Generalized Okounkov bodies, hyperbolicity-related and direct image problems

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Résumé

Dans le chapitre 1, nous développons le concept de "corps d'Okounkov" pour une (1,1)-classe pseudo-effective sur une variété kählerienne compacte. Nous démontrons la formule de différentiabilité des volumes de classes grosses pour les varétés kähleriennes sur lesquelles les cônes nef modifiés et les cônes nef coïncident. Comme conséquence, nous démontrons l'inégalité de Morse transcendante de Demailly pour ces variétés kähleriennes particulières, y compris les surfaces kähleriennes. Ensuite, nous construisons le corps d'Okounkov généralisé pour toute (1,1)-classe grosse, et nous donnons une caractérisation complète des corps d'Okounkov généralisés sur les surfaces. Nous démontrons que le volume euclidien standard du corps d'Okounkov calcule le volume d'une classe grosse, tel que défini par Boucksom, ce qui permet de résoudre un problème proposé par Lazarsfeld et Mustaţă dans le cas des surfaces. Nous étudions aussi le comportement des corps d'Okounkov généralisés sur le bord du cône gros.

Dans la deuxième partie, nous abordons des problèmes liés à l'hyperbolicité en géométrie complexe. Dans le chapitre 2, nous étudions la dégénérescence des courbes entières qui sont les feuilles de feuilletages sur des variétés projectives. La première partie du chapitre 2 généralise l'approximation diophantienne de McQuillan pour les feuilletages de dimension 1 avec des singularités absolument isolées. Comme application, nous donnons une nouvelle preuve du théorème de Brunella, à savoir que toutes les feuilles d'un feuilletage générique de degré $d \geqslant 2$ dans $\mathbb{C}P^n$ sont hyperboliques. Dans la deuxième partie du chapitre 2, nous introduisons la notion de *singularités faiblement réduites* pour les feuilletages de dimension 1. L'hypothèse de singularités faiblement réduites est moins exigeante que celle de *singularités réduites*, mais joue le même rôle dans l'étude de la conjecture de Green-Griffiths-Lang. Finalement, nous discutons d'une stratégie pour démontrer la conjecture de Green-Griffiths-Lang pour les surfaces complexes.

Dans le chapitre 3, nous démontrons la non-dégénérescence de la mesure de volume au sens de Kobayashi-Eisenman pour une variété dirigée singulière (X, V), c'est-à-dire l'hyperbolicité de la mesure au sens de Kobayashi de (X, V), lorsque le faisceau canonique de V est gros au sens de Demailly.

Dans le chapitre 4, notre premier objectif est de traiter des questions d'effectivité liées aux conjectures de Kobayashi et Debarre, en nous appuyant sur les travaux de Brotbek et de Brotbek-Darondeau. Ensuite, nous combinons ces techniques pour étudier la conjecture sur l'amplitude des fibrés de Demailly-Semple proposée par Diverio et Trapani, et nous obtenons des estimations effectives liées à ce problème. Notre résultat contient à la fois les conjectures de Kobayashi et Debarre, en plus de certaines estimations effectives.

Le but du chapitre 5 est double: d'une part, nous étudions une conjecture du type Fujita proposée par Popa et Schnell, et nous donnons une borne effective linéaire sur la génération globale générique de l'image directe du faisceau pluricanonique tordu. Nous abordons également la relation qui existe entre la valeur de la constante de Seshadri et la borne optimale. D'autre part, nous donnons une réponse affirmative à une question de Demailly-Peternell-Schneider dans un cadre plus général. Comme application, nous généralisons les théorèmes de Fujino et Gongyo sur les images des variétés de Fano faibles au cas KLT, et nous raffinons un résultat de Broustet et Pacienza sur la connexité rationnelle de l'image.

Dans le chapitre 6, nous donnons une preuve concrète et constructive de l'équivalence entre la catégorie de fibrés de Higgs semi-stables de classes de Chern nulles, et celle des représentations linéaires du groupe fondamental d'une variété kählerienne compacte lisse. Ce chapitre est rédigé en particulier pour les lecteurs qui ne sont pas familiers avec la terminologie de la catégorie graduée différentielle, telle qu'elle a été utilisée par Simpson pour démontrer l'équivalence ci-dessus sur les variétés projectives lisses. Il est aussi destiné à exposer une preuve élémentaire de la correspondance de Corlette-Simpson pour les faisceaux de Higgs semi-stables.

Abstract

In Part 1 of this thesis, we construct "Okounkov bodies" for an arbitrary pseudo-effective $(1,1)$-class on a Kähler manifold. We prove the differentiability formula of volumes of big classes for Kähler manifolds on which modified nef cones and nef cones coincide. As a consequence we prove Demailly's transcendental Morse inequality for these particular Kähler manifolds; this includes Kähler surfaces. Then we construct the generalized Okounkov body for any big (1,1)-class, and give a complete characterization of generalized Okounkov bodies on surfaces. We show that this relates the standard Euclidean volume of the body to the volume of the corresponding big class as defined by Boucksom; this solves a problem raised by Lazarsfeld and Mustaţă in the case of surfaces. We also study the behavior of the generalized Okounkov bodies on the boundary of the big cone.

Part 2 deals with Kobayashi hyperbolicity-related problems. Chapter 2's goal is to study the degeneracy of leaves of the one-dimensional foliations on higher dimensional manifolds, along the lines of [**McQ98**, **Bru99**, **McQ08**, **PS14**]. The first part of Chapter 2 generalizes McQuillan's Diophantine approximations for one-dimensional foliations with absolutely isolated singularities, on higher dimensional manifolds. As an application, we give a new proof of Brunella's hyperbolicity theorem, that is, all the leaves of a generic foliation of degree $d \geqslant 2$ in $\mathbb{C}P^n$ is hyperbolic. In the second part of Chapter 2 we introduce the so-called *weakly reduced singularities* for one-dimensional foliations on higher dimensional manifolds. The "weakly reduced singularities" assumption is less demanding than the one required for "reduced singularities", but play the same role in studying the Green-Griffiths-Lang conjecture. Finally we discuss a strategy to prove the Green-Griffiths-Lang conjecture for complex surfaces.

In Chapter 3, assuming that the canonical sheaf $\mathcal{K}_V$ is big in the sense of Demailly, we prove the Kobayashi volume-hyperbolicity for any (possibly singular) directed variety (X, V).

In Chapter 4, our first goal is to deal with effective questions related to the Kobayashi and Debarre conjectures, relying on the work of Damian Brotbek [**Bro16**] and his joint work with Lionel Darondeau [**BD15**]. We then combine these techniques to study the conjecture on the ampleness of the Demailly-Semple bundles raised by Diverio and Trapani [**DT10**], and also obtain some effective estimates related to this problem. Our result integrates both the Kobayashi and Debarre conjectures, with some (non-optimal) effective estimates.

The purpose of Chapter 5 is twofold: on the one hand we study a Fujita-type conjecture by Popa and Schnell, and give an effective (linear) bound on the generic global generation of the direct image of the twisted pluricanonical bundle. We also point out the relation between the Seshadri constant and the optimal bound. On the other hand, we give an affirmative answer to a question by Demailly-Peternell-Schneider in a more general setting. As applications, we generalize the theorems by Fujino and Gongyo on images of weak Fano manifolds to the Kawamata log terminal cases, and refine a result by Broustet and Pacienza on the rational connectedness of the image.

In Chapter 6, we give a concrete and constructive proof of the equivalence between the category of semistable Higgs bundles with vanishing Chern classes and the category of all representations of the fundamental groups [**Cor88**, **Sim88**] on smooth Kähler manifolds. This chapter is written for the complex geometers who are not familiar with the language of differential graded category used by Simpson to prove the above equivalence on smooth projective manifolds, and for those who would like to see an elementary proof of Corlette-Simpson correspondence for semistable Higgs bundles.

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Introduction (français)

0.1. Le corps d'Okounkov généralisé

La théorie des corps d'Okounkov, développée indépendamment par Lazarsfeld et Mustaţă [**LM09**] et par Kaveh et Khovanskii [**KK09**], systématise une construction due à Okounkov [**Oko96**] ; elle généralise le lien entre variétés toriques et polytopes rationnels, en associant un corps convexe à tout fibré en droites sur une variété algébrique projective, via l'introduction d'une valuation adéquate sur le corps de fonctions de cette variété.

Tous les résultats mentionnés ci-dessus concernent principalement les fibrés en droites. Comme l'ont demandé Lazarsfeld et Mustaţă [**LM09**], une question naturelle est de savoir comment construire des corps Okounkov pour les classes de cohomologie transcendantes dans le contexte de la géométrie kählerienne, et comment relier les volumes de ces classes à ceux des corps convexes associés. Dans le Chapitre 1, nous étudions ce problème de manière systématique, et nous résolvons complètement ce problème dans le cas des surfaces kähleriennes.

Rappelons que, dans la construction des corps d'Okounkov pour les fibrés en droites gros, on doit d'abord définir des fonctions de valuation des systèmes linéaires gradués, à valeur dans un domaine euclidien, relativement à un drapeau

$$Y_\bullet : X = Y_0 \supset Y_1 \supset Y_2 \supset \ldots \supset Y_{n-1} \supset Y_n = \{p\}$$

où Y_i est une sous-variété irréductible lisse de codimension i dans X. Compte tenu de l'ensemble des vecteurs de valuation normalisés, le corps d'Okounkov est obtenu par son enveloppe convexe. Cependant, pour les classes transcendantes générales, il n'existe pas d'analogue holomorphe des fibrés en droites ; pour combler ce manque, nous prenons l'ensemble des courants kähleriens à singularités analytiques dans les classes transcendantes. Grâce à la décomposition de Siu, nous sommes en mesure de définir de manière similaire une fonction de valuation.

Soient $\alpha \in H^{1,1}(X,\mathbb{R})$ une classe grosse sur une variété kählerienne X de dimension n, et $Y_\bullet$ un drapeau sur X. Nous définissons $\mathcal{S}_\alpha$ comme l'ensemble des courants kähleriens dans α à singularités analytiques. Nous définissons la fonction de valuation

$$\begin{array}{rcl} \nu : \mathcal{S}_\alpha & \to & \mathbb{R}^n \\ T & \mapsto & \nu_{Y_\bullet}(T) = (\nu_1(T), \ldots \nu_n(T)) \end{array}$$

comme suit. Tout d'abord, nous définissons

$$\nu_1(T) = \sup\{\lambda \mid T - \lambda[Y_1] \geqslant 0\},$$

où $[Y_1]$ est le courant d'intégration sur Y_1. D'après la décomposition de Siu, nous savons que $\nu_1(T)$ est le coefficient $\nu(T, Y_1)$ du courant positif $[Y_1]$ apparaissant dans la décomposition de Siu de T. Puisque T a des singularités analytiques, la restriction $T_1 := (T - \nu_1[Y_1])|_{Y_1}$ est bien définie sur Y_1, qui est encore un courant kählerien à singularités analytiques. Puis nous prenons

$$\nu_2(T) = \sup\{\lambda \mid T_1 - \lambda[Y_2] \geqslant 0\},$$

et nous continuons ainsi de définir les valeurs restantes $\nu_i(T) \in \mathbb{R}^+$.

Définition 0.1.1. Le corps d'Okounkov généralisé $\Delta_{Y_\bullet}(\alpha) \subset \mathbb{R}^n$ par rapport au drapeau $Y_\bullet$ est défini comme l'adhérence de l'ensemble des vecteurs de valuation $\nu_{Y_\bullet}(T)$.

Lorsque cette classe de cohomologie se trouve dans le groupe de Néron-Severi, en appliquant le théorème d'extension d'Ohsawa-Takegoshi, nous prouvons que le corps convexe nouvellement défini coïncide avec le corps d'Okounkov défini antérieurement.

Théorème 1. (= Theorem A) Soient X une variété projective lisse de dimension n, L un fibré en droites gros sur X et $Y_\bullet$ un drapeau admissible fixé. Alors nous avons

$$\Delta_{Y_\bullet}(c_1(L)) = \Delta_{Y_\bullet}(L) = \overline{\bigcup_{m=1}^{\infty} \frac{1}{m}\nu(mL)}.$$

En outre, dans la définition du corps d'Okounkov $\Delta_{Y_\bullet}(L)$, il suffit de prendre l'adhérence de l'ensemble des vecteurs de valuation normalisés au lieu de l'enveloppe convexe.

Un fait important pour les corps d'Okounkov est qu'on peut relier le volume d'un fibré en droites gros au volume euclidien standard du corps d'Okounkov. Il est tout à fait naturel de se demander si notre corps convexe nouvellement défini pour les classes grosses se comporte de la même manière que celui d'origine. Dans le cas des surfaces kähleriennes, nous donnons une caractérisation complète des corps d'Okounkov généralisés, et nous montrons que ce sont des polygones. En outre, nous obtenons une description explicite de la “ finitude” des polygones apparaissant comme les corps d'Okounkov généralisés. En particulier, cela s'applique également aux cas originaux. Notre théorème principal est le suivant :

THÉORÈME 2. (= Theorem B) Soient X une surface compacte kählerienne, $\alpha \in H^{1,1}(X, \mathbb{R})$ une classe grosse. Si C est un diviseur irréductible de X, il existe des fonctions continues linéaires par morceaux

$$f \ , \ g \ : \ [a, s] \mapsto \mathbb{R}_+$$

où f est convexe, g est concave et $f \leqslant g$, telles que $\Delta(\alpha) \subset \mathbb{R}^2$ soit la région bornée par les graphes de f et g :

$$\Delta(\alpha) = \{(t, y) \in \mathbb{R}^2 \mid a \leqslant t \leqslant s, \text{et } f(t) \leqslant y \leqslant g(t)\},$$

où $\Delta(\alpha)$ est le corps d'Okounkov généralisé par rapport au drapeau fixé

$$X \supseteq C \supseteq \{x\},$$

et $s = \sup\{t > 0 \mid \alpha - tC \text{ grosse}\}$. Si C est nef, $a = 0$ et f est croissante ; sinon, $a = \sup\{t > 0 \mid C \subseteq E_{nK}(\alpha - tC)\}$, où $E_{nK} := \bigcap_T E_+(T)$ est le lieu base non-kählerien de T. En outre, $\Delta(\alpha)$ est un polygone fini dont le nombre de sommets est borné par $2\rho(X) + 2$, où $\rho(X)$ est le nombre de Picard de X et

$$\mathrm{vol}_X(\alpha) = 2 \, \mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha)).$$

La preuve du théorème ci-dessus est basée sur l'inégalité de Morse transcendante de Demailly pour les surfaces kähleriennes, et nous le montrons dans le cas des surfaces kähleriennes, en utilisant la décomposition divisorielle de Zariski dûe à Boucksom. En outre, nos résultats s'appliquent également à certaines variétés kähleriennes de dimension supérieure.

THÉORÈME 3. (= Theorem F) Soit X une variété compacte kälerienne de dimension n sur lequel le cône nef modifié $\mathcal{MN}$ coïncide avec le cône nef $\mathcal{N}$. Si α et β sont des classes nefs satisfaisant l'inégalité $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$, alors $\alpha - \beta$ est grosse et $\mathrm{vol}_X(\alpha - \beta) \geqslant \alpha^n - n\alpha^{n-1} \cdot \beta$.

Nous définissons également les corps d'Okounkov généralisés pour des classes pseudo-effectives pour les surfaces kähleriennes, et étudions leurs propriétés. Nous pouvons résumer nos résultats comme suit.

THÉORÈME 4. Soient X une surface kählerienne et α une classe pseudo-effective mais non grosse :

(i) si la dimension numérique vérifie $n(\alpha) = 0$, alors pour toute courbe irréductible C qui n'est pas contenue dans la partie négative $N(\alpha)$ de la décomposition divisorielle de Zariski dûe à Boucksom, le corps d'Okounkov généralisé s'écrit

$$\Delta_{(C,x)}(\alpha) = 0 \times \nu_x(N(\alpha)|_C),$$

où $\nu_x(N(\alpha)|_C) = \nu(N(\alpha)|_C, x)$ est le nombre de Lelong de $N(\alpha)$ en x ;

(ii) si $n(\alpha) = 1$, alors pour toute courbe irréductible C satisfaisant $Z(\alpha) \cdot C > 0$, nous avons

$$\Delta_{(C,x)}(\alpha) = 0 \times [\nu_x(N(\alpha)|_C), \nu_x(N(\alpha)|_C) + Z(\alpha) \cdot C].$$

En particulier, la dimension numérique détermine la dimension du corps d'Okounkov généralisé.

0.2. Dégénérescence des courbes entières sur les variétés algébriques

Dans [**McQ98**], McQuillan a prouvé le théorème suivant, qui résout partiellement la conjecture de Green-Griffiths-Lang pour les surfaces complexes ayant un fibré cotangent gros :

THÉORÈME 0.2.1. Soient X une surface de type général et $\mathcal{F}$ un feuilletage holomorphe sur X, alors toute courbe entière $f : \mathbb{C} \to X$ tangente à $\mathcal{F}$ n'est pas Zariski-dense.

La preuve originelle du Théorème 0.2.1 est compliquée. Par la suite, de nombreux travaux [**Bru99**,**PS14**] se sont attachés à expliquer et simplifier la preuve de McQuillan. Rappelons brièvement l'idée de la preuve du théorème 0.2.1. Supposons qu'il existe une courbe entière Zariski-dense $f : \mathbb{C} \to X$ qui est tangente à $\mathcal{F}$. Alors on peut associer à f un courant positif fermé $T[f]$ de bidimension $(1, 1)$, en suivant la méthode introduite par McQuillan. Ensuite, on étudie les intersections du $T[f]$ avec le fibré tangent et le fibré normal du feuilletage $\mathcal{F}$ respectivement. Les travaux ci-dessus montrent que ces deux nombres d'intersection sont positifs. Cependant, puisque K_X est gros, on a $T[f] \cdot K_X > 0$, et par l'égalité $K_X^{-1} = T_{\mathcal{F}} + N_{\mathcal{F}}$, on aboutit à une contradiction.

Le but du Chapitre 2 est d'étudier la dégénérescence des feuilles d'un feuilletage de dimension un sur les variétés de dimension supérieure. Rappelons d'abord la formule suivante, qui est à la base de notre travail :

THÉORÈME 5. Soit $(X,\mathcal{F})$ une paire 1-feuilletée kählerienne. Si $f:\mathbb{C}\to X$ est une courbe entière tangente à $\mathcal{F}$ dont l'image n'est pas contenue dans $\mathrm{Sing}(\mathcal{F})$, alors

$$\{T[f]\}\cdot c_1(T_{\mathcal{F}})+T(f,\mathcal{J}_{\mathcal{F}})=\{T[f_{[1]}]\}\cdot c_1\big(\mathscr{O}_{X_1}(-1)\big)\geqslant 0,$$

où $\mathcal{J}_{\mathcal{F}}$ est un faisceau d'idéaux cohérent déterminé par les singularités de $\mathcal{F}$, $T(f,\mathcal{J}_{\mathcal{F}})$ est un nombre réel non négatif représentant l'intersection de $T[f]$ avec $\mathcal{J}_{\mathcal{F}}$, et $f_{[1]}$ est le relèvement de f au fibré projectivisé $P(T_X)$.

Si X est une surface complexe, comme dans la preuve de McQuillan [**McQ98**], quitte à considérer un autre modèle birationnel $(\tilde{X},\tilde{\mathcal{F}})\to(X,\mathcal{F})$, le Théorème 5 peut être amélioré comme suit :

$$T[\tilde{f}]\cdot T_{\tilde{\mathcal{F}}}\geqslant 0, \tag{0.2.1}$$

où $\tilde{f}$ est relèvement de f à $\tilde{X}$. Il s'agit d'étendre ce résultat aux dimensions supérieures. Pour cela, nous développons la théorie de McQuillan dans ce cadre.

THÉORÈME 6. (= Theorem G) Soit $(X,\mathcal{F})$ une paire 1-feuilletée kählerienne ayant des singularités simples. Pour toute courbe entière dont l'adhérence de Zariski $\overline{f(\mathbb{C})}^{\text{Zariski}}$ est de dimension au moins deux, et qui est tangente à $\mathcal{F}$, on a toujours

$$T[f]\cdot T_{\mathcal{F}}\geqslant 0.$$

Si on suppose en outre que $K_{\mathcal{F}}$ soit un fibré en droites gros, alors pour toute courbe entière f tangente à $\mathcal{F}$, ou bien f est une feuille algébrique de $\mathcal{F}$, ou bien l'image de f est contenue dans le lieu base augmenté $\mathbf{B}_+(K_{\mathcal{F}})$. En particulier, si $K_{\mathcal{F}}$ est ample, alors il n'existe pas d'application non constante $f:\mathbb{C}\to X$ tangente à $\mathcal{F}$.

Comme application du Théorème 6, nous redémontrons le théorème de Brunella [**Bru06**, Corollary] suivant.

THÉORÈME 0.2.2. (Brunella) Pour un feuilletage générique $\mathcal{F}$ de dimension un et de degré $d\geqslant 2$ sur l'espace projectif complexe $\mathbb{P}^n$, toutes les feuilles de $\mathcal{F}$ sont hyperboliques. Plus précisément, il n'existe pas d'application non constante $f:\mathbb{C}\to X$ tangente à $\mathcal{F}$.

Pour un feuilletage ayant des singularités “ absolument isolées”, on dispose du théorème de résolution des singularités de [**CCS97**, **Tom97**] qui permet de se ramener à des singularités réduites. Plus précisément, il existe une suite finie d'éclatements telle que les singularités de feuilletage soient *simples*. Nous avons le résultat suivant.

THÉORÈME 7. (= Theorem H) Soit $(X,\mathcal{F})$ une paire 1-feuilletée kählerienne ayant des singularités absolument isolées. Pour toute courbe entière dont l'adhérence de Zariski $\overline{f(\mathbb{C})}^{\text{Zariski}}$ est de dimension au moins deux, qui est tangent à $\mathcal{F}$, il existe une suite finie d'éclatements telle que pour le nouveau modèle birationnel $(\tilde{X},\tilde{\mathcal{F}})$, on ait

$$T[\tilde{f}]\cdot T_{\tilde{\mathcal{F}}}\geqslant 0,$$

où $f:\mathbb{C}\to X$ se relève en $\tilde{f}:\mathbb{C}\to\tilde{X}$.

La preuve du théorème 6 repose fortement sur la réduction des singularités. Comme la motivation originelle de McQuillan est d'étudier la conjecture de Green-Griffiths-Lang, nous suggérons d'introduire un type de singularités dites *faiblement réduites*, pour les feuilletages de dimension un sur les variétés de dimension supérieure. La condition correspondante est plus faible que celle mise en jeu par les sinularités dites réduites, mais va jouer essentiellement le même rôle dans l'étude de la conjecture de Green-Griffiths-Lang. Notre théorème est le suivant :

THÉORÈME 8. (=Theorem I) Soit X une variété projective de dimension n munie d'un feuilletage $\mathcal{F}$ de dimension un ayant des singularités faiblement réduites. Si f est une courbe entière Zariski-dense tangente à $\mathcal{F}$, satisfaisant $T[f]\cdot K_X>0$ (par exemple lorsque K_X est gros), alors on a

$$T[\hat{f}]\cdot\det N_{\hat{\mathcal{F}}}<0$$

pour une certaine paire birationnelle $(\hat{X},\hat{\mathcal{F}})$.

On remarque que le résultat suivant de Brunella [**Bru99**, Theorem 2] implique une contradiction, si on le combine avec le Théorème 8 dans le cas des surfaces complexes.

THÉORÈME 9. Soit X une surface complexe munie d'un feuilletage $\mathcal{F}$ de dimension un. On suppose que $f:\mathbb{C}\to X$ est une courbe entière Zariski-dense tangente à $\mathcal{F}$, alors on a

$$T[f]\cdot N_{\mathcal{F}}\geqslant 0.$$

Comme conséquence, nous obtenons immédiatement une autre preuve du théorème de McQuillan 0.2.1 sans utiliser son “inégalité tautologique raffinée”. En outre, cela nous permet d'obtenir une nouvelle stratégie pour étudier la conjecture de Green-Griffiths-Lang :

THÉORÈME 10. (= Theorem J) On suppose (conjecturalement) que le théorème 9 s'étend à toute paire 1-feuilletée kählerienne $(X, \mathcal{F})$, et que l'on a une suite finie d'éclatements telle que les singularités du feuilletage $\mathcal{F}$ soient faiblement réduites. Alors, toute courbe entière dans une surface projective de type général est algébriquement dégénérée.

0.3. Hyperbolicité au sens de la mesure de Kobayashi pour les variétés dirigées singulières de type général

Soit (X, V) une *variété complexe dirigée*, dans le sens de Demailly ; c'est-à-dire X est une variété complexe munie d'un sous-fibré holomorphe $V \subset T_X$ (ou éventellement, d'un sous-espace linéaire holomorphe pouvant présenter des singulariés). La philosophie de Demailly s'appuie sur le fait que certaines constructions fonctorielles fonctionnent mieux dans la catégorie des variétés complexes dirigées [**Dem95**], même dans le “cas absolu”, c'est-à-dire le cas où $V = T_X$. En général, les variétés dirigées sur lesquelles on travaille n'ont aucune raison d'être lisses. Un objectif naturel est d'étendre les résultats plus classiques de l'hyperbolicité aux variétés dirigées. Soit (X, V) une variété dirigée singulière , où X est une variété projective lisse de dimension n, et $V \subset T_X$ un sous-fibré holomorphe de rang $\mathrm{rank}(V) = r$. On définie le pseudo-volume (ou mesure de volume) de Kobayashi–Eisenman comme suit :

DÉFINITION 0.3.1. Le pseudo-volume de Kobayashi–Eisenman de (X, V) est défini par la densité infinitésimale

$$\mathbf{e}^r_{X,V}(\xi) := \inf\{\lambda > 0 \ ; \ \exists f : \mathbb{B}_r \to X,\ f(0) = x,\ \lambda f_*(\tau_0) = \xi,\ f_*(T_{\mathbb{B}_r}) \subset V\}.$$

Dans [**Dem10**] Demailly introduit la notion *faisceau canonique* $\mathcal{K}_V$ pour toute variété dirigée singulière (X, V). Il y est démontré que si $\mathcal{K}_V$ est gros, alors toutes les courbes entières non constantes $f : \mathbb{C} \to (X, V)$ doivent satisfaire certaines équations différentielles algébriques globales. Dans ce chapitre, nous étudions le pseudo-volume de Kobayashi–Eisenman d'une variété dirigée singulière (X, V), lorsque le faisceau canonique $\mathcal{K}_V$ est gros. Notre résultat est le suivant :

THÉORÈME 11. (= Theorem K) Soit (X, V) une variété dirigée singulière. On suppose que le faisceau canonique $\mathcal{K}_V$ est gros. Alors le pseudo-volume de Kobayashi–Eisenman de (X, V) est génériquement non-dégénéré.

REMARK 0.1. Dans le cas absolu, le Théorème 11 est montré dans [**Gri71**] et [**KO71**], et Demailly a énoncé et démontré le théorème 11 pour les variétés dirigées lisses [**Dem95**].

0.4. Autour de la conjecture de Diverio et Trapani

Vers 1970, S. Kobayashi a proposé les conjectures suivante pour les hypersurfaces de l'espace projectif de grand degré $d \geqslant d_n$ par rapport à la dimension. Les bornes optimales indiquées ci-dessous sont suggérées par les travaux de Zaidenberg [**Zai87**].

CONJECTURE 0.4.1. Une hypersurface générique $X_d \subset \mathbb{P}^{n+1}$ de degré d est hyperbolique pour $d \geqslant 2n+1$ si $n \geqslant 2$.

CONJECTURE 0.4.2. Le complémentaire $\mathbb{P}^n \backslash X_d$ est hyperbolique pour une hypersurface générique $X_d \subset \mathbb{P}^n$ de degré $d \geqslant 2n + 1$.

Depuis une quinzaine d'années, au moins trois techniques importantes ont été introduites pour étudier ces conjectures :

(i) Champs de vecteurs méromorphes sur les espaces de jets introduits par Siu dans [**Siu15**] afin d'obtenir davantage d'équations différentielles pour les courbes entières. L'idée consiste en une généralisation de la technique de Voisin.
(ii) La stratégie développée par Demailly dans [**Dem16**] pour étudier la conjecture de Green-Griffiths-Lang.
(iii) La construction et l'utilisation par Brotbek [**Bro16**] de familles de variétés qui sont des déformations des hypersurfaces de type Fermat (des techniques semblables ayant déjà été mises en œuvre antérieurement par Brody-Green [**BG77**] et Nadel [**Nad89**]).

Les outils introduits dans les travaux précédents ont en commun les idées de la théorie des différentielles de jets qui remontent aux travaux de Bloch [**Blo26**], et ont été développée dans les travaux de Green et Griffiths. Dans [**GG79**], à chaque variété X est associé une famille de fibrés maintenant appelés fibrés de différentielles de jets de Green-Griffiths $E^{\mathrm{GG}}_{k,m}T^*_X$, qui sont, grosso modo, des faisceaux d'équations différentielles d'ordre k et de degré m pour les courbes holomorphes.

Les travaux [**Bro16**, **Dem16**] utilisent une version raffinée des fibrés introduits par Demailly dans [**Dem95**], appelés maintenant fibrés de différentielles de jets de Demailly-Semple ou fibrés de differentielles de jets invariants $E_{k,m}T_X^*$. L'une des idées principales est que pour étudier les courbes entières tracées dans X, la chose qui importe seulement est le lieu géometrique des courbes en question et non la façon dont elles sont paramétrées. Demailly a ainsi été amené à considérer le sous-fibré $E_{k,m}T_X^* \subset E_{k,m}^{\mathrm{GG}}T_X^*$ constitué des éléments différentiels invariants par reparamétrage des jets des courbes. Dans ce contexte, Diverio a montré dans [**Div08**] qu'il existe un théorème d'annulation pour les fibrés de différentielles de jets lorsque la codimension de la sous-variété dans $\mathbb{P}^N$ est petite :

Théorème 0.4.1. (Diverio) Soit $X \subset \mathbb{P}^N$ une variété projective lisse de dimension n et de codimension c. Si $1 \leqslant k < n/c$, alors

$$H^0(X, E_{k,m}^{GG}T_X^*) = 0$$

pour tous $m \geqslant 1$.

Rappelons quelques propriétés des variétés dont le fibré cotangent est ample. Une des propriétés remarquables est qu'une variété à fibré cotangent ample est hyperbolique. Déterminer si une variété est hyperbolique ou non, ou si une variété a un fibré cotangent ample sont, en général, des questions très difficiles. De plus, il n'y a que relativement peu d'exemples connus de variétés à fibré cotangent ample. Nous rappelons donc ci-dessous quelques situations où cela se produit. Miyaoka, se basant sur des idées de Bogomolov, a construit des exemples de surfaces à fibré cotangent ample comme intersection complète dans un produit de deux surfaces à fibré cotangent presque ample. Bogomolov a construit des variétés à fibré cotangent ample comme l'intersection complète dans un produit de variétés à fibré cotangent faiblement gros (la construction a été détaillée par Debarre dans [**Deb05**]). Debarre a construit des variétés à fibré cotangent ample comme intersection complète dans une variété abélienne. Motivé par ce résultat, Debarre a conjecturé un résultat analogue dans l'espace projectif. Récemment, en s'appuyant principalement sur les idées et méthodes explicites découlant d'une série d'articles de Brotbek [**Bro14**, **Bro15**], Brotbek et Darondeau [**BD15**] et indépendamment S.-Y. Xie [**Xie15**, **Xie16**] sont parvenus à demontrer la conjecture de Debarre :

Théorème 0.4.2. (Brotbek-Darondeau, Xie) Soient X une variété projective lisse de dimension N, et A un fibré en droites très ample sur X. Alors il existe un nombre positif d_N dépendant de la dimension N, telle que pour tout $c \geqslant \frac{N}{2}$, l'intersection complète de c hypersurfaces générique dans $|A^\delta|$ a un fibré cotangent ample, dès lors que $\delta \geqslant d_N$.

En outre, Xie a donné une borne inférieure effective pour le degré $d_N := N^{N^2}$. Bien que le travail de Brotbek et Darondeau ne soit pas effectif en ce qui concerne le degré, leur méthode renforce les calculs cohomologiques faits dans [**Bro15**], et donne une construction géometrique élégante. Celle-ci consiste à définir une application Ψ du fibré cotangent relatif projectivisé $\mathbb{P}(\Omega_{\mathscr{X}/S})$ vers une certaine famille $\mathscr{Y} \to \mathbf{G}$, appelée "Grassmannianne universelle" dans la section 4.4. Elle est utilisée pour construire beaucoup de formes différentielles symétriques globales avec une torsion négative, en prenat le tiré en arrière pour récupérer de la positivité sur $\mathscr{Y}$. Afin de contrôler le lieu base, nous avons été conduits à utiliser le théorème de Nakamaye (voir [**Laz04**, Théorème 10.3.5] ou [**Bir13**, Théorème 1.3])), qui affirme que pour un fibré en droites gros et nef L, le lieu base augmenté $\mathbf{B}_+(L)$ coïncide avec le lieu base " nul" Null(L). Dans ce cas, L est choisi comme étant le fibré en droites tautologique $\mathscr{L}$ sur la Grasssmannienne universelle $\mathscr{Y}$. Dans le Chapitre 4, nous obtenons un théorème de Nakamaye effectif pour ce fibré L. Comme conséquence, nous obtenons une borne inférieure assez nettement meilleure que celle de Xie :

Théorème 12. Avec les mêmes notations que dans le Théorème 0.4.2, on peut prendre

$$d_N = 4c_0(2N-1)^{2c_0+1} + 6N - 3,$$

où $c_0 := \lfloor \frac{N+1}{2} \rfloor$.

D'autre part, en introduisant une nouvelle compactification de l'ensemble des jets réguliers $J_kT_X^{\mathrm{reg}}/\mathbb{G}_k$, Brotbek est parvenu à développer les idées dans [**BD15**] et à démontrer la conjecture de Kobayashi [**Bro16**]. L'énoncé est le suivant :

Théorème 0.4.3. (Brotbek) Soient X une variété projective lisse de dimension n, et A un fibré en droites très ample sur X. Alors il existe un nombre positif $d_{K,n}$ dépendant de la dimension n, tel que pour tout $d \geqslant d_{K,n}$, un hypersurface générique dans $|A^d|$ soit hyperbolique au sens de Kobayashi.

Le principal outil nouveau introduit par Brotbek est une construction de wronskiens liés à la tour de Demailly-Semple, qui associe des sections d'un fibré en droites aux jets invariants globaux. En fait, avant le travail de Brotbek, on n'avait pas de façon efficace de construire des differentielles de jets invariants globaux, sauf par la technique des connexions méromorphes introduites par Nadel [**Nad89**] et développées plus loin par Demailly-El Goul [**DEG97**]. Cependant, il existe certaines obstructions difficiles à surmonter pour obtenir la positivité du fibré en droites tautologique dans la tour de Demailly-Semple, en raison de la nature géométrique " tordue" de la compactification des fibrés de jets (ref. [**Dem95**]). Dans ce contexte, Brotbek a introduit une façon astucieuse d'éclater les faisceaux d'idéaux définis par les wronskiens. Afin d'obtenir une borne inférieure $d_{K,n}$ dans le Théorème 0.4.3, il suffit d'obtenir la génération effective du faisceau d'idéaux défini par les wronskiens. Dans la section 4.2.3, nous étudions ce problme et obtenons ainsi une borne effective pour la conjecture de Kobayashi :

THÉORÈME 13. Avec la même notation que dans le Théorme 0.4.3, on peut prendre

$$d_{K,n} = n^{n+1}(n+1)^{2n+5}.$$

Dans la même veine que la conjecture de Debarre, dans [**DT10**], Simone Diverio et Stefano Trapani ont posé la conjecture généralisée suivante :

CONJECTURE 0.4.3. (Diverio-Trapani) Soit $X \subset \mathbb{P}^N$ l'intersection complète de c hypersurfaces génériques de degré suffisamment grand. Alors, $E_{k,m}T^*_X$ est ample lorsque $k \geqslant \frac{N}{c} - 1$. En particulier, X est hyperbolique au sens de Kobayashi.

Dans le Chapitre 4, en suivant les méthodes géométriques élégantes de [**BD15**] et [**Bro16**] pour la preuve des conjectures Debarre et Kobayashi, nous démontrons le théorème suivant :

THÉORÈME 14. (= Theorem L) Soient X une variété projective lisse de dimension n, et A un fibré en droites très ample sur X. Soit $Z \subset X$ l'intersection complète de c hypersurfaces génériques dans $|H^0\big(X, \mathscr{O}_X(dA)\big)|$. Alors pour tout entier positif $k \geqslant \frac{n}{c} - 1$, Z est *quasi-k-jet ample* (voir Définition 4.2.1 ci-dessous) lorsque $d \geqslant 2c(\lceil \frac{n}{c} \rceil)^{n+c+2} n^{n+c}$. En particulier, Z est hyperbolique au sens de Kobayashi.

Comme notre définition de "quasi-k-jet ample" coïncide avec l'amplitude du fibré cotangent lorsque $k = 1$, le Théorème 14 contient les conjectures de Kobayashi ($c = 1$) et Debarre ($c \geqslant \frac{n}{2}$), avec certaines estimations effectives (non-optimales).

Quitte à prendre une borne inférieure légèrement plus grande, en nous appuyant sur une astuce de factorisation due à Xie [**Xie15**], nous pouvons obtenir une borne inférieure uniforme :

THÉORÈME 15. (= Theorem M) Soient X une variété projective lisse de dimension n, et A un fibré en droites très ample sur X. Pour tout c-tuple $\mathbf{d} := (d_1, \ldots, d_c)$ tel que $d_p \geqslant c^2 n^{2n+2c}(\lceil \frac{n}{c} \rceil)^{2n+2c+4}$, pour tout p tel que $1 \leqslant p \leqslant c$, et pour toutes hypersurfaces générales $H_p \in |A^{d_p}|$, l'intersection complète $Z := H_1 \cap \ldots \cap H_c$ est quasi-k-jet ample lorsque $k \geqslant k_0$.

En utilisant la relation qui existe entre les fibrés tautologiques dans les tours de Demailly-Semple et les fibrés de différentielles de jets invariants, nous prouvons le théorème suivant sur la conjecture de Diverio-Trapani :

THÉORÈME 16. (= Theorem N) Définissons $q := \mathscr{X}_{\mathbf{d}} \to \prod_{p=1}^{c} |A^{d_p}|$ comme la famille universelle des c-intersections complètes d'hypersurfaces dans $\prod_{p=1}^{c} |A^{d_p}|$, où $d_p \geqslant c^2 n^{2n+2c}(\lceil \frac{n}{c} \rceil)^{2n+2c+4}$ pour chaque $1 \leqslant p \leqslant c$. Définissons $U \subset \prod_{p=1}^{c} |A^{d_p}|$ comme l'ouvert de Zariski de $\prod_{p=1}^{c} |A^{d_p}|$ au dessus duquel $q : \mathscr{X} := q^{-1}(U) \to U$ est une fibration lisse.

Alors pour chaque $j \gg 0$, il existe un sous-fibré $V_j \subset E_{k,jm}T^*_{\mathscr{X}/U}$ défini sur $\mathscr{X}$, dont la restriction de V_j à la fibre générale Z de q est un fibré ample. De plus, pour tout $x \in Z$ fixé et tout k-jet de courbe holomorphe *régulier* $[f] : (\mathbb{C}, 0) \to (Z, x)$, il existe $P_j \in H^0(Z, V_j|_Z \otimes A^{-1})$ tel que

$$P_j(f', \ldots, f^{(k)}) \neq 0.$$

En d'autres termes, ce théorème montre que nous pouvons trouver un sous-fibré des fibrés de différentielles de jets invariants qui est ample et tel que son *lieu de base de Demailly-Semple* défini dans [**DR13**, Section 2.1] est vide.

0.5. Applications des théorèmes d'extension de L^2 aux problèmes d'images directes

Une des applications marquantes de la technique de Bochner-Kodaira-Nakano-Hörmander est le théorème d'extension L^2 de Ohsawa-Takegoshi [**OT87**]. Par la suite, de nombreux travaux se sont attachés à améliorer les bornes effectives obtenues et à développer des approches plus algébriques. On peut citer ainsi les travaux de Manivel, Siu, Berndtsson, Popovici, Błocki, Guan-Zhou, Junyan Cao [**Man93**, **Siu95**, **Ber96**, **Pop05**, **Blo13**, **GZ15**, **Cao14**]. Plus récemment, Demailly a généralisé le théorème d'extension aux sous-variétés non nécessairement réduites, sous des conditions (probablement optimales) de courbure [**Dem15a**]. Il faut mentionner ici que les résultats d'extension de Ohsawa-Takegoshi ont de nombreuses conséquences fondamentales en géométrie algébrique et analytiques : approximation des courants positifs fermés, résultats de positivité de la courbure de la métrique de Bergman, invariance des plurigenres par déformation, confirmation de la conjecture de Iitaka dans le cas où la base est une variété abélienne, etc.

Dans le Chapitre 5, nous utilisons le résultat d'extension de Demailly et le théorème d'extension pluricanonique de Berndtsson et Păun [**BP08**] pour étudier deux problèmes sur la positivité des images directes. Tout d'abord, nous étudions une conjecture de Popa et Schnell :

Conjecture 0.5.1. (Popa-Schnell) Soient $f : X \to Y$ une application surjective entre deux variétés projectives non singulières X et Y, où $\dim(Y) = n$, et L un fibré en droites ample sur Y. Alors, pour tout $k \geqslant 1$, le faisceau

$$f_*(K_X^{\otimes k}) \otimes L^l$$

est engendré par ses sections pour $l \geqslant k(n+1)$.

Dans [**PS14**], Popa et Schnell ont prouvé la conjecture dans le cas où L est un fibré en droites ample et engendré par ses sections globales, lorsque $\dim(X) = 1$. Dans une prépublication récente [**Dut17**], en appliquant le travail d'Angehrn et Siu, Dutta a amélioré le résultat de Popa et Schnell, mais avec une borne quadratique de l en termes de la dimension n :

$$l \geqslant k\Big(\binom{n+1}{2} + 1\Big).$$

Dans le Chapitre 5, en appliquant le résultat d'extension de Demailly ainsi que la méthode de l'invariance des plurigenres de Păun [**Pau07**], nous montrons le théorème suivant :

Théorème 17. (= Theorem O) Soient $f : X \to Y$ une application surjective entre deux variétés projectives non singulières X et Y, où $\dim(Y) = n$, et L un fibré en droites ample sur Y. Si y est une valeur régulière de f, alors pour tout $k \geqslant 1$, le faisceau

$$f_*(K_X^{\otimes k}) \otimes L^l$$

est, en point donné $y \in Y$, engendré par ses sections pourvu que

$$l \geqslant k(\Big\lfloor \frac{n}{\epsilon(L,y)} \Big\rfloor + 1),$$

où $\epsilon(L,y) > 0$ est la constante de Seshadri de L au point y. En particulier, lorsque $\epsilon(L,y) \geqslant 1$ aux points en position très générale $y \in Y$, alors la conjecture 0.5.1 est valable pour les points en position générale en Y ; c'est-à-dire que l'image directe

$$f_*(K_X^{\otimes k}) \otimes L^l$$

est engendrée par ses sections aux points de Y en position générale, dès que $l \geqslant k(n+1)$.

D'après un résultat de Ein-Küchle-Lazarsfeld [**EKL95**], il existe une borne inférieure universelle pour la constante de Seshadri. Plus précisément, pour un point en position très générale $y \in Y$, $\epsilon(L,y) \geqslant \frac{1}{\dim(Y)}$. En appliquant ce résultat, nous obtenons une estimation effective pour la conjecture 0.5.1 :

Théorème 18. (= Theorem P) Soient $f : X \to Y$ une application surjective entre deux variétés projectives non singulières X et Y, où $\dim(Y) = n$, et L un fibré en droites ample sur Y. Alors pour chaque $k \geqslant 1$, l'image directe

$$f_*(K_X^{\otimes k}) \otimes L^l$$

est engendrée par ses sections aux points de Y en position générale pour tout $l \geqslant k(n^2+1)$.

Par rapport au résultat de Dutta, notre borne de l est aussi quadratique en termes de la dimension n mais légèrement plus faible que la sienne. Cependant, si nous utilisons un résultat bien connu dans la théorie de Mori que $K_Y + (n+1)L$ est semi-ample pour tout fibré en droites ample L et le théorème d'extension pluricanonique de Berndtsson-Păun [**BP08**], nous obtenons une borne pour l linéaire en termes de la dimension n.

Théorème 19. (= Theorem Q) Soient $f : X \to Y$ une application surjective entre deux variétés projectives non singulières X et Y, où $\dim(Y) = n$, et L un fibré en droites ample sur Y. Alors pour tout $k \geqslant 1$, l'image directe

$$f_* K_X^{\otimes k} \otimes L^{\otimes l}$$

est engendrée par ses sections aux points de Y en position générale, dès que $l \geqslant k(n+1) + n^2 - n$.

La deuxième partie du Chapitre 5 est consacrée à l'étude d'une question de Demailly-Peternell-Schneider posée dans [**DPS01**] :

PROBLÈME 1. Soient $f : X \to Y$ une application surjective entre deux variétés projectives normales $\mathbb{Q}$-Gorenstein. Lorsque $-K_X$ est pseudo-effectif et que son lieu de base non-nef ne se projette pas surjectivement sur Y, $-K_Y$ est-il pseudo-effectif?

Inspiré par le travail récent de J. Cao sur l'isotrivialité locale de l'application d'Albanese d'une variété à fibré anticanonique nef [**Cao16**], nous répondons positivement au problème 1 lorsque X et Y sont remplacées par des paires :

THÉORÈME 20. (= Theorem R) Soit $f : X \to Y$ une application surjective entre deux paires (X, D) et (Y, Δ), où (X, D) est log-canonique et Δ est un $\mathbb{Q}$-diviseur (pas nécessairement effectif) sur Y. On suppose que $-(K_X + D) - f^*\Delta$ est pseudo-effectif, et que le lieu base non-nef $\mathbf{B}_-\big(-(K_X + D) - f^*\Delta\big)$ ne se projette pas surjectivement sur Y. Alors $-K_Y - \Delta$ est pseudo-effectif et le lieu base non-nef est contenu dans $f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \cup Z \cup Z_D$, où Z est la sous-variété minimale de Y telle que $f : X\backslash f^{-1}(Z) \to Y\backslash Z$ soit une fibration lisse, et où Z_D est une union au plus dénombrable de sous-variétés propres contenant Z telle que pour tout $y \notin Z_D$, la paire $\big(f^{-1}(y), D_{\restriction f^{-1}(y)}\big)$ soit aussi log-canonique.

Le théorème suivant de Fujino et Gongyo [**FG14**] est une conséquence directe du Théorème 20 :

THÉORÈME 0.5.1. (Fujino-Gongyo) Soit $f : X \to Y$ une fibration lisse entre deux variétés projectives lisses. Soient D un $\mathbb{Q}$-diviseur effectif sur X tel que (X, D) soit log-canonique, avec $\mathrm{Supp}(D)$ un diviseur à croisements normaux, et $\mathrm{Supp}(D)$ à croisements normaux relativement au dessus de Y. Soit Δ un $\mathbb{Q}$-diviseur (pas nécessairement effectif) sur Y. Si $-(K_X + D) - f^*\Delta$ est nef, alors $-K_Y - \Delta$ est aussi nef.

En outre, nous montrons le théorème suivant :

THÉORÈME 21. (= Theorem S) Avec les même notations dans le Théorème 20, on suppose que (X, D) est klt, que $-K_X - D - f^*\Delta$ est gros et que son lieu de base non-nef $\mathbf{B}_-(-K_X - D - f^*\Delta)$ ne domine pas Y. Alors $-K_Y - \Delta$ est gros et son lieu base non-nef est contenu dans $f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \cup Z \cup Z_D$.

En combinant les Théorèmes 20 et 21, nous prouvons le théorème suivant, qui est une généralisation d'un théorème de Fujino et Gongyo [**FG12**] sur l'image des variétés de Fano faibles :

THÉORÈME 22. (= Theorem T) Soit $f : X \to Y$ une fibration lisse entre deux variétés projectives lisses, et soit D un $\mathbb{Q}$-diviseur effectif sur X tel que (X, D) soit klt, avec $(X_y, D_{\restriction X_y})$ klt pour tout $y \in Y$. Soit Δ un $\mathbb{Q}$-diviseur (pas nécessairement effectif) sur Y. Si $-K_X - D - f^*\Delta$ est gros et nef, alors $-K_Y - \Delta$ est aussi gros et nef.

Nous utilisons le Théorème 21 afin de raffiner un résultat de Broustet et Pacienza sur la connexité rationnelle de l'image [**BrP11**, Théorème 1.2] :

THÉORÈME 23. (= Theorem U) Avec les mêmes notations que dans le Théorème 20, on suppose que (X, D) et (Y, Δ) sont deux paires klt. Si $-(K_X + \Delta + f^*\Delta)$ est gros et si son lieu base non-nef $\mathbf{B}_-(-K_X - D - f^*\Delta)$ ne domine pas Y, alors Y est rationnellement connexe modulo $f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \cup Z \cup Z_D$, ce qui signifie que pour tout point général $y \in Y$, il existe une courbe rationnelle R_y joignant y à un point de $f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \cup Z \cup Z_D$.

REMARQUE 1. Avec les mêmes hypothèses et notations que dans le Théorème 23, dans [**BrP11**], Broustet et Pacienza ont montré que Y est uniréglée.

0.6. Une remarque sur la correspondance de Corlette-Simpson

Sur une variété kählerienne X, on appelle *fibré de Higgs* la donnée d'un couple (E, θ), où E est un fibré holomorphe sur X et θ une forme différentielle régulière de degré 1 à valeurs dans le fibré des endomorphismes $End(E)$, satisfaisant l'identité $\theta \wedge \theta = 0$. En 1965, Narasimhan et Seshadri établissaient une correspondance bijective entre l'ensemble des classes d'équivalence de représentations unitaires irréductibles du groupe fondamental π d'une surface de Riemann compacte X, et l'ensemble des classes d'isomorphisme de fibrés vectoriels stables de degré 0 sur X. La correspondance fut étendue à toute variété projective lisse par Donaldson [**Don85**], puis à toute variété kählérienne compacte par Uhlenbeck et Yau [**UY86**]. Le résultat fondamental de C. Simpson [**Sim88**] donne une caractérisation des fibrés de Higgs sur lesquels il existe une métrique de Yang-Mills.

Par la suite, C. Simpson a trouvé un analogue pour les représentations linéaires quelconques. Le résultat essentiel de Simpson [**Sim92**], qui repose en partie sur les résultats de Corlette [**Cor88**], et surtout sur ceux de Donaldson [**Don87**], consiste à établir une équivalence de catégories entre la catégorie des représentations linéaires du groupe fondamental d'une variété projective lisse, celle des fibrés plats, et celle des fibrés de Higgs semi-stables de classes de Chern nulles. Ceci se traduit, quand on fixe le rang r, par l'existence de trois espaces de modules grossiers $M_{\mathrm{B}}(r)$, $M_{\mathrm{DR}}(r)$ et $M_{\mathrm{Dol}}(r)$ associés à de tels objets ; ces variétés algébriques ont même ensemble de points fermés, mais les structures algébriques diffèrent.

Le but du chapitre 6 est de donner une preuve concrète et constructive de la correspondance de Simpson entre la catégorie des représentations linéaires du groupe fondamental d'une variété compacte kählerienne lisse, et celle des fibrés de Higgs semi-stables de classes de Chern nulles. Le résultat est le suivant :

Théorème 24. (= Theorem V) Soit X une variété compacte kählerienne lisse. Alors les énoncés suivants sont équivalents :
(i) E est un fibré plat sur X ;
(ii) il existe une structure de fibré de Higgs $(E, \overline{\partial}, \theta)$ sur E, et (E, θ) possède une filtration :
$$\{0\} = (E_0, \theta_0) \subset (E_1, \theta_1) \subset \ldots \subset (E_m, \theta_m) = (E, \theta)$$
où les (E_i, θ_i) sont des sous-faisceaux de Higgs de (E, θ) tels que chaque gradué $(E_i, \theta_i)/(E_{i-1}, \theta_{i-1})$ soit un fibré de Higgs stable de classes de Chern nulles.
(iii) E est un fibré de Higgs semi-stable de classes de Chern nulles.

Dans le chapitre 6, nous ne prouvons que l'équivalence entre (i) et (ii) dans le Théorème 24. L'implication (ii) $\Rightarrow$ (iii) est triviale. Pour montrer que (iii) implique (ii), il suffit de montrer que tous les gradués dans la filtration de Jordan-Hölder d'un fibré de Higgs semi-stable de classes de Chern nulles sont des faisceaux localement libres. Dans [**DPS94**], les auteurs ont prouvé ce résultat pour les fibrés sans champ de Higgs θ. Dans [**Sim92**, Théorème 2], lorsque X est projectif, Simpson a utilisé le théorème de restriction de Mehta et Ramanathans afin de montrer le Théorème 24. Dans une prépublication récente [**NZ15**], en appliquant le flot de Yang-Mills-Higgs pour construire la métrique approximative de Hermite-Einstein pour les fibrés de Higgs semi-stables, et en combinant ceci avec les techniques de [**DPS94**], Y.-C. Nie et X. Zhang ont prouvé l'implication (iii) $\Rightarrow$ (ii).

En particulier, si le champ de Higgs disparaît, le résultat de Simpson suivant est une conséquence directe du Théorème 24 :

Théorème 0.6.1. (Simpson) Soit X une variété compacte kählerienne lisse. On suppose que le fibré holomorphe E sur X possède une filtration

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_p = E \tag{0.6.1}$$

tel que tous les gradués E_k/E_{k-1} soient hermitiens plats. Alors la connexion de Gauss-Manin D_E de E est compatible avec la connexion hermitienne naturelle sur le gradué E_k/E_{k-1} pour chaque $k = 1, \ldots, p$.

Introduction

0.7. Generalized Okounkov Bodies

The theory of Okounkov bodies, first introduced by Okounkov, and systematically developed by Lazarsfeld and Mustaţă [**LM09**] as well as by Kaveh and Khovanskii [**KK09**], aims at generalizing the notion of the Newton polytope of a projective toric variety. Its main purpose is to associate a convex body in the Euclidean space to a big line bundle on a projective manifold.

All the results mentioned above are mainly concerned with the line bundles. As was asked by Lazarsfeld and Mustaţă [**LM09**], a natural question would be to construct Okounkov bodies for transcendental cohomology classes in the Kähler geometry setting, and to realize the volumes of these classes by convex bodies as well. In Chapter 1, we have studied this problem in a systematic way, and we have solved this problem completely in the case of Kähler surfaces.

It should be stressed that in the construction of Okounkov bodies for big line bundles, one first has to define valuation-like functions from the graded linear systems of the line bundle to the Euclidean domain, with respect to a fixed flag

$$Y_\bullet : X = Y_0 \supset Y_1 \supset Y_2 \supset \ldots \supset Y_{n-1} \supset Y_n = \{p\},$$

where Y_i is a smooth irreducible subvariety of codimension i in X. Then the Okounkov body is obtained by taking the convex hull of the set of normalized valuation vectors. However, for general transcendental classes, there is no holomorphic analogue of the linear system of the line bundle; instead of this, we consider the set of Kähler currents with analytic singularities in the transcendental classes. Thanks to the Siu decomposition theorem, it is possible to define a valuation-like function very much as in the algebraic situation.

Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class on a n-dimensional Kähler manifold X, and let $Y_\bullet$ be a fixed flag on X. Set $\mathcal{S}_\alpha$ to be the set of Kähler currents in α with analytic singularities. We define the valuation-like function

$$\begin{aligned} \nu : \mathcal{S}_\alpha &\to \mathbb{R}^n \\ T &\mapsto \nu_{Y_\bullet}(T) = (\nu_1(T), \ldots \nu_n(T)) \end{aligned}$$

as follows. First, set

$$\nu_1(T) = \sup\{\lambda \mid T - \lambda[Y_1] \geqslant 0\},$$

where $[Y_1]$ is the current of integration over Y_1. By Siu's decomposition, we know that $\nu_1(T)$ is the coefficient $\nu(T, Y_1)$ of the positive current $[Y_1]$ appearing in the Siu decomposition of T. Since T has analytic singularities, the restriction $T_1 := (T - \nu_1[Y_1])|_{Y_1}$ is well-defined over Y_1, which is still a Kähler current with analytic singularities. Then take

$$\nu_2(T) = \sup\{\lambda \mid T_1 - \lambda[Y_2] \geqslant 0\},$$

and continue in this manner to define the remaining values $\nu_i(T) \in \mathbb{R}^+$.

Definition 0.7.1. The generalized Okounkov bodies $\Delta_{Y_\bullet}(\alpha) \subset \mathbb{R}^n$ with respect to the flag $Y_\bullet$ is defined to be closure of the set of valuation vectors $\nu_{Y_\bullet}(T)$.

When this cohomology class happens to lie in the Néron-Severi group, by applying the Ohsawa-Takegoshi extension Theorem, we prove that the newly defined convex body coincides with the original Okounkov body.

Theorem I. *(= Theorem A) Let X be a smooth projective variety of dimension n, L be a big line bundle on X and $Y_\bullet$ be a fixed admissible flag. Then we have*

$$\Delta_{Y_\bullet}(c_1(L)) = \Delta_{Y_\bullet}(L) = \overline{\bigcup_{m=1}^{\infty} \frac{1}{m}\nu(mL)}.$$

Moreover, in the definition of Okounkov body $\Delta_{Y_\bullet}(L)$, it suffices to take the closure of the set of normalized valuation vectors instead of the closure of the convex hull.

An important fact for the Okounkov bodies is that one can relate the volume of a given big line bundle to the standard Euclidean volume of its Okounkov body. It is quite natural to wonder whether our newly defined convex body for big classes behaves similarly as the original Okounkov body. In the case of Kähler surfaces, we give a complete characterization of generalized Okounkov bodies, and show that they must always be finite polygons. Moreover, we obtain an explicit description for the "finiteness" of the polygons appearing as generalized Okounkov bodies of big classes. In particular, this also holds for the original Okounkov bodies. Our main theorem is the following

Theorem II. *(= Theorem B) Let X be a compact Kähler surface, $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class. If C is an irreducible divisor of X, there are piecewise linear continuous functions*

$$f \ , \ g \ : \ [a, s] \mapsto \mathbb{R}_+$$

with f convex, g concave, and $f \leqslant g$, such that $\Delta(\alpha) \subset \mathbb{R}^2$ is the region bounded by the graphs of f and g:

$$\Delta(\alpha) = \{(t, y) \in \mathbb{R}^2 \mid a \leqslant t \leqslant s, \text{and } f(t) \leqslant y \leqslant g(t)\}.$$

Here $\Delta(\alpha)$ is the generalized Okounkov body with respect to the fixed flag

$$X \supseteq C \supseteq \{x\},$$

and $s = \sup\{t > 0 \mid \alpha - tC \text{ is big}\}$. If C is nef, $a = 0$ and $f(t)$ is increasing; otherwise, $a = \sup\{t > 0 \mid C \subseteq E_{nK}(\alpha - tC)\}$, where $E_{nK} := \bigcap_T E_+(T)$ for T ranging among the Kähler currents in α, which is the non-Kähler locus. Moreover, $\Delta(\alpha)$ is a finite polygon whose number of vertices is bounded by $2\rho(X) + 2$, where $\rho(X)$ is the Picard number of X, and

$$\operatorname{vol}_X(\alpha) = 2 \operatorname{vol}_{\mathbb{R}^2}(\Delta(\alpha)).$$

The proof of the above theorem is based on Demailly's transcendental Morse inequality for Kähler surfaces, and we achieved this in the case of Kähler surface using Boucksom's Divisorial Zariski decomposition. Moreover, our results also hold for some special Kähler manifolds of higher dimensions.

THEOREM III. *(= Theorem 1.3.5) Let X be a compact Kähler manifold of dimension n on which the modified nef cone $\mathcal{MN}$ coincides with the nef cone $\mathcal{N}$. If α and β are nef classes satisfying the inequality $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$, then $\alpha - \beta$ is big and $\operatorname{vol}_X(\alpha - \beta) \geqslant \alpha^n - n\alpha^{n-1} \cdot \beta$.*

We also define the generalized Okounkov bodies for pseudo-effective classes in Kähler surfaces, and study their properties. We summarize our results as follows

THEOREM IV. *(= Theorem F) Let X be a Kähler surface and α be a pseudo-effective that is not big.*

(i) *If the numerical dimension $n(\alpha) = 0$, then for any irreducible curve C which is not contained in the negative part $N(\alpha)$ of Boucksom's divisorial Zariski decomposition, we have a generalized Okounkov body of the form*

$$\Delta_{(C,x)}(\alpha) = 0 \times \nu_x(N(\alpha)|_C),$$

where $\nu_x(N(\alpha)|_C) = \nu(N(\alpha)|_C, x)$ is the Lelong number of $N(\alpha)$ at x;

(ii) *if $n(\alpha) = 1$, then for any irreducible curve C satisfying $Z(\alpha) \cdot C > 0$, we have*

$$\Delta_{(C,x)}(\alpha) = 0 \times [\nu_x(N(\alpha)|_C), \nu_x(N(\alpha)|_C) + Z(\alpha) \cdot C].$$

In particular, the numerical dimension determines the dimension of the generalized Okounkov body.

0.8. Dengeneracy of Entire Curves on Higher dimensional Manifolds

In [**McQ98**], McQuillan proved the following theorem, which partially solved the Green-Griffiths-Lang conjecture for complex surfaces with big cotangent bundle:

THEOREM 0.8.1. *(McQuillan) Let X be a surface of general type and $\mathcal{F}$ a holomorphic foliation on X. Then no entire curve $\mathcal{F} : \mathbb{C} \to X$ tangent to $\mathcal{F}$ can be Zariski dense.*

The original proof of Theorem 0.8.1 is rather involved. Somewhat later, several works appeared (*e.g.* [**Bru99**, **PS14**]), attempting to explain and simplify McQuillan's proof. Let us recall briefly the idea in proving Theorem 0.8.1. Assume that there exists a Zariski dense entire curve $f : \mathbb{C} \to X$ which is tangent to $\mathcal{F}$. Then, one can study the intersection properties of the Ahlfors current $T[f]$, which is a representative of a $(n-1, n-1)$-cohomology class in X, with the tangent bundle and the normal bundle of the foliation $\mathcal{F}$ respectively. The above works provided that both of the intersections numbers are non-negative. However, since K_X is big, then $T[f] \cdot K_X > 0$, and by the equality $K_X^{-1} = T_{\mathcal{F}} + N_{\mathcal{F}}$, a contradiction is obtained.

The goal of this paper is to study the degeneration of leaves of the one-dimensional foliations on higher dimensional manifolds, inspired by the work [**McQ98**,**Bru99**,**McQ08**,**PS14**]. Let us first recall the following *fundamental intersection formula* [**McQ98**, **Bru99**, **PS14**], which is the basis of our work:

THEOREM 0.8.2. *(Brunella-McQuillan-Păun-Sibony) Let $(X, \mathcal{F})$ be a Kähler 1-foliated pair. If $f : \mathbb{C} \to X$ is an entire curve tangent to $\mathcal{F}$ whose image is not contained in* $\operatorname{Sing}(\mathcal{F})$*, then*

$$\{T[f]\} \cdot c_1(T_{\mathcal{F}}) + T(f, \mathcal{J}_{\mathcal{F}}) = \{T[f_{[1]}]\} \cdot c_1\big(\mathscr{O}_{X_1}(-1)\big) \geqslant 0,$$

where $\mathcal{J}_{\mathcal{F}}$ is a coherent ideal sheaf determined by the singularity of $\mathcal{F}$, and $T(f, \mathcal{J}_{\mathcal{F}})$ is a non-negative real number representing the "intersection" of $T[f]$ with $\mathcal{J}_{\mathcal{F}}$ (see Remark 2.2.1 for the Definition), $f_{[1]}$ is the lift of f to the projectivized bundle $P(T_X)$.

If X is a complex surface, as is proved by McQuillan [**McQ98**], after passing to some birational model $(\tilde{X}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$, Theorem 0.8.2 can be improved to the extent that

$$T[\tilde{f}] \cdot T_{\tilde{\mathcal{F}}} \geqslant 0, \tag{0.8.1}$$

where $\tilde{f}$ is the lift of f to $\tilde{X}$. By pursuing his philosophy of "Diophantine approximation" for foliations, we can generalize (0.8.1) to higher dimensional manifolds:

THEOREM V. *(= Theorem G) Let $(X, \mathcal{F})$ be a 1-foliated pair with simple singularities (see Definition 2.3.2, and if X is a complex surface, simple singularities are reduced ones). For any entire curve whose Zariski closure $\overline{f(\mathbb{C})}^{\text{Zariski}}$ is of dimension at least 2, which is also tangent to $\mathcal{F}$, one always has*

$$T[f] \cdot T_{\mathcal{F}} \geqslant 0.$$

If one further assumes that $K_{\mathcal{F}}$ is a big line bundle, then for any entire curve f tangent to $\mathcal{F}$, either f is an algebraic leaf of $\mathcal{F}$, or the image of f is contained in the augmented base locus $\mathbf{B}_+(K_{\mathcal{F}})$ of $K_{\mathcal{F}}$. In particular, if $K_{\mathcal{F}}$ is ample, then there exists no nonconstant transcendental entire curve $f : \mathbb{C} \to X$ tangent to $\mathcal{F}$.

As an application of Theorem V, we can give a new proof of the following elegant theorem by Brunella [**Bru06**, Corollary]

THEOREM 0.8.3. *(Brunella) For a generic foliation by curves $\mathcal{F}$ of degree $d \geqslant 2$ on the complex projective space $\mathbb{P}^n$, that is, $\mathcal{F}$ is generated by a generic holomorphic section (a rational vector field)*

$$s \in H^0\big(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathscr{O}(d-1)\big),$$

all the leaves of $\mathcal{F}$ are hyperbolic. More precisely, there exists no nonconstant $f : \mathbb{C} \to \mathbb{P}^n$ tangent to $\mathcal{F}$ (and possibly passing through $\mathrm{Sing}(\mathcal{F})$*).*

For any one-dimensional foliation with absolutely isolated singularities (see Definition 2.3.1), by the reduction theorems [**CCS97**, **Tom97**], one can take a finite sequence of blow-up's to make the singularities *simple*. We thus have the following result:

THEOREM VI. *(= Theorem H) Let $\mathcal{F}$ be a foliation by curves on the n-dimensional complex manifold X, such that the singular set* $\mathrm{Sing}(\mathcal{F})$ *of the foliation $\mathcal{F}$ is a set of absolutely isolated singularities. If $f : \mathbb{C} \to X$ is an entire curve whose Zariski closure $\overline{f(\mathbb{C})}^{\text{Zariski}}$ is of dimension at least 2, which is also tangent to $\mathcal{F}$, then one can blow-up X a finite number of times to get a new birational model $(\widetilde{X}, \widetilde{\mathcal{F}})$ such that*

$$T[\widetilde{f}] \cdot T_{\widetilde{\mathcal{F}}} \geqslant 0,$$

where $\tilde{f}$ is the lift of f to $\tilde{X}$.

The proof of Theorem V relies heavily on the reduction of singularities. According to the original motivation of McQuillan is to study the Green-Griffiths-Lang conjecture, we introduce the so-called *weakly reduced singularities* (see Definition 2.3.4) for one-dimensional foliations on higher dimensional manifolds. They are less demanding than the reduced ones, but play the same role in studying the Green-Griffiths-Lang conjecture. Our theorem is as follows:

THEOREM VII. *(=Theorem I) Let X be a projective manifold of dimension n endowed with a one-dimensional foliation $\mathcal{F}$ with weakly reduced singularities. If f is a Zariski dense entire curve tangent to $\mathcal{F}$, satisfying $T[f] \cdot K_X > 0$ (e.g. K_X is big), then we have*

$$T[\hat{f}] \cdot \det N_{\hat{\mathcal{F}}} < 0$$

for some birational model $(\hat{X}, \hat{\mathcal{F}})$.

REMARK 0.8.1. Our definition of "weakly reduced singularities" is actually weaker than the usual concept of reduced singularities, which always requires a lot of checking (e.g. through a classification of singularities). We only need to focus on the multiplier ideal sheaf of $\mathscr{J}_{\mathcal{F}}$, instead of trying to understand the exact behavior of singularities.

It is notable that the following result due to Brunella [**Bru99**, Theorem 2] implies a conclusive contradiction in combination with Theorem VII, in the case of complex surfaces.

THEOREM 0.8.4. *(Brunella) Let X be a complex surface endowed with a foliation $\mathcal{F}$ (no assumption is made for singularities of $\mathcal{F}$ here). If $f : \mathbb{C} \to X$ is a Zariski dense entire curve tangent to $\mathcal{F}$, then we have*

$$T[f] \cdot N_{\mathcal{F}} \geqslant 0.$$

Therefore, we get another proof of McQuillan's Theorem 0.8.1 without using the refined formula (0.8.1) immediately. This leads us to observe that if one could resolve arbitrary singularities of one-dimensional foliations into weakly reduced ones, and generalize the previous Brunella Theorem to higher dimensional manifolds, one could infer the Green-Griffiths conjecture for surfaces of general type.

THEOREM VIII. *(= Theorem J) Assume that Theorem 0.8.4 holds for any directed variety $(X, \mathcal{F})$ where X is a base of arbitrary dimension and $\mathcal{F}$ has rank 1, and that one can resolve the singularities of $\mathcal{F}$ into weakly reduced ones. Then every entire curve drawn in a projective surface of general type must be algebraically degenerate.*

0.9. Kobayashi Volume-Hyperbolicity for (Singular) Directed Varieties

Let (X, V) be a *complex directed manifold*, i.e X is a complex manifold equipped with a holomorphic subbundle $V \subset T_X$. The philosophy behind the introduction of directed manifolds, as initially suggested by J.-P. Demailly, is that, there are certain fonctorial constructions which work better in the category of directed manifolds [**Dem95**]. This is so even in the "absolute case", i.e. in the case $V = T_X$. In general, singularities of V cannot be avoided, even after blowing-up, and V can be seen as a coherent subsheaf of T_X such that T_X/V is torsion free. Such a sheaf V is a subbundle of T_X outside of an analytic subset of codimension at least 2, which we denote here by $\mathrm{Sing}(V)$. The Kobayashi-Eisenman volume measure can also be defined for such (singular) directed pairs (X, V).

DEFINITION 0.9.1. Let (X, V) be a directed manifold with $\dim(X) = n$ and let $\mathrm{rank}(V) = r$. Then the Kobayashi-Eisenman volume measure of (X, V) is the pseudometric defined on any $\xi \in \Lambda^r V_x$ for $x \notin \mathrm{Sing}(V)$, by

$$\mathbf{e}^r_{X,V}(\xi) := \inf\{\lambda > 0;\ \exists f : \mathbb{B}_r \to X,\ f(0) = x,\ \lambda f_*(\tau_0) = \xi,\ f_*(T_{\mathbb{B}_r}) \subset V\},$$

where $\mathbb{B}_r$ is the unit ball in $\mathbb{C}^r$ and $\tau_0 = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}$ is the unit r-vector of $\mathbb{C}^r$ at the origin. One says that (X, V) is *Kobayashi measure hyperbolic* if $\mathbf{e}^r_{X,V}$ is generically positive definite, i.e. positive definite on a Zariski open set.

In [**Dem10**] the author also introduced the concept of *canonical sheaf* $\mathcal{K}_V$ for any singular directed variety (X, V), and he showed that the "bigness" of $\mathcal{K}_V$ implies that all non constant entire curves $f : \mathbb{C} \to (X, V)$ must satisfy certain global algebraic differential equations. In this note, we study the Kobayashi-Eisenman volume measure of the singular directed variety (X, V), and give another geometric consequence of the bigness of $\mathcal{K}_V$. Our main theorem is as follows:

THEOREM IX. *(= Theorem K) Let (X, V) be a compact complex directed variety (where V is possibly singular), and let $\mathrm{rank}(V) = r$, $\dim(X) = n$. If V is of general type (see Definition 3.3 below), with a base locus $\mathrm{Bs}(V) \subsetneq X$, then (X, V) is Kobayashi measure hyperbolic.*

REMARK 0.2. In the absolute case, Theorem IX is proved in [**Gri71**] and [**KO71**]; for a smooth directed variety it is proved in [**Dem95**].

0.10. Effective Results On the Diverio-Trapani Conjecture

The famous Kobayashi conjecture states that a general hypersurface in $\mathbb{P}^n$ of sufficient large degree $d \geqslant d_{K,n}$ is Kobayashi hyperbolic. In the last 15 years, at least three important techniques were introduced to study this problem: Siu's slanted vector fields for higher order jet spaces [**Siu15**], Demailly's approach for the study of the Green-Griffiths-Lang conjecture through directed varieties *strongly of general type* [**Dem16**], and Brotbek's recent construction of families of varieties which are deformations of Fermat type hypersurfaces [**Bro16**]. In the works [**Bro16**, **Dem16**], several important techniques for the study of hyperbolicity-related problems are developed using *invariant* jet differentials $E_{k,m}T^*_X$; these were introduced by J.-P. Demailly in [**Dem95**], and can be seen as a variant of the Green-Griffiths jets $E^{\mathrm{GG}}_{k,m}T^*_X$ initialed by Green-Griffiths [**GG79**], with the additional property that they are invariant under the reparametrization; both types of jets generalize to higher orders the symmetric differentials $S^m T^*_X$. However, when trying to enforce positivity for jet bundles of the complete intersection of hypersurfaces in $\mathbb{P}^N$, one cannot expect to achieve this for lower order jet differentials if the codimension of subvariety is small, as was proved by Diverio [**Div08**]:

THEOREM 0.10.1. *(Diverio) Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of hypersurfaces of any degree in $\mathbb{P}^N$. Then*

$$H^0(X, E^{GG}_{k,m}T^*_X) = 0$$

for all $m \geqslant 1$ and $1 \leqslant k < \dim(X)/\mathrm{codim}(X)$.

On the other hand, the hyperbolicity should be enhanced by taking an intersection of a larger number of projective hypersurfaces of high degree. Debarre verified this in the case of *abelian varieties*, by proving that the intersection of at least $\frac{N}{2}$ sufficiently ample general hypersurfaces in an N-dimensional abelian variety has an ample cotangent bundle. He further conjectured that the analogous statement should also hold for complete intersections in projective space. Very recently, relying mainly on the ideas and explicit methods arising in the series of articles by Brotbek [**Bro14**, **Bro15**], Brotbek and Darondeau [**BD15**] and independently S.-Y. Xie [**Xie15**, **Xie16**] proved the Debarre conjecture::

THEOREM 0.10.2. *(Brotbek-Darondeau, Xie) Let X be any smooth projective variety of dimension N, and let A be a very ample line bundle on X. Then there exists a positive number d_N depending only on the dimension N, such that for each $c \geqslant \frac{N}{2}$, the complete intersection of c general hypersurfaces in $|A^\delta|$ has an ample cotangent bundle as soon as $\delta \geqslant d_N$.*

Moreover, Xie was able to give an effective lower bound on hypersurface degrees $d_N := N^{N^2}$. Although the work by Brotbek and Darondeau is not effective as far as the lower bound d_N is concerned, they were able to strengthen the cohomological computations of [**Bro15**], and produced an elegant geometric construction, which defines a map Ψ from the projectivized relative cotangent bundle $\mathbb{P}(\Omega_{\mathcal{X}/S})$ to a certain family $\mathscr{Y} \to \mathbf{G}$. We called $\mathscr{Y}$ *the universal Grassmannian* in Section 4.4. It is used to construct a lot of global symmetric differential forms with a negative twist, by pulling-back the positivity on $\mathscr{Y}$. In order to make the base locus empty, they apply Nakamaye's Theorem (see [**Laz04**, Theorem 10.3.5] or [**Bir13**, Theorem 1.3]) which asserts that for a big and nef line bundle L on a projective variety, the augmented base locus $\mathbb{B}_+(L)$ coincides with the null locus $\mathrm{Null}(L)$. In this case, L is taken to be the tautological line bundle $\mathscr{L}$ on the universal Grasssmannian $\mathscr{Y}$. In Chapter 4, we obtain an effective result for a slightly weaker statement than the Nakamaye result used by Brotbek and Darondeau, that is still sufficient to complete the argument. In this way, as a consequence of their work, we obtain a better lower bound than Xie's:

THEOREM X. *With the same notation in Theorem 0.10.2, one can take*
$$d_N = 4c_0(2N-1)^{2c_0+1} + 6N - 3,$$
where $c_0 := \lfloor \frac{N+1}{2} \rfloor$.

On the other hand, by introducing a new compactification of the set of regular jets $J_k T_X^{\mathrm{reg}}/\mathbb{G}_k$, Brotbek was able to fully develop the ideas in [**BD15**] to prove the Kobayashi conjecture [**Bro16**]. His statement is the following:

THEOREM 0.10.3. *(Brotbek) Let X be a smooth projective variety of dimension n. For any very ample line bundle A on X and any $d \geqslant d_{K,n}$, a general hypersurface in $|A^d|$ is Kobayashi hyperbolic. Here $d_{K,n}$ depends only the dimension n.*

The main new tool Brotbek introduced is a Wronskian construction related to the Demailly-Semple tower, which associates sections of the line bundle to global invariant jet differentials. In fact, before Brotbek's work, we had few ways of constructing invariant jet differentials except the technique of meromorphic connections introduced by Nadel [**Nad89**] and developed further by Demailly-El Goul [**DEG97**]. However, there are certain insuperable obstructions to the positivity of the tautological line bundle on the Demailly-Semple towers, due to the compactification of the jet bundles (ref. [**Dem95**]). Brotbek introduced a clever way to blow-up the ideal sheaves defined by the Wronskians, which behaves well in families; as a consequence he was able to apply the openness property of ampleness for the higher order jet bundles to prove the hyperbolicity for general hypersurfaces. In order to make the lower bound $d_{K,n}$ in Theorem 0.10.3 effective, one needs to make some noetherianity arguments effective as well. Along with the Nakmaye theorem, there is another constant $m_\infty(X_k, L)$ which reflects the stability of Wronskian ideal sheaf when the positivity of the line bundle L increases. In Section 4.2.3 we study Brotbek's Wronskians and prove an effective generation result for Wronskian ideal sheaves. In this way, by adapting Brotbek's result, we have been able obtain an effective bound for the Kobayashi conjecture.

THEOREM XI. *With the same notation in Theorem 0.10.3, one can take*
$$d_{K,n} = n^{n+1}(n+1)^{2n+5}.$$

REMARK 0.3. By using Siu's technique of slanted vector fields on higher jet spaces outlined in the survey [**Siu02**], and the *Algebraic Morse Inequality* of Demailly and Trapani, the first effective lower bound for degrees of hypersurfaces that are weakly hyperbolic (one says that a variety X is weakly hyperbolic if all its entire curves lie in a proper subvariety $Y \subsetneq X$) was given by Diverio, Merker and Rousseau [**DMR10**]. They indeed confirmed the Green-Griffiths-Lang conjecture for generic hypersurfaces in $\mathbb{P}^n$ of degree $d \geqslant 2^{(n-1)^5}$. Later on, by means of a very elegant combination of his holomorphic Morse inequalities and of non probabilistic interpretation of higher order jets, Demailly was able to improve the lower bound to $d \geqslant \left\lfloor \frac{n^4}{3}\Big(n\log\big(n\log(24n)\big)\Big)^n \right\rfloor$ [**Dem10**]. The latest best known bound is $d \geqslant (5n)^2 n^n$ by Darondeau [**Dar15**]. In the recent published paper [**Siu15**], Siu provided more details to his strategy in [**Siu02**] to complete his proof of the Kobayashi conjecture, but it seems quite difficult to derive explicit degree bounds from [**Siu15**].

In the same vein as the Debarre conjecture, in [**DT10**], Simone Diverio and Stefano Trapani raised the following generalized conjecture:

CONJECTURE 0.10.1. (Diverio-Trapani) Let $X \subset \mathbb{P}^N$ be the complete intersection of c general hypersurfaces of sufficiently high degree. Then, $E_{k,m}T_X^*$ is ample provided that $k \geqslant \frac{N}{c} - 1$, and therefore X is hyperbolic.

In Chapter 4, following the elegant geometric methods in [**BD15**] and [**Bro16**] on the Debarre and Kobayashi conjectures, we prove the following theorem:

THEOREM XII. *(= Theorem L) Let X be a projective manifold of dimension n endowed with a very ample line bundle A. Let $Z \subset X$ be the complete intersection of c general hypersurfaces in $|H^0\big(X, \mathscr{O}_X(dA)\big)|$. Then for any positive integer $k \geqslant \frac{n}{c} - 1$, Z has the almost k-jet ampleness property (see Definition 4.2.1 below) provided that $d \geqslant 2c(\lceil \frac{n}{c} \rceil)^{n+c+2} n^{n+c}$. In particular, Z is Kobayashi hyperbolic.*

Since our definition of almost 1-jet ampleness coincides with the ampleness of the cotangent bundle, Theorem XII contains both the Kobayashi ($c = 1$) and Debarre conjectures ($c \geqslant \frac{n}{2}$), with some (non-optimal) effective estimates.

At the expense of a slightly larger bound, based on a factorization trick due to Xie [**Xie15**], we are able to prove the following stronger result:

THEOREM XIII. *(= Theorem M) Let X be a projective manifold of dimension n and A a very ample line bundle on X. For any c-tuple $\mathbf{d} := (d_1, \ldots, d_c)$ such that $d_p \geqslant c^2 n^{2n+2c}(\lceil \frac{n}{c} \rceil)^{2n+2c+4}$ for each $1 \leqslant p \leqslant c$, for general hypersurfaces $H_p \in |A^{d_p}|$, their complete intersection $Z := H_1 \cap \ldots \cap H_c$ is almost k-jet ample provided that $k \geqslant k_0$.*

Moreover, there exists a uniform $(e_1, \ldots, e_k) \in \mathbb{N}^k$ which only depends on n, such that $\mathscr{O}_{Z_k}(e_1, \ldots, e_k)$ is big and such that its augmented base locus satisfies

$$\mathbb{B}_+\big(\mathscr{O}_{Z_k}(e_1, \ldots, e_k)\big) \subset Z_k^{\mathrm{Sing}}$$

where Z_k^{Sing} is the set of points in Z_k which can not *be reached by the k-th lift $f_{[k]}(0)$ of any regular germ of curves $f : (\mathbb{C}, 0) \to Z$.*

From the relation between tautological bundles on the Demailly-Semple towers and invariant jet bundles, we prove the following theorem on the Diverio-Trapani conjecture:

THEOREM XIV. *(= Theorem N) Set $q := \mathscr{Z}_{\mathbf{d}} \to \prod_{p=1}^c |A^{d_p}|$ to be the universal family of c-complete intesections of hypersurfaces in $\prod_{p=1}^c |A^{d_p}|$, where $d_p \geqslant c^2 n^{2n+2c}(\lceil \frac{n}{c} \rceil)^{2n+2c+4}$ for each $1 \leqslant p \leqslant c$. Set $U \subset \prod_{p=1}^c |A^{d_p}|$ to be a Zariski open set of $\prod_{p=1}^c |A^{d_p}|$ such that when restricted to $\mathscr{X} := q^{-1}(U)$, q is a smooth fibration. Then for every $j \gg 0$, there exists a subbundle $V_j \subset E_{k,jm}T^*_{\mathscr{X}/U}$ defined on $\mathscr{X}$, whose restriction to the general fiber Z of q is an ample vector bundle. Moreover, fix any $x \in Z$, and any* regular *k-jet of holomorphic curve $[f] : (\mathbb{C}, 0) \to (Z, x)$; then for every $j \gg 0$ there exists global jet differentials $P_j \in H^0(Z, V_j|_Z \otimes A^{-1})$ (hence they are of order k and weighted degree jm)* that do not *vanish when evaluated on the k-jet defined by $(f', f'', \ldots, f^{(k)})$.*

In other words, this theorem shows that, one can find a subbundle of the invariant jet bundle, which is ample, and such that its Demailly-Semple locus defined in [**DR13**, Section 2.1] is empty.

0.11. Applications of the L^2-Extension Theorems to Direct Image Problems

One of the most important achievements of the Bochner-Kodaira-Nakano-Hörmander L^2 theory is the extension result established by T. Ohsawa and K. Takegoshi [**OT87**]. Later on, this theorem was subsequently refined by Manivel, Siu, Berndtsson, Popovici, Błocki, Guan-Zhou, Junyan Cao [**Man93**, **Siu95**, **Ber96**, **Pop05**, **Blo13**, **GZ15**, **Cao14**], and very recently Demailly proved a very general extension theorem for non necessarily reduced subvarieties, under (probably) optimal curvature conditions [**Dem15a**]. The application of L^2-extension theorems to both algebraic and analytic geometry yields fundamental results in many circumstances: various forms of approximation of closed positive currents, study of the adjoint linear systems, deformational invariance of plurigenera, positivity of direct images and recent results on the Iitaka conjecture, to quote only a few.

In Chapter 5, we apply Demailly's recent extension theorem, and the pluricanonical extension theorem by Berndtsson and Păun [**BP08**] to study two open problems on the positivity of direct images. First, we study a Fujita-type conjecture by Popa and Schnell:

CONJECTURE 0.11.1. (Popa-Schnell) Let $f : X \to Y$ be a morphism of smooth projective varieties, with $\dim(Y) = n$, and let L be an ample line bundle on Y. Then, for every $k \geqslant 1$, the sheaf

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is globally generated for any $l \geqslant k(n+1)$.

In [**PS14**], Popa and Schnell proved the conjecture in the case when L is an ample and globally generated line bundle, and in general when $\dim(X) = 1$. In a recent preprint [**Dut17**], applying the work of Angehrn and Siu, Dutta was able to remove the global generation assumption on L, by making a statement about generic global generation, with a quadratic bound on l in terms of the dimension n:

$$l \geqslant k(\binom{n+1}{2} + 1).$$

In Chapter 5 by applying Demailly's recent work on the Ohsawa-Takegoshi type extension theorem [**Dem15a**] as well as Păun's proof of Siu's invariance of plurigenera [**Pau07**], we are able to prove the following theorem:

THEOREM XV. *(= Theorem O) Let $f : X \to Y$ be a surjective morphism between smooth projective varieties, with $\dim(Y) = n$, and let L be an ample line bundle on Y. If y is a regular value of f, then for every $k \geqslant 1$, the sheaf*

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at y for any

$$l \geqslant k(\left\lfloor \frac{n}{\epsilon(L,y)} \right\rfloor + 1).$$

Here $\epsilon(L,y) > 0$ is the Seshadri constant of L at the point y. In particular, if the Seshadri constant satisfies $\epsilon(L,y) \geqslant 1$ at a very general point $y \in Y$, then Conjecture 0.11.1 holds true for general points in Y; that is, the direct image

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at the generic point of Y for any $l \geqslant k(n+1)$.

By a result of Ein-Küchle-Lazarsfeld [**EKL95**], there is a universal generic bound for the Seshadri constant depending only on the dimension of the manifold, namely, for a very generic point $y \in Y$, $\epsilon(L,y) \geqslant \frac{1}{n}$. Applying their result, we can get an effective estimate for Conjecture 0.11.1:

THEOREM XVI. *(= Theorem P) Let $f : X \to Y$ be a surjective morphism between smooth projective varieties, with $\dim(Y) = n$, and let L be an ample line bundle on Y. Then for any $k \geqslant 1$, the direct image*

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at the generic point of Y for any $l \geqslant k(n^2+1)$.

Compared to the bound on l obtained by Dutta, ours is also quadratic on n but slightly weaker than hers. However, if we apply a well-known result in the Mori-theory that $K_Y + (n+1)L$ is semi-ample for any ample line bundle L, and use the pluricanonical extension theorem by Berndtsson-Păun [**BP08**] instead, we can obtain a linear bound for l.

THEOREM XVII. *(= Theorem Q) Let $f : X \to Y$ be a surjective morphism between smooth projective manifolds, and let L be an ample line bundle on Y. Then for every $k \geqslant 1$, the sheaf*

$$f_* K_X^{\otimes k} \otimes L^{\otimes l}$$

is generated by global sections at the generic $y \in Y$ for any $l \geqslant k(n+1) + n^2 - n$.

The goal of second part of Chapter 5 is to study a question by Demailly-Peternell-Schneider in [**DPS01**]:

PROBLEM 0.11.1. Let X and Y be normal projective $\mathbb{Q}$-Gorenstein varieties. Let $f : X \to Y$ be a surjective morphism. If $-K_X$ is pseudo-effective and its non-nef locus does not project onto Y, is $-K_Y$ pseudo-effective?

Inspired by the recent work of J. Cao on the local isotriviality on the Albanese map of projective manifolds with nef anticanonical bundles [**Cao16**], we give an affirmative answer to the above problem when X and Y are replaced by pairs:

THEOREM XVIII. *(= Theorem R) Let $f : X \to Y$ be a surjective morphism from a log-canonical pair (X, D) to the smooth projective manifold Y. Let Δ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y. Suppose that $-(K_X+D)-f^*\Delta$ is pseudo-effective, and the non-nef locus $\mathbf{B}_-\big(-(K_X+D)-f^*\Delta\big)$ does not project onto Y. Then $-K_Y-\Delta$ is pseudo-effective with its non-nef locus contained in $f\big(\mathbf{B}_-(-K_X-D-f^*\Delta)\big)\cup Z\cup Z_D$, where Z is the minimal proper subvariety on Y such that $f : X\backslash f^{-1}(Z) \to Y\backslash Z$ is a smooth fibration, and Z_D is an at most countable union of proper subvarieties containing Z such that for every $y \notin Z_D$, the pair $\big(f^{-1}(y), D_{\upharpoonright f^{-1}(y)}\big)$ is also lc.*

The following theorem by Fujino and Gongyo [**FG14**] is a direct consequence of our Theorem XVIII.

THEOREM 0.11.1. *(Fujino-Gongyo) Let $f : X \to Y$ be a smooth fibration between smooth projective varieties. Let D be an effective $\mathbb{Q}$-divisor on X such that (X, D) is lc,* Supp(D) *is a simple normal crossing divisor, and* Supp(D) *is relatively normal crossing over Y. Let Δ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y. Assume that $-(K_X + D) - f^*\Delta$ is nef. Then so is $-K_Y - \Delta$.*

Moreover, we can also use analytic methods to prove the following theorem.

THEOREM XIX. *(= Theorem S) With the same notations in Theorem XVIII. Assume further that (X, D) is klt, $-K_X - D - f^*\Delta$ is big and its non-nef locus $\mathbf{B}_-(-K_X - D - f^*\Delta)$ does not dominate Y, then $-K_Y - \Delta$ is big with its non-nef locus contained in $f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \cup Z \cup Z_D$.*

As a combination of Theorem XVIII and XIX, we prove the following Theorem, which is a generalization of a theorem by Fujino and Gongyo [**FG12**] on the image of weak Fano manifolds.

THEOREM XX. *(= Theorem T) Let $f : X \to Y$ be a smooth fibration between two smooth manifolds X and Y. Let D be an effective $\mathbb{Q}$-divisor such that (X, D) is klt, and $(X_y, D_{\upharpoonright X_y})$ is also klt for every $y \in Y$. Let Δ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y. If $-K_X - D - f^*\Delta$ is big and nef, then $-K_Y - \Delta$ is also big and nef.*

Finally, we apply Theorem XIX to strengthen a result by Broustet and Pacienza on the rational connectedness of the image [**BrP11**, Theorem 1.2]:

THEOREM XXI. *(= Theorem U) With the same notation in Theorem XVIII. Assume that (X, D) and (Y, Δ) are both klt pairs. If $-(K_X + \Delta + f^*\Delta)$ is big and its non-nef locus $\mathbf{B}_-(-K_X - D - f^*\Delta)$ does not dominate Y, then Y is rational connected modulo $f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \cup Z \cup Z_D$, that is, there exists an irreducible component V of $\mathbf{B}_-(-K_Y - \Delta)$ such that for any general point y of Y, there exists a rational curve R_y passing through y and intersecting V.*

REMARK 0.4. With the same assumption and notation as in Theorem XXI, Broustet and Pacienza proved in [**BrP11**] that the image is uniruled.

0.12. A Remark on the Corlette-Simpson Correspondence

A Higgs bundle is a pair (E, θ) consisting of a holomorphic vector bundle E and a Higgs field θ, that is a holomorphic 1-form taking values in End(E) such that $\theta \wedge \theta = 0$. In [**Sim88**] Simpson generalized the Donaldson-Uhlenbeck-Yau Theorem [**Don85, UY86**]: the latter states that holomorphic vector bundles on Kähler manifolds admit Hermitian-Einstein metrics if and only if they are stables; Simpson's generalization instead deals with Higgs bundles over the (possibly non-compact) Kähler manifolds with certain boundary conditions.

Later on, in [**Sim92**], by introducing the *differential graded categories* [**Sim92**, Section 3], plus the *formality isomorphism* [**Sim92**, Lemma 2.2], Simpson extended the equivalence between the category of polystable Higgs bundles with vanishing Chern classes and the category of semi-simple representations of fundamental groups [**Cor88, Sim88**], to extensions of irreducible objects on smooth projective manifolds [**Sim92**, Corollary 3.10].

The purpose of Chapter 6 is to give a concrete and constructing proof of Simpson's correspondence for semistable Higgs bundles on *Kähler manifolds*. Our presentation is also written for complex geometers who are not familiar with the language of differential graded categories, and for readers who want an elementary proof of the Simpson correspondence for semistable Higgs bundles. The result is the following

THEOREM XXII. *(= Theorem V) Let X be a compact Kähler manifold. Then the following statements are equivalent* *(i)* E *is a flat vector bundle over* X*;*

(ii) *there is a structure of Higgs bundle* $(E, \overline{\partial}, \theta)$ *over* E*, such that it admits a filtration of Higgs bundles*

$$\{0\} = (E_0, \theta_0) \subset (E_1, \theta_1) \subset \ldots \subset (E_m, \theta_m) = (E, \theta)$$

where $\theta_i := \theta|_{E_i}$*, such that the grade terms* $(E_i, \theta_i)/(E_{i-1}, \theta_{i-1})$ *are stable Higgs bundles with vanishing Chern classes.*

(iii) E *is a semistable Higgs bundle with vanishing Chern classes.*

In Chapter 6, we only (re)prove the equivalence between (i) and (ii) in Theorem V. The implication (ii) $\Rightarrow$ (iii) is trivial. To show that (iii) implies (ii), one only needs to prove that the Jordan-Hölder filtrations of the semistable Higgs bundles with vanishing Chern classes are still a filtration of Higgs bundles rather than Higgs sheaves. In [**DPS94**], the authors proved this result for pure vector bundles, *i.e.* when the Higgs field θ vanishes. In [**Sim92**, Theorem 2], if X is projective, Simpson proved a a slightly stronger result, namely that any reflexive semistable Higgs bundle with vanishing Chern classes is an extension of stable Higgs bundles with vanishing Chern classes. His proof uses arguments that are similar to those employed in Mehta-Ramanathan's work about restriction of semistable sheaves to hyperplane sections. In a recent paper [**NZ15**], using the Yang-Mills-Higgs flow to construct the approximate Hermitian-Einstein structure for semistable Higgs bundles, combined with the techniques in [**DPS94**], Y.-C. Nie and X. Zhang proved the implication (iii) $\Rightarrow$ (ii).

In particular, if the Higgs field vanishes, one obtains a direct proof of the following result also due to Simpson.

THEOREM 0.12.1. *(Simpson) Let* X *be a compact Kähler manifold. Suppose that the holomorphic vector bundle* E *on* X *admits a filtration*

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_p = E \tag{0.12.1}$$

such that the quotients E_k/E_{k-1} *are hermitian flat vector bundles. Then the natural Gauss-Manin connection* D_E *on* E *is compatible with the natural hermitian flat connection on the quotient* E_k/E_{k-1} *for every* $k = 1, \ldots, p$*.*

In [**DPS94**], the authors introduced the definition of a *numerically flat vector bundle* (see also Definition 6.2.2), and proved that every numerically flat vector bundle admits a filtration as (0.12.1). Recently, using Theorem 0.12.1, J. Cao proved the conjecture that, for any smooth projective manifold with $-K_X$ nef, the Albanese map of X is locally isotrivial [**Cao17**]. For this, he applied an elegant criterion of [**CH13**] which relates the numerical flatness property to the local isotriviality of the fibration.

Part 1

Generalized Okounkov Bodies

CHAPTER 1

Transcendental Morse Inequality and Generalized Okounkov Bodies

1.1. Introduction

In [**Oko96**] Okounkov introduced a natural procedure to associate a convex body $\Delta(D)$ in $\mathbb{R}^n$ to any ample divisor D on an n-dimensional projective variety. Relying on the work of Okounkov, Lazarsfeld and Mustaţă [**LM09**], and Kaveh and Khovanskii [**KK09**, **KK10**], have systematically studied Okounkov's construction, and associated to any big divisor and any fixed flag of subvarieties a convex body which is now called the Okounkov body.

We now briefly recall the construction of the Okounkov body. We start with a complex projective variety X of dimension n. Fix a flag

$$Y_\bullet : X = Y_0 \supset Y_1 \supset Y_2 \supset \ldots \supset Y_{n-1} \supset Y_n = \{p\}$$

where Y_i is a smooth irreducible subvariety of codimension i in X. For a given big divisor D, one defines a valuation-like function

$$\mu = \mu_{Y_\bullet, D} : (H^0(X, \mathcal{O}_X(D)) - \{0\}) \to \mathbb{Z}^n.$$

as follows. First set $\mu_1 = \mu_1(s) = \mathrm{ord}_{Y_1}(s)$. Dividing s by a local equation of Y_1, we obtain a section

$$\widetilde{s}_1 \in H^0(X, \mathcal{O}_X(D - \mu_1 Y_1))$$

that does not vanish identically along Y_1. We restrict $\widetilde{s}_1$ on Y_1 to get a non-zero section

$$s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \mu_1 Y_1)),$$

then we write $\mu_2(s) = \mathrm{ord}_{Y_2}(s_1)$, and continue in this fashion to define the remaining integers $\mu_i(s)$. The image of the function μ in $\mathbb{Z}^n$ is denoted by $\mu(D)$. With this in hand, we define the *Okounkov body of D with respect to the fixed flag $Y_\bullet$* to be

$$\Delta(D) = \Delta_{Y_\bullet}(D) = \text{closed convex hull}\left(\bigcup_{m \geqslant 1} \frac{1}{m} \cdot \mu(mD)\right) \subseteq \mathbb{R}^n.$$

According to the open question raised in the final part of [**LM09**], it is quite natural to wonder whether one can construct "arithmetic Okounkov bodies" for an arbitrary pseudo-effective (1,1)-class α on a Kähler manifold, and realize the volumes of these classes by convex bodies as well. In this chapter, using positive currents in a natural way, we give a construction of a convex body $\Delta(\alpha)$ associated to such a class α, and show that this newly defined convex body coincides with the Okounkov body when $\alpha \in \mathrm{NS}_{\mathbb{R}}(X)$.

Theorem A. *Let X be a smooth projective variety of dimension n, L be a big line bundle on X and $Y_\bullet$ be a fixed admissible flag. Then we have*

$$\Delta(c_1(L)) = \Delta(L) = \overline{\bigcup_{m=1}^{\infty} \frac{1}{m}\nu(mL)}.$$

Moreover, in the definition of Okounkov body $\Delta(L)$, it suffices to take the closure of the set of normalized valuation vectors instead of the closure of the convex hull.

By Theorem A, we know that our definition of the Okounkov body for any pseudo-effective class could be treated as a generalization of the original Okounkov body. A very interesting problem is to find out exactly which points in the Okounkov body $\Delta(L)$ are given by valuations of sections. This is expressed by saying that a rational point of $\Delta(L)$ is "valuative". By Theorem A we can give some partial answers to this question which have been given in [**KL14**] in the case of surfaces.

Corollary 1.1.1. Let X be a projective variety of dimension n and $Y_\bullet$ be an admissible flag. If L is a big line bundle, then any rational point in $\mathrm{int}(\Delta(L))$ is a valuative point.

It is quite natural to wonder whether our newly defined convex body for big classes behaves similarly as the original Okounkov body. In the situation of complex surfaces, we give an affirmative answer to the question raised in [**LM09**], as follows:

THEOREM B. *Let X be a compact Kähler surface, $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big class. If C is an irreducible divisor of X, there are piecewise linear continuous functions*

$$f\ ,\ g\ :\ [a,s] \mapsto \mathbb{R}_+$$

with f convex, g concave, and $f \leqslant g$, such that $\Delta(\alpha) \subset \mathbb{R}^2$ is the region bounded by the graphs of f and g:

$$\Delta(\alpha) = \{(t,y) \in \mathbb{R}^2 \mid a \leqslant t \leqslant s, \text{and } f(t) \leqslant y \leqslant g(t)\}.$$

Here $\Delta(\alpha)$ is the generalized Okounkov body with respect to the fixed flag

$$X \supseteq C \supseteq \{x\},$$

and $s = \sup\{t > 0 \mid \alpha - tC \text{ is big}\}$. If C is nef, $a = 0$ and $f(t)$ is increasing; otherwise, $a = \sup\{t > 0 \mid C \subseteq E_{nK}(\alpha - tC)\}$, where $E_{nK} := \bigcap_T E_+(T)$ for T ranging among the Kähler currents in α, which is the non-Kähler locus. Moreover, $\Delta(\alpha)$ is a finite polygon whose number of vertices is bounded by $2\rho(X)+2$, where $\rho(X)$ is the Picard number of X, and

$$\mathrm{vol}_X(\alpha) = 2\,\mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha)).$$

In [**LM09**], it was asked whether the Okounkov body of a divisor on a complex surface could be an infinite polygon. In [**KLM10**], it was shown that the Okounkov body is always a finite polygon. Here we give an explicit description for the "finiteness" of the polygons appearing as generalized Okounkov bodies of big classes, and conclude that it also holds for the original Okounkov bodies by Theorem A.

As one might suspect from the construction of Okounkov bodies, the Euclidean volume of $\Delta(D)$ has a strong connection with the growth of the groups $H^0(X,\mathcal{O}_X(mD))$. In [**LM09**], the following precise relations were shown:

$$n! \cdot \mathrm{vol}_{\mathbb{R}^n}(\Delta(D)) = \mathrm{vol}_X(D) := \lim_{k\to\infty} \frac{n!}{k^n} h^0(X,\mathcal{O}_X(kD)). \tag{1.1.1}$$

The proof of (1.1.1) relies on properties of sub-semigroups of $\mathbb{N}^{n+1}$ constructed from the graded linear series $\{H^0(X,\mathcal{O}_X(mD))\}_{m\geqslant 0}$. However, when α is a big class which does not belong to $\mathrm{NS}_{\mathbb{R}}(X)$, there are no such algebraic objects which correspond to $\mathrm{vol}_X(\alpha)$, and we only have the following analytic definition due to Boucksom [**Bou02**]:

$$\mathrm{vol}_X(\alpha) := \sup_T \int_X T_{ac}^n,$$

where T ranges among all positive $(1,1)$-currents. Therefore, it is quite natural to propose the following conjecture:

CONJECTURE 1.1.1. *Let X be a compact Kähler manifold of dimension n. For any big class $\alpha \in H^{1,1}(X,\mathbb{R})$, we have*

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha)) = \frac{1}{n!} \cdot \mathrm{vol}_X(\alpha).$$

In Theorem B, we prove this conjecture in dimension 2. Our method is to relate the Euclidean volume of the slice of the generalized Okounkov body to the differential of the volume of the big class. We prove the following differentiability formula for volumes of big classses.

THEOREM C. *Let X be a compact Kähler surface and α be a big class. If β is a nef class or $\beta = \{C\}$ where C is an irreducible curve, we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}_X(\alpha + t\beta) = 2Z(\alpha)\cdot\beta,$$

where $Z(\alpha)$ is the divisorial Zariski decomposition of α defined in Section 1.2.6.

A direct corollary of this formula is the *transcendental Morse inequality*:

THEOREM D. *Let X be a compact Kähler surface. If α and β are nef classes satisfying the inequality $\alpha^2 - 2\alpha\cdot\beta > 0$, then $\alpha - \beta$ is big and $\mathrm{vol}_X(\alpha-\beta) \geqslant \alpha^2 - 2\alpha\cdot\beta$.*

In higher dimension, we also have a differentiability formula for big classes on some special Kähler manifolds.

THEOREM E. *Let X be a compact Kähler manifold of dimension n on which the modified nef cone $\mathcal{MN}$ coincides with the nef cone $\mathcal{N}$. If $\alpha \in H^{1,1}(X,\mathbb{R})$ is a big class, $\beta \in H^{1,1}(X,\mathbb{R})$ is a nef class, then*

$$\mathrm{vol}_X(\alpha+\beta) = \mathrm{vol}_X(\alpha) + n\int_0^1 Z(\alpha+t\beta)^{n-1}\cdot\beta\ \mathrm{d}t. \tag{1.1.2}$$

As a consequence, $\mathrm{vol}_X(\alpha + t\beta)$ *is* $\mathcal{C}^1$ *for* $t \in \mathbb{R}^+$ *and we have*

$$\frac{d}{dt}\bigg|_{t=t_0} \mathrm{vol}_X(\alpha + t\beta) = nZ(\alpha + t_0\beta)^{n-1} \cdot \beta \tag{1.1.3}$$

for $t_0 \geqslant 0$.

Finally, we study the generalized Okounkov bodies for pseudo-effective classes in Kähler surfaces. We summerize our results as follows

THEOREM F. *Let* X *be a Kähler surface,* α *be any pseudo-effective but not big class,*

(i) *if the numerical dimension* $n(\alpha) = 0$*, then for any irreducible curve* C *which is not contained in the negative part* $N(\alpha)$*, we have the generalized Okounkov body*

$$\Delta_{(C,x)}(\alpha) = 0 \times \nu_x(N(\alpha)|_C),$$

where $\nu_x(N(\alpha)|_C) = \nu(N(\alpha)|_C, x)$ *is the Lelong number of* $N(\alpha)$ *at* x*;*

(ii) *if* $n(\alpha) = 1$*, then for any irreducible curve* C *satisfying* $Z(\alpha) \cdot C > 0$*, we have*

$$\Delta_{(C,x)}(\alpha) = 0 \times [\nu_x(N(\alpha)|_C), \nu_x(N(\alpha)|_C) + Z(\alpha) \cdot C].$$

In particular, the numerical dimension determines the dimension of the generalized Okounkov body.

1.2. Technical preliminaries

1.2.1. Siu decomposition. Let T be a closed positive current of bidegree (p, p) on a complex manifold X. We denote by $\nu(T, x)$ its Lelong number at a point $x \in X$. For any $c > 0$, the Lelong upperlevel sets are defined by

$$E_c(T) := \{x \in X, \nu(T, x) \geqslant c\}.$$

In [**Siu74**], Siu proved that $E_c(T)$ is an analytic subset of X, of codimension at least p. Moreover, T can be written as a convergent series of closed positive currents

$$T = \sum_{k=1}^{+\infty} \nu(T, Z_k)[Z_k] + R$$

where $[Z_k]$ is a current of integration over an irreducible analytic set of dimension p, and R is a residual current with the property that $\dim E_c(R) < p$ for every $c > 0$. This decomposition is locally and globally unique: the sets Z_k are precisely the p-dimensional components occurring in the upperlevel sets $E_c(T)$, and $\nu(T, Z_k) := \inf\{\nu(T, x) | x \in Z_k\}$ is the generic Lelong number of T along Z_k.

1.2.2. Currents with analytic singularities. A closed positive (1,1) current T on a compact complex manifold X is said to have analytic (resp. algebraic) singularities along a subscheme $V(\mathcal{I})$ defined by an ideal $\mathcal{I}$ if there exists some $c \in \mathbb{R}_{>0}$ (resp. $\mathbb{Q}_{>0}$) such that locally we have

$$T = \frac{c}{2} dd^c \log(|f_1|^2 + \ldots + |f_k|^2) + dd^c v$$

where $f_1, \ldots, f_k$ are local generators of $\mathcal{I}$ and $v \in L^\infty_{\mathrm{loc}}$ (resp. and additionally, X and $V(\mathcal{I})$ are algebraic). Moreover, if v is smooth, T will be said to have mild analytic singularities. In these situations, we call the sum $\sum \nu(T, D)D$ which appears in the Siu decomposition of T the divisorial part of T. Using the Lelong-Poincaré formula, it is straightforward to check that the divisorial part $\sum \nu(T, D)D$ of a closed (1,1)-current T with analytic singularities along the subscheme $V(\mathcal{I})$ is just the divisorial part of $V(\mathcal{I})$, times the constant $c > 0$ appearing in the definition of analytic singularities. The residual part R has analytic singularities in codimension at least 2. If we denote $E_+(T) := \{x \in X | \nu(T, x) > 0\}$, then $E_+(T)$ is exactly the support of $V(\mathcal{I})$. Moreover, if $V \nsubseteq E_+(T)$ for some smooth variety V, $T|_V := \frac{c}{2} dd^c \log(|f_1|^2 + \ldots + |f_k|^2)|_V + dd^c v|_V$ is well defined, for $|f_1|^2 + \ldots + |f_k|^2$ and v are not identically equal to $-\infty$ on V. It is easy to check that this definition does not depend on the choice of the local potential of T.

DEFINITION 1.2.1. *If* $\alpha \in H^{1,1}_{\partial\bar\partial}(X, \mathbb{R})$ *is a big class, we define its* non-Kähler locus *as* $E_{nK} := \bigcap_T E_+(T)$ *for* T *ranging among the Kähler currents in* α.

We will usually use the following theorem due to Collins and Tosatti.

THEOREM 1.2.2 ([**CT13**]). *Let* X *be a compact Kähler manifold of dimension* n*. Given a nef and big class* α*, we define a subset of* X *which measures its non-Kählerianity, namely the null locus*

$$\mathrm{Null}(\alpha) := \bigcup_{\int_V \alpha^{\dim V} = 0} V,$$

where the union is taken over all positive dimensional irreducible analytic subvarieties of X*. Then we have*

$$\mathrm{Null}(\alpha) = E_{nK}(\alpha).$$

1.2.3. Regularization of currents. We will need Demailly's regularization theorem ([**Dem92**]) for closed (1,1)-currents, which enables us to approximate a given current by currents with analytic singularities, with a loss of positivity that is arbitrary small. In particular, we could approximate a Kähler current T inside its cohomology class by Kähler currents T_k with algebraic singularities, with a good control of the singularities. A big class therefore contains plenty of Kähler currents with analytic singularities.

Theorem 1.2.3. *Let T be a closed almost positive (1,1)-current on a compact complex manifold X, and fix a Hermitian form ω. Suppose that $T \geqslant \gamma$ for some real (1,1)-form γ on X. Then there exists a sequence T_k of currents with algebraic singularities in the cohomology class $\{T\}$ which converges weakly to T, such that $T_k \geqslant \gamma - \epsilon_k\omega$ for some sequence $\epsilon_k > 0$ decreasing to 0, and $\nu(T_k, x)$ increases to $\nu(T, x)$ uniformly with respect to $x \in X$.*

1.2.4. Currents with minimal singularities. Let $T_1 = \theta_1 + dd^c\varphi_1$ and $T_2 = \theta_2 + dd^c\varphi_2$ be two closed almost positive (1,1)-currents on X, where θ_i are smooth forms and φ_i are almost pluri-subharmonic functions, we say that T_1 is less singular than T_2 (write $T_1 \preceq T_2$) if we have $\varphi_2 \leqslant \varphi_1 + C$ for some constant C.

Let α be a class in $H^{1,1}_{\partial\bar\partial}(X, \mathbb{R})$ and γ be a smooth real (1,1)-form, we denote by $\alpha[\gamma]$ the set of closed almost positive (1,1)-currents $T \in \alpha$ with $T \geqslant \gamma$. Since the set of potentials of such currents is stable by taking a supremum, we conclude by standard pluripotential theory that there exists a closed almost positive (1,1)-current $T_{\min,\gamma} \in \alpha[\gamma]$ which has minimal singularities in $\alpha[\gamma]$. $T_{\min,\gamma}$ is well defined modulo dd^cL^∞. For each $\epsilon > 0$, denote by $T_{\min,\epsilon} = T_{\min,\epsilon}(\alpha)$ a current with minimal singularities in $\alpha[-\omega]$, where ω is some reference Hermitian form. The minimal multiplicity at $x \in X$ of the pseudo-effective class $\alpha \in H^{1,1}_{\partial\bar\partial}(X, \mathbb{R})$ is defined as

$$\nu(\alpha, x) := \sup_{\epsilon>0} \nu(T_{\min,\epsilon}, x).$$

For a prime divisor D, we define the generic minimal multiplicity of α along D as

$$\nu(\alpha, D) := \inf\{\nu(\alpha, x) | x \in D)\}.$$

We then have $\nu(\alpha, D) = \sup_{\epsilon>0} \nu(T_{\min,\epsilon}, D)$.

1.2.5. Lebesgue decomposition. A current T can be locally seen as a form with distribution coefficients. When T is positive, the distributions are positive measures which admit a Lebesgue decomposition into an absolutely continuous part (with respect to the Lebesgue measure on X) and a singular part. Therefore we obtain the decomposition $T = T_{\mathrm{ac}} + T_{\mathrm{sing}}$, with T_{ac} (resp. T_{sing}) globally determined thanks to the uniqueness of the Lebesgue decomposition.

Now we assume that T is a (1,1)-current. The absolutely continuous part T_{ac} is considered as a (1,1)-form with L^1_{loc} coefficients, and more generally we have $T_{\mathrm{ac}} \geqslant \gamma$ whenever $T \geqslant \gamma$ for some real smooth real form γ. Thus we can define the product T^k_{ac} of T_{ac} almost everywhere. This yields a positive Borel (k, k)-form.

1.2.6. Modified nef cone and divisorial Zariski decomposition. In this subsection, we collect some definitions and properties of the modified nef cone and divisorial Zariski decomposition. See [**Bou04**] for more details.

Definition 1.2.4. *Let X be compact complex manifold, and ω be some reference Hermitian form. Let α be a class in $H^{1,1}_{\partial\bar\partial}(X, \mathbb{R})$.*

- (i) *α is said to be a* modified Kähler class *iff it contains a Kähler current T with $\nu(T, D) = 0$ for all prime divisors D in X.*
- (ii) *α is said to be a* modified nef class *iff, for every $\epsilon > 0$, there exists a closed (1,1)-current $T_\epsilon \geqslant -\epsilon\omega$ and $\nu(T_\epsilon, D) = 0$ for every prime D.*

Remark 1.2.1. The modified nef cone $\mathcal{MN}$ is a closed convex cone which contains the nef cone $\mathcal{N}$. When X is a Kähler manifold, $\mathcal{MN}$ is just the interior of the modified Kähler cone $\mathcal{MK}$.

Remark 1.2.2. For a complex surface, the Kähler (nef) cone and the modified Kähler (modified nef) cone coincide. Indeed, analytic singularities in codimension 2 of a Kähler current T are just isolated points. Therefore the class $\{T\}$ is a Kähler class.

Definition 1.2.5. *The* negative part *of a pseudo-effective class $\alpha \in H^{1,1}_{\partial\bar\partial}(X, \mathbb{R})$ is defined as $N(\alpha) := \sum \nu(\alpha, D)D$. The* Zariski projection *of α is $Z(\alpha) := \alpha - \{N(\alpha)\}$. We call the decomposition $\alpha = Z(\alpha) + \{N(\alpha)\}$ the* divisorial Zariski decomposition *of α.*

Remark 1.2.3. We claim that the volume of $Z(\alpha)$ is equal to the volume of α. Indeed, if T is a positive current in α, then we have $T \geqslant N(\alpha)$ since $T \in \alpha[-\epsilon\omega]$ for each $\epsilon > 0$ and we conclude that $T \mapsto T - N(\alpha)$ is a bijection between the positive currents in α and those in $Z(\alpha)$. Furthermore, we notice that $(T - N(\alpha))_{\mathrm{ac}} = T_{\mathrm{ac}}$, and thus by the definition of volume of the pseudo-effective classes we conclude that $\mathrm{vol}_X(\alpha) = \mathrm{vol}_X(Z(\alpha))$.

DEFINITION 1.2.6. (i) *A family $D_1, \dots, D_q$ of prime divisors is said to be an* exceptional family *iff the convex cone generated by their cohomology classes meets the modified nef cone at 0 only.*

(ii) *An effective $\mathbb{R}$-divisor E is said to be* exceptional *iff its prime components constitute an exceptional family.*

We have the following properties of exceptional divisors:

THEOREM 1.2.7. (i) *An effective $\mathbb{R}$-divisor E is exceptional iff $Z(E) = 0$.*

(ii) *If E is an exceptional effective $\mathbb{R}$-divisor, we have $E = N(\{E\})$.*

(iii) *If $D_1, \dots, D_q$ is an exceptional family of primes, then their classes $\{D_1\}, \dots, \{D_q\}$ are linearly independent in $\mathrm{NS}_{\mathbb{R}}(X) \subset H^{1,1}(X, \mathbb{R})$. In particular, the length of the exceptional families of primes is uniformly bounded by the Picard number $\rho(X)$.*

(iv) *Let X be a surface, a family $D_1, \dots, D_r$ of prime divisors is exceptional iff its intersection matrix $(D_i \cdot D_j)$ is negative definite.*

In this chapter, we need the following properties of the modified nef cone $\mathcal{MN}$ and the divisorial Zariski decomposition due to Boucksom (ref. [**Bou04**]). We state these properties without proofs.

THEOREM 1.2.8. *Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective class. Then we have:*

(i) *Its Zariski projection $Z(\alpha)$ is a modified nef class.*

(ii) *$Z(\alpha) = \alpha$ iff α is modified nef.*

(iii) *$Z(\alpha)$ is big iff α is.*

REMARK 1.2.4. Let X be a complex Kähler surface. For a big class $\alpha \in H^{1,1}(X, \mathbb{R})$, $Z(\alpha)$ is a big and modified nef class. By Remark 1.2.1, any modified nef class is nef, it follows that $Z(\alpha)$ is big and nef.

THEOREM 1.2.9. (i) *The map $\alpha \mapsto N(\alpha)$ is convex and homogeneous on pseudo-effective class cone $\mathcal{E}$. It is continuous on the interior of $\mathcal{E}$.*

(ii) *The Zariski projection $Z : \mathcal{E} \to \mathcal{MN}$ is concave and homogeneous. It is continuous on the interior of $\mathcal{E}$.*

THEOREM 1.2.10. *Let p be a big and modified nef class. Then the primes $D_1, \dots, D_q$ contained in the non-Kähler locus $E_{nK}(p)$ form an exceptional family A, and the fiber of Z over p is the simplicial cone $Z^{-1}(p) = p + V_+(A)$, where $V_+(A) := \sum_{D \in A} \mathbb{R}_+\{D\}$.*

THEOREM 1.2.11. *Let X be a compact surface. If $\alpha \in H^{1,1}(X, \mathbb{R})$ is a pseudo-effective class, its divisorial Zariski decomposition $\alpha = Z(\alpha) + \{N(\alpha)\}$ is the unique orthogonal decomposition of α with respect to the non-degenerate quadratic form $q(\alpha) := \int \alpha^2$ into the sum of a modified nef class and the class of an exceptional effective $\mathbb{R}$-divisor.*

REMARK 1.2.5. Let X be a surface, α is the class of an effective $\mathbb{Q}$-divisor D on a projective surface, the divisorial Zariski decomposition of α is just the original Zariski decomposition of D.

1.3. TRANSCENDENTAL MORSE INEQUALITY

1.3.1. PROOF OF THE TRANSCENDENTAL MORSE INEQUALITY FOR COMPLEX SURFACES. The main goal of this section is to prove the differentiability of the volume function and the transcendental Morse inequality for complex surfaces. In fact, in the next subsection we will give a more general method to prove the transcendental Morse inequality for Kähler manifolds on which modified nef cones $\mathcal{MN}$ coincide with the nef cones $\mathcal{N}$; this includes Kähler surfaces. However, since the methods and results here are very special in studying generalized Okounkov bodies, we will treat complex surface and higher dimensional Kähler manifolds separately. Throughout this subsection, if not specially mentioned, X will stand for a complex Kähler surface. We denote by $q(\alpha) := \int \alpha^2$ the quadratic form on $H^{1,1}(X, \mathbb{R})$. By the Hodge index theorem, $(H^{1,1}(X, \mathbb{R}), q)$ has signature $(1, h^{1,1}(X) - 1)$. The open cone $\{\alpha \in H^{1,1}(X, \mathbb{R}) | q(\alpha) > 0\}$ has thus two connected components which are convex cones, and we denote by $\mathcal{P}$ the component containing the Kähler cone $\mathcal{K}$.

LEMMA 1.3.1. *Let X be a compact Kähler manifold of dimension n. If $\alpha \in H^{1,1}(X, \mathbb{R})$ is a big class, $\beta \in H^{1,1}(X, \mathbb{R})$ is a nef class, then $N(\alpha + t\beta) \leqslant N(\alpha)$ as effective $\mathbb{R}$-divisors for $t \geqslant 0$. Furthermore, when t is small enough, the prime components of $N(\alpha + t\beta)$ will be the same as those of $N(\alpha)$.*

PROOF. Since β is nef, by Theorem 1.2.9, we have

$$N(\alpha + t\beta) \leqslant N(\alpha) + tN(\beta) = N(\alpha).$$

Since the map $\alpha \mapsto N(\alpha)$ is convex on pseudo-effective class cone $\mathcal{E}$, it is continuous on the interior of $\mathcal{E}$, and thus the theorem follows. □

THEOREM 1.3.1. *If $\alpha \in H^{1,1}(X,\mathbb{R})$ is a big class and $\beta \in H^{1,1}(X,\mathbb{R})$ is a nef class, then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}_X(\alpha + t\beta) = 2Z(\alpha)\cdot\beta \tag{1.3.1}$$

PROOF. By Lemma 1.3.1, there exists an $\epsilon > 0$ such that when $0 \leqslant t < \epsilon$, we can write $N(\alpha + t\beta) = \sum_{i=1}^r a_i(t)N_i$, where $0 < a_i(t) \leqslant a_i(0) =: a_i$, and each $a_i(t)$ is a continuous and decreasing function with respect to t. According to the orthogonal property of divisorial Zariski decomposition (ref. Theorem 1.2.11), $Z(\alpha+t\beta)\cdot N(\alpha+t\beta) = 0$ for $t \geqslant 0$. Since $Z(\alpha+t\beta)$ is modified nef and thus nef (by Remark 1.2.2), we have $Z(\alpha + t\beta)\cdot N_i \geqslant 0$ for every i. When $0 \leqslant t < \epsilon$, we have $a_i(t) > 0$ for $i = 1,\ldots,r$, therefore, $Z(\alpha + t\beta)$ is orthogonal to each $\{N_i\}$ with respect to q. We denote by $V \subset H^{1,1}(X,\mathbb{R})$ the finite vector space spanned by $\{N_1\},\ldots,\{N_r\}$, by $V^{\perp}$ the orthogonal space of V with respect to q. Thus $\alpha+t\beta = Z(\alpha+t\beta)+\sum_{i=1}^r a_i(t)\{N_i\}$ is the decomposition in the direct sum $V^{\perp}\oplus V$. We decompose $\beta = \beta^{\perp} + \beta_0$ in the direct sum $V^{\perp}\oplus V$, and we have

$$Z(\alpha + t\beta) = Z(\alpha) + t\beta^{\perp},$$

$$\sum_{i=1}^{r} a_i(t)\{N_i\} = \sum_{i=1}^{r} a_i\{N_i\} + t\beta_0.$$

Since $\mathrm{vol}_X(\alpha + t\beta) = \mathrm{vol}_X(Z(\alpha + t\beta)) = Z(\alpha + t\beta)^2$ (by Remark 1.2.3), it is easy to deduce that

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}_X(\alpha + t\beta) = 2Z(\alpha)\cdot\beta^{\perp} = 2Z(\alpha)\cdot\beta.$$

The last equality follows from $\beta_0 \in V$ and $Z(\alpha) \in V^{\perp}$. We get the first half of Theorem C. □

To prove the transcendental Morse inequality for complex surfaces, we will need a criterion for bigness of a class:

THEOREM 1.3.2. *Let α and β be two nef classes such that $\alpha^2 - 2\alpha\cdot\beta > 0$, then $\alpha - \beta$ is a big class.*

PROOF. We denote by $\mathcal{P}$ the connected component of the open cone $\{\alpha \in H^{1,1}(X,\mathbb{R}) \mid q(\alpha) > 0\}$ containing the Kähler cone $\mathcal{K}$, then $\mathcal{P} \subset \mathcal{E}^0$. As a consequence of the Nakai-Moishezon criterion for surfaces (ref. [**Lam99**]), we know that, if γ is a real (1,1)-class with $\gamma^2 > 0$, then γ or $-\gamma$ is big. Since α and β are both nef, we have that $(\alpha - t\beta)^2 > 0$ for $0 \leqslant t \leqslant 1$. This means that $\alpha - t\beta$ is contained in some component of the open cone $\{\alpha \in H^{1,1}(X,\mathbb{R}) | q(\alpha) > 0\}$. But since α is big, $\alpha - t\beta$ is contained in $\mathcal{P} \subset \mathcal{B}$, and *a fortiori* $\alpha - \beta$ is. □

Now we are ready to prove the transcendental Morse inequality for complex surfaces.

PROOF OF THEOREM D. By Theorem 1.3.2, when $\alpha^2 - 2\alpha\cdot\beta > 0$, the cohomology class $\alpha - \beta$ is big. By the differentiability formula (1.3.1), we have

$$\mathrm{vol}_X(\alpha - \beta) = \alpha^2 - 2\int_0^1 Z(\alpha - t\beta)\cdot\beta\, dt.$$

Since the Zariski projection $Z : \mathcal{E} \to \mathcal{MN}$ is concave and homogeneous by Theorem 1.2.9, we have

$$\alpha = Z(\alpha) \geqslant Z(\alpha - t\beta) + Z(t\beta) \geqslant Z(\alpha - t\beta).$$

Since β is nef, we have

$$\alpha\cdot\beta \geqslant Z(\alpha - t\beta)\cdot\beta,$$

and thus

$$\mathrm{vol}_X(\alpha - \beta) \geqslant \alpha^2 - 2\alpha\cdot\beta.$$

□

In the last part of this subsection, we prove the second half of Theorem C.

THEOREM 1.3.3. *Let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big class and C be an irreducible divisor, then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}_X(\alpha + tC) = 2Z(\alpha)\cdot C. \tag{1.3.2}$$

PROOF. It suffices to prove the theorem for C not nef. Thus we have $C^2 < 0$. Write $N(\alpha) = \sum_{i=1}^r a_i N_i$, where each N_i is prime divisor. If $C \subseteq E_{nK}(Z(\alpha))$, we deduce that $Z(\alpha) \cdot C = 0$ by Theorem 1.2.2, and $\{C, N_1, \ldots, N_r\}$ forms an exceptional family by Theorem 1.2.10. Thus we have

$$Z(\alpha + tC) = Z(\alpha),$$

and

$$N(\alpha + tC) = N(\alpha) + tC$$

for $t \geqslant 0$. The theorem is thus proved in this case.

From now on we assume $C \nsubseteq E_{nK}(Z(\alpha))$, thus we have $Z(\alpha) \cdot C > 0$ and $C \nsubseteq \mathrm{Supp}(N(\alpha))$. We define

$$\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = -S^{-1} \cdot \begin{pmatrix} C \cdot N_1 \\ \vdots \\ C \cdot N_r \end{pmatrix},$$

where $S = (s_{ij})$ denotes the intersection matrix of $\{N_1, \ldots, N_r\}$. By Theorem 1.2.11 we know that S is negative definite satisfying $s_{ij} \geqslant 0$ for all $i \neq j$. We claim that $Z(\alpha) + t(\{C\} + \sum_{i=1}^r b_i \{N_i\})$ is big and nef if $0 \leqslant t < -\frac{Z(\alpha) \cdot C}{C^2}$. We need the following lemma from [**BKS03**] to prove our claim.

LEMMA 1.3.2. *Let A be a negative definite $r \times r$-matrix over the reals such that $a_{ij} \geqslant 0$ for all $i \neq j$. Then all entries of the inverse matrix A^{-1} are $\leqslant 0$.*

By Lemma 1.3.2 we know that all entries of S^{-1} are $\leqslant 0$, thus $b_j \geqslant 0$ for all $1 \leqslant j \leqslant r$ and we get the bigness of $Z(\alpha) + t(\{C\} + \sum_{i=1}^r b_i \{N_i\})$. By the construction of b_j, we have

$$(Z(\alpha) + t(\{C\} + \sum_{i=1}^r b_i \{N_i\})) \cdot N_j = 0$$

for $1 \leqslant j \leqslant r$, and

$$(Z(\alpha) + t(\{C\} + \sum_{i=1}^r b_i \{N_i\})) \cdot C > 0$$

for $0 \leqslant t < -\frac{Z(\alpha) \cdot C}{C^2}$. Thus we have the nefness and our claim follows. Since the divisorial Zariski decomposition is orthogonal and unique (see Theorem 1.2.11), we conclude that

$$N(\alpha + t\{C\}) = \sum_{i=1}^r (a_i - tb_i) N_i, \tag{1.3.3}$$

$$Z(\alpha + t\{C\}) = Z(\alpha) + t\{C\} + \sum_{i=1}^r tb_i \{N_i\}, \tag{1.3.4}$$

for t small enough. Since $\mathrm{vol}_X(\alpha + tC) = Z(\alpha + t\{C\})^2$, we have thus also obtained formula (1.3.2) in this case.

□

1.3.2. TRANSCENDENTAL MORSE INEQUALITY FOR SOME SPECIAL KÄHLER MANIFOLDS. One can modify the proof of Theorem D a little bit, to extend the transcendental Morse inequality to Kähler manifolds whose modified nef cone $\mathcal{MN}$ coincides with the nef cone $\mathcal{N}$. In this subsection, we assume X to be a compact Kähler manifold of dimension n which satisfies this condition.

LEMMA 1.3.3. *If $\alpha \in \mathcal{E}^\circ$, then the divisorial Zariski decomposition $\alpha = Z(\alpha) + N(\alpha)$ is such that*

$$Z(\alpha)^{n-1} \cdot N(\alpha) = 0.$$

REMARK 1.3.1. Lemma 1.3.3 is very similar to the Corollary 4.5 in [**BDPP13**]: If $\alpha \in \mathcal{E}_{\mathrm{NS}}$, then the divisorial Zariski decomposition $\alpha = Z(\alpha) + N(\alpha)$ is such that $\langle Z(\alpha)^{n-1} \rangle \cdot N(\alpha) = 0$. However, the proof of [**BDPP13**] is based on the orthogonal estimate for divisorial Zariski decomposition of $\mathcal{E}_{\mathrm{NS}}$, which is still a conjecture for $\alpha \in \mathcal{E}$. Here we will use Theorem 1.2.2 to prove this lemma directly.

PROOF OF LEMMA 1.3.3. By Theorem 1.2.8, if α is big, then $Z(\alpha)$ is big and modified nef, thus nef by the assumption for X. By Theorem 1.2.10, the primes $D_1, \ldots, D_q$ contained in the non-Kähler locus $E_{nK}(Z(\alpha))$ form an exceptional family, and $\alpha = Z(\alpha) + \sum_{i=1}^r a_i D_i$ for $a_i \geqslant 0$. Since $\mathrm{Null}(Z(\alpha)) = E_{nK}(Z(\alpha))$ by Theorem 1.2.2, we have $Z(\alpha)^{n-1} \cdot D_i = 0$ for each i, and thus $Z(\alpha)^{n-1} \cdot N(\alpha) = 0$. The lemma is proved. □

Proof of Theorem E. By Lemma 1.3.1, there exists $\epsilon > 0$ such that the prime components of $N(\alpha + t\beta)$ will be the same when $0 \leqslant t \leqslant \epsilon$. Moreover if we denote $N(\alpha + t\beta) = \sum_{i=1}^{r} a_i(t) N_i$, then each $a_i(t)$ is continuous and decreasing satisfying $a_i(t) > 0$. By Lemma 1.3.3, we have

$$Z(\alpha + t\beta)^{n-1} \cdot N(\alpha + t\beta) = \sum_{i=1}^{r} a_i(t) Z(\alpha + t\beta)^{n-1} \cdot N_i = 0.$$

Since $Z(\alpha + t\beta)$ is modified nef thus nef, we deduce that $Z(\alpha + t\beta)^{n-1} \cdot N_i = 0$ for $0 \leqslant t \leqslant \epsilon$ and $i = 1, \ldots, r$.

Since $a_i(t)$ is continuous and decreasing, it is almost everywhere differentiable. Thus $Z(\alpha + t\beta) = \alpha + t\beta - \sum_{i=1}^{r} a_i(t) N_i$ is an a.e. differentiable and continuous curves in the finite dimensional space $H^{1,1}(X, \mathbb{R})$ parametrized by t. Meanwhile, since $\alpha \mapsto \alpha^n$ is a polynomial in $H^{1,1}(X, \mathbb{R})$, we thus deduce that $\mathrm{vol}_X(\alpha + t\beta) = Z(\alpha + t\beta)^n$ is an a.e. differentiable function with respect to t. Therefore, if $\mathrm{vol}_X(\alpha + t\beta)$ and $a_i(t)$ are both differentiable at $t = t_0$, we have

$$\left.\frac{d}{dt}\right|_{t=t_0} \mathrm{vol}_X(\alpha + t\beta) = nZ(\alpha + t_0\beta)^{n-1} \cdot (\beta - \sum_{i=1}^{r} {a_i}'(t_0) N_i) = nZ(\alpha + t_0\beta)^{n-1} \cdot \beta.$$

Since $\mathrm{vol}_X(\alpha + t\beta)$ is increasing and continuous, it is also a.e. differentiable and thus we have

$$\begin{aligned} \mathrm{vol}_X(\alpha + s\beta) &= \mathrm{vol}_X(\alpha) + \int_0^s \frac{d}{dt} \mathrm{vol}_X(\alpha + t\beta) dt \\ &= \mathrm{vol}_X(\alpha) + n \int_0^s Z(\alpha + t\beta)^{n-1} \cdot \beta \, dt. \end{aligned} \tag{1.3.5}$$

for $0 \leqslant s \leqslant \epsilon$. Since $Z(\alpha + t\beta)$ is continuous (by Theorem 1.2.9), by (1.3.5) we deduce that $\mathrm{vol}_X(\alpha + t\beta)$ is differentiable with respect to t and its derivative

$$\left.\frac{d}{dt}\right|_{t=t_0} \mathrm{vol}_X(\alpha + t\beta) = nZ(\alpha + t_0\beta)^{n-1} \cdot \beta.$$

□

In order to prove transcendental Morse inequality, we will need the following bigness criterion obtained in [**Xia13**] and [**Pop14**].

Theorem 1.3.4. *Let X be an n-dimensional compact Kähler manifold. Assume α and β are two nef classes on X satisfying $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$, then $\alpha - \beta$ is a big class.*

The proof of the next theorem is similar to that of Theorem D and is therefore omitted.

Theorem 1.3.5. *Let X be a compact Kähler manifold on which the modified nef cone $\mathcal{MN}$ and the nef cone $\mathcal{N}$ coincide. If α and β are nef cohomology classes of type (1,1) on X satisfying the inequality $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$. Then $\alpha - \beta$ contains a Kähler current and* $\mathrm{vol}_X(\alpha - \beta) \geqslant \alpha^n - n\alpha^{n-1} \cdot \beta$.

Remark 1.3.2. In [**BFJ09**], the authors proved the following differentiability theorem:

$$\left.\frac{d}{dt}\right|_{t=t_0} \mathrm{vol}_X(L + tD) = n\langle L^{n-1}\rangle \cdot D, \tag{1.3.6}$$

where L is a big line bundle on the smooth projective variety X and D is a prime divisor. The right-hand side of the equation above involves the *positive intersection product* $\langle L^{n-1}\rangle \in H^{n-1,n-1}_{\geqslant 0}(X, \mathbb{R})$, first introduced in the analytic context in [**BDPP13**]. Theorem E could be seen as a transcendental version of (1.3.6) for some special Kähler manifolds. In the general Kähler situation, we propose the following conjecture:

Conjecture 1.3.6. *Let X be a Kähler manifold of dimensional n, α be a big class. If β is a pseudo-effective class, then we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}_X(\alpha + t\beta) = n\langle \alpha^{n-1}\rangle \cdot \beta.$$

1.4. Generalized Okounkov bodies on Kähler manifolds

1.4.1. Definition and relation with the algebraic case. Throughout this subsection, X will stand for a Kähler manifold of dimensional n. Our main goal in this subsection is to generalize the definition of Okounkov body to any pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$. First of all, we define a valuation-like function. For any positive current $T \in \alpha$ with analytic singularities, we define the valuation-like function

$$T \to \nu(T) = \nu_{Y_\bullet}(T) = (\nu_1(T), \ldots \nu_n(T))$$

as follows. First, set

$$\nu_1(T) = \sup\{\lambda \mid T - \lambda[Y_1] \geqslant 0\},$$

where $[Y_1]$ is the current of integration over Y_1. By Section 1.2.1 we know that $\nu_1(T)$ is the coefficient $\nu(T, Y_1)$ of the positive current $[Y_1]$ appearing in the Siu decomposition of T. Since T has analytic singularities, by the arguments in Section 1.2.2, $T_1 := (T - \nu_1[Y_1])|_{Y_1}$ is a well-defined positive current in the pseudo-effective class $(\alpha - \nu_1\{Y_1\})|_{Y_1}$ and it also has analytic singularities. Then take

$$\nu_2(T) = \sup\{\lambda \mid T_1 - \lambda[Y_2] \geqslant 0\},$$

and continue in this manner to define the remaining values $\nu_i(T) \in \mathbb{R}^+$.

REMARK 1.4.1. If one assumes $\alpha \in \mathrm{NS}_{\mathbb{Z}}(X)$, there exists a holomorphic line bundle L such that $\alpha = c_1(L)$. If D is the divisor of some holomorphic section $s_D \in H^0(X, \mathcal{O}_X(L))$, then we have

$$\nu([D]) = \mu(s_D),$$

where μ is the valuation-like function appeared in the definition of the original Okounkov body. Roughly speaking our definition of valuation-like function has a bigger domain of definition and thus the image of our valuation-like function contains $\bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL)$.

For any big class α, we define a $\mathbb{Q}$-*convex body* $\Delta_{\mathbb{Q}}(\alpha)$ (resp. $\mathbb{R}$-*convex body* $\Delta_{\mathbb{R}}(\alpha)$) to be the set of valuation vectors $\nu(T)$, where T ranges among all the Kähler (resp. positive) currents with algebraic (resp. analytic) singularities. Then $\Delta_{\mathbb{Q}}(\alpha) \subseteq \Delta_{\mathbb{R}}(\alpha)$. It is easy to check that this is a convex set in $\mathbb{Q}^n$ (resp. $\mathbb{R}^n$). Indeed, for any two positive currents T_0 and T_1 with algebraic (resp. analytic) singularities, we have $\nu(\epsilon T_0 + (1-\epsilon)T_1) = \epsilon\nu(T_0) + (1-\epsilon)\nu(T_1)$ for $0 \leqslant \epsilon \leqslant 1$ rational (resp. real). It is also obvious to see the homogeneous property of $\Delta_{\mathbb{Q}}(\alpha)$, that is, for all $c \in \mathbb{Q}^+$, we have

$$\Delta_{\mathbb{Q}}(c\alpha) = c\Delta_{\mathbb{Q}}(\alpha).$$

Indeed, since we have $\nu(cT) = c\nu(T)$ for all $c \in \mathbb{R}^+$, the claim follows directly.

EXAMPLE 1.1. Let L be a line bundle of degree $c > 0$ on a smooth curve C of genus g. Then we have

$$\Delta_{\mathbb{Q}}(c_1(L)) = \mathbb{Q} \cap [0, c).$$

Since $\mathrm{NS}_{\mathbb{R}}(C) = H^{1,1}(C, \mathbb{R})$, for any ample class α on C we have

$$\Delta_{\mathbb{Q}}(\alpha) = \mathbb{Q} \cap [0, \alpha \cdot C).$$

LEMMA 1.4.1. *Let α be a big class, then the $\mathbb{R}$-convex body $\Delta_{\mathbb{R}}(\alpha)$ lies in a bounded subset of $\mathbb{R}^n$.*

PROOF. It suffices to show that there exists a $b > 0$ large enough such that $\nu_i(T) < b$ for any positive current T with analytic singularities. We fix a Kähler class ω. Choose first of all $b_1 > 0$ such that

$$(\alpha - b_1 Y_1) \cdot \omega^{n-1} < 0.$$

This guarantees that $\nu_1(T) < b_1$ since $\alpha - b_1 Y_1 \notin \mathcal{E}$. Next choose b_2 large enough so that

$$((\alpha - aY_1)|_{Y_1} - b_2 Y_2) \cdot \omega^{n-2} < 0$$

for all real numbers $0 \leqslant a \leqslant b_1$. Then $\nu_2(T) \leqslant b_2$ for any positive current T with analytic singularities. Continuing in this manner we construct $b_i > 0$ for $i = 1, \ldots, n$ such that $\nu_i(T) \leqslant b_i$ for any positive current T with analytic singularities. We take $b = \max\{b_i\}$. □

LEMMA 1.4.2. *For any big class α, $\Delta_{\mathbb{Q}}(\alpha)$ is dense in $\Delta_{\mathbb{R}}(\alpha)$, in particular we have $\overline{\Delta_{\mathbb{Q}}(\alpha)} = \overline{\Delta_{\mathbb{R}}(\alpha)}$.*

PROOF. It is easy to verify that if T is a Kähler current with analytic singularities, then for any $\epsilon > 0$, there exists a Kähler current S_ϵ with algebraic singularities such that $\|\nu(S_\epsilon) - \nu(T)\| < \epsilon$ with respect to the standard norm in $\mathbb{R}^n$. For the general case, We fix a Kähler current $T_0 \in i\Theta(L)$ with algebraic singularities. Then for any positive current T with analytic singularities, $T_\epsilon := (1-\epsilon)T + \epsilon T_0$ is still a Kähler current. By Lemma 1.4.1, $\|\nu(T_\epsilon) - \nu(T)\| = \epsilon\|(\nu(T_0) - \nu(T))\|$ will tend to 0 since $\nu(T)$ is uniformly bounded for any positive current T with analytic singularities. Thus $\Delta_{\mathbb{Q}}(\alpha)$ is dense in $\Delta_{\mathbb{R}}(\alpha)$. □

Now we study the relations between $\Delta_{\mathbb{Q}}(c_1(L))$ and $\Delta(L)$ for L a big line bundle on X. First we begin with the following two lemmas.

LEMMA 1.4.3. *Let L be a big line bundle on the projective variety X of dimension n, with a singular Hermitian metric $h = e^{-\varphi}$ satisfying*

$$i\Theta_{L,h} = dd^c\varphi \geqslant \epsilon\omega$$

for some $\epsilon > 0$ and a given Kähler form ω. If the restriction of φ on a smooth hypersurface Y is not identically equal to $-\infty$, then there exists a positive integer m_0 which depends only on Y so that any holomorphic section $s_m \in H^0(Y, \mathcal{O}_Y(mL) \otimes \mathcal{I}(m\varphi|_Y))$ can be extended to $S_m \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(m\varphi))$ for any $m \geqslant m_0$.

We need the following Ohsawa-Takegoshi extension theorem to prove Lemma 1.4.3.

THEOREM 1.4.1. *Let X be a smooth projective variety, Y be a smooth divisor defined by a holomorphic section of the line bunle H with a smooth metric $h_0 = e^{-\psi}$. If L is a holomorphic line bunle with a singular metric $h = e^{-\phi}$, satisfying the curvature assumptions*

$$dd^c\phi \geqslant 0$$

and

$$dd^c\phi \geqslant \delta dd^c\psi$$

with $\delta > 0$, then for any holomorphic section $s \in H^0(Y, \mathcal{O}_Y(K_Y + L) \otimes \mathcal{I}(h|_Y))$, there exists a global holomorphic section $S \in H^0(X, \mathcal{O}_X(K_X + L + Y) \otimes \mathcal{I}(h))$ such that $S|_Y = s$.

PROOF OF LEMMA 1.4.3. Taking a smooth metric $e^{-\psi}$ and $e^{-\eta}$ on Y and K_X, we can choose m_0 large enough satisfying the curvature assumptions

$$dd^c(m\phi - \eta - \psi) \geqslant 0$$

and

$$dd^c(m\phi - \eta - \psi) \geqslant dd^c\psi$$

for any $m \geqslant m_0$.

By Theorem 1.4.1, any holomorphic section $s \in H^0(Y, \mathcal{O}_Y(K_Y + (mL - K_X - Y)|_Y) \otimes \mathcal{I}(h^m|_Y))$ can be extended to a global holomorphic section $S \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m))$ such that $S|_Y = s$. By the adjunction theorem, we have $(K_X + Y)|_Y = K_Y$, thus the lemma is proved.

□

LEMMA 1.4.4. *Let L be a big line bundle on the Riemann surface C with a singular Hermitian metric $h = e^{-\varphi}$ such that φ has algebraic singularities and*

$$i\Theta_{L,h} = dd^c\varphi \geqslant \epsilon\omega$$

for some $\epsilon > 0$. Then for a fixed point p, there exists an integer $k > 0$ such that we have a holomorphic section $s_k \in H^0(C, \mathcal{O}_C(kL) \otimes \mathcal{I}(h^k))$ satisfying $\mathrm{ord}_p(s_k) = k\nu(i\Theta_{L,h}, p)$.

PROOF. Since φ has algebraic singularities, we have the following Lebsegue decomposition

$$i\Theta_{L,h} = (i\Theta_{L,h})_{\mathrm{ac}} + \sum_{i=1}^{r} c_i x_i,$$

where each $c_i > 0$ is rational and $x_1, \ldots, x_r$ are the log poles of $i\Theta_{L,h}$ (possibly p is among them). Since we have

$$\int_C i(\Theta_{L,h})_{\mathrm{ac}} + \sum_{i=1}^{r} c_i = \deg(L),$$

thus

$$\sum_{i=1}^{r} c_i < \deg(L).$$

By Riemann-Roch theorem there exists an integer $k > 0$ satisfying

(i) kc_i is integer,
(ii) there is a holomorphic section $s_k \in H^0(C, \mathcal{O}_C(kL))$ such that $\mathrm{ord}_{x_i}(s_k) \geqslant kc_i$ and $\mathrm{ord}_p(s_k) = k\nu(i\Theta_{L,h}, p)$.

Thus s_k is locally integrable with respect to the weight $e^{-k\varphi}$. The theorem is proved.

□

THEOREM 1.4.2. *Let X be a smooth projective manifold of dimension n. For any Kähler current $T \in c_1(L)$ with algebraic singularities, there exists a holomorphic section $s \in H^0(X, \mathcal{O}_X(kL))$ such that $\mu(s) = k\nu(T)$, i.e., we have*

$$\nu(T) \in \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL).$$

In particular,

$$\Delta_{\mathbb{Q}}(c_1(L)) \subseteq \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL) \subseteq \Delta(L).$$

PROOF. First, set $\nu_i = \nu_i(T)$ and define

$$T_0 := T,\ T_1 := (T_0 - \nu_1[Y_1])|_{Y_1},\ \dots\ , T_{n-1} := (T_{n-2} - \nu_{n-1}[Y_{n-1}])|_{Y_{n-1}};$$
$$L_0 := L - \nu_1 Y_1,\ L_1 := L_0|_{Y_1} - \nu_2 Y_2,\ \dots\ , L_{n-2} := L_{n-3}|_{Y_{n-2}} - \nu_{n-1} Y_{n-1}.$$

Since $T_0 \geqslant \epsilon\omega$, we have $T_1 \geqslant \epsilon\omega|_{Y_1}, \dots, T_{n-1} \geqslant \epsilon\omega|_{Y_{n-1}}$. Since each ν_i is rational, we could find an integer m to make each $m\nu_i$ be integer so that each mL_i is a big line bundle on Y_i. If we can prove

$$\nu(mT) \in \bigcup_{k=1}^{\infty} \frac{1}{k}\mu(kmL),$$

then we will have

$$\nu(T) \in \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL),$$

by the homogeneous property $\frac{1}{m}\nu(mT) = \nu(T)$. Thus we can assume that each $\nu_i(T)$ is an integer after we replace L by mL and T by mT.

Firstly, since $T_0 \in c_1(L)$ is a Kähler current with algebraic singularities, there exists a singular metric $h = e^{-\varphi_0}$ on L whose curvature current is T_0 and φ has algebraic singularities; on the other hand, there is a canonical metric $e^{-\eta_0}$ on $\mathcal{O}_{Y_0}(Y_1)$ such that $dd^c\eta_0 = [Y_1]$ in the sense of currents, thus by the definition of ν_1 we deduce that $h_0 := e^{-\varphi_0+\nu_1\eta_0}$ is a singular metric of L_0 such that $-\varphi_0+\nu_1\eta_0$ does not vanish identically on Y_1, and $h_0|_{Y_1}$ is a singular metric of $L_0|_{Y_1}$ with algebraic singularities whose curvature current is $T_1 \geqslant \epsilon\omega|_{Y_1}$.

Secondly, there is a canonical singular metric $e^{-\eta_1}$ of $\mathcal{O}_{Y_1}(Y_2)$ on Y_1 with the curvature current $[Y_2]$. Thus the singular metric $h_1 := h_0|_{Y_1} + e^{\nu_2\eta_1}$ of the big line bundle L_1 gives a curvature current $T_1 - \nu_2[Y_2] \geqslant \epsilon\omega|_{Y_1}$. We continue in this manner to define the remaining singular metrics $h_i := h_{i-1}|_{Y_i} + e^{\nu_{i+1}\eta_i}$ of the big line bundle L_i on Y_i with curvature current $T_i - \nu_{i+1}[Y_{i+1}] \geqslant \epsilon\omega|_{Y_i}$ for $i = 0, \dots, n-1$. It is easy to see that $h_i|_{Y_{i+1}}$ is well-defined.

By Lemma 1.4.3, there exists a k_0 such that for each $k \geqslant k_0$, the following short sequence is exact

$$H^0(Y_{i-1}, \mathcal{O}_{Y_{i-1}}(kL_{i-1}) \otimes \mathcal{I}(h_{i-1}^k)) \longrightarrow H^0(Y_i, \mathcal{O}_{Y_i}(kL_{i-1}) \otimes \mathcal{I}(h_{i-1}^k|_{Y_i})) \longrightarrow 0 \tag{1.4.1}$$

for $i = 1, \dots, n-1$.

Now we begin our construction. T_{n-1} is the curvature current of the singular metric $h_{n-2}|_{Y_{n-1}}$ of $L_{n-2}|_{Y_{n-1}}$ over the Riemann surface Y_{n-1}. By Lemma 1.4.4, there exists a $k \geqslant k_0$ and a holomorphic section $s_{n-1} \in H^0(Y_{n-1}, \mathcal{O}_{Y_{n-1}}(kL_{n-2}) \otimes \mathcal{I}(h_{n-2}^k|_{Y_{n-1}}))$, such that $\mathrm{ord}_p(s_{n-1}) = k\nu(T_{n-1}, p) = k\nu_n$.

By the exact sequence (1.4.1), s_{n-1} could be extend to

$$\widetilde{s}_{n-2} \in H^0(Y_{n-2}, \mathcal{O}_{Y_{n-2}}(kL_{n-2}) \otimes \mathcal{I}(h_{n-2}^k)).$$

Now we choose a canonical section t_{n-2} of $H^0(Y_{n-2}, \mathcal{O}_{Y_{n-2}}(Y_{n-1}))$ such that the divisor of t_{n-2} is Y_{n-1}. We define $s_{n-2} := \widetilde{s}_{n-2} t_{n-2}^{\otimes \nu_{n-1}}$, by the construction of $h_{n-2} := h_{n-3}|_{Y_{n-2}} + e^{\nu_{n-1}\eta_{n-2}}$, we obtain that

$$s_{n-2} \in H^0(Y_{n-2}, \mathcal{O}_{Y_{n-2}}(kL_{n-3}) \otimes \mathcal{I}(h_{n-3}^k|_{Y_{n-2}}).$$

We can continue in this manner to construct a section $s_0 \in H^0(X, \mathcal{O}_X(kL))$ and by our construction we have

$$\mu(s_0) = (k\nu_1, \dots, k\nu_n) = k\nu(T),$$

this concludes the theorem.

□

PROPOSITION 1.4.1. *For any big line bundle L and any admissible flag $Y_\bullet$, one has $\overline{\Delta_{\mathbb{Q}}(c_1(L))} = \Delta(L)$. In particular,*

$$\Delta(L) = \overline{\bigcup_{m=1}^{\infty} \frac{1}{m}\nu(mL)}.$$

PROOF. Firstly, since $\Delta_{\mathbb{Q}}(c_1(L))$ is a convex set in $\mathbb{Q}^n$, its closure in $\mathbb{R}^n$ denoted by $\overline{\Delta_{\mathbb{Q}}(c_1(L))}$ is also a closed convex set. By Proposition 1.4.2, we have

$$\Delta_{\mathbb{Q}}(c_1(L)) \subset \bigcup_{m=1}^{\infty} \frac{1}{m} \cdot \nu(mL),$$

thus

$$\overline{\Delta_{\mathbb{Q}}(c_1(L))} \subseteq \Delta(L).$$

By Remark 1.4.1, we have $\bigcup_{m=1}^{\infty} \frac{1}{m}\nu(mL) \subseteq \Delta_{\mathbb{R}}(c_1(L))$, thus by the definition of Okounkov body $\Delta(L)$, we deduce that

$$\Delta(L) \subseteq \overline{\Delta_{\mathbb{R}}(c_1(L))}.$$

By Lemma 1.4.2, we have $\overline{\Delta_{\mathbb{Q}}(c_1(L))} = \overline{\Delta_{\mathbb{R}}(c_1(L))}$, thus the theorem is proved. □

REMARK 1.4.2. By Proposition 1.4.1, in the definition of the Okounkov body $\Delta(L)$, it suffices to close up the set of normalized valuation vectors instead of the closure of the convex hull of this set.

REMARK 1.4.3. It is easy to reprove that the Okounkov body $\Delta(L)$ depends only on the numerical equivalence class of the big line bundle L. Indeed, if L_1 and L_2 are numerically equivalent, we have $c_1(L_1) = c_1(L_2)$ thus

$$\Delta_{\mathbb{Q}}(c_1(L_1)) = \Delta_{\mathbb{Q}}(c_1(L_2)).$$

By Proposition 1.4.1, we have

$$\Delta(L_1) = \Delta(L_2).$$

Now we are ready to find some valuative points in the Okounkov bodies.

PROOF OF COROLLARY 1.1.1. In [**LM09**] we know that $\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)) = \mathrm{vol}_X(L) > 0$ by the bigness of L. Since we have $\Delta(L) = \overline{\Delta_{\mathbb{Q}}(c_1(L))}$ by Proposition 1.4.1, then for any $p \in \mathrm{int}(\Delta(L)) \cap \mathbb{Q}^n$, there exists an n-simplex Δ_n containing p with all the vertices lying in $\Delta_{\mathbb{Q}}(c_1(L))$. Since $\Delta_{\mathbb{Q}}(c_1(L))$ is a convex set in $\mathbb{Q}^n$, we have $\Delta_n \cap \mathbb{Q}^n \subseteq \Delta_{\mathbb{Q}}(c_1(L))$, and thus

$$\Delta_{\mathbb{Q}}(c_1(L)) \supseteq \mathrm{int}(\Delta(L)) \cap \mathbb{Q}^n.$$

From Theorem 1.4.2 we have $\Delta_{\mathbb{Q}}(c_1(L)) \subseteq \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL)$, thus we get the inclusion

$$\mathrm{int}(\Delta(L)) \cap \mathbb{Q}^n \subseteq \bigcup_{m=1}^{\infty} \frac{1}{m}\mu(mL),$$

which means that all rational interior points of $\Delta(L)$ are valuative. □

Pursuing the same philosophy as in Proposition 1.4.1, it is natual to extend results related to Okounkov bodies for big line bundles, to the more general case of an arbitrary big class $\alpha \in H^{1,1}(X, \mathbb{R})$. We propose the following definition.

DEFINITION 1.4.3. *Let X be a Kähler manifold of dimension n. We define the* generalized Okounkov body *of a big class $\alpha \in H^{1,1}(X, \mathbb{R})$ with respect to the fixed flag $Y_\bullet$ by*

$$\Delta(\alpha) = \overline{\Delta_{\mathbb{R}}(\alpha)} = \overline{\Delta_{\mathbb{Q}}(\alpha)}.$$

We have the following properties for generalized Okounkov bodies:

PROPOSITION 1.4.2. *Let α and β be big classes, ω be any Kähler class. Then:*

(i) $\Delta(\alpha) + \Delta(\beta) \subseteq \Delta(\alpha + \beta)$.
(ii) $\mathrm{vol}_{\mathbb{R}^n}(\Delta(\omega)) > 0$.
(iii) $\Delta(\alpha) = \bigcap_{\epsilon>0} \Delta(\alpha + \epsilon\omega)$.

PROOF. (i) is obvious from the definition of generalized Okounkov body. To prove (ii), we use induction for dimension. The result is obvious if $n = 1$, assume now that (ii) is true for $n - 1$. We choose $t > 0$ small enough such that $\omega - tY_1$ is still a Kähler class. By the main theorem of [**CT14**], any Kähler current $T \in (\omega - tY_1)|_{Y_1}$ with analytic singularities can be extended to a Kähler current $\widetilde{T} \in \omega - tY_1$, thus we have

$$\Delta(\omega) \bigcap t \times \mathbb{R}^{n-1} = t \times \Delta((\omega - tY_1)|_{Y_1}),$$

where $\Delta((\omega - tY_1)|_{Y_1})$ is the generalized Okounkov body of $(\omega - tY_1)|_{Y_1}$ with respect to the flag

$$Y_1 \supset Y_2 \supset \ldots \supset Y_n = \{p\}.$$

By the induction, we have $\mathrm{vol}_{\mathbb{R}^{n-1}}(\Delta((\omega - tY_1)|_{Y_1})) > 0$. Since $\Delta(\omega)$ contains the origin, we have $\mathrm{vol}_{\mathbb{R}^n}(\Delta(\omega)) > 0$.

Now we are ready to prove (iii). By the concavity we have

$$\Delta(\alpha + \epsilon_1\omega) + \Delta((\epsilon_2 - \epsilon_1)\omega) \subseteq \Delta(\alpha + \epsilon_2\omega)$$

if $0 \leqslant \epsilon_1 < \epsilon_2$. Since $\Delta(\omega)$ contains the origin, we have

$$\Delta(\alpha) \subseteq \bigcap_{\epsilon>0} \Delta(\alpha + \epsilon\omega),$$

and

$$\Delta(\alpha + \epsilon_1\omega) \subseteq \Delta(\alpha + \epsilon_2\omega).$$

From the concavity property, we conclude that $\mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha+t\omega))$ is a concave function for $t \geqslant 0$, thus continuous. Then we have

$$\mathrm{vol}_{\mathbb{R}^n}(\bigcap_{\epsilon>0}\Delta(\alpha+\epsilon\omega)) = \mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha)) > 0.$$

Since $\bigcap_{\epsilon>0}\Delta(\alpha+\epsilon\omega)$ and $\Delta(\alpha)$ are both closed and convex, we have

$$\Delta(\alpha) = \bigcap_{\epsilon>0}\Delta(\alpha+\epsilon\omega).$$

□

REMARK 1.4.4. We don't know whether $\mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha))$ is independent of the choice of the admissible flag. However, in the next subsection we will prove that in the case of surfaces we have

$$\mathrm{vol}_X(\alpha) = 2\,\mathrm{vol}_{\mathbb{R}^2}(\Delta((\alpha)),$$

in particular the Euclidean volume of the generalized Okounkov body is independent of the choice of the flag. We conjecture that

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(\alpha)) = \frac{1}{n!}\cdot \mathrm{vol}_X(\alpha),$$

as we proposed in the introduction.

1.4.2. GENERALIZED OKOUNKOV BODIES ON COMPLEX SURFACES. Now we will mainly focus on generalized Okounkov bodies of compact Kähler surfaces. In this section, X denotes a compact Kähler surface. We fix henceforth an admissible flag

$$X \supseteq C \supseteq \{x\},$$

on X, where $C \subset X$ is an irreducible curve and $x \in C$ is a smooth point.

DEFINITION 1.4.4. *For any big class $\alpha \in H^{1,1}(X,\mathbb{R})$, we denote the* restricted $\mathbb{R}$-convex body *of α along C by $\Delta_{\mathbb{R},X|C}(\alpha)$, which is defined to be the set of Lelong numbers $\nu(T|_C, x)$, where $T \in \alpha$ ranges among all the positive currents with analytic singularities such that $C \nsubseteq E_+(T)$. The* restricted Okounkov body *of α along C is defined as*

$$\Delta_{X|C}(\alpha) := \overline{\Delta_{\mathbb{R},X|C}(\alpha)}.$$

When $\alpha = c_1(L)$ for some big line bunle L on X, it is noticeable that $\Delta_{X|C}(\alpha) = \Delta_{X|C}(L)$, where $\Delta_{X|C}(L)$ is defined in [**LM09**]. When L is ample, we have $\Delta_{X|C}(L) = \Delta(L|_C)$. Indeed, it is suffice to show that for any section $s \in H^0(C, \mathcal{O}_C(L))$, there exists an integer m such that $s^{\otimes m}$ can be extended to a section $S_m \in H^0(X, \mathcal{O}_X(mL))$. This can be garanteed by Kodaira vanishing theorem. When α is any ample class, there is a very similar theorem which has appeared in the proof of Proposition 1.4.2. However, the proof there relies on the difficult extension theorem in [**CT14**]. Here we give a simple and direct proof when X is a complex surface. Anyway, the idea of proof here is borrowed from [**CT14**].

PROPOSITION 1.4.3. *If α is an ample class, then we have*

$$\Delta_{X|C}(\alpha) = \Delta(\alpha|_C) = [0, \alpha\cdot C].$$

PROOF. From Definition 1.4.4, we have $\Delta_{X|C}(\alpha) \subseteq \Delta(\alpha|_C)$. It suffices to prove that for any Kähler current $T \in \alpha|_C$ with mild analytic singularities, we have a positive current $\widetilde{T} \in \alpha$ with analytic singularites such that $\widetilde{T}|_C = T$. First we choose a Kähler form $\omega \in \alpha$. By assumption, we can write $T = \omega|_V + dd^c\varphi$ for some quasi-plurisubharmonic function φ on C which has mild analytic singularities. Our goal is to extend φ to a function Φ on X such that $\omega + dd^c\Phi$ is a Kähler current with analytic singularities.

Choose $\epsilon > 0$ small enough so that

$$T = \omega|_C + idd^c\varphi \geqslant 3\epsilon\omega,$$

holds as currents on C. We can cover C by finitely many charts $\{W_j\}_{1\leqslant j\leqslant N}$ satisfying the following properties:

(i) On each $W_j (j \leqslant k)$ there are local coordinates $(z_1^{(j)}, z_2^{(j)})$ such that $C\bigcap W_j = \{z_2^{(j)} = 0\}$ and

$$\varphi = \frac{c_j}{2}\log|z_1^{(j)}|^2 + g_j(z_1^{(j)})$$

where $g_j(z_1^{(j)})$ is smooth and bounded on $W_j\bigcap C$. We denote the single pole of T in $W_j(j\leqslant k)$ by x_j;

(ii) on each $W_j(j > k)$ the local potential φ is smooth and bounded on $W_j\bigcap C$;

(iii) $x_i \notin \overline{W_j}$ for $i = 1,\ldots,k$ and $j \neq i$.

Define a function φ_j on W_j (with analytic singularities) by

$$\varphi_j(z_1^{(j)}, z_2^{(j)}) = \begin{cases} \varphi(z_1^{(j)}) + A|z_2^{(j)}|^2 & \text{if} \quad j > k, \\ \frac{c_j}{2}\log(|z_1^{(j)}|^2 + |z_2^{(j)}|^2) + g_j(z_1^{(j)}) + A|z_2^{(j)}|^2 & \text{if} \quad j \leqslant k, \end{cases}$$

where $A > 0$ is a constant. If we shrink the W_j's slightly, still preserving the property that $C \subseteq \bigcup W_j$, we can choose A sufficiently large so that

$$\omega + dd^c\varphi_j \geqslant 2\epsilon\omega$$

holds on W_j for all j. We also need to construct slightly smaller open sets $W_j' \subset\subset U_j \subset\subset W_j$ such that $\bigcup W_j'$ is still a covering of C.

By construction φ_j is smooth when $j > k$, and φ_j is smooth outside the log pole x_j when $j \leqslant k$. By property (iii) above, we can glue the functions φ_j together to produce a Kähler current

$$\widetilde{T} = \omega|_U + dd^c\widetilde{\varphi} \geqslant \epsilon\omega$$

defined in a neighborhood U of C in X, thanks to Richberg's gluing procedure. Indeed, φ_i is smooth on $W_i \bigcap W_j$ for any $j \neq i$, which is a sufficient condition in using the Richberg technique. From the construction of $\widetilde{\varphi}$, we know that $\widetilde{\varphi}|_C = \varphi$, $\widetilde{\varphi}$ has log poles at every x_i and is continuous outside $x_1, \dots, x_k$.

On the other hand, since α is an ample class, there exists a rational number $\delta > 0$ such that $\alpha - \delta\{C\}$ is still ample, thus we have a Kähler form $\omega_1 \in \alpha - \delta\{C\}$. We can write $\omega_1 + \delta[C] = \omega + dd^c\phi$, where ϕ is smooth outside C, and for any point $x \in C$, we have

$$\phi = \frac{\delta}{2}\log|z_2|^2 + O(1),$$

where z_2 is the local equation of C.

Since ϕ is continuous outside C, we can choose a large constant $B > 0$ such that $\phi > \widetilde{\varphi} - B$ in a neighborhood of ∂U. Therefore we define

$$\Phi = \begin{cases} \max\{\widetilde{\varphi}, \phi + B\} & \text{on } U \\ \phi + B & \text{on } X - U, \end{cases}$$

which is well defined on the whole of X, and satisfies $\omega + dd^c\Phi \geqslant \epsilon'\omega$ for some $\epsilon' > 0$. Since $\phi = -\infty$ on C, while $\widetilde{\varphi}|_C = \varphi$, it follows that $\Phi|_C = \varphi$.

We claim that Φ also has analytic singularities. Since around x_j, we have

$$\widetilde{\varphi}(z_1, z_2) = \frac{c_j}{2}\log(|z_1|^2 + |z_2|^2) + O(1),$$

and

$$\phi(z_1, z_2) = \frac{\delta}{2}\log|z_2|^2 + O(1),$$

for some local coordinates (z_1, z_2) of x_j. Thus locally we have

$$\max\{\widetilde{\varphi}, \phi + A\} = \frac{1}{2}\log(|z_1|^{2c_j} + |z_2|^{2c_j} + |z_2|^{2\delta}) + O(1).$$

Since Φ is continuous outside $x_1, \dots, x_k$, our claim is proved. □

LEMMA 1.4.5. *Let α be a big and nef class on X, then for any $\epsilon > 0$, there exsists a Kähler current $T_\epsilon \in \alpha$ with analytic singularities such that the Lelong number $\nu(T_\epsilon, x) < \epsilon$ for any point in X. Moreover, T_ϵ also satisfies*

$$E_+(T) = E_{nK}(\alpha).$$

PROOF. Since α is big, there exists a Kähler current with analytic singularities such that $E_+(T_0) = E_{nK}(\alpha)$ and $T_0 > \omega$ for some Kähler form ω. Since α is also a nef class, for any $\delta > 0$, there exists a smooth form θ_δ in α such that $\theta_\delta \geqslant -\delta\omega$. Thus $T_\delta := \delta T_0 + (1-\delta)\theta_\delta \geqslant \delta^2\omega$ is a Kähler current with analytic singularities satisfying that

$$E_+(T_\delta) = E_+(T_0) = E_{nK}(\alpha),$$

and

$$\nu(T_\delta, x) = \delta\nu(T_0, x)$$

for any $x \in X$. Since the Lelong number $\nu(T_0, x)$ is an upper continuous function (thus bounded from above), $\nu(T_\delta, x)$ converges uniformly to zero as δ tends to 0. The lemma is proved. □

PROPOSITION 1.4.4. *Let α be a big and nef class, $C \nsubseteq E_{nK}(\alpha)$. Then we have*

$$\Delta_{X|C}(\alpha) = \Delta(\alpha|_C) = [0, \alpha \cdot C].$$

PROOF. Asumme $E_{nK}(\alpha) = \bigcup_{i=1}^r C_i$, where each C_i is an irreducible curve. By Lemma 1.4.5, for any $\epsilon > 0$ there exists a Kähler current $T_\epsilon \in \alpha$ with analytic singularities such that

$$E_+(T_\epsilon) = E_{nK}(\alpha) = \mathrm{Null}(\alpha) = \bigcup_{i=1}^r C_i$$

and $\nu(T_\epsilon, x) < \epsilon$ for all $x \in X$. Thus the Siu decomposition

$$T_\epsilon = R_\epsilon + \sum_{i=1}^r a_{i,\epsilon} C_i$$

satisfies $0 \leqslant a_{i,\epsilon} < \epsilon$, and R_ϵ is a Kähler current whose analytic singularities are isolated points. By Remark 1.2.1, the cohomology class $\{R_\epsilon\}$ is a Kähler class and converges to α as $\epsilon \to 0$. In particular, $|\{R_\epsilon\} \cdot C - \alpha \cdot C| < A\epsilon$, where A is a constant.

By Proposition 1.4.3, there exists a Kähler current $S_\epsilon \in \{R_\epsilon\}$ with analytic singularities such that $C \nsubseteq E_+(S_\epsilon)$ and $-\epsilon < \nu(S_\epsilon|_C, x) - \{R_\epsilon\} \cdot C < 0$. Thus $T'_\epsilon := S_\epsilon + \sum_{i=1}^r a_{i,\epsilon} C_i$ is a Kähler current in α with analytic singularities, and $-(1+A)\epsilon < \nu(T'_\epsilon|_C, x) - \alpha \cdot C$. Since α is big and nef, there exists a Kähler current P_ϵ in α with analytic singularities such that $\nu(P_\epsilon|_C, x) < \epsilon$. Therefore, by the definition of $\Delta_{X|C}(\alpha)$ and the convexity property we deduce that $[0, \alpha \cdot C] \subseteq \Delta_{X|C}(\alpha)$. On the other hand, $\Delta_{X|C}(\alpha) \subseteq \Delta(\alpha|_C) = [0, \alpha \cdot C]$ by definition. The proposition is proved.

□

LEMMA 1.4.6. *Let α be a big class on X with divisorial Zariski decomposition $\alpha = Z(\alpha) + N(\alpha)$. Assume that $C \nsubseteq E_{nK}(Z(\alpha))$, so that $C \nsubseteq \mathrm{Supp}(N(\alpha))$ by Theorem 1.2.10. Moreover, set*

$$f(\alpha) = \nu_x(N(\alpha)|_C), \quad g(\alpha) = \nu_x(N(\alpha)|_C) + Z(\alpha) \cdot C,$$

where $\nu_x(N(\alpha)|_C) = \nu(N(\alpha)|_C, x)$. Then the restricted Okounkov body of α along C is the interval

$$\Delta_{X|C}(\alpha) = [f(\alpha), g(\alpha)]$$

PROOF. First, by Remark 1.2.3 we conclude that $T \mapsto T - N(\alpha)$ is a bijection between the positive currents in α and those in $Z(\alpha)$, thus we have

$$E_{nK}(\alpha) = E_{nK}(Z(\alpha)) \bigcup \mathrm{supp}(N(\alpha)),$$

and

$$C \nsubseteq E_{nK}(Z(\alpha)) \iff C \nsubseteq E_{nK}(\alpha). \tag{1.4.2}$$

By the assumption of theorem, $N(\alpha)|_C$ is a well-defined positive current with analytic singularites on C. By the definition of $\Delta_{\mathbb{R},X|C}(\alpha)$, we have

$$\Delta_{\mathbb{R},X|C}(\alpha) = \Delta_{\mathbb{R},X|C}(Z(\alpha)) + \nu_x(N(\alpha)|_C).$$

We take the closure of the sets to get

$$\Delta_{X|C}(\alpha) = \Delta_{X|C}(Z(\alpha)) + \nu_x(N(\alpha)|_C).$$

Since α is big, thus $Z(\alpha)$ is big and nef, and by Proposition 1.4.4 we have $\Delta_{X|C}(Z(\alpha)) = [0, Z(\alpha) \cdot C]$. We have proved the lemma.

□

DEFINITION 1.4.5. *If α is big and β is pseudo-effective, then the slope of β with respect to α is defined as*

$$s = s(\alpha, \beta) = \sup\{t > 0 \mid \alpha - t\beta \textit{ is big}\}.$$

REMARK 1.4.5. Since the big cone is open, we know that $\{t > 0 \mid \alpha > t\beta\}$ is an open set in $\mathbb{R}^+$. Thus $\alpha - s\beta$ belongs to the boundary of the big cone denoted by $\partial\mathcal{E}$, and $\mathrm{vol}_X(\alpha - s\beta) = 0$.

PROOF OF THEOREM B. For $t \in [0, s)$, we put $\alpha_t = \alpha - t\{C\}$, and let $Z_t := Z(\alpha_t)$ and $N_t := N(\alpha_t)$ be the positive and negative part of the divisorial Zariski decomposition of α_t.

(i) First we assume C is nef. By Theorem 1.2.10, the prime divisors in $E_{nK}(Z(\alpha_t))$ form an exceptional family, thus $C \nsubseteq E_{nK}(Z(\alpha_t))$ and $C \nsubseteq E_{nK}(\alpha_t)$ by (1.4.2). By Lemma 1.4.6 we have $\Delta_{X|C}(\alpha_t) = [\nu_x(N_t|_C), Z_t \cdot C + \nu_x(N_t|_C)]$.

By the definition of $\mathbb{R}$-convex body and restrict $\mathbb{R}$-convex body, we have

$$\Delta_{\mathbb{R}}(\alpha) \bigcap t \times \mathbb{R} = t \times \Delta_{\mathbb{R},X|C}(\alpha_t).$$

Thus

$$t \times \overline{\Delta_{\mathbb{R},X|C}(\alpha_t)} \subseteq \overline{\Delta_{\mathbb{R}}(\alpha)} \bigcap t \times \mathbb{R}.$$

However, since both $\Delta_{\mathbb{R},X}(\alpha)$ and $\Delta_{\mathbb{R},X|C}(\alpha_t)$ are closed convex sets in $\mathbb{R}^2$ and $\mathbb{R}$, we have

$$t\times\overline{\Delta_{\mathbb{R},X|C}(\alpha_t)}=\overline{\Delta_{\mathbb{R}}(\alpha)}\bigcap t\times\mathbb{R},$$

therefore

$$t\times\Delta_{X|C}(\alpha_t)=\Delta(\alpha)\bigcap t\times\mathbb{R}. \tag{1.4.3}$$

Let

$$f(t)=\nu_x(N_t|_C)\ ,\ g(t)=Z_t\cdot C+\nu_x(N_t|_C),$$

then $\Delta(\alpha)\bigcap[0,s)\times\mathbb{R}$ is the region bounded by the graphs of $f(t)$ and $g(t)$.

Now we prove the piecewise linear property of $f(t)$ and $g(t)$. By Lemma 1.3.1, we have $N_{t_1}\leqslant N_{t_2}$ if $0\leqslant t_1\leqslant t_2<s$, thus $f(t)$ is increasing. Since N_t is an exceptional divisor by Theorem 1.2.11, the number of the prime components of N_t is uniformly bounded by the Picard number $\rho(X)$. Thus we can denote $N_t=\sum_{i=1}^r a_i(t)N_i$, where $a_i(t)\geqslant 0$ is an increasing and continuous function. Moreover, there exsists $0=t_0<t_1<\ldots<t_k=s$ such that the prime components of N_t are the same when t lies in the interval (t_i,t_{i+1}) for $i=0,\ldots,k-1$, and the number of prime components of N_t will increase at every t_i for $i=1,\ldots,k-1$. We write $s_i=\frac{t_{i-1}+t_i}{2}$ for $i=1,\ldots,k$.

We denote the linear subspace of $H^{1,1}(X,\mathbb{R})$ spanned by the prime components of N_{s_i} by V_i, and let $V_i^{\perp}$ be the orthogonal space of V_i with respect to q. By the proof of Lemma 1.3.1, for $t\in(t_{i-1},t_i)$ we have

$$Z_t=Z_{s_i}+(s_i-t)\{C\}_i^{\perp} \tag{1.4.4}$$

$$N_t=N_{s_i}+(s_i-t)C_i^{\|}, \tag{1.4.5}$$

where $\{C\}_i^{\perp}$ is the projection of $\{C\}$ to $V_i^{\perp}$, and $C_i^{\|}$ is a linear combination of the prime components of N_{s_i} satisfying that the cohomology class $\{C_i^{\|}\}$ is equal to the projection of $\{C\}$ to V_i. By Theorem 1.2.10, the cohomology classes of prime components of N_{s_i} are all independent, thus $C_i^{\|}$ is uniquely defined. The piecewise linearity property of $f(t)$ and $g(t)$ follows directly from (1.4.4) and (1.4.5), and thus $f(t)$ and $g(t)$ can be continuously extended to s. Therefore we conclude that $\Delta(\alpha)$ is the region bounded by the graphs of $f(t)$ and $g(t)$ for $t\in[0,s]$, and the vertices of $\Delta(\alpha)$ are contained in the set $\{(t_i,f(t_i)),(t_j,g(t_j))\in\mathbb{R}^2\mid i,j=0,\ldots,k\}$. This means that a vertex of $\Delta(\alpha)$ may only occur for those $t\in[0,s]$, where a new curve appears in N_t. Since $r\leqslant\rho(X)$, the number of vertices is bounded by $2\rho(X)+2$. The fact that $f(t)$ is convex and $g(t)$ concave is a consequence of the convexity of $\Delta(\alpha)$.

By (1.4.3), we have

$$\begin{aligned}2\,\mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha))&=2\int_0^s\mathrm{vol}_{\mathbb{R}}(\Delta_{X|C}(\alpha_t))\mathrm{d}t\\&=2\int_0^s Z_t\cdot C\mathrm{d}t\\&=\mathrm{vol}_X(\alpha)-\mathrm{vol}_X(\alpha-sC)\\&=\mathrm{vol}_X(\alpha).\end{aligned}$$

where the second equality follows by Proposition 1.4.4, the third one by Theorem 1.3.1 and the last one by Remark 1.4.5. We have proved the theorem under the assumption that C is nef.

(ii) Now we prove the theorem when C is not nef, i.e., $C^2<0$. Recall that $a:=\sup\{t>0\mid C\subseteq E_{nK}(\alpha_t)\}$. By (1.4.2), if $C\subseteq E_{nK}(\alpha_t)$ for some $t\in[0,s)$, we have $C\subseteq E_{nK}(Z(\alpha_t))$. By the proof in Theorem 1.3.3 we have

$$\begin{aligned}Z(\alpha_\tau)\cdot C&=0,\\Z(\alpha_\tau)&=Z(\alpha_t),\end{aligned}$$

for $0\leqslant\tau\leqslant t$. Thus we have

$$\{0\leqslant t<s\mid C\nsubseteq E_{nK}(\alpha_t)\}=(a,s),$$

and $\Delta(\alpha)$ is contained in $[a,s]\times\mathbb{R}$. By Theorem 1.3.3 we also have

$$\begin{aligned}2\,\mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha))&=2\int_a^s\mathrm{vol}_{\mathbb{R}}(\Delta_{X|C}(\alpha_t))\mathrm{d}t\\&=2\int_a^s Z_t\cdot C\mathrm{d}t\\&=\mathrm{vol}_X(\alpha_a)-\mathrm{vol}_X(\alpha_s)\\&=\mathrm{vol}_X(\alpha).\end{aligned}$$

Since the prime components of N_{t_1} is contained in that of N_{t_2} if $a < t_1 \leqslant t_2 < s$, using the same arguments above, we obtain the piecewise linear property of $f(t)$ and $g(t)$ which can also be extended to s. The theorem is proved completely. □

REMARK 1.4.6. If X is a projective surface, by the main result in [**BKS03**], the cone of big divisors of X admits a locally finite decomposition into locally polyhedral subcones such that the support of the negative part in the Zariski decomposition is constant on each subcone. It is noticeable that if we only assume X to be Kähler, this decomposition still holds if we replace the cone of big divisors by the cone of big classes and use divisorial Zariski decomposition instead. This property ensures that the generalized Okounkov bodies should also be polygons.

1.4.3. GENERALIZED OKOUNKOV BODIES FOR PSEUDO-EFFECTIVE CLASSES. Throughout this subsection, X will stand for a Kähler surface if not specially mentioned. Our main goal in this subsection is to study the behavior of generalized Okounkov bodies on the boundary of the big cone.

DEFINITION 1.4.6. *Let X be any Kähler manifold, if $\alpha \in H^{1,1}(X, \mathbb{R})$ is any pseudo-effective class. We define the* generalized Okounkov body $\Delta(\alpha)$ *with respect to the fixed flag by*

$$\Delta(\alpha) := \bigcap_{\epsilon > 0} \Delta(\alpha + \epsilon\omega),$$

where ω is any Kähler class.

It is easy to check that our definition does not depend on the choice of ω, and if α is big, by Proposition 1.4.2, the definition is consistent with Definition 1.4.3. Now we recall the definition of numerical dimension for any real (1,1)-class.

DEFINITION 1.4.7. *Let X be a compact Kähler manifold. For a class $\alpha \in H^{1,1}(X, \mathbb{R})$, the* numerical dimension $n(\alpha)$ *is defined to be $-\infty$ if α is not pseudo-effective, and*

$$n(\alpha) = \max\{p \in \mathbb{N}, \langle \alpha^p \rangle \neq 0\},$$

if α is pseudo-effective.

We recall that the right-hand side of the equation above involves the *positive intersection product* $\langle \alpha^p \rangle \in H^{p,p}_{\geqslant 0}(X, \mathbb{R})$ defined in [**BDPP13**]. When X is a Kähler surface, we simply have

$$n(\alpha) = \max\{p \in \mathbb{N}, Z(\alpha)^p \neq 0\}, \quad p \in \{0, 1, 2\}.$$

If $n(\alpha) = 2$, α is big and the situation is studied in the last subsection. Throughout this subsection, we assume $\alpha \in \partial\mathcal{E}$.

LEMMA 1.4.7. *Let $\{N_1, \ldots, N_r\}$ be an exceptional family of prime divisors, ω be any Kähler class. Then there exists unique positive numbers $b_1, \ldots, b_r$ such that $\omega + \sum_{i=1}^r b_i N_i$ is big and nef satisfying* $\mathrm{Null}(\omega + \sum_{i=1}^r b_i N_i) = \bigcup_{i=1}^r N_i$.

PROOF. If we set

$$\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = -S^{-1} \cdot \begin{pmatrix} \omega \cdot N_1 \\ \vdots \\ \omega \cdot N_r \end{pmatrix},$$

where S denotes the intersection matrix of $\{N_1, \ldots, N_r\}$, we have $(\omega + \sum_{i=1}^r b_i N_i) \cdot N_j = 0$ for $j = 1, \ldots, r$. By Lemma 1.3.2, we conclude that all b_i are positive and thus $\omega + \sum_{i=1}^r b_i N_i$ is big and nef. □

PROPOSITION 1.4.5. *Let α be any pseudo-effective class with $N(\alpha) = \sum_{i=1}^r a_i N_i$, ω be a Kähler class. Then for $\epsilon > 0$ small enough, we have the divisorial Zariski decomposition*

$$Z(\alpha + \epsilon\omega) = Z(\alpha) + \epsilon(\omega + \sum_{i=1}^r b_i N_i),$$

$$N(\alpha + \epsilon\omega) = \sum_{i=1}^r (a_i - \epsilon b_i) N_i,$$

where b_i is the positive number defined in Lemma 1.4.7.

PROOF. Since $Z(\alpha) + \epsilon(\omega + \sum_{i=1}^r b_i N_i)$ is nef and orthogonal to all N_i by Lemma 1.4.7, by Theorem 1.2.11, if ϵ satisfies that $a_i - \epsilon b_i > 0$ for all i, the divisorial decomposition in the proposition holds. □

If $n(\alpha) = 0$, we have $Z(\alpha) = 0$ and thus $\alpha = \sum_{i=1}^r a_i N_i$ is an exceptional effective $\mathbb{R}$-divisor. We fix a flag

$$X \supseteq C \supseteq \{x\},$$

where $C \neq N_i$ for all i. Then we have

THEOREM 1.4.8. *For any pseudo-effective class α whose numerical dimension $n(\alpha) = 0$, we have*

$$\Delta_{(C,x)}(\alpha) = 0 \times \nu_x(N(\alpha)|_C).$$

PROOF. We asumme $N(\alpha) = \sum_{i=1}^r a_i N_i$. Fix a Kähler class ω, by Proposition 1.4.5, for ϵ small enough we have

$$Z(\alpha + \epsilon\omega) = \epsilon(\omega + \sum_{i=1}^{r} b_i N_i), \tag{1.4.6}$$

$$N(\alpha + \epsilon\omega) = \sum_{i=1}^{r}(a_i - \epsilon b_i)N_i, \tag{1.4.7}$$

where b_i is the positive number defined in Lemma 1.4.7. Since $T \mapsto T - N(\alpha + \epsilon\omega)$ is a bijection between the positive currents in $\alpha + \epsilon\omega$ and those in $Z(\alpha + \epsilon\omega)$, we have

$$\Delta(\alpha + \epsilon\omega) = \epsilon\Delta(\omega + \sum_{i=1}^{r} b_i N_i) + \nu(\sum_{i=1}^{r}(a_i - \epsilon b_i)N_i),$$

where $\nu(\sum_{i=1}^r (a_i - \epsilon b_i)N_i) = \nu_{(C,x)}(\sum_{i=1}^r (a_i - \epsilon b_i)N_i)$ is the valuation-like function defined in Section 1.4.1. Thus the diameter of $\Delta(\alpha + \epsilon\omega)$ converges to 0 when ϵ tends to 0, and we conclude that $\Delta(\alpha)$ is a single point in $\mathbb{R}^2$. Since

$$\begin{aligned} \Delta(\alpha + \epsilon\omega)\bigcap 0 \times \mathbb{R} &= 0 \times \Delta_{X|C}(\alpha + \epsilon\omega) \\ &= 0 \times [\nu_x(N(\alpha + \epsilon\omega)|_C), \nu_x(N(\alpha + \epsilon\omega)|_C) + Z(\alpha + \epsilon\omega) \cdot C], \end{aligned}$$

by (1.4.6) and (1.4.7) we have

$$\Delta(\alpha)\bigcap 0 \times \mathbb{R} = 0 \times \nu_x(\sum_{i=1}^{r} a_i N_i|_C),$$

and we prove the first part of Theorem F.. □

If $n(\alpha) = 1$, $Z(\alpha)$ is nef but not big. If there exists one irreducible curve C such that $Z(\alpha) \cdot C > 0$, we fix the flag

$$X \supseteq C \supseteq \{x\},$$

then we have

THEOREM 1.4.9. *For any pseudo-effective class α whose numerical dimension $n(\alpha) = 1$, we have*

$$\Delta(\alpha) = 0 \times [\nu_x(N(\alpha)|_C), \nu_x(N(\alpha)|_C) + Z(\alpha) \cdot C].$$

PROOF. By the assumption $Z(\alpha) \cdot C > 0$ we know that $C \nsubseteq \mathrm{Supp}(N(\alpha))$. By Proposition 1.4.5, when ϵ small enough, the divisorial Zariski decomposition for $\alpha + \epsilon\omega$ is

$$Z(\alpha + \epsilon\omega) = Z(\alpha) + \epsilon(\omega + \sum_{i=1}^{r} b_i N_i), \tag{1.4.8}$$

$$N(\alpha + \epsilon\omega) = \sum_{i=1}^{r}(a_i - \epsilon b_i)N_i, \tag{1.4.9}$$

where b_i is the positive number defined in Lemma 1.4.7. Combine (1.4.8) and (1.4.9), we have

$$\begin{aligned} \Delta(\alpha)\bigcap 0 \times \mathbb{R} &= \bigcap_{\epsilon>0} \Delta(\alpha + \epsilon\omega)\bigcap 0 \times \mathbb{R} \\ &= \bigcap_{\epsilon>0} 0 \times [\nu_x(N(\alpha + \epsilon\omega)|_C), \nu_x(N(\alpha + \epsilon\omega)|_C) + Z(\alpha + \epsilon\omega) \cdot C] \\ &= 0 \times [\nu_x(\sum_{i=1}^{r} a_i N_i|_C), \nu_x(\sum_{i=1}^{r} a_i N_i|_C) + Z(\alpha) \cdot C]. \end{aligned}$$

Since we have

$$\mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha)) = \lim_{\epsilon \to 0} \mathrm{vol}_{\mathbb{R}^2}(\Delta(\alpha + \epsilon\omega)) = \lim_{\epsilon \to 0} Z(\alpha + \epsilon\omega)^2 = 0,$$

and $\Delta(\alpha)$ is a closed convex set, we conclude that there are no points of $\Delta(\alpha)$ which lie outside $0 \times \mathbb{R}$ as $\mathrm{vol}_{\mathbb{R}}(\Delta(\alpha) \bigcap 0 \times \mathbb{R}) = Z(\alpha) \cdot C > 0$. We finish the proof of Theorem F. $\square$

Part 2

On the Hyperbolicity-Related Problems

CHAPTER 2

Degeneracy of Entire Curves on Higher Dimensional Manifolds

2.1. Introduction

In [**McQ98**], McQuillan proved the following striking theorem, which partially solved the Green-Griffiths-Lang conjecture for complex surfaces with big cotangent bundle:

THEOREM 2.1.1. *(McQuillan) Let X be a surface of general type and $\mathcal{F}$ a holomorphic foliation on X. Then any entire curve $\mathcal{F}:\mathbb{C}\to X$ tangent to $\mathcal{F}$ can not be Zariski dense.*

The original proof of Theorem 2.1.1 is rather involved. Later on, there are several works (*e.g.* [**Bru99**, **PS14**]) appeared to explain and simplify McQuillan's proof. Let us recall the idea in proving Theorem 2.1.1 briefly. Assumes that there exists a Zariski dense entire curve $f:\mathbb{C}\to X$ which is tangent to $\mathcal{F}$. Then, one studies the intersection of the Ahlfors current $T[f]$, which is a representative of a $(n-1,n-1)$-cohomology class in X, with the tangent bundle and the normal bundle of the foliation $\mathcal{F}$ respectively. The above works proved that both of the intersections numbers are positive. However, since K_X is big, then $T[f]\cdot K_X>0$, and by the equality $K_X^{-1}=T_{\mathcal{F}}+N_{\mathcal{F}}$, a contradiction is obtained.

The goal of the chapter is to study the entire curves tangent to the foliation with certain singularities on higher dimensional manifolds, by pursuing the same philosophy in [**McQ98**]. Let us first recall the following *fundamental intersection formula* [**Bru97**, **McQ98**, **PS14**], which is the basis of our work:

THEOREM 2.1.2. *(Brunella-McQuillan-Păun-Sibony) Let $(X,\mathcal{F})$ be a Kähler 1-foliated pair. If $f:\mathbb{C}\to X$ is an entire curve tangent to $\mathcal{F}$ whose image is not contained in $\mathrm{Sing}(\mathcal{F})$, then*

$$\langle T[f],c_1(T_{\mathcal{F}})\rangle+T(f,\mathcal{J}_{\mathcal{F}})=\langle T[f_{[1]}],\mathcal{O}_{X_1}(-1)\rangle\geqslant 0,$$

where $\mathcal{J}_{\mathcal{F}}$ is a coherent ideal sheaf determined by the singularity of $\mathcal{F}$, and $T(f,\mathcal{J}_{\mathcal{F}})$ is a non-negative real number representing the "intersection" of $T[f]$ with $\mathcal{J}_{\mathcal{F}}$; this number will be defined later.

If X is a complex surface, as is proved by McQuillan [**McQ98**], after passing to some birational modle $(\tilde{X},\tilde{\mathcal{F}})\to(X,\mathcal{F})$, Theorem 2.1.2 can be improved to the extent that

$$T[\tilde{f}]\cdot T_{\tilde{\mathcal{F}}}\geqslant 0, \tag{2.1.1}$$

where $\tilde{f}$ is the lift of f to $\tilde{X}$. By pursuing his philosophy of "Diophantine approximation", we can generalize (2.1.1) to higher dimensional manifolds, under some assumptions on the foliation:

THEOREM G. *Let $(X,\mathcal{F})$ be a 1-foliated pair with simple singularities (see Definition 2.3.2). For any an entire curve whose Zariski closure $\overline{f(\mathbb{C})}^{\text{Zariski}}$ is of dimension at least 2, which is also tangent to $\mathcal{F}$, we always have*

$$T[f]\cdot T_{\mathcal{F}}\geqslant 0.$$

If we further assume that $K_{\mathcal{F}}$ is a big line bundle, then the image of f is contained in $\mathbf{B}_+(K_{\mathcal{F}})$. In particular, if $K_{\mathcal{F}}$ is ample, then there exists no nonconstant transcendental entire curve $f:\mathbb{C}\to X$ tangent to $\mathcal{F}$.

As an application of Theorem G, we can give a new proof of the following elegant theorem by Brunella [**Bru06**, Corollary]

THEOREM 2.1.3. *(Brunella) For a generic foliation by curves $\mathcal{F}$ of degree $d\geqslant 2$ on the complex projective space $\mathbb{P}^n$, that is, $\mathcal{F}$ is generated by a generic holomorphic section (a rational vector field)*

$$s\in H^0\big(\mathbb{P}^n,T_{\mathbb{P}^n}\otimes\mathcal{O}(d-1)\big),$$

all the leaves of $\mathcal{F}$ are hyperbolic. More precisely, there exists no nonconstant $f:\mathbb{C}\to\mathbb{P}^n$ tangent to $\mathcal{F}$ (and possibly passing through $\mathrm{Sing}(\mathcal{F})$).

Since for any one-dimensional foliation with absolutely isolated singularities (see Definition 2.3.1), by the reduction theorems [**CCS97**, **Tom97**] one can take a finite sequence of blowing-up's to make the singularities *simple*. We thus have the following result:

THEOREM H. *Let $\mathcal{F}$ be a foliation by curves on the n-dimensional complex manifold X, such that the singular set* $\mathrm{Sing}(\mathcal{F})$ *of the foliation $\mathcal{F}$ is a set of absolutely isolated singularities. If $f : \mathbb{C} \to X$ is an entire curve whose Zariski closure $\overline{f(\mathbb{C})}^{\mathrm{Zariski}}$ is of dimension at least 2, which is also tangent to $\mathcal{F}$, then one can blow-up X a finite number of times to get a new birational model $(\widetilde{X}, \widetilde{\mathcal{F}})$ such that*

$$T[\widetilde{f}] \cdot T_{\widetilde{\mathcal{F}}} \geqslant 0.$$

On the other hand, Theorem 2.1.2 leads us to the fact that the error term $T(f, \mathcal{J}_{\mathcal{F}})$ is controllable, if the singularities of $\mathcal{F}$ are not too "bad" (they are called *weakly reduced singularities* in Section 2.3.2). The theorem is as follows:

THEOREM I. *Let X be a projective manifold of dimension n endowed with a 1-dimensional foliation $\mathcal{F}$ with weakly reduced singularities. If f is a Zariski dense entire curve tangent to $\mathcal{F}$, satisfying $T[f] \cdot K_X > 0$ (e.g. K_X is big), then we have*

$$T[\hat{f}] \cdot \det N_{\hat{\mathcal{F}}} < 0$$

for some birational pair $(\hat{X}, \hat{\mathcal{F}})$.

REMARK 2.1.1. Our definition of "weakly reduced singularities" is actually weaker than the usual concept of reduced singularities, which always requires a lot of checking (e.g. a classification of singularities). We only need to focus on the multiplier ideal sheaf of $\mathcal{J}_{\mathcal{F}}$, instead of trying to understand the exact behavior of singularities.

It is notable that the following strong result due to Brunella [**Bru99**, Theorem 2] implies a conclusive contradiction in combination with Theorem I, in the case of complex surfaces.

THEOREM 2.1.4. *(Brunella) Let X be a complex surface endowed with a foliation $\mathcal{F}$ (no assumption is made for singularities of $\mathcal{F}$ here). If $f : \mathbb{C} \to X$ is a Zariski dense entire curve tangent to $\mathcal{F}$, then we have*

$$T[f] \cdot N_{\mathcal{F}} \geqslant 0.$$

Therefore, we get another proof of McQuillan's Theorem 2.1.1 without using the refined formula (2.1.1) immediately. This leads us to observe that if one can resolve any singularities of the 1-dimensional foliation $\mathcal{F}$ into weakly reduced ones, and generalize the previous Brunella Theorem to higher dimensional manifolds, one could infer the Green-Griffiths conjecture for surfaces of general type.

THEOREM J. *Assume that Theorem 2.1.4 holds for a directed variety $(X, \mathcal{F})$ where X is a base of arbitrary dimension and $\mathcal{F}$ has rank 1, and that one can resolve the singularities of $\mathcal{F}$ into weakly reduced ones. Then every entire curve drawn in a projective surface of general type must be algebraically degenerate.*

2.2. TECHNICAL PRELIMINARIES

2.2.1. NOTATIONS AND DEFINITIONS. In this subsection, we briefly recall the value distribution theory for coherent ideal sheaves [**NW14**, Setion 2.4], and some basic knowledge for foliation by curves [**Bru11**, Chapter 1 and 2].

For any coherent ideal sheaf $\mathcal{J}$ on a complex manifold X, one can construct a global quasi-plurisubarmonic function $\varphi_{\mathcal{J}}$ on X such that

$$\varphi_{\mathcal{J}} = \log(\sum_i |g_i|^2) + \mathcal{O}(1)$$

where (g_i) are local holomorphic functions that generate the ideal $\mathcal{J}$. We call $\varphi_{\mathcal{J}}$ the *characteristic function associated to the coherent sheaf* $\mathcal{J}$, which is well-defined up to some bounded function on X. For any entire curve $f : \mathbb{C} \to X$ whose image is not contained in the subscheme $Z(\mathcal{J})$, one writes

$$\varphi_{\mathcal{J}} \circ f(\tau)|_{\Delta(r)} = \sum_{|\tau_j|<r} \nu_j \log|\tau - \tau_j|^2 + \mathcal{O}(1),$$

where $\Delta(r)$ denotes the disk of radius r in $\mathbb{C}$, and ν_j is called the *multiplicity of f along $\mathcal{J}$*.

In a related way, we define the *proximity function* of f with respect to $\mathcal{J}$ by

$$m_{f,\mathcal{J}}(r) := -\frac{1}{2\pi} \int_0^{2\pi} \varphi_{\mathcal{J}} \circ f(re^{i\theta}) d\theta,$$

and the *counting function* of f with respect to $\mathcal{J}$ by

$$N_{f,\mathcal{J}}(r) := \sum_{|\tau_j|<r} \nu_j \log \frac{r}{|\tau_j|}.$$

Let us take a log resolution $p : \hat{X} \to X$ of $\mathcal{J}$ such that $p^{-1}(\mathcal{J}) = \mathcal{O}_{\hat{X}}(-D)$, let $\hat{f}$ denote to be the lift of f to $\hat{X}$ so that $p \circ \hat{f} = f$, and let Θ_D be the curvature form of D with respect to some smooth hermitian metric on $\mathcal{O}(D)$.

Now we recall the following formula, which will be very useful in what follows.

THEOREM 2.2.1. *(Jensen formula) For $r \geqslant 1$ we have*

$$\int_1^r \frac{dt}{t} \int_{\Delta(t)} dd^c \varphi = \frac{1}{2\pi} \int_0^{2\pi} \varphi(re^{i\theta}) d\theta - \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta}) d\theta, \tag{2.2.1}$$

in particular if φ is a quasi-plurisubharmonic function, then for r large enough we have

$$\int_1^r \frac{dt}{t} \int_{\Delta(t)} dd^c \varphi = \frac{1}{2\pi} \int_0^{2\pi} \varphi(re^{i\theta}) d\theta + \mathcal{O}(1).$$

Then the following *First Main Theorem* due to Nevanlinna is an immediate consequence of (2.2.1) [**PS14**, Theorem 3.6]

THEOREM 2.2.2. *As $r \to \infty$, one has*

$$T_{\hat{f}, \Theta_D}(r) = N_{f,\mathcal{J}}(r) + m_{f,\mathcal{J}}(r) + \mathcal{O}(1).$$

Let $\mathcal{F}$ be a 1-dimension foliation on a complex manifold X of dimension n. Then we can take an open covering $\{U_\alpha\}_{\alpha \in I}$ such that on each U_α there exists $v_\alpha \in H^0(U_\alpha, T_X|_{U_\alpha})$ which generates $\mathcal{F}$, and such that the v_α coincide up to multiplication by nowhere vanishing holomorphic functions $\{g_{\alpha\beta}\}$:

$$v_\alpha = g_{\alpha\beta} v_\beta$$

if $U_\alpha \cap U_\beta \neq \varnothing$. The functions $\{g_{\alpha\beta}\}$ define a Cěch cohomology class $H^1(X, \mathcal{O}_X^*)$, which is a line bundle over X. It is called the cotangent bundle of $\mathcal{F}$, and denoted here by $T_{\mathcal{F}}^*$ (or $K_{\mathcal{F}}$).

If we take any smooth hermitian metric ω on X, then ω induces a natural singular metric h_s on $T_{\mathcal{F}}$. Indeed, on each U_α, the local weight φ_α of $h_s = e^{-\varphi_\alpha}$ is given by

$$\varphi_\alpha = -\log |v_\alpha|_\omega^2 = -\log \sum a_\alpha^i \overline{a_\alpha^j} \omega_{\alpha, i\bar{j}}, \tag{2.2.2}$$

where $v_\alpha = \sum_{i=1}^n a_\alpha^i(z_\alpha) \frac{\partial}{\partial z_{\alpha,i}}$ with respect to the coordinate system $z_\alpha = (z_{\alpha,1}, \ldots, z_{\alpha,n})$ on U_α, and $\omega = \sqrt{-1} \sum \omega_{\alpha, i\bar{j}} dz_{\alpha,i} \wedge d\bar{z}_{\alpha,j}$.

We are going to define a coherent ideal sheaf $\mathcal{J}_{\mathcal{F}}$ on X reflecting the behavior of the singularies of $\mathcal{F}$. On each U_α the generators of $\mathcal{J}_{\mathcal{F}}$ are precisely the coefficients (a_α^i) of the vector v_α defining $\mathcal{F}$, and it is easy to see that this does not depend on the choice of the local coordinate charts (U_α, z_α).

If we fix a smooth metric h on $T_{\mathcal{F}}$, then there exists a globally defined function φ_s such that

$$h = h_s e^{-\varphi_s}$$

We know that

$$\varphi_s = \log |v_\alpha|_\omega^2 \tag{2.2.3}$$

modulo a bounded function, and by the very definition, φ_s is the characteristic function associated to the coherent sheaf $\mathcal{J}_{\mathcal{F}}$.

All the constructions explained above can be generalized to log pairs naturally. Let us recall the following definition.

DEFINITION 2.2.1. Let X be a smooth Kähler manifold, D a simple normal crossing divisor and $\mathcal{F}$ a foliation by curves defined on X. We say that $\mathcal{F}$ is defined on the log pair (X, D) if each component of D is invariant by $\mathcal{F}$. Such $(X, \mathcal{F}, D)$ is called a *Kähler 1-foliated triple.*

The logarithmic tangent bundle $T_X\langle -\log D \rangle$ with respect to the pair (X, D) is the locally free sheaf generated by the vector fields $(z_i \frac{\partial}{\partial z_i})_{i=1,\ldots,k}$ and $(\frac{\partial}{\partial z_i})_{i=k+1,\ldots,n}$ with respect to some local coordinates $(z_1, \ldots, z_n)$ such that D is locally defined by $\{z | z_1 z_2 \cdots z_k = 0\}$. Dually, the logarithmic cotangent bundle $\Omega_X\langle \log D \rangle$ is locally free $\mathcal{O}_X$-module generated by $(\frac{dz_i}{z_i})_{i=1,\ldots,k}$ and $(dz_i)_{i=k+1,\ldots,n}$. Hence, any smooth hermitian metric $\omega_{X,D}$ on $T_X\langle -\log D \rangle$ can be locally written as

$$\omega_{X,D} = \sqrt{-1} \sum_{i,j=1}^k \omega_{i\bar{j}} \frac{dz_i \wedge d\bar{z}_j}{z_i \bar{z}_j} + 2\mathrm{Re}\sqrt{-1} \sum_{i > k \geqslant j} \omega_{i\bar{j}} \frac{dz_i \wedge d\bar{z}_j}{\bar{z}_j} + \sqrt{-1} \sum_{i,j \geqslant k+1} \omega_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $(\omega_{i\bar{j}})$ is a smooth positive definite hermitian matrix. If $(X, \mathcal{F}, D)$ is a Kähler 1-foliated triple, such that on some coordinate charts (U_α, z_α),

$$v_\alpha = \sum_{j=1}^{k} z_{\alpha,j} a_\alpha^j \frac{\partial}{\partial z_{\alpha,j}} + \sum_{i=k+1}^{n} a_\alpha^i \frac{\partial}{\partial z_{\alpha,i}}$$

is the generator of $\mathcal{F}$ on U_α. Then $\omega_{X,D}$ induces a singular hermitian metric $h_{s,D}$ on $T_\mathcal{F}$ whose local weight is given by

$$\varphi_{\alpha,D} = -\log |v_\alpha|^2_{\omega_{X,D}} = -\log \sum_{i,j} a_\alpha^i \overline{a_\alpha^j} \omega_{i\bar{j}}.$$

We denote by $\mathcal{J}_{\mathcal{F},D}$ the coherent ideal sheaf on X generated by the functions (a_α^j). In general we have

$$\mathcal{J}_\mathcal{F} \subset \mathcal{J}_{\mathcal{F},D},$$

and the inclusion may be strict. If we find a smooth metric $h = h_{s,D} e^{-\varphi_{s,D}}$ on $T_\mathcal{F}$, then it is easy to check that $\varphi_{s,D}$ is the characteristic function associated with $\mathcal{J}_{\mathcal{F},D}$.

We also denote by $\bar{X}_1 := P(T_X\langle -\log D\rangle)$ the projectivized bundle, and $\omega_{X,D}$ induces a natural smooth metric h_1 on the tautological bundle $\mathcal{O}_{\bar{X}_1}(-1)$.

2.2.2. Basic results about Ahlfors currents. In this subsection, we will briefly recall the definitions and properties of Ahlfors currents, which were first introduced and studied by McQuillan [**McQ98**].

Let X be a Kähler manifold with ω the Kähler form, and let f be an entire curve on X. Then we can associate to f a closed positive current of $(n-1, n-1)$ type as follows. First for any $r > 0$, one defines a positive $(n-1, n-1)$-current $T_r[f]$ by

$$\langle T_r[f], \eta\rangle := \frac{T_{f,\eta}(r)}{T_{f,\omega}(r)}, \tag{2.2.4}$$

where η is any smooth 2-form on X, and $T_{f,\eta}(r) := \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*\eta$ is the *Nevanlinna's order function*. From [**McQ98**, **Bru99**], one can find a suitable sequence of (r_k) that tends to infinity, such that the weak limit of $T_{r_k}[f]$ is a closed positive current. It is denoted by $T[f]$ and called the *Ahlfors current of f*. For any line bundle L, we always use the notation $T[f] \cdot L$ to denote the cohomology intersection $\{T[f]\} \cdot c_1(L)$.

It is noticeable that $T[f]$ depends on not only the choice of ω, but also the sequence (r_k). On the other hand, as is proved in [**McQ98**, **Pau03**], it is not indispensable to assume that ω is Kähler in (2.2.4). It suffice to assume that ω is a closed semi-positive form satisfying

$$\lim_{r_k\to\infty} \frac{T_{f,\omega}(r_k)}{T_{f,\widetilde{\omega}}(r_k)} > C > 0,$$

with respect to some Kähler form $\widetilde{\omega}$.

The following "strongly nef" property of Ahlfors current is a direct consequence of the First Main Theorem:

Proposition 2.2.1. Let L be a big line bundle on a Kähler manifold X. If $f : \mathbb{C} \to X$ is an entire curve on X such that its image is not contained in the augmented base locus $\mathbf{B}_+(L)$ of L [**Laz04**, Definition 10.3.2], then $\langle T[f], c_1(L)\rangle > 0$.

Proof. Since the image of f is not contained in $\mathbf{B}_+(L)$, by the definition of the augmented base locus, one can find an effective divisor E whose support does not contain $f(\mathbb{C})$, such that

$$L \equiv A + E,$$

where A is an $\mathbb{Q}$-ample divisor, and "$\equiv$" means *numerically equivalent*. We take a smooth hermitian metric h_E on E such that the proximity function of f with respect to E is also non-negative. Since the counting function of f with respect to E is always non-negative, then by the First Main Theorem

$$T[f] \cdot E = \langle T[f], \Theta_{h_E}(E)\rangle \geqslant 0.$$

By the ampleness of A, we have

$$T[f] \cdot A > 0,$$

and thus

$$T[f] \cdot L = T[f] \cdot A + T[f] \cdot E > 0.$$

□

Definition 2.2.2. An entire curve $f : \mathbb{C} \to X$ is said to be *rational* iff f admits a factorization in the form $f = g \circ R$, where $R : \mathbb{C} \to \mathbb{P}^1$ is a rational function and $g : \mathbb{P}^1 \to X$ is a rational curve. f is said to be *transcendental* if the Zariski closure $\overline{f(\mathbb{C})}^{\text{Zariski}}$ of f is of dimension at least 2.

We have the following criterion for an entire curve to be rational [**Dem97**, Corollaire 1.7]:

PROPOSITION 2.2.2. Any entire curve $f : \mathbb{C} \to X$ is rational if and only if $T_{f,\omega}(r) = \mathcal{O}(\log r)$. In particular, if f is transcendental, then

$$\lim_{r\to\infty} \frac{T_{f,\omega}(r)}{\log r} = +\infty.$$

Let us recall the following logarithmic derivative lemma [**Dem97**, Lemme 3.7], which will be very useful in our arguments.

LEMMA 2.2.1. (logarithmic derivative lemma) Let f be a meromorphic function on $\mathbb{C}$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |\frac{f'(re^{i\theta})}{f(re^{i\theta})}|d\theta \leqslant \mathcal{O}(\log T_{f,\omega}(r) + \log r). \tag{2.2.5}$$

Let (X, V) be a smooth *directed variety*, that is, a complex manifold X equipped with a subbundle $V \subset T_X$. Denote by X_1 the projectivized bundle $P(V)$, and $\pi : X_1 \to X$ the natural projection map. Fix a hermitian metric ω on X. It induces a smooth metric h on the tautological line bundle $\mathcal{O}_{X_1}(1)$. Then for any $0 < \delta \ll 1$, $\omega_1 := \pi^*\omega + \delta\Theta_h(\mathcal{O}_{X_1}(1))$ is a hermitian metric on X_1. For any entire curve $f : \mathbb{C} \to X$ tangent to V, it is easy to see that there is a canonical lift of f to X_1, defined by

$$f_{[1]}(t) := (f(t), [f'(t)])$$

such that $\pi(f_{[1]}) = f$, and we have the following lemma:

LEMMA 2.2.2. Assume that (X, ω) is a Kähler manifold and f is transcendental. Then

$$\liminf_{r\to+\infty} \frac{T_{f_{[1]},\pi^*\omega}(r)}{T_{f_{[1]},\omega_1}(r)} \geqslant 1.$$

In particular, we can define the Ahlfors current $T[f_{[1]}]$ with respect to the semi-positive form $\pi^*\omega$ in such a way that $\pi_*T[f_{[1]}] = T[f]$.

PROOF. Since $f'(\tau)$ can be seen as a section of the bundle $f_{[1]}^*(\mathcal{O}_{X_1}(-1))$, by the Lelong-Poincaré formula we have

$$dd^c \log |f'(\tau)|^2_\omega = \sum_{|\tau_j|<r} \mu_j\delta_{\tau_j} - f_{[1]}^*\Theta_{h^*}\big(\mathcal{O}_{X_1}(-1)\big) \tag{2.2.6}$$

on $\Delta(r)$, where μ_j is the vanishing order of $f'(\tau)$ at τ_j. Thus we get

$$\int_1^r \frac{dt}{t}\int_{\Delta(t)} f_{[1]}^*\Theta_{h^*}(\mathcal{O}_{X_1}(-1)) = \sum_{|t_j|<r} \mu_j \log\frac{r}{|t_j|} - \int_1^r \frac{dt}{t}\int_{\Delta(t)} dd^c \log |f'(\tau)|^2_\omega$$

$$= \sum_{|t_j|<r} \mu_j \log\frac{r}{|t_j|} - \frac{1}{2\pi}\int_0^{2\pi} \log |f'(re^{i\theta})|^2_\omega d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log |f'(e^{i\theta})|^2_\omega d\theta, \tag{2.2.7}$$

where the last equality is a consequence of the Jensen formula (2.2.1). Let $(\varphi_\alpha)_{\alpha\in J}$ be a partition of unity subcoordinate to the covering $(U_\alpha)_{\alpha\in J}$ of X. We can take a finite family of logarithms of global meromorphic fuctions $(\log u_{\alpha j})_{\alpha\in J,1\leqslant j\leqslant n}$ as local coordinates for U_α, and by the logarithmic derivative lemma (2.2.5), we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f'(re^{i\theta})|^2_\omega d\theta &= \sum_{\alpha\in J}\frac{1}{2\pi}\int_0^{2\pi} \varphi_\alpha \log^+ |f'(re^{i\theta})|^2_\omega d\theta \\
&\leqslant \sum_{\alpha\in J}\sum_{j=1}^n C\int_0^{2\pi} \log^+ |\frac{u'_{\alpha j}(re^{i\theta})}{u_{\alpha j}(re^{i\theta})}|^2 d\theta \\
&\leqslant \mathcal{O}(\log^+ T_{f,\omega}(r) + \log r),
\end{aligned}$$

where C is some constant. Since f is transcendental, by Proposition 2.2.2 we have

$$\lim_{r\to+\infty} \frac{T_{f,\omega}(r)}{\log r} = +\infty,$$

and thus

$$\liminf_{r\to+\infty} -\frac{1}{T_{f,\omega}(r)}\int_0^{2\pi} \log |f'(re^{i\theta})|^2_\omega d\theta \geqslant 0.$$

By (2.2.7) we have

$$\liminf_{r\to+\infty} \frac{1}{T_{f,\omega}(r)}\int_1^r \frac{dt}{t}\int_{\Delta(t)} f_{[1]}^*\Theta_{h^*}\big(\mathcal{O}_{X_1}(-1)\big) \geqslant \liminf_{r\to+\infty} -\frac{1}{T_{f,\omega}(r)}\int_0^{2\pi} \log |f'(re^{i\theta})|^2_\omega d\theta \geqslant 0. \tag{2.2.8}$$

Since $\omega_1 = \pi^*\omega + \delta\Theta_h(\mathcal{O}_{X_1}(1))$ is a hermitian metric on X_1, we have

$$T_{f_{[1]},\omega_1}(r) = \int_1^r \frac{dt}{t}\int_{\Delta(t)} f_{[1]}^*\omega_1 = T_{f_{[1]},\pi^*\omega}(r) + \delta\int_1^r \frac{dt}{t}\int_{\Delta(t)} f_{[1]}^*\Theta_h\big(\mathcal{O}_{X_1}(1)\big).$$

By $T_{f,\omega}(r) = T_{f_{[1]},\pi^*\omega}(r)$ we obtain

$$\liminf_{r\to+\infty} \frac{T_{f_{[1]},\pi^*\omega}(r)}{T_{f_{[1]},\omega_1}(r)} \geqslant 1.$$

Hence we can replace ω_1 by $\pi^*\omega$ in the definition of Ahlfors current $T[f_{[1]}]$, and the equality $T_{f_{[1]},\pi^*\eta}(r) = T_{f,\eta}(r)$ for any smooth $(1,1)$-form η on X then yields

$$\pi_* T[f_{[1]}] = T[f].$$

□

Similarly we have the following lemma in [**Bru99**, p. 198]:

LEMMA 2.2.3. Let $p : \tilde{X} \to X$ be a bimeromorphic morphism between Kähler manifolds $\tilde{X}$ and X. Fix a Kähler metric ω on X. If $f : \mathbb{C} \to X$ is an entire curve whose image is not contained in the exceptional locus, then for the lift $\tilde{f}$ on $\tilde{X}$, and we can define the Ahlfors current $T[\tilde{f}]$ with respect to the semi-positive Kähler form $p^*\omega$ such that

$$p_* T[\tilde{f}] = T[f].$$

REMARK 2.2.1. For any coherent ideal sheaf $\mathcal{J}$ whose zero scheme does not contain the image of $f : \mathbb{C} \to X$, one can take a log resolution $p : \hat{X} \to X$ of $\mathcal{J}$ with $p^*\mathcal{J} = \mathcal{O}_{\hat{X}}(-D)$, and by Lemma 2.2.3 one can find a suitable sequence (r_k) such that

$$T[\hat{f}]\cdot D = \lim_{r_k\to\infty} \frac{T_{\hat{f},\Theta(D)}(r_k)}{T_{\hat{f},p^*\omega}(r_k)} = \lim_{r_k\to\infty} \frac{T_{\hat{f},\Theta(D)}(r_k)}{T_{f,\omega}(r_k)},$$

where $\Theta(D)$ is a curvature form of D with respect to some smooth metric. By Theorem 2.2.2, we know that $T[\hat{f}]\cdot D$ does not depend on the log resolution of $\mathcal{J}$, which is denoted by $T(f,\mathcal{J})$.

Finally let us recall the following *Tautological Inequality* by McQuillan [**McQ98**, Theorem 0.2.5], which can be seen as a geometric interpretation of the logarithmic derivative lemma:

THEOREM 2.2.3. *(tautological inequality) Let $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ be a transcendental entire curve in X, where (X, V) is a smooth directed variety. Then we have*

$$T[f_{[1]}]\cdot \mathcal{O}_{P(V)}(-1) \geqslant 0.$$

PROOF. By (2.2.8) we have

$$\liminf_{r\to+\infty} \frac{1}{T_{f,\omega}(r)}\int_1^r \frac{dt}{t}\int_{\Delta(t)} f_{[1]}^*\Theta_{h^*}\big(\mathcal{O}_{P(V)}(-1)\big) \geqslant 0.$$

From Lemma 2.2.2 we can take $\pi^*\omega$ as the semi-positive form used in the definition of the Ahlfors current of $T[f_{[1]}]$, where $\pi : P(V) \to X$ is the natural projection. The equality $T_{f,\omega}(r) = T_{f_{[1]},\pi^*\omega}(r)$ then implies

$$T[f_{[1]}]\cdot \mathcal{O}_{P(V)}(-1) = \lim_{r_k\to+\infty} \frac{1}{T_{f_{[1]},\pi^*\omega}(r)}\int_1^r \frac{dt}{t}\int_{\Delta(t)} f_{[1]}^*\Theta_{h^*}\big(\mathcal{O}_{P(V)}(-1)\big) \geqslant 0.$$

□

2.2.3. INTERSECTION WITH THE TANGENT BUNDLE.

THEOREM 2.2.4. *Let X be a Kähler manifold equipped with a 1-dimensional foliation $\mathcal{F}$, and let $f : \mathbb{C} \to X$ be a transcendental entire curve tangent to $\mathcal{F}$ such that its image is not contained in* Sing$(\mathcal{F})$. *Then we have*

$$T[f]\cdot T_{\mathcal{F}} + T(f,\mathcal{J}_{\mathcal{F}}) = T[f_{[1]}]\cdot \mathcal{O}_{X_1}(-1) \geqslant 0. \tag{2.2.9}$$

PROOF. Let (U_α) be a partition of unit of X such that $\mathcal{F}$ is generated by some vector fields $v_\alpha \in \Gamma(U_\alpha, T_X|_{U_\alpha})$ on U_α. Denote by $\Omega_\alpha = f^{-1}(U_\alpha)$. Since $f(\mathbb{C})$ is not contained in Sing$(\mathcal{F})$, then on each Ω_α,

$$f'(\tau) = \lambda_\alpha(\tau) v_\alpha|_{f(\tau)} \tag{2.2.10}$$

for some *meromorphic* function $\lambda_\alpha(\tau)$. We denote by η_j the multiplicities of $\lambda_\alpha(\tau)$ at τ_j, and they may be negative only if $f(\tau_j) \in \mathrm{Sing}(\mathcal{F})$. $\lambda_\alpha(\tau)$ can not have essential singularity since $\eta_j + \nu_j \geqslant 0$, where ν_j is the multiplicity of f along $\mathcal{J}_\mathcal{F}$, that is,

$$dd^c \log |v_\alpha|^2_\omega \circ f(\tau)|_{\Delta(r)} = \sum_{|\tau_j|<r} \nu_j \log|\tau - \tau_j|^2 + \mathcal{O}(1).$$

Since $v_\alpha = g_{\alpha\beta} v_\beta$ for some nowhere vanishing function $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, then if $\tau_j \in \Omega_\alpha \cap \Omega_\beta$, λ_α and λ_β have the same multiplicity at τ_j, and thus η_j does not depend on the partition on unity. By the Lelong-Poincaré formula and (2.2.10) we have

$$\begin{aligned} dd^c \log|f'(\tau)|^2_\omega &= \sum_{|\tau_j|<r} \eta_j \delta_{\tau_j} + dd^c \log|v_\alpha|^2_\omega \circ f(\tau) \\ &= \sum_{|\tau_j|<r} \eta_j \delta_{\tau_j} - f^*\Theta_{h_s}, \end{aligned} \tag{2.2.11}$$

where h_s is the singular metric on $T_\mathcal{F}$ defined in (2.2.2) whose local weight is $\varphi_\alpha = -\log|v_\alpha|^2_\omega$. If we fix a smooth metric h on $T_\mathcal{F}$, then there exists a globally defined function φ_s such that

$$h = h_s e^{-\varphi_s},$$

and φ_s is the characteristic function associated to the coherent sheaf $\mathcal{J}_\mathcal{F}$. By (2.2.11), we have

$$dd^c \log|f'(\tau)|^2_\omega - f^* dd^c \varphi_s = \sum_{|\tau_j|<r} \eta_j \delta_{\tau_j} - f^*\Theta_h(T_\mathcal{F}). \tag{2.2.12}$$

on $\Delta(r)$. Combining (2.2.6) with (2.2.12) we have

$$\sum_{|\tau_j|<r} \mu_j \delta_{\tau_j} - f^*_{[1]}\Theta_{h_1}(\mathcal{O}_{X_1}(-1)) - f^* dd^c \varphi_s = \sum_{|\tau_j|<r} \eta_j \delta_{\tau_j} - f^*\Theta_h(T_\mathcal{F}),$$

where h_1 is the smooth metric on $\mathcal{O}_{X_1}(-1)$ induced by ω, and μ_j is the multiplicity of $f'(\tau)$ at τ_j. Hence, $\mu_j - \eta_j = \nu_j$, and we have

$$f^*\Theta_h(T_\mathcal{F}) = -\sum_{|\tau_j|<r} \nu_j \delta_{\tau_j} + f^*_{[1]}\Theta_{h_1}(\mathcal{O}_{X_1}(-1)) + f^* dd^c \varphi_s$$

on each $\Delta(r)$. By the definition (2.2.4),

$$\begin{aligned} \langle T_r[f], \Theta_h(T_\mathcal{F})\rangle &:= \frac{1}{T_{f,\omega}(r)} \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*\Theta_h(T_\mathcal{F}) \\ &= \frac{1}{T_{f,\omega}(r)} \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*_{[1]}\Theta_{h_1}(\mathcal{O}_{X_1}(-1)) - \frac{1}{T_{f,\omega}(r)} \sum_{|\tau_j|<r} \nu_j \log \frac{r}{|\tau_j|} \\ &+ \frac{1}{T_{f,\omega}(r)} \frac{1}{2\pi} \int_0^{2\pi} \varphi_s \circ f(re^{i\theta}) d\theta - \frac{1}{T_{f,\omega}(r)} \frac{1}{2\pi} \int_0^{2\pi} \varphi_s \circ f(e^{i\theta}) d\theta \\ &= \frac{1}{T_{f,\omega}(r)} \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*_{[1]}\Theta_{h_1}(\mathcal{O}_{X_1}(-1)) \\ &- \frac{1}{T_{f,\omega}(r)} N_{f,\mathcal{J}_\mathcal{F}}(r) - \frac{1}{T_{f,\omega}(r)} m_{f,\mathcal{J}_\mathcal{F}}(r), \end{aligned} \tag{2.2.13}$$

where $N_{f,\mathcal{J}_\mathcal{F}}(r)$ and $m_{f,\mathcal{J}_\mathcal{F}}(r)$ are the counting and proximity function of f with respect to $\mathcal{J}_\mathcal{F}$, and the second equality above is a consequence of the Jensen formula. Since $T[f]$ is the weak limit of the positive current $T_{r_k}[f]$ for some sequence $r_k \to +\infty$, we have

$$T[f] \cdot T_\mathcal{F} = \lim_{r_k \to +\infty} \langle T_{r_k}[f], \Theta_h(T_\mathcal{F})\rangle.$$

From Theorem 2.2.2, Remark 2.2.1 and Lemma 2.2.2 we conclude that

$$\langle T[f], \Theta_h(T_\mathcal{F})\rangle + T(f, \mathcal{J}_\mathcal{F}) = T[f_{[1]}] \cdot \mathcal{O}_{X_1}(-1) \geqslant 0.$$

□

In fact, $\mathcal{K}_\mathcal{F} := K_\mathcal{F} \otimes \mathcal{J}_\mathcal{F}$ is the *canonical sheaf* of the foliation $\mathcal{F}$ defined by J.-P. Demailly in studying the Green-Griffiths-Lang conjecture [**Dem15b**, Definition 1.4], by using *admissible metric*. We recall his definition for foliations of *general type*:

DEFINITION 2.2.3. (Demailly) We say that the rank 1 sheaf $\mathcal{K}_{\mathcal{F}}$ is "big" if if the invertible sheaf $\mu^*\mathcal{K}_{\mathcal{F}}$ is big in the usual sense for any log resolution $\mu : \hat{X} \to X$ of $\mathcal{K}_{\mathcal{F}}$. Finally, we say that $(X, \mathcal{F})$ is of general type if there exists some birational model $(X', \mathcal{F}')$ of $(X, \mathcal{F})$ such that $\mathcal{K}_{\mathcal{F}'}$ is big. The *base locus* $\mathrm{Bs}(\mathcal{F})$ of $\mathcal{F}$ is defined to be

$$\mathrm{Bs}(\mathcal{F}) := \bigcap_{\alpha} \nu_\alpha \circ \mu_\alpha\big(\mathbf{B}_+(\mu_\alpha^*\mathcal{K}_{\mathcal{F}_\alpha})\big),$$

where $\nu_\alpha : (X_\alpha, \mathcal{F}_\alpha) \to (X, \mathcal{F})$ varies among all the birational morphisms of X, and $\mu_\alpha : (\tilde{X}_\alpha, \tilde{\mathcal{F}}_\alpha) \to (X_\alpha, \mathcal{F}_\alpha)$ is some log resolution of $\mathcal{K}_{\mathcal{F}_\alpha}$.

The above Theorem 2.2.4 then gives another proof of the *Generalized Green-Griffiths conjecture* for rank 1 foliations formulated in [**Dem12**]. Moreover we can specify more precisely the subvariety containing the images of all transcendental curves tangent to the foliation. The theorem is as follows:

COROLLARY 2.2.1. Let $(X, \mathcal{F})$ be a projective 1-foliated manifold of general type. If $f : \mathbb{C} \to X$ is a transcendental entire curve tangent to $\mathcal{F}$, its image must be contained in $\mathrm{Sing}(\mathcal{F}) \cup \mathrm{Bs}(\mathcal{F})$. In particular, any entire curve tangent to $\mathcal{F}$ must be algebraically degenerate.

PROOF. Assume that the image of f is not contained in $\mathrm{Sing}(\mathcal{F}) \cup \mathrm{Bs}(\mathcal{F})$, and we proceed by contradiction. By Definition 2.2.3, there exists a birational morphism $\nu_\alpha : (X_\alpha, \mathcal{F}_\alpha) \to (X, \mathcal{F})$ such that the invertible sheaf $\mu_\alpha^*\mathcal{K}_{\mathcal{F}_\alpha}$ is big in the usual sense, for some log resolution $\mu_\alpha : \tilde{X}_\alpha \to X_\alpha$ of $\mathcal{K}_{\mathcal{F}_\alpha}$, and such that the image of f_α is not contained in $\mathbf{B}_+(\mu_\alpha^*\mathcal{K}_{\mathcal{F}_\alpha})$, where $\tilde{f}_\alpha$ is the lift of f to $\tilde{X}_\alpha$. We denote by f_α the lift of f to X_α. By Proposition 2.2.1 we have

$$\langle T[f_\alpha], c_1(\mu_\alpha^*\mathcal{K}_{\mathcal{F}_\alpha})\rangle > 0.$$

By Remark 2.2.1 and the fact that $(\mu_\alpha)_* T[\tilde{f}_\alpha] = T[f_\alpha]$, we get

$$T[f_\alpha] \cdot K_{\mathcal{F}_\alpha} - T(f_\alpha, \mathcal{J}_{\mathcal{F}_\alpha}) = T[f_\alpha] \cdot \mu_\alpha^*\mathcal{K}_{\mathcal{F}_\alpha} > 0.$$

However, since f is transcendental, by Theorem 2.2.4 we infer

$$T[f_\alpha] \cdot T_{\mathcal{F}_\alpha} + T(f_\alpha, \mathcal{J}_{\mathcal{F}_\alpha}) \geqslant 0,$$

and the contradiction is obtained by observing that $c_1(K_{\mathcal{F}_\alpha}) = -c_1(T_{\mathcal{F}_\alpha})$. □

REMARK 2.2.2. In Chapter 3 we generalize the above theorem to any singular directed variety (X, V) (without assuming V to be involutive), by applying the Ahlfors-Schwarz Lemma. In the proof, the canonical sheaf plays a crucial role (and it arises in a natural way).

By a result due to Takayama [**Tak08**, Theorem 1.1], for any projective manifold X of general type, every irreducible component of $\mathbf{B}_+(K_X)$ is uniruled. It is natural to ask the following similar question:

PROBLEM 2.2.1. Let $(X, \mathcal{F})$ be a projective 1-foliated manifold of general type. Is every irreducible component of $\mathrm{Bs}(\mathcal{F})$ uniruled?

We also need the following logarithmic version of Theorem 2.2.4:

THEOREM 2.2.5. *Let $(X, \mathcal{F}, D)$ be a Kähler 1-foliated triple, and let $f : \mathbb{C} \to X$ be a transcendental entire curve tangent to $\mathcal{F}$ such that its image is not contained in* $\mathrm{Sing}(\mathcal{F}) \cup |D|$, *where $|D|$ is the support of D. Then we have*

$$T[f] \cdot T_{\mathcal{F}} + T(f, \mathcal{J}_{\mathcal{F},D}) = \langle T[\bar{f}_1], \mathcal{O}_{\bar{X}_1}(-1)\rangle \geqslant -\liminf_{r \to +\infty} \frac{N_{f,D}^{(1)}(r)}{T_{f,\omega}(r)} =: -N^{(1)}(f, D),$$

where $\bar{f}_1$ is the lift of f on $\bar{X}_1 := P(T_X\langle -\log D\rangle)$, and $N_{f,D}^{(1)}(r)$ is the truncated counting function *of f with respect to D defined by*

$$N_{f,D}^{(1)}(r) := \sum_{|\tau_j| < r, f(\tau_j) \in D} \log \frac{r}{|\tau_j|}.$$

PROOF. We adopt the same notation and concepts introduced in Section 2.2.1. Let (U_α) be a partition of unity on X. On each U_α we have

$$v_\alpha = \sum_{j=1}^{k} z_j a_\alpha^j \frac{\partial}{\partial z_j} + \sum_{i=k+1}^{n} a_\alpha^i \frac{\partial}{\partial z_i}$$

as the generator of $\mathcal{F}$, where $z_1 \cdots z_k = 0$ is the local equation of D in U_α. The hermitian metric $\omega_{X,D}$ induces a singular metric $h_{s,D}$ on $T_{\mathcal{F}}$ with local weight

$$\varphi_{\alpha,D} = -\log |v_\alpha|^2_{\omega_{X,D}} = -\log \sum_{i,j} a_\alpha^i \overline{a_\alpha^j} \omega_{i\bar{j}}.$$

If $h = h_s e^{-\varphi_{s,D}}$ is a smooth metric on $T_{\mathcal{F}}$, then $\varphi_{s,D}$ is the characteristic function associated with $\mathscr{J}_{\mathcal{F},D}$.

Since the image of f is not contained in $\mathrm{Sing}(\mathcal{F}) \cup |D|$, on $\Omega_\alpha := f^{-1}(U_\alpha)$ we have

$$\bar{f}'(\tau) := (\frac{f_1'}{f_1}, \ldots, \frac{f_k'}{f_k}, f_{k+1}', \ldots, f_n') = \lambda_\alpha(\tau)\big(a_\alpha^1(f), \ldots, a_\alpha^n(f)\big), \tag{2.2.14}$$

where $\lambda_\alpha(\tau)$ is the meromorphic functions with poles only contained in $f^{-1}\big(\mathrm{Sing}(\mathcal{F}) \cup |D|\big)$. By (2.2.14) we know that $f(\tau_j) \in D$ implies $f(\tau_j) \in \mathrm{Sing}(\mathcal{F})$. Indeed, if $f(\tau_j) \in D$, then λ_α has a pole of order at least 1 at τ_j, and such poles can only occur when $f(\tau_j) \in \mathrm{Sing}(\mathcal{F})$.

Observe that $\bar{f}'(\tau)$ can be seen as a *meromorphic* section of $\bar{f}_1^*\mathcal{O}_{\bar{X}_1}(-1)$, where

$$\bar{f}_1(\tau) := \big(f(\tau), [\bar{f}'(\tau)]\big)$$

is the canonical lift of f to $\bar{X}_1$. Then on $\Omega_\alpha \cap \Delta(r)$ we have

$$dd^c \log |f'(\tau)|^2_{\omega_{X,D}} = \sum_{|\tau_j|<r, \tau_j \in \Omega_\alpha} \eta_j \delta_{\tau_j} + f^* dd^c \log |v_\alpha|^2_{\omega_{X,D}},$$

where η_j is the vanishing order of $\lambda_\alpha(\tau)$ on τ_j. Since $v_\alpha = g_{\alpha\beta} v_\beta$, we see that η_j does not depend on the partition of unity, and thus on $\Delta(r)$ we have

$$dd^c \log |f'(\tau)|^2_{\omega_{X,D}} = \sum_{|\tau_j|<r} \eta_j \delta_{\tau_j} - f^*\Theta_h(T_{\mathcal{F}}) + f^* dd^c \varphi_{s,D}. \tag{2.2.15}$$

On the other hand, since $\omega_{X,D}$ induces a natural smooth metric on $T_X\langle -\log D\rangle$, as well as a smooth hermitian metric $\bar{h}_1$ on $\mathcal{O}_{\bar{X}_1}(-1)$, thus

$$dd^c \log |f'(\tau)|^2_{\omega_{X,D}} = dd^c \log |\bar{f}'(\tau)|^2_{\bar{h}_1} = \sum_{0<|\tau_j|<r} \mu_j \delta_{\tau_j} - \bar{f}_1^*\Theta_{\bar{h}_1}(\mathcal{O}_{\bar{X}_1}(-1)) \tag{2.2.16}$$

on $\Delta(r)$, where μ_j is the vanishing order of $\bar{f}'(t)$. By (2.2.14) we know that $\mu_j = -1$ if and only if $f(\tau_j) \in |D|$, and otherwise $\mu_j \geqslant 0$. Then by using the logarithmic derivative lemma again as in Lemma 2.2.2, we find

$$T[\bar{f}_1] \cdot \mathcal{O}_{\bar{X}_1}(-1) \geqslant -\liminf_{r \to +\infty} \frac{N^{(1)}_{f,D}(r)}{T_{f,\omega}(r)}. \tag{2.2.17}$$

We can combine (2.2.15) and (2.2.16) together to obtain

$$f^*\Theta_h(T_{\mathcal{F}}) = -\sum_{0<|\tau_j|<r} \mu_j \delta_{\tau_j} + \sum_{|\tau_j|<r} \eta_j \delta_{\tau_j} + \bar{f}_1^*\Theta_{\bar{h}_1}(\mathcal{O}_{\bar{X}_1}(-1)) + f^* dd^c \varphi_{s,D}$$

on $\Delta(r)$. By arguments very similar to those in the proof of Theorem 2.2.4, we get

$$T[f] \cdot \Theta_h(T_{\mathcal{F}}) + T(f, \mathscr{J}_{\mathcal{F},D}) = T[\bar{f}_1] \cdot \mathcal{O}_{\bar{X}_1}(-1),$$

and the theorem follows from (2.2.17). □

2.3. Degeneracy of Leaves of Foliations: Theories and Applications

2.3.1. “Diophantine Approximations" in Higher Dimensional Manifolds and Applications to Brunella's Hyperbolcity Theorem. In this subsection, we study the entire curves tangent to foliations on higher dimensional manifolds. We can generalize McQuillan's “Diophantine Approximations" for foliations with absolutely isolated singularties. First let us start with some relevant definitions and properties in [**CCS97**, **Tom97**].

Definition 2.3.1. Let $\mathcal{F}$ be a foliation by curves on a n-dimensional complex manifold X. An isolated singularity $p \in \mathrm{Sing}(\mathcal{F})$ is said to be *absolutely isolated singularity* (A.I.S.) if all the singularities of the blowing up tree of p_0 are isolated. More precisely, if we consider an arbitrary sequence of blowing-up's

$$(X, \mathcal{F}) \xleftarrow{\pi_1} (X_{(1)}, \mathcal{F}_{(1)}) \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} (X_{(n)}, \mathcal{F}_{(n)})$$

where the center of each blow-up π_i is a singular point $p_{i-1} \in \mathrm{Sing}\mathcal{F}_{(i-1)}$, then all singularties of $\mathcal{F}_{(n)}$ over the exceptional fiber are isolated.

Since locally the foliation $\mathcal{F}$ is generated by a holomorphic vector field $v = \sum_{i=1}^n a_i(z)\frac{\partial}{\partial z_i}$, the linear part of $\mathcal{F}$ at p is defined by

$$\mathcal{L}_v : m_p/m_p^2 \to m_p/m_p^2.$$

A singular point $p \in \mathrm{Sing}(\mathcal{F})$ is called *reduced* if $m_p(\mathcal{F}) = 1$ and the linear part of $\mathcal{F}$ at p has at least one non-zero eigenvalue. We shall say that $p \in \mathrm{Sing}(\mathcal{F})$ is a *non-dicritical singularity* of $\mathcal{F}$ if $\pi^{-1}(p)$ is invariant by $\mathcal{F}$, where π is the blow-up of p. Otherwise p is called a *dicritical singularity.*

Let $(X, \mathcal{F}, D)$ be a 1-foliated triple with absolutely isolated singularities. We assume that all singularities of $\mathcal{F}$ lie on D (this can always be achieved after one blowing-up). Fix a point $p \in \mathrm{Sing}(\mathcal{F})$ and denote by $e = e(D, p)$ the number of irreducible components of D through p. Then the vector field generating $\mathcal{F}$ can locally be written as

$$v = \sum_{j=1}^{e} z_j a_j(z) \frac{\partial}{\partial z_j} + \sum_{i=e+1}^{n} a_i(z) \frac{\partial}{\partial z_i},$$

where $z_1 z_2 \dots z_e = 0$ is the local equation of D at p.

DEFINITION 2.3.2. Assume that p is an absolutely isolated singularity of $\mathcal{F}$ and $e = 1$. Then p is a *simple point* if one of the following two possibilities occurs:

(A) $a_1(0) = 0$, the curve $(z_2 = \dots = z_n = 0)$ is invariant by $\mathcal{F}$ (up to an adequate formal choice of coordinates) and the linear part $\mathcal{L}_{v|_D}$ of $v|_D$ is of rank $n-1$.

(B) $a_1(0) = \lambda \neq 0$, the multiplicity of the eigenvalue λ is one and if μ is another eigenvalue of the linear part of $\mathcal{L}_v$, then $\frac{\mu}{\lambda} \notin \mathbb{Q}_+$.

Assume that $e \geqslant 2$. Then p is a *simple corner* if (up to a reordering of $(z_1, \dots, z_n)$), we have $a_1(0) = \lambda \neq 0, a_2(0) = \mu$ and $\frac{\mu}{\lambda} \notin \mathbb{Q}_+$.

We say that p is a *simple singularity* if it is a simple point or a simple corner.

The simple singularities for an n-dimensional vector field can be thought of a *final forms* in the sense that they are persistent under new blowing-up. This generalizes those obtained by Seidenberg [**Sei68**] in the two-dimensional cases.

PROPOSITION 2.3.1. Assume that p is a simple singularity of the 1-foliated triple $(X, \mathcal{F}, D)$. Let $\mu : \widetilde{X} \to X$ be the blowing-up of X with the center p and $E := \mu^{-1}(p)$ the exceptional divisor. Set $\widetilde{D} := \mu^{-1}(D \cup \{p\})$ and $\widetilde{\mathcal{F}}$ to be the induced foliation. Then:

(a) Each irreducible component of $\widetilde{D}$ is invariant by $\widetilde{\mathcal{F}}$.

(b) If $\tilde{p} \in \mathrm{Sing}(\widetilde{\mathcal{F}}) \cap E$, then $\tilde{p}$ is also a simple singularity of $\widetilde{\mathcal{F}}$ with respect to the induced 1-foliated triple $(\widetilde{X}, \widetilde{\mathcal{F}}, \widetilde{D})$. More precisely:
  - (b-1) if p is a simple point, there is exactly one simple point in $\mathrm{Sing}(\widetilde{\mathcal{F}}) \cap E$. The other points in $\mathrm{Sing}(\widetilde{\mathcal{F}}) \cap E$ are simple corners. Moreover, p and $\tilde{p}$ have the same type (A) or (B) of Definition 2.3.2.
  - (b-2) If p is a simple corner, then all points in $\mathrm{Sing}(\widetilde{\mathcal{F}}) \cap E$ are simple corners.

In [**CCS97**] and [**Tom97**], they proved the following reduction theorem for foliations with absolutely isolated singularities, which extended the Seidenberg's Theorem to higher dimensional manifolds:

THEOREM 2.3.1 (Camacho-Cano-Sad-Tom). *Let $(X, \mathcal{F}, D)$ be a 1-foliated triple with absolutely isolated singularities. Then there exists a finite sequence of blowing-up's*

$$(X, \mathcal{F}) \xleftarrow{\pi_1} (X_{(1)}, \mathcal{F}_{(1)}) \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} (X_{(n)}, \mathcal{F}_{(n)})$$

satisfying the following property:

(i) the center of each blow-up π_i is a singular point $p_{i-1} \in \mathrm{Sing}(\mathcal{F}_{(i-1)})$.

(ii) $(X_{(n)}, \mathcal{F}_{(n)})$ is a 1-foliated triple with only simple singularities.

(iii) All the singularities of $\mathcal{F}_{(n)}$ are non-dicritical.

In [**CCS97**, Theorem 6], the authors gives the final form of the simple singularities (which is, of course, absolutely isolated), and we summarize the properties of simple singularities as follows:

PROPOSITION 2.3.2. Let $(X, \mathcal{F}, D)$ be a 1-foliated triple with simple singularities, such that all singularities of $\mathcal{F}$ lie on D. For any $p \in \mathrm{Sing}(\mathcal{F})$, one can take a local coordinates $(z_1, \dots, z_n)$ of a neighborhood of p such that $\{z_1 z_2 \dots z_e = 0\}$ is the local equation of D at p, and the linear part $\mathcal{L}_v$ of a generator of $\mathcal{F}$ at p

$$v = \sum_{j=1}^{e} z_j a_j \frac{\partial}{\partial z_j} + \sum_{i=e+1}^{n} a_i \frac{\partial}{\partial z_i},$$

can be written in the following Jordan form:

$$\mathcal{L}_v = \sum_{i=1}^{s} \lambda_i z_i \frac{\partial}{\partial z_i} + \sum_{j=1}^{k} (\lambda_{s+j} z_{s+2j-1} + z_{s+2j}) \frac{\partial}{\partial z_{s+2j-1}} + \lambda_{s+j} z_{s+2j} \frac{\partial}{\partial z_{s+2j}},$$

where $e \leqslant s$, $\lambda_j \neq 0$ for $j = 2, \dots, s+k$; $\frac{\lambda_i}{\lambda_j} \notin \mathbb{Q}_+$ for $i \neq j, i, j = 1, \dots, s+k, j \neq 1$.

If we denote by $(\tilde{X}, \tilde{\mathcal{F}}, \tilde{D})$ the induced 1-foliated triple obtained by the blowing-up $\mu : \tilde{X} \to X$ at p with the exceptional divisor E, then

(i) E is invariant by $\tilde{\mathcal{F}}$.
(ii) The singularities of the induced foliation

$$\mathrm{Sing}(\tilde{\mathcal{F}}) \cap E = \{\tilde{p}_1, \ldots, \tilde{p}_s, \tilde{p}_{s+1}, \tilde{p}_{s+3}, \ldots, \tilde{p}_{s+2k-1}\},$$

where $\tilde{p}_j$ is the origin of the affine coordinate $(\tilde{z}_1, \ldots, \tilde{z}_n)$ given by

$$z_1 = \tilde{z}_1\tilde{z}_j, \ldots, \tilde{z}_j = z_j, \ldots, z_n = \tilde{z}_n\tilde{z}_j.$$

(iii) If p is a simple point, then $\tilde{p}_1$ is the only simple point among $\mathrm{Sing}(\tilde{\mathcal{F}}) \cap E$, and other $\tilde{p}_j$'s are simple corners. Moreover, $\tilde{p}_1$ and p have the same type (A) or (B) of Definition 2.3.2.
(iv) $T_{\tilde{\mathcal{F}}} = \mu^* T_{\mathcal{F}}$.

Let us recall the following definition of separatrix of the foliation.

DEFINITION 2.3.3. Let $\mathcal{F}$ be a foliation by curves defined on some open domain $U \subset \mathbb{C}^n$. A *separatrix* of the singular holomorphic foliation $\mathcal{F}$ at the point $p \in \mathrm{Sing}(\mathcal{F})$ is a local leaf $L \subset (U,p) \backslash \mathrm{Sing}(\mathcal{F})$ whose closure $L \cup p$ is a germ of analytic curve.

Based on the properties of the simple singularities in Definition 2.3.2 and Proposition 2.3.2, we can prove that, after any blowing-up, the separatrix can only pass to certain singularities.

THEOREM 2.3.2. *With the same notation as in Proposition 2.3.2, we have*
(i) if p is a simple corner (i.e. $e \geqslant 2$), each separatrix of $\mathcal{F}$ at p must be contained in D.
(ii) Assume that p is a simple point. Let $(\tilde{X}, \tilde{\mathcal{F}}, \tilde{D})$ be the induced 1-foliated triple obtained by the blowing-up at p with the exceptional divisor E, and C be a separatrix at p which is not contained in D, then the lift $\tilde{C}$ intersects with E only at $\tilde{p}$, which is the unique simple point in $\mathrm{Sing}(\tilde{\mathcal{F}}) \cap E$ *by Proposition 2.3.1.*

PROOF. Assume that we have a separatrix C of $\mathcal{F}$ at p which is not contained in D. We take a local parametrization $f : (\mathbb{C}, 0) \to (C, p)$ for this separatrix, then in the local coordinate system $(z_1, \ldots, z_n)$ introduced in Proposition 2.3.2, we have

$$\big(f_1'(\tau), \ldots, f_n'(\tau)\big) = \eta(\tau) \cdot (f_1(\tau)a_1(f), \ldots, f_e(\tau)a_e(f), a_{e+1}(f), \ldots, a_n(f))$$

for some meromorphic function $\eta(\tau)$ whose poles only appear at 0. By the assumption that C is not contained in D, for each $i = 1, \ldots, e$, $f_i(\tau)$ is not identically equal to zero. We denote by ν_i the vanishing order of $f_i(\tau)$ at 0 for $i = 1 \ldots, n$, which are all non-negative integers.

If p is a simple corner, then $e \geqslant 2$ and $\lambda_2 = a_2(0) \neq 0$, and we have

$$\eta(\tau)a_2(\tau) = \frac{f_2'(\tau)}{f_2(\tau)}.$$

This implies that the order of pole of $\eta(\tau)$ at 0 must be 1. If we denote by $\eta(\tau) = \frac{b(\tau)}{\tau}$ with $b(\tau)$ some germ of holomorphic function satisfying $b(0) \neq 0$, then

$$b(0) \cdot \lambda_2 = \nu_2 > 0.$$

Similarly we also have

$$b(0) \cdot \lambda_1 = \nu_1 > 0,$$

thus $\frac{\lambda_1}{\lambda_2} = \frac{\nu_1}{\nu_2} \in \mathbb{Q}_+$, which is a contradiction. Thus any separatrix at the simple corner must be contained in D, and we proved the claim (i).

Suppose that p is a simple point (i.e. $e = 1$). For the lift $\tilde{C}$ of C on $\tilde{X}$, we have

$$\tilde{C} \cap E \in \mathrm{Sing}(\tilde{\mathcal{F}}) \cap E = \{\tilde{p}_1, \ldots, \tilde{p}_s, \tilde{p}_{s+1}, \tilde{p}_{s+3}, \ldots, \tilde{p}_{s+2k-1}\}.$$

First we assume that $\tilde{C} \cap E = \{\tilde{p}_j\}$ for some $j \geqslant 2$. Then $\tilde{p}_j$ is the origin of the affine coordinate $(\tilde{z}_1, \ldots, \tilde{z}_n)$

$$z_1 = \tilde{z}_1\tilde{z}_j, \ldots, \tilde{z}_j = z_j, \ldots, z_n = \tilde{z}_n\tilde{z}_j,$$

and thus the lift $\tilde{f}(\tau) : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ of $f(\tau)$ is given by

$$\big(\tilde{f}_1(\tau) \ldots, \tilde{f}_n(\tau)\big) = \Big(\frac{f_1(\tau)}{f_j(\tau)}, \ldots, \frac{f_{j-1}(\tau)}{f_j(\tau)}, f_j(\tau), \frac{f_{j+1}(\tau)}{f_j(\tau)}, \ldots, \frac{f_n(\tau)}{f_j(\tau)}\Big).$$

Hence for any $i \neq j$

$$\nu_i > \nu_j,$$

and we have

$$f_j'(\tau) = \lambda_j \eta(\tau)\big(f_j(\tau) + o(\tau)\big).$$

Then we can set $\eta(\tau) = \frac{b(\tau)}{\tau}$ with $b(\tau)$ some germ of holomorphic function satisfying $b(0) \neq 0$, such that

$$b(0) \cdot \lambda_j = \nu_j > 0.$$

Since

$$f_1'(\tau) = \eta(\tau) f_1(\tau) a_1\big(f(\tau)\big) = \frac{b(\tau)}{\tau} f_1(\tau) a_1\big(f(\tau)\big),$$

thus $\lambda_1 = a_1(0) = \nu_1 / b(0)$ which implies that $\frac{\lambda_1}{\lambda_j} = \frac{\nu_1}{\nu_j} \in \mathbb{Q}_+$. This is a contradiction.

We can also derive Claim (ii) from Claim (i) directly. Indeed, since E is invariant under $\tilde{\mathcal{F}}$, by the fact that $\tilde{C}$ is not contained in E, we see that $\tilde{C} \cap E$ must be contained in $\mathrm{Sing}(\tilde{\mathcal{F}}) \cap E$. However, $\tilde{p}_1$ is the only simple point by Proposition 2.3.2, by Claim (i) we conclude that

$$\tilde{C} \cap E = \{\tilde{p}_1\}.$$

We finish the proof of the Theorem. □

LEMMA 2.3.1. Let X be a smooth projective manifold, and L an ample line bundle over X. Let $f : \mathbb{C} \to X$ be any entire curve whose Zariski closure $Z := \overline{f(\mathbb{C})}^{\mathrm{Zar}}$ is of dimension $r \geqslant 2$, and $f'(0) \neq 0$. Take any open neighborhood U containing $p := f(0)$ with coordinates $(z_1, \ldots, z_n)$ such that $f(t) = (t, 0, \ldots, 0)$ in U. For any $m \in \mathbb{N}$, we define the ideal sheaf $\mathcal{I}_m := (z_1^m, z_2, \ldots, z_n)$. Then for any $c \in \mathbb{Q}^+$ with $\frac{1}{r} < c < 1$, we have

$$m^c T[f] \cdot L \geqslant T(f, \mathcal{I}_m) \tag{2.3.1}$$

for $m \gg 0$.

PROOF. The lemma will be proved if $H^0(Z, \lfloor m^c \rfloor L \otimes \mathcal{I}_m|_Z) \neq 0$. Indeed, since L is ample, by Serre Vanishing Theorem, for $m \gg 0$ we have $H^1(X, \lfloor m^c \rfloor L \otimes \mathcal{I}_Z)$, and thus any section

$$s \in H^0(Z, \lfloor m^c \rfloor L \otimes \mathcal{I}_m|_Z) \neq 0$$

can be extended to a section

$$S \in H^0(X, \lfloor m^c \rfloor L \otimes \mathcal{I}_m).$$

Let $\mu : \tilde{X} \to X$ be a log resolution of $\mathcal{I}_m$ with $\mu^* \mathcal{I}_m = \mathcal{O}_{\tilde{X}}(-D)$. Then there is an effective divisor

$$E \sim \lfloor m^c \rfloor \mu^* L - D,$$

which does not contain Z. Here " $\sim$ " represents the linear equivalence. Hence we have

$$\lfloor m^c \rfloor T[f] \cdot L - T(f, \mathcal{I}_m) = \lfloor m^c \rfloor T[\tilde{f}] \cdot (\mu^* L - D) = T[\tilde{f}] \cdot E \geqslant 0,$$

where the last inequality is due to the First Main Theorem.

By Riemann-Roch Theorem, we have

$$\mathrm{dim} H^0(Z, \lfloor m^c \rfloor L|_Z) \sim \lfloor m^c \rfloor^r > m,$$

and the defining equations for sections given by $\mathcal{I}_m$ is m. Thus if $m \gg 0$, there always exists a non zero section in $H^0(Z, \lfloor m^c \rfloor L \otimes \mathcal{I}_m|_Z) \neq 0$. The lemma is proved. □

The geometric understanding of Lemma 2.3.1 is the following interpretation:

PROPOSITION 2.3.3. Let X be a smooth projective manifold, and let $f : \mathbb{C} \to X$ be a transcendental entire curve. We take an infinite sequence of blowing-up's

$$X \xleftarrow{\pi_1} X_{(1)} \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} X_{(n)} \xleftarrow{\pi_{n+1}} \cdots$$

such that each $\pi_i : X_{(i)} \to X_{(i-1)}$ is obtained by the blowing-up at $p_{i-1} := f_{(i-1)}(0)$ with the exceptional divisor E_i, where $f_{(i-1)} : \mathbb{C} \to X_{(i-1)}$ is the lift of f to $X_{(i-1)}$. Then we have

$$\lim_{k \to +\infty} T[f_{(k)}] \cdot E_k = 0. \tag{2.3.2}$$

PROOF. After taking finite blowing-ups, we can assume that $f'(0) \neq 0$. Take any open neighborhood U containing $p_0 := f(0)$ with coordinates $(z_1, \ldots, z_n)$ such that $f(t) = (t, 0, \ldots, 0)$ in U. Then p_1 is the origin of the affine coordinates $(z_1^{(1)}, \ldots, z_n^{(1)})$ for $\pi_1^{-1}(U)$ defined by

$$z_1 = z_1^{(1)}, z_2 = z_1^{(1)} z_2^{(1)}, \ldots, z_n = z_1^{(1)} z_n^{(1)}.$$

Set $\mu_k := \pi_k \circ \pi_{k-1} \circ \cdots \circ \pi_1$. We can inductively define the coordinates $(z_1^{(k)}, \ldots, z_n^{(k)})$, such that under this coordinates p_k is the origin, and $E_k = \{z_1^{(k)} = 0\}$. Then we have

$$\mu_k(z_1^{(k)}, \ldots, z_n^{(k)}) = (z_1^{(k)}, z_1^{(k)k} z_2^{(k)}, \ldots, z_1^{(k)k} z_n^{(k)}),$$

which implies
$$\mu_k^*\mathcal{I}_k \subset \mathcal{O}_{X_k}(-kE_k).$$
Then for any ample line bundle L, when $k \gg 0$,
$$T[f_{(k)}]\cdot E_k \leqslant \frac{1}{k}T(f_{(k)},\mu_k^*\mathcal{I}_k) = \frac{1}{k}T(f,\mathcal{I}_k) \leqslant \frac{k^{\frac{2}{3}}}{k}T[f]\cdot L,$$
where the last inequality is due to Lemma 2.3.1 by setting $c = \frac{2}{3}$. By passing k to infinity we obtain our result.

Moreover, we will prove a more general result. Fix two positive integers k and s. We denote by $\pi_{k+s,k+1} = \pi_{k+s}\circ\pi_{k+s-1}\circ\ldots\circ\pi_{k+1} : X_{(k+s)} \to X_{(k)}$ the composition of blowing-ups, and $E_k^{(k+s)} := (\pi_{k+s,k+1})_*^{-1}(E_k)$ the strict transform of E_k. Set $\mathcal{I}_s^{(k)} \subset \mathcal{O}_{X_{(k)}}$ to be the ideal sheaf defined by
$$(z_1^{(k)s}, z_2^{(k)}, \ldots, z_n^{(k)}).$$
Then we have
$$(\pi_{k+s,k+1})^*\mathcal{I}_s^{(k)} = \mathcal{O}_{X_{(k+s)}}(-E_{k+1}^{(k+s)} - 2E_{k+2}^{(k+s)} - \ldots - sE_{k+s}^{(k+s)}),$$
and
$$(\pi_{k+s,k+i+1})^*E_{k+i} = E_{k+i}^{(k+s)} + E_{k+i+1}^{(k+s)} + \ldots + E_{k+s}^{(k+s)}.$$
Then
$$\begin{aligned} T(f_{(k)},\mathcal{I}_s^{(k)}) &= T[f_{(k+s)}]\cdot(E_{k+1}^{(k+s)} + 2E_{k+2}^{(k+s)} + \ldots + sE_{k+s}^{(k+s)}) \\ &= T[f_{(k+s)}]\cdot\Big(\sum_{i=1}^{s}(\pi_{k+s,k+i+1})^*E_{k+i}\Big) \\ &= \sum_{i=1}^{s}T[f_{(k+i)}]\cdot E_{k+i}. \end{aligned}$$
By (2.3.2) we also have

(2.3.3)
$$\lim_{k\to+\infty} T(f_{(k)},\mathcal{I}_s^{(k)}) = 0.$$
□

Indeed, Proposition 2.3.3 can be extended to a more general form:

PROPOSITION 2.3.4. Let X be a projective manifold and Z be an irreducible analytic subset of X of dimension $r \geqslant 2$. For some $p_0 \in Z$, assume that there exists an infinite sequence of blowing-up's
$$X \xleftarrow{\pi_1} X_{(1)} \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} X_{(n)} \xleftarrow{\pi_{n+1}} \cdots$$
such that each $\pi_i : X_{(i)} \to X_{(i-1)}$ is obtained by the blowing-up at some $p_{i-1} \in E_{i-1}\cap Z_{i-1}\backslash E'_{i-2}$, where E_{i-1} is the exceptional divisor of π_{i-1}, E'_{i-2} and Z_{i-1} are the strict transforms $(\pi_{i-1})_*^{-1}E_{i-2}$ and $(\pi_{i-1})_*^{-1}(Z_{i-2})$. Then for any $\frac{1}{r} < c < 1$ and some ample line bundle L on X, when $k \gg 0$, there always exists sections $s_k \in H^0\big(X_{(k)}, \lfloor k^c\rfloor\mu_k^*L\otimes\mathcal{O}_{X_{(k)}}(-kE_k)\big)$ which do not vanish on Z_k. Here we set $\mu_k := \pi_k\circ\pi_{k-1}\circ\cdots\circ\pi_1$.

Thanks to Proposition 2.3.3, we can prove the following result, which can be seen as a generalization of McQuillan's “Diophantine approximation" for foliations on higher dimensional manifolds.

THEOREM 2.3.3. *Let $(X,\mathcal{F})$ be a 1-foliated pair with simple singularities. For any transcendental entire curve $f:\mathbb{C}\to X$ tangent to $\mathcal{F}$, we always have*
$$T[f]\cdot T_{\mathcal{F}} \geqslant 0.$$
If we further assume that $K_{\mathcal{F}}$ is a big line bundle, then the image of f is contained in $\mathbf{B}_+(K_{\mathcal{F}})$. In particular, if $K_{\mathcal{F}}$ is ample, then there exists no nonconstant transcendental entire curve $f:\mathbb{C}\to X$ tangent to $\mathcal{F}$.

PROOF. By Property (iv) in Proposition 2.3.2, for $(\tilde{X},\tilde{\mathcal{F}})$ obtained by blowing-up of any simply singularity p, we have
$$T[\tilde{f}]\cdot T_{\tilde{\mathcal{F}}} = T[\tilde{f}]\cdot\mu^*T_{\mathcal{F}} = T[f]\cdot T_{\mathcal{F}},$$
where μ is the blowing-up and $\tilde{f}$ is the lift. Thus after one blowing-up, we can assume that $(X,\mathcal{F},D)$ is a 1-foliated triple with simple singularities, such that all singularities of $\mathcal{F}$ lie on D. We adopt the same notation as in Proposition 2.3.2.

Fix a simple point $p_0 \in \mathrm{Sing}(\mathcal{F})$. We denote by η_0 the least non-negative integer k such that $\frac{\partial^k a_1}{\partial z_1^k}(0) \neq 0$. Take an infinite sequence of blowing-up's

$$(X, \mathcal{F}, D) \xleftarrow{\pi_1} (X_{(1)}, \mathcal{F}_{(1)}, D_{(1)}) \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_k} (X_{(k)}, \mathcal{F}_{(k)}, D_{(k)}) \xleftarrow{\pi_{k+1}} \cdots \tag{2.3.4}$$

such that the center of each blow-up π_i is the *unique* simple point $p_{i-1} \in \mathrm{Sing}(\mathcal{F}_{i-1}) \cap E_{i-1}$, where E_{i-1} is the exceptional divisor of the blowing-up π_{i-1}. In other words, this sequence (2.3.4) is the iteration of the blowing-up's in Proposition 2.3.2. For each $i > 0$, one define the non-negative positive integer η_i for the simple p_i as η_0 for p_0, and it is easy to verify that they are invariant under the blowing-up's:

$$\eta_0 = \eta_1 = \cdots = \eta_k = \cdots .$$

By Theorem 2.2.5, we have

$$T[f] \cdot T_{\mathcal{F}} = T[f_{(k)}] \cdot T_{\mathcal{F}_{(k)}} \geqslant -T(f_{(k)}, \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}}) - N^{(1)}(f_{(k)}, \mathrm{Sing}(\mathcal{F}_{(k)}) \cap D_{(k)}), \tag{2.3.5}$$

where $N^{(1)}(f_k, \mathrm{Sing}(\mathcal{F}_k) \cap D_k)$ is the truncated counting function, and the first equality in (2.3.5) is due to Property (iv) in Proposition 2.3.2. Therefore, in order to show that $T[f] \cdot T_{\mathcal{F}} \geqslant 0$, we need to prove that the right-hand-side (RHS for short) of (2.3.5) tends to 0 as $k \to +\infty$. By Definition 2.3.2, we know that the coherent ideal sheaf $\mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}}$ is not trivial at $p \in \mathrm{Sing}(\mathcal{F}_{(k)})$ if and only if p is a simple point of type (A). On the other hand, from Claim (i) in Theorem 2.3.2, the entire curve $f_{(k)}$ intersects with $D_{(k)}$ only at simple points, which implies that, only simple points in $\mathrm{Sing}(\mathcal{F}_{(k)})$ contributes to the truncated counting function $N^{(1)}(f_{(k)}, D_{(k)})$. In conclusion, we only need to study the contributions of simple points to RHS of (2.3.5).

Fix any simple point $p_0 \in \mathrm{Sing}(\mathcal{F})$. If $f(\mathbb{C})$ does not contain p_0, then by the very definition,

$$N^{(1)}\big(f_{(k)}, \mu_k^{-1}(p_0)\big) = 0$$

for all $k > 0$, where $\mu_k := \pi_k \circ \pi_{k-1} \circ \cdots \circ \pi_1$ is the composition of the blowing-up's. If f passes to p_0, then by Property (ii) in Theorem 2.3.2, the blowing-up's procedure introduced in Theorem 2.3.3 coincides with that in (2.3.4). Thus we can apply the result in Theorem 2.3.3 to show that

$$T[f_{(k)}] \cdot E_k \to 0.$$

By Claim (ii) in Theorem 2.3.2 again, $f(\mathbb{C}) \cap \mu_k^{-1}(p_0) \subset E_k$, then by the First Main Theorem, $T[f_{(k)}] \cdot E_k$ dominates the counting function $N^{(1)}\big(f_{(k)}, \mu_k^{-1}(p_0)\big)$. Thus we conclude that

$$\lim_{k \to +\infty} N^{(1)}(f_{(k)}, \mathrm{Sing}(\mathcal{F}_{(k)}) \cap D_{(k)}) = 0. \tag{2.3.6}$$

However, from Claim (ii) in Theorem 2.3.2 we see that the truncated counting function is stable under the blowing-up's in (2.3.4), that is, for each $k > 0$, we have

$$N^{(1)}(f_{(k)}, \mathrm{Sing}(\mathcal{F}_{(k)}) \cap D_{(k)}) = N^{(1)}(f_{(k+1)}, \mathrm{Sing}(\mathcal{F}_{(k+1)}) \cap D_{(k+1)}).$$

Then by (2.3.6), the truncated counting functions $N^{(1)}(f_{(k)}, D_{(k)})$ are always zero.

Take an open set U containing p_0 as the unique singularity in $\mathrm{Sing}(\mathcal{F})$. If p_0 is of type (B), then $\eta_0 = 0$, and for any $k > 0$, on the open set $U_k := \mu_k^{-1}(U)$, we have

$$\mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}} = \mathcal{O}_{X_{(k)}}.$$

Thus all the singularities in $\mathrm{Sing}(\mathcal{F}_{(k)}) \cap U_k$ have non contribution to $T(f_{(k)}, \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}})$.

If p_0 is of type (A), then $s := \eta_0 > 0$. If f passes to p_0, then then by Property (ii) in Theorem 2.3.2 again, the blowing-up's procedure introduced in Theorem 2.3.3 coincides with that in (2.3.4). After taking finite blowing-up's, we can assume that $f : (\mathbb{C}, 0) \to (X, p_0)$ with $f'(0) \neq 0$. First, we will show that on U_k

$$\mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}} = \mathcal{I}_s^{(k)},$$

where $\mathcal{I}_s^{(k)}$ is the ideal introduced in the proof of Theorem 2.3.3. Then all the contributions of all singularities in U_k to $T(f_{(k)}, \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}})$ are equal to $T(f_{(k)}, \mathcal{I}_s^{(k)})$, which tends to 0 as $k \to +\infty$ by (2.3.3).

In the coordinates $(z_1^{(k)}, \ldots, z_n^{(k)})$ introduced in Proposition 2.3.1, E_k is defined by $\{z_1^{(k)} = 0\}$, the local vector field v generating $\mathcal{F}_{(k)}$ has the following form

$$v = z_1^{(k)} b_1(z^{(k)}) \frac{\partial}{\partial z_1^{(k)}} + \sum_{i=2}^{n} b_i(z^{(k)}) \frac{\partial}{\partial z_i^{(k)}}.$$

Since $f_{(k)}(t) = (t, 0, \ldots, 0)$ is tangent to $\mathcal{F}_{(k)}$, thus for each $2 \leqslant i \leqslant n$,

$$b_i(t, 0, \ldots, 0) = 0.$$

By Proposition 2.3.1, p_k is of the same type as p_0, and thus by the definition, the linear part $\mathcal{L}_v|_{E_k}$ of $v|_D$ is of rank $n-1$. That is, the *Jacobian determinant* of the functions

$$\big(b_2(0, z_2^{(k)}, \ldots, z_n^{(k)}), \ldots, b_n(0, z_2^{(k)}, \ldots, z_n^{(k)})\big)$$

with respect to $(z_2^{(k)}, \ldots, z_n^{(k)})$ is non-zero locally. Then

$$\{z^{(k)} | b_2(z^{(k)}) = b_3(z^{(k)}) = \ldots = b_n(z^{(k)}) = 0\}$$

defines the local separatrix $f_{(k)} : (\mathbb{C}, 0) \to (X_{(k)}, p_k)$. Since the vanishing order of $b_1\big(f_{(k)}(t)\big)$ at 0 is s, we see that the ideal defined by $(b_1, b_2, \ldots, b_n)$ is $\mathcal{I}_s^{(k)}$. We thus proved that $\mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}} = \mathcal{O}_{X_{(k)}}$.

If $f(\mathbb{C})$ does not contain the type (A) simple point p_0, then $f_{(k)}(\mathbb{C})$ also avoids p_k, and the counting function $N(f_{(k)}, p_k) = 0$. By the First Main Theorem 2.2.2, the contributions made by all the singularities in U_k to $T(f_{(k)}, \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}})$ are equal to the proximity function $m(f_{(k)}, \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}})$. If we can show that

$$\lim_{k \to +\infty} m(f_{(k)}, \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}}) = 0,$$

we will finish the proof that the RHS of (2.3.5) tends to 0 as $k \to +\infty$ (may pass to a subsequence).

Case 1: there is a subsequence k_i tends to infinity such that the closure (in the Euclidean topology) $\overline{f_{(k_i)}(\mathbb{C})}$ omits p_{k_i}. Then by the definition of the proximity function, we have

$$m(f_{(k_i)}, \mathcal{J}_{\mathcal{F}_{(k_i)}, D_{(k_i)}}) = 0$$

for such k_i's tending to infinity.

Case 2: Assume that when $k \geqslant k_0$, we always have $p_k \in \overline{f_{(k)}(\mathbb{C})}^{\text{Zariski}}$. Set $Z_{k_0} := \overline{f_{(k_0)}(\mathbb{C})}^{\text{Zariski}}$. We denote by $\pi_{k+s,k+1} = \pi_{k+s} \circ \pi_{k+s-1} \circ \ldots \circ \pi_{k+1} : X_{(k+s)} \to X_{(k)}$ the composition of blowing-ups, and $E_k^{(k+s)} := (\pi_{k+s,k+1})_*^{-1}(E_k)$ the strict transform of E_k. Then for each $s > 0$, the strict transform $Z_{k+s} := (\pi_{k+s,k_0+1})_*^{-1}(Z)$ contains p_{k+s}. The pair (X_{k_0}, Z_{k_0}) and the blowing-up's (2.3.4) thus satisfy the condition in Proposition 2.3.4. We apply the result in Proposition 2.3.4 and the First Main Theorem to obtain that

$$\lfloor k^c \rfloor T[f] \cdot L = T[f_k] \cdot \lfloor k^c \rfloor \mu_k^* L \geqslant T[f_k] \cdot k E_k$$

for any $k \geqslant k_0$. This proves that

$$\lim_{k \to +\infty} T[f_k] \cdot E_k = 0. \tag{2.3.7}$$

Since when restricted on U_k we have

$$(\pi_{k+s,k+1})^* \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}} = \mathcal{O}_{U_{k+s}}(-E_{k+1}^{(k+s)} - 2E_{k+2}^{(k+s)} - \ldots - sE_{k+s}^{(k+s)}),$$

thus

$$m(f_{(k)}, \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}}) = m(f_{(k+s)}, (\pi_{k+s,k+1})^* \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}}) = \sum_{j=1}^{s} j \cdot m(f_{(k+s)}, E_{k+j}^{(k+s)}). \tag{2.3.8}$$

Set $\mathcal{I}_{(k)}$ to be the maximal ideal at p_k, then

$$(\pi_{k+s,k+1})^* \mathcal{I}_{(k)} = \mathcal{O}_{U_{k+s}}(-E_{k+1}^{(k+s)} - E_{k+2}^{(k+s)} - \ldots - E_{k+s}^{(k+s)}),$$

and thus

$$m(f_{(k)}, \mathcal{I}_{(k)}) = m(f_{(k+s)}, (\pi_{k+s,k+1})^* \mathcal{I}_{(k)}) = \sum_{j=1}^{s} m(f_{(k+s)}, E_{k+j}^{(k+s)}). \tag{2.3.9}$$

One combines (2.3.7), (2.3.8) and (2.3.9) to obtain

$$\lim_{k \to +\infty} m(f_{(k)}, \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}}) = \lim_{k \to +\infty} \sum_{i=0}^{s-1} m(f_{(k+i)}, \mathcal{I}_{(k+i)}) \leqslant \lim_{k \to +\infty} \sum_{i=1}^{s} T[f_{k+i}] \cdot E_{k+i} = 0.$$

In conclusion, we prove that, after taking a subsequence, we always have

$$\lim_{k \to +\infty} T(f_{(k)}, \mathcal{J}_{\mathcal{F}_{(k)}, D_{(k)}}) = 0.$$

Since the truncated functions $N^{(1)}(f_{(k)}, D_{(k)})$ are always zero, by (2.3.5) we see that

$$T[f] \cdot c_1(T_{\mathcal{F}}) \geqslant 0.$$

If we further assume that $K_{\mathcal{F}}$ is a big line bundle, then by Proposition 2.2.1, $f(\mathbb{C})$ is contained in the augmented base locus $\mathbf{B}_+(K_{\mathcal{F}})$. In particular, if $K_{\mathcal{F}}$ is ample, then $\mathbf{B}_+(K_{\mathcal{F}}) = \varnothing$, and thus there is no non-constant transcendental entire curve tangent to $\mathcal{F}$. This completes the proof of the theorem. □

Say that a foliation by curves on the complex projective space $\mathbb{C}P^n$ is of degree $d(\geqslant 0)$ if it is generated by a nontrivial holomorphic section (a rational vector field)

$$s \in H^0\big(\mathbb{C}P^n, T_{\mathbb{C}P^n} \otimes \mathcal{O}(d-1)\big).$$

Let $\mathrm{Fol}(d,n)$ denote the space of one-dimensional holomorphic foliations of degree d on $\mathbb{C}P^n$. As an application of Theorem 2.3.3, we give a new proof of the following theorem by Brunella [**Bru06**]:

THEOREM 2.3.4. *For a* generic *foliation $\mathcal{F}$ by curves of degree d in the complex projective space $\mathbb{C}P^n$, if $n \geqslant 2$ and $d \geqslant 2$, then there exists no nonconstant $f : \mathbb{C} \to \mathbb{C}P^n$ tangent to $\mathcal{F}$ (and possibly passing through the singularities* $\mathrm{Sing}(\mathcal{F})$*). In particular, all the leaves of $\mathcal{F}$ are hyperbolic.*

PROOF. Based a result by Lins Neto and Soares [**LS96**, Theorem II], for $d \geqslant 2$, there exists an open and Zariski dense $\mathcal{U} \subset \mathrm{Fol}(d,n)$, such that for any $\mathcal{F} \in \mathcal{U}$, it has the following two properties:

(i) all the singularities of $\mathcal{F}$ are *isolated* and *hyperbolic*, that is, around the singular point p of $\mathcal{F}$, the linear part of the vector field v generating the foliation $\mathcal{F}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfying

$$\lambda_j \neq \mathbb{R}\lambda_k, \quad \forall j \neq k.$$

(ii) No algebraic curve is invariant by $\mathcal{F}$.

Then by Proposition 2.3.2, all the singularities of $\mathcal{F}$ are absolutely isolated and moreover they are simple singularities.

Assume that there exists an entire curve $f : \mathbb{C} \to \mathbb{C}P^n$ tangent to $\mathcal{F}$. Then by Property (ii) above f is *transcendental*. However, since $K_{\mathcal{F}} = \mathcal{O}_{\mathbb{C}P^n}(d-1)$, if $d \geqslant 2$, then it is ample, and by Theorem 2.3.3 f must be constant. This proves the theorem. □

REMARK 2.3.1. In [**Bru06**], Brunella prove that if there exists an entire curve tangent to a generic foliation $\mathcal{F} \in \mathcal{U}$, then

$$T[f] \cdot N_{\mathcal{F}} = 0.$$

His methods rely on his deep intuition on the dynamical properties of the leaf space around the hyperbolic singularities of the foliation. He proved that either the invariant measure associated to the Ahlfors currents is concentrated on the periodic trajectories of the induced real 1-dimension foliation on the sphere around the isolated singularity, which is identically zero since the general foliation by curves in $\mathbb{C}P^n$ has no invariant algebraic curves; or the leaf space is parametrized by a real analytic subvariety on which the residue of the foliated one-form representing the normal bundle $N_{\mathcal{F}}$ measured with the Alhfors current is zero.

2.3.2. INTERSECTION WITH THE NORMAL BUNDLE. In Section 2.3.1, we spent a lot of effort in proving McQuillan's Diophantine approximation for foliations on higher dimensional manifolds, which relies heavily on the reduction of singularities. The original motivation of McQuillan is to study the Green-Griffiths-Lang conjecture. In this subsection, we will introduce so-called *weakly reduced singularities* which play the same role in the McQuillan Theory, but are less demanding. As an application for Theorem 2.2.4, we will study the intersection of $T[f]$ with the normal bundle; i.e. with $c_1(N_{\mathcal{F}})$. Before anything else, we begin with the following definition.

DEFINITION 2.3.4. Let X be a Kähler manifold endowed with a foliation $\mathcal{F}$ by curves. We say that $\mathcal{F}$ has *weakly reduced singularities* if

(i) for *some* log resolution $\pi : \hat{X} \to X$ of $\mathcal{J}_{\mathcal{F}}$, we have $T_{\hat{\mathcal{F}}} = \pi^* T_{\mathcal{F}}$, where $\hat{\mathcal{F}}$ is the induced foliation of $\pi^*\mathcal{F}$;

(ii) the L^2 multiplier ideal sheaf $\mathcal{I}(\mathcal{J}_{\mathcal{F}})$ of $\mathcal{J}_{\mathcal{F}}$ is equal to $\mathcal{O}_X$, i.e., at each $p \in X$, assume that the vector field v is the local generator of $\mathcal{F}$ around p, then for any $f \in \mathcal{O}_{X,p}$, we have

$$\frac{|f|^2}{|v|^2_{\omega}} \in L^1_{\mathrm{loc}},$$

where ω is any smooth hermitian metric on X.

With the previous definition, we have the following theorem.

THEOREM 2.3.5. *Let X be a projective manifold of dimension n endowed with a foliation $\mathcal{F}$ by curves with weakly reduced singularities. If $f : \mathbb{C} \to X$ is a transcendental entire curve tangent to $\mathcal{F}$, whose image is not contained in* $\mathrm{Sing}(\mathcal{F})$ *and satisfies $\langle T[f], K_X\rangle > 0$ (e.g. K_X is big), then we have*

$$T[\hat{f}] \cdot \det N_{\hat{\mathcal{F}}} < 0$$

for some birational modification $(\hat{X}, \hat{\mathcal{F}})$ of $(X, \mathcal{F})$.

PROOF. From the standard short exact sequence

$$0 \longrightarrow T_{\mathcal{F}} \longrightarrow T_X \longrightarrow N_{\mathcal{F}} \longrightarrow 0$$

that holds outside of a codimension 2 subvariety, we have

$$K_X + T_{\mathcal{F}} = -\det N_{\mathcal{F}}.$$

By the definition of multiplier ideal sheaves [**Laz04**, Definition 9.2.3], we have

$$\mathcal{I}(\mathcal{J}_{\mathcal{F}}) = \pi_*(K_{\hat{X}/X} - D),$$

where $\pi : \hat{X} \to X$ is a log resolution of $\mathcal{J}_{\mathcal{F}}$ satisfying the condition in Definition 2.3.4, such that $\pi^*\mathcal{J}_{\mathcal{F}} = \mathcal{O}_{\hat{X}}(-D)$. Since $K_{\hat{X}/X} - D$ is an $\mathbb{Z}$-divisor, we know that $\mathcal{I}(\mathcal{J}_{\mathcal{F}}) = \mathcal{O}_X$ if and only if $K_{\hat{X}/X} - D$ is effective. By the assumption that the image of f is not contained in $\mathrm{Sing}(\mathcal{F})$, i.e. in the zero scheme of $\mathcal{J}_{\mathcal{F}}$, we know that the image of $\hat{f}$ is not contained in the support of the exceptional divisor, and thus

$$T[\hat{f}] \cdot (K_{\hat{X}/X} - D) \geqslant 0.$$

Therefore we have

$$\begin{aligned} T[\hat{f}] \cdot (K_{\hat{X}} + T_{\hat{\mathcal{F}}}) &= T[\hat{f}] \cdot (\pi^* K_X + \pi^* T_{\mathcal{F}} + K_{\hat{X}/X}) \\ &\geqslant T[\hat{f}] \cdot (\pi^* K_X + \pi^* T_{\mathcal{F}} + D) \\ &= T[f] \cdot (K_X + T_{\mathcal{F}}) + T(f, \mathcal{J}_{\mathcal{F}}), \end{aligned}$$

where the last equality follows from the fact that $\pi_* T[\hat{f}] = T[f]$ and $T(f, \mathcal{J}_{\mathcal{F}}) = T[\hat{f}] \cdot D$. Since by Theorem 2.2.4 we have

$$T[f] \cdot T_{\mathcal{F}} + T(f, \mathcal{J}_{\mathcal{F}}) \geqslant 0,$$

then

$$-T[\hat{f}] \cdot \det N_{\hat{\mathcal{F}}} = T[\hat{f}] \cdot (K_{\hat{X}} + T_{\hat{\mathcal{F}}}) \geqslant T[f] \cdot K_X > 0.$$

The theorem is proved. □

From Definition 2.3.2, we see that any foliation $\mathcal{F}$ with simple singularities is also weakly reduced. Since for any complex surface equipped with a foliation $(X, \mathcal{F})$, the singularity is always absolutely isolated, and thus after taking a finite sequence of blowing-up's the singularities of the foliation are weakly reduced. Then we can get another proof of McQuillan's theorem without using his "Diophantine approximation":

THEOREM 2.3.6. *Let X be a complex surface of general type endowed with a foliation $\mathcal{F}$. Then any entire curve tangent to $\mathcal{F}$ is algebraically degenerate.*

PROOF. Assume that we have a Zariski dense entire curve $f : \mathbb{C} \to X$ tangent to $\mathcal{F}$. We proceed by contradiction.

By Seidenberg's theorem [**Sei68**] there is a finite sequence of blowing-up's $\pi : \tilde{X} \to X$ such that the singularities of the induced foliation $\tilde{\mathcal{F}}$ are weakly reduced, and the lift $\tilde{f}$ of f to $\tilde{X}$ is still Zariski dense. Thus by Theorem 2.3.5 we have

$$T[\hat{f}] \cdot N_{\hat{\mathcal{F}}} < 0$$

for some birational pair $(\hat{X}, \hat{\mathcal{F}})$ which is obtained by resolving the ideal $\mathcal{J}_{\tilde{\mathcal{F}}}$. However, Theorem 2.1.4 tells us that

$$T[\hat{f}] \cdot N_{\hat{\mathcal{F}}} \geqslant 0$$

and we get a contradiction. □

2.3.3. SIU'S REFINED TAUTOLOGICAL INEQUALITY. In [**Siu02**] Y-T. Siu proved McQuillan's "refined tautological inequality" by applying the traditional function-theoretical formulation. We will give here an improvement of this result. First we begin with the following lemma due to Siu.

LEMMA 2.3.2. Let U be an open neighborhood of 0 in $\mathbf{C}^n$ and $\pi : \widetilde{U} \to U$ be the blow-up at 0. Then $\pi^*(\mathcal{O}_U(\Omega^1_U)) \subset \mathcal{I}_E \otimes (\Omega^1_{\widetilde{U}}\langle -\log E \rangle)$, where $\mathcal{I}_E$ is the ideal sheaf of the exceptional divisor E.

THEOREM 2.3.7. *Let H be an ample line bundle on a projective manifold X of dimension n. Let Z be a finite subset of X and $f : \mathbb{C} \to X$ be an entire curve. Let $\sigma \in H^0(X, S^l\Omega_X \otimes (klH))$ be such that $f^*\sigma$ is not identically zero on $\mathbb{C}$. Let W be the zero divisor of σ in $X_1 := P(T_X)$, and $\pi : Y \to X$ be the blow-up of Z with $E := \pi^{-1}(Z)$. Then we have*

$$\frac{1}{l} N_{f_1, W}(r) + T_{\hat{f}, \Theta_E}(r) - N^{(1)}_{f, m_Z}(r) \leqslant k T_{f, \Theta_H}(r) + \mathcal{O}(\log T_{f, \Theta_H}(r) + \log r), \tag{2.3.10}$$

where $N^{(1)}_{f,m_Z}(r)$ is the truncated counting function with respect to the ideal m_Z, and Θ_H (resp. Θ_E) is the curvature of H (resp. Θ_H) with respect to some smooth metric h_H (resp. h_E).

REMARK 2.3.2. In [**Siu02**] and [**McQ98**], a slightly weaker inequality is obtained comparatively to (2.3.10), with $m_{f,m_Z}(r)$ in place of $T_{\hat{f},\Theta_E}(r) - N^{(1)}_{f,m_Z}(r)$.

PROOF OF THEOREM 2.3.7. Let $\tau = \pi^*\sigma$. By Lemma 2.3.2, τ is a holomorphic section of $S^l(\Omega_Y(\log E))\otimes \pi^*(klH)$ over Y and τ vanishes to order at least l on E. Let s_E be the canonical section of E. If we divide τ by $s_E^{\otimes l}$, then $\widetilde{\tau} := \frac{\tau}{s_E^{\otimes l}}$ is a holomorphic section of $S^l(\Omega_Y(\log E)) \otimes \pi^*(klH) \otimes (-lE)$ over Y. We now prove that

$$\|\tau(\bar{\hat{f}}'(t))\|_{\pi^* h_H^{\otimes kl}} = \|\sigma(f'(t))\|_{h_H^{\otimes kl}}, \tag{2.3.11}$$

where $\bar{\hat{f}}'(t)$ is the derivative of $\hat{f}$ in $T_{\hat{X}}\langle -\log E\rangle$ (see Definition 2.2.14). To make things simple we assume $l = 1$. Let p be a point in Z and let U be a small open set containing p such that locally we have

$$\sigma = \sum_{i=1}^{n} a_i dz_i \otimes e^{\otimes k},$$

where e is the local section of H and p is the origin. The blow-up at p is the complex submanifold of $U\times\mathbb{P}^{n-1}$ defined by $w_j z_k = w_k z_j$ for $1 \leqslant j \neq k \leqslant n$, where $[w_1 : \ldots : w_n]$ are the homogeneous coordinates of $\mathbb{P}^n$. In the affine coordinate chart $w_1 \neq 0$ we have the relation

$$(z_1, z_1w_2, \ldots, z_1w_n) = (z_1, z_2, \ldots, z_n),$$

thus

$$\tau = z_1\left((a_1 + \sum_{i=2}^{n} a_i w_i)d\log z_1 + \sum_{i=2}^{n} a_i dw_i\right) \otimes (\pi^* e)^{\otimes k},$$

and $\hat{f}(t) = (f_1, \frac{f_2}{f_1}, \ldots, \frac{f_n}{f_1})$ in the local coordinate $(z_1, w_2, \ldots, w_n)$. Thus

$$\bar{\hat{f}}'(t) := \left(\frac{f_1'}{f_1}, (\frac{f_2}{f_1})', \ldots, (\frac{f_n}{f_1})'\right)$$

with respect to the local section $(z_1\frac{\partial}{\partial z_1}, \frac{\partial}{\partial w_2}, \ldots, \frac{\partial}{\partial w_n})$ of $T_{\hat{X}}\langle -\log E\rangle$. It is easy to check the equality (2.3.11).

By the logarithmic derivative lemma again we know that $\frac{1}{2\pi}\int_0^{2\pi} \log^+\|\tau(\bar{\hat{f}}'(re^{i\theta}))\|_{\pi^* h_H^{\otimes kl}}$ and $\frac{1}{2\pi}\int_0^{2\pi} \log^+\|\widetilde{\tau}(\bar{\hat{f}}'(re^{i\theta}))\|_{\pi^* h_H^{\otimes kl}\otimes h_E^{*\otimes l}}$ are both of the order $\mathcal{O}(\log T_{f,\Theta_H}(r) + \log r)$. Using $\log x = \log^+ x - \log^+ \frac{1}{x}$ for any $x > 0$, we obtain

$$\begin{aligned}
2lm_{\hat{f},E}(r) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\|s_E^{\otimes l}\circ \hat{f}(re^{i\theta})\|^2_{h_E}} \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\|\tau(\bar{\hat{f}}'(re^{i\theta}))\|^2_{\pi^* h_H^{\otimes kl}}} + \frac{1}{2\pi}\int_0^{2\pi} \log^+\|\widetilde{\tau}(\bar{\hat{f}}'(re^{i\theta}))\|^2_{\pi^* h_H^{\otimes kl}\otimes h_E^{*\otimes l}} + \mathcal{O}(1) \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \log\|\tau(\bar{\hat{f}}'(re^{i\theta}))\|^2_{\pi^* h_H^{\otimes kl}} + \frac{1}{2\pi}\int_0^{2\pi} \log^+\|\tau(\bar{\hat{f}}'(re^{i\theta}))\|^2_{\pi^* h_H^{\otimes kl}} \\
&+ \frac{1}{2\pi}\int_0^{2\pi} \log^+\|\widetilde{\tau}(\bar{\hat{f}}'(re^{i\theta}))\|^2_{\pi^* h_H^{\otimes kl}\otimes h_E^{*\otimes l}} + \mathcal{O}(1) \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \log\|\tau(\bar{\hat{f}}'(re^{i\theta}))\|^2_{\pi^* h_H^{\otimes kl}} + \mathcal{O}(\log T_{f,\Theta_H}(r) + \log r) \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \log\|\sigma(f'(re^{i\theta}))\|^2_{h_H^{\otimes kl}} + \mathcal{O}(\log T_{f,\Theta_H}(r) + \log r),
\end{aligned} \tag{2.3.12}$$

where the last equality is due to equality (2.3.11). Observe that there is a natural isomorphism betweem $H^0(X, S^l(\Omega_X) \otimes (klH))$ and $H^0(X_1, \mathcal{O}_{X_1}(l) \otimes p^*(klH))$, where $X_1 := P(T_X)$ and p is its projection to X. We denote by P_σ the corresponding section of σ in $H^0(X_1, \mathcal{O}_{X_1}(l) \otimes p^*(klH))$, whose zero divisor is W. Then we have

$$\|P_\sigma(f_{[1]}(t)) \cdot (f'(t))^l\|_{p^* h_H^{\otimes kl}} = \|\sigma(f'(t))\|_{h_H^{\otimes kl}},$$

and thus on $\mathit{\Delta}(r)$ we have

$$N(\sigma(f'(t)), r) = N_{f_{[1]},W}(r) + l\sum_{|\tau_j|<r} \mu_j \frac{r}{|\tau_j|},$$

where $N(\sigma(f'(t)), r)$ is the counting function of $\sigma(f'(t))$ and μ_j is the vanishing order of $f'(t)$ at τ_j. Therefore, by applying the Jensen formula to the last term in (2.3.12) we obtain

$$2lm_{\hat{f},E}(r) + 2N(\sigma(f'(t)), r) \leqslant 2klT_{f,\Theta_H}(r) + \mathcal{O}(\log T_{f,\Theta_H}(r) + \log r)). \tag{2.3.13}$$

Since we have

$$N^{(1)}_{f,m_Z}(r) + \sum_{|\tau_j|<r, f(\tau_j)\in Z} \mu_j \frac{r}{|\tau_j|} = N_{f,m_Z}(r) = N_{\hat{f},E}(r),$$

then by applying Nevanlinna's First Main Theorem to $(\hat{f}, E)$ we get

$$T_{\hat{f},\Theta_E}(r) = N_{\hat{f},E}(r) + m_{\hat{f},E}(r) + \mathcal{O}(1),$$

and we can combine this with (2.3.13) to obtain

$$T_{\hat{f},\Theta_E}(r) - N^{(1)}_{f,m_Z}(r) + \frac{1}{l}N_{f_{[1]},W}(r) \leqslant kT_{f,\Theta_H}(r) + \mathcal{O}(\log T_{f,\Theta_H}(r) + \log r)).$$

□

Now we have the following refined tautological equality:

THEOREM 2.3.8. *Let X be a Kähler manifold of dimension n and $f : \mathbb{C} \to X$ be a transcendental entire curve. Then for any finite set Z we have*

$$T[f_{[1]}] \cdot \mathcal{O}_{X_1}(-1) \geqslant T(f, m_Z) - N^{(1)}_{f,m_Z} \geqslant m(f, m_Z) := \lim_{r\to\infty} \frac{m_{f,m_Z}(r)}{T_{f,\Theta_H}(r)}.$$

PROOF. First we choose k large enough, in such a way that $\mathcal{O}_{X_1}(1) \otimes p^*(kH)$ is ample over X_1. When we choose l sufficient large, $\mathcal{O}_{X_1}(l) \otimes p^*(lkH)$ will be very ample over X_1. Hence there exists a section $\sigma \in H^0(X_1, \mathcal{O}_{X_1}(l) \otimes p^*(lkH))$ whose defect is zero, i.e.

$$N(f_{[1]}, W) := \lim_{r_k\to\infty} \frac{N_{f_{[1]},W}(r_k)}{T_{f,\Theta_H}(r_k)} = \langle T[f_{[1]}], \mathcal{O}_{X_1}(l) \otimes p^*(lkH)\rangle = \langle T[f_{[1]}], \mathcal{O}_{X_1}(l)\rangle + kl,$$

where W is the zero divisor of σ, and where the last equality comes from $\langle T[f], \Theta_H\rangle = 1$. Since f is transcendental, by Theorem 2.2.2 we have

$$\lim_{r\to\infty} \frac{T_{f,\Theta_H}(r)}{\log r} = +\infty.$$

We can thus divide both sides in (2.3.10) by $T_{f,\Theta_H}(r)$ and take $r \to \infty$ to obtain

$$\frac{1}{l}N(f_{[1]}, W) + T(f, m_Z) - N^{(1)}_{f,m_Z} \leqslant k,$$

and we obtain the formula in the theorem. □

2.4. Towards the Green-Griffiths Conjecture

In order to pursue the similar strategy and prove the Green-Griffiths conjecture for any complex surface X of general type, one needs to know the existence of a 1-dimensional foliation directing any given Zariski dense entire curve $f : \mathbb{C} \to X$. The condition of $c_1(X)^2 - c_2(X) > 0$ ensures the existence of multi-foliation on X such that any entire curve should be tangent to it. The difficulty in proving the general case is that, we can not ensure that there exists such a (multi)-foliation on X itself. However, inspired by a very recent work of Demailly [**Dem15b**], we believe that his definition of a variety "strongly of general type" is in some sense akin to the construction of foliations. Although one cannot construct foliations on X directly, one can prove the existence of some special multi-foliations in certain Demailly-Semple tower of X. Indeed, in [**Dem10**] the following theorem has been proved:

THEOREM 2.4.1. *Let (X, V) be a directed variety of "general type" (cf. [**Dem12**] for the definition of general type when V is singular), then $\mathcal{O}_{X_k}(m) \otimes \pi^*_{k,0}\mathcal{O}(-\frac{m}{kr}(1 + \frac{1}{2} + \ldots + \frac{1}{k})A)$ is big thus has sections for $m \gg k \gg 1$, where X_k is the k-th stage of Demailly-Semple tower of X and A is an ample divisor on X.*

By the Fundamental Vanishing theorem we know that for every entire curve $f : \mathbb{C} \to X$, the k-jet $f_{[k]} : \mathbb{C} \to X_k$ satisfies

$$f_{[k]}(\mathbb{C}) \subset \mathrm{Bs}(H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi^*_{k,0}A^{-1})) \subsetneq X_k.$$

Assume that we have an entire curve $f : \mathbb{C} \to X$ such that its image in X is Zariski dense. By the above theorem of Demailly, there exists an $N > 0$ such that the lift of f on the Nth-stage Demailly-Semple tower can not be Zariski dense in X_N, therefore we can find an integer $k \geqslant 0$ such that $f_{[j]}$ is Zariski dense in X_j for each $0 \leqslant j \leqslant k$, while the Zariski closure of the image of $f_{[k+1]}$ is $Z \subsetneq X_{k+1}$ which project onto X_k. Since $\mathrm{rank}(T_X) = 2$, Z is thus a divisor of X_{k+1}. From the relation between $\mathrm{Pic}(X_k)$ and $\mathrm{Pic}(X)$ we know that $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{k,0}^*(B)$, for some $B \in \mathrm{Pic}(X)$, $\mathbf{a} \in \mathbb{Z}^k$ and $a_k = m$. Therefore the projection $\pi_{k+1,k} : Z \to X_k$ is a ramified $m : 1$ cover, which defines a rank 1 multi-foliation $\mathcal{F}_k \subset V_k$ on X_k, and $f_k : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is tangent to this foliation. We define the linear subspace $W \subset T_Z \subset T_{X_{k+1}}|_Z$ to be the closure

$$W := \overline{T_{Z'} \cap V_{k+1}}$$

taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection $T_{Z'} \cap V_{k+1}$ has constant rank and is a subbundle of $T_{Z'}$. As is observed in [**Dem15b**], we know that rank$W = 1$ which is an 1-dimensional foliation. We first resolve the singularities of Z to get a birational model $(\tilde{Z}, \tilde{\mathcal{F}})$ of (Z, W) such that $\tilde{Z}$ is smooth, then by the assumption in Theorem J we take a further finite sequence of blow-ups to get a new birational model $(Y, \mathcal{F})$ of $(\tilde{Z}, \tilde{\mathcal{F}})$, such that $\mathcal{F}$ has only weakly reduced singularities. We now obtain a generically finite morphism $p : Y \to X_k$, and the lift of f to Y denoted by $g : \mathbb{C} \to Y$ is still a Zariski dense curve tangent to $\mathcal{F}$ satisfying $g = p \circ f_k$. Then we have

$$K_Y \sim p^* K_{X_k} + R,$$

where R is an effective divisor whose support is contained in the ramification locus of p. We will call X_k the *critical Demailly-Semple tower for f*.

Now we state our conjectures about reduction of singularities to weakly reduced ones, and the generalization of Brunella Theorem to higher dimensional manifolds:

CONJECTURE 2.4.1. Let $(X, \mathcal{F})$ be a Kähler 1-foliated pair. Then one can obtain a new birational model $(\tilde{X}, \tilde{\mathcal{F}})$ of $(X, \mathcal{F})$ by taking finite blowing-ups such that $\tilde{\mathcal{F}}$ has weakly reduced singularities.

REMARK 2.4.1. From Proposition 2.3.2 it is easy to show that foliations with absolutely isolated singularities can be resolved into weakly reduced ones after finite blowing-up's.

CONJECTURE 2.4.2. Let $(X, \mathcal{F})$ be a Kähler 1-foliated pair. Suppose that there is a Zariski dense entire curve $f : \mathbb{C} \to X$ tangent to $\mathcal{F}$, then we have

$$T[f] \cdot \det N_{\mathcal{F}} \geqslant 0.$$

REMARK 2.4.2. If the singular set of $\mathcal{F}$ is not discrete, it is difficult to construct a smooth 2-form in $c_1(\det N_{\mathcal{F}})$ as that appearing in Baum-Bott Formula [**Bru04**, Chapter 3]. Probably we should find some representation in the leafwise cohomology, i.e. cohomology group for laminations.

We can show that Conjecture 2.4.1 and 2.4.2 suffice to prove the Green-Griffiths conjecture for complex surfaces:

PROOF OF THEOREM J. Since we have

$$\det T_{X_k} = k\pi_{k,0}^* \det T_X \otimes \mathcal{O}_{X_k}(k+1, k, \ldots, 2),$$
$$(\pi_{k,j})_* T[f_{[k]}] = T[f_{[j]}] \quad \text{for} \quad k \geqslant j,$$

by the tautological inequality and the condition of general type we have

$$T[f_{[k]}] \cdot \det T_{X_k} = -\sum_{j=1}^{k} (k - j + 2) T[f_{[j]}] \cdot \mathcal{O}_{X_j}(-1) - k\langle T[f], K_X \rangle < 0.$$

Thus we obtain

$$T[g] \cdot K_Y = T[f_k] \cdot K_{X_k} + T[g] \cdot R > 0.$$

Conjecture 2.4.1 tells us that we can find a new birational pair $(\hat{Y}, \hat{\mathcal{F}})$ of $(Y, \mathcal{F})$ with weakly reduced singularities, then by Theorem I we have

$$T[\hat{g}] \cdot \det N_{\hat{\mathcal{F}}} < 0,$$

which is a contradiction to Conjecture 2.4.2, thus any entire curve must be algebraic degenerate. □

CHAPTER 3

Kobayashi Volume-Hyperbolicity for Directed Varieties

3.1. Introduction

Let (X, V) be a *complex directed manifold*, i.e X is a complex manifold equipped with a holomorphic subbundle $V \subset T_X$. The philosophy behind the introduction of directed manifolds, as initially suggested by J.-P. Demailly, is that, there are certain fonctorial constructions which work better in the category of directed manifolds (ref. [**Dem12**]). This is so even in the "absolute case", i.e. in the case $V = T_X$. In general, singularities of V cannot be avoided, even after blowing-up, and V can be seen as a coherent subsheaf of T_X such that T_X/V is torsion free. Such a sheaf V is a subbundle of T_X outside of an analytic subset of codimension at least 2, which we denote here by $\mathrm{Sing}(V)$. The Kobayashi-Eisenman volume measure can also be defined for such (singular) directed pairs (X, V).

Definition 3.1. Let (X, V) be a directed manifold with $\dim(X) = n$ and let $\mathrm{rank}(V) = r$. Then the Kobayashi-Eisenman volume measure of (X, V) is the pseudometric defined on any $\xi \in \Lambda^r V_x$ for $x \notin \mathrm{Sing}(V)$, by

$$\mathbf{e}^r_{X,V}(\xi) := \inf\{\lambda > 0;\ \exists f : \mathbb{B}_r \to X,\ f(0) = x,\ \lambda f_*(\tau_0) = \xi,\ f_*(T_{\mathbb{B}_r}) \subset V\},$$

where $\mathbb{B}_r$ is the unit ball in $\mathbb{C}^r$ and $\tau_0 = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}$ is the unit r-vector of $\mathbb{C}^r$ at the origin. One says that (X, V) is *Kobayashi measure hyperbolic* if $\mathbf{e}^r_{X,V}$ is generically positive definite, i.e. positive definite on a Zariski open set.

In [**Dem12**] the author also introduced the concept of *canonical sheaf* $\mathcal{K}_V$ for any singular directed variety (X, V), and he showed that the "bigness" of $\mathcal{K}_V$ implies that all non constant entire curves $f : \mathbb{C} \to (X, V)$ must satisfy certain global algebraic differential equations. In this note, we study the Kobayashi-Eisenman volume measure of the singular directed variety (X, V), and give another geometric consequence of the bigness of $\mathcal{K}_V$. Our main theorem is as follows:

Theorem K. *Let (X, V) be a compact complex directed variety (where V is possibly singular), and let $\mathrm{rank}(V) = r$, $\dim(X) = n$. If V is of general type (see Definition 3.3 below), with a base locus $\mathrm{Bs}(V) \subsetneq X$ (see also Definition 3.3), then (X, V) is Kobayashi measure hyperbolic.*

Remark 3.2. In the absolute case, Theorem K is proved in [**Gri71**] and [**KO71**]; for a smooth directed variety it is proved in [**Dem12**].

3.2. Proof of the main theorem

Proof. Since the singular set $\mathrm{Sing}(V)$ of V is an analytic set of codimension $\geqslant 2$, the top exterior power $\Lambda^r V$ of V is a coherent sheaf of rank 1, and it admits a generically injective morphism to its bidual $(\Lambda^r V)^{**}$, which is an invertible sheaf (and therefore, can be seen as a line bundle). We give below an explicit construction of the multiplicative cocycle which represents the line bundle $(\Lambda^r V)^{**}$.

Since $V \subset T_X$ is a coherent sheaf, we can take a covering by open coordinate balls $\{U_\alpha\}$ satisfying the following property: on each U_α, there exist sections $e^{(\alpha)}_1, \ldots, e^{(\alpha)}_{k_\alpha} \in \Gamma(U_\alpha, T_X|_{U_\alpha})$ which generate the coherent sheaf V on U_α. Thus the sections $e^{(\alpha)}_{i_1} \wedge \cdots \wedge e^{(\alpha)}_{i_r} \in \Gamma(U_\alpha, \Lambda^r T_X|_{U_\alpha})$ with $(i_1, \ldots, i_r)$ varying among all r-tuples of $(1, \ldots, k_\alpha)$ generate the coherent sheaf $\Lambda^r V|_{U_\alpha}$, which is a subsheaf of $\Lambda^r T_X|_{U_\alpha}$. Denote $v^{(\alpha)}_I := e^{(\alpha)}_{i_1} \wedge \cdots \wedge e^{(\alpha)}_{i_r}$. Then, since $\mathrm{codim}(\mathrm{Sing(V)}) \geqslant 2$ we know that the common zero set of the family of sections $v^{(\alpha)}_I$ is contained in $\mathrm{Sing}(V)$, and thus all tensors $v^{(\alpha)}_I$ are proportional via meromorphic factors. By simplifying in a given section $v^{(\alpha)}_{I_0}$ the common zero divisor of the various meromorphic quotients $v^{(\alpha)}_{I_0}/v^{(\alpha)}_I$, one obtains a section $v_\alpha \in \Gamma(U_\alpha, \Lambda^r T_X|_{U_\alpha})$ (uniquely defined up to an invertible factor), and holomorphic functions $\{\lambda_I \in \mathcal{O}(U_\alpha)\}$ which do not have common factors, such that $v^{(\alpha)}_I = \lambda_I v_\alpha$ for all I. From this construction we can see that on $U_\alpha \cap U_\beta$, v_α and v_β coincide up to multiplication by a nowhere vanishing holomorphic function, i.e.

$$v_\alpha = g_{\alpha\beta} v_\beta$$

on $U_\alpha \cap U_\beta \neq \varnothing$, where $g_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$. This multiplicative cocycle $\{g_{\alpha\beta}\}$ defines the line bundle $(\Lambda^r V)^{***}$. If we take a Kähler metric ω on X, it induces a smooth hermitian metric H_r on $\Lambda^r T_X$, and from the natural inclusion $\Lambda^r V \to \Lambda^r T_X$, ω also induces a *singular* hermitian metric h_s of $(\Lambda^r V)^{***}$ whose local weight φ_α is equal to $\log |v_\alpha|^2_{H_r}$. It is easy to show that h_s has *analytic singularities*, and that its set of singularities satisfies $\mathrm{Sing}(h_s) \subset \mathrm{Sing}(V)$. Indeed, we have $\mathrm{Sing}(h_s) = \bigcup_\alpha \{p \in U_\alpha | v_\alpha(p) = 0\}$. Now, one gives the following definition.

DEFINITION 3.3. With the notations above, (X, V) is called to be *of general type* if there exists a singular hermitian metric h on the invertible sheaf $(\Lambda^r V)^{***}$ with analytic singularities satisfying the following two conditions:

(1) The curvature current $\Theta_h \geqslant \epsilon\omega$, i.e., it is a Kähler current.
(2) h is more singular than h_s, that is, there exists a globally defined quasi-psh function χ which is bounded from above such that
$$e^\chi \cdot h = h_s.$$

Since h and h_s has both analytic singularities, χ also has analytic singularities, and thus e^χ is a continuous function. Moreover, $e^{\chi(p)} > 0$ if $p \notin \mathrm{Sing}(h)$. We define *the base locus of* V to be
$$\mathrm{Bs}(V) := \bigcap_h \mathrm{Sing}(h),$$
where h varies among all the singular metrics on $(\Lambda^r V)^{***}$ satisfying Properties (1) and (2) above.

Now fix a point $p \notin \mathrm{Bs}(V) \cup \mathrm{Sing}(V)$; then by Definition 3.3 we can find a singular metric h on $(\Lambda^r V)^{***}$ with analytic singularities satisfying Properties (1) and (2) above, and $p \notin \mathrm{Sing}(h)$. Let f be any holomorphic map from the unit ball $\mathbb{B}_r \subset \mathbb{C}^r$ to (X, V) such that $f(0) = p$, then on each $f^{-1}(U_\alpha)$ we have
$$f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right) = a^{(\alpha)}(t) \cdot v_\alpha|_f,$$
where $a^{(\alpha)}(t)$ is meromorphic functions, with poles contained in $f^{-1}\big(\mathrm{Sing}(V) \cap U_\alpha\big)$, and satisfies
$$\left|f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)\right|^2_{H_r} = |a^{(\alpha)}(t)|^2 \cdot |v_\alpha|^2_{H_r} = |a^{(\alpha)}(t)|^2 \cdot e^{\varphi_\alpha \circ f},$$
which is bounded on any relatively compact set.

Therefore, $\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}$ can be seen as a (meromorphic!) section of $f^*(\Lambda^r V)^{**}$, and thus we set
$$\delta(t) := \left|\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right|^2_{f^*h^{-1}} = |a^{(\alpha)}(t)|^2 \cdot e^{\phi_\alpha \circ f}, \tag{1}$$
where ϕ_α is the local weight of h. By Property (2) above, there exists a globally defined quasi-psh function χ on X which is bounded from above such that
$$\delta(t) = e^{\chi \circ f} \cdot \left|f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)\right|^2_{H_r}. \tag{2}$$

Now we define a semi-positive metric $\widetilde{\gamma}$ on $\mathbb{B}_r$ by putting $\widetilde{\gamma} := f^*\omega$, then we have
$$\frac{\left|f_*(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r})\right|_{H_r}}{\det \widetilde{\gamma}} \leqslant C_0(f(t)) \leqslant C_1, \tag{3}$$
where $C_0(z)$ is a bounded function on X which does not depend on f, and we take C_1 to be its upper bound. One can find a conformal factor $\lambda(t)$ so that $\gamma := \lambda\widetilde{\gamma}$ satisfies
$$\det \gamma = \delta(t)^{\frac{1}{2}}.$$
Combining (2) and (3) together, we obtain
$$\lambda \leqslant C_1^{\frac{1}{r}} \cdot e^{\frac{\chi \circ f}{2r}}.$$
Since $\Theta_h \geqslant \epsilon\omega$, by (1) and (2) we have
$$dd^c \log \det \gamma \geqslant \frac{\epsilon}{2} f^*\omega = \frac{\epsilon}{2\lambda}\gamma \geqslant \frac{\epsilon}{2C_1^{\frac{1}{r}}} e^{-\frac{\chi \circ f}{2r}} \gamma.$$
By Property (2) in Definition 3.3 applied to h, there exists a constant $C_2 > 0$ such that
$$e^{-\frac{\chi}{2r}} \geqslant C_2.$$

Denote $A := \frac{\epsilon C_2}{2C_1^{\frac{1}{r}}}$, and it is a universal constant which does not depend on f. Then by Ahlfors-Schwarz Lemma (see Lemma 3.4 below) we have

$$\delta(0) \leqslant \left(\frac{r+1}{A}\right)^{2r}.$$

Since $p \notin \mathrm{Sing}(h) \cup \mathrm{Sing}(V)$, then we have $e^{\chi(p)} > 0$, and thus

$$\left| f_* \left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)\right|^2_{H_r} (0) \leqslant e^{-\chi(p)}\delta(0) = e^{-\chi(p)} \cdot \left(\frac{r+1}{A}\right)^{2r}.$$

Since f is taken to be arbitrary, we conclude by Definition 3.1 that the Kobayashi-Eisenman volume measure $\mathbf{e}^r_{X,V}$ is positive definite outside of $\mathrm{Bs}(V) \cup \mathrm{Sing}(V)$, and therefore, (X, V) is Kobayashi measure hyperbolic. □

LEMMA 3.4 (Ahlfors-Schwarz). *Let* $\gamma = \sqrt{-1}\sum \gamma_{jk}(t)\mathrm{d}t_j \wedge \mathrm{d}t_k$ *be an almost everywhere positive hermitian form on the ball* $B(0, R) \subset \mathbb{C}^r$ *of radius* R*, such that*

$$-\mathrm{Ricci}(\gamma) := \sqrt{-1}\partial\bar{\partial} \log \det \gamma \geqslant \mathrm{A}\gamma$$

in the sense of currents, for some constant $A > 0$*. Then*

$$\det(\gamma)(t) \leqslant \left(\frac{r+1}{AR^2}\right)^r \frac{1}{(1 - \frac{|t|^2}{R^2})^{r+1}}.$$

REMARK 3.5. If V is regular, then V is of general type if and only if $\Lambda^r V^*$ is a big line bundle. In this situation, the base locus $\mathrm{Bs}(V) = \mathbf{B}_+(\Lambda^r V)$, where $\mathbf{B}_+(\Lambda^r V^*)$ is the *augmented base locus* for the big line bundle $\Lambda^r V^*$ (ref. [**Laz04**]).

With the notations above, we define the coherent ideal sheaf $\mathcal{I}(V)$ to be germ of holomorphic functions which is locally bounded with respect to h_s, i.e., $\mathcal{I}(V)$ is the integral closure of the ideal generated by the coefficients of v_α in some local trivialization of $\Lambda^r T_X$. We denote $K_V := \Lambda^r V^{***}$ and $\mathcal{K}_V := K_V \otimes \mathcal{I}(V)$: the sheaf $\mathcal{K}_V$ is defined in [**Dem12**] to be the *canonical sheaf of* (X, V). It is easy to show that the zero scheme of $\mathcal{I}(V)$ is equal to $\mathrm{Sing}(h_s) = \mathrm{Sing}(V)$. The sheaf $\mathcal{K}_V$ is said to be a *big sheaf* iff for some log resolution $\mu : \widetilde{X} \to X$ of $\mathcal{I}(V)$ with $\mu^*\mathcal{I}(V) = \mathcal{O}_{\widetilde{X}}(-D)$, the invertible sheaf $\mu^* K_V - D$ is big in the usual sense. Now we have the following statement:

PROPOSITION 3.2.1. V is of general type if and only if $\mathcal{K}_V$ is big. Moreover, we have

$$\mathrm{Bs}(V) \subset \mu(\mathbf{B}_+(\mu^* K_V - D)) \cup \mathrm{Sing}(h_s) \subset \mu(\mathbf{B}_+(\mu^* K_V - D)) \cup \mathrm{Sing}(V).$$

PROOF. By Definition 3.3, the condition that $\mathcal{K}_V$ is a big sheaf implies that K_V and $\mu^* K_V - D$ are both big line bundles. For $m \gg 0$, we have an isomorphism

$$\mu^* : H^0(X, (mK_V - A) \otimes \mathcal{I}(V)^m) \xrightarrow{\simeq} H^0(\widetilde{X}, m\mu^* K_V - \mu^* A - mD). \tag{4}$$

Let us fix a very ample divisor A. Then for $m \gg 0$, the base locus (in the usual sense) $\mathbf{B}(m\mu^* K_V - mD - \mu^* A)$ is stably contained in $\mathbf{B}_+(\mu^* K_V - D)$ (ref. [**Laz04**]). Thus we can take a $m \gg 0$ to choose a basis $s_1, \ldots, s_k \in H^0(\widetilde{X}, m\mu^* K_V - mD - \mu^* A)$, whose common zero is contained in $\mathbf{B}_+(\mu^* K_V - D)$. By the isomorphism (4) there exists $\{e_i\}_{1 \leqslant i \leqslant k} \subset H^0(X, (mK_V - A) \otimes \mathcal{I}(V)^m)$ such that

$$\mu^*(e_i) = s_i.$$

We define a singular metric h_m on $mK_V - A$ by putting

$$|\xi|^2_{h_m} := \frac{|\xi|^2}{\sum_{i=1}^k |e_i|^2} \quad \text{for} \quad \xi \in (mK_V - A)_x.$$

Choose a smooth metric h_A on A such that the curvature $\Theta_A \geqslant \epsilon\omega$ is a smooth Kähler form. Then $h := (h_m h_A)^{\frac{1}{m}}$ defines a singular metric on K_V with analytic singularities, such that its curvature current $\Theta_h \geqslant \frac{1}{m}\Theta_A \geqslant \frac{\epsilon}{m}\omega$. From the construction we know that h is more singular than h_s, and $\mathrm{Sing}(h) \subset \mu(\mathbf{B}_+(\mu^* K_V - D)) \cup \mathrm{Sing}(h_s)$. □

REMARK 3.6. Thanks to Proposition 3.2.1 we could have taken Definition 3.3 as another equivalent definition of the bigness of $\mathcal{K}_V$, one that is more analytic. By Theorem K we can replace the condition that V is of general type by the bigness of $\mathcal{K}_V$, and we see in this way that the definition of the canonical sheaf of a singular directed variety is very natural.

A direct consequence of Theorem K is the following corollary, which was suggested in [**GPR13**]:

COROLLARY 3.7. Let (X, V) be directed varieties with $\mathrm{rank}(V) = r$, and f be a holomorphic map from $\mathbb{C}^r$ to (X, V) with generic maximal rank. Then if $\mathcal{K}_V$ is big, the image of f is contained in $\mathrm{Bs}(V) \subsetneq X$.

The famous conjecture by Green-Griffiths states that in the absolute case the converse of Theorem K should be true. It is natural to ask whether we have similar results for arbitrary directed varieties. A result by Marco Brunella [**Bru11**] gives a weak converse of Theorem K for every 1-directed variety:

THEOREM 3.8. *Let X be a compact Kähler manifold equipped with a singular holomorphic foliation $\mathcal{F}$ by curves. Suppose that $\mathcal{F}$ contains at least one leaf which is hyperbolic, then the canonical bundle $K_{\mathcal{F}}$ is pseudoeffective.*

Indeed, Brunella proved more than the results stated in the above theorem. By putting on $K_{\mathcal{F}}$ precisely the Poincaré metric of hyperbolic leaves, he constructed a singular hermitian metric h on $K_{\mathcal{F}}$ (possibly not with analytic singularities), such that the set of points where h is locally unbounded is the polar set $\mathrm{Sing}(\mathcal{F}) \cup \mathrm{Parab}(\mathcal{F})$, where $\mathrm{Parab}(\mathcal{F})$ is the union of parabolic leaves, and such that the curvature Θ_h of the metric h is a positive current. In this vein, a natural question is:

QUESTION 3.9. Can Brunella's theorem be strengthened by stating that when a foliation $(X, \mathcal{F})$ admits a hyperbolic leaf, then not only $K_{\mathcal{F}}$ is pseudo-effective, but also the canonical sheaf $\mathcal{K}_{\mathcal{F}} = K_{\mathcal{F}} \otimes \mathcal{I}(\mathcal{F})$ is pseudoeffective? In other words, can we find a singular hermitian metric h on $K_{\mathcal{F}}$ with the curvature Θ_h is a positive current, and h is more singular than h_s? (Recall that h_s is the singular metric on $K_{\mathcal{F}}$ induced by a hermitian metric on T_X).

REMARK 3.10. In [**McQ08**] the author introduces the definition of *canonical singularities* for foliations, in dimension 2 this definition is equivalent to *reduced singularities* in the sense of Seidenberg. The generic foliation by curves of degree d in $\mathbb{C}P^n$ is another example of canonical singularities. In this situation, one cannot expect to improve the "bigness" of the canonical sheaf $\mathcal{K}_{\mathcal{F}}$ by blowing-up. Indeed, this birational model is "stable" in the sense that, $\pi_*\mathcal{K}_{\widetilde{\mathcal{F}}} = \mathcal{K}_{\mathcal{F}}$ for any birational model $\pi : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$. However, on a complex surface endowed with a foliation $\mathcal{F}$ with reduced singularities, if f is an entire curve tangent to the foliation, and $T[f]$ is the Ahlfors current associated with f, then in [**McQ98**] it is shown that the lower bound for $T[f] \cdot c_1(T_{\mathcal{F}})$ can be improved by an infinite sequence of blowing-ups. Indeed, for certain singularities, the separatrices containing them are rational curves, and thus the lifted entire curve will not pass through these singularities. In the literature [**Bru99, McQ98**] this type of singularities is sometimes called "*small*", i.e. the lifted entire curve will not pass to these singularities. Since $T[f] \cdot c_1(T_{\mathcal{F}})$ is related to value distribution, these "small" singularities do not have any negative contribution to the lower bound for $T[f] \cdot c_1(T_{\mathcal{F}})$, which will be substantially increased by the effect of performing blow-ups. In Chapter 2 This "Diophantine approximation" idea has been generalized to higher dimensions.

CHAPTER 4

Effective Results on The Diverio-Trapani Conjecture

Abstract. The aim of this work is to study the conjecture on the ampleness of Demailly-Semple bundles raised by Diverio and Trapani, and also obtain some effective estimates related to this problem.

4.1. Introduction

In recent years, an important technique in studying hyperbolicity-related problems is *invariant* jet differentials $E_{k,m}T_X^*$ introduced by J.-P. Demailly, which can be seen as a generalization to higher order of symmetric differentials, but invariant under the reparametrization. To prove hyperbolicity-type statements for projective manifolds, one needs to construct (many) global jet differentials vanishing on an ample divisor on the given manifold X (cf. Theorem 4.2.3 below). If one deal the with positivity for jet bundles of the complete intersection of hypersurfaces in $\mathbb{P}^N$, as was proved in [**Div08**], one cannot expect to achieve this for lower order jet differentials if the codimension of subvariety is small:

Theorem 4.1.1. *(Diverio) Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of hypersurfaces of any degree in $\mathbb{P}^N$. Then*

$$H^0(X, E_{k,m}^{GG}T_X^*) = 0$$

for all $m \geqslant 1$ and $1 \leqslant k < \dim(X)/\mathrm{codim}(X)$.

On the other hand, in principle, the positivity (or hyperbolicity) of a generic complete intersection in the projective space should be increased by cutting more and more with projective hypersurfaces of high degree. In [**Deb05**], Debarre verified this in the case of *abelian variety*, in which he proved that the intersection of at least $\frac{N}{2}$ sufficiently ample general hypersurfaces in an N-dimensional abelian variety has ample cotangent bundle. Motivated by this result, he conjectured that the analogous statement holds in the projective space:

Conjecture 4.1.1. (Debarre) The cotangent bundle of the intersection in $\mathbb{P}^N$ of at least $\frac{N}{2}$ general hypersurfaces of sufficiently high degree is ample.

The first important result in this direction was obtained by Brotbek in [**Bro14**], where he was able to prove the Debarre conjecture for complete intersection surfaces in $\mathbb{P}^4$. Later, in [**Bro15**] he proved the ampleness of the cotangent bundle of the intersection of at least $\frac{3n-2}{4}$ general hypersurfaces of high degree in $\mathbb{P}^n$. Very recently, based on the ideas and explicit methods arising in [**Bro15**], Brotbek and Darondeau [**BD15**] and independently S.-Y. Xie [**Xie15, Xie16**] proved the Debarre conjecture:

Theorem 4.1.2. *(Brotbek-Darondeau, Xie) Let X be any smooth projective variety of dimension N, and A a very ample line bundle on X, there exists a positive number d_N depending only on the dimension N, such that for each $c \geqslant \frac{N}{2}$, the complete intersection of c general hypersurfaces in $|A^\delta|$ has ample cotangent bundle.*

Moreover, Xie was able to give an effective lower bound on hypersurface degrees $d_N := N^{N^2}$. Although the work by Brotbek and Darondeau is not effective on the lower bound d_N, growing from some interpretation of the cohomological computations in [**Bro15**], they established an elegant geometric construction, which defines a map Ψ from the projectivized relative cotangent bundle $\mathbb{P}(\Omega_{\mathscr{X}/S})$ to a certain family $\mathscr{Y} \to \mathbf{G}$, which we called *the universal Grassmannian* in Section 4.4, to construct a lot of global symmetric differential forms with a negative twist by pulling-back the positivity on $\mathscr{Y}$. In order to make the base locus empty, they applied the Nakamaye Theorem, which asserts that for a big and nef line bundle L on a projective variety, the augmented base locus $\mathbb{B}_+(L)$ coincides with the null locus $\mathrm{Null}(L)$, to the tautological line bundle $\mathscr{L}$ on the universal Grasssmannian $\mathscr{Y}$. In Section 4.4, we obtain an *effective* result (see Theorem 4.4.3) related to the Nakamaye Theorem they used, which is a bit weaker but still valid in their proof. Thus based on their work we can obtain a better lower bound

$$d_N = 4c_0(2N-1)^{2c_0+1} + 6N - 3,$$

where $c_0 := \lfloor \frac{N+1}{2} \rfloor$.

On the other hand, by introducing a new compactification of the set of regular jets $J_kT_X^{\mathrm{reg}}/\mathbb{G}_k$, Brotbek was able to fully develop the ideas in [**BD15**] to prove the Kobayashi conjecture [**Bro16**]. His statement is thus the following:

THEOREM 4.1.3. *(Brotbek) Let X be a smooth projective variety of dimension n. For any very ample line bundle A on X and any $d \geqslant d_{K,n}$, a general hypersurface in $|A^d|$ is Kobayashi hyperbolic. Here $d_{K,n}$ depends only the dimension n.*

In [**Bro16**], the main new tool he constructed is the *Wronskians* on the Demailly-Semple tower, which associates sections of the line bundle to global invariant jet differentials. As there are certain insuperable obstructions to the positivity of the tautological line bundle on the Demailly-Semple towers, due to the compactification of the jet bundles (ref. [**Dem95**]), Brotbek introduced a clever way to blow-up the ideal sheaves defined by the Wronskians, which behaves well in families, so that he was able to apply the openness property of ampleness for the higher order jet bundles to prove the hyperbolicity for general hypersurfaces. In order to make the lower bound $d_{K,n}$ in Theorem 4.1.3 effective, one needs to obtain some effective estimates arising in some noetherianity arguments. As well as the Nakamaye Theorem, there is another constant $m_\infty(X_k, L)$ (see Section 4.2.3) which reflects the stability of Wronskian ideal sheaf when the positivity of the line bundle L increases. In Section 4.2.3 we study Brotbek's Wronskians and prove the *effective finite generation* for Wronskian ideal sheaf (Theorem 4.2.4), and thus based on Brotbek's result we were able to obtain an effective bound for the Kobayasi conjecture

$$d_{K,n} = n^{n+1}(n+1)^{2n+5}.$$

REMARK 4.1.1. By using Siu's technique of slanted vector fields on higher jet spaces outlined in his survey [**Siu02**], and the *Algebraic Morse Inequality* by Demailly and Trapani, the first effective lower bound for the degree of the general hypersurface which is weakly hyperbolic (say that a variety X is weakly hyperbolic if all its entire curves lie in a proper subvariety $Y \subsetneq X$) was given by Diverio, Merker and Rousseau [**DMR10**], where they confirmed the Green-Griffiths-Lang conjecture for generic hypersurfaces in $\mathbb{P}^n$ of degree $d \geqslant 2^{(n-1)^5}$. Later on, by means of a very elegant combination of his holomorphic Morse inequalities and a probabilistic interpretation of higher order jets, Demailly was able to improve the lower bound to $d \geqslant \left\lfloor \frac{n^4}{3}\Big(n\log\big(n\log(24n)\big)\Big)^n \right\rfloor$ [**Dem10**]. The latest best known bound was $d \geqslant (5n)^2n^n$ by Darondeau [**Dar15**]. In the recent published paper [**Siu15**], Siu provided more details to his strategy in [**Siu02**] to complete his proof of the Kobayashi conjecture, and the bound on the degree following [**Siu15**] are very difficult to make explicit.

In the same vein as the Debarre conjecture, in [**DT10**], Simone Diverio and Stefano Trapani raised the following generalized conjecture:

CONJECTURE 4.1.2. (Diverio-Trapani) Let $X \subset \mathbb{P}^N$ be the complete intersection of c general hypersurfaces of sufficiently high degree. Then, $E_{k,m}T_X^*$ is ample provided that $k \geqslant \frac{N}{c} - 1$, and therefore X is hyperbolic.

In this chapter, based mainly on the elegant geometric methods in [**BD15**] and [**Bro16**] on the Debarre and Kobayashi conjectures, we prove the following theorem:

THEOREM L. *Let X be a projective manifold of dimension n endowed with a very ample line bundle A. Let $Z \subset X$ be the complete intersection of c general hypersurfaces in $|H^0\big(X, \mathscr{O}_X(dA)\big)|$. Then for any positive integer $k \geqslant \frac{n}{c} - 1$, Z has almost k-jet ampleness (see Definition 4.2.1 below) provided that $d \geqslant 2c(\lceil \frac{n}{c} \rceil)^{n+c+2}n^{n+c}$. In particular, Z is Kobayashi hyperbolic.*

Since our definition for almost 1-jet ampleness coincides with ampleness of cotangent bundle, then our Main Theorem integrates both the Kobayashi ($c = 1$) and Debarre conjectures ($c \geqslant \frac{n}{2}$), with some (non-optimal) effective estimates.

At the expense of a slightly larger bound, based on a factorization trick due to Xie [**Xie15**], we are able to prove the following stronger result:

THEOREM M. *Let X be a projective manifold of dimension n and A a very ample line bundle on X. For any c-tuple $\mathbf{d} := (d_1, \ldots, d_c)$ such that $d_p \geqslant c^2n^{2n+2c}(\lceil \frac{n}{c} \rceil)^{2n+2c+4}$ for each $1 \leqslant p \leqslant c$, for general hypersurfaces $H_p \in |A^{d_p}|$, their complete intersection $Z := H_1 \cap \ldots \cap H_c$ has almost k-jet ampleness provided that $k \geqslant k_0$.*

Moreover, there exists a uniform $(e_1, \ldots, e_k) \in \mathbb{N}^k$ which only depends on n, such that $\mathscr{O}_{Z_k}(e_1, \ldots, e_k)$ is big and its augmented base locus

$$\mathbb{B}_+\big(\mathscr{O}_{Z_k}(e_1, \ldots, e_k)\big) \subset Z_k^{\mathrm{Sing}}$$

where Z_k^{Sing} *is the set of points in* Z_k *which* can not *be reached by the* k*-th lift* $f_{[k]}(0)$ *of any regular germ of curves* $f : (\mathbb{C}, 0) \to Z$.

From the relation between tautological bundles on the Demailly-Semple towers and invariant jet bundles, we prove the following theorem on the Diverio-Trapani conjecture:

THEOREM N. *Set* $q := \mathscr{X}_{\mathbf{d}} \to \prod_{p=1}^{c} |A^{d_p}|$ *to be the universal family of* c*-complete intesections of hypersurfaces in* $\prod_{p=1}^{c} |A^{d_p}|$*, where* $d_p \geqslant c^2 n^{2n+2c}(\lceil \frac{n}{c} \rceil)^{2n+2c+4}$ *for each* $1 \leqslant p \leqslant c$. *Set* $U \subset \prod_{p=1}^{c} |A^{d_p}|$ *to be a Zariski open set of* $\prod_{p=1}^{c} |A^{d_p}|$ *such that when restricted to* $\mathscr{X} := q^{-1}(U)$*,* q *is a smooth fibration. Then for every* $j \gg 0$*, there exists a subbundle* $V_j \subset E_{k,jm} T^*_{\mathscr{X}/U}$ *defined on* $\mathscr{X}$*, whose restriction to the general fiber* Z *of* q *is an ample vector bundle. Moreover, fix any* $x \in Z$*, and any* regular k*-jet of holomorphic curve* $[f] : (\mathbb{C}, 0) \to (Z, x)$*, then for every* $j \gg 0$ *there exists global jet differentials* $P_j \in H^0(Z, V_j|_Z \otimes A^{-1})$ *(hence they are of order* k *and weighted degree* jm*)* does not *vanish when evaluated on the* k*-jet defined by* $(f', f'', \ldots, f^{(k)})$.

In other words, this theorem shows that, we can find a subbundle of the invariant jet bundle, which is ample, and its Demailly-Semple locus defined in [**DR13**, Section 2.1] is empty.

4.2. Technical Preliminaries and Lemmas

4.2.1. Invariant Jet Differentials. Let (X, V) be a directed manifold, i.e. a pair where X is a complex manifold and $V \subset T_X$ a holomorphic subbundle of the tangent bundle. One defines $J_k V \to X$ to be the bundle of k-jets of germs of parametrized curves in X, that is, the set of equivalent classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$ which are tangent to V, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leqslant j \leqslant k$, when computed in some local coordinate system of X near x. From now no, if not specially mentioned, we always assume that $V = T_X$. The projection map $p_k : J_k T_X \to X$ is simply taken to be $[f] \mapsto f(0)$. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, the elements $[f]$ of any fiber $J_{k,x}, x \in \Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$f = (f_1, \ldots, f_n) : (\mathbb{C}, 0) \to \Omega \subset \mathbb{C}^{nk},$$

and they are completetely determined by their Taylor expansion of order k at $t = 0$:

$$f(t) = x + t f'(0) + \frac{t^2}{2!} f''(0) + \ldots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1}).$$

In these coordinates, the fiber $J_{k,x}$ can thus be identified with the set of k-tuples of vectors

$$(\xi_1, \ldots, \xi_k) = \big(f'(0), f''(0), \ldots, f^{(k)}(0)\big) \in \mathbb{C}^n.$$

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \to \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \quad a_1 \in \mathbb{C}^*, a_j \in \mathbb{C}, j > 2,$$

in which the composition law is taken modulo terms t_j of degree $j > k$. Then $\mathbb{G}_k$ is a k-dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on $J_k T_X$. The action consists of reparametrizing k-jets of maps $f : (\mathbb{C}, 0) \to X$ by a biholomorphic change of parameter $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ defined by $(f, \varphi) \mapsto f \circ \varphi$. The corresponding $\mathbb{C}^*$-action on k-jets is described in coordinates by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

Green-Griffiths introduced the vector bundle $E_{k,m}^{\mathrm{GG}} T_X^*$ whose fibers are complex valued polynomials $Q(f', f'', \ldots, f^{(k)})$ on the fibres of $J_k T_X$, of weighted degree m with respect to the $\mathbb{C}^*$-action, *i.e.*, $Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)})$, for all $\lambda \in \mathbb{C}^*$ and $(f', f'', \ldots, f^{(k)}) \in J_k V$. One calls $E_{k,m}^{\mathrm{GG}} T_X^*$ the bundle of jet differentials of order k and weighted degree m. Let $U \subset X$ be an open set with local coordinate $(z_1, \ldots, z_n)$, then any local section $P \in \Gamma(U, E_{k,m}^{\mathrm{GG}} T_X^*|_U)$ can be written as

$$\sum_{|\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k| = m} c_\alpha(z) (d^1 z)^{\alpha_1} (d^2 z)^{\alpha_2} \cdots (d^k z)^{\alpha_k},$$

where $c_\alpha(z) \in \Gamma(U, \mathscr{O}_U)$ for any $\alpha := (\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}^n)^k$, such that for any holomorphic curve $\gamma : \Omega \to U$ with $\Omega \subset \mathbb{C}$, we have

$$P([\gamma])(t) = \sum_{|\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k| = m} c_\alpha\big(\gamma(t)\big) \big(\gamma'(t)\big)^{\alpha_1} \big(\gamma''(t)\big)^{\alpha_2} \cdots \big(\gamma^{(k)}(t)\big)^{\alpha_k} \in \Gamma(\Omega, \mathscr{O}_\Omega),$$

where $[\gamma] : \Omega \to J_k T_X|_U$ is the natural lifted holomorphic curve on $J_k T_X$ induced by γ.

However, we are more interested in the more geometric context introduced by J.-P. Demailly in [**Dem95**]: the subbundle $E_{k,m}V^* \subset E_{k,m}^{GG}V^*$ which is a set of polynomial differential operators $Q(f', f'', \ldots, f^{(k)})$ which are invariant under arbitrary changes of parametrization, that is, for any $\varphi \in \mathbb{G}_k$, we have

$$Q\big((f\circ\varphi)', (f\circ\varphi)'', \ldots, (f\circ\varphi)^{(k)}\big) = \varphi'(0)^m Q(f', f'', \ldots, f^{(k)}).$$

The bundle $E_{k,m}V^*$ is called the bundle of *invariant jet differentials* of order k and degree m. A very natural construction for invariant jet differentials is *Wronskians*. In [**Bro16**] Brotbek introduced a type of Wronskians induced by global sections in some linear system. We will recall briefly his constructions in Section 4.2.3.

4.2.2. Demailly-Semple Jet Bundles. Let X be a complex manifold of dimension n. If V is a subbundle of rank r, one constructs a tower of *Demailly-Semple k-jet bundles* $\pi_{k-1,k} : (X_k, V_k) \to (X_{k-1}, V_{k-1})$ that are $\mathbb{P}^{r-1}$-bundles, with $\dim X_k = n + k(r-1)$ and $\mathrm{rank}(V_k) = r$. For this, we take $(X_0, V_0) = (X, V)$, and for every $k \geqslant 1$, inductively we set $X_k := P(V_{k-1})$ and

$$V_k := (\pi_{k-1,k})_*^{-1}\mathscr{O}_{X_k}(-1) \subset T_{X_k},$$

where $\mathscr{O}_{X_k}(1)$ is the tautological line bundle on $X_k = P(V_{k-1})$, $\pi_{k-1,k} : X_k \to X_{k-1}$ the natural projection and $(\pi_{k-1,k})_* = d\pi_{k-1,k} : T_{X_k} \to \pi^*_{k-1,k}T_{X_{k-1}}$ its differential. By composing the projections we get for all pairs of indices $0 \leqslant j \leqslant k$ natural morphisms

$$\pi_{j,k} : X_k \to X_j, \quad (\pi_{j,k})_* = (d\pi_{j,k})|_{V_k} : V_k \to (\pi_{j,k})^*V_j,$$

and for every k-tuple $\mathbf{a} = (a_1, ..., a_k) \in \mathbb{Z}^k$ we define

$$\mathscr{O}_{X_k}(\mathbf{a}) = \otimes_{1\leqslant j\leqslant k}\pi^*_{j,k}\mathscr{O}_{X_j}(a_j).$$

We also have an inductively defined k-th lifting for germs of holomorphic curves such that $f_{[k]} : (\mathbb{C}, 0) \to X_k$ is obtained as $f_{[k]}(t) = \big(f_{[k-1]}(t), [f'_{[k-1]}(t)]\big)$. Moreover, if one denote by

$$J_k^{\mathrm{reg}}V := \{[f]_k \in J_kV | f'(0) \neq 0\}$$

the space of *regular k-jets* tangent to V, then there exists a morphism

$$\begin{array}{ccc} J_k^{\mathrm{reg}}V & \to & X_k \\ {[f]} & \mapsto & f_{[k]}(0) \end{array}$$

whose image is an open set in X_k denote by X_k^{reg}, which can be identified with the quotient $J_k^{\mathrm{reg}}/\mathbb{G}_k$ [**Dem95**, Theorem 6.8]. In other words, $X_k^{\mathrm{reg}} \subset X_k$ is the set of elements $f_{[k]}(0)$ in X_k which can be reached by all regular germs of curves f, and set $X_k^{\mathrm{sing}} := X_k \backslash X_k^{\mathrm{reg}}$, which is a divisor in X_k. Thus X_k is a relative compactification of $J_k^{\mathrm{reg}}/\mathbb{G}_k$ over X. Dealing with hyperbolicity problems, we are allowed to have small base locus contained in X_k^{sing} [**Dem95**, Section 7].

We will need the following parametrizing theorem due to J.-P. Demailly [**Dem95**, Corollary 5.12]:

Theorem 4.2.1. *Let (X, V) be a directed variety. For any $w_0 \in X_k$, there exists an open neighborhood U_{w_0} of w_0 and a family of germs of curves $(f_w)_{w\in U_{w_0}}$, tangent to V depending holomorphically on w such that*

$$(f_w)_{[k]}(0) = w \quad \text{and} \quad (f_w)'_{[k-1]}(0) \neq 0, \quad \forall w \in U_{w_0}.$$

In particular, $(f_w)'_{[k-1]}(0)$ gives a local trivialization of the tautological line bundle $\mathscr{O}_{X_k}(-1)$ on U_{w_0}.

By [**Dem95**, Theorem 6.8], we have the following isomorphism between Demailly-Semple jet bundles and invariant jet differentials:

Theorem 4.2.2. *(Direct image formula) Let (X, V) be a directed variety. The direct image sheaf*

$$(\pi_{0,k})_*\mathscr{O}_{X_k}(m) \cong E_{k,m}V^* \tag{4.2.1}$$

can be identified with the sheaf of holomorphic sections of $E_{k,m}V^*$. In particular, for any line bundle L, we have the following isomorphism induced by $(\pi_{0,k})_*$:

$$(\pi_{0,k})_* : H^0\big(X_k, \mathscr{O}_{X_k}(m) \otimes \pi^*_{0,k}L\big) \xrightarrow{\cong} H^0(X, E_{k,m}V^* \otimes L). \tag{4.2.2}$$

Moreover, let $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ and $m = a_1 + \ldots + a_k$, then we have

$$(\pi_{0,k})_*\mathscr{O}_{X_k}(\mathbf{a}) \cong \overline{F}^{\mathbf{a}}E_{k,m}V^* \tag{4.2.3}$$

where $\overline{F}^a E_{k,m}V^*$ is the subbundle of polynomials $Q(f', f'', \ldots, f^{(k)}) \in E_{k,m}V^*$ involving only monomials $(f^{(\bullet)})^l$ such that

$$l_{s+1} + 2l_{s+2} + \ldots + (k-s)l_k \leqslant a_{s+1} + \ldots + a_k$$

for all $s = 0, \ldots, k-1$.

Therefore, with the notations in Theorem 4.2.1, for any given local invariant jet differential $P \in \Gamma(U, E_{k,m}V^*|_U)$, the inverse image under $(\pi_{0,k})_*$ is the section in

$$\sigma_P \in \Gamma\big(U_{w_0}, \mathscr{O}_{X_k}(m)|_{U_{w_0}}\big)$$

defined by

$$\sigma_P(w) := P(f'_w, f''_w, \dots, f^{(k)}_w)\big((f_w)'_{[k-1]}(0)\big)^{-m}. \tag{4.2.4}$$

The general philosophy of the theory of (invariant) jet differentials is that their global sections with values in an anti-ample divisor provide algebraic differential equations which every entire curve must satisfy, which is an application of Ahlfors-Schwarz lemma. The following *Fundamental Vanishing Theorem* shows the obstructions to the existence of entire curves:

THEOREM 4.2.3. *(Demailly, Green-Griffiths, Siu-Yeung) Let (X, V) be a directed projective variety and $f : \mathbb{C} \to (X, V)$ an entire curve tangent to V. Then for every global section $P \in H^0\big(X, E_{k,m}V^* \otimes \mathscr{O}(-A)\big)$ where A is an ample divisor of X, one has $P(f', f'', \dots, f^{(k)}) = 0$. In other word, if we denote by s the unique section in $H^0\big(X_k, \mathscr{O}_{X_k}(m) \otimes \pi^*_{0,k}(-A)\big)$ corresponding to P induced by the isomorphism (4.2.2), and $Z(s) \subset X_k$ the base locus of this section, then $f_{[k]}(\mathbb{C}) \subset Z(s)$.*

Now we state the following definition which describes the positivity of the the invariant jet bundles:

DEFINITION 4.2.1. Let X be a projective manifold. We say that X has *almost k-jet ampleness* if and only if there exists a k-tuple of positive integers $(a_1, \dots, a_k)$ such that $\mathscr{O}_{X_k}(a_1, \dots, a_k)$ is big and its augmented base locus satisfies the condition

$$\mathbb{B}_+\big(\mathscr{O}_{X_k}(a_1, \dots, a_k)\big) \subset X_k^{\mathrm{sing}}.$$

By applying Theorem 4.2.3, we can quickly conclude that, if X has almost k-jet ampleness, then its Demailly-Semple locus [**DR13**, Section 2.1] is an empty set, and thus X is Kobayashi hyperbolic.

4.2.3. BROTBEK'S WRONSKIANS. In this subsection, we will study the property of the Wronskians constructed by Brotbek in [**Bro16**], which associates any $k+1$ sections of a given line bundle L to invariant k-jet differentials of weighted degree $k' := \frac{(k+1)k}{2}$, that is, sections in $H^0(X, E_{k,k'}T^*_X \otimes L^{k+1})$. We prove that, the Wronskians factorizes through a natural morphism from the bundle $J^k\mathscr{O}_X(L)$ of k-jet sections of L to the invariant jet bundles $E_{k,k'}T^*_X \otimes L^{k+1}$. Moreover, we obtain an "*effective finite generation*" of the k-th Wronskian ideal sheaf $\mathfrak{w}(X_k, L)$ (see also Theorem 4.2.4 below).

Let X be an n-dimensional compact complex manifold. If $(z_1, \dots, z_n)$ are local holomorphic coordinates on an open set $U \subset X$, then since J_kT_X is a locally trivial holomorphic fiber bundle, we have the homeomorphism

$$J_kT_X|_U \sim U \times \mathbb{C}^{nk},$$

which is given by $[f] \mapsto \big(f(0), f'(0), \dots, f^{(k)}(0)\big)$.

For any holomorphic function $g \in \mathscr{O}(U)$, and $1 \leqslant j \leqslant k$, there exists an induced holomorphic function $d^{[j]}_U(g)$ on $\mathscr{O}(p_k^{-1}(U))$, defined by

$$d^{[j]}_U(g)\big(f'(0), f''(0), \dots, f^{(k)}(0)\big) := (g \circ f)^{(j)}(0).$$

Moreover, we have the following lemma

LEMMA 4.2.1. For any $k \geqslant 1$, we have $d^{[k]}_U(g) \in \Gamma(U, E^{\mathrm{GG}}_{k,k}T^*_U)$, and

$$d^{[k]}_U(g) = \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=k} c_\alpha(z)(d^1z)^{\alpha_1}(d^2z)^{\alpha_2}\cdots(d^kz)^{\alpha_k}, \tag{4.2.5}$$

such that for each $\alpha := (\alpha_1, \dots, \alpha_k) \in (\mathbb{N}^n)^k$, $c_\alpha(z) \in \Gamma(U, \mathscr{O}_U)$ is a $\mathbb{Z}$-linear combination of $\frac{\partial^{|\beta|}g}{\partial z^\beta}(z)$ with $|\beta| \leqslant k$.

PROOF. We will prove the lemma by induction on k. For $k = 1$, we simply have

$$d^{[1]}_U(g) = \sum_{i=1}^n \frac{\partial g}{\partial z_i}(z)\mathrm{d}z^i \in \Gamma(U, T^*_U),$$

and thus the statements are true for $k = 1$.

Suppose that $d_U^{[k]}(g)$ has the form (4.2.5), then we have

$$\begin{aligned} d_U^{[k+1]}(g) &= \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=k} \sum_{i=1}^{k}\sum_{j=1}^{n} c_\alpha(z)(d^1z)^{\alpha_1}\cdots(d^iz)^{\alpha_i-\mathbf{e}_j}(d^{i+1}z)^{\alpha_{i+1}+\mathbf{e}_j}\cdots(d^kz)^{\alpha_k} \\ &+ \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=k} \sum_{j=1}^{n} \frac{\partial c_\alpha(z)}{\partial z^j}(d^1z)^{\alpha_1+\mathbf{e}_j}\cdots(d^kz)^{\alpha_k}, \end{aligned}$$

where $\mathbf{e}_1=(1,0,\dots,0),\dots,\mathbf{e}_n=(0,\dots,0,1)$ is the standard basis in $\mathbb{Z}^n$. If the lemma is true for k, so is $k+1$. Thus the lemma holds for any $k\in\mathbb{N}$. □

Since the bundle

$$E_{k,\bullet}^{\mathrm{GG}}T_X^* := \bigoplus_{m\geqslant 0} E_{k,m}^{\mathrm{GG}}T_X^*$$

is a bundle of graded algebras (the product is obtained simply by taking the product of polynomials). There are natural inclusions $E_{k,\bullet}^{\mathrm{GG}}\subset E_{k+1,\bullet}^{\mathrm{GG}}$ of algebras, hence

$$E_{\infty,\bullet}^{\mathrm{GG}}T_X^* := \bigcup_{k\geqslant 0} E_{k,\bullet}^{\mathrm{GG}}T_X^*$$

is also an (commutative) algebra. Then for any $(k+1)$ holomorphic functions $g_0,\dots,g_k\in\mathscr{O}(U)$, one can associate them to a natural k-jet differentials of order k and weighted degree $k':=\frac{k(k+1)}{2}$, say *Wronskians*, in the following way

$$W_U(g_0,\dots,g_k) := \begin{vmatrix} d_U^{[0]}(g_0) & \cdots & d_U^{[0]}(g_k) \\ \vdots & \ddots & \vdots \\ d_U^{[k]}(g_0) & \cdots & d_U^{[k]}(g_k)\end{vmatrix} \in \Gamma(U, E_{k,k'}^{\mathrm{GG}}T_U^*).$$

If we set

$$W_U(g_0,\dots,g_k) = \sum_{|\alpha_1|+2|\alpha_2|+\ldots+k|\alpha_k|=k'} b_\alpha(z)(d^1z)^{\alpha_1}(d^2z)^{\alpha_2}\cdots(d^kz)^{\alpha_k},$$

then for each $\alpha:=(\alpha_1,\dots,\alpha_k)\in(\mathbb{N}^n)^k$ with $|\alpha_1|+2|\alpha_2|+\ldots+k|\alpha_k|=k'$, by Lemma 4.2.1 there exists $\{a_{\alpha\boldsymbol{\beta}}\in\mathbb{Z}\}_{\boldsymbol{\beta}=(\beta_0,\dots,\beta_k),|\beta_j|\leqslant k}$, such that we have

$$b_\alpha(z) = \sum_{|\beta_j|\leqslant k} a_{\alpha\boldsymbol{\beta}} \frac{\partial^{|\beta_0|}g_0(z)}{\partial z^{\beta_0}}\cdots\frac{\partial^{|\beta_k|}g_k(z)}{\partial z^{\beta_k}}. \tag{4.2.6}$$

By the properties of the Wronskian, for any permutation $\sigma\in\mathrm{Sym}(\{0,1,\dots,k\})$, we always have

$$W_U(g_{\sigma(0)},\dots,g_{\sigma(k)}) = (-1)^{\mathrm{sign}(\sigma)}W_U(g_0,\dots,g_k),$$

and thus $a_{\alpha\boldsymbol{\beta}}=(-1)^{\mathrm{sign}(\sigma)}a_{\alpha\sigma(\boldsymbol{\beta})}$. Here $\sigma(\boldsymbol{\beta}):=(\beta_{\sigma(0)},\dots,\beta_{\sigma(k)})$.

On the other hand, for any holomorphic line bundle A on X, one can define the bundle J^kA of *k-jet sections* of A by $(J^kA)_x=\mathscr{O}_x(A)/\big(\mathcal{M}_x^{k+1}\cdot\mathscr{O}_x(A)\big)$ for every $x\in X$, where $\mathcal{M}_x$ is the maximal ideal of $\mathscr{O}_x$. Then J^kA has a filtration whose graded bundle is $\oplus_{0\leqslant p\leqslant k}S^pT_X^*\otimes\mathcal{O}(A)$. Set e_U to be a holomorphic frame of A and $(z_1,\dots,z_n)$ analytic coordinates on an open subset $U\subset X$. The fiber $(J^kA)_x$ can be identified with the set of Taylor developments of order k:

$$\sum_{|\alpha|\leqslant k} c_\beta(z-x)^\beta\cdot e_U,$$

and the coefficients c_β define coordinates along the fibers of J^kA. Thus one has a natural local trivialization of J^kA given by

$$\begin{aligned} \Psi_U : U\times\mathbb{C}^{I_k} &\to J^kA|_U, \\ (x,c_\beta) &\mapsto \sum_{\beta\in I_k} c_\beta(z-x)^\beta\cdot e_U, \end{aligned}$$

where

$$I_k := \{\beta=(\beta_1,\dots,\beta_n)\in\mathbb{N}^n \,|\, |\beta|\leqslant k\}.$$

For any local section $s=s_U\cdot e_U\in\Gamma(U,A)$, one has a natural map (no more a $\mathscr{O}_U$-module morphism!)

$$i_{[k]} : \Gamma(U,A)\to\Gamma(U,J^kA),$$

which is given by

$$\Psi_U^{-1} \circ i_{[k]}(s)(x) = (x, \frac{\partial^{|\alpha|} s_U}{\partial z^\alpha}(x)).$$

The local coordinates $(z_1, \ldots, z_n)$ on U also induces a natural local trivialization of the bundle of jet differentials $E_{k,m}^{\mathrm{GG}} T_U^* \to U$. Indeed, as any local section of $P \in \Gamma(U, E_{k,m}^{\mathrm{GG}} T_X^*|_U)$ is given by

$$\sum_{|\alpha_1|+2|\alpha_2|+\ldots+k|\alpha_k|=m} c_\alpha(z)(d^1 z)^{\alpha_1}(d^2 z)^{\alpha_2} \cdots (d^k z)^{\alpha_k},$$

where $c_\alpha(z) \in \Gamma(U, \mathscr{O}_U)$ for any α, one has the natural local trivialization of $E_{k,m}^{\mathrm{GG}} T_X^* \to X$ given by

$$\begin{aligned} \Phi_U : U \times \mathbb{C}^{I_{k,m}} &\to E_{k,m}^{\mathrm{GG}} T_U^*, \\ (z, c_\alpha) &\mapsto \sum_{|\alpha_1|+2|\alpha_2|+\ldots+k|\alpha_k|=m} c_\alpha (d^1 z)^{\alpha_1}(d^2 z)^{\alpha_2} \cdots (d^k z)^{\alpha_k}, \end{aligned}$$

where

$$I_{k,m} := \{\alpha = (\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}^n)^k \,||\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k| = m\}.$$

Now we define a multi-linear map

$$\tilde{\mu} : \prod^{k+1} \mathbb{C}^{I_k} \to \mathbb{C}^{I_{k,k'}} \tag{4.2.7}$$

$$(c_{0,\beta_0}, \ldots, c_{k,\beta_k}) \mapsto (\sum_{\boldsymbol{\beta}:=(\beta_0,\ldots,\beta_k)} a_{\alpha\boldsymbol{\beta}} c_{0,\beta_0} c_{1,\beta_1} \cdots c_{k,\beta_k})_{\alpha \in I_{k,k'}}, \tag{4.2.8}$$

where $a_{\alpha\boldsymbol{\beta}} \in \mathbb{Z}$ arises from (4.2.6). By the property that $a_{\alpha\boldsymbol{\beta}} = (-1)^{\mathrm{sign}(\sigma)} a_{\alpha\sigma(\boldsymbol{\beta})}$ for any permutation σ, the multi-linear map $\tilde{\mu}$ is alternating, and thus there exists a unique linear map

$$\mu : \wedge^{k+1} \mathbb{C}^{I_k} \to \mathbb{C}^{I_{k,k'}},$$

such that $\tilde{\mu} = \mu \circ w$. Here the map

$$w : \prod^{k+1} \mathbb{C}^{I_k} \to \wedge^{k+1} \mathbb{C}^{I_k}$$

which associates to $k+1$ vectors from $\mathbb{C}^{I_k}$ their exterior product.

By the local trivialization Ψ_U and Φ_U, μ induces a bundle morphism

$$\tilde{W}_U : \wedge^{k+1}(J^k \mathscr{O}_U) \to E_{k,k'} T_U^*$$

defined by

$$\begin{array}{ccc} U \times \wedge^{k+1} \mathbb{C}^{I_k} & \xrightarrow{\mathbb{1} \times \mu} & U \times \mathbb{C}^{I_{k,k'}} \\ \downarrow \Psi_U & & \downarrow \Phi_U \\ \wedge^{k+1} J^k \mathscr{O}_U & \xrightarrow{\tilde{W}_U} & E_{k,k'}^{\mathrm{GG}} T_U^*. \end{array}$$

Composing with $i_{[k]} : \Gamma(U, \mathscr{O}_U) \to \Gamma(U, J^k \mathscr{O}_U)$, we recover Brotbek's Wronskians W_U

$$W_U : \wedge^{k+1} H^0(U, \mathscr{O}_U) \xrightarrow{i_{[k]}} \wedge^{k+1} H^0(U, J^k \mathscr{O}_U) \to H^0(U, \wedge^{k+1} J^k \mathscr{O}_U) \xrightarrow{\tilde{W}_U} H^0(U, E_{k,k'}^{\mathrm{GG}} T_U^*).$$

An important fact for the Wronskian is that, it is invariant under the $\mathbb{G}_k$ action [**Bro16**, Proposition 2.2]:

LEMMA 4.2.2. With the notation as above, $W_U(g_0, \ldots, g_k) \in E_{k,k'} T_U^*$, where $k' := \frac{k(k+1)}{2}$.

In other words, the bundle morphism $\tilde{W}_U$ factors through the subbundle

$$E_{k,k'} T_U^* \subset E_{k,k'}^{\mathrm{GG}} T_U^*.$$

Now we consider the Demailly-Semple k-jet bundle of (X_k, V_k) of the direct variety (X, T_X) constructed in Section 4.2.2. Fix coordiantes $(z_1, \ldots, z_n)$ on U, $T_X|_U$ can be trivialized with the basis $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\}$. Set $U_k := X_k \cap \pi_{0,k}^{-1}(U)$, and under that trivialization we have

$$U_k = U \times \mathcal{R}_{n,k},$$

where $\mathcal{R}_{n,k}$ is some rational variety introduced in [**Dem95**, Theorem 9.1]. Moreover, the tautological bundle

$$\mathscr{O}_{X_k}(1)|_{U_k} = \mathrm{pr}_2^*(\mathscr{O}_{\mathcal{R}_{n,k}}(1)), \tag{4.2.9}$$

where $\mathrm{pr}_2 : U_k \to \mathcal{R}_{n,k}$ is the projection on the factor $\mathcal{R}_{n,k}$. By the direct image formula (4.2.1)

$$(\pi_{0,k})_*\mathscr{O}_{X_k}(m) \cong E_{k,m}T_X^*,$$

we conclude that, under the above trivialization, the direct image $(\pi_{0,k})_*$ induces a natural isomorphism (or a local trivialization of the vector bundle $E_{k,m}T_U^*$)

$$\varphi_U : U \times H^0\big(\mathcal{R}_{n,k}, \mathscr{O}_{\mathcal{R}_{n,k}}(m)\big) \to E_{k,m}T_U^*. \tag{4.2.10}$$

Moreover, under the trivialization Φ_U, the inclusion $E_{k,m}T_X^* \subset E_{k,m}^{\mathrm{GG}}T_X^*$ is also a constant linear injective map, that is, there exists an injective linear map $\nu : F^{k,m} \to \mathbb{C}^{I_{k,m}}$ such that

$$\begin{array}{ccc} U \times F^{k,m} & \xrightarrow{\mathbb{1}\times\nu} & U \times \mathbb{C}^{I_{k,m}} \\ \Big\downarrow{\scriptstyle \varphi_U} & & \Big\downarrow{\scriptstyle \Phi_U} \\ E_{k,m}T_X^*|_U & \hookrightarrow & E_{k,m}^{\mathrm{GG}}T_X^*|_U. \end{array}$$

Here we denote $F^{k,m} := H^0\big(\mathcal{R}_{n,k}, \mathscr{O}_{\mathcal{R}_{n,k}}(m)\big)$.

Therefore, under the trivializations φ_U and Ψ_U, the factorised bundle morphism $\tilde{W}_U$ is still a constant linear map. That is, there exists a linear map $\tilde{\nu} : \wedge^{k+1}\mathbb{C}^{I_k} \to F^{k,k'}$ such that $\mu = \nu \circ \tilde{\nu}$ and we have the following diagram:

$$\begin{array}{ccccc} U \times \wedge^{k+1}\mathbb{C}^{I_k} & \xrightarrow{\mathbb{1}\times\tilde{\nu}} & U \times F^{k,k'} & \xrightarrow{\mathbb{1}\times\nu} & U \times \mathbb{C}^{I_{k,m}} \\ \Big\downarrow{\scriptstyle \Psi_U} & & \Big\downarrow{\scriptstyle \varphi_U} & & \Big\downarrow{\scriptstyle \Phi_U} \\ \wedge^{k+1}J^k\mathscr{O}_U & \xrightarrow{\tilde{W}_U} & E_{k,k'}T_U^* & \hookrightarrow & E_{k,m}^{\mathrm{GG}}T_X^*|_U. \end{array}$$

We set

$$S := \mathrm{Image}(\tilde{\nu}) \subset H^0(\mathcal{R}_{n,k}, \mathscr{O}_{\mathcal{R}_{n,k}}(k')),$$

and denote by $\mathcal{I}_{n,k} \subset \mathscr{O}_{\mathcal{R}_{n,k}}$ the base ideal of the linear system S. Denote $\mathfrak{w}_U$ to be the ideal sheaf $\mathrm{pr}_2^*(\mathcal{I}_{n,k})$ on U_k.

On the other hand, one has a natural global construction for the invariant jet differentials on X: let L be any holomorphic line bundle on X, for any $s_0, \ldots, s_k \in H^0(X, L)$, if we choose a local trivialization of L above U, we define

$$W_U(s_0, \ldots, s_k) := W_U(s_{0,U}, \ldots, s_{k,U}) \in \Gamma(U, E_{k,k'}T_U^*),$$

and if gluing together, we have the global section [**Bro16**, Proposition 2.3]:

PROPOSITION 4.2.1. For any $s_0, \ldots, s_k \in H^0(X, L)$, the locally defined jet differential equations $W_U(s_0, \ldots, s_k)$ glue together into a global section

$$W(s_0, \ldots, s_k) \in H^0(X, E_{k,k'}T_X^* \otimes L^{k+1}).$$

The proof of the proposition follows from the fact that for any $s_U \in \Gamma(U, \mathscr{O}_U)$, we have

$$W_U(s_U s_{0,U}, \ldots, s_U s_{k,U}) = s_U^{k+1} W_U(s_{0,U}, \ldots, s_{k,U}).$$

We will denote by

$$\omega(s_0, \ldots, s_k) = (\pi_{0,k})_*^{-1} W(s_0, \ldots, s_k) \in H^0(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{k+1}) \tag{4.2.11}$$

the inverse image of the Wronskian $W(s_0, \ldots, s_k)$ under the global isomorphism (4.2.2) induced by the direct image $(\pi_{0,k})_*$.

Now let

$$\mathbb{W}(X_k, L) := \mathrm{Span}\{\omega(s_0, \ldots, s_n) | s_0, \ldots, s_n \in H^0(X, L)\} \subset H^0\big(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^*(L^{k+1})\big)$$

be the associated sublinear system of $H^0\big(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^*(L^{k+1})\big)$. One defines the *k-th Wronskian ideal sheaf* of L, denoted by $\mathfrak{w}(X_k, L)$, to be the base ideal $\mathfrak{b}\big(\mathbb{W}(X_k, L)\big)$ of the linear system $\mathbb{W}(X_k, L)$.

By the definition, if A is any line bundle on X, and $s \in H^0(X, A)$, we have

$$W(s \cdot s_0, \ldots, s \cdot s_k) = s^{k+1} W(s_0, \ldots, s_k) \in H^0(X, E_{k,k'}T_X^* \otimes L^{k+1} \otimes A^{k+1}).$$

Thus if L is very ample we have a chain of inclusions

$$\mathfrak{w}(X_k, L) \subset \mathfrak{w}(X_k, L^2) \subset \ldots \subset \mathfrak{w}(X_k, L^m) \subset \ldots.$$

By the Noetherianity, this increasing sequence stabilizes after some $m_\infty(X_k, L)$, and we denote the obtained asymptotic ideal by

$$\mathfrak{w}_\infty(X_k, L) := \mathfrak{w}(X_k, L^m) \quad \text{for any } m \geqslant m_\infty(X_k, L). \tag{4.2.12}$$

An important property for $\mathfrak{w}(X_k, L)$ is the following in [**Bro16**, Lemma 2.4]:

LEMMA 4.2.3. If L generates k-jets at every point of X, that is, for any $x \in X$, the map

$$H^0(X, L) \to L \otimes \mathscr{O}_{X,x}/\mathcal{M}_{X,x}^{k+1} = (J^k L)_x$$

is surjective, where $\mathcal{M}_x$ is the maximal ideal of $\mathscr{O}_x$, then

$$\operatorname{Supp}(\mathscr{O}_{X_k}/\mathfrak{w}(X_k, L)) \subset X_k^{\text{sing}}.$$

For any very ample line bundle L, assume that $L|_U$ can be trivialized. Now we will compare the globally defined asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(X_k, L)$ with our locally defined $\mathfrak{w}_U$.

When restricted to $U_k := \pi_{0,k}^{-1}(U)$, the global map

$$\omega(\bullet) : \wedge^{k+1} H^0(X, \mathscr{O}_X(L)) \to H^0\big(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^*(L^{k+1})\big)$$

defined in (4.2.11) can be localized as the following

$$\begin{array}{ccccccc}
\omega_U : \ \wedge^{k+1} H^0(X, \mathscr{O}_X(L)) & \xrightarrow{i_{[k]}} & \wedge^{k+1} H^0(U, J^k L|_U) & \longrightarrow & H^0(U, \wedge^{k+1} J^k \mathscr{O}_U) & \xrightarrow{\tilde{W}_U} & H^0(U, E_{k,k'} T_U^*) \\
\downarrow \mathbb{1} & & \wr\!\downarrow \Psi_U^{-1} & & \wr\!\downarrow \Psi_U^{-1} & & \wr\!\downarrow \varphi_U^{-1} \\
\wedge^{k+1} H^0(X, \mathscr{O}_X(L)) & \xrightarrow{\Psi_U^{-1} \circ i_{[k]}} & \wedge^{k+1} H^0(U, U \times \mathbb{C}^{I_k}) & \xrightarrow{l_k} & H^0(U, U \times \wedge^{k+1} \mathbb{C}^{I_k}) & \xrightarrow{\tilde{\nu}} & H^0(U, U \times F^{k,k'}).
\end{array}$$

where $H^0(U, U \times \wedge^{k+1}\mathbb{C}^{I_k})$, $H^0(U, U \times \mathbb{C}^{I_k})$ and $H^0(U, U \times F^{k,k'})$ are the sets of sections of the trivial bundles, and we also use the relation $\mathscr{O}_{X_k}(k')|_{U_k} = \operatorname{pr}_2^*(\mathscr{O}_{\mathcal{R}_{n,k}}(k'))$ in (4.2.9) to identify

$$H^0(U, U \times F^{k,k'}) \cong H^0\big(U_k, \mathscr{O}_{X_k}(k')|_{U_k}\big).$$

Then by the definition we have

$$\mathfrak{w}(X_k, L)|_{U_k} = \mathfrak{b}\big(\{\omega_U(s_0 \wedge \ldots \wedge s_k) | s_0, \ldots, s_k \in H^0(X, L)\}\big).$$

Now we choose arbitrary sections $s_0, \ldots, s_k \in H^0\big(X, \mathscr{O}_X(L)\big)$, we have

$$h(s_0 \wedge \ldots \wedge s_k) := l_k \circ \Psi_U^{-1} \circ i_{[k]}(s_0 \wedge \ldots \wedge s_k) \in \Gamma(U, U \times \wedge^{k+1}\mathbb{C}^{I_k}),$$

which is a holomorphic section of the trivial bundle $U \times \wedge^{k+1}\mathbb{C}^{I_k} \to U$. Thus

$$\omega_U(s_0 \wedge \ldots \wedge s_k) = \tilde{\nu} \circ h(s_0 \wedge \ldots \wedge s_k)$$

is a a holomorphic section of the trivial bundle $U \times F^{k,k'} \to U$, where $\tilde{\nu} : \wedge^{k+1}\mathbb{C}^{I_k} \to F^{k,k'}$ is a $\mathbb{C}$-linear map.

Recall that

$$\mathcal{I}_{n,k} := \mathfrak{b}(\operatorname{Image}(\tilde{\nu})) \subset \mathscr{O}_{\mathcal{R}_{n,k}}$$

is the base ideal of the linear system $\operatorname{Image}(\tilde{\nu}) \subset H^0\big(\mathcal{R}_{n,k}, \mathscr{O}_{\mathcal{R}_{n,k}}(k')\big)$, and $\mathfrak{w}_U$ is defined to be the ideal sheaf $\operatorname{pr}_2^*(\mathcal{I}_{n,k})$ on U_k. Thus the zero scheme of $\tilde{\nu} \circ h(s_0 \wedge \ldots \wedge s_k)$ is contained in $\mathfrak{w}_U$. As $s_0, \ldots, s_k$ are arbitrary, we always have

$$\mathfrak{w}(X_k, L)|_{U_k} \subset \mathfrak{w}_U. \tag{4.2.13}$$

On the other hand, suppose that the line bundle L generates k-jets, *i.e.*, the $\mathbb{C}$-linear map

$$H^0(X, L) \to (J^k L)_x$$

is surjective for any $x \in X$. Then for any $x \in U$, any vector $e \in \wedge^{k+1}\mathbb{C}^{I_k}$, there exists $r(k+1)$ sections $\{s_{ji}\}_{0 \leqslant j \leqslant k, 1 \leqslant i \leqslant r} \in H^0(X, L)$ such that

$$e = \sum_{i=1}^{r} h(s_{0i} \wedge \ldots \wedge s_{ki})(x).$$

Therefore, the set of images $\omega_U(\bullet)(x) = F^{k,k'}$ for any $x \in U$, and thus the ideal sheaf $\mathfrak{w}(X_k, L)$, when restricted to each fiber $x \times \mathcal{R}_{n,k} \subset U_k$, is equal to $\mathcal{I}_{n,k}$. That is, if we denote by

$$i_x : \mathcal{R}_{n,k} \to U_k$$

which is induced the inclusive map $x \times \mathcal{R}_{n,k} \to U \times \mathcal{R}_{n,k}$, the inverse image of $\mathfrak{w}(X_k, L)$ under i_x

$$i_x^*(\mathfrak{w}(X_k, L)) := i_x^{-1}\mathfrak{w}(X_k, L) \otimes_{i_x^{-1}\mathscr{O}_{U_k}} \mathscr{O}_{\mathcal{R}_{n,k}}$$

is the same as $\mathcal{I}_{n,k}$. Thus we have

$$\mathfrak{w}(X_k, L)|_{U_k} = \mathfrak{w}_U. \tag{4.2.14}$$

As U is any open set on X with local coordinates $(z_1, \ldots, z_n)$ such that $L|_U$ can be trivialized, from the inclusive relation (4.2.13) we see that

$$\mathfrak{w}(X_k, L) = \mathfrak{w}(X_k, L^2) = \ldots = \mathfrak{w}(X_k, L^k) = \ldots,$$

and thus we conclude that, for any ample line bundle L which generates k-jets everywhere, the k-th Wronskian ideal sheaf of L coincides with the asymptotic ideal sheaf

$$\mathfrak{w}(X_k, L) = \mathfrak{w}_\infty(X_k, L).$$

Moreover, from the local relation (4.2.14), we see that this asymptotic ideal sheaf does not depend on the choice of the very ample line bundle L, which was also proved by Brotbek in [**Bro16**, Lemma 2.6]. We denote by $\mathfrak{w}_\infty(X_k)$ the asymptotic Wronskian ideal sheaf.

In conclusion, we have the following theorem:

THEOREM 4.2.4. *If L generates k-jets at each point of X, then $\mathfrak{w}(X_k, L) = \mathfrak{w}_\infty(X_k)$ and $m_\infty(X_k, L) = 1$. In particular, if L is known to be only very ample, we have $\mathfrak{w}(X_k, L^k) = \mathfrak{w}_\infty(X_k)$ and $m_\infty(X_k, L) = k$.*

As was shown in [**Bro16**, Lemma 2.6], $\mathfrak{w}_\infty(X_k)$ behaves well under restriction, that is, for any directed variety (Y, V_Y) with $Y \subset X$ and $V_Y \subset V_X|_Y$, under the induced inclusion $Y_k \subset X_k$ one has

$$\mathfrak{w}_\infty(X_k)|_{Y_k} = \mathfrak{w}_\infty(Y_k).$$

4.2.4. BLOW-UPS OF THE WRONSKIAN IDEAL SHEAF. This subsection are mainly borrowed from [**Bro16**]. We will state some important results without proof, and the readers who are interested in the details are encouraged to refer to [**Bro16**, Section 2.4].

From [**Dem95**, Theorem 6.8], $\mathscr{O}_{X_k}(1)$ is only relatively big, and X_k^{Sing} is the obstruction to the ampleness of $\mathscr{O}_{X_k}(1)$. However, for the hyperbolicity problems, X_k^{Sing} is negligible since X_k is a relative compactification of $J_k^{\mathrm{reg}}/\mathbb{G}_k = X_k^{\mathrm{reg}}$ over X, and for every non-constant entire curve f on X, its k-th lift $f_{[k]} : \mathbb{C} \to X_k$ can not be contained in X_k^{Sing}. Thus we want to find a *good and fonctorial* compactification of X_k^{reg} such that the tautological line bundle is ample. Brotbek introduced a clever way to overcome this difficulty.

For any directed manifold (X, V), we denote by

$$\hat{X}_k := \mathrm{Bl}_{\mathfrak{w}_\infty(X_k)}(X_k) \to X_k$$

the blow-up of X_k along $\mathfrak{w}_\infty(X_k)$, and F the effective Cartier divisor on $\hat{X}_k$ such that

$$\mathscr{O}_{\hat{X}_k}(-F) = \nu_k^{-1}\mathfrak{w}_\infty(X_k).$$

Take a very ample line bundle L on X, for any $m \geqslant 0$, and any $s_0, \ldots, s_k \in H^0(X, L^m)$, there exists

$$\hat{\omega}(s_0, \ldots, s_k) \in H^0\Big(\hat{X}_k, \nu_k^*\big(\mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{m(k+1)}\big) \otimes \mathscr{O}_{\hat{X}_k}(-F)\Big),$$

such that

$$\nu_k^*\omega(s_0, \ldots, s_k) = s_F \cdot \hat{\omega}(s_0, \ldots, s_k).$$

Here $s_F \in H^0(\hat{X}_k, F)$ is the tautological section. Then by Theorem 4.2.4, for any $\hat{w} \in \hat{X}_k$ and any $m \geqslant k$, there exists $s_0, \ldots, s_k \in H^0(X, L^m)$ such that

$$\hat{\omega}(s_0, \ldots, s_k)(\hat{w}) \neq 0.$$

The blow-ups is fonctorial thanks to the fact that the asymptotic Wronskian ideal sheaf behaves well under restriction. Namely, if $(Y, V_Y) \subset (X, V_X)$ is a sub-directed variety, then $\hat{Y}_k$ is the strict transform of Y_k in X_k under the blowing-up morphism $\nu_k : \hat{X}_k \to X_k$. This fonctorial property also holds for families [**Bro16**, Proposition 2.7]:

THEOREM 4.2.5. *Let $\mathscr{X} \xrightarrow{\rho} T$ be a smooth and projective morphism between non-singular varieties. We denote by $\mathscr{X}_k^{\mathrm{rel}}$ the k-th Demailly-Semple tower of the relative directed variety $(\mathscr{X}, T_{\mathscr{X}/T})$. Take $\nu_k : \hat{\mathscr{X}}_k^{\mathrm{rel}} \to \mathscr{X}_k^{\mathrm{rel}}$ to be the blow-ups of the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(\mathscr{X}_k^{\mathrm{rel}})$. Then for any $t_0 \in T$ writing $X_{t_0} := \rho^{-1}(t_0)$, we have*

$$\nu_k^{-1}(X_{t_0,k}) = \hat{X}_{t_0,k}.$$

4.3. Proof of the Main Theorems

4.3.1. Families of Complete Intersections of Fermat-type Hypersurfaces. Let X be a projective manifold of dimension n endowed with a very ample line bundle A. We first construct a family of complete intersection subvarieties in X cut out by certain Fermat-type hypersurfaces. For an integer $N \geqslant n$, we fix $N+1$ sections in general position $\tau_0, \ldots, \tau_N \in H^0(X, A)$. By "general position" we mean that the hypersurfaces $\{\tau_i = 0\}_{i=0,\ldots,N}$ are all smooth and irreducibles ones, and they are simple normal crossing. For any $1 \leqslant c \leqslant n-1$, and two c-tuples of positive integers $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_c), \boldsymbol{\delta} = (\delta_1, \ldots, \delta_c)$, we construct the family $\mathscr{X}$ as follows: For any $p = 1, \ldots, c$, set $\mathbb{I}^p := \{I = (i_0, \ldots, i_N) \| I| = \delta_p\}$ and $\mathbf{a}^p := \Big(a_I^p \in H^0\big(X, \mathscr{O}_X(\epsilon_p A)\big)\Big)_{|I|=\delta_p}$. For the positive integers r and k fixed later according to our needs, we define the bihomogenous sections of $\mathscr{O}_X\Big(\big(\epsilon_p + (r+k)\delta_p\big)A\Big)$ over X by

$$\boldsymbol{F}^p(\mathbf{a}^p)(x) : x \mapsto \sum_{|I|=\delta_p} a_I^p(x)\boldsymbol{\tau}(x)^{(r+k)I},$$

where $\mathbf{a}^p$ varies in the parameter space $S_p := \bigoplus_{I \in \mathbb{I}^p} H^0\big(X, \mathscr{O}_X(\epsilon_p A)\big)$, and $\boldsymbol{\tau} := (\tau_0, \ldots, \tau_N)$.

We then consider the family $\overline{\mathscr{X}} \subset S_1 \times \ldots \times S_c \times X$ of complete intersection varieties in X defined by those sections:

$$\overline{\mathscr{X}} := \{(\mathbf{a}^1, \ldots, \mathbf{a}^c, x) \in S_1 \times \ldots \times S_c \times X | \boldsymbol{F}^1(\mathbf{a}^1)(x) = \ldots = \boldsymbol{F}^c(\mathbf{a}^c)(x) = 0\}. \tag{4.3.1}$$

We know that there is a non-empty Zariski open set $S \subset S_1 \times \ldots \times S_c$ parametrizing smooth varieties and we will work on $\mathscr{X} := q_1^{-1}(S) \cap \overline{\mathscr{X}}$, where q_1 is the natural projection from $S_1 \times \ldots \times S_c \times X$ to $S_1 \times \ldots \times S_c$. Set $\mathscr{X}_k$ to be the k-th Demailly-Semple tower of the relative tangent bundle $(\mathscr{X}, T_{\mathscr{X}/S})$, and $\hat{\mathscr{X}}_k$ the blowing-up of the asmptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(\mathscr{X}_k)$, and we would like to construct a regular morphism from $\hat{\mathscr{X}}_k$ (after shrinking a bit) to a suitable generically finite to one family and to "pull-back" the positivity from the parameter space of this family. First we begin with a technical lemma by Brotbek [**Bro16**, Lemma 3.2]:

Lemma 4.3.1. *Let U be an open subset of X on which both A and T_X can be trivialized. Fix any $1 \leqslant p \leqslant c$. For any $I = (i_0, \ldots, i_N) \in \mathbb{I}^p$, there exists a $\mathbb{C}$-linear map*

$$d_{I,U}^{[j]} : H^0(X, \epsilon_p A) \to \Gamma(U, E_{k,k}^{\mathrm{GG}} T_U^*)$$

such that for any $a \in H^0(X, \epsilon_p A)$, $d_U^{[j]}(a\boldsymbol{\tau}^{(r+k)I}) = \boldsymbol{\tau}_U^{rI} d_{I,U}^{[j]}(a)$, where $\boldsymbol{\tau}_U := (\tau_{0,U}, \ldots, \tau_{N,U})$ is the local trivialization of $\boldsymbol{\tau}$ over U.

Therefore, for any $I_0, \ldots, I_k \in \mathbb{I}^p$ and any $a_{I_0}, \ldots, a_{I_k} \in H^0(X, \epsilon_p A)$ one can define

$$W_{U,I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}) := \begin{vmatrix} d_{I_0,U}^{[0]}(a_{I_0}) & \cdots & d_{I_k,U}^{[0]}(a_{I_k}) \\ \vdots & \ddots & \vdots \\ d_{I_0,U}^{[k]}(a_{I_0}) & \cdots & d_{I_k,U}^{[k]}(a_{I_k}) \end{vmatrix} \in \Gamma(U, E_{k,k'}^{\mathrm{GG}} T_U^*), \tag{4.3.2}$$

and by Lemma 4.3.1 we obtain

$$W_U(a_{I_0}\boldsymbol{\tau}^{(r+k)I_0}, \ldots, a_{I_k}\boldsymbol{\tau}^{(r+k)I_k}) = \boldsymbol{\tau}_U^{r(I_0+\ldots+I_k)} W_{U,I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}).$$

From Proposition 4.2.1 one can also glue them together

Lemma 4.3.2. *For any $I_0, \ldots, I_k \in \mathbb{I}^p$ and any $a_{I_0}, \ldots, a_{I_k} \in H^0(X, \epsilon_p A)$, the locally defined functions $W_{U,I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k})$ can be glued together into a global section*

$$W_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}) \in H^0(X, E_{k,k'} T_X^* \otimes A^{(k+1)(\epsilon_p + k\delta_p)})$$

such that

$$W(a_{I_0}\boldsymbol{\tau}^{(r+k)I_0}, \ldots, a_{I_k}\boldsymbol{\tau}^{(r+k)I_k}) = \boldsymbol{\tau}^{r(I_0+\ldots+I_k)} W_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}). \tag{4.3.3}$$

We denote by

$$\omega_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}) \in H^0\big(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^* A^{(k+1)(\epsilon_p + k\delta_p)}\big) \tag{4.3.4}$$

the inverse image of $W_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k})$ under the isomorphism (4.2.2), then by (4.3.3) we have

$$\omega(a_{I_0}\boldsymbol{\tau}^{(r+k)I_0}, \ldots, a_{I_k}\boldsymbol{\tau}^{(r+k)I_k}) = (\pi_{0,k}^*\boldsymbol{\tau})^{r(I_0+\ldots+I_k)} \omega_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}). \tag{4.3.5}$$

Hence for every $1 \leqslant p \leqslant c$ we can construct a rational map given by the Wronskians

$$\begin{aligned} \Phi^p : S_p \times X_k &\dashrightarrow P(\Lambda^{k+1}\mathbb{C}^{\mathbb{I}^p}) \\ (\mathbf{a}, w) &\mapsto ([\omega_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k})(w)])_{I_0,\ldots,I_k \in \mathbb{I}^p}, \end{aligned}$$

where $\mathbb{C}^{\mathbb{I}^p} := \oplus_{I\in\mathbb{I}^p}\mathbb{C} \simeq \mathbb{C}^{\binom{N+\delta_p}{\delta_p}}$.

<u>**Claim**</u> 4.3.1. Φ^p factors through the Plücker embedding

$$\mathrm{Pluc} : \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \hookrightarrow P(\Lambda^{k+1}\mathbb{C}^{\mathbb{I}^p}).$$

<u>Proof:</u> For any $w_0 \in X_k$, by Theorem 4.2.1, one can find an open neighborhood U_{w_0} of w_0 with $U_{w_0} \subset \pi_{0,k}^{-1}(U)$, where $A|_U$ can be trivialized; and a family of germs of curves $(f_w)_{w\in U_{w_0}}$ depending holomorphically on w with $(f_w)_{[k]}(0) = w$. Then for any $\mathbf{a} = (a_I)_{I\in\mathbb{I}^p} \in S_p$ and any $0 \leqslant j \leqslant k$, we denote by

$$d^{[j]}_{\bullet,w_0}(\mathbf{a}, w) := \big(d^{[j]}_{I,U}(a_I)(f'_w, f''_w, \ldots, f^{(k)}_w)\big)_{I\in\mathbb{I}^p} \in \mathbb{C}^{\mathbb{I}^p},$$

and the local rational map

$$\begin{array}{rcl} \Phi^p_{w_0} : S_p \times U_{w_0} & \dashrightarrow & \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \\ (\mathbf{a}, w) & \mapsto & \mathrm{Span}\big(d^{[0]}_{\bullet,w_0}(\mathbf{a}, w), \ldots, d^{[k]}_{\bullet,w_0}(\mathbf{a}, w)\big). \end{array} \tag{4.3.6}$$

We will show that this definition does not depend on the choice of w_0. Indeed, by Definition 4.3.2 one has $\Phi^p = \mathrm{Pluc}\circ\Phi^p_{w_0}$, which shows that Φ^p factor through Pluc and we still denote by $\Phi^p : S_p \times X_k \dashrightarrow \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$ by abuse of notation. ■

Recall that $\hat{X}_k$ is denoted to be the blow-up $\nu_k : \hat{X}_k \to X_k$ of the asymptotic k-th Wronskian ideal sheaf $\mathfrak{w}_\infty(X_k)$, such that $\nu_k^{-1}\mathfrak{w}_\infty(X_k) = \mathscr{O}_{\hat{X}_k}(-F)$ for some effective cartier divisor F on $\hat{X}_k$. First, we have the following

<u>**Claim**</u> 4.3.2. $\hat{\nu}_k$ induces a rational map

$$\hat{\Phi}^p : S_p \times \hat{X}_k \dashrightarrow \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}),$$

such that

$$\begin{array}{ccc} S_p \times \hat{X}_k & & \\ {\scriptstyle \mathbb{1}\times\nu_k}\downarrow & \overset{\hat{\Phi}^p}{\searrow} & \\ S_p \times X_k & \overset{\Phi^p}{\dashrightarrow} & \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \end{array}$$

<u>Proof:</u> By the definition of the asmptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(X_k)$, we have

$$\omega(a_{I_0}\boldsymbol{\tau}^{(r+k)I_0}, \ldots, a_{I_k}\boldsymbol{\tau}^{(r+k)I_k}) \in H^0\big(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^* A^{(k+1)\left(\epsilon_p+(k+r)\delta_p\right)} \otimes \mathfrak{w}_\infty(X_k)\big).$$

Since $(\pi_{0,k}^*\boldsymbol{\tau})^{r(I_0+\ldots+I_k)}$ does not vanish along any irreducible component of the zero scheme of $\mathfrak{w}_\infty(X_k)$, by (4.3.5) we see that

$$\omega_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}) \in H^0\big(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^* A^{(k+1)(\epsilon_p+k\delta_p)} \otimes \mathfrak{w}_\infty(X_k)\big),$$

and thus there exists

$$\hat{\omega}_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}) \in H^0\Big(\hat{X}_k, \nu_k^*\big(\mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^* A^{(k+1)(\epsilon_p+k\delta_p)}\big) \otimes \mathscr{O}_{\hat{X}_k}(-F)\Big),$$

such that

$$\nu_k^*\omega_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}) = s_F \cdot \hat{\omega}_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k}).$$

Therefore, if we define the rational map

$$\begin{array}{rcl} \Phi^p : S_p \times \hat{X}_k & \dashrightarrow & P(\Lambda^{k+1}\mathbb{C}^{\mathbb{I}^p}) \\ (\mathbf{a}, \hat{w}) & \mapsto & ([\hat{\omega}_{I_0,\ldots,I_k}(a_{I_0}, \ldots, a_{I_k})(\hat{w})])_{I_0,\ldots,I_k\in\mathbb{I}^p}, \end{array}$$

then on $\hat{X}_k\backslash F$ we have $\hat{\Phi}^p = \Phi^p \circ \nu_k$, and thus $\hat{\Phi}^p$ also factors through the Plücker embedding

$$\mathrm{Pluc} : \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \hookrightarrow P(\Lambda^{k+1}\mathbb{C}^{\mathbb{I}^p}).$$

■

We are going to show that ν_k partially resolves the indeterminacy of Φ^p. To clarify this, we need to introduce some notations. For any $x \in X$, we set

$$N_x := \#\{j \in \{0, \ldots, N\} \big| \tau_j(x) \neq 0\} \text{ and } \mathbb{I}^p_x := \{I \in \mathbb{I}^p, |\tau^I(x) \neq 0\}.$$

Since the τ_j's are in general position, and $N \geqslant n$, we have $N_x \geqslant 1$ for all $x \in X$. Then we define

$$\Sigma := \{x \in X \big| N_x = 1\} \text{ and } X^\circ := X\backslash\Sigma.$$

Observe that if $N > n$, then $X^\circ = X$, and if $N = n$, then Σ is a finite set of points. We denote by $\hat{X}^\circ_k := (\pi_{0,k} \circ \nu_k)^{-1}(X^\circ)$. We have the following crucial lemma of resolution of indeterminacy due to Brotbek [**Bro16**, Proposition 3.8]:

LEMMA 4.3.3. (Brotbek) Suppose that

$$(\star)\qquad N \geqslant n \geqslant 2, \quad k \geqslant 1, \quad \epsilon_p \geqslant m_\infty(X_k, A) = k \text{ and } \delta_p \geqslant n(k+1).$$

Then there exists a non-empty Zariski open subset $U_{\mathrm{def},p} \subset S_p$ such that the restriction $\hat{\Phi}^p|_{U_{\mathrm{def},p}\times \hat{X}_k^\circ}$ is a morphism:

$$\begin{array}{ccc} U_{\mathrm{def},p} \times \hat{X}_k^\circ & & \\ {\scriptstyle \mathbb{1}\times\nu_k}\downarrow & \overset{\hat{\Phi}^p}{\searrow} & \\ U_{\mathrm{def},p} \times X_k^\circ & \overset{\Phi^p}{\dashrightarrow} & \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \end{array}$$

In Lemma **??**rational, we have applied our Theorem 4.2.4 to set $m_\infty(X_k, A) = k$.

4.3.2. MAPPING TO THE UNIVERSAL GRASSMANNIAN. Set $U_{\mathrm{def}} := U_{\mathrm{def},1}\times\ldots\times U_{\mathrm{def},c}\bigcap S$. We suppose from now on that $N \geqslant n \geqslant 2$, $\epsilon_p \geqslant k \geqslant 1$ and that $\delta_p \geqslant n(k+1)$ for any $1 \leqslant p \leqslant c$. Then by Lemma 4.3.3 we get a regular morphism

$$\begin{array}{rcl} \Psi : U_{\mathrm{def}} \times \hat{X}_k^\circ & \to & \mathbf{G}_{k+1}(\delta_1) \times \ldots \times \mathbf{G}_{k+1}(\delta_c) \times \mathbb{P}^N \\ (\mathbf{a}, \xi) & \mapsto & \big(\hat{\Phi}^1(\mathbf{a}^1, \xi), \ldots, \hat{\Phi}^c(\mathbf{a}^c, \xi), [\tau^r(\xi)]\big). \end{array}$$

$[\tau^r(\xi)] := \big[\tau_0^r\big(\pi_{0,k}\circ\nu_k(\xi)\big) : \ldots : \tau_N^r\big(\pi_{0,k}\circ\nu_k(\xi)\big)\big]$, and we write $\mathbf{G}_{k+1}(\delta_p) := \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$ and $\mathbf{G} := \mathbf{G}_{k+1}(\delta_1)\times\ldots\mathbf{G}_{k+1}(\delta_c)$ for brevity.

From now on we *always* assume that $(k+1)c \geqslant N$. Using the natural identification

$$\begin{array}{rcl} \mathbb{C}^{\mathbb{I}^p} & \to & H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\delta_i)\big) \\ (a_I)_{I\in\mathbb{I}^p} & \mapsto & \sum\limits_{I\in\mathbb{I}^p} a_I z^I, \end{array}$$

we set $\mathscr{Y}$ to be the *universal Grassmannian* defined by

$$\mathscr{Y} := \{(\Delta_1, \ldots, \Delta_c, [z]) \in \mathbf{G}\times\mathbb{P}^N | \forall 1 \leqslant i \leqslant c, \forall P \in \Delta_i : P([z]) = 0\}.$$

If we denote by $p : \mathscr{Y} \to \mathbf{G}$ the first projection map, then p is a generically finite to one (may not surjective) morphism. Set G^∞ to be the set of points in $\mathbf{G} := \mathbf{G}_{k+1}(\delta_1)\times\ldots\mathbf{G}_{k+1}(\delta_c)$ such that the fiber of $p : \mathscr{Y} \to \mathbf{G}$ is not a finite set, and we say that G^∞ is the *non-finite loci* of $\mathbf{G}$.

We need to cover X by a natural stratification induced by the vanishing of the τ_j's. For any $J \subset \{0, \ldots, N\}$ and $1 \leqslant p \leqslant c$ we define

$$\begin{array}{rl} X_J & := \{x \in X \big| \tau_j(x) = 0 \Leftrightarrow j \in J\}, \\ \mathbb{I}_J^p & := \{I \in \mathbb{I}^p \big| \mathrm{Supp}(I) \subset \{0, \ldots, N\}\backslash J\}, \\ \mathbb{P}_J & := \{[z] \in \mathbb{P}^N \big| z_j = 0 \text{ iff } j \in J\}, \end{array}$$

$\hat{X}_{k,J} := (\pi_{0,k}\circ\nu_k)^{-1}(X_J)$ and $\hat{X}_{k,J}^\circ := \hat{X}_{k,J}\cap\hat{X}_k^\circ$. Set $\mathscr{Y}_J := \mathscr{Y}\cap\big(\mathbf{G}\times\mathbb{P}_J\big) \subset \mathbf{G}\times\mathbb{P}^N$, and G_J^∞ also the set of points in $\mathbf{G}$ such that the fiber of the first projection map $p_J : \mathscr{Y}_J \to \mathbf{G}$ is not a finite set.

Now set

$$U_{\mathrm{def},p}^\circ := U_{\mathrm{def},p}\cap\big\{\mathbf{a}^p \in S_p \big| \{\boldsymbol{F}^p(\mathbf{a}^p)(x) = 0\}\cap\Sigma = \varnothing\big\} \text{ and } U_{\mathrm{def}}^\circ := U_{\mathrm{def},1}^\circ\times\ldots\times U_{\mathrm{def},c}^\circ\cap U_{\mathrm{def}}.$$

Since Σ is a finite set, $U_{\mathrm{def},p}^\circ$ is a non-empty Zariski open subset of $U_{\mathrm{def},p}$ for each p. Consider the universal family of codimension c smooth varieties $\mathscr{H} := (U_{\mathrm{def}}^\circ\times X)\cap\mathscr{X}$, then

$$\mathscr{H}\cap\{U_{\mathrm{def}}^\circ\times\Sigma\} = \varnothing. \tag{4.3.7}$$

We denote by $\mathscr{H}_k^{\mathrm{rel}}$ the k-th Demailly-Semple tower of the relative directed variety $(\mathscr{H}, T_{\mathscr{H}/U_{\mathrm{def}}^\circ})$. If $\hat{\mathscr{H}}_k^{\mathrm{rel}}$ is obtained by the blowing-up of the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(\mathscr{H}_k^{\mathrm{rel}})$, then by the arguments in Section 4.2.4 we have

$$(\mathbb{1}\times\nu_k)^{-1}(\mathscr{H}_k^{\mathrm{rel}}) = \hat{\mathscr{H}}_k^{\mathrm{rel}}.$$

Moreover for any $\mathbf{a}\in U_{\mathrm{def}}^\circ$, if we denote by $H_{\mathbf{a},k} := \mathscr{H}_k^{\mathrm{rel}}\cap(\{\mathbf{a}\}\times X_k)$ and $\hat{H}_{\mathbf{a},k} := \hat{\mathscr{H}}_k^{\mathrm{rel}}\cap(\{\mathbf{a}\}\times\hat{X}_k)$, then $\nu_k|_{\hat{H}_{\mathbf{a},k}} : \hat{H}_{\mathbf{a},k} \to H_{\mathbf{a},k}$ is indeed the blowing-up of the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(H_{\mathbf{a},k})$. By (4.3.7), $\Psi|_{\hat{\mathscr{H}}_k^{\mathrm{rel}}}$ is a regular morphsim. Set

$$\hat{\mathscr{H}}_{k,J}^{\mathrm{rel}} := \hat{\mathscr{H}}_k^{\mathrm{rel}}\cap(U_{\mathrm{def}}^\circ\times\hat{X}_{k,J}),$$

and we have the following

PROPOSITION 4.3.1. For any $J \subset \{0,\dots,N\}$, when restricted to $\hat{\mathscr{H}}^{\mathrm{rel}}_{k,J}$ the morphism Ψ factors through $\mathscr{Y}_J$:

$$\Psi|_{\hat{\mathscr{H}}^{\mathrm{rel}}_{k,J}} : \hat{\mathscr{H}}^{\mathrm{rel}}_{k,J} \to \mathscr{Y}_J \subset \mathbf{G} \times P_J.$$

PROOF. Since when restricted to $U^{\circ}_{\mathrm{def}} \times \hat{X}^{\circ}_{k,J}$, Ψ factors through $\mathbf{G} \times \mathbb{P}_J$. Thus it suffices to prove that $\Psi|_{\hat{\mathscr{H}}^{\mathrm{rel}}_{k}}$ factors through $\mathscr{Y}$. Since $\Phi^p = \hat{\Phi}^p \circ \nu_k$, it suffices to prove that the rational map

$$\begin{array}{rcl} \tilde{\Psi} : S \times X_k & \dashrightarrow & \mathbf{G} \times \mathbb{P}^N \\ (\mathbf{a}, w) & \mapsto & \big(\Phi^1(\mathbf{a}^1, w), \dots, \Phi^c(\mathbf{a}^c, w), [\tau^r(w)]\big) \end{array}$$

factor through $\mathscr{Y}$ when restricted to $\mathscr{H}^{\mathrm{rel}}_k$. Take any $(\mathbf{a}, w_0) \in \mathscr{H}^{\mathrm{rel}}_k$ outside the indeterminacy of $\tilde{\Psi}$, and by Lemma 4.2.1 one can find a germ of curve $f : (\mathbb{C}, 0) \to \big(X, x := \pi_{0,k}(w_0)\big)$ with $f_{[k]}(0) = w_0$. Recall that $H_{\mathbf{a},k} := \mathscr{H}^{\mathrm{rel}}_k \cap (\{\mathbf{a}\} \times X_k)$ is the k-th Demailly-Semple tower of $(H_{\mathbf{a}}, T_{H_{\mathbf{a}}})$. Therefore, we have $\big(f(0), f'(0), \dots, f^{(k)}(0)\big) \in J_k H_{\mathbf{a}}$.

Take an open subset $U \subset X$ containing x such that $A|_U$ can be trivialized. Since $H_{\mathbf{a}}$ is defined by the equations

$$\begin{cases} \boldsymbol{F}^1(\mathbf{a}^1)(x) := \sum_{|I|=\delta_1} a^1_I(x) \boldsymbol{\tau}(x)^{(r+k)I} = 0, \\ \vdots \\ \boldsymbol{F}^c(\mathbf{a}^c)(x) := \sum_{|I|=\delta_c} a^c_I(x) \boldsymbol{\tau}(x)^{(r+k)I} = 0, \end{cases}$$

then $d^{[j]}_U \boldsymbol{F}^p(\mathbf{a}^p)(f', f'', \dots, f^{(k)}) = 0$ for any $1 \leqslant p \leqslant c$ and $0 \leqslant j \leqslant k$. By Lemma 4.3.1 we have $d^{[j]}_U \boldsymbol{F}^p(\mathbf{a}^p) = \boldsymbol{\tau}^{rI}_U \cdot \sum_{|I|=\delta_p} d^{[j]}_{I,U}(a^p_I)$. By the definition for Φ^p (4.3.6), we see that $\tilde{\Psi}(\mathbf{a}, w_0) \in \mathscr{Y}$. This completes the proof of the Proposition. □

To proceed further, we need another important technical lemma in [**Bro16**, Lemma 3.4] as follows

LEMMA 4.3.4. Suppose that $\epsilon \geqslant m_\infty(X_k, A) = k$. Fix any $1 \leqslant p \leqslant c$. For any $\hat{w}_0 \in \hat{X}_k$, there exists an open neighborhood $\hat{U}_{\hat{w}_0} \subset \hat{X}_k$ of $\hat{w}_0$ satisfying the following. For any $I \in \mathbb{I}^p$ and $0 \leqslant i \leqslant k$ there exists a linear map

$$g^p_{i,I} : H^0(X, A^\epsilon) \to \mathscr{O}(\hat{U}_{\hat{w}_0})$$

such that for any $(\mathbf{a}^p, \hat{w}) \in S_p \times \hat{U}_{\hat{w}_0}$, writting $g^p_{i,\bullet}(\mathbf{a}^p, \hat{w}) = \big(g^p_{i,I}(a^p_I)(\hat{w})\big)_{I \in \mathbb{I}^p} \in \mathbb{C}^{\mathbb{I}^p}$ one has

(i) The Plücker coordinates of $\hat{\Phi}^p(\mathbf{a}^p, \hat{w})$ are all vanishing if and only if

$$\dim \mathrm{Span}\big(g^p_{0,\bullet}(\mathbf{a}^p, \hat{w}), \dots, g^p_{k,\bullet}(\mathbf{a}^p, \hat{w})\big) < k+1.$$

(ii) If $\dim \mathrm{Span}\big(g^p_{0,\bullet}(\mathbf{a}^p, \hat{w}), \dots, g^p_{k,\bullet}(\mathbf{a}^p, \hat{w})\big) = k+1$, then

$$\hat{\Phi}^p(\mathbf{a}^p, \hat{w}) = \mathrm{Span}\big(g^p_{0,\bullet}(\mathbf{a}^p, \hat{w}), \dots, g^p_{k,\bullet}(\mathbf{a}^p, \hat{w})\big) \in \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}).$$

(iii) Define the linear map

$$\begin{array}{rcl} \hat{\varphi}^p_{\hat{w}_0} : S_p & \to & (\mathbb{C}^{\mathbb{I}^p})^{k+1} \\ \mathbf{a}^p & \mapsto & \big(g^p_{0,\bullet}(\mathbf{a}^p, \hat{w}_0), \dots, g^p_{k,\bullet}(\mathbf{a}^p, \hat{w}_0)\big). \end{array} \tag{4.3.8}$$

Set $x := \pi_{0,k} \circ \nu_k(\hat{w}_0)$ and $\rho^p_x : (\mathbb{C}^{\mathbb{I}^p})^{k+1} \to (\mathbb{C}^{\mathbb{I}^p_x})^{k+1}$ the natural projection map, then one has

$$\mathrm{rank} \rho^p_x \circ \hat{\varphi}^p_{\hat{w}_0} = (k+1)\#\mathbb{I}^p_x.$$

Here $\mathbb{I}^p_x := \{I \in \mathbb{I}^p | \boldsymbol{\tau}^I(x) \neq 0\}$.

Now we are ready to prove the following lemma, which is a variant of [**Bro16**, Lemma 3.9]:

LEMMA 4.3.5. (Avoiding exceptional locus) For any $J \subset \{0,\dots,N\}$. If $\delta_p \geqslant (n-1)(k+1)+1$ for any $p = 1,\dots,c$, then there exists a non-empty Zariski open subset $U_J \subset U^{\circ}_{\mathrm{def}}$ such that

$$\hat{\Phi}^{-1}(G^{\infty}_J) \cap (U_J \times \hat{X}^{\circ}_{k,J}) = \varnothing.$$

Here we define the map (which is a morphism by Lemma 4.3.3)

$$\begin{array}{rcl} \hat{\Phi} : U^{\circ}_{\mathrm{def}} \times \hat{X}^{\circ}_k & \to & \mathbf{G}_{k+1}(\delta_1) \times \ldots \times \mathbf{G}_{k+1}(\delta_c) \\ (\mathbf{a}, \xi) & \mapsto & \big(\hat{\Phi}^1(\mathbf{a}^1, \xi), \dots, \hat{\Phi}^c(\mathbf{a}^c, \xi)\big), \end{array}$$

which is the composition $\pi_1 \circ \Psi$. Here $\pi_1 : \mathbf{G} \times \mathbb{P}^N \to \mathbf{G}$ is the first projection.

PROOF. Fix any $\hat{w}_0 \in \hat{X}_k^\circ$, we set $x := \pi_{0,k} \circ \nu_k(\hat{w}_0)$. Then there exists a unique $J \subset \{0, \ldots, N\}$ such that $x \in X_J$, and we define the following analogues of $\mathscr{Y}$ parametrized by affine spaces

$$\widetilde{\mathscr{Y}} := \{\big(\alpha_{10}, \ldots, \alpha_{1k}, \ldots, \alpha_{c0} \ldots, \alpha_{ck}, [z]\big) \in \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p})^{k+1} \times \mathbb{P}_J \big| \forall 1 \leqslant p \leqslant c, 0 \leqslant j \leqslant k, \alpha_{pi}([z]) = 0\},$$

$$\widetilde{\mathscr{Y}_x} := \{\big(\alpha_{10}, \ldots, \alpha_{1k}, \ldots, \alpha_{c0} \ldots, \alpha_{ck}, [z]\big) \in \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p_J})^{k+1} \times \mathbb{P}_J \big| \forall 1 \leqslant p \leqslant c, 0 \leqslant j \leqslant k, \alpha_{pi}([z]) = 0\},$$

here we use the identification $\mathbb{C}^{\mathbb{I}^p} \cong H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\delta_p)\big)$ and $\mathbb{C}^{\mathbb{I}^p_J} \cong H^0\big(\mathbb{P}_J, \mathscr{O}_{\mathbb{P}_J}(\delta_p)\big)$. By analogy with G_J^∞, we denote by $V_{1,J}^\infty$ (resp. $V_{2,J}^\infty$) the set of points in $\prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p})^{k+1}$ (resp. $\prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p_J})^{k+1}$) at which the fiber in $\widetilde{\mathscr{Y}}$ (resp. $\widetilde{\mathscr{Y}_x}$) is positive dimensional.

For every $1 \leqslant p \leqslant c$ we take the linear map $\hat{\varphi}^p_{\hat{w}_0} : S_p \to (\mathbb{C}^{\mathbb{I}^p})^{k+1}$ as in (4.3.8). By Lemma 4.3.4, for any $\mathbf{a} \in U_{\mathrm{def}}^\circ$ we have

$$\hat{\Phi}(\mathbf{a}, \hat{w}_0) = \big([\hat{\varphi}^1_{\hat{w}_0}(\mathbf{a}^1)], \ldots, [\hat{\varphi}^c_{\hat{w}_0}(\mathbf{a}^c)]\big),$$

here $[\hat{\varphi}^p_{\hat{w}_0}(\mathbf{a}^p)] := \mathrm{Span}\big(g^p_{0,\bullet}(\mathbf{a}^p, \hat{w}_0), \ldots, g^p_{k,\bullet}(\mathbf{a}^p, \hat{w}_0)\big) \in \mathrm{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$. Then we have

$$\hat{\Phi}^{-1}(G_J^\infty) \cap (U_{\mathrm{def}}^\circ \times \{\hat{w}_0\}) = \hat{\varphi}^{-1}_{\hat{w}_0}(V_{1,J}^\infty) \cap U_{\mathrm{def}}^\circ = (\rho_x \circ \hat{\varphi}_{\hat{w}_0})^{-1}(V_{2,J}^\infty) \cap U_{\mathrm{def}}^\circ,$$

where we denote by

$$\begin{array}{rcl} \hat{\varphi}_{\hat{w}_0} : S_1 \times \ldots S_c & \to & \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p})^{k+1} \\ \mathbf{a} = (\mathbf{a}^1, \ldots, \mathbf{a}^c) & \mapsto & \big(\hat{\varphi}^1_{\hat{w}_0}(\mathbf{a}^1), \ldots, \hat{\varphi}^c_{\hat{w}_0}(\mathbf{a}^c)\big), \end{array}$$

and

$$\rho_x : \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p})^{k+1} \to \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p_x})^{k+1}$$

is the natural projection map. By the above notations we have $\mathbb{I}^p_J = \mathbb{I}^p_x$ for any $p = 1, \ldots, c$. Since the linear map $\rho_x \circ \hat{\varphi}_{\hat{w}_0}$ is diagonal by blocks, by Lemma 4.3.4 we have

$$\mathrm{rank} \rho_x \circ \hat{\varphi}_{\hat{w}_0} = \sum_{p=1}^{c} \mathrm{rank} \rho^p_x \circ \hat{\varphi}^p_{\hat{w}_0} = \sum_{p=1}^{c} (k+1) \# \mathbb{I}^p_x.$$

Therefore

$$\begin{array}{rcl} \dim\big(\hat{\Phi}^{-1}(G_J^\infty) \cap (U_{\mathrm{def}}^\circ \times \{\hat{w}_0\})\big) & \leqslant & \dim\big((\rho_x \circ \hat{\varphi}_{\hat{w}_0})^{-1}(V_{2,J}^\infty)\big) \\ & \leqslant & \dim(V_{2,J}^\infty) + \dim \ker(\rho_x \circ \hat{\varphi}_{\hat{w}_0}) \\ & \leqslant & \dim(V_{2,J}^\infty) + \dim(S_1 \times \ldots \times S_c) - \mathrm{rank}(\rho_x \circ \hat{\varphi}_{\hat{w}_0}) \\ & = & \dim(V_{2,J}^\infty) + \dim(S_1 \times \ldots \times S_c) - \sum_{p=1}^{c} (k+1) \# \mathbb{I}^p_x. \end{array}$$

Since

$$\begin{array}{rcl} \dim(V_{2,J}^\infty) & = & \dim(\prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p_J})^{k+1}) - \mathrm{codim}(V_{2,J}^\infty, \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p_J})^{k+1}) \\ & = & \sum_{p=1}^{c} (k+1) \# \mathbb{I}^p_J - \mathrm{codim}(V_{2,J}^\infty, \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p_J})^{k+1}) \\ & = & \sum_{p=1}^{c} (k+1) \# \mathbb{I}^p_x - \mathrm{codim}(V_{2,J}^\infty, \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p_J})^{k+1}), \end{array}$$

then we have

$$\dim\big(\hat{\Phi}^{-1}(G_J^\infty) \cap U_{\mathrm{def}}^\circ \times \{\hat{w}_0\}\big) \leqslant \dim(S_1 \times \ldots \times S_c) - \mathrm{codim}(V_{2,J}^\infty, \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p_J})^{k+1}),$$

which yields

$$\dim\big(\hat{\Phi}^{-1}(G_J^\infty) \cap U_{\mathrm{def}}^\circ \times \hat{X}_{k,J}^\circ\big) \leqslant \dim(S_1 \times \ldots \times S_c) - \mathrm{codim}(V_{2,J}^\infty, \prod_{p=1}^{c} (\mathbb{C}^{\mathbb{I}^p_J})^{k+1}) + \dim \hat{X}_k.$$

By a result due to Olivier Benoist [**BD15**, Corollary 3.2], we have

$$\operatorname{codim}(V_{2,J}^{\infty}, \prod_{p=1}^{c}(\mathbb{C}^{\mathbb{I}_J^p})^{k+1}) \geqslant \min_{1\leqslant p\leqslant c} \delta_p + 1.$$

Therefore, if

$$(\clubsuit) \qquad \dim \hat{X}_k < \min_{1\leqslant p\leqslant c} \delta_p + 1,$$

$\hat{\Phi}^{-1}(G_J^{\infty})$ doesn't dominate $U_{\rm def}^{\circ}$ via the projection $U_{\rm def}^{\circ} \times \hat{X}_{k,J}^{\circ} \to U_{\rm def}^{\circ}$, and thus we can find a non-empty Zariski open subset $U_J \subset U_{\rm def}^{\circ}$ such that

$$\hat{\Phi}^{-1}(G_J^{\infty}) \cap (U_J \times \hat{X}_{k,J}^{\circ}) = \varnothing.$$

Thus if $\min_{1\leqslant p\leqslant c} \delta_p \geqslant (n-1)(k+1)+1$, Condition ♣ is always satisfied. We finish the proof of the lemma. □

4.3.3. Pull-back of the Positivity. For any c-tuple of positive integers $\mathbf{e} = (e_1, \ldots, e_c)$, we denote by

$$\mathcal{L}(\mathbf{e}) := \mathscr{O}_{\mathbf{G}_{k+1}(\delta_1)}(e_1) \boxtimes \ldots \boxtimes \mathscr{O}_{\mathbf{G}_{k+1}(\delta_c)}(e_c),$$

which is a very ample line bundle on $\mathbf{G}$. Since $p_J : \mathscr{Y}_J \to \mathbf{G}$ is a generically finite to one morphism, by the Nakamaye Theorem (see [**Laz04**, Theorem 10.3.5] for smooth projective varieties, and [**Bir13**, Theorem 1.3] for any projective scheme over any field), the augmented base locus $\mathbb{B}_+\big(p_J^*\mathcal{L}(\mathbf{e})\big)$ for $p_J^*\mathcal{L}(\mathbf{e})$ coincides with its *exceptional locus* (or say *null locus*)

$$E_J := \{y \in \mathscr{Y} | \dim_y \Big(p_J^{-1}\big(p_J(y)\big)\Big) > 0\},$$

which is contained in $p_J^{-1}(G_J^{\infty})$. Thus if $e_i \gg 0$ for each $1 \leqslant i \leqslant c$, we have

$$E_J = \operatorname{Bs}\big(p_J^*\mathcal{L}(\mathbf{e}) \otimes q_J^*\mathscr{O}_{P_J}(-1)\big) \subset p_J^{-1}(G_J^{\infty}), \tag{4.3.9}$$

where $q_J : \mathscr{Y}_J \to \mathbb{P}_J$ is denoted to be the second projection map. In Section 4.4, we obtain an effective estimate for $\mathbf{e}$ such that the inclusive relation in (4.3.9) holds. The theorem is the following

Theorem 4.3.1. *With the above notations, set* $b_p := \frac{\prod_{j=1}^{c}\delta_j^{k+1}}{\delta_p}$, *then for any* $J \subset \{0, \ldots, N\}$ *and any* $\mathbf{a} \in \mathbb{Z}^c$ *with* $a_p \geqslant b_p$ *for every* $1 \leqslant p \leqslant c$, *we have*

$$\operatorname{Bs}\big(p_J^*\mathcal{L}(\mathbf{a}) \otimes q_J^*\mathscr{O}_{\mathbb{P}^N}(-1)\big) \subset \operatorname{Bs}\big(p_J^*\mathcal{L}(\mathbf{b}) \otimes q_J^*\mathscr{O}_{\mathbb{P}^N}(-1)\big) \subset p_J^{-1}(G_J^{\infty}).$$

Since the technique in proving this theorem is of independent interest, we will leave the proof to Section 4.4.

Remark 4.3.1. Since $p_J^{-1}(G_J^{\infty})$ may strictly contain the null locus $\operatorname{Null}(p^*\mathscr{L}|_{\mathscr{Y}_J}) = E_J$, Theorem 4.3.1 does not imply the Nakamaye Theorem used in [**BD15**] and [**Bro16**]. That is, for some J and $\mathbf{a} \in \mathbb{N}^c$ with $a_j \geqslant b_j$ for every j, the Null locus E_J may be strictly contained in $p_J^*\mathcal{L}(\mathbf{a}) \otimes q_J^*\mathscr{O}_{\mathbb{P}^N}(-1)$. However, as we will see later, our proof of the Main Theorem only relies on the inclusive relation in (4.3.9). We thank Brotbek for pointing this important reduction to us.

By (4.3.4) we have

$$\Psi^*\big(\mathcal{L}(\mathbf{b}) \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)\big) = \nu_k^*\big(\mathscr{O}_{X_k}(\sum_{p=1}^{c} b_p k') \otimes \pi_{0,k}^* A^{-q(\boldsymbol{\epsilon},\boldsymbol{\delta},r)}\big) \otimes \mathscr{O}_{\hat{X}_k}(-\sum_{p=1}^{c} b_p F). \tag{4.3.10}$$

Here we set $q(\boldsymbol{\epsilon},\boldsymbol{\delta},r) := r - \sum_{p=1}^{c} b_p(k+1)(\epsilon_p + k\delta_p)$. Observe that if we take

$$(\spadesuit) \qquad \sum_{p=1}^{c} b_p(k+1)(\epsilon_p + k\delta_p) < r,$$

then (4.3.10) becomes an *invariant* k-jet differential with a negative twist, which enables us to apply Theorem 4.2.3 to constrain all the entire curves. More precisely, we have the following theorem:

Theorem 4.3.2. *On an* n*-dimensional smooth projective variety* X*, equipped with a very ample line bundle* A*. Let* c *be any integer satisfying* $1 \leqslant c \leqslant n-1$*. If we take* $k_0 = \lceil \frac{n}{c} \rceil - 1$ *and* $N = n$*, then for any degrees* $(d_1, \ldots, d_c) \in (\mathbb{N})^c$ *satisfying*

$$\exists \boldsymbol{\delta}_{\left(\delta_p \geqslant \delta_0 := n(k_0+1)\right)}, \ \exists \boldsymbol{\epsilon}_{(\epsilon_p \geqslant k_0)}, \ \exists r > \sum_{p=1}^{c} b_p(k_0+1)(\epsilon_p + k_0\delta_p), \ \ s.t.$$

$$d_p = \delta_p(r + k_0) + \epsilon_p \qquad (p = 1, \ldots, c), \tag{4.3.11}$$

the complete intersection $\mathbf{H} := H_1 \cap \ldots H_c$ *of general hypersurfaces* $H_1 \in |A^{d_1}|, \ldots, H_c \in |A^{d_c}|$ *has almost* k*-jet ampleness.*

PROOF. We will prove the theorem in several steps. First observe that, the choice for $(\boldsymbol{\epsilon}, \boldsymbol{\delta}, r, c, N, k)$ in the Theorem fufills all the requirements in Condition ★, ♠ and ♣, and thus we are free to apply all the corresponding theorems above. Based on the same vein in [**BD15**, **Bro16**], we have the following result

CLAIM 4.3.3. Set $U_{\rm nef} := \cap_J U_J$. For any $\mathbf{a} \in U_{\rm nef}$, the line bundle

$$\nu_k^*\big(\mathscr{O}_{X_k}(\sum_{p=1}^{c} b_p k') \otimes \pi_{0,k}^* A^{-q(\boldsymbol{\epsilon},\boldsymbol{\delta},r)}\big) \otimes \mathscr{O}_{\hat{X}_k}(-\sum_{p=1}^{c} b_p F)|_{\hat{H}_{\mathbf{a},k}}$$

is nef on $\hat{H}_{\mathbf{a},k}$. Recall that we denote by $q(\boldsymbol{\epsilon},\boldsymbol{\delta},r) := r - \sum_{p=1}^{c} b_p(k_0+1)(\epsilon_p + k_0\delta_p) > 0$.

Proof: In order to prove nefness, it suffices to show that for any irreducible curve, its intersection with the line bundle is non-negative. For any fixed $\mathbf{a} \in U_{\rm nef}$, and any irreducible curve $C \subset \hat{H}_{\mathbf{a},k}$, one can find the unique $J \subset \{0, \ldots, N\}$ such that $\hat{X}_{k,J} \cap C =: C^\circ$ is a non-empty Zariski open subset of C, and thus $C^\circ \subset \hat{\mathscr{H}}_{k,J}$. From Proposition 4.3.1, Ψ factors through $\mathscr{Y}_J$ when restricted to $\hat{\mathscr{H}}_{k,J}$, thus $\Psi|_{C^\circ}$ also factors through $\mathscr{Y}_J$, and by the properness of $\mathscr{Y}_J$, $\Psi|_C$ factors through $\mathscr{Y}_J$ as well. By Lemma 4.3.5 and the definition of $U_{\rm nef}$, we have

$$\hat{\Phi}(C^\circ) \cap G_J^\infty = \varnothing,$$

and thus

$$\Psi(C) \not\subset p_J^{-1}(G_J^\infty).$$

From Theorem 4.3.1 we know that

$$\mathrm{Bs}\big(p_J^*\mathcal{L}(\mathbf{b}) \otimes q_J^*\mathscr{O}_{P_J}(-1)\big) \subset p_J^{-1}(G_J^\infty),$$

which yields

$$\Psi(C) \cdot \big(p_J^*\mathcal{L}(\mathbf{b}) \otimes q_J^*\mathscr{O}_{P_J}(-1)\big) \geqslant 0.$$

From the relation (4.3.10) we obtain that

$$C \cdot \Big(\nu_k^*\big(\mathscr{O}_{X_k}(\sum_{p=1}^{c} b_p k') \otimes \pi_{0,k}^* A^{-q(\boldsymbol{\epsilon},\boldsymbol{\delta},r)}\big) \otimes \mathscr{O}_{\hat{X}_k}(-\sum_{p=1}^{c} b_p F)\Big) \geqslant 0,$$

which proves the claim. ■

By [**Dem95**, Proposition 6.16], we can find an ample line bundle on $\hat{X}^k$ of the form

$$\tilde{A} := \nu_k^*\big(\mathscr{O}_{X_k}(a_1, \ldots, a_k) \otimes \pi_{0,k}^* A^{a_0}\big) \otimes \mathscr{O}_{\hat{X}_k}(-F)$$

for some $a_0, \ldots, a_k \in \mathbb{N}$. Therefore, for any $m > a_0$, the line bundle

$$\nu_k^*\big(\mathscr{O}_{X_k}(a_1, \ldots, a_{k-1}, a_k + \sum_{p=1}^{c} m b_p k') \otimes \pi_{0,k}^* A^{a_0 - mq(\boldsymbol{\epsilon},\boldsymbol{\delta},r)}\big) \otimes \mathscr{O}_{\hat{X}_k}\big(-(\sum_{p=1}^{c} m b_p + 1)F\big)|_{\hat{H}_{\mathbf{a},k}}$$

is ample for any $\mathbf{a} \in U_{\rm nef}$, and thus there exists $e_0, \ldots, e_k \in \mathbb{N}$ such that

$$\nu_k^*\mathscr{O}_{X_k}(e_1, \ldots, e_k) \otimes \mathscr{O}_{\hat{X}_k}(-e_0 F)|_{\hat{H}_{\mathbf{a},k}}$$

is ample. By the openness property of ampleness, one has a non-empty Zariski open subset $U_{\rm ample} \subset \prod_{i=1}^{c} |A^{d_i}|$ such that for any $(H_1, \ldots, H_c) \in U_{\rm ample}$, their intersection $\mathbf{H} := H_1 \cap \ldots \cap H_c$ is a reduced smooth variety of codimension c in X, and the restriction of the line bundle $\nu_k^*\mathscr{O}_{X_k}(e_1, \ldots, e_k) \otimes \mathscr{O}_{\hat{X}_k}(-qF)|_{\hat{\mathbf{H}}_k}$ is ample (recall that $\hat{\mathbf{H}}_k$ is denoted to be the blow-up of $\mathbf{H}_k$ along $\mathfrak{w}_\infty(\mathbf{H}_k)$). Since the exceptional locus of the blow-up $\nu_k : \hat{X}_k \to X_k$ is contained in $X_k^{\rm sing}$, then for the complete intersection $\mathbf{H} := H_1 \cap \ldots H_c$ of general hypersurfaces $H_1 \in |A^{d_1}|, \ldots, H_c \in |A^{d_c}|$, the augmented base locus of the line bundle

$$\mathscr{O}_{\mathbf{H}_k}(e_1, \ldots, e_k) = \mathscr{O}_{X_k}(e_1, \ldots, e_k)|_{\mathbf{H}_k}$$

is contained in $X_k^{\rm sing} \cap \mathbf{H}_k$, and we conclude that H_k has almost k-jet ampleness by the fact that $X_k^{\rm sing} \cap \mathbf{H}_k = \mathbf{H}_k^{\rm sing}$.

□

Now we make some effective estimates based on Theorem 4.3.2. If we take

$$d_0 := \delta_0\big(c(k_0+1)(k_0+\delta_0+k_0\delta_0-1)\delta_0^{c(k_0+1)-1} + k_0 + 1\big) + k_0,$$

then any $d \geqslant d_0$ has a decomposition

$$d = (t + k_0)\delta_0 + \epsilon$$

with $k_0 \leqslant \epsilon < \delta_0 + k_0$ and $t \geqslant c(k_0+1)(k_0+\delta_0+k_0\delta_0-1)\delta_0^{c(k_0+1)-1}+1$, satisfying the conditions in Theorem 4.3.2. Therefore, the complete intersection $H_1 \cap \ldots \cap H_c$ of general hypersurfaces $H_1, \ldots, H_c \in |A^d|$ with $d \geqslant d_0$ has almost k_0-jet ampleness. By [**Dem95**, Lemma 7.6], if a complex manifold Y has almost k-jet ampleness, then it will also has almost l-jet ampleness for any $l \geqslant k$. A computation gives a rough estimate $d_0 \leqslant 2c(\lceil \frac{n}{c} \rceil)^{n+c+2} n^{n+c}$, and this completes the proof of Theorem L.

4.3.4. Uniform Estimates for the Lower Bounds on the Degree. In Theorem 4.3.2, the lower bound on the degrees is not uniform and it depends on the directions. In this subsection, we will adopt a factorization trick due to Xie [**Xie15**] to overcome this difficulty, but in the loss of slightly worse bound. First, we began with the following lemma observed by Xie:

Lemma 4.3.6. For all positive integers $\tilde{d}_0$ every integer $d \geqslant \tilde{d}_0^2 + \tilde{d}_0$ can be decomposed into

$$d = (\tilde{d}_0 + 1)a + (\tilde{d}_0 + 2)b$$

where a and b are nonnegative integers.

Let X be an n-dimensional smooth projective variety, equipped with a very ample line bundle A. Let c be any integer satisfying $1 \leqslant c \leqslant \lceil \frac{n}{2} \rceil$. Set $k_0 = \lceil \frac{n}{c} \rceil - 1$, $\delta_0 := n(k_0+1)$, $r_0 := c(k_0+1)\delta_0^{c(k_0+1)-1}(1+k_0+k_0\delta_0)+1$, and $\tilde{d}_0 := \delta_0(r_0+k_0)+k_0-1$. Then any c-tuple of integers in the form $(\tilde{d}_0+1, \ldots, \tilde{d}_0+1, \tilde{d}_0+2, \ldots, \tilde{d}_0+2)$ satisfies the condition (4.3.11). Take Z to be *any* complete intersection of c *general* hypersurfaces in $|A^{\tilde{d}_0+1}|$ *or* $|A^{\tilde{d}_0+2}|$, and $\hat{Z}_k$ is the variety obtained by the blow-up of Z_k along the k-th asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(Z_k)$. From Section 4.2.4 we see that, the Wronskian ideal sheaf is fonctorial under restrictions and thus $\hat{Z}_k = \nu_k^{-1}(Z_k)$, where $\nu_k : \hat{X}_k \to X_k$ is also the blow-up of the Wronskian ideal sheaf $\mathfrak{w}_\infty(X_k)$. From Theorem 4.3.2 and Claim 4.3.3 we see that, the line bundle

$$\nu_k^*\big(\mathscr{O}_{X_k}(c\delta_0^{c(k_0+1)-1}k') \otimes \pi_{0,k}^* A^{-1}\big) \otimes \mathscr{O}_{\hat{X}_k}(-c\delta_0^{c(k_0+1)-1}F)|_{\hat{Z}_k}$$

is nef. Take an ample line bundle on $\hat{X}_k$ of the form

$$\nu_k^*\big(\mathscr{O}_{X_k}(a_1, \ldots, a_k) \otimes \pi_{0,k}^* A^{a_0}\big) \otimes \mathscr{O}_{\hat{X}_k}(-F)$$

where $a_0, \ldots, a_k \in \mathbb{N}$. Then the line bundle

$$\nu_k^*\big(\mathscr{O}_{X_k}(a_1, \ldots, a_{k-1}, a_k + a_0 c\delta_0^{c(k_0+1)-1}k')\big) \otimes \mathscr{O}_{\hat{X}_k}\big(-(a_0 c\delta_0^{c(k_0+1)-1}+1)F\big)|_{\hat{Z}_k}$$

is ample. Within this setting, we have

Theorem 4.3.3. *For any c-tuple $\mathbf{d} := (d_1, \ldots, d_c)$ such that $d_p \geqslant \tilde{d}_0^2 + \tilde{d}_0$ for each $1 \leqslant p \leqslant c$, for general hypersurfaces $H_p \in |A^{d_p}|$, their complete intersection $Z := H_1 \cap \ldots \cap H_c$ has almost k-jet ampleness provided that $k \geqslant k_0$.*

Moreover, there exists a uniform $(e_0, e_1, \ldots, e_c) \in \mathbb{N}^{c+1}$ which does not depend on $\mathbf{d}$, such that

$$\nu_k^*\big(\mathscr{O}_{Z_k}(e_1, \ldots, e_k)\big) \otimes \mathscr{O}_{\hat{Z}_k}\big(-e_0 F_{Z_k}\big)$$

is an ample line bundle, where where $\nu_k : \hat{Z}_k \to Z_k$ is also the blow-up of the Wronskian ideal sheaf $\mathfrak{w}_\infty(Z_k)$, and F_{Z_k} is the effective cartier divisor on $\hat{Z}_k$ such that $\mathscr{O}_{\hat{Z}_k}(-F_{Z_k}) = \nu_k^*\mathfrak{w}_\infty(Z_k)$.

Proof. Let us denote by $q : \mathscr{X}_\mathbf{d} \to \prod_{p=1}^c |A^{d_p}|$ the universal family of c-complete intesections of hypersurfaces in $\prod_{p=1}^c |A^{d_p}|$, *i.e.*

$$\mathscr{X}_\mathbf{d} := \{(s_1, \ldots, s_c; x) \in \prod_{p=1}^c |A^{d_p}| \times X | \forall p, s_p \in |A^{d_p}| \text{ and } s_p(x) = 0\}. \tag{4.3.12}$$

By Lemma 4.3.6 we have the following decompositions

$$d_p = (\tilde{d}_0 + 1)a_p + (\tilde{d}_0 + 2)b_p$$

for each $1 \leqslant p \leqslant c$. Consider the linear system $V_p \subset |A^{d_p}|$ generated by sections in $\mathrm{Sym}^{a_p}|A^{\tilde{d}_0+1}| \times \mathrm{Sym}^{b_p}|A^{\tilde{d}_0+2}|$, then for a generic choice of $(s_1, \ldots, s_c) \in V_1 \times \ldots \times V_c$, their complete intersection $Y = \sum_{s=1}^l n_s Z^s$ (may not be reduced) is a union of smooth codimension c subvarieties $Z^1, \ldots, Z^l$ which are complete intersections of c general hypersurfaces in $|A^{\tilde{d}_0+1}|$ or $|A^{\tilde{d}_0+2}|$. By the arguments above the line bundle

$$\nu_k^*\big(\mathscr{O}_{X_k}(a_1, \ldots, a_{k-1}, a_k + a_0 c\delta_0^{c(k_0+1)-1}k')\big) \otimes \mathscr{O}_{\hat{X}_k}\big(-(a_0 c\delta_0^{c(k_0+1)-1}+1)F\big)|_{\hat{Z}_k^s}$$

is ample for each $s = 1, \ldots, l$, and so is for Y. Since ampleness is open in families, this also holds for the general fiber Z of $q : \mathscr{Z}_{\mathbf{d}} \to \prod_{p=1}^{c} |A^{d_p}|$, that is, for the complete intersection $Z := H_1 \cap \ldots \cap H_c$ of any general hypersurfaces in $\prod_{p=1}^{c} |A^{d_p}|$, the line bundle

$$\mathscr{O}_{Z_k}(a_1, \ldots, a_{k-1}, a_k + a_0 c\delta_0^{c(k_0+1)-1}k') \otimes \mathscr{O}_{\hat{Z}_k}\big(-(a_0 c\delta_0^{c(k_0+1)-1}+1)F_{Z_k}\big) =$$
$$\nu_k^*\big(\mathscr{O}_{X_k}(a_1, \ldots, a_{k-1}, a_k + a_0 c\delta_0^{c(k_0+1)-1}k')\big) \otimes \mathscr{O}_{\hat{X}_k}\big(-(a_0 c\delta_0^{c(k_0+1)-1}+1)F\big)|_{\hat{Z}_k}$$

is ample. As the choice of $\mathbf{a}$ is independant of $\mathbf{d}$, we obtain our theorem. □

Roughly, we can take the lower bound to be $c^2 n^{2n+2c}(\lceil \frac{n}{c} \rceil)^{2n+2c+4} \geqslant \tilde{d}_0^2 + \tilde{d}_0$, and we finish the proof of Theorem M.

4.3.5. On the Diverio-Trapani Conjecture. In this subsection we will prove Theorem N. Let X be a projective manifold of dimension n and A a very ample line bundle on X. Recall that we denote by $\hat{X}_k$ is the blow-up of X_k along the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(X_k)$, and F the effective Cartier divisor such that $\mathscr{O}_{\hat{X}_k}(-F) = \nu_k^*\big(\mathfrak{w}_\infty(X_k)\big)$. From the proof of Theorem 5.3.6, one can find a uniform $\mathbf{e} := (e_0, \ldots, e_c) \in \mathbb{N}^{c+1}$ such that, for the generic fiber Z of the universal family $q : \mathscr{Z}_{\mathbf{d}} \to \prod_{p=1}^{c} |A^{d_p}|$ defined in (4.3.12), where $d_p \geqslant \tilde{d}_0^2 + \tilde{d}_0$ for each $1 \leqslant p \leqslant c$, the line bundle

$$\nu_k^*\mathscr{O}_{X_k}(e_1, \ldots, e_c) \otimes \mathscr{O}_{\hat{X}_k}(-e_0 F)|_{\hat{Z}_k} = \nu_k^*\mathscr{O}_{Z_k}(e_1, \ldots, e_c) \otimes \mathscr{O}_{\hat{Z}_k}(-e_0 F_{Z_k}) \tag{4.3.13}$$

is *very* ample. From Section 4.2.3 we can take an open covering $\{U_\alpha\}$ of Z such that:

a) each $U_{\alpha k} := \pi_{0,k}^{-1}(U_\alpha)$ is a trivial product $U_\alpha \times \mathcal{R}_{n-c,k}$, where $\mathcal{R}_{n-c,k}$ is some smooth rational variety.
b) Set $\mathrm{pr}_2 : U_\alpha \times \mathcal{R}_{n-c,k} \to \mathcal{R}_{n-c,k}$ to be the projection map. There exists an ideal sheaf $\mathcal{I}_{n-c,k}$ on $\mathcal{R}_{n-c,k}$ such that
$$\mathfrak{w}_\infty(Z_k) = \mathrm{pr}_2^*(\mathcal{I}_{n-c,k}).$$

Let us denote by $\mu_k : \hat{\mathcal{R}}_{n-c,k} \to \mathcal{R}_{n-c,k}$ the blow-up of $\mathcal{R}_{n-c,k}$ along $\mathcal{I}_{n-c,k}$, and E is the effective divisor on $\hat{\mathcal{R}}_{n-c,k}$ such that

$$\mathscr{O}_{\hat{\mathcal{R}}_{n-c,k}}(-E) := \mu_k^*(\mathcal{I}_{n-c,k}).$$

Set $\hat{U}_{\alpha k} := \nu_k^{-1}(U_{\alpha k})$, then we have

$$\begin{array}{ccc} U_\alpha \times \hat{\mathcal{R}}_{n-c,k} & \xrightarrow{\cong} & \hat{U}_{\alpha k} \\ \downarrow {\scriptstyle \mathbb{1}\times\mu_k} & & \downarrow {\scriptstyle \nu_k} \\ U_\alpha \times \mathcal{R}_{n-c,k} & \xrightarrow{\cong} & U_{\alpha k}. \end{array}$$

Therefore, $\pi_{0,k} \circ \nu_k : \hat{Z}_k \to Z$ is a local isotrivial family with fiber $\hat{\mathcal{R}}_{n-c,k}$, and thus for any $j > 0$ the direct image $(\pi_{0,k} \circ \nu_k)_*(jL)$ is always locally free on Z, here we denote by $L := \nu_k^*\mathscr{O}_{Z_k}(e_1, \ldots, e_c) \otimes \mathscr{O}_{\hat{Z}_k}(-e_0 F_{Z_k})$. Since

$$(\nu_k)_*(jL) = \mathscr{O}_{Z_k}(je_1, \ldots, je_k) \otimes \mathcal{I}_j,$$

where $\mathcal{I}_j = (\nu_k)_*\mathscr{O}_{\hat{Z}_k}(-je_0 F_{Z_k})$ is some ideal sheaf of Z_k supported on Z_k^{Sing}, by the Direct image formula (4.2.3) we have

$$(\pi_{0,k} \circ \nu_k)_*(jL) \subset \overline{F}^{j\mathbf{e}} E_{k,jm} T_Z^* \tag{4.3.14}$$

where $m = e_1 + \ldots + e_k$.

<u>**Claim**</u> 4.3.4. There exists a positive integer j_1 such that for each $j \geqslant j_1$, the direct image $(\pi_{0,k} \circ \nu_k)_*(jL) \subset \mathscr{O}(\overline{F}^{j\mathbf{e}} E_{k,jm} T_Z^*)$ is an ample vector bundle on Z.

<u>Proof:</u> Let us denote by $A_Z := A|_Z$. As L is ample, one can find an integer $j_0 \gg 0$ such that for each $j \geqslant j_0$, all higher direct image sheaf $R^i(\pi_{0,k} \circ \nu_k)_*(jL)$ vanishes, and $jL - (\pi_{0,k} \circ \nu_k)^* A_Z$ is ample.

Set $V_j := (\pi_{0,k} \circ \nu_k)_*\big(jL - (\pi_{0,k} \circ \nu_k)^* A_Z\big)$ which is a local free sheaf for any $j \geqslant 0$. Consider any coherent $\mathcal{F}$ on Z. Then by the degeneration of the Leray spectral sequence, for each $j \geqslant j_0$, we have

$$H^i(Z, V_j \otimes \mathcal{F}) = H^i(\hat{Z}_k, L^j \otimes (\pi_{0,k} \circ \nu_k)^* A_Z^{-1} \otimes (\pi_{0,k} \circ \nu_k)^*\mathcal{F}) \tag{4.3.15}$$

for any $i > 0$. Fix a point $y \in Z$, with maximal ideal $\mathcal{M}_y \subset \mathscr{O}_Z$. Then we have the exact sequence

$$0 \to \mathcal{M}_y \to \mathscr{O}_Z \to \mathscr{O}_Z/\mathcal{M}_y \to 0.$$

As L is ample, there exists a positive integer $j(y) \geqslant j_0$ such that

$$H^1(Z, V_j \otimes \mathcal{M}_y) = H^1\big(\hat{Z}_k, L^j \otimes (\pi_{0,k} \circ \nu_k)^* A_Z^{-1} \otimes (\pi_{0,k} \circ \nu_k)^*\mathcal{M}_y\big) = 0$$

for $j \geqslant j(y)$, and so we see using the exact sequence above that V_j is generated by its global sections at y. The same therefore holds in a Zariski open neighborhood of y, and by the compactness of Z we can find a integer $j_1 \geqslant j_0$ such that V_j is globally generated when $j \geqslant j_1$. Thus $V_j \otimes A_Z = (\pi_{0,k} \circ \nu_k)_*(jL)$ is an ample vector bundle for any $j \geqslant j_1$. ∎

Since the ampleness is open in families, then Claim 4.3.4 holds for general fibers of $q : \mathscr{Z}_{\mathbf{d}} \to \prod_{p=1}^{c} |A^{d_p}|$. Set $U \subset \prod_{p=1}^{c} |A^{d_p}|$ to be a Zariski open set of $\prod_{p=1}^{c} |A^{d_p}|$ such that when restricted to $\mathscr{Y} := q^{-1}(U)$, q is a smooth fibration. Denote by $\mathscr{Y}_k$ the k-th Demailly-Semple tower of $(\mathscr{Y}, T_{\mathscr{Y}/U})$, and $\nu_k : \hat{\mathscr{Y}}_k \to \mathscr{Y}_k$ the blowing-up of the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(\mathscr{Y}_k)$ with $\mathscr{O}_{\hat{\mathscr{Y}}_k}(-F_{\mathscr{Y}_k}) = \nu_k^* \mathfrak{w}_\infty(\mathscr{Y}_k)$. Then for every $j \gg 0$, we define the vector bundle V_j on $\mathscr{Y}$ by

$$V_j := (\pi_{0,k} \circ \nu_k)_* \big(\nu_k^* \mathscr{O}_{\mathscr{Y}_k}(je_1, \ldots, je_n) \otimes \mathscr{O}_{\hat{\mathscr{Y}}_k}(-je_0 F_{\mathscr{Y}_k})\big),$$

and its restriction to the general fiber Z of q is

$$(\pi_{0,k} \circ \nu_k)_* \big(\nu_k^* \mathscr{O}_{Z_k}(je_1, \ldots, je_n) \otimes \mathscr{O}_{\hat{Z}_k}(-je_0 F_{Z_k})\big),$$

which is ample by Claim 4.3.4. We finish the proof of the first part in Theorem N. Since $L = \nu_k^* \mathscr{O}_{Z_k}(e_1, \ldots, e_c) \otimes \mathscr{O}_{\hat{Z}_k}(-e_0 F_{Z_k})$ is very ample on $\hat{Z}_k$, we can take $j \gg 0$ such that $jL - (\pi_{0,k} \circ \nu_k)^* A_Z^{-1}$ is still very ample, and by the relation

$$(\nu_k)_*(jL) = \mathscr{O}_{Z_k}(je_1, \ldots, je_k) \otimes \mathcal{I}_j,$$

we see that the base locus of

$$H^0\big(Z_k, \mathscr{O}_{Z_k}(je_1, \ldots, je_k) \otimes (\pi_{0,k})^* A_Z^{-1} \otimes \mathcal{I}_j\big)$$

is contained in Z_k^{Sing}. We finish the proof of Theorem N.

4.4. Effective Estimates Related to the Nakamaye Theorem

In this section we prove Theorem 4.3.1. For simplicity and to make this part readable, we give a complete proof for $c = 1$. The proof for the general cases is exact the same and we will show the general ideas for that. We begin with some definitions and notations of the *universal Grassmannian*.

We consider $V := H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\delta)\big)$, that is, the space of homogeneous polynomials of degree δ in $\mathbb{C}[z_0, \ldots, z_n]$, and for any $J \subset \{0, \ldots, N\}$, we set

$$\mathbb{P}_J := \{[z_0, \ldots, z_N] \in \mathbb{P}^N | z_j = 0 \text{ if } j \in J\}.$$

Given any $\Delta \in \mathrm{Gr_{k+1}(V)}$ and $[z] \in \mathbb{P}^N$, we write $\Delta([z]) = 0$ if and only if $P(z) = 0$ for all $P \in \Delta \subset V$. We then define the universal Grassmannian to be

$$\mathscr{Y} := \{(\Delta, [z]) \in \mathrm{Gr}_{k+1}(V) \times \mathbb{P}^N | \Delta([z]) = 0\}, \tag{4.4.1}$$

and for any $J \subset \{0, \ldots, N\}$, set

$$\mathscr{Y}_J := \mathscr{Y} \cap (\mathrm{Gr}_{k+1}(V) \times \mathbb{P}_J). \tag{4.4.2}$$

From now on we always assume that $k + 1 \geqslant N$, then $p : \mathscr{Y} \to \mathrm{Gr_{k+1}(V)}$ is a generically finite to one morphism. Denote $q : \mathscr{Y} \to \mathbb{P}^N$ to be the projection on the second factor. Let $\mathscr{L}$ be the very ample line bundle on $\mathrm{Gr}_{k+1}(V)$ which is the pull back of $\mathscr{O}(1)$ under the Plücker embedding. Then $p^*\mathscr{L}|_{\mathscr{Y}_J}$ is a big and nef line bundle on $\mathscr{Y}_J$ for any J. For any $J \subset \{0, \ldots, N\}$ we denote by $p_J : \mathscr{Y}_J \to \mathrm{Gr}_{k+1}(V)$, and $q_J : \mathscr{Y}_J \to \mathbb{P}_J$ the projections. Similarly we set

$$E_J := \{y \in \mathscr{Y} | \dim_y(p_J^{-1}(p_J(y))) > 0\}$$

$$G_J^\infty := p_J(E_J) \subset \mathrm{Gr}_{k+1}(V),$$

then $E_J = \mathrm{Null(p^*\mathscr{L}|_{\mathscr{Y}_J})}$. For $J = \varnothing$ we have $\mathscr{Y}_J = \mathscr{Y}$ and denote by $E := E_\varnothing$ and $G^\infty := G_\varnothing^\infty$.

Now we begin to prove Theorem 4.3.1. First of all suppose that $c = 1$ and $k + 1 = N$. Then in this case $p : \mathscr{Y} \to \mathrm{Gr_N(V)}$ is a generically finite to one surjective morphism. We first deal with the case $J = \varnothing$.

Let us pick a smooth curve C in $\mathrm{Gr_N(V)}$ of degree 1, given by

$$\Delta([t_0, t_1]) := \mathrm{Span}(z_1^\delta, z_2^\delta, \ldots z_{N-1}^\delta, t_0 z_N^\delta + t_1 z_0^\delta),$$

where $[t_0, t_1] \in \mathbb{P}^1$. Indeed, the curve C is the line in the projective space $\mathbb{P}(\Lambda^N V)$ defined by two vectors $z_0^\delta \wedge z_1^\delta \wedge \cdots \wedge z_{N-1}^\delta$ and $z_1^\delta \wedge z_2^\delta \wedge \cdots \wedge z_N^\delta$ in $\Lambda^N V$, which is of degree 1 with respect to the tautological line bundle $\mathscr{L}$. That is,

$$\mathscr{L} \cdot C = 1.$$

Now consider the hyperplane D in $\mathbb{P}^N$ given by $\{[z_0, \ldots, z_N] | z_0 + z_N = 0\}$. We have

LEMMA 4.4.1. The intersection number of the curve p^*C and the divisor q^*D in $\mathscr{Y}$ is δ^{N-1}. Moreover, $p_*q^*D \sim \delta^{N-1}\mathscr{L}$, where "$\sim$" stands for linear equivalence.

PROOF. An easy computation shows that p^*C and q^*D intersect only at the point

$$\mathrm{Span}(z_1^\delta, z_2^\delta, \ldots z_{N-1}^\delta, z_N^\delta + (-1)^{\delta+1}z_0^\delta) \times [1, 0 \ldots, 0, -1] \in \mathscr{Y}$$

with multiplicity δ^{N-1}. The first statement follows. By the projection formula we have

$$p_*q^*D \cdot C = p_*(q^*D \cdot p^*C) = \delta^{N-1}.$$

As $\mathrm{Pic}(\mathrm{Gr_N(V)}) \cong \mathbb{Z}$ with the generator $\mathscr{L}$, then we get $p_*q^*D \sim \delta^{N-1}\mathscr{L}$ by the fact that $\mathscr{L} \cdot C = 1$. □

We first observe that $p^*p_*q^*D - q^*D$ is an effective divisor of $\mathscr{Y}$, and by Lemma 4.4.1 we conclude that $\delta^{N-1}p^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)$ is effective. We also have a good control of the base locus as follows:

CLAIM 4.4.1. For any $m \geqslant \delta^N$, we always have

$$\mathrm{Bs}\big(mp^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)\big) \subset p^{-1}(G^\infty). \tag{4.4.3}$$

Proof: Pick any $\Delta_0 \notin G^\infty$, $p^{-1}(\Delta_0)$ is a finite set by the definition of G^∞. Thus one can choose a hyperplane $D \in H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1))$ such that $D \cap q\big(p^{-1}(\Delta_0)\big) = \varnothing$. From Lemma 4.4.1 we know that the divisor $p^*p_*q^*D - q^*D$ is effective and lies in the linear system $|\delta^{N-1}p^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)|$ of $\mathscr{Y}$.

For any $\Delta \in \mathrm{Gr_N(V)}$, if we denote by

$$\mathrm{Int}(\Delta) := \{[z] \in \mathbb{P}_N | \Delta([z]) = 0\},$$

then $q\big(p^{-1}(\Delta)\big) = \mathrm{Int}(\Delta)$. Hence the condition that $D \cap q\big(p^{-1}(\Delta_0)\big) = \varnothing$ is equivalent to that $\mathrm{Int}(\Delta_0) \cap D = \varnothing$. However, for any $\Delta \in p_*q^*D$, we must have $\mathrm{Int}(\Delta) \cap D \neq \varnothing$, therefore $\Delta_0 \notin p_*q^*D$. As Δ_0 was arbitrary, we conclude that

$$\mathrm{Bs}\big(\delta^{N-1}p^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)\big) \subset p^{-1}(G^\infty).$$

As $\mathscr{L}$ is very ample on $\mathrm{Gr_N(V)}$, for any $m \geqslant \delta^{N-1}$, we have

$$\mathrm{Bs}(mp^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)) \subset \mathrm{Bs}\big(\delta^{N-1}p^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)\big) \subset p^{-1}(G^\infty).$$

The Claim is thus proved. ■

Now we deal with the general case $J \subset \{0, \ldots, N\}$. Without loss of generality we can assume that $J = \{n+1, \ldots, N\}$. First recall our previous notation $p_J : \mathscr{Y}_J \to \mathrm{Gr_N(V)}$, and let $q_J : \mathscr{Y}_J \to \mathbb{P}_J$ be the second projection. For any $\Delta_0 \notin G_J^\infty$, the set $p_J^{-1}(\Delta_0) = \mathrm{Int}(\Delta_0) \cap \mathbb{P}_J$ is finite. Thus one can choose a generic hyperplane $D \in H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1))$ such that $\mathrm{Int}(\Delta_0) \cap D \cap \mathbb{P}_J = \varnothing$. One can further choose a proper coordinate for $\mathbb{P}^N$ such that $D = \{z_n = 0\}$.

Observe that $\mathscr{Y} \xrightarrow{q} \mathbb{P}^N$ is a *local trivial fibration*. Indeed, any linear transformation $g \in \mathrm{GL}(\mathbb{C}^{N+1})$ induces a natural action $\tilde{g} \in \mathrm{GL}(V)$, hence also a biholomorphism $\hat{g}$ of $\mathrm{Gr}_N(V)$. For any $e \in \mathbb{P}^N$, $\hat{g}$ maps the fiber $q^{-1}(e)$ to $q^{-1}\big(g(e)\big)$ bijectively. Since $\mathrm{GL}_{N+1}(\mathbb{C})$ acts transitively on $\mathbb{P}^N$, the fibration $\mathscr{Y} \xrightarrow{q} \mathbb{P}^N$ can then be locally trivialized. Therefore $q_J^*(D \cap \mathbb{P}_J)$ is a *reduced* divisor in $\mathscr{Y}_J$. Set $E := p_J\big(q_J^{-1}(D \cap \mathbb{P}_J)\big)$ set-theoretically. Then for any divisor $\tilde{H} \in |m\mathscr{L}|$ on $\mathrm{Gr}_N(V)$ such that $E \subset \tilde{H}$ and $p_J(\mathscr{Y}_J) \not\subset \tilde{H}$, $p_J^*(\tilde{H}) - q_J^*(D \cap \mathbb{P}_J)$ is an effective divisor in $|mp_J^*\mathscr{L} - q_J^*\mathscr{O}_{\mathbb{P}_J}(1)|$. However, it may happen that for any hyperplane $\tilde{D} \in \mathbb{P}^N$, all constructed divisors of the form $p_*q^*(\tilde{D})$ will always contain Δ_0.

Choose a decomposition of $V = V_1 \oplus V_2$ such that V_1 is spanned by the vectors $\{z^\alpha \in V | \alpha_n = \ldots = \alpha_N = 0\}$ and V_2 is spanned by other z^α's. Let us denote G to be the subgroup of the general linear group $GL(V)$ which is the lower triangle matrix with respect to the decomposition of $V = V_1 \oplus V_2$ as follows:

$$\{g \in GL(V) | g = \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}, B \in GL(V_2), A \in Hom(V_1, V_2)\}. \tag{4.4.4}$$

The subgroup G also induced a natural group action on the Grassmannian $\mathrm{Gr}_N(V)$, and we have the following

LEMMA 4.4.2. Set $H := p_*(q^*D)$. Then for any $g \in G$, $E \subset g(H)$ and there exists a $g_0 \in G$ such that $\Delta_0 \notin g_0(H)$.

PROOF. For any $\Delta \in \mathrm{Gr}_N(V)$, choose $\{s_1, \ldots, s_N\} \subset V$ which spans Δ. Let $s_i = u_i + v_i$ be the unique decomposition of s_i under $V = V_1 \oplus V_2$. Then by $E := p_J\big(q_J^{-1}(D \cap \mathbb{P}_J)\big)$ we see that $\Delta \in E$ if and only if $\cap_{i=1}^N \{u_i = 0\} \cap \mathbb{P}^{n-1} \neq \varnothing$, where $\mathbb{P}^{n-1} := \{[z_0 : \cdots : z_N] \in \mathbb{P}^N | z_j = 0 \text{ for } j \geqslant n\}$. For any $g \in GL(V)$, $g(\Delta)$ is spanned by $\{g(s_1), \ldots, g(s_N)\}$. By the definition of G, for any $g \in G$, we have the decomposition $g(s_i) = u_i + v_i'$ with respect to $V = V_1 \oplus V_2$ which keeps the V_1 factors invariant. Thus we prove the first part of the claim.

Set $\{t_1, \ldots, t_N\} \subset V$ which spans Δ_0 and $t_i = u_i + v_i$ be the unique decomposition of t_i under $V = V_1 \oplus V_2$. Since $\mathrm{Int}(\Delta_0) \cap \mathbb{P}^{n-1} = \varnothing$, we have $\cap_{i=1}^{N}\{u_i = 0\} \cap \mathbb{P}^{n-1} = \varnothing$. We can then choose the basis $\{t_1, \ldots, t_N\}$ spanning Δ_0 properly, so that

(i) $\bigcap_{i=1}^{n}\{u_i = 0\} \cap \mathbb{P}^{n-1} = \varnothing$;
(ii) for some $m \geqslant n$, $\{u_1, \ldots, u_m\}$ is a set of vectors in V_1 which is linearly independant;
(iii) $u_{m+1} = \ldots = u_N = 0$.

Then $\cap_{i=1}^{n}\{u_i = 0\} \cap \{z_n = 0\} = \mathbb{P}^{N-n-1} := \{[z_0 : \cdots : z_N] \in \mathbb{P}^N | z_j = 0 \text{ for } j \leqslant n\}$, and $\{v_{m+1}, \ldots, v_N\}$ is a set of linearly independant vectors in V_2.

Let us denote by $\Delta' \in \mathrm{Gr}_N(V)$ spanned by

$$\begin{cases} \tilde{u}_1 := u_1 \\ \vdots \\ \tilde{u}_n := u_n \\ \tilde{u}_{n+1} := u_{n+1} + z_{n+1}^{\delta} \\ \vdots \\ \tilde{u}_m := u_m + z_m^{\delta} \\ \tilde{u}_{m+1} := u_{m+1} + z_{m+1}^{\delta} = z_{m+1}^{\delta} \\ \vdots \\ \tilde{u}_N := u_N + z_N^{\delta} = z_N^{\delta} \end{cases}.$$

Then $\mathrm{Int}(\Delta') \cap \{z_n = 0\} = \varnothing$, which is equivalent to that $\Delta' \notin H := p_*q^*(D)$. By the choice of Δ' one can find a $g_0 \in G$ such that $g_0(\Delta') = \Delta_0$. Indeed, by linear independances of $\{v_{m+1}, \ldots, v_N\}$ in V_2 and $\{u_1, \ldots, u_m\}$ in V_1, we can find a $B \in GL(V_2)$ satisfying that $B(z_i^{\delta}) = v_i$ for all $i \geqslant m+1$, and $A \in \mathrm{Hom}(V_1, V_2)$ such that $A(u_i) = v_i$ for $i \leqslant n$ and $A(u_j) = v_j - B(z_j^{\delta})$ for $n+1 \leqslant j \leqslant m$. Set $g_0 := \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}$ which is the type (4.4.4), and by the construction of g_0 we have that $g_0(\Delta') = \Delta_0$. Thus $\Delta_0 \notin g_0(H)$ and we finish the proof of the claim. □

Since $H \in |\delta^{N-1}\mathscr{L}|$, $g_0(H)$ still lies in $|\delta^{N-1}\mathscr{L}|$. Indeed, since the complex general linear group $GL(V)$ is connected, the automorphism map of $\mathrm{Gr_N(V)}$ induced by g_0-action is homotopic to the identity map, and thus the g_0 action induces the identity on the cohomology groups. By Lemma 4.4.2, $E \subset g_0(H)$ and $\Delta_0 \notin g_0(H)$. As $q_J^*(D \cap \mathbb{P}_J)$ is a reduced (Cartier) divisor on $\mathscr{Y}_J$, the divisor

$$p_J^*\big(g_0(H)\big) - q_J^*(D \cap \mathbb{P}_J) \in |\delta^{N-1}p_J^*\mathscr{L} - q_J^*\mathscr{O}_{\mathbb{P}_J}(1)|$$

is effective and avoids the finite set $p_J^{-1}(\Delta_0)$.

Since $\Delta_0 \in \mathrm{Gr}_N(V)$ is any arbitrary point not contained in G_J^{∞}, thus the base locus of $\delta^{N-1}p_J^*\mathscr{L} - q_J^*\mathscr{O}_{\mathbb{P}_J}(1)$ is totally contained in $p_J^{-1}(G_J^{\infty})$. In conclusion we have the following theorem:

THEOREM 4.4.1. *Let $\mathscr{Y} \subset \mathrm{Gr}_N(V) \times \mathbb{P}^N$ and $\mathscr{Y}_J$ be the universal families defined in (4.4.1) and (4.4.2). For any $J \subset \{0, \ldots, N\}$, we have*

$$\mathrm{Bs}(mp^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)|_{\mathscr{Y}_J}) \subset p_J^{-1}(G_J^{\infty})$$

for any $m \geqslant \delta^{N-1}$.

Fix any positive integer $n < N$. Consider $\mathbb{P}^n$ as a subspace of $\mathbb{P}^N$ defined by $z_{n+1} = \ldots = z_N = 0$. Set $V_n := H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\delta))$, and we have a natural inclusion $\mathrm{Gr}_N(V_n) \subset \mathrm{Gr}_N(V)$. For any $J \subset \{0, \ldots, n\}$, we denote by $\tilde{J} := J \cup \{n+1, \ldots, N\}$, and $\mathbb{P}_{\tilde{J}} := \{[z_0, \ldots, z_N] \in \mathbb{P}^N | z_j = 0 \text{ if } j \in \tilde{J}\}$. Set

$$\tilde{\mathscr{Y}}_J := \{(\Delta, [z]) \in \mathrm{Gr}_N(V_n) \times \mathbb{P}_{\tilde{J}} | \Delta([z]) = 0\}.$$

Define $\tilde{p}_J : \tilde{\mathscr{Y}}_J \to \mathrm{Gr_N(V_n)}$ and $\tilde{q}_J : \tilde{\mathscr{Y}}_J \to \mathbb{P}_{\tilde{J}}$ the respective projections. Set

$$\tilde{G}_J^{\infty} := \{\Delta \in \mathrm{Gr}_N(V_n) | \tilde{p}_J^{-1}(\Delta) \text{ is not finite set}\}.$$

Let $\mathscr{Y} \subset \mathrm{Gr}_N(V) \times \mathbb{P}^N$ and $\mathscr{Y}_{\tilde{J}}$ be the universal families defined in (4.4.1) and (4.4.2). There is a natural inclusion $i_n : \mathrm{Gr}_N(V_n) \hookrightarrow \mathrm{Gr}_N(V)$, which induces the following inclusions:

$$\begin{array}{ccc} \tilde{\mathscr{Y}}_J & \hookrightarrow & \mathrm{Gr}_N(V_n) \times \mathbb{P}_{\tilde{J}} \\ \big\downarrow & & \big\downarrow{\scriptstyle i_n \times \mathbb{1}} \\ \mathscr{Y}_{\tilde{J}} & \hookrightarrow & \mathrm{Gr}_N(V) \times \mathbb{P}_{\tilde{J}} \end{array}$$

Under the inclusion i_n, we have

$$\tilde{G}_J^{\infty} = G_{\tilde{J}}^{\infty} \cap \mathrm{Gr}_N(V_n).$$

From Theorem 4.4.1, for $m \geqslant \delta^{N-1}$ we also have

$$\begin{aligned} \mathrm{Bs}(mp^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)|_{\tilde{\mathscr{Y}}_J}) &\subset \mathrm{Bs}(mp^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)|_{\mathscr{Y}_{\tilde{J}}}) \cap \tilde{\mathscr{Y}}_J \\ &\subset p_{\tilde{J}}^{-1}(G_{\tilde{J}}^{\infty}) \cap \tilde{\mathscr{Y}}_J \\ &= \tilde{p}_J^{-1}(\tilde{G}_J^{\infty}), \end{aligned} \tag{4.4.5}$$

where $p_{\tilde{J}} : \mathscr{Y}_{\tilde{J}} \to \mathrm{Gr}_N(V)$ and $q_{\tilde{J}} : \mathscr{Y}_{\tilde{J}} \to \mathbb{P}_{\tilde{J}}$ are the projection maps. Since the pull back

$$i_n^* : \mathrm{Pic}\big(\mathrm{Gr_N(V)}\big) \xrightarrow{\approx} \mathrm{Pic}\big(\mathrm{Gr_N(V_n)}\big)$$

is an isomorphism between the Picard groups, and $\mathscr{L}_n := i_n^*\mathscr{L}$ is still the tautological line bundle on $\mathrm{Gr}_N(V_n)$. Then

$$mp^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)|_{\tilde{\mathscr{Y}}_J} = m\tilde{p}_J^*(\mathscr{L}_n) - \tilde{q}_J^*\mathscr{O}_{\mathbb{P}_J}(1),$$

and by (4.4.5) we have

$$\mathrm{Bs}\big(m\tilde{p}_J^*(\mathscr{L}_n) - \tilde{q}_J^*\mathscr{O}_{\mathbb{P}_J}(1)\big) \subset \tilde{p}_J^{-1}\big(\tilde{G}_J^{\infty}\big). \tag{4.4.6}$$

We are in the situation to prove Theorem 4.3.1 for $c = 1$ and general $k+1 \geqslant N$:

THEOREM 4.4.2. *For any $k+1 \geqslant N$, set $\mathscr{Y} \subset \mathrm{Gr}_{k+1}(V) \times \mathbb{P}^N$ and $\mathscr{Y}_J$ to be the universal families defined in (4.4.1) and (4.4.2). For any $J \subset \{0, \ldots, N\}$, and $k+1 \geqslant N$, we have*

$$\mathrm{Bs}(mp^*\mathscr{L} - q^*\mathscr{O}_{\mathbb{P}^N}(1)|_{\mathscr{Y}_J}) \subset p_J^{-1}(G_J^{\infty}) \tag{4.4.7}$$

for any $m \geqslant \delta^k$.

PROOF. Indeed, if we consider $\mathbb{P}^N$ as a subspace in $\mathbb{P}^{k+1}$ defined by $z_{N+1} = \ldots = z_{k+1} = 0$, the theorem follows from (4.4.6) directly. □

The above theorem can be generalized to the case of products of Grassmannians. We first set $N := c(k+1)$, and denote $V_i := H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\delta_i)\big)$ and $\mathbf{G} := \prod_{i=1}^c \mathrm{Gr}_{k+1}(V_i)$ for simplicity. Let $\mathscr{Y}$ be the generalized universal Grassmannian defined by

$$\mathscr{Y} := \{(\Delta_1, \ldots, \Delta_c, z) \in \mathbf{G} \times \mathbb{P}^N | \forall\ i, \Delta_i([z]) = 0\}.$$

Let $p : \mathscr{Y} \to \mathbf{G}$, $q : \mathscr{Y} \to \mathbb{P}^N$ and $p_i : \mathscr{Y} \to \mathrm{Gr}_{k+1}(\delta_i)$ be the canonical projections to each factor; then p is a generically finite to one morphism. Define a group homeomorphism

$$\begin{array}{rcl} \mathscr{L} : \mathbb{Z}^c & \to & \mathrm{Pic}(\mathbf{G}) \\ \mathbf{a} = (a_1, \ldots, a_c) & \mapsto & \mathscr{O}_{\mathrm{Gr}_{k+1}(V_1)}(a_1) \boxtimes \ldots \boxtimes \mathscr{O}_{\mathrm{Gr}_{k+1}(V_c)}(a_c) \end{array}$$

which is moreover an isomorphism.

We then define smooth lines $\{C_i\}_{i=1,\ldots,c}$ in $\mathbf{G}$, given by

$$\begin{aligned} \Delta_i([t_0, t_1]) := {} & \mathrm{Span}(z_1^{\delta_1}, z_{c+1}^{\delta_1} \ldots, z_{kc+1}^{\delta_1}) \times \mathrm{Span}(z_2^{\delta_2}, z_{c+2}^{\delta_2} \ldots, z_{kc+2}^{\delta_2}) \times \ldots \\ & \times \mathrm{Span}(t_0 z_i^{\delta_i} + t_1 z_0^{\delta_i}, z_{c+i}^{\delta_i}, \ldots, z_{kc+i}^{\delta_i}) \times \ldots \times \mathrm{Span}(z_c^{\delta_c}, z_{2c}^{\delta_c}, \ldots, z_{(k+1)c}^{\delta_c}) \end{aligned}$$

for $[t_0, t_1] \in \mathbb{P}^1$. It is easy to verify that $\mathcal{L}(\mathbf{a}) \cdot C_i = a_i$ for each i. Consider the hyperplane D_i of $\mathbb{P}^n$ given by $\{[z_0, \ldots, z_N] | z_i + z_0 = 0\}$. Then we have

LEMMA 4.4.3. The intersection number of the curve p^*C_i and the divisor q^*D_i in $\mathscr{Y}$ is $b_i := \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$. Moreover, $p_*q^*\mathscr{O}_{\mathbb{P}^N}(1) \equiv \mathscr{L}(\mathbf{b})$, where $\mathbf{b} = (b_1, \ldots, b_c)$.

PROOF. It is easy to show that p^*C_i and q^*D_i intersect only at one point with multiplicity b_i. By the projection formula we have

$$p_*q^*D_i \cdot C_i = p_*(q^*D_i \cdot p^*C_i) = b_i.$$

Since

$$\mathscr{L}(\mathbf{a}) \cdot C_i = a_i$$

for any $\mathbf{a} \in \mathbb{Z}^c$. Thus

$$p_*q^*D_i \equiv p_*q^*\mathscr{O}_{\mathbb{P}^N}(1) \equiv \mathcal{L}(\mathbf{b}).$$

□

Then by similar arguments above, $p^*\mathcal{L}(\mathbf{b}) \otimes q^*\mathscr{O}_{\mathbb{P}^N}(-1)$ is effective, and its base locus

$$\mathrm{Bs}(p^*\mathcal{L}(\mathbf{b}) \otimes q^*\mathscr{O}_{\mathbb{P}^N}(-1)) \subset p^{-1}(G^\infty), \tag{4.4.8}$$

where G^∞ is the set of points in $\mathbf{G}$ such that their p-fiber is not a finite set. We can then apply the methods already used above to show that (4.4.8) also holds for all the strata $\mathscr{Y}_I$ of $\mathscr{Y}$, and for general k with $c(k+1) \geqslant N$. In conclusion, we have the following theorem

THEOREM 4.4.3. *Let $\mathscr{Y}$ be the generalized universal Grassmannian defined by*

$$\mathscr{Y} := \{(\Delta_1, \ldots, \Delta_c, z) \in \mathrm{Gr}_{k+1}(V_1) \times \ldots \times \mathrm{Gr}_{k+1}(V_c) \times \mathbb{P}^N | \forall\ i, \Delta_i([z]) = 0\}$$

here $V_i := H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\delta_i)\big)$, and $(k+1)c \geqslant N$. Then for any strata $\mathscr{Y}_J := (\mathbf{G} \times \mathbb{P}_J) \cap \mathscr{Y}$, any $\mathbf{a} \in \mathbb{Z}^c$ with $a_i \geqslant \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$ for each i, we have

$$\mathrm{Bs}\big(p^*\mathcal{L}(\mathbf{a}) \otimes q^*\mathscr{O}_{\mathbb{P}^N}(-1)|_{\mathscr{Y}_J}\big) \subset p^{-1}(G_J^\infty),$$

where G_J^∞ is the set of points in $\mathbf{G} := \prod_{i=1}^c \mathrm{Gr}_{k+1}(V_i)$ with positive dimension fibers in $\mathscr{Y}_J$.

Part 3

On the Direct Image Problems

CHAPTER 5

Applications of the L^2 Extension Theorems to Direct Image Problems

Abstract. In the first part of the chapter, we study a Fujita-type conjecture by Popa and Schnell, and give an effective bound on the generic global generation of the direct image of the twisted pluricanonical bundle. We also point out the relation between the Seshadri constant and the optimal bound. In the second part, we give an affirmative answer to a question by Demailly-Peternell-Schneider in a more general setting. As an application, we generalize the theorems by Fujino and Gongyo on images of weak Fano manifolds to the Kawamata log terminal cases, and refine a result by Broustet and Pacienza on the rational connectedness of the image.

5.1. Introduction

The first goal of this chapter is to study the following conjecture by Popa and Schnell:

Conjecture 5.1.1. (Popa-Schnell) Let $f : X \to Y$ be a morphism of smooth projective varieties, with $\dim(Y) = n$, and let L be an ample line bundle on Y. Then, for every $k \geqslant 1$, the sheaf

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is globally generated for any $l \geqslant k(n+1)$.

In [**PS14**], Popa and Schnell proved the conjecture in the case when L is an ample and globally generated line bundle, and in general when $\dim(X) = 1$. In a recent preprint [**Dut17**], Dutta was able to remove the global generation assumption on L making a statement about generic global generation with weaker bound on the twist, as in the work of Angehrn and Siu [**AS95**], on the effective freeness of adjoint bundles. Her theorem is as follows:

Theorem 5.1.1. *(Dutta) Let $f : X \to Y$ be a morphism of smooth projective varieties, with* $\dim(Y) = n$*, and let L be an ample line bundle on Y. Then, for every $m \geqslant 1$, the sheaf*

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at a general point $y \in Y$, either

(a) for all $l \geqslant k(\binom{n+1}{2} + 1)$

or

(b) for all $l \geqslant k(n+1)$ when $n \leqslant 4$.

Here $\binom{n+1}{2}$ is the Angehrn-Siu type bound in their work on the Fujita conjecture [**AS95**].

Inspired by Demailly's recent work on the Ohsawa-Takegoshi type extension theorem [**Dem15a**] and Păun's proof of Siu's invariance of plurigenera [**Pau07**], we are able to prove the following theorem:

Theorem O. *Let $f : X \to Y$ be a morphism of smooth projective varieties, with* $\dim(Y) = n$*, and let L be an ample line bundle on Y. If y is a regular value of f, then for every $k \geqslant 1$, the sheaf*

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at y for any $l \geqslant k(\left\lfloor \frac{n}{\epsilon(L,y)} \right\rfloor + 1)$. Here $\epsilon(L, y) > 0$ is the Seshadri constant of L at the point y.

Motivated in part by his study of linear series in connection with the Fujita conjecture, Demailly introduced the *Seshadri constant* to measure the local positivity of the ample line bundle at a point [**Dem92**]. After that Ein and Lazarsfeld systematically studied the Seshadri constant, and they first proved that for any ample line bundle L on a projective surface Y, the Seshadri constant

$$\epsilon(L, y) \geqslant 1$$

for a very general point on Y [**EL93**]. Inspired by this result, they further raised the following conjecture:

CONJECTURE 5.1.2. (Ein-Lazarsfeld) Let Y be any projective manifold, and L any ample line bundle on Y. Then the Seshadri constant

$$\epsilon(L, y) \geqslant 1$$

at a very general point $y \in Y$.

In [**EKL95**], they proved the existence of universal generic bound in a fixed dimension. However, the bound is suboptimal by a factor of $n = \dim(Y)$.

THEOREM 5.1.2. *(Ein-Küchle-Lazarsfeld) Let Y be a projective variety, and L an ample line bundle on Y. Then for any given $\delta > 0$, the locus*

$$\{y \in Y | \epsilon(L, y) > \frac{1}{n+\delta}\}$$

contains a Zariski-dense open set in Y.

Applying Theorem 5.1.2 to our Theorem O, we have the following general result:

THEOREM P. *Let $f : X \to Y$ be a morphism of smooth projective varieties, with $\dim(Y) = n$, and let L be an ample line bundle on Y. Then for any $k \geqslant 1$, the direct image*

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at the generic points of Y for any $l \geqslant k(n^2+1)$. In particular, if the manifold Y satisfies Conjecture 5.1.2, then Conjecture 5.1.1 holds true for general points in Y; that is, the direct image

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at the generic points of Y for any $l \geqslant k(n+1)$.

Compared to Theorem 5.1.1 by Dutta, our bound for l is also quadratic on n but slightly weaker than hers. However, if we apply the result that $K_Y + (n+1)L$ is semi-ample for any ample line bundle L, we can obtain a linear bound for l.

THEOREM Q. *Let $f : X \to Y$ be a morphism of smooth projective varieties, with $\dim(Y) = n$, and let L be an ample line bundle on Y. Then for every $k \geqslant 1$, the sheaf*

$$f_*(K_X^{\otimes k}) \otimes L^{\otimes l}$$

is generated by global sections at the generic $y \in Y$ for any $l \geqslant k(n+1) + n^2 - n$.

The second part of the chapter is to study a question by Demailly-Peternell-Schneider in [**DPS01**]:

PROBLEM 5.1.1. Let X and Y be normal projective $\mathbb{Q}$-Gorenstein varieties. Let $f : X \to Y$ be a surjective morphism. If $-K_X$ is pseudo-effective and its non-nef locus does not project onto Y, is $-K_Y$ pseudo-effective?

Inspired by the recent work of J. Cao on the local isotriviality on the Albanese map of projective manifolds with nef anticanonical bundles [**Cao16**], we give an affirmative answer to the above problem when X and Y are smooth pairs:

THEOREM R. *Let $f : X \to Y$ be a surjective morphism from a log-canonical (lc for short) pair (X, D) to the smooth projective manifold Y. Let Δ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y. Suppose that $-(K_X + D) - f^*\Delta$ is pseudo-effective, and the non-nef locus $\mathbf{B}_-\big(-(K_X + D) - f^*\Delta\big)$ does not project onto Y. Then $-K_Y - \Delta$ is pseudo-effective with its non-nef locus contained in $f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \cup Z \cup Z_D$, where Z is the minimal proper subvariety on Y such that $f : X \backslash f^{-1}(Z) \to Y \backslash Z$ is a smooth fibration, and Z_D is an at most countable union of proper subvarities containing Z such that for every $y \notin Z_D$, the pair $\big(f^{-1}(y), D_{\restriction f^{-1}(y)}\big)$ is also lc.*

The following theorem by Fujino and Gongyo [**FG14**] is a direct consequence of our Theorem R.

THEOREM 5.1.3. *(Fujino-Gongyo) Let $f : X \to Y$ be a smooth fibration between smooth projective varieties. Let D be an effective $\mathbb{Q}$-divisor on X such that (X, D) is lc,* Supp(D) *is a simple normal crossing divisor, and* Supp(D) *is relatively normal crossing over Y. Let Δ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y. Assume that $-(K_X + D) - f^*\Delta$ is nef. Then so is $-K_Y - \Delta$.*

Moreover, we can also use analytic methods to prove the following theorem.

THEOREM S. *With the same notations in Theorem R. Assume further that (X, D) is klt, $-K_X - D - f^*\Delta$ is big and its non-nef locus $\mathbf{B}_-(-K_X - D - f^*\Delta)$ does not dominate Y, then $-K_Y - \Delta$ is big with its non-nef locus contained in $f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \cup Z \cup Z_D$.*

As a combination of Theorem R and S, we prove the following Theorem, which is a generalization of a theorem by Fujino and Gongyo [**FG12**].

THEOREM T. *With the same notations in Theorem 5.1.3, if* $-K_X - D - f^*\Delta$ *is big and nef, then* $-K_Y - \Delta$ *is also big and nef.*

Finally, we apply Theorem S to refine a result by Broustet and Pacienza on the rational connectedness of the image (compared to Theorem 5.5.5 below):

THEOREM U. *Let* X *be a normal projective variety and* D *an effective* $\mathbb{Q}$*-divisor on* X *such that* $K_X + D$ *is* $\mathbb{Q}$*-Cartier. Let* Y *be a normal and* $\mathbb{Q}$*-Gorenstein projective variety with klt singularities. If* $f : X \to Y$ *is a surjective morphism such that* $-(K_X + D)$ *is big and the restriction of* f *to* $\mathrm{NNef}(-K_X - D) \bigcup \mathrm{Nklt}(X, D)$ *does not dominate* Y*, then* Y *is rational connected modulo* $\mathrm{NNef}(-K_Y)$*, that is, there exists an irreducible component* V *of* $\mathrm{NNef}(-K_Y)$ *such that for any general point* y *of* Y *there exists a rational curve* R_y *passing through* y *and intersecting* V*.*

5.2. Preliminary Techniques

5.2.1. SESHADRI CONSTANTS. In the work [**Dem92**], Demailly define the following *Seshadri constant*:

DEFINITION 5.2.1. Le L be a nef line bundle over a projective algebraic manifold X. To every point $x \in X$, one defines the number

$$\epsilon(L, x) := \inf \frac{L \cdot C}{\nu(C, x)}$$

where the infimum is taken over all reduced irreducible curves C passing through x and $\nu(C, x)$ is the multiplicity of C at x. $\epsilon(L, x)$ will be called the *Seshadri constant* L at x.

On the other hand, Demailly also introduced another constant $\gamma(L, x)$ for any nef line bundle L. First, we begin with the following definition.

DEFINITION 5.2.2. A function $\psi : X \to]-\infty, +\infty]$ on a complex manifold X of dimension m is said to be quasi-plurisubharmonic (quasi-psh for short) if ψ is locally the sum of a psh function and of a smooth function (or equivalently, if $\sqrt{-1}\partial\overline{\partial}\psi$ is locally bounded from below) . In addition, we say that ψ has neat analytic singularities if every point $x \in X$ possesses an open neighborhood U on which ψ can be written

$$\psi = c \log \sum_{j=1}^{N} |g_j|^2 + w(z)$$

where $g_j \in \mathscr{O}(U)$, $c \geqslant 0$ and $w(z) \in \mathscr{C}^\infty(U)$.

DEFINITION 5.2.3. A singular metric h on the line bundle L is said to have a *logarithmic pole of coefficient* ν at a point $x \in X$, if on a neighborhood U of x, the local weight φ of h can be written

$$\varphi = \nu \log \sum |z - x|^2 + w(z)$$

where $\nu > 0$ and $w(z) \in \mathscr{C}^\infty(U)$. In this setting, we set $\nu(h, x) := \nu$.

Then we set

$$\gamma(L, x) := \sup_h \nu(h, x),$$

where the supremum is taken over all singular hermitian metrics h of L with positive curvature current, whose local weight φ has neat singularities and logarithmic poles at x.

The numbers $\epsilon(L, x)$ and $\gamma(L, x)$ will be seen to carry a lot of useful information about the local positivity of L. In case L is big and nef, these two constants coincide outside a certain proper subvariety of X (see [**Dem92**, Theorem 6.4])

THEOREM 5.2.1. *(Demailly) Let* L *be a big and nef line bundle over* X*. Then we have*

$$\epsilon(L, x) = \gamma(L, x)$$

for any $x \notin \mathbf{B}_+(L)$*, where* $\mathbf{B}_+(L)$ *is the augmented base locus of* L *(see [**Laz04**, Definition 10.2.2]). In particular, if* L *is ample, then* $\epsilon(L, x) = \gamma(L, x)$ *holds everywhere.*

As we mentioned in Section 5.1, in [**EKL95**], Ein, Küchle and Lazarsfeld gave the existence of universal generic bounds for the Seshadri constants in a fixed dimension.

THEOREM 5.2.2. *(Ein-Küchle-Lazarsfeld) Let Y be an irreducible projective variety of dimension n, and L a nef line bundle on Y. Suppose there exists a countable union $\mathcal{B} \subset Y$ of proper subvarieties of Y plus a positive real number $\alpha > 0$ such that*

$$L^r \cdot Z \geqslant (\alpha \cdot r)^r \tag{5.2.1}$$

for every irreducible subvariety $Z \subset Y$ of dimension r $(1 \leqslant r \leqslant n)$ with $Z \nsubseteq \mathcal{B}$. Then

$$\epsilon(L, y) \geqslant \alpha$$

for all $y \in Y$ outside a countable union of proper subvarieties in Y. In particular, for any ample line bundle L on Y,

$$\epsilon(L, y) \geqslant \frac{1}{n} \tag{5.2.2}$$

for a very general point y.

The above theorem gives a lower bound on the Seshadri constant of a nef and big line bundle at a very general point. However, as was also proved in [**EKL95**], for the ample line bundle, the above theorem is valid on a Zariski-open set by the semi-continuity of the Seshadri constant of the ample line bundle. In other word, let L be an ample line bundle on an irreducible projective variety Y. Suppose that there is a positive rational number B and a smooth point $y \in Y$ for which one knows that

$$\epsilon(L, y) > B.$$

Then the locus

$$\{z \in Y | \epsilon(L, z) > B\}$$

contains a Zariski-open dense set in Y.

5.2.2. L^2 EXTENSION THEOREM. Before we state Demailly's Ohsawa-Takegoshi type Extension Theorem, we begin with a definition in [**Dem15a**].

DEFINITION 5.2.4. If ψ is a quasi-psh function on a complex manifold X, the multiplier ideal sheaf $\mathscr{I}(\psi)$ is the coherent analytic subsheaf of $\mathscr{O}_X$ defined by

$$\mathscr{I}(\psi)_x := \{f \in \mathscr{O}_{X,x}; \exists U \ni x, \int_U |f|^2 e^{-\psi} \mathrm{d}\lambda < +\infty\}$$

where U is an open coordinate neighborhood of x, and $\mathrm{d}\lambda$ the standard Lebesgue measure in the corresponding open chart of $\mathbb{C}^n$. We say that the singularities of ψ are log canonical along the zero variety $Y := V(\mathscr{I}(\psi))$ if $\mathscr{I}\big((1-\epsilon)\psi\big)_{\restriction Y} = \mathscr{O}_{X \restriction Y}$ for every $\epsilon > 0$.

If ψ possesses both neat and log canonical singularities, it is easy to show that the zero scheme $V(\mathscr{I}(Y))$ is a reduced variety. In this case one can also associate in a natural way a measure $dV_{Y^\circ,\omega}[\psi]$ on the set $Y^\circ := Y^{\mathrm{reg}}$ of regular points of Y as follows. If $g \in \mathscr{C}_c(Y^\circ)$ is a compactly supported continuous function on Y°, and $\tilde{g}$ compactly supported extension of g to X, we set

$$\int_{Y^\circ} g dV_{Y^\circ,\omega}[\psi] := \limsup_{t \to -\infty} \int_{x \in X, t < \psi(x) < t+1} \tilde{g}(x) dV_{X,\omega}. \tag{5.2.3}$$

Here ω is a Kähler metric on X, and $dV_{X,\omega} = \frac{\omega^m}{m!}$. In [**Dem15a**] Demailly proved that the limit does not depend on the continuous extension $\tilde{g}$, and one gets in this way a measure with smooth positive density with respect to the Lebesgue measure, at least on an (analytic) Zariski open set in Y°.

We are ready to recall the Ohsawa-Takegoshi type extension Theorem by Demailly. We only need a special case of his very general statement:

THEOREM 5.2.3. *(Demailly) Let X be a smooth projective manifold, and ω a Kähler metric on X. Let L be a holomorphic line bundle equipped with a (singular) hermitian metric h on X, and let $\psi : X \to]-\infty, +\infty]$ be a quasi-psh function on X with neat analytic singularities. Let Y be the analytic subvariety of X defined by $Y = V(\mathscr{I}(Y))$ and assume that ψ has log canonical singularities along Y, so that Y is reduced. Finally, assume that the curvature current*

$$\mathrm{i}\Theta_{L,h} + \alpha\sqrt{-1}\partial\bar{\partial}\psi \geqslant 0$$

for all $\alpha \in [1, 1+\delta]$ and some $\delta > 0$. Then for every section $s \in H^0\big(Y^\circ, (K_X \otimes L)_{\restriction Y^\circ}\big)$ on $Y^\circ := Y^{\mathrm{reg}}$ such that

$$\int_{Y^\circ} |s|^2_{\omega,h} dV_{Y^\circ,\omega}[\psi] < +\infty,$$

there is an extension of $S \in H^0(X, K_X \otimes L)$ whose restriction to Y° is equal to s, such that

$$\int_X \gamma(\delta\psi)|S|^2_{\omega,h} e^{-\psi} dV_{X,\omega} \leqslant \frac{34}{\delta} \int_{Y^\circ} |s|^2_{\omega,h} dV_{Y^\circ,\omega}[\psi].$$

Here we set

$$\gamma = \begin{cases} e^{-\frac{x}{2}} & \text{if} \quad x \geqslant 0, \\ \dfrac{1}{1+x^2} & \text{if} \quad x < 0. \end{cases}$$

A direct consequence of Theorem 5.2.3 is the following extension theorem for fibrations:

COROLLARY 5.2.1. Let $f : X \to Y$ be a surjective morphism between smooth manifolds. For any ample line bundle L on Y, any regular value y of f, if the Seshadri constant of L satisfies that

$$\epsilon(L, y) > \dim(Y) = n, \tag{5.2.4}$$

then for any pseudo-effective line bundle L_1 over X with a singular hermitian metric h such that $\Theta_{L_1,h} \geqslant 0$, and the restriction of h to X_y is not identically zero, any section s of

$$H^0\big(X_y, (K_X \otimes f^*L \otimes L_1)_{\restriction X_y} \otimes \mathscr{J}(h_{\restriction X_y})\big).$$

can always be extended to a global one

$$S \in H^0\big(X, K_X \otimes f^*L \otimes L_1\big)$$

with certain L^2 estimates which do not depend on L_1.

PROOF. Since L is ample over Y, one can find a smooth hermitian metric h_0 on L with the curvature form $\mathrm{i}\Theta_{L,h_0} \geqslant \omega$, where ω is some Kähler form on Y.

By the lower bound of Seshadri constant $\epsilon(L, y) > n$, we can find a global quasi-psh function φ with neat singularities on Y such that

(a) $\mathrm{i}\Theta_{L,h_0} + \sqrt{-1}\partial\bar\partial\varphi \geqslant 0$;
(b) φ is smooth outside y;
(c) on a neighborhood W of y, we have

$$\varphi = (1+\delta)n \log \sum |z - y|^2 + w(z)$$

where $\delta > 0$ and $w(z) \in \mathscr{C}^\infty(W)$ with $w(y) = 0$

Now set $\psi := \frac{1}{1+\delta}\varphi \circ f$, which is a quasi-psh function with neat singularities on X. Moreover, since y is the regular value of f, the inverse image $X_y := f^{-1}\{y\}$ is a closed smooth submanifold of codimension n in X, and the multiplier ideal sheaf

$$\mathscr{J}(\psi) = \mathscr{J}(\mathcal{I}^{\langle n\rangle}_{X_y}) = \mathcal{I}_{X_y}.$$

Here $\mathcal{I}^{\langle n\rangle}_{X_y}$ is the ideal sheaf consisting of germs of functions that have multiplicity $\geqslant n$ at a general point of X_y:

$$\mathcal{I}^{\langle n\rangle}_{X_y} := \{f \in \mathscr{O}_X \,|\, \mathrm{ord}_x(f) \geqslant n \text{ for a general point } x \in X\}.$$

Thus $\mathscr{J}(\psi)$ has log canonical singularities, and we have

$$\mathrm{i}\Theta_{L_1,h} + \mathrm{i}\Theta_{f^*L, f^*h_0} + \alpha\sqrt{-1}\partial\bar\partial\psi \geqslant 0$$

for all $\alpha \in [1, 1+\delta]$. Then for any section s of

$$H^0\big(X_y, (K_X \otimes f^*L \otimes L_1)_{\restriction X_y} \otimes \mathscr{J}(h_{\restriction X_y})\big),$$

we can apply Theorem 5.2.3 to extend s to a global section

$$S \in H^0\big(X, K_X \otimes f^*L \otimes L_1 \otimes \mathscr{J}(h)\big)$$

such that

$$\int_X \gamma(\delta\psi)|S|^2_{\omega,f^*h_0h_1} e^{-\psi} dV_{X,\omega} \leqslant \frac{34}{\delta} \int_{X_y} |s|^2_{\omega,f^*h_0h_1} dV_{X_y,\omega}[\psi].$$

Assume that $\dim(X) = m + n$. From (5.2.3) one can then check that $dV_{X_y,\omega}[\psi]$ is the smooth measure supported on X_y, such that

$$dV_{X_y,\omega}[\psi] = C_0 \frac{\omega^m_{\restriction X_y}}{m!},$$

where C_0 is some contant depending only on m, n. Since δ depends only on $\epsilon(L, y)$, write $C := \frac{34}{\delta}C_0$ which does not depend on L_1. We thus obtain

$$\int_X \gamma(\delta\psi)|S|^2_{\omega,f^*h_0h_1}e^{-\psi}dV_{X,\omega} \leqslant C_0\int_{X_y}|s|^2_{\omega,f^*h_0h_1}\frac{\omega^m_{\restriction X_y}}{m!}, \tag{5.2.5}$$

where the L^2 estimate does not depend on L_1. □

5.2.3. The Extension Theorem for Twisted Pluricanonical Bundles. We recall the following twisted pluricanonical extension theorem, which was inspired by that used by J. Cao to prove the local triviality of Albanese maps of projective manifolds with nef anticanonical bundles [**Cao16**]. It is a consequence of [**BP08**, Section A.2].

Theorem 5.2.4. *Let Y be a n-dimensional projective manifold and let A_Y be any line bundle on Y such that the difference $A_Y - K_Y$ is an ample line bundle. Let $f : X \to Y$ be a surjective morphism from a smooth projective manifold X to Y and L be a pseudo-effective line bundle on X with a possible singular metric h_L such that*

$$\mathrm{i}\Theta_{h_L}(L) \geqslant 0.$$

Assume that for some regular value z, we have

(i) all the sections of the bundle $mK_{X_z} + L$ extend near z,

(ii) $H^0\big(X_z, (mK_{X_z} + L_{\restriction X_z}) \otimes \mathscr{I}(h^{\frac{1}{m}}_{L\restriction X_z})\big) \neq \varnothing$.

Then for $y \in Y$ such that

(a) y is the regular value of f,

(b) the Seshadri constant $\epsilon(A_Y - K_Y, y) > n$,

the restriction map

$$H^0(X, mK_{X/Y} + L + f^*A_Y) \to H^0\big(X_y, (mK_{X_y} + L_{\restriction X_y}) \otimes \mathscr{I}(h^{\frac{1}{m}}_{L\restriction X_y})\big)$$

is surjective. In particular, the choice of A_Y depends only on Y and is independent of f, L, m.

Proof. Thanks to [**BP10**, A.2.1], the conditions (i) and (ii) imply that there exists a m-relative Bergman type metric $h_{m,B}$ on $mK_{X/Y} + L$ with respect to h_L such that $i\Theta_{h_{m,B}}(mK_{X/Y} + L) \geqslant 0$. Thus $h := \frac{m-1}{m}h_{m,B} + \frac{1}{m}h_L$ defines a possible singular metric on

$$\widetilde{L} := \frac{m-1}{m}(mK_{X/Y} + L) + \frac{1}{m}L = (m-1)K_{X/Y} + L,$$

with $i\Theta_h(\widetilde{L}) \geqslant 0$.

Take any $s \in H^0\big(X_y, (mK_{X_y} + L_{\restriction X_y}) \otimes \mathscr{I}(h^{\frac{1}{m}}_{L\restriction X_y})\big)$, by the construction of the m-relative Bergman kernel metric, $|s|^2_{h_{m,B}}$ is $\mathscr{C}^0$-bounded. Then we see that

$$\begin{aligned}\int_{X_y}|s|^2_{\omega,h}dV_{X_y,\omega} &= \int_{X_y}|s|^{\frac{2(m-1)}{m}}_{h_{m,B}}|s|^{\frac{2}{m}}_{\omega,h_L^{\frac{1}{m}}}dV_{X_y,\omega}\\ &\leqslant C\int_{X_y}|s|^{\frac{2}{m}}_{\omega,h_L^{\frac{1}{m}}}dV_{X_y,\omega} < +\infty.\end{aligned}$$

We then can apply Corollary 5.2.1 to $K_X + \widetilde{L} + f^*(A_Y - K_Y)$, to extend s to a section in $H^0(X, K_{X/Y} + \widetilde{L} + f^*A_Y)$. In conclusion, the restriction

$$H^0(X, mK_{X/Y} + L + f^*A_Y) \to H^0\big(X_y, (mK_{X_y} + L_{\restriction X_y}) \otimes \mathscr{I}(h^{\frac{1}{m}}_{L\restriction X_y})\big)$$

is surjective and the theorem is proved. □

5.3. On the Conjecture of Popa and Schnell

Let $f : X \to Y$ be the surjective morphism between smooth projective manifolds, and let L be an ample line bundle on Y with a smooth hermitian metric h_0 such that the curvature form $\mathrm{i}\Theta_{h_0} \geqslant \omega$ for some Kähler metric ω on Y. Assume that $\dim(Y) = n$ and $\dim(X) = m + n$. Fix any point y on Y which is the regular value of f. Take any positive real number ν such that

$$\epsilon(L, y) > \frac{1}{\nu}.$$

Then we have

$$\epsilon(\lceil n\nu\rceil L, y) > n.$$

Set $\tilde{L} := \lceil n\nu \rceil f^*L$ with the smooth hermitian metric $\tilde{h} := f^*h_0^{\lceil n\nu \rceil}$, then we can restate Corollary 5.2.1 in the following variant form:

PROPOSITION 5.3.1. There is a globally defined quasi-psh function ψ_0 defined over X and a positive number δ such that, for any pseudo-effective line bundle L_1 equipped with the possible singular hermitian metric h_1, whose curvature current $i\Theta_{L_1,h_1} \geqslant 0$ and h_1 is not identically zero when restricted on X_y, for any section

$$s \in H^0\big(X_y, (K_X \otimes \tilde{L} \otimes L_1)_{\upharpoonright X_y} \otimes \mathscr{I}(h_{1\upharpoonright X_y})\big),$$

there is a global section

$$S \in H^0(X, K_X \otimes \tilde{L} \otimes L_1)$$

whose restriction to X_y is s, such that

$$\int_X \gamma(\delta\psi_0)|S|^2_{\omega,\tilde{h}h_1} e^{-\psi_0} dV_{X,\omega} \leqslant C \int_{X_y} |s|^2_{\omega,\tilde{h}h_1} dV_{X_y,\omega}.$$

Here $dV_{X_y,\omega} := \frac{\omega^m_{\upharpoonright X_y}}{m!}$, and C is some constant which does not depend on L_1.

Thus from Proposition 5.3.1, if we set L_1 to be the trivial bundle on X, we see that the following morphism

$$H^0(X, K_X \otimes f^*L^{\otimes \lceil n\nu \rceil}) \to H^0\big(X_y, (K_X \otimes f^*L^{\otimes \lceil n\nu \rceil})_{\upharpoonright X_y}\big)$$

is always surjective. As one can take ν to be arbitrary close to $\frac{1}{\epsilon(L,y)}$ so that $\lceil n\nu \rceil = \left\lfloor \frac{n}{\epsilon(L,y)} \right\rfloor + 1$, we see that the direct image $f_*K_X \otimes L^{\otimes \lfloor \frac{n}{\epsilon(L,y)} \rfloor + 1}$ is generated by global sections at y. Since y is an arbitrary regular value of f, we thus prove Theorem O for $k = 1$. In order to prove the theorem for any $k \geqslant 2$, we need to apply the techniques in proving Siu's invariance of plurigenera [**Siu97**] by Păun [**Pau07**].

PROOF. (Proof of Theorem O) Fix any $k \geqslant 2$ and any $\sigma \in H^0\big(X_y, k(K_X + \tilde{L})_{\upharpoonright X_y}\big)$. We want to find a global section $\Sigma \in H^0\big(X, k(K_X + \tilde{L})\big)$ whose restriction to X_y is σ.

Choose a very ample line bundle A on X such that for every $r = 0, \ldots, k-1$, the line bundle $F_{0,r} := r(K_X + \tilde{L}) + A$ is globally generated by sections

$$\{u_j^{(0,r)}\}_{j=1,\ldots,N_r} \subset H^0(X, F_{0,r}).$$

We then define inductively a sequence of line bundles

$$F_{q,r} := (qk + r)(K_X + \tilde{L}) + A$$

for any $q \geqslant 0$, and $0 \leqslant r \leqslant k-1$. By constructions we have

$$\begin{cases} F_{q,r+1} = K_X + F_{q,r} + \tilde{L} & \text{if} \quad r < k-1, \\ F_{q+1,0} = K_X + F_{q,k-1} + \tilde{L} & \text{if} \quad r = k-1. \end{cases} \tag{5.3.1}$$

We are going to construct inductively families of sections, say $\{u_j^{(q,r)}\}_{j=1,\ldots,N_r}$, of $F_{q,r}$ over X, together with ad hoc L^2 estimates, such that each $u_j^{(q,r)}$ is an extension of $v_j^{(q,r)}$, where we set

$$v_j^{(q,r)} := \sigma^q u^{(0,r)}_{j\upharpoonright X_y} \in H^0(X, F_{q,r}).$$

Now, by induction, assume that such $\{u_j^{(q,r)}\}_{j=1,\ldots,N_r}$ above can be constructed. Then $F_{p,r}$ can be equipped with a natural singular hermitian metric $h_{q,r}$ defined by

$$|\xi|^2_{h_{q,r}} := \frac{|\xi|^2}{\sum_{j=1}^{N_r} |u_j^{(q,r)}|^2},$$

such that $i\Theta_{h_{q,r}} \geqslant 0$. Let h_{K_X} be the smooth hermitian metric of the canonical bundle K_X induced by the volume form $dV_{X,\omega}$, and set $\hat{h} := h_{K_X}\tilde{h}$ to be the smooth metric on $K_X + \tilde{L}$, then by construction the pointwise norm with respect to the metric $h_{q,r}$ is

$$\begin{cases} |v_j^{(q,r+1)}|^2_{\omega,h_{q,r}\tilde{h}} = \dfrac{|v_j^{(0,r+1)}|^2_{\hat{h}^{r+1}h_A}}{\sum_{i=1}^{N_r} |v_i^{(0,r)}|^2_{\hat{h}^r h_A}} & \text{if} \quad r < k-1, \\ |v_j^{(q+1,0)}|^2_{\omega,h_{q,r}\tilde{h}} = \dfrac{|\sigma|_{\hat{h}^k} |v_j^{(0,0)}|^2_{h_A}}{\sum_{j=1}^{N_r} |v_i^{(0,r)}|^2_{\hat{h}^{k-1}h_A}} & \text{if} \quad r = k-1. \end{cases} \tag{5.3.2}$$

where h_A is a smooth hermitian metric on A with strictly positive curvature. Since the sections $\{v_i^{(0,r)}\}_{i=1,N_r}$ generates $F_{0,r\restriction X_y}$, there is a constant $C_1 > 0$ such that (5.3.2) is uniformly $\mathscr{C}^0$ bounded above by C_1. From (5.3.1), it then follows from Proposition 5.3.1 that one can extend $v_j^{(q,r+1)}$ (or $v_j^{(q+1,0)}$ if $r = k-1$) into a section $u_j^{(q,r+1)}$ ($u_j^{(q+1,0)}$ respectively) over X such that

$$(5.3.3)\qquad \begin{cases} \displaystyle\int_X \gamma(\delta\psi_0)e^{-\psi_0}\sum_{j=1}^{N_{r+1}}|u_j^{(q,r+1)}|^2_{\omega,h_{q,r}\tilde{h}}dV_{X,\omega} \leqslant C_2 & \text{if} \quad r < k-1,\\ \displaystyle\int_X \gamma(\delta\psi_0)e^{-\psi_0}\sum_{j=1}^{N_0}|u_j^{(q+1,0)}|^2_{\omega,h_{q,r}\tilde{h}}dV_{X,\omega} \leqslant C_2 & \text{if} \quad r = k-1. \end{cases}$$

for some uniform constant C_2. From (5.3.2), (5.3.3) is equivalent to

$$(5.3.4)\qquad \begin{cases} \displaystyle\int_X \gamma(\delta\psi_0)e^{-\psi_0}\frac{\sum_{i=1}^{N_{r+1}}|u_i^{(q,r+1)}|^2_{\hat{h}^{qk+r+1}h_A}}{\sum_{i=1}^{N_r}|u_i^{(q,r)}|^2_{\hat{h}^{qk+r}h_A}}dV_{X,\omega} \leqslant C_2 & \text{if} \quad r < k-1,\\ \displaystyle\int_X \gamma(\delta\psi_0)e^{-\psi_0}\frac{\sum_{i=1}^{N_0}|u_i^{(q,r+1)}|^2_{\hat{h}^{qk+k}h_A}}{\sum_{i=1}^{N_r}|u_i^{(q,r)}|^2_{\hat{h}^{qk+k-1}h_A}}dV_{X,\omega} \leqslant C_2 & \text{if} \quad r = k-1. \end{cases}$$

Let us denote by

$$a_{qk+r}(x) := \sum_{i=1}^{N_r}|u_i^{(q,r)}|_{\hat{h}^{qk+r}h_A},$$

which is a quasi-psh and bounded non-negative smooth function on X. By the integrability of $\log\gamma(\delta\psi_0)$ and ψ_0 with respect to the standard Lebesgue measure over X, combined with the concavity property of the logarithmic function as well as the Jensen inequality, we can find some constant C_3 and C_4 such that

$$(5.3.5)\qquad \int_X \log\frac{a_l}{a_{l-1}}dV_{X,\omega} \leqslant C_3 - \int_X \log\gamma(\delta\psi_0)dV_{X,\omega} + \int_X \psi_0 dV_{X,\omega} \leqslant C_4$$

for any $l \geqslant 1$. Since $a_1(x)$ is a bounded smooth function on X, we can also find a constant $C_5 \geqslant C_4$ such that

$$\int_X \log a_1 dV_{X,\omega} \leqslant C_5.$$

Combined these inequalities together we obtain

$$\int_X \frac{\log a_l}{l}dV_{X,\omega} \leqslant C_5$$

for any $l \geqslant 1$. Set $f_q := \frac{\log a_{qk}}{q}$, and we have the following properties:

(a) for any $q \geqslant 1$, we have

$$\int_X f_q dV_{X,\omega} \leqslant C_5;$$

(b) the inequality

$$k\Theta_{\hat{h}}(K_X + \tilde{L}) + \sqrt{-1}\partial\overline{\partial}f_q \geqslant -\frac{1}{q}\Theta_{h_A}(A)$$

holds true in the sense of currents on X;

(c) on X_y the following equality is satisfied

$$f_{q\restriction X_y} = \log|\sigma|^2_{\hat{h}^k} + a_0(x)_{\restriction X_y}$$

where $a_0(x) = \log\sum_{i=1}^{N_0}|u_i^{(0,0)}|_{h_A}$ is a smooth function on X.

By the mean value inequality for the psh functions, as a consequence of the properties (a) and (b), one can show the existence of a uniform upper bound for the functions f_q over X. Thus the sequence $f_q(z)$ must have some subsequence which converges in L^1 topology on X to the potential f_∞, in the form of the regularized limit

$$f_\infty(z) := \limsup_{\zeta\to z}\lim_{q_\nu\to+\infty} f_{q_\nu}(\zeta),$$

which satisfies

$$k\Theta_{\hat{h}}(K_X + \tilde{L}) + \sqrt{-1}\partial\overline{\partial}f_\infty \geqslant 0$$

as a current on X. Moreover, by Property (c) f_∞ is not identically $-\infty$ on X_y, as well as

$$(5.3.6)\qquad f_\infty \geqslant \log|\sigma|^2_{\hat{h}^k} + \mathscr{O}(1)$$

pointwise on X_y.

Now we construct a singular hermitian metric h_∞ on $(k-1)(K_X+L)$ defined by

$$h_\infty := \hat{h}^{k-1}e^{-\frac{k-1}{k}f_\infty}.$$

Then $\Theta_{h_\infty}\big((k-1)(K_X+\tilde{L})\big) \geqslant 0$. Write $k(K_X+L) = K_X+(k-1)(K_X+\tilde{L})+\tilde{L}$, where $(k-1)(K_X+\tilde{L})$ is equipped with the singular hermitian metric h_∞. Since

$$|\sigma|^2_{\omega,\tilde{h}h_\infty} = |\sigma|^2_{\hat{h}h_\infty} = |\sigma|^{\frac{2(k-1)}{k}}_{h_\infty} \cdot |\sigma|^{\frac{2}{k}}_{\hat{h}}$$

which is $\mathscr{C}^0$ bounded, we then can apply Proposition 5.3.1 to extend σ to a global section $\Sigma \in H^0(X, k(K_X+\tilde{L}))$.

In conclusion, for any regular value y of the morphism f, the following morphism

$$H^0(X, K_X^{\otimes k}\otimes f^*L^{\otimes l}) \to H^0\big(X_y, (K_X^{\otimes k}\otimes f^*L^{\otimes l})_{\upharpoonright X_y}\big)$$

is always surjective for any $l > \frac{n}{\epsilon(L,y)}$. Thus Theorem O is proved. □

In order to improve the above quadratic bound to linear, we need to apply the twisted pluricanonical extension theorem in Section 5.2.3 instead. First, we recall the following result arising from birational geometry:

THEOREM 5.3.1. *Let L be an ample line bundle over a projective n-fold Y, then the adjoint line bundle $K_Y+(n+1)L$ is semi-ample.*

Based on the Mori theory, one observes that $n+1$ is the maximal length of extremal rays of smooth projective n-folds, which shows that $K_Y+(n+1)L$ is nef. By the base-point-free theorem, one can even show that $K_Y+(n+1)L$ is semiample. In his work on the Fujita conjecture [**Dem96**], Demailly also gave an analytic proof for the fact that $K_Y+(n+1)L$ is nef.

PROOF. (Proof of Theorem Q) By Theorem 5.2.2, the Seshadri constant

$$\epsilon\big((n^2+1)L, y\big) > n$$

for a generic $y \in Y$. From Theorem 5.3.1, we see that $K_Y+(n+1)L$ can be equipped with a smooth hermitian metric h with semi-positive curvature. Applying Theorem 5.2.4, we see that for any $m \geqslant 1$, the restriction map

$$H^0\Big(X, mK_{X/Y}+(m-1)f^*\big(K_Y+(n+1)L\big)+f^*\big(K_Y+(n^2+1)L\big)\Big) \to H^0(X_y, mK_{X\upharpoonright X_y})$$

is surjective for a generic y in Y. In other words, for any $k \geqslant 1$ and any $l \geqslant k(n+1)+n^2-n$, the direct image

$$f_*(K_X^{\otimes k})\otimes L^{\otimes l}$$

is generated by global sections at the generic points of Y. This completes the proof of Theorem Q. □

5.4. On a Question of Demailly-Peternell-Schneider

In this section, we prove Theorem R and thus give an affirmative answer to Problem 5.1.1 in the case that both X and Y are smooth manifolds.

PROOF. (Proof of Theorem R) Take a sufficient ample line bundle A_X on X such that A_X+D is ample, and the direct image $f_*(A_X)$ is a torsion free coherent sheaf which is not only locally free but also globally generated over the Zariski open set $X^\circ := X\backslash f^{-1}(Z)$. Then $f_*(A_X)$ is locally free outside a subvariety $W \subset Z$ of codimension at least 2. Set r to be the generic rank of $f_*(A_X)$, and denote by

$$\det f_*(A_X) := \wedge^r f_*(A_X)^{\star\star}$$

to be the bidual of $\wedge^r f_*(A_X)$ which is an invertible sheaf over Y, then there is coherent ideal sheaf $\mathcal{I}$ supported on W such that

$$\wedge^r f_*(A_X) = \det f_*(A_X)\otimes\mathcal{I}.$$

Take a smooth hermitian metric h on A_X+D such that $\mathrm{i}\Theta_h \geqslant 3\omega$ for some Kähler metric ω. Let us also choose a very ample line bundle A_Y on Y such that A_Y-K_Y generates $n+1$ jets everywhere and $A_Y+\det f_*(A_X)$ is also an ample line bundle on Y. In particular, the Seshadri constant $\epsilon(A_Y-K_Y, y) > n$ for any y.

By the definition of non-nef locus, for any pseudo-effective line bundle E on X, we have

$$\mathbf{B}_-(E) = \bigcup_{m\in\mathbb{N}}\bigcap_{k\in\mathbb{N}} \mathrm{Bs}(kA_X+kmE).$$

Equivalently, in [**BDPP13**], it is shown that

$$\mathbf{B}_-(E) = \bigcup_{m\in\mathbb{N}} \bigcap_T E_+(T),$$

where T runs over the set $c_1(E)[-\frac{1}{m}\omega]$ of all closed real $(1,1)$-currents $T \in c_1(E)$ such that $T \geqslant -\frac{1}{m}\omega$, and $E_+(T)$ denotes the locus where the Lelong numbers of T are strictly positive. By [**Bou02**], there is always a current $T_{\min,m}$ which achieves minimum singularities and minimum Lelong numbers among all members of $c_1(E)[-\frac{1}{m}\omega]$ hence

$$\mathbf{B}_-(E) = \bigcup_{m\in\mathbb{N}} E_+(T_{\min,m}).$$

By Demailly's regularization theorem in [**Dem92b**], for every $m \in \mathbb{N}$, we can find a closed (1,1)-current $T_m \in c_1(E)$ with neat singularities such that $T_m \geqslant -\frac{2}{m}\omega$, and

$$E_+(T_{\min,2m}) \subset E_+(T_m) \subset E_+(T_{\min,m}).$$

Equivalently, there exists a singular hermitian metric $\tilde{h}_m$ on E with neat singularities, such that the curvature current

$$\mathrm{i}\Theta_{\tilde{h}_m} = T_m \geqslant -\frac{2}{m}\omega.$$

Set $E := -(K_X + D) - f^*\Delta$. Since $\mathbf{B}_-\big(-(K_X + D) - f^*\Delta\big)$ does not project onto Y, thus for any $m \in \mathbb{N}$, $Z_m := f\big(E_+(T_m)\big)$ is a proper subvariety of Y, and the singular hermitian metric $\tilde{h}_m^{\otimes m}h$ on $-m(K_X + D) - mf^*\Delta + A_X + D$ is smooth on $X\backslash f^{-1}(Z_m)$.

For the $\mathbb{Q}$-effective divisor $D = \sum_{i=1}^t a_i D_i$, there is a canonical singular hermitian metric h_D defined on D, with the local weight

$$\varphi_D = \sum_{i=1}^t a_i \log|g_i|,$$

where $g_i \in \Gamma(U, \mathscr{O}_U)$ is a holomorphic function locally defining D_i on some open set $U \subset X$. Therefore, the curvature current

$$\mathrm{i}\Theta_{h_D} = [D] \geqslant 0,$$

and thus h_D is a singular hermitian metric with neat singularities.

Recall that Z_D is denoted to be the minimal set containing Z, such that for every $y \notin Z_D$, the pair $(X_y, D_{\restriction X_y})$ is also lc. Here we denote by $X_y := f^{-1}(y)$. Since (X, D) is lc, thus Z_D is an at most countable union of proper subvarieties of Y. Indeed, the set

$$Y_m := \{y \notin Z | (X_y, (1 - \frac{1}{m})D_{\restriction X_y}) \text{ is klt}\}$$

is an Zariski open set of Y. Therefore, we have

$$Z_D = \bigcup_{m=1}^{\infty} Y\backslash Y_m.$$

Thus for the singular hermitian metric $h_m := \tilde{h}_m^{\otimes m} h h_D^{\otimes m-1}$ on $-mK_X + A_X - mf^*\Delta$, the multiplier ideal sheaf

$$\mathscr{I}(h_{m\restriction X_y}^{\frac{1}{m}}) = \mathscr{I}((1 - \frac{1}{m})D_{\restriction X_y}) = \mathscr{O}_{X_y}$$

for any $y \notin Z_m \cup Y\backslash Y_m$. Moreover, the curvature current $\mathrm{i}\Theta_{h_m} \geqslant \omega$.

By Theorem 5.2.4 applied with $L = -mK_X + A_X - mf^*\Delta$ equipped with the hermitian metric h_m, the restriction is surjective:

$$H^0(X, mK_{X/Y} - mK_X + A_X - mf^*\Delta + f^*A_Y) \to H^0(X_y, A_{X\restriction X_y})$$

for any $y \notin Z_m \cup Y\backslash Y_m$. In other words, the direct image sheaf

$$f_*\big(mK_{X/Y} - mK_X + A_X - mf^*\Delta + f^*A_Y\big) = (-K_Y - \Delta)^m \otimes A_Y \otimes f_*(A_X) \tag{5.4.1}$$

is generated by global sections over $Y_m\backslash Z_m$, and by the assumption that $f_*(A_X)$ is locally free over $Y\backslash Z$, we conclude that the top exterior power

$$\wedge^r\big((-K_Y - \Delta)^m \otimes A_Y \otimes f_*(A_X)\big) = (-K_Y - \Delta)^{rm} \otimes A_Y^r \otimes \det f_*(A_X) \otimes \mathcal{I}$$

is also generated by global sections over $Y_m\backslash Z_m$. In particular, for every $m \in \mathbb{N}$, the base locus

$$\mathrm{Bs}\big((-K_Y - \Delta)^{rm} \otimes A_Y^r \otimes \det f_*(A_X)\big) \subset Z_m \bigcup Y\backslash Y_m. \tag{5.4.2}$$

By our choice of A_Y, $rA_Y + \det f_*(A_X)$ is an ample line bundle on Y, thus let m tends to infinity, we obtain the pseudo-effectivity of $-K_Y - \Delta$. Moreover, from (5.4.2) we see that the non-nef locus

$$\mathbf{B}_-(-K_Y - \Delta) \subset \bigcup_{m=1}^{\infty} Z_m \bigcup Y \backslash Y_m = f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \bigcup Z_D.$$

Hence Theorem R is proved. □

If f is a smooth fibration, Supp(D) is a simple normal crossing divisor, and Supp(D) is relatively normal crossing over Y, then the condition that (X, D) is lc implies that $(X_y, D_{\upharpoonright X_y})$ is also lc for every $y \in Y$. Thus $Z_D = \varnothing$. If $-K_X - D - f^*\Delta$ is nef, then $\mathbf{B}_-(-K_X - D - f^*\Delta) = \varnothing$. Thus from Theorem R, $\mathbf{B}_-(-K_Y - \Delta)$ is also empty which implies that $-K_Y - \Delta$ is nef. This completes our proof of Theorem 5.1.3.

By setting $D = 0$ and $\Delta = 0$ in Theorem 5.1.3, the following theorem by Miyaoka is a direct consequence.

THEOREM 5.4.1. *(Miyaoka) Let $f : X \to Y$ be a smooth morphism between smooth projective manifolds X and Y. If $-K_X$ is nef, then so is $-K_Y$.*

REMARK 5.4.1. The original proof of Miyaoka [**Miy93**] relies on the mod p reduction arguments. There is also another Hodge theoretic proof by Fujino and Gongyo without using the mod p reduction arguments [**FG14**].

REMARK 5.4.2. In [**BBP13**], for any pseudo-effective line bundle L, $\mathbf{B}_-(L)$ is called *restricted base locus* of L, and the *non-nef locus* NNef(L) [**BBP13**, Definition 1.7] is defined in terms of the asymptotic or numerical vanishing orders attached to $|L|$. If the underlying projective variety X is smooth, then we have

$$\mathbf{B}_-(L) = \mathrm{NNef}(L).$$

Since in this chapter we always assume that X and Y are smooth projective manifolds, we do not distinguish these two equivalent objects.

REMARK 5.4.3. In [**CZ13**], M. Chen and Q. Zhang proved the similar result that, for the surjective morphism from the log canonical pair (X, D) onto a $\mathbb{Q}$-Gorenstein variety Y, if $-(K_X + D)$ is nef, then $-K_Y$ is pseudo-effective. In a very recent preprint [**Ou17**], W. Ou extended the theorem by Chen-Zhang to the rational dominant maps, which was a crucial step in his proof of the *generic nefness conjecture* for tangent sheaves by T. Peternell [**Pet12**, Conjecture 1.5].

5.5. On the Inheritance of the Image

5.5.1. On the Images of Weak KLT Fano Manifolds. One says that a projective manifold X is weak Fano if $-K_X$ is big and nef. In the series of articles [**FG12**] and [**FG14**], Fujino and Gongyo studied the image of weak Fano manifolds. They proved the following theorem:

THEOREM 5.5.1. *(Fujino-Gongyo) Let $f : X \to Y$ be a smooth fibration between two smooth manifolds X and Y. If X is weak Fano, then so is Y.*

In this section, we are going to prove a more general theorem as follows:

THEOREM 5.5.2. *Let $f : X \to Y$ be a surjective morphism between two smooth manifolds X and Y. Let D be an effective $\mathbb{Q}$-divisor such that (X, D) is klt. Let Δ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y. If $-K_X - D - f^*\Delta$ is big and its non-nef locus $\mathbf{B}_-(-K_X - D - f^*\Delta)$ does not project onto Y, then $-K_Y - \Delta$ is big.*

PROOF. Take a very ample line bundle A_Y over Y such that A_Y generates $n+1$ jets everywhere. Since $-K_X - D - f^*\Delta$ is big, we can find a positive integer a such that $-a(K_X + D + f^*\Delta) - 2f^*A_Y$ is effective. Fix any effective divisor $E \in |-a(K_X + D + f^*\Delta) - 2f^*A_Y|$. Since (X, D) is klt, then there exists a positive integer $m > a$ such that the multiplier ideal sheaf

$$\mathscr{I}\left(\frac{1}{m-1}E_{\upharpoonright X_y}\right) = \mathscr{I}\left(\frac{m}{m-1}D_{\upharpoonright X_y}\right) = \mathscr{O}_{X_y} \tag{5.5.1}$$

for the generic fiber X_y. We can also find a singular hermitian metric h_1 with neat singularities on $-(m^2 - a)(K_X + D + f^*\Delta)$ such that $i\Theta_{h_1} \geqslant \tilde{\omega}$ for some Kähler metric $\tilde{\omega}$ on X. Take some small rational number $\epsilon > 0$ such that $\mathscr{I}(h^{\epsilon}_{1\upharpoonright X_y}) = \mathscr{O}_{X_y}$ for the generic fiber X_y.

On the other hand, since the non-nef locus $\mathbf{B}_-(-K_X - D - f^*\Delta)$ does not project onto Y, from the proof of Theorem R in Section 5.4, we can find a singular hermitian metric h_ϵ over $-(m^2 - a)(K_X + D + f^*\Delta)$ with neat singularities, such that $i\Theta_{h_\epsilon} \geqslant -\epsilon\tilde{\omega}$ and the singularities of h_ϵ does not project onto Y. Set $h := h_1^{\epsilon} h_\epsilon^{1-\epsilon}$

which is also a hermitian metric on $-(m^2-a)(K_X+D+f^*\Delta)$, then we have $\mathrm{i}\Theta_h \geqslant \epsilon^2\tilde{\omega}$ and the multiplier ideal sheaf

$$\mathscr{J}(h_{\upharpoonright X_y}) = \mathscr{O}_{X_y} \tag{5.5.2}$$

for the generic fiber X_y.

Take a generic fiber X_y of f such that y is the regular value of f, and both (5.5.1) and (5.5.2) are satisfied. We equip the line bundle $-m^2(K_X+D+f^*\Delta)-2f^*A_Y+m^2D$ with the singular hermitian metric $h_0 := h_E h h_D^{\otimes m^2}$, where h_E (resp. h_D) is the tautological singular hermitian metric on $-a(K_X+D+f^*\Delta)-2f^*A_Y$ (resp. D) induced by the effective divisor E (resp. D), such that

$$\mathrm{i}\Theta_{h_E} = [E](\text{ resp. } \mathrm{i}\Theta_{h_D} = [D]).$$

Then we claim that the multiplier ideal sheaf $\mathscr{J}(h_{0\upharpoonright X_y}^{\frac{1}{m^2}}) = \mathscr{O}_{X_y}$. Indeed, for any $s \in \mathscr{O}_{X_y,z}$, let φ_E, φ_D and φ be the weights of the metric h_E, h_D and h on a small neighborhood $U \subset X_y$ of a point $z \in X_y$. Then by the Hölder inequality we have

$$\int_U |s|^2 e^{-\frac{\varphi_E+\varphi}{m^2}+\varphi_D} \leqslant \Big(\int_U |s|^2 e^{-\varphi}\Big)^{\frac{1}{m^2}} \cdot \Big(\int_U |s|^2 e^{-\frac{\varphi_E}{m-1}}\Big)^{\frac{m-1}{m^2}} \cdot \Big(\int_U |s|^2 e^{-\frac{m}{m-1}\varphi_D}\Big)^{\frac{m-1}{m}} < +\infty.$$

Here we use the conditions (5.5.1) and (5.5.2). By applying Theorem 5.2.4 with $L = -m^2(K_X+f^*\Delta) - 2f^*A_Y$ endowed with the singular hermitian metric h_0, we obtained the desired surjectivity:

$$H^0\big(X, m^2K_{X/Y} + (-m^2K_X - m^2f^*\Delta - 2f^*A_Y) + f^*A_Y\big) \to H^0\big(X_y, f^*(-m^2K_Y - m^2\Delta - A_Y)_{\upharpoonright X_y}\big) = \mathbb{C}^l,$$

where l is the number of the connected components of X_y. In particular, we have the non-vanishing

$$H^0\big(X, f^*(-m^2K_Y - m^2\Delta - A_Y)\big) \neq 0.$$

Now we claim that $-m^2K_Y - m^2\Delta - A_Y$ is a pseudo-effective line bundle over Y. Indeed, we first take a stein factorization of f

$$X \xrightarrow{f'} Y' \xrightarrow{p} Y,$$

where $p : Y' \to Y$ is a finite surjective morphism and the morphism $f' : X \to Y'$ has connected fibers. Then we have an isomorphism

$$f'_* : H^0\big(X, f^*(-m^2K_Y - m^2\Delta - A_Y)\big) \xrightarrow{\cong} H^0\big(Y', p^*(-m^2K_Y - m^2\Delta - A_Y)\big),$$

which implies that the line bundle $p^*(-m^2K_Y - m^2\Delta - A_Y)$ is effective. Since $p : Y' \to Y$ is a finite surjective morphism, by a result of S. Boucksom [**Bou02**, Proposition 4.2], $-m^2K_Y - m^2\Delta - A_Y$ is a pseudo-effective line bundle, which also shows that $-K_Y - \Delta$ is big. □

Therefore, we can extend Theorem 5.5.1 to the weak klt Fano cases:

PROOF OF THEOREM T. Sicen f is a smooth fibration, (X, D) is klt, and $(X_y, D_{\upharpoonright X_y})$ is also klt for every $y \in Y$, from the very definition of Z_D in Theorem R we see that $Z_D = \varnothing$. By the nefness of $-(K_X + D) - f^*\Delta$, the set

$$\mathbf{B}_-\big(-(K_X + D) - f^*\Delta\big) = \varnothing.$$

Thus from Theorem R we conclude that $-K_Y - \Delta$ is nef. The bigness of $-K_Y - \Delta$ follows from Theorem T directly. This completes the proof. □

By setting $D = 0$ and $\Delta = 0$ in Theorem T, we obtain Theorem 5.5.1 directly.

REMARK 5.5.1. If we only assume that $-K_X$ is big, then the following example given in [**FG12**] shows that, even if f is smooth, $-K_Y$ is not big.

EXAMPLE 5.1. Let $E \subset \mathbb{P}^2$ be a smooth cubic curve. Consider $f : X = \mathbb{P}_E(\mathscr{O}_E \oplus \mathscr{O}_E(1)) \to E = Y$. Then, we see that $-K_X$ is big. However, $-K_Y$ is not big since E is a smooth elliptic curve.

It is noticeable that, in [**Pau12**] S. Boucksom pointed out that the following theorem, which is a special case of Theorem 1.2 in [**Ber09**], implies [**FG12**, Theorem 4.1] or [**KMM92**, Corollary 2.9]:

THEOREM 5.5.3. *(Boucksom-Păun) Let $f : X \to Y$ be a smooth fibration between two smooth manifolds. If $-K_X$ is semi-positive (strictly positive), then $-K_Y$ is semi-positive (strictly positive).*

Finally, let us mention that, in [**FG12**], the authors raised the following conjecture, which was solved very recently by C. Birkar and Y. Chen [**BC16**]:

THEOREM 5.5.4. *(Fujino-Gongyo, Birkar-Chen) Let $f : X \to Y$ be a smooth fibration between two smooth projective manifolds. If $-K_X$ is semi-ample, then so is $-K_Y$.*

The proof in [**BC16**] relies on very deep consequences of the minimal model program in birational geometry and of Hodge theory. It is an interesting question to know whether we can use pure analytic methods to give a new proof of this theorem.

5.5.2. On the Rational Connectedness of the Image. By Mori's bend-and-break, Fano varieties are uniruled; in fact by [**Cam92**, **KMM92**] a stronger result holds: the projective Fano variety is rationally connected. Later on Q. Zhang and Hacon-McKernan proved that the same conclusion holds for a klt pair (X, D) such that $-(K_X + D)$ is big and nef [**Zha06**, **HM07**]. This was generalized by Broustet and Pacienza [**BrP11**, Theorem 1.2], who proved that a klt pair (X, D) with $-(K_X + D)$ big is rationally connected modulo the non-nef locus of $-(K_X + D)$, that is, there exists an irreducible component V of $\mathbf{B}_-\big(-(K_X + D)\big)$ such that for any general point x of X there exists a rational curve R_x passing through X and intersecting V. Moreover, they also proved the following result for the image:

Theorem 5.5.5. *(Broustet-Pacienza) Let (X, D) be a klt pair such that $-(K_X + D)$ is big. Let $f : X \dashrightarrow Y$ be a dominant rational map with connected fibers such that the non-nef locus of $-(K_X + D)$ does not dominate Y, then Y is uniruled.*

In this subsection, we will refine their results in a more general setting. First, we need to extend Theorem 5.5.2 to surjective morphisms between singular varieties.

Theorem 5.5.6. *Let X be a normal projective variety and D an effective $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let Y be a normal and $\mathbb{Q}$-Gorenstein projective variety. If $f : X \to Y$ is a surjective morphism such that $-(K_X + D)$ is big and the restriction of f to $\mathrm{NNef}(-K_X - D) \bigcup \mathrm{Nklt}(X, D)$ does not dominate Y, then $-K_Y$ is also big.*

Proof. Let $p : Y' \to Y$ be a log-resolution of singularities of Y. Let $\pi : X' \to X$ be a log resolution of (X, D), such that the induced rational map $f' : X' \to Y'$ is in fact a morphism. We have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\ \pi\ } & X \\ {\scriptstyle f'}\downarrow & \overset{g}{\searrow} & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\ p\ } & Y. \end{array}$$

Let D' be an effective $\mathbb{Q}$-divisor on X' such that $\pi_*(D') = D$ and $K_{X'} + D' = \pi^*(K_X + D) + F$, with F effective and not having common components with D'. Note that

$$\pi\big(\mathrm{Nklt}(X', D')\big) \subset \mathrm{Nklt}(X, D).$$

By [**BBP13**, Lemma 2.6], we also have

$$\pi\Big(\mathrm{NNef}\big(-\pi^*(K_X + D)\big)\Big) \subset \mathrm{NNef}\big(-(K_X + D)\big).$$

Thus by the assumption of the theorem, we have

$$g\big(\mathrm{NNef}(-K_{X'} - D' + F) \bigcup \mathrm{Nklt}(X', D')\big) \subsetneq Y \tag{5.5.3}$$

Take a very ample line bundle A_Y over Y such that A_Y generates $n + 1$ jets everywhere. We can take an ample line $A_{Y'} := p^*A_Y - E'$ over Y', where $E_Y = \sum_j c_j E_j'$'s are exceptional divisors of p. Since $-K_X - D$ is big, so is $-K_{X'} - D' + F$, and we can find a positive integer a such that $-a(K_{X'} + D' - F) - 2g^*A_Y$ is effective. Fix any effective divisor $E \in |-a(K_{X'} + D' - F) - 2g^*A_Y|$. By (5.5.3), then there exists a positive integer $m > a$ such that the multiplier ideal sheaf

$$\mathscr{I}\big(\frac{1}{m-1} E_{\restriction X_y'}\big) = \mathscr{I}\big(\frac{m}{m-1} D'_{\restriction X_y'}\big) = \mathscr{O}_{X_y'} \tag{5.5.4}$$

for the generic (smooth) fiber X_y' of $g = p \circ f'$. We can also find a singular hermitian metric h_1 with neat singularities on $-(m^2 - a)(-K_{X'} - D' + F)$ such that $\mathrm{i}\Theta_{h_1} \geqslant \tilde{\omega}$ for some Kähler metric $\tilde{\omega}$ on X'. Take some small rational number $\epsilon > 0$ such that $\mathscr{I}(h^{\epsilon}_{1\restriction X_y'}) = \mathscr{O}_{X_y'}$ for the generic fiber X_y'.

On the other hand, from (5.5.3) the non-nef locus $\mathrm{NNef}(-K_{X'} - D' + F)$ does not project onto Y, from the proof of Theorem R in Section 5.4, we can find a singular hermitian metric h_ϵ over $-(m^2 - a)(K_{X'} + D' - F)$ with neat singularities, such that $\mathrm{i}\Theta_{h_\epsilon} \geqslant -\epsilon\tilde{\omega}$ and the singularities of h_ϵ does not project onto Y. Set $h := h_1^{\epsilon} h_\epsilon^{1-\epsilon}$ which is also a hermitian metric on $-(m^2 - a)(K_{X'} + D' - F)$, then we have $\mathrm{i}\Theta_h \geqslant \epsilon^2\tilde{\omega}$ and the multiplier ideal sheaf

$$\mathscr{I}(h_{\restriction X_y'}) = \mathscr{O}_{X_y'} \tag{5.5.5}$$

for the generic fiber X_y' of $g : X' \to Y$.

Take a generic $y \in Y^{\mathrm{reg}}$ such that the fiber X'_y of g is reduced and smooth, and both (5.5.4) and (5.5.5) are satisfied. We equip the line bundle $-m^2(K_{X'} + D' - F) - 2g^*A_Y + m^2 D'$ with the singular hermitian metric $h_0 := h_E h h_{D'}^{\otimes m^2}$, where h_E (resp. $h_{D'}$) is the tautological singular hermitian metric on $-a(K_{X'} + D' - F) - 2g^*A_Y$ (resp. D) induced by the effective divisor E (resp. D), such that

$$\mathrm{i}\Theta_{h_E} = [E](\text{ resp. } \mathrm{i}\Theta_{h_{D'}} = [D]).$$

Then we claim that the multiplier ideal sheaf $\mathscr{J}(h_{0\restriction X'_y}^{\frac{1}{m^2}}) = \mathscr{O}_{X'_y}$. Indeed, for any $s \in \mathscr{O}_{X'_y,z}$, let φ_E, $\varphi_{D'}$ and φ be the weights of the metric h_E, $h_{D'}$ and h on a small neighborhood $U \subset X'_y$ of a point $z \in X'_y$. Then by the Hölder inequality we have

$$\int_U |s|^2 e^{-\frac{\varphi_E+\varphi}{m^2}+\varphi_{D'}} \leqslant (\int_U |s|^2 e^{-\varphi})^{\frac{1}{m^2}} \cdot (\int_U |s|^2 e^{-\frac{\varphi_E}{m-1}})^{\frac{m-1}{m^2}} \cdot (\int_U |s|^2 e^{-\frac{m}{m-1}\varphi_{D'}})^{\frac{m-1}{m}} < +\infty.$$

Here we use the conditions (5.5.1) and (5.5.2). By applying Theorem 5.2.4 to the surjective morphism $g : X' \to Y$ with $L = -m^2(K_{X'} - F) - 2g^*A_Y$ endowed with the singular hermitian metric h_0, we obtained the desired surjectivity:

$$H^0\big(X', m^2K_{X'/Y} + (-m^2K_{X'} - 2g^*A_Y + m^2F) + g^*A_Y\big) \to H^0\big(X'_y, (-m^2g^*K_Y - g^*A_Y + m^2F)_{\restriction X'_y}\big) \neq 0.$$

In particular, we have the non-vanishing

$$H^0\big(X', g^*(-m^2K_Y - A_Y) + m^2F\big) \neq 0.$$

Note that $g = f\circ\pi$. Since X is normal with F the exceptional divisors of the birational morphism $\pi : X' \to X$, we have the following isomorphism

$$H^0\big(X', \pi^*f^*(-m^2K_Y - A_Y) + m^2F\big) \xrightarrow{\cong} H^0\big(X, f^*(-m^2K_Y - A_Y)\big).$$

Now we repeat the same proof of Theorem 5.5.2 to show that $-m^2K_Y - A_{Y'}$ is a pseudo-effective line bundle. Thus $-K_Y$ is a big line bundle. □

Proof of Theorem U. The proof is more or less direct. By Theorem 5.5.6 we see that $-K_Y$ is big. By Broustet-Pacienza's Theorem [**BrP11**, Theorem 1.2], Y is rationally connected modulo the non-nef locus $\mathbf{B}_-(-K_Y)$. The theorem is thus proved. □

Part 4

A Remark on the Corlette-Simpson Correspondence

CHAPTER 6

Semi-stable Higgs Bundles with Vanishing Chern Classes On Kähler Manifolds

6.1. Introduction

Recently, J. Cao proved the conjecture that, for any smooth projective manifold whose anticanonical bundle is nef, the Albanese map of X is locally isotrivial [**Cao17**]. He applied an elegant criteria in [**CH13**] for the local isotriviality of the fibration, relying on the deep results for the numerically flat vector bundles (see Definition 6.2.2 below) in [**DPS94**] and [**Sim92**]:

Theorem 6.1.1. *Let E be a numerically flat vector bundle over a Kähler manifold X. Then*

*(i) [**DPS94**, Theorem 1.18] E admits a filtration*

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_p = E \tag{6.1.1}$$

by vector subbundles such that the quotients E_k/E_{k-1} are hermitian flat, that is, given by unitary representations $\pi_1(X) \to U(r_k)$. In particular, a vector bundle is numerically flat if and only if E is semistable and all the Chern classes of E vanishes.

*(ii) [**Sim92**, Section 3] E is a local system V, and the underlying holomorphic vector bundles of E and V are the same; in the other word, any semistable vector bundle with vanishing Chern classes has a holomorphic flat structure which is an extension of unitary flat bundles.*

Moreover, in [**Sim92**], by introducing the notation of *differential graded category* [**Sim92**, Section 3], plus the *formality isomorphism* [**Sim92**, Lemma 2.2], Simpson can extend the equivalence between the category of polystable Higgs bundles with vanishing Chern classes and the category of semi-simple representations of the fundamental groups, to extensions of irreducible objects on smooth projective manifolds [**Sim92**, Corollary 3.10].

The purpose of this chapter is to give a concrete and constructing proof of Simpson's correspondence for semistable Higgs bundles, for the complex geometers who are not familiar with the language of category.

Theorem V. *Let X be a compact Kähler manifold. Then the following statements are equivalent*

(i) E is a flat vector bundle over X;

(ii) there is a structure of Higgs bundle $(E, \overline{\partial}, \theta)$ over E, such that it admits a filtration of Higgs bundles

$$\{0\} = (E_0, \theta_0) \subset (E_1, \theta_1) \subset \ldots \subset (E_m, \theta_m) = (E, \theta)$$

where $\theta_i := \theta|_{E_i}$, such that the grade terms $(E_i, \theta_i)/(E_{i-1}, \theta_{i-1})$ are stable Higgs bundles with vanishing Chern classes.

(iii) E is a semistable Higgs bundle with vanishing Chern classes;

Remark 6.1.1. In this chapter, we only (re)prove the equivalence between (i) and (ii) in Theorem V. The implication (ii) ⇒ (iii) is trivial. To show that (iii) implies (ii), one only needs to prove that the Jordan-Hölder filtrations of the semistable Higgs bundles with vanishing Chern classes are still a filtration of Higgs bundles, rather than Higgs sheaves. In [**DPS94**], they proved this result for pure vector bundles, *i.e.* the Higgs fields θ vanish. In [**Sim92**, Theorem 2], if X is projective, Simpson proved a slightly stronger result that any reflexive semistable Higgs bundle with vanishing Chern classes is an extension of stable Higgs bundles with vanishing Chern classes, by the similar arguments of Mehta-Ramanathan's work about restriction of semistable sheaves to hyperplane sections. In a recent paper [**NZ15**], using the Yang-Mills-Higgs flow to construct the approximate Hermitian-Einstein structure for semistable Higgs bundles, combined with the techniques in [**DPS94**], Y.-C. Nie and X. Zhang proved the implication (iii) ⇒ (ii).

In particular, we give a direct proof of Part (ii) in Theorem 6.1.1.

Theorem W. *Let X be a compact Kähler manifold. Suppose that E be a numerically flat vector bundle over X. Then the natural Gauss-Manin connection D_E on E is compatible with the natural hermitian flat connection on the quotient E_k/E_{k-1} for every $k = 1, \ldots, p$, where E_k is the vector bundle appearing in the filtration* (6.1.1).

6.2. Technical Preliminaries

6.2.1. Higgs bundles. Let us recall the following definition of Higgs bundles and the stability.

Definition 6.2.1. Let X be a n-dimensional Kähler manifold with a fixed Kähler metric ω. A ***Higgs bundle*** is a triple $(E, \overline{\partial}_E, \theta)$, where E is a smooth vector bundle on X, $\overline{\partial}_E$ is a $(0,1)$-connection satisfying the integrability condition $\overline{\partial}_E^2 = 0$, and θ is a map $\theta : E \to E \otimes \mathcal{A}^{1,0}(X, E)$ such that

$$(\overline{\partial}_E + \theta)^2 = 0. \tag{6.2.1}$$

By Koszul-Malgrange theorem, $\overline{\partial}_E$ gives rise to a holomorphic structure on E. Thus (6.2.1) is equivalent to that

$$\overline{\partial}_E(\theta) = 0, \quad \text{and} \quad \theta \wedge \theta = 0.$$

We say that a Higgs bundle $(E, \overline{\partial}_E, \theta)$ is stable (resp. semistable) if for all θ-invariant torsin-free coherent subsheaves $F \subset E$, say ***Higgs subsheaves*** of (F, θ), we have

$$\mu_\omega(F) := \frac{c_1(\det F) \cdot \omega^{n-1}}{\mathrm{rank} F} < (\text{resp.} \leqslant) \frac{c_1(\det E) \cdot \omega^{n-1}}{\mathrm{rank} E} =: \mu_\omega(E)$$

where $\det F = (\wedge^{\mathrm{rank} F} F)^{\star\star}$ is the determinant bundle of F, and we say that $\mu_\omega(F)$ is the *slope* of F with respect to ω. A Higgs bundle $(E, \overline{\partial}_E, \theta)$ is *polystable* if it is a direct sum of stable Higgs bundles with the same slope.

For any Higgs bundle (E, θ) over a Kähler manifold X, if h is a metric on E, set $D(h)$ to be its Chern connection with $D(h)^{0,1} = \overline{\partial}$. Consider furthermore the connection

$$D_h = D(h) + \theta + \theta_h^\star,$$

where $\theta_h^\star$ is the adjoint of θ with respect to h, and let F_h denote its curvature. Then the metric h on (E, θ) is called *Hermitian-Yang-Mills* if

$$\Lambda F_h = \mu_\omega(E).$$

6.2.2. Higher Order Kähler Identities for Harmonic Bundles. Let (V, D) is a flat bundle with a metric h. Decompose $D = d' + d''$ into connections of type $(1,0)$ and $(0,1)$ respectively. Let δ' and δ'' be the unique $(1,0)$ and $(0,1)$ connections respectively, such that the connections $\delta' + d''$ and $d' + \delta''$ preserve the metric K. Set

$$\theta = \frac{d' - \delta'}{2}, \quad \overline{\partial} = \frac{d'' + \delta''}{2}, \quad \partial = \frac{d' + \delta'}{2}, \tag{6.2.2}$$

then we can decompose the connection D into

$$D = \overline{\partial} + \theta + \partial + \theta_h^*,$$

here θ_h^* is the adjoint of θ with respect to h, and it is easy to verify that $\overline{\partial} + \partial$ is also a metric connection. In general, the triple $(V, \overline{\partial}, \theta)$ might not be a Higgs bundle.

However, since the hermitian metric h on V can be thought of as a map

$$\Phi_h : X \to GL(n, \mathbb{C})/U(n),$$

a deep theorem by Siu-Sampson shows that, if Φ_h happens to be a harmonic map, then $(V, \overline{\partial}, \theta)$ is a Higgs bundle. Such a metric on V is called *harmonic* metric, and we say that (V, D, h) is a *harmonic* bundle.

Suppose (E, D, h) is a harmonic bundle. We denote the *harmonic decomposition*

$$D'' = \overline{\partial} + \theta, \quad D' = \partial + \theta_h^*. \tag{6.2.3}$$

Define the Laplacians

$$\begin{aligned} \Delta &= DD^* + D^*D \\ \Delta'' &= D''(D'')^* + (D'')^*D'' \end{aligned}$$

and similarly Δ'. Then by [**Sim92**, Section 2] we have

$$\Delta = 2\Delta' = 2\Delta'', \tag{6.2.4}$$

and thus the spaces of harmonic forms with coefficients in E are all the same, which are denoted by $\mathscr{H}^\bullet(E)$. We also have the following orthogonal decompositions of the space of E-valued forms with respect to the L^2-inner product:

$$\begin{aligned} A^p(E) &= \mathscr{H}^p(E) \oplus \mathrm{Im}(D) \oplus \mathrm{Im}(D^*) \\ &= \mathscr{H}^p(E) \oplus \mathrm{Im}(D'') \oplus \mathrm{Im}\big((D'')^*\big) \\ &= \mathscr{H}^p(E) \oplus \mathrm{Im}(D') \oplus \mathrm{Im}\big((D')^*\big). \end{aligned}$$

Then we also have the following $\partial\bar{\partial}$-lemma for harmonic bundles [**Sim92**, Lemma 2.1]:

LEMMA 6.2.1. ($\partial\bar{\partial}$-lemma) If (E, D, h) is a harmonic bundle, then

$$\operatorname{Ker}(D') \cap \operatorname{Ker}(D'') \cap \big(\operatorname{Im}(D'') + \operatorname{Im}(D')\big) = \operatorname{Im}(D'D''). \tag{6.2.5}$$

We will define the *de Rham cohomology* $H^i_{\mathrm{DR}}(X, E)$ for the flat bundles. We identify E with the locally constant sheaf of flat sections of E. Consider the sheaves of $\mathscr{C}^\infty$ differential forms with coefficients in E:

$$E \to \big(\mathscr{A}^0(E) \xrightarrow{D} \mathscr{A}^1(E) \xrightarrow{D} \ldots\big),$$

which are fine, and thus the cohomology $H^i_{\mathrm{DR}}(X, E)$ is naturally isomorphic to the cohomology of the complex of global sections

$$\big(A^\bullet(E), D_E\big) = A^0(E) \xrightarrow{D} A^1(E) \xrightarrow{D} A^2(E) \xrightarrow{D} \ldots$$

which we denote by $H^i(X, E) = \frac{\operatorname{Ker} D_E}{\operatorname{Im} D_E}$.

Let us recall the following famous Corlette-Simpson Correspondence:

THEOREM 6.2.1. *Let X be a compact Kähler manifold of dimension n.*

*(i) [**Cor88, Don87**] A flat bundle V has a harmonic metric if and only if it arises from a semisimple representation of the fundamental group $\pi_1(X)$.*
*(ii) [**Sim88**] A Higgs bundle E has a Hermitian-Yang-Mills metric if and only if it is polystable. Such a metric is harmonic if and only if $\mathrm{ch}_1(E)\cdot\omega^{n-1} = \mathrm{ch}_2(E)\cdot\omega^{n-2} = 0$.*

Then from Part (i) in Theorem 6.2.1 we see that, if the flat bundle arises from a semisimple representation of the fundamental group $\pi_1(X)$, there is a metric h on V such that the pair $(\bar{\partial}, \theta)$ constructed in (6.2.2) is a Higgs bundle.

6.2.3. NUMERICALLY FLAT VECTOR BUNDLE. Let E be a holomorphic vector bundle of rank r over a compact complex manifold X. We denote by $\mathbb{P}(E)$ the projectivized bundle of hyperplanes of E and by $\mathcal{O}_E(1)$ the tautological line bundle over $\mathbb{P}(E)$. Recall the following definition in [**DPS94**].

DEFINITION 6.2.2. Let X be a compact complex manifold.

(i) We say that a line bundle L is nef, if for any $\epsilon > 0$, there exists a smooth hermitian metric h_ϵ on L such that $i\Theta_{h_\epsilon}(L) \geqslant -\epsilon\omega$, where ω is a fixed Kähler metric on X.
(ii) A holomorphic vector bundle E is said to be nef if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef over $\mathbb{P}(E)$.
(iii) We say that a holomorphic vector bundle E is *numerically flat* if both E and its dual $E^\star$ is nef.

From Part (i) in Theorem 6.1.1 we see that, a vector bundle E is numerically flat if and only if it is a representation of $\pi_1(X)$ which is a successive extension of unitary representations.

6.3. PROOFS OF MAIN THEOREMS

PROOF OF THEOREM V. Let $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ be the representation of the fundamental group corresponding to the flat vector bundle E. After taking some conjugation, one can put the representation in block upper triangular form

$$\begin{bmatrix} \rho_1 & * & \ldots & * \\ 0 & \rho_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \rho_m \end{bmatrix} \tag{6.3.1}$$

such that for every $i = 1, \ldots, r$, $\rho_i : \pi_1(X) \to GL(r_i, \mathbb{C})$ is an irreducible representations. Thus there is a filtration of flat vector bundles

$$\{0\} = E_0 \subset E_1 \subset \ldots \subset E_m = E$$

such that

(i) each E_i is invariant under the flat connection D_E, that is, $D_E(E_i) \subset E_i \otimes \mathscr{A}^1(X)$;
(ii) the flat connection D_E on E, when restricted to each E_i, induces also a flat connection D_i on E_i;
(iii) the quotient connection D_i on $Q_i := E_i/E_{i-1}$ induced by D_E is also flat, which corresponds to the irreducible representation $\rho_i : \pi_1(X) \to GL(r_i, \mathbb{C})$.

Thus by Corlette's Theorem (Theorem 6.2.1 (i)), we can find a (unique) harmonic metric h_i such that (Q_i, D_i, h_i) is an harmonic bundle. Thus for each i, there is a unique harmonic decomposition as (6.2.3)

$$D_i' = \overline{\partial}_i + \theta_i, \quad D_i'' = \partial_i + \theta_i^*, \tag{6.3.2}$$

where θ_i^* is the adjoint of θ_i with respect to h_i. Moreover, Q_i cam be equipped with a Higgs bundle structure $(Q_i, \overline{\partial}_i, \theta_i)$.

For simplicity we first consider the case that E is an extension of an irreducible representation by another one, that is, $m = 2$ and we have an exact sequence of flat vector bundles over X:

$$0 \to Q_1 \to E \to Q_2 \to 0,$$

and thus there is $\eta \in A^1\big(X, \mathrm{Hom}(Q_2, Q_1)\big)$ such that D_E is given by

$$D_E = \begin{bmatrix} D_1 & \eta \\ 0 & D_2 \end{bmatrix}.$$

We denote by $D_{2,1}$ the corresponding flat connection on the bundle $\mathrm{Hom}(Q_2, Q_1)$ induced by D_1 and D_2. Since both (Q_1, D_1, h_1) and (Q_2, D_2, h_2) are both harmonic bundles, so is $\big(\mathrm{Hom}(Q_2, Q_1), D_{2,1}, h_1 h_2^*\big)$. Set $D_{2,1}'$ and $D_{2,1}''$ to be the harmonic decomposition of $D_{2,1}$ as (6.2.3), and let $\Delta_{2,1}$ and $\Delta_{2,1}''$ to be the Laplacians of $D_{2,1}$ and $D_{2,1}''$ respectively. Then by (6.2.4) we have

$$\Delta_{2,1} = 2\Delta_{2,1}''.$$

Since $D_E^2 = 0$, we have

$$D_{2,1}(\eta) = 0.$$

If there is another $\eta' \in A^1\big(X, \mathrm{Hom}(Q_2, Q_1)\big)$ such that $\eta' = \eta + D_{2,1}(a)$ for some $a \in A^0\big(X, \mathrm{Hom}(Q_2, Q_1)\big)$, then there exists a gauge transformation $g \in \mathrm{Aut}(E)$ such that

$$g \circ D_E \circ g^{-1} = D_E + D_{2,1}(a). \tag{6.3.3}$$

Indeed, we can define

$$g = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}, \tag{6.3.4}$$

and it is easy to show that g satisfies (6.3.3). Thus D_E and $D_E + D_{2,1}(a)$ defines the same flat bundle as E. In other words, $\{\eta\} \in H^1_{\mathrm{DR}}\big(X, \mathrm{Hom}(Q_2, Q_1)\big)$ is the extension class corresponding to E.

Take a harmonic representation η' in the extension class $\{\eta\}$, then

$$\Delta_{2,1}(\eta') = 2\Delta_{2,1}'(\eta'),$$

in particular

$$D_{2,1}'(\eta') = 0. \tag{6.3.5}$$

Let η_1' and η_2' to be the $(1,0)$ and $(0,1)$-part of η' respectively. Set

$$\overline{\partial}_E := \begin{bmatrix} \overline{\partial}_1 & \eta_2' \\ 0 & \overline{\partial}_2 \end{bmatrix},$$

and

$$\theta_E := \begin{bmatrix} \theta_1 & \eta_1' \\ 0 & \theta_2 \end{bmatrix},$$

then from (6.3.5) it is easy to see that $(\overline{\partial}_E, \theta_E)$ is a Higgs bundle over X, which is compatible with the Higgs bundle structures $(Q_i, \overline{\partial}_i, \theta_i)$. We prove the theorem when $m = 2$.

For general $m \geqslant 2$, we will prove the theorem by inductions. Set $\nabla_j := D_1 \oplus \ldots \oplus D_j$ to be the flat connection on $Q_1 \oplus \ldots \oplus Q_j$, and

$$\nabla_j' := D_1' \oplus D_2' \oplus \ldots \oplus D_j', \quad \nabla_j'' := D_1'' \oplus D_2'' \oplus \ldots \oplus D_j'',$$

where $D_i = D_i' + D_i''$ is the harmonic decomposition (6.3.2).

Then by induction, we can show that there is a flat connection $\tilde{D}_j$ on the smooth vector bundle $Q_1 \oplus \ldots \oplus Q_j$ of the following form

$$\tilde{D}_j = \begin{bmatrix} D_1 & & B_j \\ & \ddots & \\ 0 & & D_j \end{bmatrix}, \tag{6.3.6}$$

such that the pair $(Q_1\oplus\ldots\oplus Q_j,\tilde{D}_j)$ defines the same flat vector bundle as E_j. Here $B_j\in A^1\big(X,\mathrm{End}(Q_1\oplus\ldots\oplus Q_j)\big)$ which is strictly upper-triangle such that

$$\nabla_j(B_j)+B_j\wedge B_j=0. \tag{6.3.7}$$

<u>Claim</u> 6.3.1. Assume that we can find a proper $B_{j-1}\in A^1\big(X,\mathrm{End}(Q_1\oplus\ldots\oplus Q_{j-1})\big)$ which is strictly upper-triangle such that

(i) $\nabla_{j-1}(B_{j-1})+B_{j-1}\wedge B_{j-1}=0$.
(ii) for $\tilde{D}_{j-1}$ defined in (6.3.6), the pair $(Q_1\oplus\ldots\oplus Q_{j-1},\tilde{D}_{j-1})$ defines a flat vector bundle which is isomorphic to E_{j-1}.
(iii) $\nabla'_{j-1}(B_{j-1})=0$.

Then so is true for j.

<u>Proof:</u> Since E_j is an extension of E_{j-1} by Q_j

$$0\to E_{j-1}\to E_j\to Q_j\to 0,$$

we denote by $\beta\in H^1_{\mathrm{DR}}\big(X,\mathrm{Hom}(Q_j,E_{j-1})\big)$ the extension class. Choose any representative $A\in\beta$, then the pair $(Q_1\oplus\ldots\oplus Q_j,\tilde{D}_j)$ defining the same holomorphic structure as E_j has the following form:

$$\tilde{D}_j=\left[\begin{array}{ccc:c} D_1 & & B_{j-1} & a_1\\ & \ddots & & \vdots\\ 0 & & D_{j-1} & a_{j-1}\\ \hdashline 0 & \ldots & 0 & D_j\end{array}\right] \tag{6.3.8}$$

where $A=a_1\oplus\ldots\oplus a_{j-1}$ with $a_i\in A^1\big(X,\mathrm{Hom}(Q_j,Q_i)\big)$. Then by $\tilde{D}_j^2=0$, one has

$$\tilde{D}_{j-1}\circ A+A\circ D_j=0. \tag{6.3.9}$$

In particular, $D_{j,j-1}(a_{j-1})=0$, where $D_{j,i}$ the connection on $\mathrm{Hom}(Q_j,Q_i)$ induced by D_j and D_i. Since $\big(\mathrm{Hom}(Q_j,Q_i),D_{j,i},h_ih_j^*\big)$ is also a harmonic bundle, we set $D'_{j,i}$ and $D''_{j,i}$ to be the harmonic decomposition of $D_{j,i}$ as (6.2.3). From (6.2.4) we can find

$$c_{j-1}\in\mathscr{C}^\infty\big(X,\mathrm{Hom}(Q_j,Q_{j-1})\big)$$

such that $\Delta_{j,j-1}(a_{j-1}+D_{j,j-1}c_{j-1})=0$, where $\Delta_{j,i}$ (resp. $\Delta'_{j,i}$) is the Laplacian of $D_{j,i}$ (resp. $D'_{j,i}$). Set

$$A_1:=a_1\oplus\ldots\oplus a_{j-1}+(\tilde{D}_{j-1}\circ c_{j-1}+c_{j-1}\circ D_j)$$

then $A_1\in\beta$. Indeed, if we denote by $D_{j,\widetilde{j-1}}$ the induced flat connection on $\mathrm{Hom}(Q_j,E_{j-1})$ by the connections D_j and $\tilde{D}_{j-1}$, then $A_1=A+D_{j,\widetilde{j-1}}(c_{j-1})$, and thus A and A_1 belong to the same extension class, and define the same flat vector bundle E_j.

If we write $A=a'_1\oplus\ldots\oplus a'_{j-1}$, then $a'_{j-1}=a_{j-1}+D_{j,j-1}(c_{j-1})$. Moreover, since $\Delta_{j,j-1}=2\Delta'_{j,j-1}$, we have

$$D'_{j,j-1}(a'_{j-1})=0.$$

This gives us hints that we can use some *ad hoc* methods to find the proper A.

Assume now for some $A=a_1\oplus\ldots\oplus a_{j-1}\in\beta$ such that $D''_{j,i}(a_i)=0$ for all $i=k+1,\ldots,j-1$. By (6.3.9) we have

$$D_{j,k}(a_k)+\sum_{i=k+1}^{j-1}b_{ki}a_i=0,$$

here b_{ki} is the projection of $B_{j-1}\in A^1\big(X,\mathrm{End}(Q_1\oplus\ldots\oplus Q_{j-1})\big)$ to the component $A^1\big(X,\mathrm{Hom}(Q_i,Q_k)\big)$. By the assumption that $\nabla'_{j-1}B_{j-1}=0$, we have $D''_{i,k}(b_{ki})=0$, then

$$\begin{aligned}0&=D''_{j,k}D_{j,k}(a_k)+D''_{j,k}\Big(\sum_{i=k+1}^{j-1}b_{ki}a_i\Big)\\&=D''_{j,k}D_{j,k}(a_k)+\sum_{i=k+1}^{j-1}\big(D''_{i,k}(b_{ki})a_i-b_{ki}D''_{j,i}(a_i)\big)\\&=D''_{j,k}D'_{j,k}(a_k).\end{aligned}$$

By applying Lemma 6.2.1 for $D'_{j,k}(a_k)$, there exists $c_k \in \mathscr{C}^\infty\big(X, \mathrm{Hom}(Q_j, Q_k)\big)$ such that

$$D'_{j,k}(a_k) = -D'_{j,k}D''_{j,k}(c_k) = -D'_{j,k}D_{j,k}(c_k).$$

Set

$$\tilde{A} := A + D_{j,\widetilde{j-1}}(c_k) = A + (\tilde{D}_{j-1} \circ c_k + c_k \circ D_j)$$

which also belongs to the extension class β, then

$$\tilde{A} = a'_1 \oplus \ldots \oplus a'_{k-1} \oplus \big(a_k + D_{j,k}(c_k)\big) \oplus a_{k+1} \oplus \ldots \oplus a_{j-1}.$$

That is, the components of $\tilde{A}$ in $A^1\big(X, \mathrm{Hom}(Q_j, Q_i)\big)$ for $i = k+1, \ldots, j-1$ are the same as that of A, and the component of $\tilde{A}$ in $A^1\big(X, \mathrm{Hom}(Q_j, Q_k)\big)$ are replaced by $a_k + D_{j,k}(c_k)$, such that $D'_{j,k}\big(a_k + D_{j,k}(c_k)\big) = 0$. Thus by the induction we can choose $A \in \beta$ properly such that $D'_{j,k}(a_k) = 0$ for all $k = 1, \ldots, j-1$. This is equivalent to

$$\nabla'_j(B_j) = 0.$$

From the construction of $\tilde{D}_j$, it is a flat connection which defines the vector bundle E_j. The claim is thus proved. ■

By the above claim we know that there exists $\eta \in A^1\big(X, \mathrm{End}(Q_1 \oplus \ldots \oplus Q_m)\big)$ which is strictly upper-triangle, such that $(Q_1 \oplus \ldots \oplus Q_m, \nabla_m + \eta)$ gives rise to the flat bundle E, and satisfies $\nabla'_m(\eta) = 0$. By the flatness of $\nabla_m + \eta$, we have

$$\nabla_m(\eta) + \eta \wedge \eta = 0.$$

Then

$$\nabla''_m(\eta) + \eta \wedge \eta = 0. \tag{6.3.10}$$

Set

$$\bar{\partial}_E := \begin{bmatrix} \bar{\partial}_1 & & \eta'' \\ & \ddots & \\ 0 & & \bar{\partial}_m \end{bmatrix},$$

and

$$\theta_E = \begin{bmatrix} \theta_1 & & \eta' \\ & \ddots & \\ 0 & & \theta_m \end{bmatrix}.$$

Here η' and η'' is the $(1,0)$ and $(0,1)$-part of η respectively, and $D''_i = \bar{\partial}_i + \theta_i$ is defined as (6.2.2). Then (6.3.10) is equivalent to that $(E, \bar{\partial}_E, \theta_E)$ is a Higgs bundle over X. In this setting, for each $1 \leqslant i \leqslant m$, E_i is a Higgs subbundle of E, and the induced Higgs bundle structure on the quotient E_i/E_{i-1} is the same as $(Q_i, \bar{\partial}_i, \theta_i)$. We thus proved that (i) implies (ii). The implication of (ii) to (i) is almost the same methods. □

We will quickly show that Theorem W is a special case of Theorem V.

Proof of Theorem W. From Part (i) in Theorem 6.1.1 we see that, a vector bundle E is numerically flat if and only if it is a representation of $\pi_1(X)$ which is a successive extension of unitary representations. Thus the connections on the quotient E_i/E_{i-1} are unitary ones, that is, there are no Higgs fields θ_i, and it is easy to show that so is for their successive extension E. □

In a forthcoming paper, we are going to study the following conjecture:

Conjecture 6.3.1. Let X be a compact Kähler manifold, and let $\rho : \pi_1(X) \to GL(r, \mathbb{C})$ be a representation, with (E, D) the corresponding flat vector bundle. Fix a hermitian metric h on E. Take the harmonic flow of the connection D introduced in [**Cor88**, (1)]

$$\frac{\partial A}{\partial t} = -D_A D_A^{+,*}\theta_A,$$

where $D_A = D_A^+ + \theta_A$ such that D_A^+ is the metric connection and θ_A is self-adjoint, and $D_A^{+,*}$ is the adjoint of D_A^+ with respect to the metric h. Then the limited connection D_∞ exists, which is also flat, and corresponds to the representation which is the semisimplification of ρ.

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