\varphi \cup \psi).$$

Below, we concentrate ourselves on the proof of the second equality. The integral Bott-Chern complex is quasi-isomorphic to the complex

$$\tilde{\mathcal{B}}^\bullet_{p,q,\mathbb{Z}} := \mathrm{Cone}(\mathcal{I}^\bullet_X \oplus s(F^{p,\bullet}\mathcal{D}'^{\bullet,\bullet}_X) \oplus s(F^{\bullet,q}\mathcal{D}'^{\bullet,\bullet}_X) \xrightarrow{(\epsilon,-\bar{\epsilon})+(-\ell,-\bar{\ell})} \mathcal{D}'^\bullet_X \oplus \mathcal{D}'^\bullet_X)[-1]$$

where ϵ is the natural map $\mathcal{I}^\bullet_X \to \mathcal{D}'^\bullet_X$, $s(F^{p,\bullet}\mathcal{D}'^{\bullet,\bullet}_X)$ is the total complex of $F^{p,\bullet}\mathcal{D}'^{\bullet,\bullet}_X$, i.e. the direct sum of spaces of currents of bidegree (k,l) $(k \leqslant p)$, and $\ell, \bar{\ell}$ are the natural maps $s(F^{p,\bullet}\mathcal{D}'^{\bullet,\bullet}_X) \to \mathcal{D}'^\bullet_X$ and $s(F^{\bullet,q}\mathcal{D}'^{\bullet,\bullet}_X) \to \mathcal{D}'^\bullet_X$. We start by defining a multiplication between $\mathcal{B}^\bullet_{p',q',\mathbb{Z}}$ and $\tilde{\mathcal{B}}^\bullet_{p',q',\mathbb{Z}}$ that is compatible with the multiplication of the integral Bott-Chern complex. In this way, we avoid the problematic weight product of two currents. We first perform a similar construction for the integral Deligne complex. One can represent the product

$$\cup_0 : A(p)^\bullet \otimes \mathrm{Cone}(\mathcal{I}^\bullet_X \oplus s(F^{q,\bullet}\mathcal{D}'^{\bullet,\bullet}_X) \xrightarrow{\epsilon-\ell} \mathcal{D}'^\bullet_X)[-1] \to \mathrm{Cone}(\mathcal{I}^\bullet_X \oplus s(F^{p+q,\bullet}\mathcal{D}'^{\bullet,\bullet}_X) \xrightarrow{\epsilon-\ell} \mathcal{D}'^\bullet_X)[-1]$$

by the following table

	a_q	f_q	ω_q
a_p	$a_p \cdot a_q$	0	$a_p \cdot \omega_q$
f_p	0	$f_p \wedge f_q$	0
ω_p	0	$\omega_p \wedge f_q$	0

representing elements of

	$\mathcal{I}^\bullet_X$	$s(F^{q,\bullet}\mathcal{D}'^{\bullet,\bullet}_X)$	$\mathcal{D}'^\bullet_X$
$\mathbb{Z}(p)$	$\mathcal{I}^\bullet_X$	0	$\mathcal{D}'^\bullet_X$
$F^p\Omega^\bullet$	0	$s(F^{p+q,\bullet}\mathcal{D}'^{\bullet,\bullet}_X)$	0
$\Omega^\bullet$	0	$\mathcal{D}'^\bullet_X$	0

.

Notice that the wedge product of smooth forms and currents is always well-defined. We also observe that since a locally integral current is represented by a generalised measure by the Riesz representation theorem, it defines a current of degree 0. We now check that the multiplication is a morphism of complex, i.e. that

$$d(x \cup_0 y) = dx \cup_0 y + (-1)^{\deg(x)} x \cup_0 dy.$$

Both sides of the equation can be represented by the following table

	a_q	f_q	ω_q
a_p	$a_p \cdot a_q + a_p \cdot da_q$	0	$a_p \cdot d\omega_q + a_p \cdot \omega_q$
f_p	0	$(-df_p \wedge f_q - (-1)^{\deg f_p} f_p \wedge df_q, -f_p \wedge f_q)$	0
ω_p	0	$d\omega_p \wedge f_q + (-1)^{\deg(\omega_p)}\omega_p \wedge df_q$	0

The calculation is different from the previous case. The difference just occurs in the first column, as a locally integral current is not necessarily closed, while the exterior differential of constant is always 0. The first object is

$$d(a_p \cup_0 a_q) = d(a_p \cdot a_q) = da_p \cdot a_q + (-1)^{\deg(a_p)} a_p \cdot da_q = da_p \cup_0 a_q + (-1)^{\deg(a_p)} a_p \cup_0 da_q.$$

The second object is

$$d(f_p \cup_0 a_q) = d(0) = 0 + 0 = (-df_p, f_p) \cup_0 a_q + (-1)^{\deg(f_p)} f_p \cup_0 (-da_q, a_q) = df_p \cup_0 a_q + (-1)^{\deg(f_p)} f_p \cup_0 da_q.$$

The third object is

$$d(\omega_p \cup_0 a_q) = d(0) = 0 + 0 = d\omega_p \cup_0 a_q + (-1)^{\deg(\omega_p)} \omega_p \cup_0 (-da_q, a_q) = d\omega_p \cup_0 a_q + (-1)^{\deg(\omega_p)} \omega_p \cup_0 da_q.$$

One can change the definition of $\cup_0$ for the modified Deligne complex by introducing a different sign for the morphism at degree 0, according to the table

	a_q	f_q	ω_q
a_p	$a_p \cdot a_q$	0	0
f_p	0	$-f_p \wedge f_q$	$(-1)^{\deg(f_p)-1} f_p \wedge \omega_q$
ω_p	$\omega_p \cdot a_q$	0	0

representing elements of

	$\mathcal{I}_X^\bullet$	$s(F^{q,\bullet}\mathcal{D}_X^{',\bullet,\bullet})$	$\mathcal{D}_X^{'\bullet}$
$\mathbb{Z}(p)$	$\mathcal{I}_X^\bullet$	0	0
$F^p\Omega^\bullet$	0	$s(F^{p+q,\bullet}\mathcal{D}_X^{',\bullet,\bullet})$	$\mathcal{D}_X^{'\bullet}$
$\Omega^\bullet$	$\mathcal{D}_X^{'\bullet}$	0	0

.

The verification that this is a morphism of complexes can be represented by the table

	a_q	f_q	ω_q
a_p	$(-d(a_p \cdot a_q), -a_p \cdot a_q)$	0	0
f_p	0	$(df_p \wedge f_q + (-1)^{\deg(f_p)} f_p \wedge df_q, f_p \wedge f_q)$	$(-1)^{\deg(f_p)-1} df_p \wedge \omega_q - f_p \wedge d\omega_q$
ω_p	$d(\omega_p \cdot a_q)$	0	0

The difference with the previous calculation just occurs in the first column. The first object is

$$d(a_p \cup_0 a_q) = (-d(a_p \cdot a_q), -a_p \cdot a_q) = (-da_p, -a_p) \cup_0 a_q + a_p \cup_0 (-da_q, -a_q) = da_p \cup_0 a_q + (-1)^{\deg(a_p)} a_p \cup_0 da_q.$$

The second object is

$$df_p \cup_0 a_q + (-1)^{\deg(f_p)} f_p \cup_0 da_q = (-df_p, -f_p) \cup_0 a_q + (-1)^{\deg(f_p)} f_p \cup_0 (-da_q, -a_q)$$
$$= -f_p \wedge a_q + (-1)^{2\deg(f_p)-1} f_p \wedge (-a_q) = 0.$$

The third object is

$$d(\omega_p \cup_0 a_q) = d(\omega_p \wedge a_q) = d\omega_p \wedge a_q + (-1)^{\deg(\omega_p)} \omega_p \wedge da_q = d\omega_p \cup_0 a_q + (-1)^{\deg(\omega_p)} \omega_p \cup_0 da_q.$$

We have the following commutative diagram of $\mathbb{Z}_X$-modules, where, as before, the multiplication of Deligne complex and the modified multiplication of the modified Deligne complex induce the multiplication of the integral Bott-Chern complex

$$\begin{array}{ccc} \mathcal{B}(p,q,\mathbb{Z})^\bullet \otimes_{\mathbb{Z}_X} \mathcal{B}(p',q',\mathbb{Z})^\bullet & \xrightarrow{\cup} & \mathcal{B}(p+p',q+q',\mathbb{Z})^\bullet \\ \downarrow & & \downarrow \\ \mathcal{B}(p,q,\mathbb{Z})^\bullet \otimes_{\mathbb{Z}_X} \tilde{\mathcal{B}}(p',q',\mathbb{Z})^\bullet & \xrightarrow{\cup_0} & \tilde{\mathcal{B}}(p+p',q+q',\mathbb{Z})^\bullet. \end{array}$$

The vertical arrow is induced by the morphism of complexes $\mathbb{Z}(p) \to \mathcal{I}_X^\bullet$. The "gluing condition" used to define the multiplication of the integral Bott-Chern complex, starting from the Deligne complex and the conjugate (modified) Deligne complex, is that $\mathbb{Z}(p) \otimes \mathcal{I}_X^\bullet \to \mathcal{I}_X^\bullet$ should be the same for both complexes. Now, the second equality comes from the straightforward check

$$f_*(f^*\psi \cup_0 \varphi) = \psi \cup_0 f_*\varphi.$$

This equality induces as follows the desired formula on the level of hypercohomology. By the algebraic Künneth formula (cf. Theorem 15.5 in [**Dem12b**]), we have a morphism

$$H^*(Ra_{Y*}\mathcal{B}(p,q,\mathbb{Z})) \otimes H^*(Ra_{Y*}Rf_*\tilde{\mathcal{B}}(p,q,\mathbb{Z})) \to H^*(Ra_{Y*}\mathcal{B}(p,q,\mathbb{Z}) \otimes^L Ra_{Y*}(Rf_*\tilde{\mathcal{B}}(p,q,\mathbb{Z}))).$$

Notice that since $\mathbb{Z}$ is a PID, $\mathcal{B}(p,q,\mathbb{Z})$, $\tilde{\mathcal{B}}(p,q,\mathbb{Z})$ are torsion free and flat. Notice also that $\tilde{\mathcal{B}}(p,q,\mathbb{Z})$ is also a soft complex. There is in fact no need to write functors R and L in the above morphism. We have proven that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}(p,q,\mathbb{Z}) \otimes f_*\tilde{\mathcal{B}}(p',q',\mathbb{Z}) & \longrightarrow & f_*(f^*\mathcal{B}(p,q,\mathbb{Z}) \otimes \tilde{\mathcal{B}}(p',q',\mathbb{Z})) \\ & \searrow & \downarrow \\ & & \tilde{\mathcal{B}}(p+p',q+q',\mathbb{Z})). \end{array} \qquad (*)$$

Let us observe that a tensor product of a soft complexe by a flat complex is soft. By taking Ra_{Y*} (equivalently a_{Y*} since all complexes are soft) the above commutative diagram induces the following commutative diagram

$$\begin{array}{ccccc} Ra_{Y*}\mathcal{B}(p,q,\mathbb{Z})\otimes^L Ra_{X*}\tilde{\mathcal{B}}(p',q',\mathbb{Z}) & \to & Ra_{Y*}(\mathcal{B}(p,q,\mathbb{Z})\otimes^L Rf_*\tilde{\mathcal{B}}(p',q',\mathbb{Z})) & \to & Ra_{X*}(f^*\mathcal{B}(p,q,\mathbb{Z})\otimes^L\tilde{\mathcal{B}}(p',q',\mathbb{Z})) \\ & \searrow & \downarrow & \swarrow & \\ & & Ra_{Y*}\tilde{\mathcal{B}}(p+p',q+q',\mathbb{Z})). & & \end{array}$$

(Remark that the symbol f^* used here is denoted f^{-1} by some authors.) The left arrow is the natural morphism and the left-down arrow is just the composition. Taking hypercohomology and composing with the morphism in the Künneth formula give the projection formula.

The order for taking the cup product is unimportant when passing to hypercohomology, since the integral Bott-Chern cohomology is anti-commutative. This finishes the proof of the projection formula. □

6.4. Chern classes of a vector bundle

In this part we give a construction of the Chern class of a vector bundle in the integral Bott-Chern cohomology. It is borrowed from Junyan Cao (personal communication). The general line is Grothendieck's construction of Chern classes of a vector bundle via the splitting principle. In particular, we prove axiom A stated in the introduction. We first recall the definition of the first Chern class of a line bundle in integral Bott-Chern cohomology, following [**Sch07**].

Let L be a holomorphic line bundle over X and $\mathcal{U} = (U_j)$ be an open covering of X with connected intersections such that on each U_j, L is locally trivial by a nowhere-vanishing section e_j. We denote g_{jk} the transition function defined on $U_j \cap U_k$ defined by the relation $e_k(x) = g_{jk}(x)e_j(x)$. Perhaps with further refinement of the open covering, we can suppose that $g_{jk} = \exp(u_{jk})$. The element

$$\{g_{jk}\} \in \check{H}^1(\mathcal{U},\mathcal{O}^*) \cong H^1(X,\mathcal{O}^*)$$

determines the isomorphic class of L. Let h be a hermitian metric on L and we denote by D the Chern connection associated with (L,h) and by Θ the curvature of the Chern connection. On U_j, the Chern connection is given by the formula

$$D(\xi_j(x)e_j(x)) = (d\xi_j(x) - \partial\varphi_j(x)\xi_j(x)) \otimes e_j(x)$$

where φ_j is the local weight function of the metric under the trivialisation defined by

$$e^{-\varphi_j(z)} = |e_j(z)|_h^2,$$

which verifies the compatibility condition on $U_j \cap U_k$:

$$-\varphi_k + \varphi_j = u_{jk} + \overline{u_{jk}}.$$

We define the Čech 2-cocycle $\delta(u_{jk})$ to be $(2\pi i c_{jkl})$ which means on U_{jkl}

$$2\pi i c_{jkl} = u_{jk} - u_{jl} + u_{kl}.$$

Taking exponential map on the both sides we know

$$\exp 2\pi i c_{jkl} = g_{jk} * g_{jl}^{-1} * g_{kl} = 1$$

which in particular shows $(2\pi i c_{jkl}) \in \check{C}^2(X,\mathbb{Z}(1))$ a $2-$Čech cocycle with value in $\mathbb{Z}(1)$. We define the first Chern class of L in the integral Bott-Chern cohomology to be

$$c_1(L)_{BC,\mathbb{Z}} := \{(2\pi i c_{jkl}),(u_{jk}),(\overline{u_{jk}})\} \in H^{1,1}_{BC}(X,\mathbb{Z}).$$

We prove in what follows that this hypercocycle also represents the Chern class of L in the complex Bott-Chern cohomology. For the complex Bott-Chern cohomology, the corresponding global representative (1,1)-form via the quasi-isomorphic complex $\mathcal{L}^\bullet_{p,q}[1]$ which is defined with $p=1, q=1$

$$\mathcal{L}^k_{p,q} = \bigoplus_{\substack{r+s=k\\ r<p,s<q}} \mathcal{E}^{r,s} \quad \text{if} k \leqslant p+q-2,$$

$$\mathcal{L}^{k-1}_{p-1,q-1} = \bigoplus_{\substack{r+s=k\\ r\geqslant p,s\geqslant q}} \mathcal{E}^{r,s} \quad \text{if} k \geqslant p+q,$$

with differential

$$\mathcal{L}^0 \xrightarrow{\mathrm{pr}_{\mathcal{L}^1}\circ d} \mathcal{L}^1 \xrightarrow{\mathrm{pr}_{\mathcal{L}^2}\circ d} \ldots \to \mathcal{L}^{k-2} \xrightarrow{\frac{i}{2\pi}\partial\overline{\partial}} \mathcal{L}^{k-1} \xrightarrow{d} \mathcal{L}^k \xrightarrow{d} \ldots$$

is just the global form with $\frac{i}{2\pi}\partial\overline{\partial}\varphi_j$ on U_j. Notice that the complex $\mathcal{L}^\bullet_{p,q}$ is acyclic. The proof of the quasi isomorphism between $\mathcal{L}^\bullet_{p,q}$ and $\mathcal{B}^\bullet_{p,q}$ can be found in section 12 Chap VI of [**Dem12b**]. (Notice that in [**Sch07**], the operator $\frac{i}{2\pi}\partial\overline{\partial}$ is changed by $\partial\overline{\partial}$. Here we take this choice so that the first Chern class of a line bundle in the integral Bott-Chern class has image as the first Chern class in the complex Bott-Chern class under the canonical morphism.)

With the same notation as in [**Sch07**], $\alpha^{0,0}$ can be chosen to be (φ_j), so the global representative is $\theta^{0,0} = \frac{i}{2\pi}\partial\overline{\partial}\alpha^{0,0}$. This is exactly the curvature form on U_j. Therefore the hypercocycle of $\mathcal{B}^{\bullet}_{1,1,\mathbb{Z}}$ viewed as a hypercocycle of $\mathcal{B}^{\bullet}_{1,1,\mathbb{C}}$ corresponding to Θ is

$$\{\Theta\} \longleftrightarrow \{(2\pi i c_{jkl}), (u_{jk}), (\overline{u_{jk}})\}.$$

Observe that the first Chern class of the complex Bott-Chern cohomology is just represented by the curvature. We denote by ϵ_{BC} the canonical map from the integral Bott-Chern complex to the complex Bott-Chern complex. We have in hypercohomology

$$\epsilon_{BC}\, c_1(L)_{BC,\mathbb{Z}} = c_1(L)_{BC}.$$

Notice that the Chern classes of a vector bundle in integral Bott-Chern cohomology (which will be defined below) and in complex Bott-Chern cohomology are both defined by means of the splitting principle, in such a way that for any d and any vector bundle E we have

$$\epsilon_{BC}\, c_d(E)_{BC,\mathbb{Z}} = c_d(E)_{BC}.$$

To construct the Chern class of a vector bundle, we use Grothendieck's splitting principle. We begin by proving a Leray-Hirsch type theorem for the integral Bott-Chern cohomology. This theorem is a direct consequence of the Hodge decomposition theorem and of the Leray-Hirsch theorem for De Rham cohomology, in case X is a compact Kähler manifold. Here we give a generalisation to arbitrary compact complex manifolds. Before giving the statement in the integral Bott-Chern cohomology, we prove lemma 6.29 below, which is a proposition of the same nature for Dolbeault cohomology, and which will be used in a further induction process. The proof also uses the following Künneth type theorem for Dolbeault cohomology.

THEOREM 6.28. *Let X, Y be any two complex manifolds, Y being compact. Then one has the Künneth isomorphism*

$$H^{p,q}(X \times Y) = \bigoplus_{k+l=p, m+n=q} H^{k,m}(X) \otimes H^{l,n}(Y).$$

PROOF. With respect to local coordinates (x^i) on X and (w^j) on Y, the sheaf $\Omega^{p,q}_{X\times Y}$ is a locally free $\mathcal{O}_{X\times Y}$-module with the basis $dx^I \wedge dw^J$ $(|I| = p, |J| = q)$. Similarly the Ω^p_X (resp. Ω^q_Y) is locally a free $\mathcal{O}_X$-module (resp. $\mathcal{O}_Y$-module) with the basis dx^I with $|I| = p$ (resp. dw^J with $|J| = q$). With this identification, the vector bundle isomorphism

$$\Omega^k_{X\times Y} \cong \bigoplus_{p+q=k} \Omega^p_X \boxtimes \Omega^q_Y$$

is just the canonical isomorphism

$$\mathcal{O}_{X\times Y} \cong \mathcal{O}_X \hat{\boxtimes} \mathcal{O}_Y.$$

The symbol ˆ means here that we take the topological tensor product of two nuclear spaces (for more details, cf. [**Dem12b**], Section 5 of Chap. IX).

By Remark (5,24) of Chap. IX in [**Dem12b**], when Y is compact we have

$$H^{p,q}(X \times Y) = H^q(X \times Y, \Omega^p_{X\times Y}) = \bigoplus_{k+l=p} H^q(X \times Y, \Omega^k_X \boxtimes \Omega^l_Y)$$

$$= \bigoplus_{k+l=p}\bigoplus_{m+n=q} H^m(X, \Omega^k_X) \otimes H^n(Y, \Omega^l_Y) = \bigoplus_{k+l=p}\bigoplus_{m+n=q} H^{k,m}(X) \otimes H^{l,n}(Y).$$

□

We can now state the relevant Leray-Hirsch type theorem for Dolbeault cohomology.

LEMMA 6.29. *Let X be a compact complex manifold and E be a vector bundle of rank r on X. One has an isomorphism*

$$\bigoplus_{s\leqslant r-1} H^{p-s,k-p-s}(X) \cdot c_1^s(\mathcal{O}(1)) \to H^{p,k-p}(\mathbb{P}(E)).$$

PROOF. We follow in general the proof of Leray-Hirsch theorem as in [**BT82**]. Take a finite open covering (U_i) of X. We do an induction on the open cover. In the following, U, V are respectively $\bigcup_{i\leqslant i_0} U_i$ and U_{i_0+1} appearing in the open covering.

We have a short exact sequence of complexes of abelian groups:

$$0 \to \mathcal{A}^{q,\bullet}(U \cup V) \to \mathcal{A}^{q,\bullet}(U) \oplus \mathcal{A}^{q,\bullet}(V) \to \mathcal{A}^{q,\bullet}(U \cap V) \to 0.$$

It induces a long exact sequence

$$\cdots \to H^{p,q}(U \cup V) \to H^{p,q}(U) \oplus H^{p,q}(V) \to H^{p,q}(U \cap V) \to H^{p+1,q}(U \cup V) \to \cdots.$$

We verify the following diagram is commutative where both lines are exact

$$\begin{array}{ccccccc}
\bigoplus_{s\leqslant r-1} H^{p-s,k-p-s}(U\cup V)\cdot c_1^s(\mathcal{O}(1)) & \to & \bigoplus_{s\leqslant r-1}(H^{p-s,k-p-s}(U)\oplus H^{p-s,k-p-s}(V))\cdot c_1^s(\mathcal{O}(1)) & \to & \bigoplus_{s\leqslant r-1} H^{p-s,k-p-s}(U\cap V)\cdot c_1^s(\mathcal{O}(1)) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H^{p,k-p}(\mathbb{P}(E|_{U\cup V})) & \longrightarrow & H^{p,k-p}(\mathbb{P}(E|_U))\oplus H^{p,k-p}(\mathbb{P}(E|_V)) & \longrightarrow & H^{p,k-p}(\mathbb{P}(E|_{U\cap V})) & \longrightarrow & \cdots
\end{array}$$

The commutativity of the diagram is clear at all places, except at

$$\begin{array}{ccc}
\bigoplus_{s\leqslant r-1} H^{p-s,k-p-s}(U\cap V)\cdot c_1^s(\mathcal{O}(1)) & \xrightarrow{\overline{\partial}^*} & \bigoplus_{s\leqslant r-1} H^{p-s,k-p-s}(U\cup V)\cdot c_1^s(\mathcal{O}(1)) \\
\downarrow & & \downarrow \\
H^{p,k-p}(\mathbb{P}(E|_{U\cap V})) & \xrightarrow{\overline{\partial}^*} & H^{p,k-p}(\mathbb{P}(E|_{U\cup V}))
\end{array}.$$

We denote by ψ the vertical maps. Let ρ_U, ρ_V be a partition of unity associated with U, V so the functions $\pi^*\rho_U, \pi^*\rho_V$ form a partition of unity associated with $\pi^{-1}(U), \pi^{-1}(V)$. For any ω a global representative of the cohomology class $H^{p-s,k-p-s}(U\cap V)$ and ϕ a global representative of the cohomology class $c_1^s(\mathcal{O}(1))$, we have on $\pi^{-1}(U)$

$$\psi(\overline{\partial}^*(\omega\otimes\phi)) = \pi^*(\overline{\partial}(\rho_U\omega))\wedge\phi.$$
$$\overline{\partial}^*\psi(\omega\otimes\phi) = \overline{\partial}^*(\pi^*\omega\wedge\phi) = \overline{\partial}(\pi^*\rho_U\cdot\pi^*\omega\wedge\phi) = \pi^*(\overline{\partial}(\rho_U\omega))\wedge\phi.$$

The last equality use the fact that the global representative is $\overline{\partial}$-closed.

By the five lemma, once we know the vertical arrows are isomorphisms for the terms involving $U\cap V, U, V$, we know the isomorphism for the terms involving $U\cup V$. We take U, V to be the local trivial open sets chosen above. If we have $\pi^{-1}(U_i)\cong U_i\times\mathbb{P}^{r-1}$ for any i, we have $\pi^{-1}(U_i\cap U_j)\cong(U_i\cap U_j)\times\mathbb{P}^{r-1}$. For any U open set on which π is locally trivial we have the following commutative diagram

$$\begin{array}{ccc}
\bigoplus_{s\leqslant r-1} H^{p-s,k-p-s}(U)\cdot c_1^s(\mathcal{O}_{\mathbb{P}^{r-1}}(1)) & \xrightarrow{\mathrm{id}\otimes s} & \bigoplus_{s\leqslant r-1} H^{p-s,k-p-s}(U)\cdot c_1^s(\mathcal{O}_{\mathbb{P}(E)}(1)) \\
 & \searrow^{\cong} & \Big\downarrow{\scriptstyle \pi^*\wedge i^*_{\pi^{-1}(U)}} \\
 & & H^{p,k-p}(\mathbb{P}(E|_U))
\end{array}$$

where the map s is associating $c_1^s(\mathcal{O}_{\mathbb{P}^{r-1}}(1))$ to $c_1^s(\mathcal{O}_{\mathbb{P}(E)}(1))$ and $i_{\pi^{-1}(U)}$ is the inclusion of $\pi^{-1}(U)$ in $\mathbb{P}(E)$. By the above Künneth type theorem for Dolbeault cohomology, we get an isomorphism as shown in the diagram. We next check that the diagram commutes. In fact, one has

$$\begin{aligned}
i^*_{\pi^{-1}(U)}(s(c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1)))) &= i^*_{\pi^{-1}(U)}c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) = c_1(\mathcal{O}_{\mathbb{P}(E|_U)}(1)) \\
&= c_1(\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{r-1}}(1)) = \mathrm{pr}_2^*c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1)).
\end{aligned}$$

Here we consider $U\times\mathbb{P}^{r-1}\xrightarrow{(\pi,\mathrm{pr}_2)}\pi^{-1}(U)$. In the calculation we have used many times the functoriality of Chern classes of line bundles, which is a direct consequence of their construction.

The commutativity of the diagram and the fact that the horizontal arrow is a linear isomorphism show that the vertical arrow is also an isomorphism. Using this argument, we have isomorphisms for the terms involving $U_i\cap U_j, U_i, U_j$.

The induction process can be done in three different cases. The case of finite union of open sets is obtained by the induction assumption. The case of a single local trivialising open chart is done as above. The case of the intersection of a union of open sets and of a local trivialising chart (which also yields a local trivialising chart) is again done as above. Since the covering is finite, the induction is achieved in finitely many steps. □

Remark 6.30. The difference between this proof and the one given in **[BT82]** is that we do not have to take a good covering since here the induction on the open covering start with the Künneth type theorem instead of Poincaré lemma which in fact shows that the De Rham cohomology is homotopy invariant. We are forced to do it because the Dolbeault cohomology is not homotopy invariant. For example $H^{0,0}$ of a point is $\mathbb{C}$ while $H^{0,0}$ of $\mathbb{C}$ are the entire functions.

Now, we prove the principal proposition of this section, namely a Leray-Hirsch type theorem for the integral Bott-Chern cohomology.

Proposition 6.4.1. *Let X be a compact complex manifold, E a vector bundle of rank r over it. Then, we have*

$$\mathbb{H}^k(\mathbb{P}(E),\mathcal{B}^\bullet_{p,q,\mathbb{Z}}) = \mathbb{H}^k(X,\mathcal{B}^\bullet_{p,q,\mathbb{Z}})\oplus\mathbb{H}^{k-2}(X,\mathcal{B}^\bullet_{p-1,q-1,\mathbb{Z}})\cdot\omega\oplus\cdots\oplus\mathbb{H}^{k-2r+2}(X,\mathcal{B}^\bullet_{p-r+1,q-r+1,\mathbb{Z}})\cdot\omega^{r-1}$$

where ω is the first Chern class of the tautological line bundle over $\mathbb{P}(E)$ in $H^{1,1}_{BC}(\mathbb{P}(E),\mathbb{Z})$ as defined above.

In the proposition we use the following notations.
If $p < 0$ (resp. $q < 0$), we denote $\mathcal{B}^\bullet_{p,q,\mathbb{Z}} = \mathcal{B}^\bullet_{0,q,\mathbb{Z}}$ (resp. $\mathcal{B}^\bullet_{p,0,\mathbb{Z}}$). The morphism

$$F : \bigoplus_{s \leqslant r-1} \mathbb{H}^{k-2s}(X, \mathcal{B}^\bullet_{p-s,q-s,\mathbb{Z}}) \cdot \omega^s \to \mathbb{H}^k(\mathbb{P}(E), \mathcal{B}_{p,q})$$

is defined as follows: let $\pi : \mathbb{P}(E) \to X$;

If $s \leqslant \min(p, q)$, $F(\alpha \cdot \omega^s) = \pi^*(\alpha) \cdot \omega^s$,

If $s \geqslant p$, $F(\alpha \cdot \omega^s) = \pi^*(\alpha) \cdot \omega^p \cdot \mathrm{pr}_{0,1}(\omega)^{s-p}$,

If $s \geqslant q$, $F(\alpha \cdot \omega^s) = \pi^*(\alpha) \cdot \omega^q \cdot \mathrm{pr}_{1,0}(\omega)^{s-q}$,

where the projection $\mathrm{pr}_{0,1}$ is induced by the canonical projection from $\mathcal{B}^\bullet_{1,1,\mathbb{Z}}$ to $\mathcal{B}^\bullet_{0,1,\mathbb{Z}}$. Similarly $\mathrm{pr}_{1,0}$ is induced by the projection to $\mathcal{B}^\bullet_{1,0,\mathbb{Z}}$.

Notice that when $p = q = r$, $k = 2r$, this is just the normal splitting principle without the complicated notations.

Proof. The idea is to use the exact sequence

$$0 \to \Omega^p[p] \to \mathcal{B}^\bullet_{p+1,q,\mathbb{Z}} \to \mathcal{B}^\bullet_{p,q,\mathbb{Z}} \to 0$$

to reduce the proof to the Dolbeault case. In this proof, we use the usual convention for differential forms that for $p < 0$, $\Omega^p[p] = 0$. We begin by proving that the following diagram is commutative and that its two lines are exact:

$$\begin{array}{ccccccc}
\bigoplus_{s\leqslant r-1} \mathbb{H}^{k-2s}(X, \Omega^{p-s}[p-s]) \cdot \omega^s & \to & \bigoplus_{s\leqslant r-1} \mathbb{H}^{k-2s}(X, \mathcal{B}^\bullet_{p+1-s,q-s,\mathbb{Z}}) \cdot \omega^s & \to & \bigoplus_{s\leqslant r-1} \mathbb{H}^{k-2s}(X, \mathcal{B}^\bullet_{p-s,q-s,\mathbb{Z}}) \cdot \omega^s & \to & \bigoplus_{s\leqslant r-1} \mathbb{H}^{k-2s+1}(X, \Omega^{p-s}[p-s]) \cdot \omega^s \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^k(\mathbb{P}(E), \Omega^p[p]) & \longrightarrow & \mathbb{H}^k(\mathbb{P}(E), \mathcal{B}^\bullet_{p+1,q,\mathbb{Z}}) & \longrightarrow & \mathbb{H}^k(\mathbb{P}(E), \mathcal{B}^\bullet_{p,q,\mathbb{Z}}) & \longrightarrow & \mathbb{H}^{k+1}(\mathbb{P}(E), \Omega^p[p])
\end{array}$$

We first check the exactness of the two lines. The exactness is just obtained from the long exact sequence associated with the short exact sequence of sheaves. We now check the commutativity of the first square.

$$\begin{array}{ccc}
\mathbb{H}^{k-2s}(X, \Omega^{p-s}[p-s]) \cdot \omega^s & \xrightarrow{i} & \mathbb{H}^{k-2s}(X, \mathcal{B}^\bullet_{p+1-s,q-s,\mathbb{Z}}) \cdot \omega^s \\
\downarrow G & & \downarrow F \\
\mathbb{H}^k(\mathbb{P}(E), \Omega^p[p]) & \xrightarrow{i} & \mathbb{H}^k(\mathbb{P}(E), \mathcal{B}^\bullet_{p+1,q,\mathbb{Z}}).
\end{array}$$

The morphism G is induced from the following morphism of complexes $\Omega^{p-s}[p-s] \otimes_{\mathbb{Z}_X} \mathcal{B}^\bullet_{1,1,\mathbb{Z}} \to \Omega^{p-s+1}[p-s+1]$. Denote the germs as $\alpha \in \Omega^{p-s}[p-s]$, $\omega = (\tilde{c}, \beta; \overline{\beta})$. We define

$$G(\alpha \otimes \beta) = \alpha \wedge (\partial\beta).$$

We take it equal to zero otherwise.
We check that this a morphism of complexes. In fact, we have

$$0 = d(G(\alpha \otimes \tilde{c})) = G(d\alpha \otimes \tilde{c}) + (-1)^{p-s} G(\alpha \otimes d\tilde{c}) = G(0 \otimes \tilde{c}) + (-1)^{p-s} \alpha \wedge \partial\tilde{c},$$
$$0 = d(G(\alpha \otimes \beta)) = G(d\alpha \otimes \beta) + (-1)^{p-s} G(\alpha \otimes d\beta) = G(0 \otimes \beta) + (-1)^{p-s} G(\alpha \otimes 0),$$
$$0 = d(G(\alpha \otimes \overline{\beta})) = G(d\alpha \otimes \overline{\beta}) + (-1)^{p-s} G(\alpha \otimes d\overline{\beta}) = G(0 \otimes \overline{\beta}) + (-1)^{p-s} G(\alpha \otimes 0).$$

Therefore G defines a morphism at the level of hypercohomology. From now on, we do not pay attention to write α or $\pi^*\alpha$ when the context should make the meaning clear. Notice that the morphism F is induced by a morphism of complexes. (It is just the cup product of the integral Bott-Chern cohomology defined in section 3.) To prove the commutativity at the level of hypercohomology, it is enough to show the commutativity at the level of complexes. It is enough to check the commutativity for the case $s \leqslant p$. We have

$$i(\alpha \wedge (\partial\beta)^s) = (0, 0, 0 \ldots \alpha^{p-s} \wedge (\partial\beta)^s; 0),$$

which is equal to the image of $F \circ i$.
We check the commutativity of the second square. Let $\alpha = (c, \alpha_0, \ldots, \alpha_{p-s}; \overline{\beta}_0, \ldots, \overline{\beta}_{q-s-1})$, $\omega = (\tilde{c}, \beta; \overline{\beta})$ be the representatives of hypercocycles. If $s \leqslant p$, the horizontal morphism just consists of forgetting the term involving α_{p-s}, thus it is commutative. Otherwise, $\alpha = (c, \overline{\beta}_0, \ldots, \overline{\beta}_{q-s-1})$ and the morphism is induced by the identity map at the level of complexes, so it is commutative.
We check the commutativity of the third square.

$$\begin{array}{ccc}
\bigoplus \mathbb{H}^{k-2s}(X, \mathcal{B}^\bullet_{p-s,q-s,\mathbb{Z}}) \cdot \omega^s & \xrightarrow{i} & \bigoplus \mathbb{H}^{k-2s+1}(X, \Omega^{p-s}[p-s]) \cdot \omega^s \\
\downarrow F & & \downarrow G \\
\mathbb{H}^k(\mathbb{P}(E), \mathcal{B}^\bullet_{p,q,\mathbb{Z}}) & \xrightarrow{i} & \mathbb{H}^{k+1}(\mathbb{P}(E), \Omega^p[p]).
\end{array}$$

If $s \leqslant p-1$, take a representative of hypercocycle $\alpha = \big(c, \alpha_0, ..., \alpha_{p-s-1}; \overline{\beta}_0, ..., \overline{\beta}_{q-s-1}\big)$, which is the image of hypercocycle of $\mathcal{B}^\bullet_{p-s+1,q-s,\mathbb{Z}}$ $\big(c, \alpha_0, ..., \alpha_{p-s-1}, 0; \overline{\beta}_0, ..., \overline{\beta}_{q-s-1}\big)$. By the definition of the connecting morphism, $i(\alpha)$ can be taken as the degree $(p-s)$ element of the hypercocycle $\check{\delta}\big(c, \alpha_0, ..., \alpha_{p-s-1}, 0; \overline{\beta}_0, ..., \overline{\beta}_{q-s-1}\big)$ which is $\partial\alpha_{p-s-1}$. Hence

$$G(i(\alpha)) = \partial\alpha_{p-s-1} \wedge (\partial\beta)^s.$$

On the other hand, $i(F(\alpha)) = \partial(\alpha_{p-s-1} \wedge (\partial\beta)^s) = \partial\alpha_{p-s-1} \wedge (\partial\beta)^s$.

If $s = p$, we take a representative of the hypercocycle $\alpha = \big(c, \overline{\beta}_0, ..., \overline{\beta}_{q-s-1}\big)$, which is the image of the hypercocycle $\big(c, 0; \overline{\beta}_0, ..., \overline{\beta}_{q-s-1}\big)$ of $\mathcal{B}^\bullet_{1,q-s,\mathbb{Z}}$. By definition of the connecting morphism, $i(\alpha)$ can be taken as the degree 0 element of the hypercocycle $\check{\delta}\big(c, 0; \overline{\beta}_0, ..., \overline{\beta}_{q-s-1}\big)$, which is c.
Therefore $i(\alpha) = c$ and $G(i(\alpha)) = c \wedge (\partial\beta)^s$. The two elements with highest degrees in the hypercocycle $F(\alpha)$ are $c \wedge \beta \wedge (\partial\beta)^{s-1}$ and $c \wedge (\partial\beta)^s$. Now, $i(F(\alpha))$ is the degree p element of the hypercocycle $\check{\delta}(F(\alpha))$, namely

$$i(F(\alpha)) = \partial(c \wedge \beta \wedge (\partial\beta)^{s-1}) = G(i(\alpha)).$$

If $s < p$, the sequence

$$0 \to \Omega^{p-s}[p] \to \mathcal{B}^\bullet_{p+1-s,q,\mathbb{Z}} \xrightarrow{\sim} \mathcal{B}^\bullet_{p-s,q,\mathbb{Z}} \to 0$$

is an isomorphism between the second and third terms, which therefore induces a zero connecting morphism. The diagram is also commutative in this case.

At this point, all the asserted commutativity properties have been checked.

Using the five lemma to perform an induction on p, we have to prove that the following morphism is an isomorphism:

$$G : \bigoplus_{s \leqslant r-1} \mathbb{H}^{k-2s}(X, \Omega^{p-s}[p-s]) \cdot \omega^s \to \mathbb{H}^k(\mathbb{P}(E), \Omega^p[p]).$$

On the Čech cohomology groups $\check{H}^p(X, \Omega^q)$, one can introduce a ring structure by the wedge product

$$\check{H}^p(X, \Omega^q) \times \check{H}^{p'}(X, \Omega^{q'}) \to \check{H}^{p+p'}(X, \Omega^{q+q'}).$$

On the other hand, using the De Rham-Weil isomorphism, we have a canonical isomorphism

$$\phi : \check{H}^p(X, \Omega^q) \to H^{q,p}(X, \mathbb{C}).$$

Lemma 6.31 below shows that the isomorphism is compatible with the ring structure of Dolbeault cohomology, possibly up to a sign.

Now we prove that G is an isomorphism. Let $\omega = (c, \beta; \overline{\beta})$, so that by definition $G(\alpha \cdot \omega^s)$ is represented by the k-hypercocycle $G(\alpha \cdot \omega^s) = \pi^*(\alpha) \wedge (\partial\beta)^s$. By the construction of the Chern class of the line bundle $\mathcal{O}(1)$, we have $\beta_{jk} + \overline{\beta_{jk}} = \phi_j - \phi_k$ which implies

$$\partial\beta_{jk} = \partial(\phi_j - \phi_k).$$

A diagram chasing procedure similar to the proof of the De Rham-Weil isomorphism gives that the image of $\partial\beta_{jk}$ in $H^{1,1}(\mathbb{P}(E), \mathbb{C})$ is $-\overline{\partial}(\partial\phi_j)$, where the later form is the curvature. The negative sign comes from the convention that if we denote δ, d the differentials of a double complex, $d\delta + \delta d = 0$. Therefore, to define a double complex from the Čech complex and $\overline{\partial}$-complex, we have to add a negative sign following the parity. In conclusion ω represents $c_1(\mathcal{O}(1))$, hence by the Leray-Hirsch type theorem for Dolbeault cohomology and by lemma 6.31, the isomorphism G is settled.

To conclude the proof of the proposition, the five lemma and an induction on p reduce the proof to the case $p = 0$. It is enough to show that

$$\mathbb{H}^k(\mathbb{P}(E), \mathcal{B}^\bullet_{0,q,\mathbb{Z}}) = \mathbb{H}^k(X, \mathcal{B}^\bullet_{0,q,\mathbb{Z}}) \oplus \mathbb{H}^{k-2}(X, \mathcal{B}^\bullet_{0,q-1,\mathbb{Z}}) \cdot \omega \oplus \cdots \oplus \mathbb{H}^{k-2r+2}(X, \mathcal{B}^\bullet_{0,q-r+1,\mathbb{Z}}) \cdot \omega^{r-1}.$$

The short exact sequence $0 \to \overline{\Omega}^q[q] \to \mathcal{B}^\bullet_{0,q+1,\mathbb{Z}} \to \mathcal{B}^\bullet_{0,q,\mathbb{Z}} \to 0$ induces the two lines of the following diagram are exact.

$$\begin{array}{ccccccc}
\bigoplus_{s\leqslant r-1} \mathbb{H}^{k-2s}(X, \overline{\Omega}^{q-s}[q-s]) \cdot \omega^s & \to & \bigoplus_{s\leqslant r-1} \mathbb{H}^{k-2s}(X, \mathcal{B}_{0,q+1-s}) \cdot \omega^s & \to & \bigoplus_{s\leqslant r-1} \mathbb{H}^{k-2s}(X, \mathcal{B}_{0,q-s}) \cdot \omega^s & \to & \bigoplus_{s\leqslant r-1} \mathbb{H}^{k-2s+1}(X, \overline{\Omega^{q-s}}[q-s]) \cdot \omega^s \;, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^k(\mathbb{P}(E), \overline{\Omega}^q[q]) & \longrightarrow & \mathbb{H}^k(\mathbb{P}(E), \mathcal{B}_{0,q+1}) & \longrightarrow & \mathbb{H}^k(\mathbb{P}(E), \mathcal{B}_{0,q}) & \longrightarrow & \mathbb{H}^{k+1}(\mathbb{P}(E), \overline{\Omega}^q[q])
\end{array}$$

Here we change the connecting morphism of the first line with a sign $(-1)^s$ on the relevant terms. This change does not affect the exactness of sequence but ensures the commutativity of the diagram. As before, we check that the diagram is commutative. To simply the sign in the cup product of Bott-Chern cohomology, we use the anti-commutativity of the integral Bott-Chern class. For any class α, $\alpha \cdot \omega = \omega \cdot \alpha$. Notice that since $p = 0$, ω is in fact $\mathrm{pr}_{0,1}\omega$. With the same notations as before, this time the morphism G is induced by the morphism of complexes $\mathcal{B}^{\bullet}_{1,1,\mathbb{Z}} \otimes_{\mathbb{Z}_X} \overline{\Omega^{q-s}}[p-s] \to \overline{\Omega^{q-s+1}}[p-s+1]$. Denote the germs by $\alpha \in \overline{\Omega^{q-s}}[p-s]$ and $\omega = (\tilde{c}, \beta; \overline{\beta})$. We define

$$G(\overline{\beta} \otimes \alpha) = (\overline{\partial\beta}) \wedge \alpha$$

and take it equal to zero otherwise.
We check that this is a morphism of complexes. Indeed, we have

$$0 = d(G(\tilde{c} \otimes \alpha)) = G(\tilde{c} \otimes d\alpha) + G(d\tilde{c} \otimes \alpha) = G(\tilde{c} \otimes 0) + \overline{\partial}\tilde{c} \wedge \alpha,$$

$$0 = d(G(\beta \otimes \alpha)) = -G(\beta \otimes d\alpha) + G(d\beta \otimes \alpha) = -G(\beta \otimes 0) + G(0 \otimes \alpha),$$

$$0 = d(G(\bar{\beta} \otimes \alpha)) = -G(\bar{\beta} \otimes d\alpha) + G(d\bar{\beta} \otimes \alpha) = -G(\bar{\beta} \otimes 0) + G(0 \otimes \alpha).$$

To check the commutativity of the first square, it is enough to check the commutativity at the level of complexes for the case $s \leqslant q$.

$$i((\overline{\partial\beta})^s \wedge \alpha) = (0, 0; 0 .... (\overline{\partial\beta})^s \wedge \alpha^{q-s})$$

which is equal to the image of $F \circ i$. The commutativity of the second square is easy.
We now check the commutativity of the third square. Take hypercocycles $\alpha = (c, \overline{v}_0, ..., \overline{v}_{q-s})$, $\mathrm{pr}_{0,1}(\omega) = (\tilde{c}, \overline{\beta})$. It is enough to check the case $s \leqslant q$, otherwise the connecting morphism is zero map. If $s \leqslant q-1$, the image of α under the connecting morphism is $\overline{\partial}\overline{v}_{q-s}$. The image at the lower right corner of the diagram is $(-\overline{\partial\beta})^s \wedge \overline{\partial}\overline{v}_{q-s}$. (The sign comes from the change of the signs in the first line.) On the other hand, the image under the connecting morphism of $F(\alpha) = (\overline{\partial\beta})^s \wedge \overline{v}_{q-s}$ is $\overline{\partial}((\overline{\partial\beta})^s \wedge \overline{v}_{q-s}) = (-\overline{\partial\beta})^s \wedge \overline{\partial}\overline{v}_{q-s}$.
If $s = q$, the image of α under the connecting morphism is $-c$. The image at the lower right corner of the diagram is $(-\overline{\partial\beta})^s \wedge -c$. On the other hand, the elements with the two highest degrees in the hypercocycle $F(\alpha)$ are $(\overline{\partial\beta})^{s-1} \wedge \overline{\beta} \wedge c$ and $(\overline{\partial\beta})^s \wedge c$. The image of the first one under the connecting morphism is $\overline{\partial}((\overline{\partial\beta})^{s-1} \wedge \overline{\beta} \wedge c) = (-\overline{\partial\beta})^s \wedge -c$.

By the five lemma, similar arguments as those given above reduce the induction on q to the case $q = 0$, $p = 0$. In the case $\mathcal{B}^{\bullet}_{p,q,\mathbb{Z}} = \mathbb{Z}$, the isomorphism is trivial. □

LEMMA 6.31. *One has the following relation:*

$$\phi(\check{H}^q(X, \Omega^p)) \wedge \phi(\check{H}^{q'}(X, \Omega^{p'})) = (-1)^{pq'} \phi(\check{H}^q(X, \Omega^p) \cdot \check{H}^{q'}(X, \Omega^{p'})).$$

PROOF. We denote by $\mathcal{E}^{(p,q)}$ the sheaf of smooth (p, q)-forms on X. We modify the definition of the wedge product so that on $\mathcal{A}^{\bullet} := \oplus_p \mathcal{E}^{p,\bullet}$ the $\overline{\partial}$ operator defines a graded derivation, instead of taking d as the graded derivation. We define $\tilde{\wedge} : \mathcal{A}^{\bullet} \otimes_{\mathbb{C}_X} \mathcal{A}^{\bullet} \to \mathcal{A}^{\bullet}$ as $\omega^{p,q} \tilde{\wedge} \tilde{\omega}^{p',q'} = (-1)^{q'p} \omega^{p,q} \wedge \tilde{\omega}^{p',q'}$.
To verify that $\overline{\partial}$ is indeed a graded derivation, we compute

$$\overline{\partial}(\omega^{p,q} \tilde{\wedge} \tilde{\omega}^{p',q'}) = (-1)^{q'p} \overline{\partial}(\omega^{p,q} \wedge \tilde{\omega}^{p',q'}) = (-1)^{q'p}(\overline{\partial}\omega^{p,q} \wedge \tilde{\omega}^{p',q'} + (-1)^{p+q}\omega^{p,q} \wedge \overline{\partial}\tilde{\omega}^{p',q'})$$

$$= \overline{\partial}\omega^{p,q} \tilde{\wedge} \tilde{\omega}^{p',q'} + (-1)^q \omega^{p,q} \tilde{\wedge} \overline{\partial}\tilde{\omega}^{p',q'}.$$

Hence, we obtain a cup product on $\mathbb{H}^{\bullet}(X, \mathcal{A}^{\bullet}) \otimes_{\mathbb{C}} \mathbb{H}^{\bullet}(X, \mathcal{A}^{\bullet}) \to \mathbb{H}^{\bullet}(X, \mathcal{A}^{\bullet})$ with respect to the Čech-differential and $\overline{\partial}$, and this endows $\mathbb{H}^{\bullet}(X, \mathcal{A}^{\bullet})$ with a $\mathbb{C}$-algebra structure.

Let $\mathcal{U}$ be an open covering of X such that any finite intersection is Stein. There exist two natural morphisms the inclusion of holomorphic forms into smooth forms $i : \check{C}^{\bullet}(\mathcal{U}, \oplus_p \Omega^p) \to \check{C}^{\bullet}(\mathcal{U}, \mathcal{A}^{\bullet})$ and the restriction $r : \mathcal{A}^{\bullet}(X) \to \check{C}^{\bullet}(\mathcal{U}, \mathcal{A}^{\bullet})$. Given the ring structure on the hypercohomology induced from the wedge product on $\oplus_p \Omega^p$, the inclusion is a $\mathbb{C}$-algebra morphism. The restriction morphism $s \mapsto (s|_{U_\alpha})_\alpha$ is also a $\mathbb{C}$-algebra morphism.

For fixed p, by a spectral sequence calculation in the double complex $\check{C}^{\bullet}(\mathcal{U}, \mathcal{A}^{\bullet})$, we get isomorphisms induced by i, r respectively

$$\check{H}^q(X, \Omega^p) \cong H^{p,q}(X, \mathbb{C}) \cong H^q_D(\check{C}^{\bullet}(\mathcal{U}, \mathcal{E}^{p,\bullet})),$$

where D is the total differential of the double complex. Hence we find a $\mathbb{C}$-algebra isomorphism

$$\oplus_{p,q} \check{H}^q(X, \Omega^p) \cong \oplus_{p,q} H^{p,q}(X, \mathbb{C}) \cong \oplus_q H^q_D(\check{C}^{\bullet}(\mathcal{U}, \mathcal{A}^{\bullet})).$$

Here the cup product on $\oplus_{p,q} H^{p,q}(X, \mathbb{C})$ is induced by $\tilde{\wedge}$ instead of $\wedge$. Therefore we obtain

$$\phi(\check{H}^q(X, \Omega^p)) \wedge \phi(\check{H}^{q'}(X, \Omega^{p'})) = (-1)^{pq'} \phi(\check{H}^q(X, \Omega^p) \cdot \check{H}^{q'}(X, \Omega^{p'}))$$

if we return to the ordinary wedge product. □

The splitting principle can thus be applied and gives the following definition of Chern classes for a vector bundle.

DEFINITION 6.32. *Taking $p = q = r$, $k = 2r$, there are unique elements $c_i \in H^{i,i}_{BC}(X, \mathbb{Z})$, such that*

$$\omega^r + \sum (-1)^i \pi^*(c_i) \cdot \omega^{r-i} = 0$$

where $\omega = c_1(\mathcal{O}(1))$ by the above proposition 6.4.1. We define the Chern classes of a vector bundle E in the integral Bott-Chern cohomology to be precisely the c_i.

We now prove some elementary properties of Chern classes in the integral Bott-Chern cohomology. In particular, we check that axiom A of the introduction holds. Let us first observe that such Chern classes are unique, since they satisfy the Grothendieck axioms for Chern classes included in Axiom A.

PROPOSITION 6.4.2. (functoriality of Chern classes) *Let $f : X \to Y$ be a holomorphic morphism between two compact complex manifolds, and E be a holomorphic vector bundle of rank r over Y, Then we have*

$$f^*(c_k(E)) = c_k(f^*(E)).$$

PROOF. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}(1)|_{\mathbb{P}(f^*(E))} & \longrightarrow & \mathcal{O}(1)|_{\mathbb{P}(E)} \\
\downarrow & & \downarrow \\
\mathbb{P}(f^*(E)) & \xrightarrow{\tilde{f}} & \mathbb{P}(E) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
X & \xrightarrow{f} & Y,
\end{array}$$

which in particular shows that $\tilde{f}^*(\mathcal{O}(1)|_{\mathbb{P}(E)}) = \mathcal{O}(1)|_{\mathbb{P}(f^*(E))}$. By the functoriality of the first Chern class (directly obtained from its construction), we get $\tilde{f}^*(c_1(\mathcal{O}(1)|_{\mathbb{P}(E)})) = c_1(\mathcal{O}(1)|_{\mathbb{P}(f^*(E))})$. By the definition of Chern classes, we have an equality

$$\sum (-1)^s c_1^{r-s}(\mathcal{O}(1)|_{\mathbb{P}(E)}) \cdot \pi^*(c_s(E)) = 0.$$

Hence $\sum (-1)^s \tilde{f}^*(c_1(\mathcal{O}(1)|_{\mathbb{P}(E)}))^{r-s} \cdot \tilde{f}^*(\pi^*(c_s(E))) = 0$, from which the definition of Chern classes yields $f^*(c_k(E)) = c_k(f^*(E))$. □

The next property is the Whitney formula.

PROPOSITION 6.4.3. *Let $0 \to E \to F \to G \to 0$ be a short exact sequence of holomorphic vector bundles. Then we have* $\mathrm{ch}(E) + \mathrm{ch}(G) = \mathrm{ch}(F)$ *and* $c(E) \cdot c(G) = c(F)$.

PROOF. On $X \times \mathbb{P}^1$, there exists a short exact sequence of holomorphic vector bundles

$$0 \to \tilde{E} \to \tilde{F} \to \tilde{G} \to 0,$$

such that the restriction of exact sequence on the complex submanifold $X \times \{0\}$ is $0 \to E \to F \to G \to 0$ and the restriction on $X \times \{\infty\}$ is $0 \to E \to E \oplus G \to G \to 0$. The existence of such a sequence can be found for example in [**Sou92**]. In the case of a direct sum, we obviously have the formulas $\mathrm{ch}(G) + \mathrm{ch}(E) = \mathrm{ch}(E \oplus G)$ and $c(E \oplus G) = c(E) \cdot c(G)$ by the splitting principle (cf. section 21 [**BT82**]).

On the other hand, we have the following commutative diagram for every point $a \in \mathbb{P}^1$:

$$\begin{array}{ccc}
X \times \mathbb{P}^1 & \xrightarrow{\pi} & X \, . \\
{\scriptstyle i_a}\uparrow & \nearrow{\scriptstyle \mathrm{Id}} & \\
X & &
\end{array}$$

The identity element of the ring $\oplus_{k,p,q} \mathbb{H}^k(X, \mathcal{B}^\bullet_{p,q,\mathbb{Z}})$ is the element in $\mathbb{H}^0(X, \mathcal{B}^\bullet_{0,0,\mathbb{Z}})$ represented by the constant $1 \in \mathbb{Z}(0)(X)$ (more precisely the 0-cocycle $1 \in \mathbb{Z}(0)(U_i)$ for each U_i in the open covering). Via the quasi-isomorphism, it can also be represented by the integral current associated with X. We denote this element by Id_X. By the projection formula we have for every $\alpha \in \mathbb{H}^\bullet(X \times \mathbb{P}^1)$ that

$$\pi_*(i_{a*}(\mathrm{Id}_X) \cdot \alpha) = \pi_*(i_{a*}(\mathrm{Id}_X \cdot i_a^*(\alpha)) = \pi_*(i_{a*}(i_a^*(\alpha)) = \mathrm{Id}_*(i_a^*(\alpha)) = i_a^*(\alpha).$$

By the functoriality of Chern classes, we thus find

$$\pi_*(i_{0*}(\mathrm{Id}_X) \cdot (\mathrm{ch}(\tilde{G}) + \mathrm{ch}(\tilde{E}) - \mathrm{ch}(\tilde{F}))) = (\mathrm{ch}(\tilde{G}) + \mathrm{ch}(\tilde{E}) - \mathrm{ch}(\tilde{F}))|_{X \times \{0\}} = \mathrm{ch}(G) + \mathrm{ch}(E) - \mathrm{ch}(F),$$

$$\pi_*(i_{\infty*}(\mathrm{Id}_X) \cdot (\mathrm{ch}(\tilde{G}) + \mathrm{ch}(\tilde{E}) - \mathrm{ch}(\tilde{F}))) = (\mathrm{ch}(\tilde{G}) + \mathrm{ch}(\tilde{E}) - \mathrm{ch}(\tilde{F}))|_{X \times \{\infty\}} = 0.$$

To prove the Whitney formula, it is enough to prove the following homotopy property: let $i_a : X \hookrightarrow X \times \mathbb{P}^1$ be the inclusion into the complex submanifold $X \times \{a\}$, then $i_a^*(\alpha)$ is independent of the choice of a.

Since $X \times \{a\}$ is a codimension 1 analytic set in $X \times \mathbb{P}^1$, its associated integral current defines a global section of $\mathcal{I}_X^2$. Since $[X \times \{a\}]$ is of type (1,1), it projects to zero in $H^0(X, \sigma_{1,\bullet}\mathcal{D}_X^{'\bullet,\bullet} \oplus \sigma_{\bullet,1}\mathcal{D}_X^{'\bullet,\bullet})$. Hence $2\pi\sqrt{-1}[X \times \{a\}]$ defines a hypercocycle for the integral Bott-Chern complex $\mathcal{B}^{\bullet}_{1,1,\mathbb{Z}}$. By the construction of the push-forward, this element represents $i_{a*}(\mathrm{Id}_X)$. In the following we denote $i_{a*}(\mathrm{Id}_X)$ as $\{2\pi\sqrt{-1}[X \times \{a\}]\}$ (which is just the cycle class defined in the next section). With this notation, we have proved the equality

$$\mathrm{ch}(G) + \mathrm{ch}(E) - \mathrm{ch}(F) = \pi_*(2\pi\sqrt{-1}(\{[X \times \{0\}]\} - \{[X \times \{\infty\}]\}) \cdot (\mathrm{ch}(\tilde{G}) + \mathrm{ch}(\tilde{E}) - \mathrm{ch}(\tilde{F}))).$$

We denote by z the parameter in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and by $[0, \infty]$ a (real) line connecting 0 and ∞ in $\mathbb{P}^1$ (for example we can take the positive real axis). Then the function $\ln z$ is well defined on $\mathbb{P}^1 \smallsetminus [0, \infty]$. $X \times [0, \infty]$ is a real codimension one real analytic set of $X \times \mathbb{P}^1$, so it well defines a locally integral current. As a current $d([X \times [0, \infty]]) = -[X \times 0] + [X \times \infty]$. For any smooth form of type $(n+1, n)$ with compact support where n is the complex dimension of X

$$\langle \bar{\partial} \ln z, \phi^{n+1,n} \rangle = -\langle \ln z, \bar{\partial}\phi^{n+1,n} \rangle = -\int_{X \times [0,\infty]^+ - X \times [0,\infty]^-} \ln z \cdot \phi = -2i\pi \int_{X \times [0,\infty]} \phi.$$

The second equality is a consequence of the Stokes formula. It shows that $\mathrm{pr}_{0,1}([X \times [0, \infty]]) = -\frac{1}{2\pi i}\bar{\partial} ln(z)$. Similarly $\mathrm{pr}_{1,0}([X \times [0, \infty]]) = -\frac{1}{2\pi i}\partial ln(\bar{z})$. Therefore, in the space of global sections of the mapping cone $\mathrm{Cone}(\Delta)^{\bullet}[-1](X \times \mathbb{P}^1)$ for $p = 1, q = 1$, we have

$$([X \times \{0\}] - [X \times \{\infty\}], 0) = \delta(X \times [0, \infty], \frac{1}{2\pi\sqrt{-1}} \ln \bar{z} \oplus -\frac{1}{2\pi\sqrt{-1}} \ln z),$$

where δ is the differential of the integral Bott-Chern complex. In other words, $[X \times \{0\}] - [X \times \{\infty\}]$ is exact, and this means that $\mathrm{ch}(G) + \mathrm{ch}(E) - \mathrm{ch}(F) = 0$ in the integral Bott-Chern cohomology class. The proof of the total Chern class formula is similar.

(It would be more direct to conclude that the class of $-[X \times 0] + [X \times \infty]$ is 0 in the complex Bott-Chern cohomology. Using a resolution by currents, this is equivalent to show that as currents on $X \times \mathbb{P}^1$, $-[X \times 0] + [X \times \infty]$ is $\partial\bar{\partial}$−exact. However, notice that

$$-[X \times 0] + [X \times \infty] = -i\partial\bar{\partial}([X] \ln |z|)$$

where we view z as a meromorphic function on $\mathbb{P}^1$ with a single zero at 0 and a single pole at infinity.) □

6.5. Cohomology class of an analytic set

To prove the other axioms, we have to study the transformation of cohomology groups under what appears to be the "wrong" direction. For example the pull back of a cohomology class represented by the closed current associated with a cycle should morally be represented by the pull back of this current, but such pull backs are not always well defined. In this section, given an irreducible analytic cycle Z of codimension k in X, we will associate to it a cycle class in the integral Bott-Chern cohomology $H^{k,k}_{BC}(X, \mathbb{Z})$. Then we will prove a number of elementary properties of this type of cycle classes. In particular, the projection formula, the transformation formula of a cycle class under a morphism will be established (Axiom B (3)). At the end, we will deduce the commutativity property of pull back and push forward by projections and inclusions, according to Axiom B (4). The excess formula (Axiom B(5)) is a direct consequence, using the standard deformation technique of the normal bundle.

To show that, in certain cases, the pull back of a current representing a class induces a well defined map in cohomology, we bypass the difficulty by showing a corresponding formula for the Bloch cycle class, which takes values in local cohomology. We make this choice since locally the Bloch cycle class can be given explicitly, and its pull back can also be made explicit.

Cohomology with support is involved since cycle classes can be represented in a natural way by currents associated which the cycle. These are in fact supported in the given analytic sets, whence the appearance of cohomology with support.

With this refinement, technically, we can show that before taking the hypercohomology, the complex $R\Gamma_Z(X, \mathcal{O}_X)$ can be centered at the degree we want. Hence the related spectral sequences degenerate. This allows us to glue local sections into global ones to define the Bloch cycle class.

Attention should be paid to the fact that the Bloch cycle class lies in the derived category of $\mathcal{O}_X$-modules, while the integral Bott-Chern complex lies in the derived category of sheaves of abelian groups $D(\mathrm{Sh}(X))$.

In this section, we denote $\mathbb{H}^{\bullet}_{|Z|}(X, \bullet)$ or $\mathbb{H}^{\bullet}_{Z}(X, \bullet)$ the local hypercohomology class of some complex on X with support in Z.

6.5.1. Definition of cycle classes. We start by defining a cycle class in the integral Bott-Chern cohomology. This is an analogue of the cycle class in integral Deligne cohomology that has been defined in [**ZZ84**]. As before, we denote by $\Delta : \mathbb{C}_X \to \sigma_p\Omega_X^\bullet \oplus \sigma_q\overline{\Omega_X}^\bullet$.

For any p, q, we have the following commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{B}^\bullet_{p,q,\mathbb{Z}} & \longrightarrow & \mathcal{B}^\bullet_{p,q,\mathbb{C}} & \longrightarrow & \mathbb{C}_X/\mathbb{Z}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_X & \longrightarrow & \mathbb{C}_X & \longrightarrow & \mathbb{C}_X/\mathbb{Z}_X & \longrightarrow & 0. \end{array}$$

The vertical morphism of complexes consists of forgetting the terms with degree > 0. It induces the following diagram with exact lines for $p = q = k$.

$$\begin{array}{ccccccc} \mathbb{H}^{2k-1}_{|Z|}(X, \mathbb{C}_X/\mathbb{Z}_X) & \longrightarrow & \mathbb{H}^{2k}_{|Z|}(X, \mathcal{B}^\bullet_{k,k,\mathbb{Z}}) & \longrightarrow & \mathbb{H}^{2k}_{|Z|}(X, \mathcal{B}^\bullet_{k,k,\mathbb{C}}) & \longrightarrow & \mathbb{H}^{2k}_{|Z|}(X, \mathbb{C}_X/\mathbb{Z}_X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^{2k-1}_{|Z|}(X, \mathbb{C}_X/\mathbb{Z}_X) & \longrightarrow & \mathbb{H}^{2k}_{|Z|}(X, \mathbb{Z}_X) & \longrightarrow & \mathbb{H}^{2k}_{|Z|}(X, \mathbb{C}_X) & \longrightarrow & \mathbb{H}^{2k}_{|Z|}(X, \mathbb{C}_X/\mathbb{Z}_X). \end{array}$$

The first and fourth vertical arrow are the identity map. By the Poincaré duality for cohomology with support we know

$$\mathbb{H}^{2k-1}_{|Z|}(X, \mathbb{C}_X/\mathbb{Z}_X) \cong \mathbb{H}_{2n-2k+1}(Z, \mathbb{C}_X/\mathbb{Z}_X) = 0$$

where the second equality comes from the fact that the real dimension of Z is $2n - 2k$.

By chasing the diagram, we know for any elements $a \in \mathbb{H}^{2k}_{|Z|}(X, \mathcal{B}^\bullet_{k,k,\mathbb{C}})$ and $b \in \mathbb{H}^{2k}_{|Z|}(X, \mathbb{Z}_X)$ such that their images in $\mathbb{H}^{2k}_{|Z|}(X, \mathbb{C}_X)$ are the same, then there exists a unique element in $H^{2k}_{|Z|}(X, \mathcal{B}^\bullet_{k,k,\mathbb{Z}})$ such that the image of this element is a, b respectively.

To define the cycle class, it is thus enough to associate the cycle two elements in $\mathbb{H}^{2k}_{|Z|}(X, \mathcal{B}^\bullet_{k,k,\mathbb{C}})$, $\mathbb{H}^{2k}_{|Z|}(X, \mathbb{Z}_X)$ such that their image in $\mathbb{H}^{2k}_{|Z|}(X, \mathbb{C}_X)$ is the same. The cycle Z defines a global section in $H^0(X, \mathcal{I}^{2k}_X)$ so it represents an element in $\mathbb{H}^{2k}_{|Z|}(X, \mathbb{Z}_X)$. The inclusion $\mathbb{Z}_X \to \mathbb{C}_X$ induces in the derived category a morphism $\mathcal{I}^\bullet_X \to \mathcal{D}'^\bullet_X$. These two quasi-isomorphic morphisms induce the same morphism when passing to hypercohomology. The cycle class in $\mathbb{H}^{2k}_{|Z|}(X, \mathbb{Z}_X)$ associated with Z has an image in $\mathbb{H}^{2k}_{|Z|}(X, \mathbb{C}_X)$ represented also by the integral current associated with Z.

On the other hand, $\mathbb{C}_X$ is quasi-isomorphic to the complex $\mathcal{D}'^\bullet_X$. The complex Bott-Chern complex is quasi isomorphic to the mapping cone $C(q)^\bullet[-1]$ with the natural map $q : \mathcal{D}'^\bullet_X \to \sigma_{k,\bullet}\mathcal{D}'^{\bullet,\bullet}_X \oplus \sigma_{\bullet,k}\mathcal{D}'^{\bullet,\bullet}_X$ with a negative sign on the second component. The integral current associated with Z defines a global section of $H^0(X, \mathcal{D}'^\bullet_X)$ of bidegree (k, k). And its image in $H^0(X, \sigma_{k,\bullet}\mathcal{D}'^{\bullet,\bullet}_X \oplus \sigma_{\bullet,k}\mathcal{D}'^{\bullet,\bullet}_X)$ is 0. This means in particular that the integration current defines a hypercocycle. Here the hypercohomology class can be represented by this global section since the sheaf of currents is acyclic. Hence the integration current $([Z], 0 \oplus 0)$ represents an element in $\mathbb{H}^{2k}_{|Z|}(X, \mathcal{B}^\bullet_{k,k,\mathbb{C}})$. Under the forgetting map $\mathcal{B}^\bullet_{k,k,\mathbb{C}} \to \mathbb{C}_X$, its image in $\mathbb{H}^{2k}_{|Z|}(X, \mathbb{C}_X)$ can also be represented by the same integration current $[Z]$.

In conclusion, the cycle class associated with Z in $\mathbb{H}^{2k}_{|Z|}(X, \mathcal{B}^\bullet_{k,k,\mathbb{Z}})$ is exactly the class of the integral current associated with Z view as an element in $H^{2k}_{|Z|}(X, \mathrm{Cone}(\tilde{q})^\bullet[-1])$ with $\tilde{q} : \mathcal{I}^\bullet_X \to \sigma_{k,\bullet}\mathcal{D}'^{\bullet,\bullet}_X \oplus \sigma_{\bullet,k}\mathcal{D}'^{\bullet,\bullet}_X$. The image under the canonical map $\mathbb{H}^{2k}_{|Z|}(X, \mathcal{B}^\bullet_{k,k,\mathbb{Z}}) \to \mathbb{H}^{2k}(X, \mathcal{B}^\bullet_{k,k,\mathbb{Z}})$ defines finally the cycle space associated with Z represented by the same integration current. (This construction is already used in the proof of the Whitney formula.) We denote in the following the cycle class associated with Z as $\{[Z]\}$.

Notice that $i_{Z*}1 = \{[Z]\}$ where $1 \in H^{0,0}_{BC}(Z, \mathbb{Z})$ the identity in $\oplus_{p,q} H^{p,q}_{BC}(Z, \mathbb{Z})$. The identity in $\oplus_{p,q} H^{p,q}_{BC}(Z, \mathbb{Z})$ corresponds a global constant section $1 \in \Gamma(Z, \mathbb{Z}_Z)$ whose image under i_{Z*} in the hypercohomology is defined by locally integral current $[Z]$ by the construction of the push forward. This global current represents the cycle space $\{[Z]\}$ on X.

Now we prove some properties of cycle classes. We start by the following lemma which expresses the push forward of a cohomology class by an arbitrary morphism in terms of the pull back and push forward of its projection, and a multiplication by the cycle class associated with the graph of the morphism.

LEMMA 6.33. *Let $f : X \to Y$ be a holomorphic map between complex manifolds. Assume X to be compact. Let α be an integral Bott-Chern cohomology class. Denote by Γ the graph of f in $X \times Y$ and by p_1, p_2 the two canonical projections. Then one has*

$$f_*\alpha = p_{2*}(p_1^*\alpha \cdot \{[\Gamma]\}).$$

PROOF. This can be checked directly using the multiplication structure as in the Deligne-Beilinson complex. The compactness condition is just used to ensure that the push-forward is well defined. Taking $[\Gamma]$ as the global representative of the cohomology class, the cup product is induced by the wedge product between the forms and locally integral currents at the level of complexes. We prove at the level of complexes that

$$f_*(\alpha) = p_{2*}(p_1^*(\alpha) \cup_0 [\Gamma]).$$

It suffices to check on germs on Y. Let U be an open set of X such that $U = f^{-1}(V)$ for some connected open set V of Y. There are two kinds of sheaves in the Deligne-Beilinson complex: locally constant sheaf in $\mathbb{Z}$ and sheaves of holomorphic forms.

Let $\alpha \in \Omega_Y^p(U)$. Let $\omega \in C^\infty_{(n-p,n),c}(U)$ be a smooth form with compact support in U. Then we have

$$\langle f_*\alpha, \omega\rangle = \langle \alpha, f^*\omega\rangle = \int_U \alpha \wedge f^*\omega = \int_{\Gamma\cap p_1^{-1}(U)} p_1^*\alpha \wedge p_1^* f^*\omega$$

$$= \int_{\Gamma\cap p_1^{-1}(U)} p_1^*\alpha \wedge p_2^*\omega = \langle p_{2*}(p_1^*(\alpha) \cup_0 [\Gamma]), \omega\rangle.$$

Notice that p_1 induces a biholomorphism between $\Gamma \cap p_1^{-1}(U)$ and U.

For $c \in \mathbb{Z}_X(U)$, its image under f_* via the quasi-isomorphism is the local integral current $cf_*[U]$. The equality at the level of complexes is just

$$cf_*[U] = p_{2*}(c[\Gamma \cap p_2^{-1}(U)]) = p_{2*}(p_1^*(c) \cup_0 [\Gamma]).$$

Passing to hypercohomology gives the desired equality. □

As in [**Gri10**], we have the following property. It is a combination of the above lemma and the pull back of the cycle class under a closed immersion (the proof will be postponed to the next subsection).

PROPOSITION 6.5.1. *Let $f : X \to Y$ be a surjective proper map between compact manifolds, and let D be a divisor of Y. We denote $f^*D = m_1\tilde{D}_1 + \cdots + m_N\tilde{D}_N$. Let $\tilde{f}_i : \tilde{D}_i \to D$ $(1 \leqslant i \leqslant N)$ be the restriction of f to $\tilde{D}_i$. Then we have*

$$f^* i_{D*} = \sum_{i=1}^N m_i\, i_{\tilde{D}_i *}\tilde{f}_i^*.$$

PROOF. The proof is identical to the case of the Deligne complex. For self-containedness, we give briefly the details. The idea consists of passing to the graph and using the above lemma. Since all spaces are compact, the push-forward is always well-defined. Let Γ be the graph of $i_D : D \hookrightarrow Y$ and let $\tilde{\Gamma'}_i$ be the graph of $i_{\tilde{D'}_i} : \tilde{D'}_i \hookrightarrow X$. We denote all terms involving X with a prime symbol $'$ and all other terms without that symbol. By definition, $[\Gamma'_i] := (\tilde{f}_i, \mathrm{id})_*[\tilde{\Gamma}'_i]$ as current which induces as cycle class $\{[\Gamma'_i]\} = (\tilde{f}_i, \mathrm{id})_*\{[\tilde{\Gamma}'_i]\}$. $[\Gamma'_i]$ is supported in the image of $(\tilde{f}_i, \mathrm{id})$. We denote by p_j $(j = 1, 2)$ the natural projections of $D \times Y$, by p'_j projections of $D \times X$, and by $\tilde{p}'_{j,i}$ projections of $\tilde{D}_i \times X$.

In terms of currents, we have $(\mathrm{id}, f)^*[\Gamma] = \sum_{i=1}^N m_i[\Gamma'_i]$. We can prove the Bloch cycle class equality $(\mathrm{id}, f)^*\{[\Gamma]\} = \sum_{i=1}^N m_i\{[\Gamma'_i]\}$. The proof will be given in Lemma 6.46. Then we have

$$f_* i_{D*}\alpha = f^* p_{2*}(p_1^*\alpha \cdot \{[\Gamma]\}) = p'_{2*}(\mathrm{id}, f)^*(p_1^*\alpha \cdot \{[\Gamma]\})$$

$$= p'_{2*}((\mathrm{id}, f)^* p_1^*\alpha \cdot (\mathrm{id}, f)^*\{[\Gamma]\}) = \sum_{i=1}^N m_i p'_{2*}(p_1'^*\alpha \cdot \{[\Gamma'_i]\})$$

$$= \sum_{i=1}^N m_i p'_{2*}(\tilde{f}_i, \mathrm{id})_*((\tilde{f}_i, \mathrm{id})^* p_1'^*\alpha \cdot \{[\tilde{\Gamma}'_i]\}) = \sum_{i=1}^N m_i \tilde{p'}_{2,i*}(\tilde{p'}^*_{1,i}\tilde{f}_i^{\,*}\alpha \cdot \{[\tilde{\Gamma}'_i]\})$$

$$= \sum_{i=1}^N m_i\, i_{\tilde{D}_i *}\tilde{f}_i^*\alpha.$$

The first equality uses the lemma 6.33. The second formula uses the proposition 6.2.2 for $f \circ p'_2 = p_2 \circ (\mathrm{id}, f)$. The third equality uses the fact that pull-back is a ring morphism. The fourth equality uses the fact that $p'_1 = p_1 \circ (\mathrm{id}, f)$. The fifth equality uses the projection formula. The sixth equality uses the fact that $\tilde{p'}_{2,i} = (\tilde{f}_i, \mathrm{id}) \circ p'_2$ and $\tilde{f}_i \circ \tilde{p'}_{1,i} = p'_1 \circ (\tilde{f}_i, \mathrm{id})$. The last equality uses another time lemma 6.33. The surjectivity of f is just used to ensure that the pull-back of a divisor is a divisor. □

We give an easy generalisation of a lemma in [**Sch07**]. It gives the expected relation between the integral Bott-Chern cohomology and the Deligne cohomology. In particular, one can reduce the relevant properties of cycle classes in the integral Bott-Chern cohomology to the Deligne complex case, when they only involve the group structure.

LEMMA 6.34. *For any $p \geqslant 1$, we have a $\mathbb{Z}$-module isomorphism*

$$H^{p,p}_{BC}(X,\mathbb{Z}) \simeq H^{2p}_D(X,\mathbb{Z}(p)) \oplus \overline{\mathbb{H}^{2p-1}(X,\Omega^\bullet_{<p})}.$$

Moreover, via the isomorphism, for any proper cycle Z in X, the cycle class $\{[Z]\}_{BC}$ associated with Z in the integral Bott-Chern cohomology corresponds to $(\{[Z]\}_D, 0)$, where $\{[Z]\}_D$ is the cycle class associated with Z in the Deligne cohomology.
This isomorphism is functorial with respect to pull backs.

PROOF. We have the short exact sequence

$$0 \to \overline{\Omega}^\bullet_{<p}[1] \to \mathcal{B}^\bullet_{p,p,\mathbb{Z}} \to \mathcal{D}(p)^\bullet \to 0.$$

We can prove as shown in [**Sch07**] that the short exact sequence is in fact split, so that we have an abelian group isomorphism

$$H^{p,p}_{BC}(X,\mathbb{Z}) \simeq H^{2p}_D(X,\mathbb{Z}(p)) \oplus \overline{\mathbb{H}^{2p-1}(X,\Omega^\bullet_{<p})}$$

by taking the hypercohomology. We have to transform the complex involving smooth forms into a cone complex involving currents. These complexes are quasi-isomorphic, so that the splitting induces a morphism of complexes in the derived category. However, we want to modify that splitting to relate the cycle spaces in our different cohomology theories (respectively Deligne and integral Bott-Chern).
Let A be the matrix

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

We use the construction for A given in the next remark which shows that the integral Bott-Chern complex is quasi-isomorphic to $\mathrm{Cone}(\mathcal{I}^\bullet_X \xrightarrow{(1,0)} \sigma_{p,\bullet}\mathcal{D}^{\bullet,\bullet}_X \oplus \sigma_{\bullet,p}\mathcal{D}^{\bullet,\bullet}_X)[-1]$. The Deligne complex is quasi-isomorphic to $\mathrm{Cone}(\mathcal{I}^\bullet_X \xrightarrow{\mathrm{pr}_{p,\bullet}} \sigma_{p,\bullet}\mathcal{D}^{\bullet,\bullet}_X)[-1]$. There exists a splitting morphism given by for any element $(a,b) \in \mathcal{I}^k_X \oplus \sigma^{k-1}_{p,\bullet}\mathcal{D}^{\bullet,\bullet}_X$ by

$$F : \mathrm{Cone}(\mathcal{I}^\bullet_X \xrightarrow{\mathrm{pr}_{p,\bullet}} \sigma_{p,\bullet}\mathcal{D}^{\bullet,\bullet}_X)[-1] \to \mathrm{Cone}(\mathcal{I}^\bullet_X \xrightarrow{(1,0)} \sigma_{p,\bullet}\mathcal{D}^{\bullet,\bullet}_X \oplus \sigma_{\bullet,p}\mathcal{D}^{\bullet,\bullet}_X)[-1]$$

$$(a,b) \mapsto (a,b,0).$$

We verify that it is a morphism of complexes:

$$F(d(a,b)) = F(-da, \mathrm{pr}_{p,\bullet}a + \partial b) = (-da, \mathrm{pr}_{p,\bullet}a + \partial b, 0)$$

$$= d(F(a,b)) = d(a,b,0) = (-da, \mathrm{pr}_{p,\bullet}a + \partial b, \bar\partial 0).$$

Via this splitting isomorphism the cycle space associated with an analytic set Z is the cohomology class represented by $[Z]$ and $([Z],0)$ respectively. Thus the image of the cycle class $\{[Z]\}_D$ under F is $\{[Z]\}_{BC}$.
The functoriality comes from the functoriality of the construction given in the remark. □

REMARK 6.35. The sign in the definition of the integral Bott-Chern complex is unimportant for the group structure of the integral Bott-Chern cohomology when $p = q$. In fact, up to an isomorphism of abelian group, we can change the vector $(1,-1)$ to be any non zero vector in $\mathbb{C}^2$. To do it, we need the following construction.

Recall that the integral Bott-Chern complex is $\mathrm{Cone}(\mathbb{Z} \xrightarrow{(+,-)} \Omega^\bullet_{<p} \oplus \overline{\Omega}^\bullet_{<p})[-1]$ the mapping cone of the morphism $\mathbb{Z} \xrightarrow{(+,-)} \Omega^\bullet_{<p} \oplus \overline{\Omega}^\bullet_{<p}$. Let $A \in GL(2,\mathbb{C})$ be any invertible matrix. We denote by $a_{ij} (1 \leqslant i,j \leqslant 2)$ the elements of A. Then we have the following isomorphism of $\mathbb{Z}_X$-complex $\Omega^\bullet_{<p} \oplus \overline{\Omega}^\bullet_{<p}$. For any k, $(\omega_1,\omega_2) \in \Omega^k \oplus \overline{\Omega}^k$ sends to $(a_{11}\omega_1 + a_{12}\overline{\omega_2}, a_{21}\overline{\omega_1} + a_{22}\omega_2)$. The conjugation transforms the holomorphic forms to the anti-holomorphic forms and vice versa. (In fact it is $\mathbb{R}_X$-morphism not $\mathbb{C}_X$-morphism.) The inverse morphism is induced by the matrix A^{-1}.
Via this isomorphism of complex of $\mathbb{Z}_X$-sheaves, the integral Bott-Chern complex is isomorphic to

$$\mathrm{Cone}(\mathbb{Z} \xrightarrow{A(1,-1)^t} \Omega^\bullet_{<p} \oplus \overline{\Omega}^\bullet_{<p})[-1].$$

For any vector $(a,b) \in \mathbb{C}^2$, if we choose adequately A so that $(a,b)^t = A(1,-1)^t$, the integral Bott-Chern complex is isomorphic to $\mathrm{Cone}(\mathbb{Z} \xrightarrow{(a,b)} \Omega^\bullet_{<p} \oplus \overline{\Omega}^\bullet_{<p})[-1]$, which induces an isomorphism by passing to hypercohomology. This construction is functorial with respect to pull-backs, since the pull-back by a holomorphic map preserves the holomorphic forms and the anti-holomorphic forms.

This construction does not work for complex Bott-Chern cohomology since the isomorphism we have constructed is not complex linear.

The integral Bott-Chern complex is quasi-isomorphic to $\mathrm{Cone}(\mathcal{I}_X^\bullet \xrightarrow{\Delta} \sigma_{p,\bullet}\mathcal{D}_X^{\bullet,\bullet} \oplus \sigma_{\bullet,p}\mathcal{D}_X^{\bullet,\bullet})[-1]$. Via this quasi-isomorphism, the above construction gives an isomorphism of complexes

$$F : \mathrm{Cone}(\mathcal{I}_X^\bullet \xrightarrow{\Delta} \sigma_{p,\bullet}\mathcal{D}_X^{\bullet,\bullet} \oplus \sigma_{\bullet,p}\mathcal{D}_X^{\bullet,\bullet})[-1] \to \mathrm{Cone}(\mathcal{I}_X^\bullet \xrightarrow{A(1,-1)^t} \sigma_{p,\bullet}\mathcal{D}_X^{\bullet,\bullet} \oplus \sigma_{\bullet,p}\mathcal{D}_X^{\bullet,\bullet})[-1]$$

sending (a,b,c) to $(a, a_{11}b + a_{12}\bar{c}, a_{21}\bar{b} + a_{22}c)$. Here $A(1,-1)^t$ is the composition of Δ with the morphism given as in the above construction for $\sigma_{p,\bullet}\mathcal{D}_X^{\bullet,\bullet} \oplus \sigma_{\bullet,p}\mathcal{D}_X^{\bullet,\bullet}$ and A. Concretely for any k, the differential of $T \in \mathcal{I}_X{}^k$ sends to $(a_{11}\mathrm{pr}_{p,\bullet}T - a_{12}\mathrm{pr}_{p,\bullet}T, a_{21}\mathrm{pr}_{\bullet,p}T - a_{22}\mathrm{pr}_{\bullet,p}T)$ with value in $\sigma_{p,\bullet}\mathcal{D}_X^{\bullet,k} \oplus \sigma_{\bullet,p}\mathcal{D}_X^{k,\bullet}$. We check that A induces a morphism of complexes.

$$\begin{aligned} F(d(a,b,c)) &= F(-da, \mathrm{pr}_{p,\bullet}a + \partial b, -\mathrm{pr}_{\bullet,p}\bar{a} + \bar{\partial}c) \\ &= (-da, a_{11}\mathrm{pr}_{p,\bullet}a + a_{11}\partial b - a_{12}\overline{\mathrm{pr}_{\bullet,p}\bar{a}} + a_{12}\partial\bar{c}, a_{21}\overline{\mathrm{pr}_{p,\bullet}a} + a_{21}\bar{\partial}b - a_{22}\mathrm{pr}_{\bullet,p}\bar{a} + a_{22}\bar{\partial}c). \\ d(F(a,b,c)) &= d(a, a_{11}b + a_{12}\bar{c}, a_{21}\bar{b} + a_{22}c) \\ &= (-da, a_{11}\mathrm{pr}_{p,\bullet}a + a_{11}\partial b - a_{12}\mathrm{pr}_{p,\bullet}a + a_{12}\partial\bar{c}, a_{21}\mathrm{pr}_{\bullet,p}\bar{a} + a_{21}\bar{\partial}b - a_{22}\mathrm{pr}_{\bullet,p}\bar{a} + a_{22}\bar{\partial}c). \end{aligned}$$

In particular, since the cycle class associated with an analytic set Z is represented by the global section $([Z], 0 \oplus 0)$ where $[Z]$ is the current associated with Z, its image under the isomorphism is represented by the same section for any matrix A.

Now we return to the transformation of a cycle class under a morphism in the integral Bott-Chern cohomology.

LEMMA 6.36. *Let X be any complex manifold, Y and Z be compact submanifolds of X which intersect transversally and let $W = Y \cap Z$. Let $i_Y : Y \to X$ be the inclusion. Then we have in the integral Bott-Chern cohomology the equality*

$$i_Y^*\{[Z]\} = \{[W]\}.$$

PROOF. In this proof we denote $\{[Z]\}_{BC}$ for the cycle class associated with an analytic set Z in the integral Bott-Chern cohomology and $\{[Z]\}_D$ for the corresponding class in the Deligne cohomology. Via the isomorphism given in Lemma 6.34 and the functoriality, the equality $i_Y^*\{[Z]\}_{BC} = \{[W]\}_{BC}$ is equivalent to the equality $i_Y^*\{[Z]\}_D = \{[W]\}_D$. The proof in the Deligne complex case is given in the following via the Bloch cycle spaces. □

6.5.2. Deligne and Bloch cycle class. For self-containedness, we present here the general line of the proof of the equality in the Deligne complex case, as given in [**Gri10**]. We also need the local formula expressing Bloch cycle classes to complete the proof of Proposition 6.5.1. We start by recalling the Bloch cycle construction made in [**Blo72**] that was mentioned at the beginning of the section. The detour through Bloch cycle classes is organised as follows. First we recall the definition of the algebraic local cohomology groups and of the topological local cohomology groups. We show that for a coherent $\mathcal{O}_X$-module, these group are the same up to the forgetting functor. Secondly, we can give locally an explicit resolution of Čech-type complex of $\mathcal{O}_X$ in the algebraic local cohomology case. Next we show that associated with a cycle using the local resolution we can glue some local sections to a global section which is the Bloch cycle class associated with this cycle. Finally, we prove that under a suitable canonical map, the image of the Deligne cycle class is the Bloch cycle class associated with the same cycle. The author expresses warm thanks to Stéphane Guillermou for very interesting discussions on this subject.

Let X be a complex manifold and Z be an irreducible analytic subset of X of codimension d. Let $\mathcal{F}$ be a coherent sheaf on X. There are two notions of local cohomology with support in Z. A topological definition is the derived functor of the function “sheaf of sections with support in Z” given on any open subset $U \subset X$ by

$$\Gamma_Z(\mathcal{F})(U) := \{s \in \mathcal{F}(U) | \mathrm{supp}(s) \subset Z\}.$$

For every $x \in X$ and $s \in \mathcal{F}_x$, we have an induced $\mathcal{O}_x$-morphism $\mathcal{O}_x \to \mathcal{F}_x$ given by $f \mapsto f \cdot s$. The annihilator $\mathrm{Ann}(s)$ of s is defined to be the kernel of this morphism. The support of s is the zero variety $V(\mathrm{Ann}(s))$ of $\mathrm{Ann}(s)$; by the Nullstellensatz theorem, saying that the support is contained in Z is equivalent to the fact that $I_Z \subset \sqrt{\mathrm{Ann}(s)}$. Since the ideal sheaf associated with an analytic set is coherent, this is equivalent to the fact that $I_Z^n \subset \mathrm{Ann}(s)$ for $n \gg 0$, which amounts to say that $I_Z^n s = 0$ for $n \gg 0$. Next, this is equivalent to say that $\mathcal{O}_x \xrightarrow{\times s} \mathcal{F}_x$ factorise through $\mathcal{O}_x/\mathcal{I}_{Z,x}^n \xrightarrow{\times s} \mathcal{F}_x$ for some n. In other words,

$$\Gamma_Z(\mathcal{F}) = \varinjlim_{n\to\infty} \mathrm{Hom}(\mathcal{O}_X/\mathcal{I}_Z^n, \mathcal{F}).$$

The construction does not give the same equality if we replace the coherent sheaf $\mathcal{F}$ by a complex of coherent sheaves (in particular, when the complex is unbounded), or by an arbitrary sheaf. In this case, we must define the algebraic local cohomology sheaf supported in Z to be the derived functor of the sheaf $\Gamma_{[Z]}(\mathcal{F}) := \varinjlim_{n\to\infty} \mathrm{Hom}(\mathcal{O}_X/\mathcal{I}_Z^n, \mathcal{F})$. Since the direct limit functor is exact, we have

$$R^i\Gamma_{[Z]}(\mathcal{F}) := \varinjlim_{n\to\infty} \mathcal{E}xt^i(\mathcal{O}_X/\mathcal{I}_Z^n, \mathcal{F}).$$

We define the algebraic local cohomology sheaf complex with the same formula, after replacing the given sheaf $\mathcal{F}$ by a complex of sheaves.

Given an $\mathcal{O}_X$-complex $\mathcal{F}^\bullet$, we still have an injective morphism (but not necessarily an isomorphism) $\Gamma_{[Z]}(\mathcal{F}^\bullet) \to \Gamma_Z(\mathcal{F}^\bullet)$. The image of an element in $\Gamma_{[Z]}(\mathcal{F}^\bullet)$ is given by the image of the constant function 1 under the composition morphism $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_Z^n \to \mathcal{F}^\bullet$ for some n large enough such that $\mathcal{O}_X/\mathcal{I}_Z^n \to \mathcal{F}^\bullet$ is defined. We have the following local-to-global spectral sequence

$$E_2^{p,q} = H^p(X, R^q\Gamma_{[Z]}(\mathcal{F})) \Rightarrow H^{p+q}_{[Z]}(X, \mathcal{F}).$$

Here $H^i_{[Z]}(X, \mathcal{F}) := \varinjlim Ext^i_{\mathcal{O}_X}(\mathcal{O}_X/I_Z^k, \mathcal{F})$ is the algebraic local cohomology. We have similar spectral sequence for complex changing the cohomology by the hypercohomology.

We prove that $R^q\Gamma_{[Z]}(\mathcal{O}_X)$ is trivial for any $q \neq d$. The easy direction is a consequence of the following proposition [**Kas02**, Prop. 2.20].

PROPOSITION 6.5.2. *Let X be a non singular variety and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then for any $k < \mathrm{codim}(\mathrm{supp}(\mathcal{F}))$, we have $\mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = 0$.*

Use the proposition for $\mathcal{F} = \mathcal{O}_X/\mathcal{I}_Z^m$ for any m. We have $R^k\Gamma_{[Z]}(\mathcal{O}_X) = 0$ for any $k < d$ passing the direct limit. The converse direction needs to resolve the sheaf $\mathcal{O}_X/\mathcal{I}_Z^m$ by Koszul type complex. Assume that Z is a smooth submanifold or a locally complete intersection (this is the only case we need in the following) from which we can suppose locally $Z = V(f_1, \cdots, f_d)$. To start with, we notice that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$

$$\Gamma_{[Z]}(\mathcal{F}) = \varinjlim_{n\to\infty} \mathrm{Hom}(\mathcal{O}_X/(f_1^n, \cdots, f_d^n), \mathcal{F}).$$

This comes from the relation below, that holds for every n

$$\mathcal{I}_Z^{d(n-1)+1} \subset (f_1^n, \cdots, f_d^n) \subset \mathcal{I}_Z^n.$$

One can resolve locally $\mathcal{O}_X/(f_1^n, \cdots, f_d^n)$ by the Koszul complex $K_*(f_1^n, \cdots, f_d^n)[-d]$. For example, when $d = 1$, $\mathcal{O}_X/(f_1)$ is quasi-isomorphic to the complex $0 \to \mathcal{O}_X \xrightarrow{\times f_1} \mathcal{O}_X \to 0$ concentrated in degrees -1 and 0. Since $\varinjlim$ is an exact functor, for any $k > d$, we have

$$R^k\Gamma_{[Z]}(\mathcal{O}_X) = \varinjlim_{n\to\infty} R^k\,\mathrm{Hom}(\mathcal{O}_X/(f_1^n, \cdots, f_d^n), \mathcal{O}_X) = 0$$

where the last equality comes from the fact that each element is 0 even before taking the limit.

We describe the Bloch cycle class associated with Z in $H^d_{[Z]}(X, \Omega^d_X)$. Since Z is a local complete intersection and Ω^d is locally free, $R^q\Gamma_{[Z]}(\Omega^d) = 0$ for any $q \neq d$. Hence the local-to-global spectral sequence degenerates, and

$$H^d_{[Z]}(X, \Omega^d) \cong \Gamma(X, R^d\Gamma_{[Z]}(\Omega^d)).$$

As a consequence, it is enough to describe the cycle class locally, as the local representatives patch into a global section.

Let (U_i) be a Stein open covering of X such that $Z \cap U_i = \{f^1_{(i)} = \cdots = f^d_{(i)} = 0\}$ for $f^j_{(i)} \in \Gamma(U_i, \mathcal{O}_X)$. We need the following result.

LEMMA 6.37. *The direct limit of the dual of the Koszul complex $\mathrm{Hom}_{\mathcal{O}_X}(K_*((f^1_{(i)})^n, \cdots, (f^d_{(i)})^n), \mathcal{O}_X)$ on U_i is the extended Čech-type complex associated with Stein open covering of $U_i \setminus Z$ given by $V^j_{(i)} = \{f^j_{(i)} \neq 0\}$. More precisely the limit is*

$$\mathcal{O}_X \to \prod_{j_0} \mathcal{O}_X[\frac{1}{f^{j_0}_{(i)}}] \to \prod_{j_0<j_1} \mathcal{O}_X[\frac{1}{f^{j_0}_{(i)} f^{j_1}_{(i)}}] \to \cdots \to \mathcal{O}_X[\frac{1}{f^1_{(i)} \cdots f^d_{(i)}}]$$

with $\mathcal{O}_X$ at degree 0. In the following we will denote this complex by $\check{C}^\bullet(\mathcal{O}_X)$.

PROOF. On the one hand, the natural morphism between the duals of the Koszul type complexes, mapping $\mathrm{Hom}_{\mathcal{O}_X}(K_*((f^1_{(i)})^n, \cdots, (f^d_{(i)})^n), \mathcal{O}_X)$ to $\mathrm{Hom}_{\mathcal{O}_X}(K_*((f^1_{(i)})^{n+1}, \cdots, (f^d_{(i)})^{n+1}), \mathcal{O}_X)$, is given by sending $(e_{i_1} \wedge \ldots \wedge e_{i_p})^*$ to $f_{i_{p+1}} \cdots f_{i_r}(e_{i_1} \wedge \ldots \wedge e_{i_p})^*$ where the indices satisfy $\{1, \ldots, r\} = \{i_1, \ldots, i_r\}$. On the other hand, in general we know that if $\mathcal{F}^\bullet$ is a complex of $\mathcal{O}_U$-sheaves for every complex space U and if $f \in \mathcal{O}_U(U)$, the direct limit of the complex system $\cdots \to \mathcal{F}^\bullet \xrightarrow{\times f} \mathcal{F}^\bullet \to \cdots$ is $\mathcal{F}^\bullet[\frac{1}{f}]$. The isomorphism is given by sending s a local section in the i-th copy of $\mathcal{F}^\bullet$ to $\frac{s}{f^i}$. This completes the proof by combining the two facts.

Notice that in the analytic setting $\mathcal{O}_X[\frac{1}{f}]$ is not the same as $j_*\mathcal{O}_{X\smallsetminus V(f)}$ where j is the open immersion of $X \smallsetminus V(f)$ into X, since a holomorphic function on $X \smallsetminus V(f)$ can have essential singularities along $V(f)$. □

REMARK 6.38. Denote by $\mathrm{Sh}(X)$ the category of sheaves of abelian groups on X and by $C(\mathrm{Sh}(X))$ the category of complex of sheaves of abelian groups on X. Notice that Γ_Z is a left exact functor from $C(\mathrm{Sh}(X))$ to $C(\mathrm{Sh}(X))$. So it induces a right derived functor from $D(\mathrm{Sh}(X))$ to $D(\mathrm{Sh}(X))$. We denote by G the forgetting functor from $C(\mathrm{Mod}(\mathcal{O}_X))$ the category of complexes of quasi-coherent $\mathcal{O}_X$-module (that is the direct limit of a sequence of coherent $\mathcal{O}_X$-module) to $C(\mathrm{Sh}(X))$. For any coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$, we have

$$G \circ R\Gamma_{[Z]}(\mathcal{F}) = R\Gamma_Z \circ G(\mathcal{F}).$$

As we have seen above, the equality

$$G \circ \Gamma_{[Z]}(\mathcal{F}) = \Gamma_Z \circ G(\mathcal{F})$$

also holds. We further observe in general that for two functors A, B the relation $R(A \circ B) = RA \circ RB$ holds if for any injective object I we have $R^iA(B(I)) = 0$ for any $i > 0$. The forgetting functor is an exact functor, hence $R^iG = 0$ for any $i > 0$. We have

$$R(G \circ \Gamma_{[Z]})(\mathcal{F}) = RG \circ R\Gamma_{[Z]}(\mathcal{F}) = G \circ R\Gamma_{[Z]}(\mathcal{F}).$$

On the other hand, if I is an injective $\mathcal{O}_X$-module, I is flasque and so is $G(I)$. By **[Har77]** Chap III exercise 2.3, $R^i\Gamma_Z(G(I)) = 0$ for any $i > 0$. Hence we have

$$R(\Gamma_Z \circ G)(\mathcal{F}) = R\Gamma_Z \circ RG(\mathcal{F}) = R\Gamma_Z \circ G(\mathcal{F}).$$

In particular, $R^i\Gamma_Z(\mathcal{F})$ is also concentrated at degree d for any locally free $\mathcal{O}_X$-module.

Now we define a global section corresponding to the Bloch cycle by patching local sections. Locally the differential form

$$\frac{df^1_{(i)} \wedge \cdots \wedge f^d_{(i)}}{f^1_{(i)} \cdots f^d_{(i)}}$$

gives rise to a $(d-1)$-Čech-cocyle with value in Ω^d with respect to the open covering $V^j_{(i)}$, so it defines a section of $R^d\Gamma_{[Z]}(\Omega^d)$ on U_i by passing to the quotient. As in **[Gro62]** exposé 149, we have the following result.

LEMMA 6.39. *These sections can be shown to patch to a global section of $R^d\Gamma_{[Z]}(\Omega^d)$ which we will denote $\{[Z]\}_{Bl}$.*

PROOF. For any $z \in Z$, let $(f_1, \cdots, f_d), (\tilde{f}_1, \cdots, \tilde{f}_d)$ be two systems of generators near a neighbourhood of z. Then there exists $A \in GL(\mathcal{O}_z)$ such that $(\tilde{f}_1, \cdots, \tilde{f}_d) = (f_1, \cdots, f_d)A$. By the Gaussian elimination, the matrix A can be generated in perhaps a small open set by row-switching transformations, row-multiplying transformations and row-addition transformations with values in $\mathcal{O}_X$. Thus we reduce the check in these three cases. The sections are invariant under the row-switching transformations by anti-commutativity of Čech-complex and the anti-commutativity of differential forms.

If $(\tilde{f}_1, \cdots, \tilde{f}_d) = (f_1, \cdots, uf_d)$ with $u \in \mathcal{O}_z^\times$, we have

$$\frac{df_1 \wedge \cdots \wedge d(uf_d)}{f_1 \cdots (uf_d)} = \frac{df_1}{f_1} \wedge \cdots \wedge (\frac{df_d}{f_d} + \frac{du}{u}).$$

The difference corresponds to a Čech coboundary $\delta(0, \cdots, (-1)^d \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_{d-1}}{f_{d-1}} \wedge \frac{du}{u})$ since $\frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_{d-1}}{f_{d-1}} \wedge \frac{du}{u} \in \Omega^d(U)[\frac{1}{f_1 \cdots f_{d-1}}]$ for some open set U.

If $(\tilde{f}_1, \cdots, \tilde{f}_d) = (f_1, \cdots, vf_1 + f_d)$ with $v \in \mathcal{O}_z$, we have

$$\frac{1}{f_d + vf_1} = \sum_{k=0}^{\infty} (-1)^k \frac{v^k f_1^k}{f_d^{k+1}}$$

$$\frac{df_1 \wedge \cdots \wedge d(vf_1 + f_d)}{f_1 \cdots (vf_1 + f_d)} = \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_d}{f_d} + \sum_{k=1} \frac{(-1)^k v^k f_1^{k-1}}{f_d^{k+1}} df_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{d-1}}{f_{d-1}} \wedge df_d$$

$$+ \sum_{k=0} \frac{(-1)^k v^k f_1^k}{f_d^{k+1}} df_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{d-1}}{f_{d-1}} \wedge dv.$$

The difference corresponds to a Čech coboundary

$$\delta(\sum_{k=1} \frac{(-1)^k v^k f_1^{k-1}}{f_d^{k+1}} df_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{d-1}}{f_{d-1}} \wedge df_d + \sum_{k=0} \frac{(-1)^k v^k f_1^k}{f_d^{k+1}} df_1 \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{d-1}}{f_{d-1}} \wedge dv, 0, \cdots, 0).$$

□

Notice that by remark 6.38, the Bloch cycle class takes values in the (topological) local cohomology under the forgetting functor.

We now give the relation between the Bloch class and the Deligne cycle class. The complex Deligne complex is the mapping cone of $\mathrm{Cone}(\mathbb{C} \to \sigma_p \Omega^\bullet)[-1]$. Via the quasi-isomorphism in $D(\mathrm{Sh}(X))$, the complex Deligne complex is also isomorphic to the mapping cone $\mathrm{Cone}(q)^\bullet[-1]$ of the quotient map $q : \Omega_X^\bullet \to \sigma_p \Omega_X^\bullet$ by Dolbeault-Grothendieck lemma. We also have short exact sequence

$$0 \to F^p \Omega_X^\bullet \to \Omega_X^\bullet \to \sigma_p \Omega_X^\bullet \to 0$$

which shows in particular $F^p \Omega_X^\bullet \cong \mathrm{Cone}(q)^\bullet[-1]$. Hence we have in the derived category of sheaves of abelian groups an isomorphism

$$\mathcal{D}(p)_{\mathbb{C}}^\bullet \cong F^p \Omega_X^\bullet.$$

The exact sequence

$$0 \to F^{d+1}\Omega^\bullet \to F^d \Omega^\bullet \to \Omega^d[-d] \to 0$$

gives

$$\mathbb{H}_Z^{2d}(X, F^{d+1}\Omega^\bullet) \to \mathbb{H}_Z^{2d}(X, F^d\Omega^\bullet) \to H_Z^d(X, \Omega^d) \to \mathbb{H}_Z^{2d+1}(X, F^{d+1}\Omega^\bullet) \to \cdots$$

In the following, we show that the Deligne cycle class sends to the Bloch cycle class in the above long exact sequence. To prove the degeneration of some spectral sequence, we need the following lemma.

LEMMA 6.40. *Let $\mathcal{E}^\bullet : \cdots \to 0 \to E^0 \to \cdots \to E^p \to 0 \to \cdots$ be a complex of locally free $\mathcal{O}_X$-module of finite length $p+1$ in the category of complexes of abelian groups. Then $R\Gamma_Z(\mathcal{E}^\bullet)$ is quasi-isomorphic to the complex*

$$(*) \qquad (\cdots \to 0 \to R^d\Gamma_Z(E^0) \to \cdots \to R^d\Gamma_Z(E^p) \to 0 \to \cdots)[-d]$$

In particular, $R\Gamma_Z(F^p\Omega^\bullet)$ is quasi-isomorphic for every p to the complex

$$(\cdots \to 0 \to R^d\Gamma_Z(\Omega^p) \to \cdots \to R^d\Gamma_Z(\Omega^n) \to 0 \to \cdots)[-d],$$

where $R^d\Gamma_Z(\Omega^p)$ is placed at degree p.

PROOF. The proof is a consequence of an induction on the length of the complex. When the length is 1, the proof is straightforward by the fact that $R^q\Gamma_Z(\Omega^p)$ is concentrated at $q = d$. Assuming the assertion to hold for i, we denote by $\mathcal{E}^i$ the concatenation of terms $\mathcal{E}^\bullet$ up to degree i. We have a short exact sequence

$$0 \to E^{i+1}[-i-1] \to \mathcal{E}^{i+1} \to \mathcal{E}^i \to 0$$

which induces a distinguished triangle

$$\mathcal{E}^{i+1} \to \mathcal{E}^i \to E^{i+1}[-i] \xrightarrow{+1} .$$

Since $R\Gamma_Z$ converts distinguished triangles into distinguished triangles, we get a distinguished triangle

$$R\Gamma_Z(\mathcal{E}^{i+1}) \to R\Gamma_Z(\mathcal{E}^i) \to R\Gamma_Z(E^{i+1}[-i]) \xrightarrow{+1} .$$

By the induction assumption we get a quasi-isomorphism $R\Gamma_Z(\mathcal{E}^i) \cong (\cdots \to 0 \to R^d\Gamma_Z(E^0) \to \cdots \to R^d\Gamma_Z(E^i) \to 0 \to \cdots)[-d]$. Therefore, we see that $R\Gamma_Z(\mathcal{E}^{i+1})$ is quasi-isomorphic to the mapping cone of $(\cdots \to 0 \to R^d\Gamma_Z(E^0) \to \cdots \to R^d\Gamma_Z(E^i) \to 0 \to \cdots)[-d]$ to $R^d\Gamma_Z(E^{i+1})[-d]$, which proves the result. The particular case comes from the fact that the differential on X has maximal degree n. □

REMARK 6.41. In fact, one can show that the differential in the complex $(*)$ is induced by the differential of the complex $\mathcal{E}^\bullet$. Stéphane Guillermou indicated to us the following proof in a more general setting.

Let $\mathcal{E}^\bullet \in C(\mathrm{Sh}(X))$ such that for any i, one has $R\Gamma_Z(E^i) = H_Z^d(E^i)[-d]$ in the derived category $D(\mathrm{Sh}(X))$ of sheaves of abelian groups. Then we have a quasi-isomorphism

$$R\Gamma_Z(\mathcal{E}^\bullet) \cong (H_Z^d(E^0) \to \cdots \to H_Z^d(E^p))[-d]$$

where the differential map on the right is induced by the differential of $\mathcal{E}^\bullet$. Take an injective resolution for each E^i so as to obtain a double complex $I^\bullet$

$$\begin{array}{ccccc} I^{0,0} & \xrightarrow{\overline{\partial}^{0,0}} & I^{0,1} & \xrightarrow{\overline{\partial}^{0,1}} & \cdots \\ \downarrow \partial^{0,0} & & \downarrow \partial^{0,1} & & \\ I^{1,0} & \xrightarrow{\overline{\partial}^{1,0}} & I^{1,1} & \xrightarrow{\overline{\partial}^{1,1}} & \cdots \\ \downarrow \partial^{1,0} & & \downarrow \partial^{1,1} & & \\ \cdots & & \cdots & & \cdots \end{array}$$

Then $R\Gamma_Z(\mathcal{E}^\bullet) \cong \Gamma_Z(\mathrm{Tot}(I^{\bullet,\bullet}))$. Take $A^{\bullet,\bullet} = \Gamma_Z(I^{\bullet,\bullet})$, $B^{\bullet,\bullet} = \tau_{\leqslant d,\bullet}A^{\bullet,\bullet}$ and $C^{\bullet,\bullet} = \tau_{\geqslant d,\bullet}B^{\bullet,\bullet}$. Here $\tau_{\leqslant d,\bullet}, \tau_{\geqslant d,\bullet}A$ are the concatenation functors. More concretely, $B^{\bullet,\bullet}$ is

$$\begin{array}{ccccc} \Gamma_Z(I^{0,0}) & \xrightarrow{\Gamma_Z(\overline{\partial}^{0,0})} & \Gamma_Z(I^{0,1}) & \xrightarrow{\Gamma_Z(\overline{\partial}^{0,1})} & \cdots \\ \downarrow \Gamma_Z(\partial^{0,0}) & & \downarrow \Gamma_Z(\partial^{0,1}) & & \\ \Gamma_Z(I^{1,0}) & \xrightarrow{\Gamma_Z(\overline{\partial}^{1,0})} & \Gamma_Z(I^{1,1}) & \xrightarrow{\Gamma_Z(\overline{\partial}^{1,1})} & \cdots \\ \downarrow \Gamma_Z(\partial^{1,0}) & & \downarrow \Gamma_Z(\partial^{1,1}) & & \\ \cdots & & \cdots & & \cdots \end{array}$$

$$\begin{array}{ccccc} \mathrm{Ker}(\Gamma_Z(\partial^{d,0})) & \xrightarrow{\Gamma_Z(\overline{\partial}^{d,0})} & \mathrm{Ker}(\Gamma_Z(\partial^{d,1})) & \xrightarrow{\Gamma_Z(\overline{\partial}^{d,1})} & \cdots \\ \downarrow 0 & & \downarrow 0 & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

and $C^{\bullet,\bullet}$ is concentrated on the $(d+1)$th-line which is $(H^d_Z(E^0) \to \cdots \to H^d_Z(E^p))$. Since the concatenation functor preserves cohomology up to degree d, one can use the following lemma twice, for the pair $A^{\bullet,\bullet}, B^{\bullet,\bullet}$ and for the pair $B^{\bullet,\bullet}, C^{\bullet,\bullet}$, to conclude the result.

LEMMA 6.42. *Let $A^{\bullet,\bullet}, B^{\bullet,\bullet}$ two double complexes of sheaves of abelian groups. Let $u : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ be a morphism of double complex which induces an isomorphism of double complex*

$$H_{\overline{\partial}}H_\partial(A^{\bullet,\bullet}) \cong H_{\overline{\partial}}H_\partial(B^{\bullet,\bullet})$$

where $\partial, \overline{\partial}$ is the two differentials of the corresponding double complexes. (Since the morphism in the double complexes $H_{\overline{\partial}}H_\partial(A^{\bullet,\bullet}), H_{\overline{\partial}}H_\partial(B^{\bullet,\bullet})$ is in fact the zero morphism, the isomorphism of the double complex is the same as isomorphism of each term in the double complex.) Then we have an isomorphism of total complexes

$$\mathrm{Tot}(A^{\bullet,\bullet}) \cong \mathrm{Tot}(B^{\bullet,\bullet}).$$

The standard spectral sequence of double complexes gives

$$H^i(X, R^d\Gamma_Z((F^{d+1}\Omega^\bullet)^j)[-d]) \Rightarrow \mathbb{H}^{i+j}(X, R\Gamma_Z(F^{d+1}\Omega^\bullet)) = \mathbb{H}^{i+j}_Z(X, F^{d+1}\Omega^\bullet).$$

By reasoning on degrees and using the above lemma 6.40 to calculate the derived functor of the complex, one sees that the spectral sequence degenerates. Thus we find

$$\mathbb{H}^{2d}_Z(X, F^{d+1}\Omega^\bullet) = 0,$$

$$\mathbb{H}^{2d+1}_Z(X, F^{d+1}\Omega^\bullet) = \Gamma(X, R^d\Gamma_Z(\Omega^{d+1})).$$

The image of $\{[Z]\}_{Bl}$ under the boundary morphism is represented by the cocycles

$$d\left(\frac{df^1_{(i)} \wedge \cdots \wedge f^d_{(i)}}{f^1_{(i)} \cdots f^d_{(i)}}\right) = 0 \in \Gamma(U_i, R^d\Gamma_Z(\Omega^{d+1})).$$

Here we use remark 6.41, which ensures that the boundary morphism is induced by the standard differential of differential forms.

By the long exact sequence before Lemma 6.40, we know that the class $\{[Z]\}_{Bl}$ lifts to a unique class $\{[Z]\}_D$. In conclusion, the image of the Deligne cycle class under the natural morphism is the Bloch cycle class and the natural morphism is injective. In this way, to evaluate the transformation of a Deligne cycle class under a morphism, it is enough to evaluate the corresponding transformation of the Bloch cycle class.

REMARK 6.43. One can show that the Bloch cycle can also be represented by the global section $(2\pi i)^d[Z]$ of the current associated with the cycle Z. This is a direct consequence of the Lelong-Poincaré formula. Consider the extended Čech complex for the open covering $(V^j_{(i)})$ of $X \setminus Z$ as in the lemma 6.37. Notice that these open sets give an open covering of $X \setminus Z$ but at the degree 0, the component of complex is $\oplus_i \Omega^d_{|U_i}$. We resolve Ω^d by complex of currents $D^{d,\bullet}_X$ and we consider the total complex $\mathrm{Tot}(\check{C}^\bullet(D^{d,\bullet}_X))$. $\{(2\pi i)^d[Z \cap U_i]\}_i$ defines a element in $\check{C}^0(D^{d,d}_X)$. The differential is given by $\check{\delta} = (-1)^l(-1)^{k+1}\delta + \overline{\partial}$ on $\check{C}^k(D^{d,l}_X)$ where δ is the Čech-differential and $\check{\delta}$ the total differential. The factor $(-1)^{k+1}$ comes from the commutativity of the double complex. The factor $(-1)^l$ comes from the fact that the extended Čech complex (as the direct limit of $\mathrm{Hom}_{\mathcal{O}_X}(K_*((f^1_{(i)})^n, \cdots, (f^d_{(i)})^n), \mathcal{D}^{d,\bullet}_X))$ differs from the ordinary Čech complex by the same factor.

The boundary of the $\check{C}^{d-1}(D^{d,0}_X)$-hypercocycle defined by $\frac{df^1_{(i)} \wedge \cdots \wedge df^d_{(i)}}{f^1_{(i)} \cdots f^d_{(i)}}$ on $\{f^2_{(i)} \neq 0, \cdots, f^d_{(i)} \neq 0\}$ 0 otherwise, is the hypercocycle defined by $(-1)^d \frac{df^1_{(i)} \wedge \cdots \wedge df^d_{(i)}}{f^1_{(i)} \cdots f^d_{(i)}}$ on $\{f^1_{(i)} \neq 0, \cdots, f^d_{(i)} \neq 0\}$ (as the component of $\check{C}^d(D^{d,0}_X)$) and 0 otherwise (as the component in $\check{C}^{d-1}(D^{d,1}_X)$). On the other hand

$$\overline{\partial}(\frac{df^1_{(i)} \wedge \cdots \wedge df^d_{(i)}}{f^1_{(i)} \cdots df^d_{(i)}}) = (2\pi i)[f^1_{(i)} = 0] \wedge \frac{df^2_{(i)} \wedge \cdots \wedge df^d_{(i)}}{f^2_{(i)} \cdots f^d_{(i)}}$$

on $\{f^2_{(i)} \neq 0, \cdots, f^d_{(i)} \neq 0\}$ by the Lelong-Poincaré formula. Hence the Bloch cycle is cohomologous to the hypercocycle defined by $(-1)^{d-1}(2\pi i)[f^1_{(i)} = 0] \wedge \frac{df^2_{(i)} \wedge \cdots \wedge df^d_{(i)}}{f^2_{(i)} \cdots f^d_{(i)}}$ on $\{f^2_{(i)} \neq 0, \cdots, f^d_{(i)} \neq 0\}$ and 0 otherwise.

By induction, it is also cohomologous for any k to the hypercocycle defined by $(-1)^{d-1}(2\pi i)^{2k+1}[f^1_{(i)} = \cdots = f^{2k+1}_{(i)} = 0] \wedge \frac{df^{2k+2}_{(i)} \wedge \cdots \wedge df^d_{(i)}}{f^{2k+2}_{(i)} \cdots f^d_{(i)}}$ on $\{f^{2k+2}_{(i)} \neq 0, \cdots, f^d_{(i)} \neq 0\}$ and 0 otherwise. Notice that when doing the induction we use the fact that the currents involving terms $[f^j_{(i)} = 0]$ are zero on the open subset $\{f^j_{(i)} \neq 0\}$. We also observe that, since Z is a locally complete intersection of X, the wedge product of the currents $[f^j_{(i)} = 0] \wedge [f^k_{(i)} = 0]$ for $j \neq k$ is well defined. The induction is pursued until one reaches $k = d$. This finishes the proof.

By a similar argument, one shows that the Deligne cycle class can also be represented by the global current $[Z]$, as in the previous subsection. In particular, the image of the Deligne cycle class under the natural morphism is the Bloch cycle class. Since the Bloch cycle class is represented by meromorphic forms, the pull back of Bloch cycle classes is much easier to express. This explains our choice of introducing Bloch cycle classes to circumvent the difficulties.

REMARK 6.44. (Functoriality of local cohomology) As in [**Inv84**] page 125, we have the following commutative diagram. Let A be a closed subset of a complex manifold X and B a closed subset of Y a complex manifold. A holomophic map $f : X \to Y$ with $f(X \setminus A) \subset Y \setminus B$ will induce for any p, q

$$\begin{array}{ccc} H^q_B(Y, F^q\Omega^\bullet_Y) & \longrightarrow & H^q(Y, F^q\Omega^\bullet_Y) \\ \downarrow & & \downarrow \\ H^q_A(X, f^*F^q\Omega^\bullet_Y) & \longrightarrow & H^q(X, f^*F^q\Omega^\bullet_Y) \\ \downarrow & & \downarrow \\ H^q_A(X, F^q\Omega^\bullet_X) & \longrightarrow & H^q(X, F^q\Omega^\bullet_X). \end{array}$$

The first diagram commutes by taking injective resolutions $F^q\Omega^\bullet_Y \to J^\bullet$ and $f^*F^q\Omega^\bullet_Y \to I^\bullet$ and the commutative morphism of complexes

$$\begin{array}{ccc} \Gamma_B(Y, J^\bullet) & \longrightarrow & \Gamma(Y, J^\bullet) \\ \downarrow & & \downarrow \\ \Gamma_A(X, I^\bullet) & \longrightarrow & \Gamma(X, I^\bullet). \end{array}$$

The second diagram is given by natural inclusion $f^*F^q\Omega^\bullet_Y \to F^q\Omega^\bullet_X$ in $C(\mathrm{Sh}(X))$. This shows the functoriality of local cohomology under pull-backs.

Now we finish the "detour" via Bloch cycle classes. In the sequel, we reduce equalities to be proved for the Deligne (or Bott-Chern) cycle classes to the case of Bloch classes, using functoriality under pull-backs. This will complete the proof of most of the properties contained in Axiom B.

LEMMA 6.45. *Let X be a complex manifold. Let Y and Z be compact submanifolds of X that intersect transversally into $W = Y \cap Z$. Let $i_Y : Y \to X$ be the inclusion. Then we have in the integral Deligne cohomology the identity*

$$i_Y^*\{[Z]\}_D = \{[W]\}_D.$$

PROOF. Using the exact sequence, $0 \to D(d)^\bullet \to D(d)^\bullet_{\mathbb{C}} \to \mathbb{C}/\mathbb{Z} \to 0$, as in the Bott-Chern case, we can reduce the integral case to the complex case by the injectivity of local cohomology. By the construction of Bloch classes, it is enough to prove the equality for Bloch classes, thanks to the injectivity of the Deligne complex into the Bloch complex after passing to hypercohomology. Since Ω^d is a coherent $\mathcal{O}_X$ or $\mathcal{O}_Y$ sheaf, the topological local cohomology $H^\bullet_Z$ is isomorphic to the algebraic local cohomology $H^\bullet_{[Z]}$. We can cover X by Stein open sets U_i such that $U_i \cap Z \neq \varnothing$ and, in ad hoc local coordinate charts,

$$U_i \cap Z = \{z_{n-k+1} = \cdots = z_n = 0\} \qquad \text{for every } i.$$

We can also suppose that in any open set U_i of the covering such that $U_i \cap Y \neq \varnothing$, we have

$$U_i \cap Y = \{z_{n-l} = \cdots = z_1 = 0\}.$$

In particular, this gives in local coordinates

$$U_i \cap W = \{z_{n-k+1} = \cdots = z_n = z_{n-l} = \cdots = z_1 = 0\}.$$

In this case, the cycle class satisfies

$$i_Y^*\{\frac{dz_{n-k+1} \wedge \cdots \wedge dz_n}{z_{n-k+1} \cdots z_n}\}_{|U_i} = \{\frac{dz_{n-k+1} \wedge \cdots \wedge dz_n}{z_{n-k+1} \cdots z_n}\}_{|U_i \cap Y}$$

which implies $i_Y^*\{[Z]\}_{Bl} = \{[W]\}_{Bl}$. □

LEMMA 6.46. *With the same notation as in Proposition 6.5.1, we have*

$$(\mathrm{id}, f)^*\{[\Gamma]\} = \sum_{i=1}^N m_i \{[\Gamma_i']\}.$$

PROOF. In ad hoc local coordinates, $(z_1, \cdots, z_n) \in U$, we can write $D = \{z_1 = 0\}$. Therefore $\Gamma = \{(w_2, \cdots, w_n, z_1, \cdots, z_n) | z_1 = 0, z_i = w_i, \forall i \geqslant 2\} \subset D \times Y$ in this coordinate. As in the previous lemma, it is enough to prove the equality for the Bloch classes. Locally, $\{[D]\}_{Bl}$ is represented by

$$\frac{dz_1 \wedge d(z_2 - w_2) \wedge \cdots \wedge d(z_n - w_n)}{z_1(z_2 - w_n) \cdots (z_n - w_n)} \quad \text{in} \quad \Gamma(U, \mathcal{O}_U[\frac{1}{z_1(z_2 - w_n) \cdots (z_n - w_n)}]).$$

Locally we may write $f = (f_1, \cdots, f_n)$ for some coordinate chart V of X such that $f_1 = x_1^{m_1} \cdots x_n^{m_n}$. Then $(\mathrm{id}, f)^*\{[\Gamma]\}_{Bl}$ on $(D \cap U) \times V$ is represented by

$$\frac{df_1 \wedge d(f_2 - w_2) \wedge \cdots \wedge d(f_n - w_n)}{f_1(f_2 - w_n) \cdots (f_n - w_n)} = \sum_{i=1}^N m_i \frac{dx_i \wedge d(f_2 - w_2) \wedge \cdots \wedge d(f_n - w_n)}{x_i(f_2 - w_n) \cdots (f_n - w_n)}$$

in

$$\Gamma((D \cap U) \times V, \mathcal{O}_{(D \cap U) \times V}[\frac{1}{f_1(f_2 - w_n) \cdots (f_n - w_n)}]).$$

On the other hand, in $(D \cap U) \times V$, Γ_i' is given by $\{(w_2, \cdots, w_n, x_1, \cdots, x_n) | x_i = 0, w_j = f_j(x), \forall j \geqslant 2\}$. This proves the equality. The cohomology groups involved are all calculated by taking support in $\tilde{D}$. Since all cohomological arguments remain valid when the closed set is a locally complete intersection, we can still reach the desired conclusion, although $\tilde{D}$ is not necessarily a submanifold. □

In fact lemma 6.36 gives as a special case the following proposition, which translates into the equality $i_Y^* i_{Z*} 1 = i_{W/Y*} i_{W/Z}^* 1$.

PROPOSITION 6.5.3. *Consider the following commutative diagram, where Y and Z are compact and intersect transversally with $W = Y \cap Z$:*

$$\begin{array}{ccc} W & \xrightarrow{i_{W/Y}} & Y \\ {\scriptstyle i_{W/Z}}\downarrow & & \downarrow{\scriptstyle i_Y} \\ Z & \xrightarrow[i_Z]{} & X \end{array}$$

Then we have $i_Y^* i_{Z*} = i_{W/Y*} i_{W/Z}^*$.

PROOF. Under the assumptions, W is compact and $i_Y, i_Z, i_{W/Y}, i_{W/Z}$ are all proper. In the following, we denote by $p_{i,X_1/X_2}(i = 1,2,\ X_1, X_2 = X, Y, Z, W)$ the natural projection of $X_1 \times X_2$ onto the i-th component. We denote Γ_{X_1/X_2} the graph of i_{X_1/X_2} with $X_1, X_2 = W, X, Y, Z$, and make substitutions $i_Y = i_{Y/X}$, $i_Z = i_{Z/X}$. We have

$$i_Y^* i_{Z*}\alpha = i_Y^*(p_{2,Z/X*}(p_{1,Z/X}^*\alpha \cdot \{[\Gamma_{Z/X}]\}))$$
$$= p_{2,Z/Y*}(\mathrm{id}_Z, i_Y)^*(p_{1,Z/X}^*\alpha \cdot \{[\Gamma_{Z/X}]\}) = p_{2,Z/Y*}((\mathrm{id}_Z, i_Y)^* p_{1,Z/X}^*\alpha.(\mathrm{id}_Z, i_Y)^*\{[\Gamma_{Z/X}]\})$$
$$= p_{2,Z/Y*}(p_{1,Z/Y}^*\alpha \cdot \{[\Gamma_{W/Y}]\}) = i_{W/Y*}i_{W/Z}^*\alpha.$$

The first equality uses the lemma 6.33. The second equality uses Proposition 6.2.2 for $p_{2,Z/X} \circ (\mathrm{id}_Z, i_{Y/X}) = i_{Y/X} \circ p_{2,Z/Y}$. The third equality uses the fact that pulling back is a ring morphism. The fourth equality uses the fact that $p_{1,Z/X} \circ (\mathrm{id}_Z, i_{Y/X}) = p_{1,Z/Y}$. It also uses the fact that $\Gamma_{Z/X}$ is transversal in $Z \times X$ with $Z \times Y$ and lemma 6.46. To prove the last equality, take ω a smooth form defined on U an open set of Z in a Čech representative of α. Take ω' any smooth form with compact support on $U \cap Y$. We have to prove that

$$\langle p_{2,Z/Y*}(p_{1,Z/Y}^*\omega \wedge [\Gamma_{W/Y}]), \omega'\rangle = \langle i_{W/Y*}i_{W/Z}^*\omega, \omega'\rangle.$$

This holds true since

$$\langle p_{2,Z/Y*}(p_{1,Z/Y}^*\omega \wedge [\Gamma_{W/Y}]), \omega'\rangle = \langle (p_{1,Z/Y}^*\omega \wedge [\Gamma_{W/Y}]), p_{2,Z/Y}^*\omega'\rangle$$
$$= \int_{\Gamma_{W/Y}} p_{1,Z/Y}^*\omega \wedge p_{2,Z/Y}^*\omega' = \int_{\Gamma_{W/Y}} p_{1,W/Y}^* i_{W/Z}^*\omega \wedge p_{2,W/Y}^*\omega'$$
$$= \langle i_{W/Y*}i_{W/Z}^*\omega, \omega'\rangle.$$

Notice that all the projections other than $p_{1,Y/X}, p_{1,Z/X}$ are proper. The terms involving the two morphisms use only the pull-back, which is well defined even for a non proper morphism. So in the assumption, we do not need to assume that X is compact. □

The transversality condition is necessary in the above proposition. Indeed, if we take $Y = Z = W$, the morphism $i_Y^* i_{Y*}$ is not equal to the identity. To calculate it, we need the following excess formula. In the reverse direction, the formula is far easier. For any smooth submanifold Z of X and any cohomology class α on X we have

$$i_{Z*}i_Z^*\alpha = \alpha \cdot \{[Z]\}.$$

This can be derived from the projection formula, which implies

$$i_{Z*}i_Z^*\alpha = i_{Z*}(i_Z^*\alpha \cdot 1) = \alpha \cdot i_{Z*}1 = \alpha \cdot \{[Z]\}.$$

PROPOSITION 6.5.4. *If Y is a smooth hypersurface of X with X a compact complex manifold, then for any α an integral Bott-Chern cohomological class,*

$$i_Y^* i_{Y*}\alpha = \alpha \cdot c_1(N_{Y/X}).$$

PROOF. We use the deformation of the normal cone (cf. [**Ful84**] chap V). Let M be the blow up of $X \times \mathbb{P}^1$ along $Y \times \{0\}$, $\tilde{X}$ be the strict transform of $X \times \{0\}$ under the blow up. Let $M^\circ = M \setminus \tilde{X}$. Then we have an injection $F : Y \times \mathbb{P}^1 \hookrightarrow M^\circ$. There exists a flat morphism $\rho : M \to \mathbb{P}^1$ such that the following diagram commutes

$$\begin{array}{ccc} Y \times \mathbb{P}^1 & \xhookrightarrow{F} & M^\circ \\ & \searrow^{\mathrm{pr}_1} & \downarrow{\scriptstyle \rho|_{M^\circ}} \\ & & \mathbb{P}^1. \end{array}$$

The fibre over ∞ is $N_{Y/X}$ and the fibre over other points is X. We denote the inclusion $N_{Y/X} \hookrightarrow M^\circ$ by j_0, the zero section $Y \hookrightarrow N_{Y/X}$ by i, the projections of $(Y \times \mathbb{P}^1) \times M^\circ$ (resp. $Y \times \mathbb{P}^1$, resp. $(Y \times \mathbb{P}^1) \times N_{Y/X}$, and resp. $Y \times N_{Y/X}$) on the first and second factor by pr_1 and pr_2 (resp. $\tilde{\mathrm{pr}}_1$, $\tilde{\mathrm{pr}}_2$, resp. pr'_1, pr'_2, resp. pr''_1, pr''_2). We denote by $\Gamma \subset Y \times \mathbb{P}^1 \times M^\circ$ the graph of F and by Γ' the graph of i. Finally, we denote the inclusion of the central fibre $i_0 : Y \to Y \times \mathbb{P}^1$ by i_0, and define $[\Gamma''] = (i_0, \mathrm{id}_{N_{Y/X}})_*[\Gamma']$.

Since Y is compact, pr'_2, pr''_2 are proper, we find $(i_0, \mathrm{id}_{N_{Y/X}})_*\{[\Gamma']\} = \{[\Gamma'']\}$, and also $(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0)^*\{[\Gamma]\} = \{[\Gamma'']\}$ since the image of $(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0)$ and Γ are transversal with intersection equal to Γ''. Let γ be the class on M° defined by $\gamma = F_*(\tilde{\mathrm{pr}}_1^*\alpha)$. Then we have

$$j_0^*\gamma = j_0^*F_*(\tilde{\mathrm{pr}}_1^*\alpha) = j_0^*\mathrm{pr}_{2*}(\mathrm{pr}_1^*\tilde{\mathrm{pr}}_1^*\alpha \cdot \{[\Gamma]\})$$
$$= \mathrm{pr}'_{2*}(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0)^*(\mathrm{pr}_1^*\tilde{\mathrm{pr}}_1^*\alpha \cdot \{[\Gamma]\}) = \mathrm{pr}'_{2*}[(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0)^*\mathrm{pr}_1^*\tilde{\mathrm{pr}}_1^*\alpha \cdot \{[\Gamma'']\}]$$
$$= \mathrm{pr}'_{2*}(i_0, \mathrm{id}_{N_{Y/X}})_*(i_0, \mathrm{id}_{N_{Y/X}})^*[(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0)^* pr_1^*\tilde{\mathrm{pr}}_1^*\alpha \cdot \{[\Gamma']\}]$$
$$= \mathrm{pr}'_{2*}(i_0, \mathrm{id}_{N_{Y/X}})_*[(i_0, \mathrm{id}_{N_{Y/X}})^*(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0)^* pr_1^*\tilde{\mathrm{pr}}_1^*\alpha \cdot \{[\Gamma']\}]$$

$$= \mathrm{pr}''_{2*}(\mathrm{pr}''^{*}_1\alpha \cdot \{[\Gamma']\}) = i_*\alpha.$$

The second equality uses lemma 6.33. The third equality uses Proposition 6.2.2 for $\mathrm{pr}_2\circ(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0) = j_0\circ\mathrm{pr}'_1$. The fourth equality uses the fact that $(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0)^*$ is a ring morphism. The fifth equality uses the projection formula. The sixth equality uses $\mathrm{pr}''_2 = \mathrm{pr}'_2\circ(i_0, \mathrm{id}_{N_{Y/X}})$ and $\tilde{\mathrm{pr}}_1\circ\mathrm{pr}_1\circ(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0)\circ(i_0,\mathrm{id}_{N_{Y/X}}) = \mathrm{pr}''_1$. The last equality uses another time lemma 6.33.

By the homotopy principle which is proven in the Whitney formula, the class $(F^*\gamma)|_{Y\times\{t\}}$ is independent of the choice of t. For $t=0$, $(F^*\gamma)|_{Y\times\{0\}} = i^*j_0^*\gamma = i^*i_*\alpha$. For $t\neq 0$,

$$(F^*\gamma)|_{Y\times\{t\}} = (F^*F_*(\tilde{\mathrm{pr}}_1^*\alpha))|_{Y\times\{t\}}$$

$$= i^*_{Y\times\{t\}/X\times\{t\}} i^*_{X\times\{t\}/M^\circ} i_{Y\times\{t\}*}(\tilde{\mathrm{pr}}_1^*\alpha) = i^*_Y i_{Y\times\{t\}/X\times\{t\}*} i^*_{Y\times\{t\}/Y\times\mathbb{P}^1}\tilde{\mathrm{pr}}_1^*\alpha$$

$$= i^*_Y i_{Y*}\alpha.$$

The third equality the proposition 6.5.3, and the fact that $Y\times\mathbb{P}^1$ and $X\times\{t\}$ intersect transversally in M° with intersection $Y\times\{t\}$. (Here M° is non compact.) The last equality uses the fact that $\tilde{\mathrm{pr}}_1\circ i_{Y\times\{t\}/Y\times\mathbb{P}^1} =$ id.

Let π be the projection of $N_{Y/X}$ onto Y. Then $\alpha = i^*\pi^*\alpha$. We have

$$i^*i_*\alpha = i^*i_*i^*\pi^*\alpha = i^*(\pi^*\alpha\cdot\overline{\{[Y]\}})$$

where the second equality uses the remark before this proposition and $\overline{\{[Y]\}}$ is the class of Y in $N_{Y/X}$. So $i^*i_*\alpha = i^*\pi^*\alpha\cdot i^*\overline{\{[Y]\}} = \alpha\cdot i^*\overline{\{[Y]\}}$. By lemma 6.47 below, $i^*\overline{\{[Y]\}} = i^*c_1(\mathcal{O}_{N_{Y/X}}(Y)) = c_1(\mathcal{O}_{N_{Y/X}}(Y)|_Y) = c_1(N_{Y/N_{Y/X}}) = c_1(N_{Y/X})$. □

LEMMA 6.47. *Let D be a simple normal crossing divisor in a complex manifold X (that need not necessarily be compact). Then we have*

$$c_1(\mathcal{O}(D)) = \{[D]\}.$$

PROOF. By an obvious additivity argument, we can suppose that the divisor is reduced. The first Chern class in complex Bott-Chern cohomology can be defined by singular metric since in the complex $\mathcal{L}^\bullet_{1,1}[1]$ the forms can be changed by currents. These two complexes are quasi-isomorphic. The line bundle of the effective divisor has a canonical section s_D which induces a singular metric on X. The image of the first Chern class in complex Bott-Chern cohomology can be represented by global section $\frac{i}{2\pi}\partial\overline{\partial}\log|s_D|^2$. A priori, $\log|s_D|^2$ is the weight function of the singular metric on some open set on which s_D can be trivialized. But in fact, $\frac{i}{2\pi}\partial\overline{\partial}\log|s_D|^2$ is independent of the choice of trivialisation. By the Lelong-Poincaré formula, the image of the first Chern class in complex Bott-Chern cohomology can be represented by the current $[D]$.

By construction, the image of Chern class in integral Bott-Chern class under the canonical map is the Chern class in integral singular cohomology. Since $\mathcal{O}(D)$ is a complex line bundle, its first Chern class is just its Euler class. Classically the Euler class of the Poincaré dual of the zeros of the smooth section s_D. It can also seen from the fact the cycle class in the hypercohomology of $\mathcal{B}^\bullet_{1,1,\mathbb{Z}}$ sends to the cycle class in the hypercohomolgy of $\mathcal{B}^\bullet_{0,0,\mathbb{Z}}$ (which is just the singular cohomology) induced from the natural morphism of complexes $\mathcal{B}^\bullet_{1,1,\mathbb{Z}}\to\mathcal{B}^\bullet_{0,0,\mathbb{Z}}$. Since we have the equality of classes

$$c_1(\mathcal{O}(D)) = \{[D]\}$$

in the complex Bott-Chern cohomology as well as in the integral singular cohomology, we deduce the equality for the integral Bott-Chern cohomology. □

6.6. Transformation under blow-up

In this part, we want to show that the integral Bott-Chern class satisfies the rest of the axioms B in [**Gri10**] (see Axiom B (5)(6)(7) in Introduction).

To start with, we prove the transformation formula of the integral Bott-Chern cohomology under blow up. The closed immersions, projections and blow ups are the most elementary morphisms in the description of Serre's proof of Riemann-Roch-Grothendieck formula. In fact, by considering the graph, any projective morphism can be written as a composition of a closed immersion and a projection. By devissage, we reduce the general closed immersion to the case of closed immersion of a smooth hypersurface. To perform this reduction, we need to blow up submanifolds, and thus a study of the cohomology of blow ups is required. To do this, we will need the following version for Dolbeault cohomology groups stated in [**RYY17**].

THEOREM 6.48. *Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n$ and $Y\subset X$ a closed complex submanifold of complex codimension $r\geqslant 2$. Suppose that $p:\tilde{X}\to X$ is the blow-up of X along Y. We* *denote by E the exception divisor and by $i : Y \to X$, $E \to \tilde{X}$ the inclusions, by $q : E \to Y$ the restriction of p on E. Then for any $0 \leqslant p, q \leqslant n$, there is an isomorphism*

$$j^* : H^{p,q}_{\bar{\partial}}(\tilde{X})/p^* H^{p,q}_{\bar{\partial}}(X) \cong H^{p,q}_{\bar{\partial}}(E)/q^* H^{p,q}_{\bar{\partial}}(Y).$$

$$j^* : H^{p,q}_{\partial}(\tilde{X})/p^* H^{p,q}_{\partial}(X) \cong H^{p,q}_{\partial}(E)/q^* H^{p,q}_{\partial}(Y).$$

PROOF. The first statement is the main theorem of [**RYY17**]. The second statement uses the fact that

$$H^{p,q}_{\partial}(X) = \ker\{\partial : \Gamma(X, C^{p,q}_{\infty}) \to \Gamma(X, C^{p+1,q}_{\infty})\}/\mathrm{Im}\{\partial : \Gamma(X, C^{p-1,q}_{\infty}) \to \Gamma(X, C^{p,q}_{\infty})\}$$

$$= \overline{\ker\{\bar{\partial} : \Gamma(X, C^{q,p}_{\infty}) \to \Gamma(X, C^{q,p+1}_{\infty})\}/\mathrm{Im}\{\bar{\partial} : \Gamma(X, C^{q,p-1}_{\infty}) \to \Gamma(X, C^{q,p}_{\infty})\}} = \overline{H^{q,p}_{\bar{\partial}}(X)}.$$

Now the second statement comes from the first statement. □

We also need the classical analogue for integral coefficient cohomology (cf. [**GH78**], page 603) by using the Mayer-Vietoris sequence involving a tubular neighbourhood of Y.

LEMMA 6.49. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and $Y \subset X$ a closed complex submanifold of complex codimension $r \geqslant 2$. Suppose that $p : \tilde{X} \to X$ is the blow-up of X along Y. We denote by E the exception divisor and by $i : Y \to X$, $E \to \tilde{X}$ the inclusions, by $q : E \to Y$ the restriction of p on E. Then for any n there is an isomorphism*

$$j^* : H^n(\tilde{X}, \mathbb{Z})/p^* H^n(X, \mathbb{Z}) \cong H^n(E, \mathbb{Z})/q^* H^n(Y, \mathbb{Z}).$$

Using these results, we can prove by induction an analogous result for integral Bott-Chern cohomology.

PROPOSITION 6.6.1. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and $Y \subset X$ a closed complex submanifold of complex codimension $r \geqslant 2$. Suppose that $p : \tilde{X} \to X$ is the blow-up of X along Y. We denote by E the exception divisor and by $i : Y \to X$, $E \to \tilde{X}$ the inclusions, by $q : E \to Y$ the restriction of p on E. Then for any n, p, q there is an isomorphism*

$$j^* : \mathbb{H}^n(\tilde{X}, \mathcal{B}^\bullet_{p,q,\mathbb{Z}})/p^* \mathbb{H}^n(X, \mathcal{B}^\bullet_{p,q,\mathbb{Z}}) \cong \mathbb{H}^n(E, \mathcal{B}^\bullet_{p,q,\mathbb{Z}})/q^* \mathbb{H}^n(Y, \mathcal{B}^\bullet_{p,q,\mathbb{Z}}).$$

PROOF. The short exact sequence

$$0 \to \Omega^{p+1}[-p-1] \to \mathcal{B}^\bullet_{p+1,q,\mathbb{Z}} \to \mathcal{B}^\bullet_{p,q,\mathbb{Z}} \to 0$$

induces a commutative diagram

$$\begin{array}{ccccccc}
H^{n-p-1,p+1}_{\bar{\partial}}(\tilde{X})/p^* H^{n-p-1,p+1}_{\bar{\partial}}(X) & \longrightarrow & \mathbb{H}^n(\tilde{X}, \mathcal{B}^\bullet_{p+1,q,\mathbb{Z}})/p^* \mathbb{H}^n(X, \mathcal{B}^\bullet_{p+1,q,\mathbb{Z}}) & \longrightarrow & \mathbb{H}^n(\tilde{X}, \mathcal{B}^\bullet_{p,q,\mathbb{Z}})/p^* \mathbb{H}^n(X, \mathcal{B}^\bullet_{p,q,\mathbb{Z}}) & \longrightarrow & \cdots \\
\downarrow j^* & & \downarrow j^* & & \downarrow j^* & & \\
H^{n-p-1,p+1}_{\bar{\partial}}(E)/q^* H^{n-p-1,p+1}_{\bar{\partial}}(Y) & \longrightarrow & \mathbb{H}^n(E, \mathcal{B}^\bullet_{p+1,q,\mathbb{Z}})/q^* \mathbb{H}^n(Y, \mathcal{B}^\bullet_{p+1,q,\mathbb{Z}}) & \longrightarrow & \mathbb{H}^n(E, \mathcal{B}^\bullet_{p,q,\mathbb{Z}})/q^* \mathbb{H}^n(Y, \mathcal{B}^\bullet_{p,q,\mathbb{Z}}) & \longrightarrow & \cdots
\end{array}$$

By the five lemma and Theorem 6.48, one can reduce the proof to the case $p = 0$ by induction. Then the short exact sequence

$$0 \to \overline{\Omega^{q+1}[-q-1]} \to \mathcal{B}^\bullet_{0,q+1,\mathbb{Z}} \to \mathcal{B}^\bullet_{0,q,\mathbb{Z}} \to 0$$

induces a commutative diagram

$$\begin{array}{ccccccc}
H^{q+1,n-q-1}_{\partial}(\tilde{X})/p^* H^{q+1,n-q-1}_{\partial}(X) & \longrightarrow & \mathbb{H}^n(\tilde{X}, \mathcal{B}^\bullet_{0,q+1,\mathbb{Z}})/p^* \mathbb{H}^n(X, \mathcal{B}^\bullet_{0,q+1,\mathbb{Z}}) & \longrightarrow & \mathbb{H}^n(\tilde{X}, \mathcal{B}^\bullet_{0,q,\mathbb{Z}})/p^* \mathbb{H}^n(X, \mathcal{B}^\bullet_{0,q,\mathbb{Z}}) & \longrightarrow & \cdots \\
\downarrow j^* & & \downarrow j^* & & \downarrow j^* & & \\
H^{q+1,n-q-1}_{\partial}(E)/q^* H^{q+1,n-q-1}_{\partial}(Y) & \longrightarrow & \mathbb{H}^n(E, \mathcal{B}^\bullet_{0,q+1,\mathbb{Z}})/q^* \mathbb{H}^n(Y, \mathcal{B}^\bullet_{0,q+1,\mathbb{Z}}) & \longrightarrow & \mathbb{H}^n(E, \mathcal{B}^\bullet_{0,q,\mathbb{Z}})/q^* \mathbb{H}^n(Y, \mathcal{B}^\bullet_{0,q,\mathbb{Z}}) & \longrightarrow & \cdots
\end{array}$$

By the five lemma and Theorem 6.48 again, one can reduce the proof to the case $p = 0$, $q = 0$ by induction. This is done directly by Lemma 6.49. □

A direct application of the proposition is the following general excess formula.

PROPOSITION 6.6.2. *With the same notation in the above proposition, if F is the excess conormal bundle on E defined by the exact sequence*

$$0 \to F \to q^* N^*_{Y/X} \to N^*_{E/\tilde{X}} \to 0,$$

one has the following excess formula for any cohomology class α on Y :

$$p^* i_* \alpha = j_*(q^* \alpha \cdot c_{d-1}(F^*)).$$

PROOF. Define $\beta = j_*(q^*\alpha \cdot c_{d-1}(F^*))$. By the excess formula for a line bundle, we have

$$j^*\beta = [q^*\alpha \cdot c_{d-1}(F^*)] \cdot c_1(N_{E/\tilde{X}}) = q^*\alpha \cdot q^*(c_d(N_{Y/X})).$$

The second equality uses the Whitney formula for Chern class of vector bundles. Hence $j^*\beta \in \mathrm{Im}(q^*)$ and by the above Proposition we know $\beta = p^*\gamma$ for some cohomology class on X. So $p_*\beta = p_*p^*\gamma = \gamma$ where the second equality uses $p_*p^* = \mathrm{id}$ proven in the second section. Then we have

$$\beta = p^*p_*\beta = p^*p_*j_*(q^*\alpha \cdot c_{d-1}(F^*)) = p^*i_*q_*(q^*\alpha \cdot c_{d-1}(F^*))$$
$$= p^*i_*(\alpha \cdot q_*c_{d-1}(F^*)) = p^*i_*\alpha.$$

The first equality on the second line uses the projection formula. The last equality uses the fact that $q_*c_{d-1}(F^*) = 1$, as follows from the next lemma. ☐

LEMMA 6.50. *Let $G \to X$ be a vector bundle of rank r which induces $\pi : \mathbb{P}(G) \to X$. Let H be the vector bundle defined by the exact sequence*

$$0 \to H \to \pi^*G \to \mathcal{O}_{\mathbb{P}(G)}(1) \to 0.$$

Then we have $\pi_*(c_{r-1}(H)) = (-1)^{r-1}$.

PROOF. We start the proof for the complex Bott-Chern cohomology such that the cohomology class can be represented by global differential forms. By the Whitney formula for the total Chern class, $c(\pi^*G) = c(H) \cdot c(\mathcal{O}_{\mathbb{P}(G)}(1))$. We denote $h := c_1(\mathcal{O}_{\mathbb{P}(G)}(1))$. Then

$$c(H) = c(\pi^*G)(1+h)^{-1} = (1 + c_1(\pi^*G) + \cdots + c_r(\pi^*G))(1 - h + h^2 + \cdots).$$

The element of degree $r-1$ on two sides is $c_{r-1}(H) = (-1)^{r-1}h^{r-1} + (-1)^{r-2}h^{r-2}c_1(\pi^*G) + \cdots + c_{r-1}(\pi^*G)$. π_* is given by integration along the fibre direction. By degree reason, $\pi_*c_{r-1}(H) = (-1)^{r-1}\pi_*h^{r-1} = (-1)^{r-1}$. The integration can be calculated by a metric on $\mathcal{O}_{\mathbb{P}(G)}(1)$ induced by a smooth Hermitian metric on G. This finishes the proof of the complex case.

Since the equality is taken in $H^{0,0}_{BC}(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \cong \mathbb{Z}$ which is a lattice in $H^{0,0}_{BC}(X,\mathbb{C}) = H^0(X,\mathbb{C}) \cong \mathbb{C}$. We deduces the integral case from the complex one. ☐

Everything we have done also works for rational Bott-Chern cohomology. In [**Gri10**], Grivaux shows that as soon as one has a good intersection theory for some cohomology theory, one can use the Riemann-Roch-Grothendieck formula to construct the Chern class of a coherent sheaf by an induction on dimension. The last axiom that remains to be proven is the Hirzebruch–Riemann–Roch theorem. It can be reduced to the case of the Deligne complex by the following observation made in lemma 7.2 of [**Sch07**].

LEMMA 6.51. *Let X be a compact Kähler manifold. Then for any $p \in \mathbb{N}^*$ and $k \in \mathbb{N}$ we have*

$$\mathbb{H}^k(X, \Omega^\bullet_{<p}) \cong \oplus_{r+s=k, r<p} H^{r,s}(X,\mathbb{C}).$$

Since $\mathbb{P}^n$ is Kähler, the lemma gives the complete description of the integral Bott-Chern cohomology for the projective spaces.

PROPOSITION 6.6.3. *The natural morphism $\oplus_k H^{k,k}_{BC}(\mathbb{P}^n,\mathbb{Z}) \to \oplus_p H^{2p}_D(\mathbb{P}^n,\mathbb{Z}(p))$ induces an isomorphism of rings. In particular, the Hirzebruch–Riemann–Roch theorem holds for integral Bott-Chern cohomology.*

PROOF. By the lemma 6.51, we have for any $p \in \mathbb{N}^*$

$$\mathbb{H}^{2p}(\mathbb{P}^n, \Omega^\bullet_{<p}) = 0 \to H^{p,p}_{BC}(\mathbb{P}^n,\mathbb{Z}) \to H^{2p}_D(\mathbb{P}^n,\mathbb{Z}(p)) \to \mathbb{H}^{2p+1}(\mathbb{P}^n, \Omega^\bullet_{<p}) = 0.$$

The second morphism is the natural morphism from Bott-Chern cohomology to Deligne cohomology which is in fact an isomorphism shown by the exact sequence. For $p = 0$, it is also an isomorphism since the complexes are the same. Since the natural morphism from Bott-Chern cohomology to Deligne cohomology is a ring morphism, we have the first statement. ☐

REMARK 6.52. As far as we know, it seems that Grivaux's method does not work for constructing Chern classes of a coherent sheaf in the integral Bott-Chern cohomology, as opposed to the rational cohomology. The main reason is that the Chern characteristic class is additive but the total Chern class is multiplicative, and switching from one to the other involves denominators. The proof given in [**Ful84**] for the Riemann-Roch-Grothendieck formula in the context of coherent sheaves and the Chow ring reduces to proving that the Riemann-Roch-Grothendieck formula holds for vector bundles. The additivity of the Chern characteristic class and the nature of the formula ensure that after proving the special case of bundles, the Riemann-Roch-Grothendieck formula will also be valid for coherent sheaves on projective manifolds. However, one needs the projectivity condition to ensure that the Grothendieck group of coherent sheaves and the Grothendieck group of vector bundles are the same.

There exists an analogue of the "integral" Riemann-Roch-Grothendieck formula given in [**Jou70**]. In this work, Jouanolou proved that for a closed embedding $f : X \to Y$ of non-singular varieties of codimension d and for any vector bundle of rank e on X, then the total Chern class in Chow groups satisfies

$$c(f_* E) = 1 + f_*(P(c_1(N), \cdots, c_d(N), c_1(E), \cdots, c_e(E)))$$

where N is the normal bundle and P is some universal polynomial depending only on d, e. This formula does not work directly for coherent sheaves by simply replacing e with the generic rank of the coherent sheaf involved, even in the projective case. This is caused by the lack of additivity and the appearance of polynomials. As a consequence, a different choice of the values of e will give a completely different class. As a matter of fact, a coherent sheaf can carry in its Chern classes some information that extend to degrees beyond its generic rank. At this point, there does not seem to exist a similar integral Riemann-Roch-Grothendieck formula for coherent sheaves.

An easy counter example is obtained by considering $f : \mathbb{P}^2 \to \mathbb{P}^3$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^2}/m_0$. The left hand side is equal to $c(\mathcal{O}_{\mathbb{P}^3}/m_0) = \frac{c(\mathcal{O}_{\mathbb{P}^3})}{c(m_0)} = 1 - c_1(\mathcal{O}_{\mathbb{P}^3}(1))^3$, but the right hand of the universal polynomial with $d = 1, e = 1$ where 1 is the generic rank of $\mathcal{O}_{\mathbb{P}^3}/m_0$ gives $1 + f_* P(c_1(N), c_1(\mathcal{O}_{\mathbb{P}^2}/m_0)) = 1 + f_* P(c_1(\mathcal{O}_{\mathbb{P}^2}(1)), c_1(\mathcal{O}_{\mathbb{P}^2}/m_0)) = 1 + f_*\left(\frac{1}{1+c_1(\mathcal{O}_{\mathbb{P}^2}/m_0)-c_1(\mathcal{O}_{\mathbb{P}^2}(1))} - 1\right) = 1 + c_1(\mathcal{O}_{\mathbb{P}^3}(1))^2 + c_1(\mathcal{O}_{\mathbb{P}^3}(1))^3$. The same example shows that the formula is not valid when we taking e to be the largest number such that the Chern class is not trivial. We do not know whether there are any substitutes of the Riemann-Roch-Grothendieck formula used in Grivaux's induction argument, that would be capable of defining Chern classes in integral Bott-Chern cohomology.

6.7. Appendix: Top degree integral Bott-Chern cohomology

In this section, using the duality between the complex Bott-Chern cohomology and the Aeppli cohomology, we give a description of the integral Bott-Chern cohomology in top degree, on any compact connected manifold X. We denote by n the complex dimension of X. We start by recalling the definition of Aeppli cohomology.

DEFINITION 6.53. (Aeppli cohomology). *For all $p, q \leqslant \dim X$, one defines*

$$H_A^{p,q}(X, \mathbb{C}) := \frac{\ker\{\partial\overline{\partial} : C_\infty^{p,q}(X) \to C_\infty^{p+1,q+1}(X)\}}{(\mathrm{Im}\{\partial : C_\infty^{p-1,q}(X) \to C_\infty^{p,q}(X)\}) + (\mathrm{Im}\{\overline{\partial} : C_\infty^{p,q-1}(X) \to C_\infty^{p,q}(X)\})}.$$

As is well known, the natural pairing between $H_A^{p,q}(X, \mathbb{C})$ and $H_{BC}^{n-p,n-q}(X, \mathbb{C})$, defined by integrating wedge products of forms on X, induces a duality between Aeppli cohomology and complex Bott-Chern cohomology. In particular

$$H_{BC}^{n,n}(X, \mathbb{C}) = (H_A^{0,0}(X, \mathbb{C}))^* = \{f \in C_\infty^{0,0}(X) | \partial\overline{\partial} f = 0\}^* = \mathbb{C}.$$

We also need the following lemma.

LEMMA 6.54.

$$\mathbb{H}^{2n-1}(X, \sigma_n \Omega_X^\bullet \oplus \sigma_n \overline{\Omega}_X^\bullet) \cong H^{2n-1}(X, \mathbb{C}).$$

PROOF. The short exact sequence

$$0 \to \sigma_n \Omega_X^\bullet \oplus \sigma_n \overline{\Omega}_X^\bullet[-1] \to \mathcal{B}_{n,n,\mathbb{C}}^\bullet \to \mathbb{C} \to 0$$

induces the long exact sequence

$$\mathbb{H}^{2n-1}(X, \mathcal{B}_{n,n,\mathbb{C}}^\bullet) \to H^{2n-1}(X, \mathbb{C}) \to \mathbb{H}^{2n-1}(X, \sigma_n \Omega_X^\bullet \oplus \sigma_n \overline{\Omega}_X^\bullet) \to H_{BC}^{n,n}(X, \mathbb{C}) \to H^{2n}(X, \mathbb{C}).$$

Since the last morphism is a linear isomorphism, we have

$$\mathbb{H}^{2n-1}(X, \sigma_n \Omega_X^\bullet \oplus \sigma_n \overline{\Omega}_X^\bullet) \cong H^{2n-1}(X, \mathbb{C}) / \mathbb{H}^{2n-1}(X, \mathcal{B}_{n,n,\mathbb{C}}^\bullet).$$

We claim that $\mathbb{H}^{2n-1}(X, \mathcal{B}_{n,n,\mathbb{C}}^\bullet) \cong \mathbb{H}^{2n-1}(X, \mathcal{B}_{1,1,\mathbb{C}}^\bullet)^*$ as topological linear spaces. The argument is as follows. Recall that the complex Bott-Chern complex $\mathcal{B}_{n,n,\mathbb{C}}^\bullet$ is quasi-isomorphic to the complex $(\mathcal{L}_{n,n}^\bullet[1], \delta[1])$ defined by

$$\mathcal{L}_{n,n}^k = \bigoplus_{p+q=k, p<n, q<n} C_\infty^{p,q} \quad \text{for } k \leqslant 2n-2;$$

$$\mathcal{L}_{n,n}^{k-1} = \bigoplus_{p+q=k, p\geqslant n, q\geqslant n} C_\infty^{p,q} \quad \text{for } k \geqslant 2n$$

(for the proof, see [**Sch07**]). The differential δ^k is chosen to be the exterior derivative d for $k \neq 2n-2$ (in the case $k \leqslant 2n-3$ we neglect the components which fall outside $\mathcal{L}_{n,n}^{k+1}$) and we set

$$\delta^{2n-2} = \partial\overline{\partial} : C_\infty^{n-1,n-1} \to C_\infty^{n,n}.$$

We view this complex in the category of sheaves of topological linear space where the differential is continuous. We denote $\tilde{\mathcal{L}}^\bullet_{1,1}$ the complex obtained by changing smooth forms by currents which is quasi-isomorphic to $\mathcal{L}^\bullet_{1,1}$. By a direct calculation, the dual of the component of the complex Bott-Chern complex $\mathcal{L}^k_{n,n}$ in degree k is $\tilde{\mathcal{L}}^{2n-1-k}_{1,1}$. By the universal coefficient theorem, we have

$$\mathbb{H}^{2n-1}(X, \mathcal{B}^\bullet_{n,n,\mathbb{C}}) \cong \mathbb{H}^{2n-1}(X, \mathcal{B}^\bullet_{1,1,\mathbb{C}})^*$$

If $n \geqslant 2$, since $\mathcal{B}^\bullet_{1,1,\mathbb{C}}$ vanishes for degree bigger than 2, $\mathbb{H}^{2n-1}(X, \mathcal{B}^\bullet_{1,1,\mathbb{C}}) = 0$ which proves the lemma in this case.

If $n = 1$, we claim that the image of $\mathbb{H}^{2n-1}(X, \mathcal{B}^\bullet_{n,n,\mathbb{C}})$ in $H^{2n-1}(X, \mathbb{C})$ is 0. This is equivalent to say that $\mathbb{H}^1(X, \mathcal{O}_X \oplus \overline{\mathcal{O}}_X[-1]) \to \mathbb{H}^1(X, \mathcal{B}^\bullet_{1,1,\mathbb{C}})$ is surjective. We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_X \oplus \overline{\mathcal{O}}_X & & \\
\downarrow & & \\
C^{0,0}_\infty \oplus C^{0,0}_\infty & \xrightarrow{(\overline{\partial},\partial)} & C^{0,1}_\infty \oplus C^{1,0}_\infty \\
\downarrow + & & \downarrow \frac{1}{2}(\partial\circ p_1 - \overline{\partial}\circ p_2) \\
C^{0,0}_\infty & \xrightarrow{\partial\overline{\partial}} & C^{1,1}_\infty
\end{array}$$

with a translation of degree 1. Hence the map $\mathbb{H}^1(X, \mathcal{O}_X \oplus \overline{\mathcal{O}}_X[-1]) \to \mathbb{H}^1(X, \mathcal{B}^\bullet_{1,1,\mathbb{C}})$ is given by

$$\mathbb{H}^0(X, C^{0,0}_\infty \oplus C^{0,0}_\infty \to C^{0,1}_\infty \oplus C^{1,0}_\infty) \to \mathbb{H}^0(X, C^{0,0}_\infty \to C^{1,1}_\infty).$$

For any constant function c on X, $(c, 0)$ defines an element of $\mathbb{H}^0(X, C^{0,0}_\infty \oplus C^{0,0}_\infty \to C^{0,1}_\infty \oplus C^{1,0}_\infty)$ whose image is c in $\mathbb{H}^0(X, C^{0,0}_\infty \to C^{1,1}_\infty)$. This completes the proof when $n = 1$. □

Another way to prove the lemma when $n = 1$ is to see that

$$\begin{aligned} H^1(X, \mathcal{O}_X) \oplus H^1(X, \overline{\mathcal{O}}_X) &\cong H^1(X, \mathcal{O}_X) \oplus \overline{H^1(X, \mathcal{O}_X)} \\ &= H^{1,0}(X) \oplus \overline{H^{1,0}}(X) = H^1(X, \mathbb{C}). \end{aligned}$$

Here we remark that a Riemann surface is Kähler so we have the Hodge decomposition theorem. Now we can give the structure of the integral Bott-Chern cohomology in top degree.

PROPOSITION 6.7.1. *Under the above assumption, we have a short exact sequence*

$$0 \to H^{2n-1}(X, \mathbb{C})/H^{2n-1}(X, \mathbb{Z}) \to H^{n,n}_{BC}(X, \mathbb{Z}) \to \mathbb{Z} \to 0.$$

PROOF. The commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \sigma_n\Omega^\bullet_X \oplus \sigma_n\overline{\Omega}^\bullet_X[-1] & \longrightarrow & \mathcal{B}^\bullet_{n,n,\mathbb{Z}} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma_n\Omega^\bullet_X \oplus \sigma_n\overline{\Omega}^\bullet_X[-1] & \longrightarrow & \mathcal{B}^\bullet_{n,n,\mathbb{C}} & \longrightarrow & \mathbb{C} & \longrightarrow & 0
\end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccc}
H^{2n-1}(X, \mathbb{Z}) & \longrightarrow & \mathbb{H}^{2n-1}(X, \sigma_n\Omega^\bullet_X \oplus \sigma_n\overline{\Omega}^\bullet_X) & \longrightarrow & H^{n,n}_{BC}(X, \mathbb{Z}) & \longrightarrow & H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z} \\
\downarrow & & \downarrow id & & \downarrow & & \downarrow \\
H^{2n-1}(X, \mathbb{C}) & \xrightarrow{\sim} & \mathbb{H}^{2n-1}(X, \sigma_n\Omega^\bullet_X \oplus \sigma_n\overline{\Omega}^\bullet_X) & \longrightarrow & H^{n,n}_{BC}(X, \mathbb{C}) \cong \mathbb{C} & \xrightarrow{\sim} & H^{2n}(X, \mathbb{C}) \cong \mathbb{C}
\end{array}$$

The rightest morphism on the first line is surjective since for any $x \in X$ the image of the cycle class associated with x in the integral Bott-Chern cohomology is the corresponding cycle class in the singular cohomology $H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$. The image is a generator in the singular cohomology. Hence we have the surjectivity in the proposition. The kernel of this morphism is $H^{2n-1}(X, \mathbb{C})/H^{2n-1}(X, \mathbb{Z})$ by lemma 6.54 and chasing into the commutative diagram. □

REMARK 6.55. This kind of description does not work in general for the integral Deligne cohomology. By the Poincaré-Grothendieck lemma, we get in the derived category $D(\mathrm{Sh}(X))$ a quasi-isomorphism

$$\mathbb{C}_X \cong \Omega^\bullet_X.$$

Hence the Deligne complex in top degree is quasi-isomorphic to Ω^n. However, in general, we do not have an isomorphism between $H^{2n}_D(X, \mathbb{C}) \cong H^n(X, \Omega^n_X)$ and $H^{2n}(X, \mathbb{C}) \cong \mathbb{C}$. If the manifold is Kähler, this is true by the Hodge decomposition theorem. If the manifold is not Kähler, the Frölicher spectral sequence does not necessarily degenerate at page 1. In this case, we only have a surjection, but not necessarily an isomorphism.

REMARK 6.56. The short exact sequence in the proposition splits in a non canonical way. Fix a point $x \in X$. Sending 1 to the cycle class associated with x in the integral Bott-Chern cohomology gives such a splitting. But a priori such a splitting depends on the choice of x.

Bibliography

[ABW19] Mats Andersson, Zbigniew Błocki, Elizabeth Wulcan, *On a Monge-Ampère operator for plurisubharmonic functions with analytic singularities,* Indiana University Mathematics Journal 68 (2019), 1217-1231.

[AH19] Fabrizio Anella, Andreas Höring, *Twisted cotangent bundles of Hyperkähler manifolds*, arXiv:1906.11528

[BBP13] Uniruledness of stable base loci of adjoint linear systems via Mori Theory (with A. Broustet and G. Pacienza), Math. Z. 275, no. 1-2, 499-507 (2013)

[BDPP13] S. Boucksom, J. -P. Demailly, M. Paun, T. Peternell: *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013) 201-248.

[Bea00] Arnaud Beauville, *Complex manifolds with split tangent bundle.* Complex analysis and algebraic geometry (in memory of M. Schneider), 61-70; de Gruyter (2000).

[BEGZ10] Sébastien Boucksom, Philippe Eyssidieux, Vincent Guedj, Ahmed Zeriahi, *Monge-Ampère equations in big cohomology classes,* Acta Math. 205 (2010), no. 2, 199–262.

[Bis11] Jean-Michel Bismut, *Laplacien hypoelliptique et cohomologie de Bott-Chern,* C. R. Math. Acad. Sci. Paris 349 (2011), no. 1-2, 75–80.

[Bis13] Jean-Michel Bismut, *Hypoelliptic Laplacian and Bott-Chern cohomology,* Birkhauser, Progress in Mathematics, 2013 ed. (24 mai 2013), 203 pages.

[BL92] Christina Birkenhake, Herbert Lange, *Complex Abelian Varieties,* Springer-Verlag Berlin and Heidelberg GmbH and Co. K; Grundlehren der mathematischen Wissenschaften; (1992); 638 pages.

[Bl56] André Blanchard. *Sur les variétés analytiques complexes*, Ann. Sci. Ec. Norm.Sup.,73 (1956), 151-202.

[Blo72] Spencer Janney Bloch, *Semi-regularity and de Rham cohomology,* Invention Math.,17, (1972), 51–66.

[Bl19] Zbigniew Błocki, *On the Monge-Ampère operator for quasi-plurisubharmonic functions with analytic singularities,* Bulletin of the London Mathematical Society 51 (2019), 431-435.

[Bog] F. Bogomolov, Unstable vector bundles on curves and surfaces,Proc. Internat. Congress of Math.Helsinki(1978), 517–524.

[Bou02a] S. Boucksom: *On the volume of a line bundle*, Internat. J. Math. **13** (2002), no. 10, 1043-1063.

[Bou02b] Sébastien Boucksom, *Cônes positifs des variétés complexes compactes* (French), Ph. D. Thesis (2002).

[Bou04] S. Boucksom: *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 45-76.

[BP08] B. Berndtsson, M. Păun: *Bergman kernels and the pseudoeffectivity of relative canonical bundles.* Duke Mathematical Journal, 2008, 145(2): 341-378.

[BS56] A. Borel, J.-P. Serre,*Le théorème de Riemann-Roch (d'après Grothendieck)*, Bull. Soc. Math. France, 86, (1958), 97–136.

[Br03] Marco Brunella, *Foliations on complex projective surfaces.* (English) Zbl 1070.32502 Dynamical systems. Part II. Topological, geometrical and ergodic properties of dynamics. Selected papers from the Research Trimester held in Pisa, Italy, February 4–April 26, 2002. Pisa: Scuola Normale Superiore. Pubblicazioni del Centro di Ricerca Matematica Ennio de Giorgi. arXiv: math.CV/0212082

[BS94] S. Bando, Y.-T. Siu, Stable sheaves and Einstein-Hermitian metrics, Geometry and analysis on complex manifolds, 39–50, World Sci. Publ., River Edge, NJ, (1994).

[BT82] E. Bedford and B.A. Taylor. A new capacity for plurisubharmonic functions, Acta Math.,149(1982), 1–41.

[BoTu82] Raoul Bott, Loring W Tu, *Differential Forms in Algebraic Topology,* Graduate Texts in Mathematics 82,(1982), Springer.

[Cam04] F. Campana. Orbifolds, special varieties and classification theory. Ann. Inst.Fourier 54 (2004), 499-665.

[Cam11] F. Campana. Orbifoldes géométriques spéciales et classification biméro-morphe des variétés Kählériennes compactes. JIMJ 10 (2011), 809-934.

[Cao13] Junyan Cao, *A remark on compact Kähler manifolds with nef anticanonical bundles and applicaions,* arXiv:1305.4397.

[Cao14] Junyan Cao, Numerical dimension and a Kawamata-Viehweg-Nadel type vanishing theorem on compact Kähler manifolds, Compos. Math. 150 (2014), n.11, pp 1869-1902.

[Cao18] Junyan Cao, *Kodaira dimension of algebraic fiber spaces over surfaces* , Algebr. Geom. 5 (2018), no. 6, 728–741.

[Cao19] J. Cao: *Albanese maps of projective manifolds with nef anticanonical bundles.* arXiv:1612.05921, Annales Scientifiques de l'École Normale Supérieure, 4e série, t. 52, (2019), 1137-1154.

[CCM19] Frédéric Campana, Junyan Cao, Shin-ichi Matsumura, Projective klt pairs with nef anti-canonical divisor. arXiv:1910.06471

[CGP13] Frédéric Campana, Henri Guenancia, Mihai Păun, *Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields,* Annales scientifiques de l'Ecole Normale Supérieure, Série 4, Tome 46 (2013) no. 6, p. 879-916.

[CH17] Junyan Cao, Andreas Höring, *Manifolds with nef anticanonical bundle,* J. Reine Angew. Math. 724 (2017), 203–244.

[CH19] Junyan Cao, Andreas Höring, *A decomposition theorem for projective manifolds with nef anticanonical bundle,* Journal of Algebraic Geometry 28 (2019), 567-597.

[CP17] Junyan Cao, Mihai Păun, *Kodaira dimension of algebraic fiber spaces over Abelian varieties,* Invent. Math. 207 (2017), no. 1, pp 345-387.

[CT15] T. C. Collins, V. Tosatti: *Kähler currents and null loci*, arXiv:1304.5216v4 [math.CV]. Inventiones mathematicae volume 202, pages1167–1198(2015).

[CT14] T. C. Collins, V. Tosatti: *An extension theorem for Kähler currents*, arXiv:1311.4485v2 [math.CV]. Annales de la faculté des sciences de Toulouse Mathématiques (2014), Volume: 23, Issue: 4, page 893-905.

[Dem82] J.-P. Demailly. —Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au dessus d'une variété kählérienne complète, Ann. Sci. Ec. Norm. Sup.,15(1982), 457–511.

[Dem92a] J.-P. Demailly: *Singular Hermitian metrics on positive line bundles.* Complex algebraic varieties. Springer Berlin Heidelberg, 1992: 87-104.

[Dem92b] J.-P. Demailly: *Regularization of closed positive currents and intersection theory.* Journal of Algebraic Geometry, 1992, 1(3): 361-409.

[Dem93] Jean-Pierre Demailly, A numerical criterion for very ample line bundles, J. Differential Geom 37 (1993) 323-374

[Dem99] Jean-Pierre Demailly, Pseudoconvex-concave duality and regularization of currents, Several Complex Variables, MSRI publications, Volume 37, Cambridge Univ. Press (1999), 233-271.

[Dem02] Jean-Pierre Demailly. *On the Frobenius integrability of certain holomorphic p-forms,* math.AG/0004067, Complex Geometry, Collection of Papers dedicated to Hans Grauert, edited by I. Bauer, F. Catanese, Y. Kawamata, T. Peternell, and Y.-T. Siu, Springer (2002) 93-98.

[Dem14] Jean-Pierre Demailly, *On the cohomology of pseudoeffective line bundles,* written exposition of a talk given at the Abel Symposium, Trondheim, July 2013; manuscript Institut Fourier, January 9, (2014).

[Dem12a] J.-P. Demailly: *Analytic methods in algebraic geometry.* International Press, 2012.

[Dem12b] Jean-Pierre Demailly, *Complex analytic and differential geometry.* online-book: https://www-fourier.ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf, (2012).

[Deng16] Ya Deng, *Simpson correspondence for semistable Higgs bundles over Kähler manifolds.* hal-02391629 . To appear in Pure and Applied Mathematics Quarterly.

[Div16] Simone Diverio. *Segre forms and Kobayashi-Lübke inequality.* Math. Z., 283(3-4) :1033–1047, 2016.

[Don85] S. K. Donaldson, Anti self-dual Yang-Mills connections over complexalgebraic surfaces and stable vector bundles,Proc. London Math. Soc. (3)50(1985), 1–26.

[deR60] George de Rham, *Variétés Différentiables,* Paris, Hermann, 1960. (3rd Ed., 1973).

[DP04] J.-P. Demailly, M. Paun. —Numerical characterization of the Kähler cone of a compactKähler manifold, arXiv: math.AG/0105176, Annals of Math,159(2004), 1247–1274.

[DP03] Jean-Pierre Demailly, Thomas Peternell. *A Kawamata-Viehweg vanishing theorem on compact Kähler manifolds.* In Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), Surv. Differ.Geom., VIII, pages 139–169. Int. Press, Somerville, MA, 2003.

[DPS93] Jean-Pierre Demailly, Thomas Peternell and Michael Schneider, *Kähler manifolds with nu-merically effective Ricci class,* Compositio Math. 89 (1993) 217-240.

[DPS94] J.-P. Demailly, T. Peternell, M. Schneider: *Compact complex manifolds with numerically effective tangent bundles.* J. Algebraic Geom. vol 3, (1994) number 2, pp. 295-345.

[DPS96] J.-P. Demailly, T. Peternell, M. Schneider: Compact Kähler manifolds with hermitian semipositive anticanonical bundle, Compositio Math. 101 (1996) 217-224.

[DPS01] J.-P. Demailly, T. Peternell, M. Schneider: *Pseudo-effective line bundles on compact Kähler manifolds.* International Journal of Mathematics, 2001, 12(06): 689-741.

[Dru18] Stéphane Druel. *A decomposition theorem for singular spaces with trivial canonical class of dimension at most five.*Inventiones mathematicae volume 211, pages245–296(2018), DOI : 10.1007/s00222-017- 0748-y.

[EM84] H. El Mir.Sur le prolongement des courants positifs fermés, Acta Math.,153(1984),1–45.

[Eno93] Ichiro Enoki. *Strong-Lefschetz-type theorem for semi-positive line bundles over compact Kähler manifolds.* In Geometry and global analysis (Sendai, 1993), pages 211–212. Tohoku Univ., Sendai, 1993.

[EV88] H. Esnault, E. Viehweg, *Deligne-Beilinson cohomology, in : Beilinson conjectures on special values of L-functions,* Perspectives in Mathematics 4, Academic press, (1988).

[Fed96] Herbert Federer, *Geometric Measure Theory,* Springer-Verlag Berlin and Heidelberg GmbH and Co. K; Reprint of the 1st ed. Berlin, Heidelberg, New York 1969 (5 janvier 1996). 677 pages.

[Ful84] William Fulton, *Intersection theory,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, (1984), ISBN 0387121765, 9780387121765.

[GH78] Phillip Griffiths, Joseph Harris , *Principles of Algebraic Geometry,* Pure and Applied Mathematics Monograph, Wiley-Blackwell (6 December 1978), 832 pages.

[GL16] Qi'an Guan and Zhenqian Li. *Cluster points of jumping coefficients and equisingularties of plurisubharmonic functions,* arXiv:1604.03406.

[GR58] Hans Grauert, *On Levi's problem and the imbedding of real-analytic manifolds.* Ann. Math. Princeton 68, 460–472 (1958).

[GR70] Hans Grauert, Oswald Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen,* Invent. Math. 11 (1970), 263–292.

[Gri10] Julien Grivaux, *Chern classes in Deligne cohomology for coherent analytic sheaves*, arXiv: 0712.2207v7. Mathematische Annalen 347 (2), (2010), p. 249-284.

[Gro62] Alexander Grothendieck, *Fondements de la géométrie algébrique,* Extraits du Séminaire Bourbaki, 1957-1962, Paris, Secrétariat mathématique, (1962).

[GP16] Henri Guenancia, Mihai Păun, *Conic singularities metrics with prescribed Ricci curvature: General cone angles along normal crossing divisors,* J. Differential Geom. Volume 103, Number 1 (2016), 15-57.

[Gue13] Henri Guenancia, *Kähler-Einstein metrics with cone singularities on klt pairs.* Int. J. Math. 24 1350035 (2013).

[Gul12] Dincer Guler.*On Segre forms of positive vector bundles.* Canad. Math. Bull., 55(1):108– 113, 2012.

[GZ14a] Qi'an Guan and Xiangyu Zhou. strong openness conjecture and related problems for plurisubhar-monic functions. ArXiv e-prints, math.CV/1401.7158, 2014.

[GZ15a] Qi'an Guan and Xiangyu Zhou. Effectiveness of demailly's strong openness conjecture and relatedproblems. ArXiv e-prints, math.CV/1403.7247, 2014. Inventiones mathematicae volume 202, 635–676(2015).

[GZ15b] Q. Guan, X. Zhou: *A solution of an L^2 extension problem with an optimal estimate and applications.* Annals of Mathematics, 2015, 181(3): 1139-1208.

[GZ15c] Qi'an Guan and Xiangyu Zhou. *A proof of Demailly's strong openness conjecture.* ArXive-prints, math.CV/1311.3781, Annals of Mathematics, 2015, 182: 605-616.

[GZ17] Vincent Guedj and Ahmed Zeriahi. *Degenerate Complex Monge–Ampère Equations.* EMS Tracts in Mathematics, Volume: 26; 496 pp; 2017.

[Har77] Robin Hartshorne, *Algebraic Geometry* (Graduate Texts in Mathematics) , Springer; 1st ed. 1977. Corr. 8th printing 1997 edition (April 1, 1997), 496 pages.

[Hat02] Allen Hatcher. *Algebraic Topology,* Cambridge University Press, (2002), 544 pages.

[HIM19] Genki Hosono, Masataka Iwai, Shin-ichi Matsumura, On projective manifolds with pseudo-effective tangent bundle. arXiv:1908.06421.

[Hir64] Heisuke Hironaka. *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.* Ann. of Math. (2) 79 (1964), 109–203; ibid. (2), 79:205–326, 1964.

[HP19] Andreas Höring, Thomas Peternell. *Algebraic integrability of foliations with numerically trivial canonical bundle.* Invent. math. 216, 395–419 (2019). https://doi.org/10.1007/s00222-018-00853-2

[HPS16] Christopher Hacon, Mihnea Popa, Christian Schnell, Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Paun, arXiv:1611.08768

[Huy03] Daniel Huybrechts, *The Kähler cone of a compact hyperkähler manifold,* Math. Ann. (2003) 326: 499. https://doi.org/10.1007/s00208-003-0433, math.AG/9909109.

[Inv84] Birger Iversen, *Cohomology of Sheaves,* Universitext, Springer-Verlag Berlin and Heidelberg GmbH and Co. K, (1984), 464 pages.

[Jou70] Jean-Pierre Jouanolou, *Riemann-Roch sans denominateurs.* (French) Invent. Math. 11, (1970) 15-26.

[Kas02] Masaki Kashiwara, *D-modules and microlocal calculus,* Translations of Mathematical Monographs, Vol. 217, AMS, (2002).

[Kin71] James R. King, *The currents defined by analytic varieties,* Acta Math. 127 (1971), pp. 185-220.

[KS02] Masaki Kashiwara, Pierre Schapira, *Sheaves on manifolds,* Series: Grundlehren der mathematischen Wissenschaften, Springer, (2002), ISBN: 9783540518617,3540518614.

[Kim07] Dano Kim. *Extension of pluriadjoint sections from a log-canonical center.* Thesis (Ph.D.)–Princeton University. 2007.

[Kim15] Dano Kim. *Equivalence of plurisubharmonic singularities and Siu-type metrics.* Monatsh. Math. 178 (2015), no. 1, 85–95.

[Kob75] S. Kobayashi, Negative vector bundles and complex Finsler structures; Nagoya math. J.57(1975) 153–166.

[Kob87] S. Kobayashi, Differential geometry of complex vector bundles, Princeton Univ. Press, 1987.

[Koi17] Takayuki Koike, *Ueda theory for compact curves with nodes.* arXiv:1507.00109, Indiana U. Math. J, Volume 66, Number 3 , 845–876, 2017.

[Kur13] Alex Küronya, *Positivity on subvarieties and vanishing of higher cohomology,* Annales de l'Institut Fourier, Tome 63 (2013) no. 5, p. 1717-1737.

[LaB14] Monica la Barbiera, *On the symmetric algebra of graded modules and torsion freeness.* Mathematical Reports 16(4):503-511, 2014.

[Lae02] Laurent Laeng, *Estimations spectrales asymptotiques en géométrie hermitienne* Ph.D thesis, 2002.

[Lel68] Lelong, P. Fonctions plurisousharmoniques et formes différentielles positives, Dunod, Paris,Gordon and Breach, New York (1968).

[LOY20] Jie Liu, Wenhao Ou, Xiaokui Yang *Projective manifolds whose tangent bundle contains a strictly nef subsheaf,* arXiv:2004.08507

[Laz04] Robert Lazarsfeld, *Positivity in Algebraic Geometry - Volume 1, Classical Setting: Line Bundles and Linear Series.* Springer-Verlag Berlin Heidelberg, (2004). ISBN 978-3-540-22528-7. https://doi.org/10.1007/978-3-642-18808-4.

[Lel57] P. Lelong: Intégration sur un ensemble analytique complexe, Bull. Soc. Math. France,85(1957), 239–262.

[Les12] John Lesieutre, *The diminished base locus is not always closed.* Compositio Mathematica 150(10). December 2012.

[LRRS18] Richard Lärkäng, Hossein Raufi, Jean Ruppenthal, Martin Sera, Chern forms of singular metrics on vector bundles, Advances in Mathematics, Volume 326, 21 February 2018, Pages 465-489

[Mat13] Shin-ichi Matsumura, A Nadel vanishing theorem for metrics with minimal singularities on big line bundles. Advances in Mathematics Volume 280, 6 August 2015, Pages 188-207.

[Mic64] Artibano Micali, *Sur les algèbres universelles.* Ann. Inst. Fourier 14 (1964), 33–88.

[Mou95] Christophe Mourougane. *Versions kählériennes du théorème d'annulation de Bogomolov-Sommese.* C. R. Acad. Sci. Paris Sér. I Math., 321(11):1459–1462, 1995.

[Mou98] Christophe Mourougane, *Versions kählériennes du théorème d'annulation de Bogomolov.* Collectanea Mathematica 49.2-3 (1998): 433-445.

[Mou04] Christophe Mourougane. *Computations of Bott-Chern classes on P(E).* Duke Math. J., 124(2):389–420, 2004.

[Nad89] A. Nadel: *Hyperbolic surfaces in* $\mathcal{CP}^3$. Duke Math. J, 1989, 58(3): 749-771.

[Nad90] Alan Michael Nadel. *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature.* Ann. of Math. (2), 132(3):549–596, 1990.

[Nak04] Noboru Nakayama, *Zariski decomposition and abundance,* MSJ Memoirs, Volume 14 Tokyo, Japan: The Mathematical Society of Japan, (2004), 277 pp.

[OSS80] Christian Okonek, Michael Schneider, Heinz Spindler, *Vector Bundles on Complex Projective Spaces: With an Appendix by S. I. Gelfand,* Modern Birkhäuser Classics, Springer, 1980, 239 pages.

[Ou17] Wenhao Ou, *On generic nefness of tangent sheaves,* arXiv:1703.03175

[Paun98] Mahai Păun, Fibrés en droites numériquement effectifs et variétés kählériennes compactes a courbure de Ricci nef. PhD thesis.

[Paun16] Mahai Păun, Singular Hermitian metrics and positivity of direct images of pluricanonical bundles. Algebraic Geometry: Salt Lake City 2015, pp.519-553. arXiv:1606.00174.

[Ram] Ramanujam, C.P. Remarks on the Kodaira vanishing theorem, J. Indian. Math. Soc., 36 (1972), 41-50.

[Pau17] Mihai Păun. *Relative adjoint transcendental classes and Albanese maps of compact Kähler manifolds with nef Ricci curvature.* Adv. Stud. Pure Math. Higher Dimensional Algebraic Geometry: In honour of Professor Yujiro Kawamata's sixtieth birthday, K. Oguiso, C. Birkar, S. Ishii and S. Takayama, eds. (Tokyo: Mathematical Society of Japan, 2017), 335 - 356.

[Rau15] H. Raufi,Singular Hermitian metrics on holomorphic vector bundles, Ark. Mat.53(2015), no. 2, 359–382.

[Rie71] Oswald Riemenschneider, *Characterizing Moishezon Spaces by Almost Positive Coherent Analytic Sheaves,* Math. Z. 123 (1971), 263–284.

[Ros68] *Hugo Rossi, Picard variety of an isolated singular point,* Rice Univ. Studies 54 (1968), no. 4, 63–73.

[RYY17] Sheng RAO, Song YANG, Xiangdong YANG, *Dolbeault cohomologies of blowing up complex manifolds,* (2017), arXiv:1712.06749.

[Sch07] Michel Schweitzer, *Autour de la cohomologie de Bott-Chern*, arXiv:0709.3528v1.

[Ser56] Jean-Pierre Serre. *Géométrie algébrique et géométrie analytique.* Ann. Inst. Fourier (Grenoble), 6, 1-42, 1956.

[Sko72] Henri Skoda. *Sous-ensembles analytiques d'ordre fini ou infini dans* $\mathbb{C}^n$. Bull. Soc. Math.France,100, 353–408, 1972.

[Sib85] N. Sibony. —Quelques problèmes de prolongement de courants en analyse complexe, Duke Math. J.,52(1985), 157–197.

[Siu74] Y.-T Siu: *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. **27** (1974), 53-156.

[Siu01] Yum-tong Siu; *Very ampleness part of Fujita's conjecture and multiplier ideal sheaves of Kohn and Nadel.* In: Complex Analysis and Geometry, ed. J. D. McNeal, Volume 9 of the Ohio State University Mathematical Research Institute Publications, Verlag Walter de Gruyter, pp.171-191, 2001.

[Sko82] Henri Skoda: Sous-ensembles analytiques d'ordre fini ou infini dans $\mathbb{C}^n$, Bull. Soc. Math.France,100(1972), 353–408.

[Som] Sommese, A.J. ,Submanifolds of abelian varieties, Math. Ann. 233 (1978), 229-250.

[Sou92] Christophe Soulé, *Lectures on Arakelov geometry,* Cambridge University Press ,(1992), ISBN: 9780511623950.

[SS] Bernard Shiffman, Andrew John Sommese. Vanishing Theorems on Complex Manifolds, Series: Progress in Mathematics 56, Publisher: Birkhäuser Basel, Year: 1985.

[Tak97] Kensho Takegoshi. *On cohomology groups of nef line bundles tensorized with multiplier ideal sheaves on compact Kähler manifolds.* Osaka J. Math., 34(4):783–802, 1997.

[Tos] Valentino Tosatti, Orthogonality of divisorial Zariski decompositions for classes with volume zero, Tohoku Math. J. 71 (2019), no.1, 1-8

[UY86] K. Uhlenbeck, S.-T. Yau: *On the existence of Hermitean Yang-Mills-connections on stable bundles over Kähler manifolds*, Comm. Pure Appl. Math. 39 (1986) 257-293.

[Voi02a] Claire Voisin, *A counterexample to the Hodge conjecture extended to Kähler varieties*, IMRN (2002), n° 20, 1057-1075.

[Voi02b] Claire Voisin, *Hodge theory and complex algebraic geometry I.* Cambridge University Press; (2002), 334 pages.

[Wang19] Juanyong Wang, *On the Iitaka Conjecture* $C_{n,m}$ *for Kähler Fibre Spaces,* https://arxiv.org/abs/1907.06705.

[Wu18] Xiaojun Wu, *On the hard Lefschetz theorem for pseudo-effective line bundles,* arXiv:1911.13253.

[Wu19a] Xiaojun Wu, *Note on Junyan Cao's vanishing theorem*, Preprint.

[Wu19b] *Note on the Bogomolov vanishing theorem,* Preprint.

[Wu19c] *Note on the Nakano vanishing theorem,* Preprint.

[Wu19d] Xiaojun Wu, *Considerations on nefness in higher codimension,* Preprint.

[Wu19e] Xiaojun Wu, *Intersection theory and Chern classes in Bott-Chern cohomology,* Preprint.

[Wu20] Xiaojun Wu, *Strongly pseudo-effective and numerically flat reflexive sheaves,* arXiv:2004.14676.

[Yau78] S.T. Yau. —On the Ricci curvature of a complex Kähler manifold and the complex Monge-Ampère equation I, Comm. Pure and Appl. Math.,31(1978), 339–411.

[Zha06] Q. Zhang: *Rational connectedness of log Q-Fano varieties.* Journal fur die reine und angewandte Mathematik (Crelles Journal), 2006, 2006(590): 131-142. IMRN (2002), n0 20, 1057-1075.

[ZZ84] Fouad El Zein, Steven Zucker, *Extendability of normal functions associated to algebraic cycles.* In: Griffiths P (ed) Topics in Transcendental Algebraic Geometry. Annals of Math. Studies, vol. 106, Princeton University Press, pp. 269–288 (1984).