Vanishing theorems and structure theorems of compact kähler manifolds

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Théorèmes d'annulation et théorèmes de structure sur les variétés kähleriennes compactes

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Introduction

0.1 Un peu d'histoire à propos des méthodes analytiques

La notion de métrique kählerienne a été introduite en 1933 par E. Kähler [K3̈3], pour l'étude des variétés à courbure négative. Cependant, il est remarquable que [1] déjà au début des années 40, il était bien connu parmi les experts que la métrique de Fubini-Study est kählerienne, et de nombreux liens entre la géométrie kählerienne et la géométrie algébrique ont ainsi été trouvés. Bien que les variétés kähleriennes partagent un grand nombre de propriétés des variétés projectives, il y a des différences importantes entre ces deux catégories (cf. [Voi05a], [Voi10] pour une explication détaillée des différences). Il est aussi très intéressant d'étudier les propriétés communes à ces deux catégories. L'objet principal de cette thèse est de généraliser un certain nombre de résultats bien connus de la géométrie algébrique au cas kählerien non nécessairement projectif.

Comme on travaille dans le cadre kählerien, les outils analytiques jouent un rôle central dans notre approche. Avant d'expliquer les techniques modernes mises en jeu, il est utile de rappeller les travaux classiques de Kodaira.

Théorème 0.1.1 (Critère de Kodaira). *Soit X une varété kählerienne. Alors X est projective si et seulement s'il existe une métrique kählerienne ω dont la classe de cohomologie est image d'une classe entière dans $H^2(X,\mathbb{Z})$.*

L'implication directe, à savoir que la projectivité implique l'existence d'une métrique kählerienne à coefficients entiers est triviale. Inversement,si la classe ω appartient à $H^{1,1}(X)\cap H^2(X,\mathbb{Z})/$ tors, on peut associer à ω un fibré ample L tel que $c_1(L)=\omega$. La difficulté est de construire un plongement $X\hookrightarrow\mathbb{P}^N$ à l'aide de L. L'idée de Kodaira est de montrer que si m est assez grand, alors $H^0(X,mL)$ engendre toutes les directions et sépare toutes les paires de points de X. Pour montrer cela, Kodaira a utilisé la technique d'éclatement et la technique de Bochner. On voit alors apparaître un problème central de la géométrie complexe : construire des sections vérifiant des propriétés supplémentaires particulières. On rappelle maintenant deux méthodes analytiques pour engendrer les sections globales, introduites après l'époque de Kodaira : technique des estimations L^2 et théorème d'extension de Ohsawa-Takegoshi.

Estimations L^2

Un jalon important de la théorie des estimations L^2 est la résolution par Hörmander des équations de type $\overline{\partial}$, qui est un des outils les plus puissants dans l'analyse complexe à plusieurs variables et la géométrie analytique. On rappelle d'abord quelques notions standards en géométire analytique.

Définition 0.1.1. *Soit X une variété complexe lisse munie d'une $(1,1)$-forme lisse ω qui est strictement positive sur X. Soit (E,h) un fibré vectoriel holomorphe au-dessus de X muni d'une métrique hermitienne h. Soit D la connexion de Chern du fibré (E,h). On peut vérifier facilement que $D\circ D\in C^\infty(X,\Lambda^{1,1}\otimes\mathrm{Hom}(E,E))$. On appelle $i\Theta_h(E)=D\circ D$ la courbure de (L,h).*

Soit φ une fonction semi-continue supérieurement : $\varphi: X\to[-\infty,+\infty[$. On dit que φ est une fonction plurisousharmonique ("psh" en abrégé) si $i\partial\overline{\partial}\varphi\geq 0$ dans le sens du courant. On dit que φ est une fonction quasi-plurisousharmonique ("quasi-psh" en abrégé) si localement, $i\partial\overline{\partial}\varphi\geq -C\omega$ au sens des courants.

1. Nous renvoyons à [Bou96] pour cette histoire intéressante.

Comme on travaille ici principalement dans le cadre des variétés compactes, on utilisera la version suivante des estimations L^2 (cf.[Dem00]).

Théorème 0.1.2 (Estimation L^2 pour $\overline{\partial}$). *Soit (X,ω) une variété kählerienne compacte de dimension n. Soit (L,φ) un fibré en droites muni d'une métrique hermitienne singulière à courbure $i\Theta_\varphi(L)\geq 0$. Soient $\gamma_1(x)\leq\cdots\leq\gamma_n(x)$ les valeurs propres de courbure de $i\Theta_\varphi(L)$ par rapport à ω. Alors pour toute (n,q)-forme g sur X à valeurs dans L telle que*

$$\overline{\partial}g=0 \qquad et \int_X(\gamma_1(x)+\cdots+\gamma_q(x))^{-1}|g|^2e^{-2\varphi}dV_\omega<+\infty,$$

il existe f telle que $\overline{\partial}f=g$ et

$$\int_X|f|^2e^{-2\varphi}dV_\omega\leq\int_X(\gamma_1(x)+\cdots+\gamma_q(x))^{-1}|g|^2e^{-2\varphi}dV_\omega. \tag{0.1}$$

Remarque 0.1.3. *Le théorème reste vrai si on remplace X par une variété faiblement pseudoconvexe. Ce fait est très important dans les applications.*

Pour avoir une version plus algébrique et des applications plus précises, on introduit maintenant le concept de faiseau d'idéaux multiplicateurs.

Définition 0.1.2. *Soit φ une fonction quasi-psh sur un ouvert U, on associe à φ le faiseau d'idéaux $\mathscr{I}(\varphi)\subset\mathscr{O}_U$, formé des germes de fonctions holomorphes $f\in\mathscr{O}_{U,x}$ telles que $|f|^2e^{-2\varphi}$ soit intégrable dans un voisinage de x.*

On a alors la version algérique du Théorème 0.1.2.

Théorème 0.1.4. *Soit (X,ω) une variété kählerienne compacte de dimension n. Soit (L,φ) un fibré en droites muni d'une métrique hermitienne singulière à courbure $i\Theta_\varphi(L)\geq 0$. Soient $\gamma_1(x)\leq\cdots\leq\gamma_n(x)$ les valeurs propres de courubre de $i\Theta_\varphi(L)$ par rapport à ω. Si*

$$\sum_{i=1}^q\gamma_i\geq c$$

pour une certaine constante $c>0$, alors

$$H^q(X,K_X\otimes L\otimes\mathscr{I}(\varphi))=0.$$

Remarque 0.1.5. *Il y a une information importante qui est perdue dans la version algébrique qualitative, à savoir le contrôle de la norme L^2 de f dans l'inégalité (0.1) du Théorème 0.1.2. Celui-ci joue un rôle très important dans certaines applications analytiques.*

Expliquons brièvement comment utiliser le Théorème 0.1.2 pour construire des sections globales. Pour simplifier, on suppose que Ω est un ouvert de Stein, et $x_0\in\Omega$ un point arbitraire. Pour trouver une section globale $f\in H^0(\Omega,\mathscr{O})$ telle que $f(x_0)=1$, on prend d'abord un germe $u\in\mathscr{O}_{x_0}$ tel que $u(x_0)=1$. Soit ψ une fonction tronquante qui vaut 1 au voisinage de x_0. On applique le Théorème 0.1.2 au fibré trivial $\mathscr{O}_\Omega$ par rapport à la métrique $e^{-\varphi}$, où φ est une fonction strictement psh, possédant la singularité $2n\cdot\ln|x-x_0|$ au voisinage de x_0. On obtient alors une fonction v telle que

$$\overline{\partial}v=\overline{\partial}(\psi\cdot u) \qquad \text{et} \qquad \int_X|v|^2e^{-2\varphi}<+\infty. \tag{0.2}$$

D'après l'estimation (0.2), on a $v(x_0)=0$. Alors $(\psi\cdot u-v)\in H^0(\Omega,\mathscr{O}_\Omega)$ et $(\psi\cdot u-v)(x_0)=1$.

Ce type de méthode a connu beaucoup de succès importants en géométrie analytique. Par exemple, le problème de Levi, les problèmes de Cousin, le problème de régularisation des courants positifs peuvent se résoudre de cette manière.... On remarque finalement dans cette partie que dans les travaux récents de Chen-Donaldson-Sun [CDS12] et Tian [Tia12] sur la résolution de la conjecture de Tian-Yau-Donaldson, on utilise l'estimation L^2 pour donner une estimation uniforme des noyaux de Bergman associés aux diviseurs anti-pluricanoniques $-m \cdot K_X$ pour des variétés de Fano qui vérifient les conditions de Cheeger-Colding.

Le théorème d'extension de Ohsawa-Takegoshi-Manivel

On explique ici une autre façon de construire des sections globales, reposant sur le théorème d'extension de Ohsawa et Takegoshi [OT87, Ohs88]. Signalons que Manivel [Man93] en a donné une version générale. Nous énoncerons ici seulement un cas particulier simple.

Théorème 0.1.6 (Extension de Ohsawa-Takegoshi)**.** *Soit Ω un domaine pseudoconvexe borné dans $\mathbb{C}^n$, et φ une fonction psh sur Ω. Soit H un sous-espace affine de $\mathbb{C}^n$ et on note $\Omega_1 = \Omega \cap H$. Alors pour tous les fonctions holomorphes f sur Ω_1, il existe une fonction holomorphe F sur Ω tel que $F = f$ en Ω_1, et*

$$\int_{\Omega} |F|^2 e^{-\varphi} \leq C(\Omega) \cdot \int_{\Omega_1} |f|^2 e^{-\varphi},$$

où la constante $C(\Omega)$ ne dépend que du domaine Ω.

Remarque 0.1.7. *L'estimation L^2 obtenue dans l'extension de Ohsawa-Takegoshi-Manivel est en fait globale : on peut aussi étendre des sections de fibrés holomorphes à partir d'une sous-variété de la variété ambiante, à condition d'introduire des déterminants jacobiens ad hoc. On remarque aussi que l'on n'a pas besoin de positivité stricte de la courbure dans l'extension de Ohsawa-Takegoshi. C'est une différence remarquable avec la technique de Kodaira-Hörmander.*

Remarque 0.1.8. *[Che11] a donné une preuve simple du Théorème 0.1.6 en utilisant seulement la technique originale de Hörmander pour les estimations L^2. Dans les travaux récents de [Bł12] et [ZGZ12], on en trouvera des applications et des développements importants.*

Il y a beaucoup d'applications du théorème d'extension de Ohsawa-Takegoshi. Dans [Siu98] et [Siu04], Siu a utilisé celui-ci pour montrer l'invariance de plurigenres dans le cas projectif [2], ce qui a des conséquences importantes en géométrie birationnelle. L'autre application importante est que l'extension de Ohsawa-Takegoshi donne un lien entre les noyaux de Bergman des sous-variétés et ceux de la variété ambiante grâce à la propriété extrémale des noyaux de Bergman. En fait, si on regarde seulement le cas où la sous-variété est un point $x \in X$, le Théorème 0.1.6 est déjà une étape essentielle dans la preuve du théorème de régularisation de Demailly (cf. [Dem92]). On voit aussi que dans les travaux de [BP08] et [BP12], l'estimation de Ohsawa-Takegoshi-Manivel est essentielle pour contrôler le comportement de la métrique canonique qui est construite au moyen des noyaux de Bergman.

Mentionnons finalement une question qui nous paraît se poser dans ce contexte : comme les noyaux de Bergman apparaissent plus naturellement dans le cadre de l'extension de Ohsawa-Takegoshi que dans celui des estimation L^2, peut-on utiliser l'extension de Ohsawa-Takegoshi pour donner une preuve simple de l'estimation uniforme des noyaux de Bergman des sections pluri-anticanoniques dans la conjecture de Tian-Yau-Donaldson, ceci sans utiliser la théorie des limites de Gromov-Hausdorff ?

Techniques de construction de métriques à courbure singulière

Si on réfléchit à la philosophie des deux méthodes précédentes, un point essentiel est de trouver des bonnes métriques à courbure positive avec un contrôle sur la singularité. On voit assez vite que, dans le cadre kählerien non nécessairement projectif, il est plus facile et raisonnable de construire des métriques à courbure positive que des sections globales. [3] En fait, une construction typique des métriques à courbure positive consiste à utiliser des sections globales :

2. Dans [Pău07], Păun en a donné une preuve simple.

3. Même dans le cas projectif, on peut voir la différence en observant que le cône des classes pseudo-effectives est parfois plus grand que le cône des classes effectives. La conjecture d'abondance illustre la difficulté de passer des métriques aux sections.

Proposition 0.1.1 (Équation de Lelong-Poincaré)**.** *Soit L un fibré en droites sur X et soit h_0 une métrique lisse sur L. Soient $\{s_1, ..., s_k\} \subset H^0(X, mL)$. Alors elles définissent une métrique $h = h_0 \cdot e^{-\varphi}$ sur L, où $\varphi = \frac{1}{m}\ln(\sum |s_i|^2_{h_0^m})$.*

D'après l'équation de Lelong-Poincaré, on a

$$\frac{i}{2\pi}\Theta_h(L) = \frac{i}{2\pi}\Theta_{h_0}(L) + dd^c\varphi \geq 0.$$

Donc la nouvelle métrique $h_0 \cdot e^{-\varphi}$ est à courbure semi-positive (dans le sens des courants).

Dans un language plus analytique, trouver des bonnes métriques à courbure positive est équivalent à la question suivante : soit $\alpha \in H^{1,1}(X, \mathbb{Q})$ (ou plus généralement $H^{1,1}(X, \mathbb{R})$), trouver de bons représentants positifs. On explique maintenant une méthode puissante permettant d'utiliser l'équation de Monge-Ampère pour construire de telles métriques. On rappelle d'abord le théorème de Calabi-Yau (aussi étudié par Aubin dans le cas Kähler-Einstein avec $c_1(X) < 0$).

Théorème 0.1.9 (Yau)**.** *Soit (X, ω) une variété kählerienne compacte de dimension n. Alors pour toute forme volume lisse $f > 0$ satisfaisant $\int_X f = \int_X \omega^n$, il existe une métrique kählerienne $\widetilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ telle que*

$$\widetilde{\omega}^n = f.$$

On explique maintenant l'idée permettant d'utiliser l'équation de Monge-Ampère pour construire des métriques singulières sur un fibré en droites positif. Cette idée a été d'abord proposée dans [Dem93]. Pour simplifier, on suppose que L est ample (mais dans les applications, L peut avoir une positivité dégénérée). On prend une suite de formes de volumes $\{f_i\}_{i=1}^{\infty}$ telle que la masse de f_i soit de plus en plus concentrée sur un point x de X. En résolvant l'équation de Monge-Ampère, on peut trouver une suite de métriques lisses $\{h_i\}_{i=1}^{\infty}$ sur L telle que $i\Theta_{h_i}(L)^n$ soit de plus en plus concentrée sur x. En passant à la limite, on peut obtenir une métrique à courbure positive sur L qui possède une singularité au point x. Bien que l'on puisse obtenir beaucoup de métriques par cette méthode, un inconvénient est que l'on a seulement une estimation sur $(i\Theta_{h_i}(L))^n$. Mais d'après les travaux de [Mou95],[DP03a], [DP04], cette difficulté peut être surmontée. Cette méthode a trouvé beaucoup d'applications en géométrie kählerienne. Signalons aussi que Dinew et Kołodziej ont montré récemment l'existence de solutions des l'équations Hessiennes (cf.[DK12], [HMW10]) :

Théorème 0.1.10. *Soit (X, ω) une variété kählerienne compacte de dimension n. Soit $m \leq n$ et f une fonction lisse strictement positive sur X telle que $\int_X f\omega^n = \int_X \omega^n$. Alors l'équation Hessienne*

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = f\omega^n$$

admet une solution.

Comme les équations Hessiennes sont plus générales que l'équation de Monge-Ampère, on peut espérer qu'il y aura des applications intéressantes en géomérie kählerienne.

Notons qu'il existe aussi d'autres méthodes plus algébriques pour construire des métriques à courbure positive : par le théorème de Riemann-Roch, par les inégalités de Morse holomorphe, par la méthode de Angehrn-Siu (cf. [AS95], [Dem85], [Dem96], [Dem12]). Nous renvoyons les lecteurs aux exposés de [Dem00] et [Siu02] pour des explications plus détaillées. Avant de finir cette section, nous rappelons une version faible des inégalités de Morse transcendantes (cf. [DP04]), qui sera utilisée plusieurs fois dans cette thèse.

Théorème 0.1.11. *Soit (X, ω) une variété kählerienne lisse compacte de dimension n et soit $\alpha \in H^{1,1}(X, \mathbb{R})$ une classe nef vérifiant*

$$\int_X \alpha^n > 0.$$

Alors il existe $\delta > 0$, tel que $\alpha - \delta\omega$ soit pseudo-effectif.

0.2 Un résumé des principaux résultats de cette thèse

Le théorème d'annulation de Nadel

La première partie de la thèse a pour but de généraliser le théorème d'annulation de Nadel, un des outils les plus importants de la géométrie birationnelle, au cas kählerien arbitraire. On rappelle d'abord le théorème d'annulation de Nadel classique :

Théorème 0.2.1. *(Nadel 89, Demailly 93) : Soit X une variété projective, $(L, h_0 \cdot e^{-\varphi})$ un fibré sur X tel que $i\Theta_\varphi(L) > \epsilon\omega$ pour certain $\epsilon > 0$. Alors*

$$H^q(X, K_X + L \otimes \mathscr{I}(\varphi)) = 0 \qquad \text{pour } q \geq 1,$$

où $\mathscr{I}(\varphi)$ est le faisceau d'idéaux multiplicateurs associé à φ.

Un fibré en droites est dit gros, s'il admet une métrique singulière dont la courbure associée est strictement positive. D'après des résultats bien connus, l'existence d'un fibré gros implique que X doit être de Moishezon. Par conséquence, pour généraliser le théorème de Nadel dans le cas kählerien arbitraire, il est naturel d'étudier le cas où la courbure est dégénérée et d'introduire une notion de dimension numérique. H.Tsuji [Tsu07] a défini une notion de dimension numérique pour des variétés projectives par une méthode purement algébrique ; rappelons ici cette définition :

Définition 0.2.1. *Soit X une variété projective, (L, φ) un fibré en droites pseudo-effectif. On définit :*

$$\nu_{\text{num}}(L, \varphi) = \sup\{\dim V \mid V \text{ sous-variété de } X \text{ telle que}$$

$$\varphi \text{ est bien définie sur } V \text{ et } (V, L, \varphi) \text{ est gros.}\}$$

Comme la définition de Tsuji dépend de l'existence de sous-variétés, on doit chercher une définition plus analytique si la variété X est non algébrique. En utilisant une "approximation quasi-equisingulière" essentiellement construite au moyen des noyaux de Bergman (cf. [DPS01]), on peut généraliser la définition de H.Tsuji dans le cas kählerien arbitraire. Plus précisement, on définit d'abord la notion d' "approximation quasi-equisingulière", telle qu'elle apparaît à peu de choses près dans [DPS01] :

Définition 0.2.2. *Soit $\theta + dd^c\varphi$ un courant positif, où θ est une $(1,1)$-forme lisse et φ est une fonction quasi-psh sur une variété kählerienne compacte (X, ω). On dit que $\{\varphi_k\}_{k=1}^\infty$ est une approximation quasi-equisingulière de φ pour le courant $\theta + dd^c\varphi$ si*

(i) $\{\varphi_k\}_{k=1}^\infty$ converge vers φ dans L^1 et

$$\theta + dd^c\varphi_k \geq -\tau_k \cdot \omega \qquad \text{où } \lim_{k\to+\infty} \tau_k \to 0.$$

(ii) toutes les φ_k sont à singularités analytiques et φ_k est moins singulière que φ_{k+1}, c'est à dire qu'il existe une constante C_k telle que

$$\varphi_{k+1} \leq \varphi_k + C_k.$$

(iii) Pour tous $\delta > 0$ et $m \in \mathbb{N}$, il existe $k_0(\delta, m) \in \mathbb{N}$ tel que

$$\mathscr{I}(m(1+\delta)\varphi_k) \subset \mathscr{I}(m\varphi) \qquad \text{si } k \geq k_0(\delta, m)$$

Il est facile de voir que si $\{\psi_i\}_{i=1}^{+\infty}$ est une autre approximation analytique de φ pour le courant $\theta + dd^c\varphi$, alors pour toute $(n-1, n-1)$-forme semi-positive u, on a

$$\lim_{i\to\infty} \int_X (dd^c\psi_i)_{\text{ac}} \wedge u \geq \lim_{i\to\infty} \int_X (dd^c\varphi_i)_{\text{ac}} \wedge u \tag{0.3}$$

où $(dd^c\varphi_i)_{\text{ac}}$ est la partie absolument continue du courant $dd^c\varphi_i$. Grâce à l'inégalité (0.3), on peut définir l'intersection des courants positifs comme la limite des intersections des parties absolument continues des approximations quasi-équisingulières. En particulier, on peut définir la dimension numérique de la façon suivante :

Définition 0.2.3. *Soit* (L,φ) *un fibré pseudo-effectif. On définit la dimension numérique* $\mathrm{nd}(L,\varphi)$ *: c'est le plus grand entier* $v\in\mathbb{N}$*, tel que*

$$\langle (i\Theta_{\varphi}(L))^{v}\rangle \neq 0,$$

où $\langle (i\Theta_{\varphi}(L))^{v}\rangle$ *est le produit défini précédemment.*

En utilisant l'extension de Ohsawa-Takegoshi et la propriété extrémale des noyaux de Bergman, on peut montrer que la Définition 0.2.3 est équivalente à la Définition 0.2.1 lorsque X est projective. Avec une bonne définition de la dimension numérique, on montre dans [Cao12b] que

Théorème 0.2.2. *Soit* X *une variété kählerienne lisse compacte, et* (L,φ) *un fibré en droites pseudo-effectif sur* X*. Alors*

$$H^{p}(X,K_X\otimes L\otimes \mathscr{I}_{+}(\varphi))=0 \qquad \text{si } p\geq n-\mathrm{nd}(L,\varphi)+1,$$

où $\mathscr{I}_{+}(\varphi)=\lim_{\epsilon\to 0^{+}}\mathscr{I}((1+\epsilon)\varphi)$.

Expliquons ici l'idée de la preuve. Soit $[u]\in H^{p}(X,K_X\otimes L\otimes\mathscr{I}_{+}(\varphi))$. Soit f une (n,p)-forme lisse à valeurs dans L qui représente $[u]$, à savoir

$$\overline{\partial} f=0 \qquad \text{et} \qquad \int_X |f|^2 e^{-2(1+s_1)\varphi}<+\infty.$$

On voudrait trouver une forme u de bidgré $(n,p-1)$ telle que

$$f=\overline{\partial} u \qquad \text{et} \qquad \int_X |u|^2 e^{-2(1+s)\varphi}<+\infty.$$

D'après la contruction de l'approximation quasi-équisingulière, on peut montrer que $\mathscr{I}_{+}(\varphi)=\mathscr{I}((1+\frac{1}{k})\varphi_k)$ et $\int_X |f|^2 e^{-2(1+\frac{1}{k})\varphi_k}<+\infty$. On applique alors la méthode de l'estimation L^2 au fibré hermitien $(L,(1+\frac{1}{k})\varphi_k)$ et à f. Comme $i\Theta_{(1+\frac{1}{k})\varphi_k}(L)\geq -\tau_k-\frac{1}{k}$, on a $f=\overline{\partial}u_k+v_k$ et

$$\int_X |u_k|^2 e^{-2(1+\frac{1}{k})\varphi_k}+\frac{1}{2p(\tau_k+C\frac{1}{k})}\int_X |v_k|^2 e^{-2(1+\frac{1}{k})\varphi_k}$$

$$\leq \int_X \frac{1}{\sum_{i=1}^{p}\lambda_{i,k}+2p(\tau_k+C\frac{1}{k})}|f|^2 e^{-2(1+\frac{1}{k})\varphi_k}$$

où $\lambda_{1,k}\leq\lambda_{2,k}\leq\cdots\leq\lambda_{n,k}$ sont les valeurs propres de $i\Theta_{\varphi_k}(L)$.

On espère que $v_k\to 0$ dans un sens convenable. Mais on n'a pas d'estimation de $\sum_{i=1}^{p}\lambda_{i,k}$ ponctuellement. On utilise alors une équation de Monge-Ampère pour reparamétriser $\sum_{i=1}^{p}\lambda_{i,k}$ ponctuellement. [4] On peut finalement trouver des potentiels $\widehat{\varphi}_k$, tels que $\mathscr{I}_{+}(\varphi)=\mathscr{I}(\widehat{\varphi}_k)$ et f peut se décomposer comme

$$f=\overline{\partial}\widehat{u}_k+\widehat{v}_k \qquad \text{et} \qquad \int_X |\widehat{v}_k|^2 e^{-2\widehat{\varphi}_k}\to 0.$$

De cette manière, en résolvant l'équation $\overline{\partial}$, on peut associer à chaque $\widehat{v}_k$ un cocycle de Čech :

$$\check{v}_k=\{\check{v}_{k,\alpha_0,\ldots,\alpha_p}\}\in \check{C}^{p}(\mathscr{U},K_X\otimes L\otimes\mathscr{I}_{+}(\varphi)).$$

On utilise le théorème de l'application ouverte dans un espace de Fréchet pour conclure le théorème. Plus précisément, on a besoin du lemme suivant :

4. Ce type de méthode est bien connu grâce aux travaux de [Mou95] et [DP03a].

Lemma 0.2.1. *Soit L un fibré en droites sur une variété Kählerienne compacte X, et φ une métrique singulière sur L. Soit $\{U_\alpha\}_{\alpha\in I}$ un recouvrement de Stein de X, et soit $u \in \check{H}^p(X, K_X + L \otimes \mathscr{I}_+(\varphi))$. S'il existe une suite $\{v_k\}_{k=1}^{\infty} \subset \check{C}^p(\mathscr{U}, K_X \otimes L \otimes \mathscr{I}_+(\varphi))$ dans la même classe de cohomologie que u, vérifiant*

$$\lim_{k\to\infty} \int_{U_{\alpha_0\cdots\alpha_p}} |v_{k,\alpha_0\ldots\alpha_p}|^2 \to 0, \tag{0.4}$$

où les normes $|v|^2$ sont prises par rapport à une métrique lisse fixée sur L, alors $u = 0$ dans $\check{H}^p(X, K_X + L \otimes \mathscr{I}_+(\varphi))$.

Avant de terminer cette partie, on doit remarquer que l'on peut aussi définir la dimension numérique par le produit non pluripolaire (cf.[BEGZ10]). Le produit des courants défini par l'approximation quasi-equisingulière met en quelque sorte en évidence l'aspect algébrique des courants. En revanche, la définition de [BEGZ10] est plus efficace pour les analystes. Donc c'est une question naturelle de se demander si les deux définitions de dimension numérique sont équivalentes. Il nous semble que c'est une question intéressante dans l'étude des fonctions plurisousharmoniques.

Dimension numérique

La deuxième partie est consacrée à l'étude de la dimension numérique du fibré anticanonique lorsque $-K_X$ est nef. Bien que l'idée principale de ce chapitre soit une partie de [Cao12a], on l'a traitée ici comme un chapitre indépendant à cause de ses relations avec tous les autres chapitres. En fait, la technique principale de ce chapitre consiste à généraliser l'annulation de Kawamata-Viehweg à certaines variétés kähleriennes non nécessairement projectives. Pour cette raison, il se situe dans le même esprit que le chapitre précédent. En outre, le résultat principal de ce chapitre joue un rôle important dans l'étude des variétés kähleriennes à fibré anticanonique numériquement effectif (“nef” en abrégé), qui est le sujet principal des deux chapitres suivants.

On rappelle d'abord le théorème d'annulation de Kawamata-Viehweg classique (cf.[Dem12, Theorem 6.25])

Théorème 0.2.3. *Soit X une variété projective lisse de dimension n et L un fibré en droites nef. Alors*

$$H^q(X, K_X + L) = 0 \qquad \text{pour } q \geq n - \mathrm{nd}(L) + 1,$$

où $\mathrm{nd}(L)$ est la dimension numérique de L.

La preuve du Théorème 0.2.3 utilise la technique des sections hyperplanes et donc le fait qu'il existe d'un fibré ample sur les variétés projectives. En utilisant un fibré ample, on peut facilement réduire ce théorème au cas où L est gros et nef. Alors l'annulation de Nadel (cf. Théorème 0.2.1) conclut la preuve. Bien que la preuve dans le cas projectif soit assez simple, il est embarrassant que l'on ne sache pas si cette annulation est encore valable dans le cas kählerien arbitraire. [5] Une approche naïve est de considérer la métrique à singularité minimale $h_{\min}$ associée à L, et d'utiliser le Théorème 0.2.2. Mais le problème est qu'en général on a seulement

$$\mathrm{nd}(L) \geq \mathrm{nd}(L, h_{\min}),$$

où $\mathrm{nd}(L, h_{\min})$ est la dimension numérique introduite dans la Définition 0.2.3. L'exemple typique est ici celui consideré dans [DPS94, Exemple 1.7] : soit E un fibré de rang 2 obtenu comme une extension non triviale sur une courbe elliptique T

$$0 \to \mathscr{O}_T \to E \to \mathscr{O}_T \to 0.$$

Alors le fibré $L = \mathscr{O}_E(1)$ sur $X = \mathbb{P}(E)$ est nef et $\mathrm{nd}(L) = 1$. Mais malheursement, $\mathrm{nd}(L, h_{\min}) = 0$.

5. La preuve de [Eno93] contient des idées intéressantes, mais elle est malheureusement incomplète.

D'autre part, il est bien connu que les tores génériques sont non projectifs. Or les tores non projectifs sont l'un des blocs importants permettant de construire des variétés kähleriennes non nécessairement projectives. Il est donc intéressant d'étudier des variétés qui admettent une fibration vers un tore (ou quotient d'un tore). Une première observation importante est que, même si un tore n'est pas nécessairement projectif, il n'est pas loin d'être projectif s'il possède un diviseur effectif. Plus concrètement, on a

Proposition 0.2.2. *Soit $T = \mathbb{C}^n/\Gamma$ un tore complexe de dimension n, et soit $\alpha \in H^{1,1}(T, \mathbb{Z})$ une classe pseudo-effective. Alors T admet une submersion*

$$\pi : T \to S$$

vers une variété abélienne S. De plus, $\alpha = \pi^* c_1(A)$ pour un fibré ample A sur S.

Dans ce chapitre, on étudie d'abord l'annulation de Kawamata-Viehweg pour les fibrés qui vérifient la condition suivante.

Définition 0.2.4. *Soit $\pi : X \to T$ une fibration et α une $(1,1)$-classe sur X. On dit que α est π-gros, si la restriction sur la fibre générique $\alpha|_F$ est grosse.*

On montre d'abord

Théorème 0.2.4. *Soit (X, ω_X) une variété kählerienne compacte de dimension n. On suppose qu'il existe une fibration*

$$X \xrightarrow{\pi} T \xrightarrow{\pi_1} S$$

où π est une surjection vers une variété lisse T de dimension r, et π_1 est une submersion vers une courbe lisse S. Soit L un fibré en droites nef, π-gros sur X, vérifiant

$$\pi_*(c_1(L)^{n-r+1}) = \pi_1^*(\omega_S). \tag{0.5}$$

Alors

$$H^p(X, K_X + L) = 0 \qquad \textit{pour } p \geq r.$$

L'idée de la preuve est la suivante. D'après l'équation (0.5), on peut montrer que $L - c \cdot \pi^* \pi_1^*(\omega_S)$ est pseudo-effective pour un certain $c > 0$, en résolvant une équation de Monge-Ampère. Comme L est aussi π-gros, on peut alors construire une métrique h sur L dont la courbure contient $n - r + 1$ valeurs propres positives. Si on observe de plus que L est nef, on voit qu'il existe une métrique lisse h_ϵ tel que

$$i\Theta_{h_\epsilon}(L) \geq -\epsilon \omega_X.$$

En combinant les trois constructions, on peut finalement montrer le Théorème 0.2.4. En utilisant la Proposition 0.2.2 et le Théorème 0.2.4, on peut facilement montrer que

Théorème 0.2.5. *Soit (X, ω_X) une variété kählerienne compacte de dimension n qui admet une surjection vers un tore T de dimension r*

$$\pi : X \to T.$$

Soit L un fibré nef et π-gros sur X. Alors on a

$$H^q(X, K_X + L) = 0 \qquad \textit{pour } q \geq \min(r, n - \mathrm{nd}(L) + 1). \tag{0.6}$$

Remarque 0.2.6. *Si on peut montrer dans le Théorème 0.2.4 que* $\mathrm{nd}(L) = n - r + 1$, *alors on peut améliorer l'annulation* (0.6) *du Théorème 0.2.5 en concluant que*

$$H^q(X, K_X + L) = 0 \qquad \textit{pour } q \geq n - \mathrm{nd}(L) + 1.$$

Comme application du Théorème 0.2.5, on a le résultat suivant qui joue un rôle important dans les deux derniers chapitres.

Théorème 0.2.7. *Soit X une variété kählerienne compacte de dimension n à fibré anticanonique nef. Soit $\pi : X \to T$ une fibration vers un tore T de dimension r. Si $-K_X$ est π-gros, alors* $\mathrm{nd}(-K_X) = n-r$.

Déformation des variétés kähleriennes

D'après le critère de Kodaira (cf. Théorème 0.1.1), on sait qu'une variété kählerienne X est projective si et seulement si il existe une 2-forme d-fermée positive dans $H^{1,1}(X) \cap H^2(X, \mathbb{Q})$. Comme les rationnels sont denses dans les réels, il est naturel de se demander si on peut obtenir une variété projective en déformant la structure complexe par une déformation arbitrairement petite.

Conjecture 0.2.3 (posée par Kodaira). *Soit X une variété kählerienne compacte, est-ce que l'on peut l'approximer par des variétés projectives ?*

Si $\dim X = 2$, Kodaira a montré la conjecture en utilisant son résultat sur la classification des surfaces kähleriennes. Plus récemment, toujours pour les surfaces, Buchdahl [Buc06] a donné d'abord une preuve plus simple pour les surfaces où il n'y a pas d'obstruction de déformation, et dans [Buc08], a démontré finalement la conjecture en dimension 2 sans utiliser la théorie de classification. Expliquons maintenant sa preuve dans le cas où il n'y a pas d'obstruction de déformation, situation qui sera utile dans la partie suivante. Dans sa démonstration, Buchdahl a utilisé la proposition suivante afin de simplifier la situation :

Proposition 0.2.4. *Soit X une variété kählerienne compacte sans obstruction de déformation. On suppose qu'il existe une classe kählerienne ω telle que*

$$\omega\wedge : H^1(X, T_X) \to H^2(X, \mathscr{O}_X) \tag{0.7}$$

soit surjective. Alors on peut approximer X par des variétés projectives.

Comme la surjectivité de (0.7) est équivalente à l'injectivité de

$$\omega\wedge : H^{n-2}(X, K_X) \to H^{n-1}(X, \Omega^1(K_X)), \tag{0.8}$$

il reste à vérifier que les surfaces qui ne vérifient pas l'injectivité de (0.8) sont déjà projectives. On observe déjà en dimension 2 que la non injectivité de (0.8) implique l'existence d'une section globale de $H^0(X, K_X)$; Buchdahl montre finalement que toutes ces surfaces sont projectives.

Si $\dim X \geq 4$, il a été démontré par C.Voisin que l'on ne peut pas toujours déformer une variété kählérienne compacte vers une variété projective, ce qui répond négativement à la conjecture 0.2.3. En fait, d'après la construction de Voisin, on peut voir qu'il existe même des obstructions topologiques. Plus concrètement, C.Voisin a construit une variété kählerienne X telle que l'anneau de cohomologie $H^*(X, \mathbb{Z})$ ne provient pas d'une variété projective. Par contre, la conjecture 0.2.3 reste ouverte si $\dim X = 3$. Il serait d'ailleurs intéressant de considérer la conjecture sous l'hypothèse que la variété de dimension 3 est minimale et n'a pas d'obstruction de déformation.

D'autre part, comme le fibré canonique contrôle la géométrie des variétés compactes, il est naturel de penser que la conjecture de Kodaira peut avoir une réponse positive si on ajoute des conditions supplémentaires sur le fibré canonique. Par exemple, D.Huybrechts a montré que tous les variétés hyperkählériennes peuvent être approximées par des variétés projectives (le fibré canonique est alors trivial).

Dans ce chapitre, on montre que dans les trois cas suivants on peut toujours approximer une variété kählerienne compacte par des variétés projectives :

(1) variétés kähleriennes compactes à fibré anticanonique semi-positif.

(2) variétés kähleriennes compactes ayant une métrique analytique dont la courbure bisectionnelle est holomorphiquement semi-négative.

(3) variétés kählériennes compactes à fibré tangent nef.

On explique maintenant l'idée de la preuve. Si X est une variété kählérienne compacte à fibré anticanonique semi-positif, il est démontré dans [DPS96] qu'après un revêtement étale fini $\pi : \widetilde{X} \to X$, la variété peut être décomposée en un produit de tores, de variétés de Calabi-Yau, hyperkählériennes et de variétés projectives. Par conséquent, la difficulté principale est de déformer $\widetilde{X}$ en gardant l'action du groupe du revêtement. D'après une idée de C.Voisin, on peut utilise le critère de densité (cf. Proposition 0.2.4) pour résoudre cette difficulté. En combinant celui-ci avec un calcul explicite de la structure de Hodge de $H^2(X, \mathbb{C})$, on peut conclure en construisant une déformation de X, de sorte que beaucoup de ses fibres soient des variétés projectives. On obtient le théorème suivant [Cao12a].

Théorème 0.2.8. *Soit X une variété kählerienne compacte telle que $-K_X$ soit semi-positif. Alors on peut approximer X par des variétés projectives.*

Si le fibré tangent est nef, c'est plus compliqué. La difficulté est que, dans ce cas, il n'y a pas nécessairement de métrique canonique à courbure semi-positive. Mais grâce au théorème principal de [DPS94], on sait qu'après un revêtement étale fini, ce type de variétés admet une fibration lisse vers un tore. L'idée ici est de déformer le tore en préservant la structure de la fibration. Grâce au Théorème 0.2.7 et à un résultat de C.Simpson [Sim92], on montre qu'après un revêtement étale fini, l'application Albanese est localement triviale. En combinant ceci avec la Proposition 0.2.4, on montre enfin que

Théorème 0.2.9. *Soit X une variété Kählerienne compacte à fibré tangent nef. Alors on peut approximer X par des variétés projectives.*

Il est plus généralement intéressant de considérer la question suivante.

Conjecture 0.2.5. *Soit X une variété Kählerienne compacte à fibré anticanonique nef. Est-ce que l'on peut approximer X par des variétés projectives ?*

Pour attaquer cette conjecture, une méthode naturelle est d'étudier la structure d'une variété kählerienne compacte à fibré anticanonique nef. Le chapitre suivant est consacré à l'étude de ce type de variétés.

Variétés à fibré anticanonique nef

Un problème central de la géométrie différentielle est d'étudier les variétés satisfaisant des conditions de courbure. Dans le domaine de la géomérie complexe, il faut d'abord mentionner les travaux célèbres de [Mor79] et [SY80], où il est démontré que les espaces projectifs sont les seules variétés à courbure bisectionnelle positive. A la suite de ces travaux, [Mok88] a classifié toutes les variétés kähleriennes compactes à courbure bisectionnelle holomorphe semi-positive. Au début des années 90, les travaux foudamentaux de [CP91] et [DPS94] ont étudié les variétés kähleriennes compactes à fibré tangent nef, une notion plus naturelle que la courbure bisectionnelle holomorphe semi-positive en géométrie algébrique. Dans ce chapitre, on étudie un cas plus général, à savoir les variétés à fibré anticanonique nef.

D'après [Miy87], on sait que si K_X est nef, alors Ω_X^1 est génériquement semipositif, c'est-à-dire que Ω_X^1 est à pentes semi-positives relativement à la filtration de Harder-Narasimhan pour la polarisation $(H_1, \cdots, H_{n-1})$, où les $\{H_i\}$ sont des diviseurs amples. Donc il est naturel de se poser la même question dans le cas dual. En utilisant la première équalité de Bianchi, on peut facilement montrer (cf.[Cao13]) que

Théorème 0.2.10. *Soit (X, ω) une variété kählerienne compacte à fibré anticanonique nef (resp. à fibré canonique nef). Soit*

$$0 \subset \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = T_X \qquad (\textit{resp. } \Omega_X^1)$$

la filtration de Harder-Narasimhan par rapport à ω. Alors

$$\int_X c_1(\mathscr{E}_{i+1}/\mathscr{E}_i) \wedge \omega^{n-1} \geq 0 \qquad \textit{pour tout } i.$$

Remarque 0.2.11. *Lorsque K_X est nef, ce théorème est démontré dans [Miy87] sous l'hypothèse que X soit projective. Lorsque $-K_X$ est nef, ce théorème est nouveau même dans le cas projectif.*

L'idée de la preuve est assez simple. Grâce au Théorème de Aubin-Yau (cf. Théorème 0.1.9), le caractère nef de $-K_X$ implique que pour tout $\epsilon > 0$, il existe une métrique lisse ω_ϵ sur T_X tel que

$$\mathrm{Ric}(\omega_\epsilon) \geq -\epsilon\omega_\epsilon.$$

Par la première équalité de Bianchi qui lie la courbure de Ricci et $\mathrm{Tr}_{\omega_\epsilon} i\Theta_{\omega_\epsilon}(T_X)$ on peut enfin montrer le Théorème 0.2.10.

Si on étudie la filtration de Harder-Narasimhan en détails, on peut obtenir que la condition

$$H^0(X, (T_X^*)^{\otimes m}) = 0 \qquad \text{pour tout } m \geq 1$$

implique que

$$\int_X c_1(\mathscr{E}_{i+1}/\mathscr{E}_i) \wedge \omega^{n-1} > 0 \qquad \text{pour tout } i.$$

En combinant ceci avec [BM01], on obtient un cas particulier de la conjecture de Mumford

Proposition 0.2.6. *Soit X une variété kählerienne compacte à fibré anticanonique nef. Si*

$$H^0(X, (T_X^*)^{\otimes m}) = 0 \qquad \textit{pour tout } m \geq 1,$$

alors X est rationnellement connexe.

Comme autre application, on peut facilement montrer que :

Proposition 0.2.7. *Soit (X, ω_X) une variété kählerienne compacte à fibré anticanonique nef. Alors*

$$\int_X c_2(T_X) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} \geq 0$$

pour tout $\epsilon > 0$ assez petit.

On doit aussi remarquer qu'il y a une conjecture plus générale de Peternell :

Conjecture 0.2.8. *Soit X une variété Kählerienne compacte à fibré anticanonique nef (resp. à fibré canonique nef). Soit*

$$0 \subset \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = T_X \qquad (\textit{resp. } \Omega_X^1)$$

la filtration de Harder-Narasimhan par rapport à $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{n-1}$. Alors

$$\int_X c_1(\mathscr{E}_{i+1}/\mathscr{E}_i) \wedge \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{n-1} \geq 0 \qquad \textit{pour tout } i.$$

D'après la preuve du Théorème 0.2.10, on sait que la conjecture est vraie dans le cas où tous les ω_i sont identiques. Pour attaquer la conjecture générale de Peternell, il est peut-être intéressant de considérer une conjecture plus générale, à savoir si la fitration Harder-Narasimhan de T_X est aussi semipositive par rapport à une polarisation par des courbes mobiles.

Dans les deux dernières sections de ce chapitre, on étudie la structure des variétés kähleriennes compactes à fibré anticanonique nef. On remarque d'abord que si $-K_X$ est semipositive, les travaux [DPS96], [CDP12] montrent qu'après un revêtement étale fini $\pi : \widetilde{X} \to X$, la variété peut être décomposée en un produit de tores, de variétés de Calabi-Yau, de variétés hyperkählériennes et de variétés rationnellement connexes. La conjecture suivante posée dans [DPS93] et [CDP12] s'inscrit dans cette perspective.

Conjecture 0.2.9. *Soit X une variété kählerienne compacte à fibré anticanonique nef. Alors l'application d'Albanese est submersive, et elle est localement triviale, c'est-à-dire qu'il n'y a pas de déformation de la structure complexe sur les fibres.*

Si X est projective, la surjectivité de l'application d'Albanese a été montrée par Q.Zhang dans [Zha96]. Toujours sous l'hypothèse que X soit projective, [LTZZ10] a montré que l'application d'Albanese est équidimensionnelle et que toutes les fibres sont réduites. Plus récemment, M.Păun [Pau12a] a montré la surjectivité dans le cas kählerien, comme corollaire d'un théorème profond sur la positivité de l'image directe. Grâce au Théorème 0.2.10, on peut donner une nouvelle preuve de la surjectivité de l'application d'Albanese.

Proposition 0.2.10. *Soit X une variété kählerienne compacte à fibré anticanonique nef. Alors l'application d'Albanese est surjective, et elle est lisse hors d'une sous-variété de codimension au moins 2. En particulier, les fibres de l'application d'Albanese sont réduites en codimension 1.*

L'idée de la preuve est de considérer la filtration de Harder-Narasimhan de T_X :

$$0 \subset \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = T_X.$$

D'après le Théorème 0.2.10, les pentes sont positives. On suppose d'abord que tous les $\mathscr{E}_i/\mathscr{E}_{i-1}$ sont localement libres. Alors le théorème de Uhlenbeck-Yau [UY86] implique qu'il existe une métrique lisse h_i sur $\mathscr{E}_{i+1}/\mathscr{E}_i$ telle que

$$\mathrm{Tr}_{\omega_X} i\Theta_{h_i}(\mathscr{E}_{i+1}/\mathscr{E}_i) \geq 0.$$

Donc pour tout $\epsilon > 0$, on peut construire une métrique h_ϵ sur T_X, telle que

$$\mathrm{Tr}_{\omega_X} i\Theta_{h_\epsilon}(T_X) \geq -\epsilon \cdot \mathrm{Id}\,.$$

On peut alors utiliser la technique de Bochner pour montrer que les éléments de $H^0(X, \Omega_X)$ sont partout non nuls. On obtient ainsi la lissité de l'application d'Albanese. Si $\mathscr{E}_i/\mathscr{E}_{i-1}$ n'est pas libre, par la même méthode, on peut montrer seulement que l'application d'Albanese est lisse hors d'une sous-variété de codimension au moins 2.

Dans un travail en commun avec A.Höring, nous étudions la structure de l'application d'Albanese avec davantage de détails. On peut ainsi obtenir :

Théorème 0.2.12. *Soit X une variété kählerienne compacte à fibré anticanonique nef. On suppose qu'il existe une surjection $\pi : X \to T$ vers un tore T et que $-K_X$ est π-gros. Alors π est lisse et localement triviale.*

Expliquons l'idée de la preuve. D'après le théorème 0.2.7 et [Anc87], on peut montrer que $E_m = \pi_*(-mK_X)$ est localement trivial pour $m \gg 1$. L'étape clé de la preuve est de montrer que E_m est un fibré numériquement plat. On montre d'abord que E_m est nef. Comme le tore T n'est pas nécessairement projectif, on ne peut pas utiliser l'argument classique de [DPS94, Lemma 3.21]. Mais comme T est un tore, on a aussi une isogénie $\varphi_k : T \to T$ de degré 2^k. On fixe une partition de l'unité sur T. En utilisant la méthode de régularisation [Dem92], on peut construire une métrique h sur $E_{m,k} = \pi_*(-mK_{X_k})$ où $X_k = T \times_{\varphi_k} X$, telle que

$$i\Theta_h(E_{m,k}) \geq -C \cdot \omega_T,$$

pour un constante C indépendant de k. Grâce à [DPS94, Proposition 1.8], on obtient une métrique lisse h_k sur E_m telle que

$$i\Theta_{h_k}(E_m) \geq -\frac{C}{2^{k-1}}\omega_T.$$

Lorsque k tend vers l'infini, on obtient alors que E_m est nef. Pour montrer que E_m est numériquement plat, il reste à montrer que $c_1(E_m) = 0$. On suppose par l'absurde que $c_1(E_m) \neq 0$. D'après la Proposition 0.2.2, $c_1(E_m)$ induit une fibration lisse $T \to S$. On considère la filtration de Harder-Narasimhan de E_m par rapport à $\omega_S + \epsilon\omega_X$, pour un ϵ assez petit :

$$0 \subset \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_s = E_m.$$

On peut montrer que tous les membres de la filtration sont des fibrés localement triviaux et semi-positifs. Alors la condition $c_1(E_m) \neq 0$ et la construction impliquent finalement que $\mathscr{F}_0$ est ample. Par un calcul un peu compliqué, on peut montrer que l'amplitude de $\mathscr{F}_0$ implique l'inclusion $X \hookrightarrow \mathbb{P}(E_m/\mathscr{F}_0)$. La contradiction s'ensuit. Le théorème s'obtient finalement en combinant ceci avec un résultat de Simpson [Sim92].

Chapitre 1

Introduction and elementary definitions

1.1 Introduction

The notion of a Kähler metric has been introduced in 1933 by E.Kähler in [K3̈3], in view of the study of negatively curved manifolds. However, it is quite remarkable that [1] already in the early 40s, it was well known among experts that the Fubini-Study metric is Kähler, and in this way many connections between Kähler geometry and algebraic geometry were found. Although Kähler manifolds share a lot of properties with projective manifolds, there are some strong differences between these two categories. We refer to [Voi05a] and [Voi10] for a detailed discussion about the differences. It is a also very interesting to study certain important properties that these two fundamental categories share together. The aim of this thesis is to generalize some well known results in algebraic geometry to Kähler geometry.

Nadel vanishing theorem

The first part of the thesis is to generalize the Nadel vanishing theorem, one of the most important tools in birational geometry, to an arbitrary Kähler manifold. We first recall the classical Nadel vanishing theorem (cf.[Nad89], [Dem93]).

Theorem 1.1.1. *Let (X, ω_X) be a projective manifold with a Kähler metric ω_X and let L be a line bundle on X with a singular metric h. Assume that $i\Theta_h(L) \geq \epsilon\omega_X$ in the sense of currents for some $\epsilon > 0$. Then*

$$H^q(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}(h)) = 0 \qquad \textit{for all } q \geq 1,$$

where $\mathscr{I}(\varphi)$ is the multiplier ideal sheaves associated to φ.

Recall that a line bundle is said to be big, if it has a singular metric such that the curvature is strictly positive. By [DP04], we know that the existence of a big line bundle on a compact Kähler manifold implies that the manifold should be projective. Therefore, to generalize the Nadel vanishing theorem to an arbitrary Kähler manifold, it is natural to consider the case of degenerate curvature and to introduce a notion of numerical dimension. H.Tsuji [Tsu07] has already defined a notion of numerical dimension on projective manifolds.

Definition 1.1.1. *Let X be a projective variety and (L, φ) a pseudo-effective line bundle, i.e. $i\Theta_\varphi(L) \geq 0$ in the sense of currents. One defines*

$$\nu_{\text{num}}(L, \varphi) = \sup\{\dim V \mid V \textit{ subvariety of } X \textit{ such that}$$

$$\varphi \textit{ is well defined on } V \textit{ and } (V, L, \varphi) \textit{ is big.}\}$$

1. We refer to [Bou96] for this interesting historical fact.

Here (V, L, φ) big means that there is a desingularization $\pi : \widetilde{V} \to V$ such that

$$\overline{\lim_{m\to\infty}} \frac{h^0(\widetilde{V}, m\pi^*(L) \otimes \mathscr{I}(m\varphi \circ \pi))}{m^n} > 0$$

where n is the dimension of V.

Using a "quasi-equisingular approximation", which can be essentially constructed by the Bergman kernel method (cf. [DPS01]), we first generalize the definition of H.Tsuji to arbitrary compact Kähler manifolds. With a well-defined notion of numerical dimension (which is proved to coincide with the definition of H.Tsuji when X is projective), we have proved in [Cao12b] that

Theorem 1.1.2. *Let (L, φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n. Then*

$$H^p(X, K_X \otimes L \otimes \mathscr{I}_+(\varphi)) = 0 \qquad \textit{for any } p \geq n - \nu_{\text{num}}(L, \varphi) + 1,$$

where $\mathscr{I}_+(\varphi)$ is the upper semicontinuous variant of the multiplier ideal sheaf associated to φ (cf. [FJ05]).

We should also remark that there is an another way of defining the numerical dimension through non-pluripolar products of closed positive currents (cf.[BEGZ10]). The product of currents defined by quasi-equisingular approximations reflects the algebraic face of currents. By contrast, the definition of [BEGZ10] is possibly more pleasant to analysts. Therefore, it is a natural question whether these two definitions of numerical dimension are equivalent. Maybe it is a profound question in the study of plurisubharmonic functions.

Numerical dimension

The second part is devoted to study of the numerical dimension of $-K_X$ when $-K_X$ is nef. Although the main idea in this chapter is just a part of [Cao12a], we treat it here as an independent chapter because of its relations to all other chapters. In fact, the main technique in this chapter is to generalize the Kawamata-Viehweg vanishing theorem to certain Kähler manifolds that are not necessarily projective. Therefore it is in the same spirit as the above chapter. Moreover, the main result in this chapter plays an important role in understanding the structure of Kähler manifolds with nef anticanonical bundles, which is the main subject of the next two chapters.

Let us first recall the well-known Kawamata-Viehweg vanishing theorem (cf.[Dem12, Theorem 6.25])

Theorem 1.1.3. *Let X be a projective manifold and let F be a nef line bundle over X. Then*

$$H^q(X, K_X + F) = 0 \qquad \textit{for all } q \geq n - \text{nd}(F) + 1.$$

It is rather embarrassing that we do not know whether this result is true for an arbitrary Kähler manifold. [2] We prove here in this chapter the following particular vanishing theorem

Theorem 1.1.4. *Let (X, ω_X) be a n-dimensional compact Kähler manifold. Assume that there is a two step fibration tower*

$$X \xrightarrow{\pi} T \xrightarrow{\pi_1} S$$

where π is surjective to a smooth variety T of dimension r, and π_1 is a submersion to a smooth curve S. Let L be a π-big, nef line bundle on X, satisfying

$$\pi_*(c_1(L)^{n-r+1}) = \pi_1^*(\omega_S)$$

for a Kähler metric ω_S on S and

$$L^{n-r+t} \wedge \pi^*\pi_1^*(\omega_S) = 0 \qquad \textit{where } n - r + t = \text{nd}(L).$$

Then

$$H^p(X, K_X + L) = 0 \qquad \textit{for } p \geq r.$$

2. The proof in [Eno93] contains some important ideas, but it is unfortunately incomplete.

Using the above weak vanishing theorem, we can prove the main result of this chapter :

Theorem 1.1.5. *Let X be a compact Kähler manifold of dimension n with nef anticanonical bundle, and let $\pi : X \to T$ be a fibration onto a torus T of dimension r. If $-K_X$ is π-big, then* $\mathrm{nd}(-K_X) = n-r$.

Deformation of Kähler manifolds

It has been shown by C.Voisin that one cannot always deform a compact Kähler manifold into a projective algebraic manifold, thereby answering negatively a question raised by Kodaira. On the other hand, since the canonical bundle controls the geometry of the variety, the Kodaira conjecture may be given a positive answer under an additional semipositivity or seminegativity condition on the canonical bundle, namely such a compact Kähler manifold can be approximated by deformations of projective manifolds. For example, Huybrechts proved that all hyperkähler manifolds (the canonical bundle is therefore trivial) can be approximated by projective manifolds.

We prove in this chapter that in the following three simple cases, compact Kähler manifolds can be approximated by projective varieties.

(1) Compact Kähler manifolds with hermitian semipositive anticanonical bundle.
(2) Compact Kähler manifolds with real analytic metrics and nonpositive bisectional curvature.
(3) Compact Kähler manifolds with nef tangent bundle.

We now explain the idea of the proof. If X is a compact Kähler manifold with hermitian semipositive anticanonical bundle, [DPS96] proved that after a finite étale covering $\pi : \widetilde{X} \to X$, the resulting manifold $\widetilde{X}$ can be decomposed as a product of tori, Calabi-Yan manifolds, hyperkähler manifolds and projective manifolds. Therefore the main difficulty is to deform $\widetilde{X}$ by keeping the group action operating on it. Thanks to an idea of C.Voisin, we can use a density criterion (cf.[Voi05a, Proposition 5.20]) to resolve this difficulty. Combining this with an explicit calculation of the Hodge structure of $H^2(X,\mathbb{C})$, we can finally construct a deformation of X so that many of its fibers are projective varieties. We obtain the following theorem [Cao12a].

Theorem 1.1.6. *Let X be a compact Kähler manifold with hermitian semipositive anticanonical bundle. Then X can be approximated by projective varieties.*

If the tangent bundle is numerically effective, the situation is more complicated. The difficulty is that in this case, there is no canonical metric with semipositive curvature. Thanks to the main theorem in [DPS94], we know that after a finite étale covering, such varieties have a smooth fibration to a complex torus. Our idea is to deform the torus by preserving the fibration structure. Thanks to Theorem 1.1.5 and Simpson's result [Sim92], we can finally prove that

Theorem 1.1.7. *Let X be a compact Kähler manifold with nef tangent bundles. Then X can be approximated by projective varieties.*

Varieties with anti-canonical bundle

One of the central questions in differential geometry is to study varieties under some constraints on the curvature. In the domain of Kähler geometry or algebraic geometry, we should first mention the pioneering works of [Mor79] and [SY80], where the projective spaces are proved to be the only varieties with positive holomorphic bisectional curvature. Later on, [Mok88] classified all compact Kähler manifolds with semipositive holomorphic bisectional curvature. In the beginning of the 90s, [CP91] and [DPS94] have studied compact Kähler (projective) manifolds with nef tangent bundles, a more algebraic notion than semipositive holomorphic bisectional curvature. In this chapter, we would like to study a more general case, namely varieties with nef anticanonical bundles.

We first prove in [Cao13] a part of a conjecture made by Peternell.

Theorem 1.1.8. *Let (X,ω) be a compact Kähler manifold with nef anticanonical bundle (resp. with nef canonical bundle). Let*

$$0 \subset \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = T_X \qquad (\textit{resp. } \Omega_X^1)$$

be the Harder-Narasimhan semistable filtration with respect to ω. *Then*

$$\int_X c_1(\mathscr{E}_{i+1}/\mathscr{E}_i)\wedge\omega^{n-1}\geq 0 \qquad \textit{for all } i.$$

Remark 1.1.9. *If* K_X *is nef, the theorem is proved in [Miy87] for algebraic manifolds. Here we prove it for arbitrary compact Kähler manifolds with nef canonical bundles. If* $-K_X$ *is nef, the theorem is a new result even for algebraic manifolds.*

As an application, we can prove a special case of Mumford's conjecture :

Proposition 1.1.10. *Let* X *be a compact Kähler manifold with nef anticanonical bundle. If*

$$H^0(X,(T_X^*)^{\otimes m})=0 \qquad \textit{for all } m\geq 1,$$

then X *is rationally connected.*

Another direct application is to give a partial answer to a conjecture of Kawamata :

Proposition 1.1.11. *Let* (X,ω_X) *be a compact Kähler manifold with nef anticanonical bundle. Then*

$$\int_X c_2(T_X)\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2}\geq 0$$

for any $\epsilon>0$ *small enough.*

In the last two sections of this chapter, we study the structure of compact Kähler manifolds with nef anticanonical bundles. We should mention first the following conjecture raised in [DPS93] and [CDP12].

Conjecture 1.1.12. *Let* X *be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map is a submersion, and the map is locally trivial, i.e., there is no deformation of complex structures on the fibers.*

If X is assumed to be projective, the surjectivity of the Albanese map was first proved by Q.Zhang in [Zha96]. Still assuming that X is projective, [LTZZ10] proved that the Albanese map is equidimensional and that all the fibers are reduced. Recently, M.Păun [Pau12a] proved the surjectivity for the Kähler case as a corollary of a powerful method based on a direct image argument. Thanks to Theorem 1.1.8, we can give a new proof of the surjectivity of the Albanese map for arbitrary Kähler manifolds with nef anticanonical bundles.

Proposition 1.1.13. *Let* X *be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map is surjective, and smooth outside a subvariety of codimension at least 2. In particular, the fibers of the Albanese map are reduced in codimension 1.*

In a joint work with A.Höring, we have studied compact Kähler manifolds with nef anticanonical bundles in more detail. We can prove :

Theorem 1.1.14. *Let* X *be a compact Kähler manifold with nef anticanonical bundle. Assume that there is a surjective morphism* $\pi : X\to T$ *to a torus* T *and that* $-K_X$ *is big on the generic fiber. Then the morphism* π *is smooth and locally trivial, i.e. there is no deformation of complex structures in this fibration.*

1.2 Elementary definitions and results

We first recall some basic definitions and results about quasi-psh functions.

Definition 1.2.1. *Let X be a complex manifold. We say that φ is a psh function (resp. a quasi-psh function) on X, if*

$$i\partial\overline{\partial}\varphi \geq 0, \qquad (\textit{resp. } i\partial\overline{\partial}\varphi \geq -c\cdot\omega_X \textit{ locally })$$

where c is a positive constant and ω_X is a smooth hermitian metric on X.

We say that a quasi-psh function φ has analytic singularities, if locally one has φ is locally of the form

$$\varphi(z) = c\cdot\ln(\sum |g_i|^2) + O(1)$$

with $c > 0$ and $\{g_i\}$ are holomorphic functions.

Let φ, ψ be two quasi-psh functions. We say that φ is less singular than ψ if

$$\psi \leq \varphi + C$$

for some constant C. We denote this relation by $\varphi \preccurlyeq \psi$.

We now recall the analytic definition of multiplier ideal sheaves.

Definition 1.2.2. *Let φ be a quasi-psh function. The multiplier ideal sheaves $\mathscr{I}(\varphi)$ is defined as*

$$\mathscr{I}(\varphi)_x = \{f \in \mathscr{O}_X | \exists U_x, \ \int_{U_x} |f|^2 e^{-2\varphi} < +\infty\}$$

where U_x is some open neighborhood of x in X.

We refer to [Dem12] and [Laz04] for a more detailed introduction to the concept of multiplier ideal sheaf.

When φ does not possess analytic singularities, we need to introduce the "upper semicontinuous regularization" of the multiplier ideal sheaf.

Definition 1.2.3. *Let φ be a quasi-psh function. We define the upper semi-continuous regularization of the multiplier ideal sheaf by*

$$\mathscr{I}_+(\varphi) = \lim_{\epsilon\to 0^+} \mathscr{I}((1+\epsilon)\varphi).$$

Remark 1.2.1. *By the Noetherian property of coherent ideal sheaves, there exists an $\epsilon > 0$ such that*

$$\mathscr{I}_+(\varphi) = \mathscr{I}((1+\epsilon')\varphi) \qquad \textit{for any } 0 < \epsilon' < \epsilon.$$

When φ has analytic singularities, it is easy to see that $\mathscr{I}_+(\varphi) = \mathscr{I}(\varphi)$. Conjecturally it is expected that the equality holds for all psh functions. [3]

We now discuss the notion of positivity in Kähler geometry (cf. [Dem12] for details).

Important Convention : In this thesis, when we talk about a line bundle L on X, we always first implicitly fix a smooth metric h_0 on L. Given a singular metric φ on L or sometimes φ for simplicity, we just mean that the new metric on L is given by $h_0 e^{-\varphi}$. Recall that the curvature of the metric φ for L is

$$\frac{i}{2\pi}\Theta_\varphi(L) = \frac{i}{2\pi}\Theta_{h_0}(L) + dd^c\varphi.$$

Definition 1.2.4. *Let L be a line bundle and φ a metric (maybe singular) on L. A pair (L,φ) is said to be a pseudo-effective line bundle if $\frac{i}{2\pi}\Theta_\varphi(L) \geq 0$ as a current.*

3. This was conjectured in [DK01]. The equality is well known in dimension 1 and is proved to be true in dimension 2 by Favre-Jonsson [FJ05]. See [DP04] for more details about $\mathscr{I}_+(\varphi)$.

Definition 1.2.5. *Let (X, ω_X) be a compact Kähler manifold, and let $\alpha \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$ be a real cohomology class of type $(1,1)$. We say that α is nef if for every $\epsilon > 0$, there is a smooth $(1,1)$-form α_ϵ in the same class of α such that $\alpha_\epsilon \geq -\epsilon\omega_X$.*

We say that α is pseudoeffective if there exists a $(1,1)$-current $T \geq 0$ in the same class of α.

We say that α is big if there exists $\epsilon > 0$ such that $\alpha - \epsilon\omega_X$ is pseudoeffective.

Definition 1.2.6. *Let α be a nef class on a compact Kähler manifold X, and let $\pi : X \to T$ be a fibration. We say that α is π-big if for a general fiber F, the restriction $\alpha|_F$ is big.*

Recall also the definition of numerical dimension for a nef cohomology class.

Definition 1.2.7. *[Dem12, Def 6.20] Let X be a compact Kähler manifold, and let $\alpha \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$ be a real cohomology class of type $(1,1)$. Suppose that α is nef. We define the numerical dimension of α by*

$$\mathrm{nd}(\alpha) := \max\{k \in \mathbb{N} \mid \alpha^k \neq 0 \text{ in } H^{2k}(X, \mathbb{R})\}.$$

Remark 1.2.2. *In the situation above, set $m = \mathrm{nd}(\alpha)$. By [Dem12, Prop 6.21] the cohomology class α^m can be represented by a non-zero closed positive (m,m)-current T. Therefore we have $\int_X \alpha^m \wedge \omega_X^{\dim X - m} \neq 0$ for any Kähler class ω_X.*

The notion of nefness can be generalized to vector bundles (cf. [DPS94] for details).

Definition 1.2.8. *A vector bundle E is said to be numerically effective (nef) if the canonical bundle $\mathscr{O}_E(1)$ is nef on $\mathbb{P}(E)$, the projective bundle of hyperplanes in the fibers of E. For a nef line bundle L on a compact Kähler manifold, the numerical dimension $\mathrm{nd}(L)$ is defined to be the largest number v, such that $c_1(L)^v \neq 0$.*

A holomorphic vector bundle E over X is said to be numerically flat if both E and E^* are nef (or equivalently if E and $(\det E)^{-1}$ are nef).

We conclude the introduction by the following well-known result due to Aubin [Aub78] and Yau [Yau78], which plays a central role in this thesis.

Theorem 1.2.3. *Let (X, ω) be a compact Kähler manifold and $\dim X = n$. Then for any smooth volume form $f > 0$ such that $\int_X f = \int_X \omega^n$, there exists a Kähler metric $\widetilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ in the same Kähler class as ω, such that $\widetilde{\omega}^n = f$.*

This theorem in some sense can be used as a replacement for ample divisors. Such ideas are well developed, for example, in [Dem93], [Eno93], [Mou95],[DP04] .... Interestingly, we should mention that the Monge-Ampère equation was already considered in the birth of Kähler manifolds. (cf. [Bou96]) [4]

4. We thank P.Eyssidieux for telling us this amazing history.

Chapitre 2

Numerical dimension and a Kawamata-Viehweg-Nadel type vanishing theorem on compact Kähler manifolds

2.1 Introduction

Let X be a compact Kähler manifold and let (L, φ) be a pseudo-effective line bundle on X. We refer to Section 2, Definition 2.2.1 for the definition of a pseudo-effective pair (L, φ). H.Tsuji [Tsu07] has defined a notion of numerical dimension of such a pair, using an algebraic method :

Definition 2.1.1. *Let X be a projective variety and let (L, φ) be a pseudo-effective line bundle. One defines the numerical dimension of (L, φ) to be*

$$\nu_{\text{num}}(L, \varphi) = \max\{\dim V \mid V \textit{ subvariety of } X \textit{ such that}$$

$$\varphi \textit{ is well defined on } V \textit{ and } (V, L, \varphi) \textit{ is big.} \}$$

Here (V, L, φ) to be big means that there exists a desingularization $\pi : \widetilde{V} \to V$ such that

$$\overline{\lim_{m\to\infty}} \frac{h^0(\widetilde{V}, m\pi^*(L) \otimes \mathscr{I}(m\varphi \circ \pi))}{m^n} > 0,$$

where n is the dimension of V. [1]

Since Tsuji's definition depends on the existence of subvarieties, it is more convenient to find a more analytic definition when the base manifold is not projective. Following a suggestion of J-P. Demailly, we first define a notion of numerical dimension $\text{nd}(L, \varphi)$ (cf. Definition 2.3.1) for a pseudo-effective line bundle (L, φ) on a manifold X which is just assumed to be compact Kähler. The definition involves a certain cohomological intersection product of positive currents, introduced in Section 2. We discuss the properties of $\text{nd}(L, \varphi)$ in Section 3 and 4. The main properties are as follows.

Proposition 2.1.1 ((=Proposition 2.3.7)). *Let (L, φ) be a pseudo-effective line bundle on a projective variety X of dimension n. If $\text{nd}(L, \varphi) = n$, then*

$$\lim_{m\to\infty} \frac{h^0(X, mL \otimes \mathscr{I}(m\varphi))}{m^n} > 0.$$

Proposition 2.1.2 ((=Proposition 2.4.2)). *Let (L, φ) be a pseudo-effective line bundle on a projective variety X. Then*

$$\nu_{\text{num}}(L, \varphi) = \text{nd}(L, \varphi).$$

1. [Tsu07] proved that the bigness does not depend on the choice of desingularizations.

Our main interest in this article is to prove a general Kawamata-Viehweg-Nadel vanishing theorem on an arbitrary compact Kähler manifold. Our statement is as follows.

Theorem 2.1.3 ((=Theorem 2.5.13))**.** *Let (L, φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n. Then*

$$H^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi)) = 0 \qquad \text{for every } p \geq n - \mathrm{nd}(L, \varphi) + 1,$$

where $\mathscr{I}_+(\varphi)$ is the upper semicontinuous variant of the multiplier ideal sheaf associated to φ (cf. [FJ05]).

The organization of the article is as follows. In Section 2, we first recall some elementary results about the analytic multiplier ideal sheaves and define our cohomological product of positive currents by quasi-equisingular approximation. In Section 3, using the product defined in Section 2, we give our definition of the numerical dimension $\mathrm{nd}(L, \varphi)$ for a pseudo-effective line bundle L equipped with a singular metric φ. The main goal of this section is to give an asymptotic estimate of sections when $\mathrm{nd}(L, \varphi) = \dim X$. In section 4, we prove that our numerical dimension coincides with Definition 2.1.1 when X is projective. We also give a numerical criterion of the numerical dimension and discuss a relationship between the numerical dimension without multiplier ideal sheaves and the numerical dimension defined here. In Section 5, we first give a quick proof of our Kawamata-Viehweg-Nadel vanishing theorem on projective varieties. We finally generalize the vanishing theorem on arbitrary compact Kähler manifolds by the methods developed in [DP03b], [Eno93] and [Mou95].
Acknowledgements : I would like to thank Professor J-P. Demailly for numerous ideas and suggestions for this article, and also for his patience and disponibility. I would also like to thank the referees for their valuable suggestions.

2.2 Cohomological product of positive currents

We first recall some basic definitions and results about quasi-psh functions (cf. [Dem12] for details). Let X be a complex manifold. We say that φ is a psh function (resp. a quasi-psh function) on X, if $\varphi : X \to [-\infty, +\infty[$ is upper semicontinuous and

$$i\partial\overline{\partial}\varphi \geq 0, \qquad (\text{resp. } i\partial\overline{\partial}\varphi \geq -c \cdot \omega_X)$$

where c is a positive constant and ω_X is a smooth hermitian metric on X. We say that a quasi-psh function φ has analytic singularities, if φ is locally of the form

$$\varphi(z) = c \cdot \ln(\sum |g_i|^2) + O(1)$$

with $c > 0$ and $\{g_i\}$ are holomorphic functions. Let φ, ψ be two quasi-psh functions. We say that φ is less singular than ψ if

$$\psi \leq \varphi + C$$

for some constant C. We denote it $\varphi \preccurlyeq \psi$.

We now recall the analytic definition of multiplier ideal sheaves. Let φ be a quasi-psh function. We can define the multiplier ideal sheaves associated to the quasi-psh function φ :

$$\mathscr{I}(\varphi)_x = \{f \in \mathscr{O}_X | \int_{U_x} |f|^2 e^{-2\varphi} < +\infty\}$$

where U_x is some open neighborhood of x in X. It is well known that $\mathscr{I}(\varphi)$ is a coherent sheaf (cf. [Dem12] for a more detailed introduction to the concept of multiplier ideal sheaf). When φ does not possess analytic singularities, one needs to introduce the "upper semicontinuous regularization" of $\mathscr{I}(\varphi)$, namely the ideal sheaf

$$\mathscr{I}_+(\varphi) = \lim_{\epsilon \to 0^+} \mathscr{I}((1+\epsilon)\varphi).$$

By the Noetherian property of coherent ideal sheaves, there exists an $\epsilon > 0$ such that

$$\mathscr{I}_+(\varphi) = \mathscr{I}((1+\epsilon')\varphi) \qquad \text{for every } 0 < \epsilon' < \epsilon.$$

When φ has analytic singularities, it is easy to see that

$$\mathscr{I}_+(\varphi) = \mathscr{I}(\varphi). \tag{2.1}$$

Conjecturally the equality (2.1) holds for all quasi-psh functions. Recently, B.Berndtsson [Ber13] proved that equality (2.1) holds for quasi-psh functions φ such that $\mathscr{I}(\varphi) = \mathscr{O}_X$. However, it is unknown whether his method can be generalized to arbitrary quasi-psh functions. [2]

Important Convention : When we talk about a line bundle L on X, we always implicitly fix a smooth metric h_0 on L. Given a quasi-psh function φ on X, we can therefore construct a new metric (maybe singular) on L by setting $h_0 \cdot e^{-\varphi}$. In a similar fashion, when we prescribe a "singular metric" φ on L, we actually mean that the metric on L is given by $h_0 \cdot e^{-\varphi}$. Recall that the curvature form for the metric φ is

$$\frac{i}{2\pi}\Theta_\varphi(L) = \frac{i}{2\pi}\Theta_{h_0}(L) + dd^c\varphi$$

by the Poincaré-Lelong formula.

Definition 2.2.1. *Let L be a pseudo-effective line bundle on a compact Kähler manifold X equipped with a metric φ. We say that (L, φ) is a pseudo-effective pair (or sometimes pseudo-effective line bundle), if the curvature form $\frac{i}{2\pi}\Theta_\varphi(L)$ is positive as a current, i.e., $\frac{i}{2\pi}\Theta_\varphi(L) \geq 0$.*

Let $\pi : \widetilde{X} \to X$ be a modification of a smooth variety X, and let φ, ψ be two quasi-psh fuctions on X such that $\mathscr{I}(\varphi) \subset \mathscr{I}(\psi)$. In general, this inclusion does not imply that $\mathscr{I}(\varphi\circ\pi) \subset \mathscr{I}(\psi\circ\pi)$. In order to compare $\mathscr{I}(\varphi\circ\pi)$ and $\mathscr{I}(\psi\circ\pi)$, we need the following lemma.

Lemma 2.2.1. *Let $E = \pi^* K_X - K_{\widetilde{X}}$. If $\mathscr{I}(\varphi) \subset \mathscr{I}(\psi)$, then*

$$\mathscr{I}(\varphi\circ\pi) \otimes \mathscr{O}(-E) \subset \mathscr{I}(\psi\circ\pi),$$

where the sheaf $\mathscr{O}(-E)$ is the germs of holomorphic functions f such that $\operatorname{div}(f) \geq E$.

Proof. It is known that $\mathscr{I}(\varphi\circ\pi) \subset \pi^*\mathscr{I}(\varphi)$ (cf. [Dem12, Prop 14.3]). Then for any $f \in \mathscr{I}(\varphi\circ\pi)_x$, we have

$$\int_{\pi(U_x)} |\pi_*(f)|^2 e^{-2\varphi} < +\infty, \tag{2.2}$$

where U_x is some open neighborhood of x (its image $\pi(U_x)$ is not necessary open). Combining (2.2) with the condition $\mathscr{I}(\varphi) \subset \mathscr{I}(\psi)$, we get

$$\int_{\pi(U_x)} \mid \pi_* f \mid^2 e^{-2\psi} < +\infty. \tag{2.3}$$

(2.3) implies that

$$\int_{U_x} \mid f \mid^2 \mid J \mid^2 e^{-2\psi\circ\pi} < +\infty, \tag{2.4}$$

where J is Jacobian of π. Since $\mathscr{O}(-E) = J \cdot \mathscr{O}_X$, (2.4) implies the lemma. □

2. The equality (2.1) is well known in dimension 1 and is proved to be true in dimension 2 by Favre-Jonsson [FJ05]. See [DP03b] for more details about $\mathscr{I}_+(\varphi)$.

Let X be a compact Kähler manifold and let T be a closed positive $(1,1)$-current. It is well known that T can be written as

$$T = \theta + dd^c\varphi,$$

where θ is a smooth $(1,1)$-closed form representing $[T] \in H^{1,1}(X,\mathbb{R})$ and φ is a quasi-psh function. Demailly's famous regularization theorem states that φ can be approximated by a sequence of quasi-psh functions with analytic singularities. Such type of approximation is said to be an analytic approximation of φ. Among all these analytic approximations, we want to deal with those which somehow preserve the information concerning the singularities of T. More precisely, we introduce the following definition.

Definition 2.2.2. *Let $\theta + dd^c\varphi$ be a positive current on a compact Kähler manifold (X,ω), where θ is a smooth form and φ is a quasi-psh function on X. We say that $\{\varphi_k\}_{k=1}^{\infty}$ is a quasi-equisingular approximation of φ for the current $\theta + dd^c\varphi$ if it satisfies the following conditions :*

(i) the sequence $\{\varphi_k\}_{k=1}^{\infty}$ converges to φ in L^1 topology and

$$\theta + dd^c\varphi_k \geq -\tau_k \cdot \omega$$

for some constants $\tau_k \to 0$ as $k \to +\infty$.

(ii) all φ_k have analytic singularities and $\varphi_k \preccurlyeq \varphi_{k+1}$ for all k.

(iii) For any $\delta > 0$ and $m \in \mathbb{N}$, there exists $k_0(\delta, m) \in \mathbb{N}$ such that

$$\mathscr{I}(m(1+\delta)\varphi_k) \subset \mathscr{I}(m\varphi) \qquad \text{for every } k \geq k_0(\delta, m)$$

Remark 2.2.2. *By condition (i), the concept of a quasi-equisingular approximation depends not only on φ but rather on the current $\theta + dd^c\varphi$.*

The existence of quasi-equisingular approximations was essentially proved in [DPS01, Thm 2.2.1] by a Bergman kernel method. Such approximations are in some sense the most singular ones asymptotically. The following proposition makes this assertion more precise.

Proposition 2.2.3. *Let $\theta + dd^c\varphi_1, \theta + dd^c\varphi_2$ be two positive currents on a compact Kähler manifold X. We assume that the quasi-psh function φ_2 is more singular than φ_1. Let $\{\varphi_{i,1}\}_{i=1}^{\infty}$ be an analytic approximation of φ_1 and let $\{\varphi_{i,2}\}_{i=1}^{\infty}$ be a quasi-equisingular approximation of φ_2. For any closed smooth $(n-1,n-1)$-semi-positive form u, we have*

$$\varliminf_{i\to\infty} \int_X (dd^c\varphi_{i,1})_{\mathrm{ac}} \wedge u \geq \varlimsup_{i\to\infty} \int_X (dd^c\varphi_{i,2})_{\mathrm{ac}} \wedge u \tag{2.5}$$

where $(dd^c\varphi_{i,1})_{\mathrm{ac}}$ denotes the absolutely continuous part of the current $dd^c\varphi_{i,1}$.

Proof. The idea of the proof is rather standard (cf. [Bou02] or [Dem12, Thm 18.12]). To prove (2.5), it is enough to show that

$$\int_X (dd^c\varphi_{s,1})_{\mathrm{ac}} \wedge u \geq \varlimsup_{i\to\infty} \int_X (dd^c\varphi_{i,2})_{\mathrm{ac}} \wedge u \tag{2.6}$$

for every $s \in \mathbb{N}$ fixed. Since $\{\varphi_{i,2}\}_{i=1}^{\infty}$ is a quasi-equisingular approximation of φ_2, for any $\delta > 0$ and $m \in \mathbb{N}$, there exists a $k_0(\delta, m) \in \mathbb{N}$ such that

$$\mathscr{I}(m(1+\delta)\varphi_{k,2}) \subset \mathscr{I}(m\varphi_2) \qquad \text{for every } k \geq k_0(\delta, m). \tag{2.7}$$

Since $\varphi_{s,1} \preccurlyeq \varphi_1 \preccurlyeq \varphi_2$ by assumption, (2.7) implies that

$$\mathscr{I}(m(1+\delta)\varphi_{k,2}) \subset \mathscr{I}(m\varphi_{s,1}) \tag{2.8}$$

for any $s \in \mathbb{N}$ and $k \geq k_0(\delta, m)$.

Using (2.8), we begin to prove (2.6). Let $\pi : \widehat{X} \to X$ be a log resolution of $\varphi_{s,1}$, i.e., $dd^c(\varphi_{s,1} \circ \pi)$ is locally of the form

$$dd^c(\varphi_{s,1} \circ \pi) = [F] + C^\infty,$$

where F is a $\mathbb{R}$-normal crossing divisor. By Lemma 2.2.1, (2.8) implies that

$$\mathscr{I}(m(1+\delta)\varphi_{k,2} \circ \pi) \otimes \mathscr{O}(-J) \subset \mathscr{I}(m\varphi_{s,1} \circ \pi) = \mathscr{O}(-\lfloor mF \rfloor) \tag{2.9}$$

for $k \geq k_0(\delta, m)$, where J is the Jacobian of the blow up π. Since F is a normal crossing divisor, (2.9) implies that $m(1+\delta)dd^c\varphi_{k,2} \circ \pi + [J] - \lfloor mF \rfloor$ is a positive current. Then

$$\int_{\widehat{X}} (m(1+\delta) \cdot dd^c\varphi_{k,2} \circ \pi)_{\mathrm{ac}} \wedge u \leq C + \int_{\widehat{X}} (m \cdot dd^c\varphi_{s,1} \circ \pi)_{\mathrm{ac}} \wedge u$$

for $k \geq k_0(\delta, m)$, where C is a constant independent of m and k. Letting $m \to +\infty$, we get

$$\int_{\widehat{X}} (dd^c\varphi_{k,2} \circ \pi)_{\mathrm{ac}} \wedge u \leq O(\frac{1}{m}) + C_1\delta + \int_{\widehat{X}} (dd^c\varphi_{s,1} \circ \pi)_{\mathrm{ac}} \wedge u \tag{2.10}$$

for $k \geq k_0(\delta, m)$, where C_1 is a constant independent of m and k. Then

$$\int_X (dd^c\varphi_{k,2})_{\mathrm{ac}} \wedge u \leq O(\frac{1}{m}) + C_1\delta + \int_X (dd^c\varphi_{s,1})_{\mathrm{ac}} \wedge u \qquad \text{for } k \geq k_0(\delta, m).$$

Letting $m \to +\infty$ and $\delta \to 0$, we get

$$\overline{\lim_{k\to\infty}} \int_X (dd^c\varphi_{k,2})_{\mathrm{ac}} \wedge u \leq \int_X (dd^c\varphi_{s,1})_{\mathrm{ac}} \wedge u.$$

(2.6) is proved.

□

Remark 2.2.4. *By taking $\varphi_1 = \varphi_2$ and $\varphi_{i,1} = \varphi_{i,2}$ in Proposition 2.2.3, we obtain that the sequence $\{\int_X (dd^c\varphi_{i,2})_{\mathrm{ac}} \wedge u\}_{i=1}^\infty$ is in fact convergent. Moreover, if $\{\varphi_{i,1}\}, \{\varphi_{i,2}\}$ are two quasi-equisingular approximations of φ, Proposition 2.2.3 implies that*

$$\lim_{i\to\infty} \int_X (dd^c\varphi_{i,1})_{\mathrm{ac}} \wedge u = \lim_{i\to\infty} \int_X (dd^c\varphi_{i,2})_{\mathrm{ac}} \wedge u. \tag{2.11}$$

Thanks to Proposition 2.2.3 and (2.11), we can define a related cohomological product of closed positive $(1,1)$-currents.

Definition 2.2.3. *Let $T_1, \cdots, T_k$ be closed positive $(1,1)$-currents on a compact Kähler manifold X. We write them by the potential forms $T_i = \theta_i + dd^c\varphi_i$ as usual. Let $\{\varphi_{i,j}\}_{j=1}^\infty$ be a quasi-equisingular approximation of φ_i. Then we can define a product*

$$\langle T_1, T_2, \cdots, T_k \rangle$$

as an element in $H^{k,k}_{\geq 0}(X)$ (cf. [Bou02] or [Dem12, Thm 18.12]) such that for all $u \in H^{n-k,n-k}(X)$,

$$\langle T_1, T_2, \cdots, T_k \rangle \wedge u$$

$$= \lim_{j\to\infty} \int_X (\theta_1 + dd^c\varphi_{1,j})_{\mathrm{ac}} \wedge \cdots \wedge (\theta_k + dd^c\varphi_{k,j})_{\mathrm{ac}} \wedge u$$

where $\wedge$ is the usual wedge product in cohomology.

Remark 2.2.5. *Let $\{\psi_{i,j}\}_{j=1}^\infty$ be an analytic approximation (not necessarily quasi-equisingular) of φ_i. Thanks to Proposition 2.2.3 and some standard arguments (cf. [Dem12, Thm 18.12]), we have*

$$\underline{\lim_{j\to\infty}} \int_X (\theta_1 + dd^c\psi_{1,j})_{\mathrm{ac}} \wedge \cdots \wedge (\theta_k + dd^c\psi_{k,j})_{\mathrm{ac}} \wedge u \geq \lim_{j\to\infty} \int_X (\theta_1 + dd^c\varphi_{1,j})_{\mathrm{ac}} \wedge \cdots \wedge (\theta_k + dd^c\varphi_{k,j})_{\mathrm{ac}} \wedge u.$$

This means that the product defined in Definition 2.2.3 is smaller than the product defined by any other analytic approximations. In particular, the product defined in Definition 2.2.3 does not depend on the choice of the quasi-equisingular approximations.

2.3 Numerical dimension

Using Definition 2.2.3, we can give our definition of the numerical dimension.

Definition 2.3.1. *Let (L, φ) be a pseudo-effective line bundle on a compact Kähler manifold X. We define the numerical dimension* $\mathrm{nd}(L, \varphi)$ *to be the largest $v \in \mathbb{N}$, such that $\langle (i\Theta_\varphi(L))^v \rangle \neq 0$, where the cohomological product $\langle (i\Theta_\varphi(L))^v \rangle$ is the v-fold product of $i\Theta_\varphi(L)$ defined in Definition 2.2.3.*

Let (L, φ) be a pseudo-effective line bundle on X of dimension n such that $\mathrm{nd}(L, \varphi) = n$. If the quasi-psh function φ has analytic singularities, it is not difficult to see that

$$\frac{h^0(X, mL \otimes \mathscr{I}(m\varphi))}{m^n}$$

admits a strictly positive limit by using the Riemann-Roch formula. When φ is just a quasi-psh function, H.Tsuji conjectured in [Tsu07] that

$$\frac{h^0(X, mL \otimes \mathscr{I}(m\varphi))}{m^n}$$

also admits a strictly positive limit. The main goal of this section is to prove Proposition 2.1.1, i.e., if $\mathrm{nd}(L, \varphi) = n$, then

$$\varliminf_{m \to \infty} \frac{h^0(X, mL \otimes \mathscr{I}(m\varphi))}{m^n} > 0.$$

To begin with, we first explain the construction of quasi-equisingular approximations by a Bergman Kernel method. Before doing this, we first give a useful estimate by using the comparison of integrals method in [DPS01, Thm 2.2.1, Step 2]. Although the proof is almost the same, we give the proof here for the sake of completeness.

Lemma 2.3.1. *Let A be a very ample line bundle on a projective manifold X and let (L, φ) be a pseudo-effective line bundle. Let φ_m be the metric on L constructed by the Bergman Kernel of $H^0(X, \mathscr{O}(A + mL) \otimes \mathscr{O}(m\varphi))$ with respect to the metric $m\varphi$. Then*

$$\mathscr{I}(\frac{sm}{m-s}\varphi_m) \subset \mathscr{I}(s\varphi) \qquad \text{for any } m, s \in \mathbb{N}.$$

Proof. First of all, we have the following estimate on X :

$$\int_{s\cdot\varphi(x) \leq \frac{sm}{m-s}\cdot\varphi_m(x)} e^{-2s\cdot\varphi(x)} = \int_{s\cdot\varphi(x) \leq \frac{sm}{m-s}\cdot\varphi_m(x)} e^{2(m-s)\cdot\varphi(x) - 2m\cdot\varphi(x)}$$

$$\leq \int_X e^{2m\cdot\varphi_m} e^{-2m\cdot\varphi} = h^0(X, \mathscr{O}(A + mL) \otimes \mathscr{I}(m\varphi)) < +\infty.$$

Using the above finiteness, for any $f \in \mathscr{I}(\frac{sm}{m-s}\varphi_m)_x$, we have

$$\int_{U_x} |\, f \,|^2 \, e^{-2s\varphi} \leq \int_{s\varphi(x) \leq \frac{sm}{m-s}\varphi_m(x)} |\, f \,|^2 \, e^{-2s\varphi} + \int_{U_x} |\, f \,|^2 \, e^{-\frac{2sm}{m-s}\varphi_m}$$

$$\leq \sup |\, f \,|^2 \cdot \int_{s\varphi(x) \leq \frac{sm}{m-s}\varphi_m(x)} e^{-2s\varphi} + \int_{U_x} |\, f \,|^2 \, e^{-2\frac{sm}{m-s}\varphi_m} < +\infty.$$

Then $f \in \mathscr{I}(s\varphi)$. The lemma is proved. □

We are going to construct a quasi-equisingular approximation to φ. Although such type of approximations was implicitly constructed in [DPS01, Thm 2.2.1] in the local case, we can easily adapt that construction to a global situation by using the same techniques.

Proposition 2.3.2. *Let X be a projective variety of dimension n and let ω be a Kähler metric in $H^{1,1}(X,\mathbb{Q})$. Let (L,φ) be a pseudo-effective line bundle on X (cf. Definition 2.2.1) such that* $\mathrm{nd}(L,\varphi)=n$.

Let (G,h_G) be an ample line bundle on X equipped with a smooth metric h_G, such that the curvature form $i\Theta_{h_G}(G)$ is positive and sufficiently large (e.g. G is very ample and $G-K_X$ is ample). Let $\{\tau_{p,q,i}\}_i$ be an orthonormal basis of

$$H^0(X,\mathscr{O}(2^pG+2^qL)\otimes\mathscr{I}(2^q\varphi))$$

with respect to the singular metric $h_G^{2^p}\cdot h_0^{2^q}\cdot e^{-2^q\varphi}$. We define

$$\varphi_{p,q}=\frac{1}{2^q}\ln\sum_i|\tau_{p,q,i}|^2_{h_G^{2^p}\cdot h_0^{2^q}}.$$

Then there exist two increasing integral sequences $p_m\to+\infty$ and $q_m\to+\infty$ with

$$\lim_{m\to+\infty}(q_m/p_m)=+\infty$$

and

$$q_m-q_{m-1}\geq p_m-p_{m-1}\qquad \textit{for all } m\in\mathbb{N},$$

such that $\{\varphi_{p_m,q_m}\}_{m=1}^{+\infty}$ is a quasi-equisingular approximation of φ for the current $\frac{i}{2\pi}\Theta_{h_0}(L)+dd^c\varphi$. Set $\varphi_m:=\varphi_{p_m,q_m}$ for simplicity.

Moreover, $\{\varphi_m\}$ satisfies the following two properties :

(i) : $H^0(X,\mathscr{O}(2^{p_m}G+2^{q_m}L)\otimes\mathscr{I}(2^{q_m}\varphi_m))=H^0(X,\mathscr{O}(2^{p_m}G+2^{q_m}L)\otimes\mathscr{I}(2^{q_m}\varphi))$ *for every $m\in\mathbb{N}^+$.*

(ii) : *There exists a constant $C>0$ independent of G, m, such that*

$$\int_X(\frac{i}{2\pi}\Theta_{\varphi_m}(L)+\epsilon\omega)^n_{\mathrm{ac}}>C$$

for all $\epsilon>0$ and $m\geq m_0(\epsilon)$ (i.e. m is larger than a constant depended on ϵ.).

Proof. By [Dem12, Thm 13.21, Thm 13.23], there exists two squences $p_m\to+\infty$ and $q_m\to+\infty$ with

$$\lim_m q_m/p_m=+\infty$$

and

$$q_m-q_{m-1}\geq p_m-p_{m-1}\text{ for all } m\in\mathbb{N},$$

such that $\{\varphi_m\}$ is an analytic approximation of φ for the current $\frac{i}{2\pi}\Theta_\varphi(L)$. Since φ_m is constructed by Bergman kernel, by using Lemma 2.3.1, $\{\varphi_m\}$ satisfies Property *(iii)* in Definition 2.2.2. To prove that $\{\varphi_m\}$ is a quasi-equisingular approximation, it remains to prove Property *(ii)* in Definition 2.2.2.

We first prove that

$$\varphi_{p-1,q-1}\preccurlyeq\varphi_{p,q}\qquad\text{and}\qquad\varphi_{p,q-1}\preccurlyeq\varphi_{p-1,q-1}\tag{2.12}$$

by using the standard diagnoal trick (cf. [Del10] or [DPS01, Thm 2.2.1, Step 3]). Let Δ be the diagonal of $X\times X$ and π_1,π_2 two projections from $X\times X$ to X. Set

$$F:=2^{p-1}\pi_1^*G+2^{p-1}\pi_2^*G+2^{q-1}\pi_1^*L+2^{q-1}\pi_2^*L$$

be a new bundle on $X\times X$ equipped with a singular metric $2^{q-1}\pi_1^*(\varphi)+2^{q-1}\pi_2^*(\varphi)$. Since $2^{p-1}G-K_X$ is enough ample, we can apply the Ohsawa-Takegoshi extension theorem from Δ to $X\times X$ for the line bundle F. Then the following map is surjective :

$$\begin{aligned}(H^0(X,\mathscr{O}(2^{p-1}G+2^{q-1}L)\otimes\mathscr{I}(2^{q-1}\varphi)))^2\\ \to H^0(X,\mathscr{O}(2^pG+2^qL)\otimes\mathscr{I}(2^q\varphi)).\end{aligned}\tag{2.13}$$

Let $\{f_{p-1,q-1,i}\}_{i=1}^{N}$ be an orthonormal basis of

$$H^0(X, \mathscr{O}(2^{p-1}G + 2^{q-1}L) \otimes \mathscr{I}(2^{q-1}\varphi))$$

with respect to the singular metric $h_G^{2^{p-1}} \cdot h_0^{2^{q-1}} \cdot e^{-2^{q-1}\varphi}$. For any

$$g \in H^0(X, \mathscr{O}(2^pG + 2^qL) \otimes \mathscr{I}(2^q\varphi)),$$

by applying the effective version of Ohsawa-Takegoshi extension theorem to (2.13), there exist constants $\{c_{i,j}\}$ such that

$$g(z) = (\sum_{i,j} c_{i,j} f_{p-1,q-1,i}(z) f_{p-1,q-1,j}(w))|_{z=w}$$

and

$$\sum_{i,j} |c_{i,j}|^2 \leq C_1 \|g\|^2,$$

where C_1 depends only on X and $\|g\|$ is the L^2-norm with respect to the singular metric $h_G^{2^p} \cdot h_0^{2^q} e^{-2^q\varphi}$. By the Cauchy-Schwarz inequality, we have

$$|g(z)|^2_{h_G^{2^p} \cdot h_0^{2^q}} \leq (\sum_{i,j} |c_{i,j}|^2)(\sum_{i,j} |f_{p-1,q-1,i}(z) f_{p-1,q-1,j}(z)|^2_{h_G^{2^p} \cdot h_0^{2^q}})$$

$$\leq C_1 \|g\|^2 (\sum_i |f_{p-1,q-1,i}(z)|^2_{h_G^{2^{p-1}} \cdot h_0^{2^{q-1}}})^2.$$

Assuming $\|g\| = 1$, we get

$$\frac{1}{2^q} \ln|g(z)|^2_{h_G^{2^p} \cdot h_0^{2^q}} \leq \frac{\ln C_1}{2^q} + \frac{1}{2^{q-1}} \ln(\sum_i |f_{p-1,q-1,i}(z)|^2_{h_G^{2^{p-1}} \cdot h_0^{2^{q-1}}})$$

$$= \frac{\ln C_1}{2^q} + \varphi_{p-1,q-1}(z).$$

By the extremal property of Bergman kernel, we finally obtain

$$\varphi_{p-1,q-1} \preccurlyeq \varphi_{p,q}.$$

The first inequality in (2.12) is proved. The second inequality in (2.12) is evident by observing that G is very ample. Thanks to the construction of p_m and q_m, (2.12) implies that $\varphi_{m-1} \preccurlyeq \varphi_m$. Therefore φ_m is a quasi-equisingular approximation of φ for the current $\frac{i}{2\pi}\Theta_\varphi(L)$.

It remains to check Property (i) and Property (ii) listed in the proposition. Property (i) comes directly from the construction of φ_m. Property (ii) follows from the fact that $\mathrm{nd}(L, \varphi) = n$ and φ_m is an quasi-equisingular approximation. □

The rest of the section is devoted to the proof of Proposition 2.1.1. The strategy is as follows. Thanks to Property (ii) of Proposition 2.3.2, we can construct a new metric on L with strictly positive curvature, that is more singular than φ in an asymptotic way (cf. (2.22)). Then Proposition 2.1.1 follows by a standard estimate for this new metric. Before giving the construction of the new metric, we need the following two preparatory propositions.

Proposition 2.3.3. *Let φ_m be the quasi-psh function constructed in Proposition 2.3.2. Then there exists another quasi-psh function $\widetilde{\varphi}_m$ such that*

(i) : $\sup_{x \in X} \widetilde{\varphi}_m(x) = 0$

(ii) : $\frac{i}{2\pi}\Theta_{\widetilde{\varphi}_m}(L) \geq \frac{\delta}{2} \cdot \omega$, *where δ is a strictly positive number independent of m.*

(iii) : $\varphi_m \preccurlyeq \widetilde{\varphi}_m$

Proof. Let $\pi : X_m \to X$ be a log resolution of φ_m. We can hence assume that

$$\frac{i}{2\pi}\Theta_{\varphi_m\circ\pi}(\pi^* L) = [E] + \beta,$$

where $[E]$ is a normal crossing divisor and $\beta \in C^\infty$. Keeping the notation used in Proposition 2.3.2, since $\omega \in H^{1,1}(X,\mathbb{Q})$, we can find a $\mathbb{Q}$-ample line bundle A on X such that $c_1(A) = \omega$. Let ϵ be a positive rational number. By property (ii) of Proposition 2.3.2, we have

$$\int_X (\frac{i}{2\pi}\Theta_{\varphi_m}(L) + \epsilon\omega)^n_{\mathrm{ac}} > C.$$

Thanks to Proposition 2.3.2,

$$(\frac{i}{2\pi}\Theta_{\varphi_m\circ\pi}(\pi^* L) + \epsilon\pi^*\omega)_{\mathrm{ac}}$$

is a $\mathbb{Q}$-nef class for m large enough. We can thus choose a $\mathbb{Q}$-nef line bundle F_m on X_m such that

$$c_1(F_m) = (\frac{i}{2\pi}\Theta_{\varphi_m\circ\pi}(\pi^* L) + \epsilon\pi^*\omega)_{\mathrm{ac}}. \tag{2.14}$$

We now prove that

$$F_m - \delta\pi^*\omega \tag{2.15}$$

is pseudo-effective for a uniform constant $\delta > 0$ independent of ϵ and m. In order to prove (2.15), we first give a uniform upper bound of $F_m^{n-1}\cdot\pi^* A$. Let C_1 be a constant such that $C_1 \cdot A - L$ is effective. Using the nefness of F_m and $\pi^* A$, (2.14) implies that

$$F_m^{n-1}\cdot\pi^* A \leq F_m^{n-2}\cdot(\pi^* L + \epsilon\pi^*\omega)\cdot\pi^* A \leq F_m^{n-2}\cdot(C_1+\epsilon)\pi^* A\cdot\pi^* A$$
$$\leq F_m^{n-3}\cdot((C_1+\epsilon)\pi^* A)^2\cdot\pi^* A \leq \cdots \leq ((C_1+\epsilon)\pi^* A)^{n-1}\cdot\pi^* A.$$

Therefore $\{F_m^{n-1}\cdot\pi^* A\}_m$ is uniformlly bounded (for $\epsilon < 1$). Combining this with Property (ii) of Proposition 2.3.2, we can thus choose a rational constant $\delta > 0$ independent of ϵ and m, such that

$$F_m^n > n\delta F_m^{n-1}\cdot\pi^* A. \tag{2.16}$$

Using the holomorphic Morse inequality (cf. [Dem12, Chapter 8] or [Tra11]) for the $\mathbb{Q}$-bundle $F_m - \delta\cdot\pi^*(A)$ on X_m, we have

$$h^0(X_m, kF_m - k\delta\cdot\pi^* A) \geq C\frac{k^n}{n!}(F_m^n - n\delta F_m^{n-1}\cdot\pi^* A) + O(k^{n-1}). \tag{2.17}$$

Combining (2.16) and (2.17), we obtain that $F_m - \delta\pi^*\omega$ is pseudo-effective.

By taking $\epsilon \leq \frac{\delta}{2}$, the pseudo-effectiveness (2.15) implies that $\frac{i}{2\pi}\Theta_{\varphi_m\circ\pi}(\pi^* L)_{\mathrm{ac}} - \frac{\delta}{2}\pi^*\omega$ is pseudo-effective. In other words, there exists a quasi-psh function ψ_m on X_m such that

$$\frac{i}{2\pi}\Theta_{\varphi_m\circ\pi}(\pi^* L) + dd^c\psi_m \geq \frac{\delta}{2}\pi^*\omega. \tag{2.18}$$

Let C_1 be a constant such that

$$\sup_{x\in X_m}(\varphi_m\circ\pi + \psi_m + C_1)(x) = 0.$$

Then (2.18) implies that $\varphi_m\circ\pi(x) + \psi_m(x) + C_1$ induces a quasi-psh function on X. We denote it $\widetilde{\varphi}_m$. It is easy to check that $\widetilde{\varphi}_m$ satisfies all the requirements in the proposition. □

Remark 2.3.4. *In the proof of Proposition 2.3.3, we assume that ϵ is rational. The reason is that we want to use the holomorphic Morse inequality* (2.17)*. However, by using the techniques in [DP04], we can get the same results without the assumption that ϵ is rational.*

Thanks to Proposition 2.3.3, we are going to construct a singular metric on L which is a type of limit of $\widetilde{\varphi}_m$. We first recall the notion of upper semicontinuous regularization. Let $\Omega \subset \mathbb{R}^n$ and let $(u_\alpha)_{\alpha\in I}$ be a family of upper semicontinuous fuctions $\Omega \to [-\infty, +\infty[$. Assume that (u_α) is locally uniformly bounded from above. Since the upper envelope

$$u = \sup_{\alpha\in I} u_\alpha$$

need not be upper semicontinuous, we consider its upper semicontinuous regularization :

$$u^*(z) = \lim_{\epsilon\to 0} \sup_{B(z,\epsilon)} u.$$

We denote this upper semicontinuous regularization by $\widetilde{\sup_\alpha}(u_\alpha)$. It is easy to prove that if $\{u_\alpha\}_{\alpha\in I}$ are psh fonctions which are locally uniformly bounded from above, then $\widetilde{\sup_\alpha}(u_\alpha)$ is also a psh function (cf. [Dem12] for details).

We need the following lemma.

Lemma 2.3.5. *Let φ be a quasi-psh fonction with normal crossing singularities, i.e., φ is locally of the form*

$$\varphi = \sum_i a_i \ln|f_i| + O(1),$$

where f_i are holomorphic fonctions and $\sum_i \operatorname{div}(f_i)$ is a normal crossing divisor. Let $\{\psi_i\}$ be quasi-psh functions such that

$$\sup_{z\in X} \psi_i(z) \leq 0 \qquad \text{and} \qquad dd^c\psi_i \geq -C\omega$$

for some uniform constant C independent of i. If $\varphi \preccurlyeq \psi_i$ for all i, then

$$\varphi \preccurlyeq \widetilde{\sup_i}(\psi_i).$$

Proof. Since φ has normal crossing singularities and φ is less singular than φ_i, the differences $\psi_i - \varphi$ are quasi-psh functions and

$$dd^c(\psi_i - \varphi) \geq -C_1\omega \tag{2.19}$$

for some uniform constant C_1 independent of i. Since $\sup_{z\in X} \psi_i(z) \leq 0$ and $dd^c\psi_i \geq -C\omega$ for a uniform constant C, the standard potential theory implies that there exists a constant M such that

$$\int_X \psi_i \leq M \qquad \text{for all } i.$$

Therefore

$$\int_X (\psi_i - \varphi) \leq M' \tag{2.20}$$

for a uniform constant M'.

Combining (2.19) with (2.20), there exists a uniform constant C_2 such that

$$\sup_{z\in X}(\psi_i(z) - \varphi(z)) \leq C_2 \qquad \text{for all } i.$$

Therefore $\varphi \preccurlyeq \widetilde{\sup_i}(\psi_i)$ and the lemma is proved. □

Thanks to Proposition 2.3.3 and Proposition 2.3.5, we can construct the following crucial metric mentioned in the paragraph before Proposition 2.3.3.

Proposition 2.3.6. *In the situation of Proposition 2.3.3, set*

$$\widetilde{\varphi}(z) := \lim_{m\to\infty} \widetilde{\sup_{s\geq 0}}((\widetilde{\varphi}_{m+s}(z))).$$

Then the new metric $\widetilde{\varphi}$ *satisfies :*

$$\frac{i}{2\pi}\Theta_{\widetilde{\varphi}}(L) \geq \frac{\delta}{2}\omega \tag{2.21}$$

and

$$\varphi_m \preccurlyeq \widetilde{\varphi} \qquad \text{for every } m \geq 1. \tag{2.22}$$

Proof. By Proposition 2.3.3, we have

$$\frac{i}{2\pi}\Theta_{\widetilde{\varphi}_m}(L) \geq \frac{\delta}{2}\omega \qquad \text{for } m \geq 1.$$

By letting $m \to +\infty$, (2.21) is proved. To check (2.22), since $\widetilde{\varphi} \leq \widetilde{\sup_{s\geq 0}}(\widetilde{\varphi}_{m+s})$ by construction, it is enough to show that

$$\varphi_m \preccurlyeq \widetilde{\sup_{s\geq 0}}(\widetilde{\varphi}_{m+s}). \tag{2.23}$$

Combining Proposition 2.3.2 with Proposition 2.3.3, we have

$$\varphi_m \preccurlyeq \varphi_{m+s} \preccurlyeq \widetilde{\varphi}_{m+s} \qquad \text{for every } m, s. \tag{2.24}$$

Let $\pi : \widehat{X} \to X$ be a log resolution of φ_m. By (2.24), we have

$$\varphi_m \circ \pi \preccurlyeq \varphi_{m+s} \circ \pi \preccurlyeq \widetilde{\varphi}_{m+s} \circ \pi. \tag{2.25}$$

Since $\varphi_m \circ \pi$ has normal crossing singularities, by Lemma 2.3.5, (2.25) implies that

$$\varphi_m \circ \pi \preccurlyeq \widetilde{\sup_{s\geq 0}}(\widetilde{\varphi}_{m+s} \circ \pi).$$

By passing to π_*, (2.23) is proved. □

Using the new metric $\widetilde{\varphi}$, we can give the following asymptotic estimate.

Proposition 2.3.7 ((= Proposition 2.1.1))**.** *Let X be a projective variety of dimension n and let (L,φ) be a pseudo-effective line bundle on X such that* $\mathrm{nd}(L,\varphi) = n$*. Then*

$$\varliminf_{m\to\infty} \frac{h^0(X, mL \otimes \mathscr{I}(m\varphi))}{m^n} > 0.$$

Proof. Let $\{\varphi_m\}$ be the quasi-equisingular approximation of φ constructed in Proposition 2.3.2. By Lemma 2.3.1, for every $m \in \mathbb{N}$, we have

$$h^0(X, mL \otimes \mathscr{I}(m\varphi)) \geq h^0(X, mL \otimes \mathscr{I}(\frac{m \cdot 2^{q_k}}{2^{q_k} - m}\varphi_k)). \tag{2.26}$$

Let $\widetilde{\varphi}$ be the metric constructed in Proposition 2.3.6. By (2.22) in Proposition 2.3.6, we have

$$h^0(X, mL \otimes \mathscr{I}(\frac{m \cdot 2^{q_k}}{2^{q_k} - m}\varphi_k)) \geq h^0(X, mL \otimes \mathscr{I}(\frac{m \cdot 2^{q_k}}{2^{q_k} - m}\widetilde{\varphi})). \tag{2.27}$$

for every k, m. Combining (2.26) with (2.27), we have

$$h^0(X, mL \otimes \mathscr{I}(m\varphi)) \geq h^0(X, mL \otimes \mathscr{I}(\frac{m \cdot 2^{q_k}}{2^{q_k} - m}\widetilde{\varphi})). \tag{2.28}$$

Since (2.28) is true for every m and k, we can take k so large that $2^{q_k} \gg m$. By applying (2.21) to (2.28), we have

$$\varliminf_{m\to\infty} \frac{h^0(X, mL \otimes \mathscr{I}(m\varphi))}{m^n} > 0.$$

The proposition is proved. □

Remark 2.3.8. *Proposition 2.3.7 implies that if* $\mathrm{nd}(L,\varphi)=\dim X$*, then* $\nu_{\mathrm{num}}(L,\varphi)=\dim X$ *(cf. Definition 2.1.1). In the next section, we will study the relation between* $\mathrm{nd}(L,\varphi)$ *and* $\nu_{\mathrm{num}}(L,\varphi)$ *in more detail.*

2.4 A numerical criterion

Until now, we have two concepts of numerical dimension for a pseudo-effective pair (L,φ) : the “algebraic" concept $\nu_{\mathrm{num}}(L,\varphi)$ and the more analytic concept $\mathrm{nd}(L,\varphi)$ (see Definition 2.1.1 and Definition 2.3.1). We prove in this section that these two definitions coincide when X is projective. Before giving the proof, we first list some properties of multiplier ideal sheaves which will be useful in our context. The essential tool here is the Ohsawa-Takegoshi extension theorem (cf. [Dem12, Chapter 12]).

Lemma 2.4.1. *Let* (L,φ) *be a pseudo-effective line bundle on a projective variety* X *of dimension* n *and let* $\{\varphi_k\}$ *be a quasi-equisingular approximation of* φ*. Let* s_1 *be a positive number such that*

$$\mathscr{I}_+(\varphi)=\mathscr{I}((1+\epsilon')\varphi) \qquad \text{for every } 0<\epsilon'\leq s_1. \tag{2.29}$$

Assume that A *is a very ample line bundle and* S *is the zero divisor of a very general global section of* $H^0(X,A)$*. We have the following properties :*

(i) The restrictions

$$\mathscr{I}(m\varphi_k)\to\mathscr{I}(m\varphi_k|_S), \qquad \mathscr{I}_+(m\varphi_k)\to\mathscr{I}_+(m\varphi_k|_S) \tag{2.30}$$

$$\mathscr{I}(m\varphi)\to\mathscr{I}(m\varphi|_S), \qquad \mathscr{I}_+(m\varphi)\to\mathscr{I}_+(m\varphi|_S) \tag{2.31}$$

are well defined for all $m\in\mathbb{N}$*, where* $\varphi|_S$ *denotes the restriction of* φ *on* S *and* $\mathscr{I}(\varphi|_S)$ *is the multiplier ideal sheaf associated to* $\varphi|_S$ *on* S.[3] *Moreover we have*

$$\mathscr{I}((1+\epsilon')\varphi|_S)=\mathscr{I}((1+s_1)\varphi|_S) \qquad \text{for every } 0<\epsilon'\leq s_1.$$

(ii) $\{\varphi_k|_S\}$ *is a quasi-equisingular approximation of* $\varphi|_S$*.*

(iii) If the restrictions are well defined, we have an exact sequence :

$$0\to\mathscr{I}_+(\varphi)\otimes\mathscr{O}(-S)\to\mathrm{adj}_S^{\epsilon}(\varphi)\to\mathscr{I}_+(\varphi|_S)\to 0$$

for every $0<\epsilon\leq s_1$*, where*

$$\mathrm{adj}_S^{\epsilon}(\varphi)_x=\{f\in\mathscr{O}_x,\int_{U_x}\frac{|\ f\ |^2}{|\ s\ |^{2(1-\frac{\epsilon}{2})}}e^{-2(1+\epsilon)\varphi}<+\infty\}.$$

(iv) $\mathrm{adj}_S^{\epsilon}(\varphi)=\mathscr{I}_+(\varphi)$ *for every* $0<\epsilon\leq s_1$*.*

Proof. (i) : First of all, since S is very general, φ_k and φ are well defined on S. Since the mutiplier ideal sheaves here are coherent and the restrictions (2.30), (2.31) contain only countable morphismes, by Fubini theorem, it is easy to see that the restrictions (2.30) and (2.31) are well defined.

For the second part of (i), since S is very general, we can suppose that

$$\mathscr{I}((1+s_1)\varphi)\to\mathscr{I}((1+s_1)\varphi|_S) \tag{2.32}$$

is well defined. Combining this with (2.29), we obtain that

$$\mathscr{I}((1+\epsilon')\varphi)\to\mathscr{I}((1+\epsilon')\varphi|_S) \tag{2.33}$$

3. Note that $\varphi|_S$ is also quasi-psh if it is well defined.

is well defined for every $0 < \epsilon' < s_1$. Let $f \in \mathscr{I}(S, (1+s_1)\varphi\mid_S)_x$. Applying the Ohsawa-Takegoshi extension theorem to (2.32), there exists a function $\widetilde{f} \in \mathscr{I}((1+s_1)\varphi)$ such that $\widetilde{f}|_S = f$. Thanks to (2.29) and (2.33), $\widetilde{f}|_S \in \mathscr{I}((1+\epsilon')\varphi|_S)$ for every $0 < \epsilon' < s_1$. (i) is proved.

(ii) : Since $\{\varphi_k\}$ is a quasi-equisingular approximation of φ, we have

$$\mathscr{I}(m(1+\delta)\varphi_k) \subset \mathscr{I}(m\varphi) \qquad \text{for every } k \geq k_0(\delta, m). \tag{2.34}$$

To prove (ii), it is enough to prove that

$$\mathscr{I}(m(1+\delta)\varphi_k|_S) \subset \mathscr{I}(m\varphi|_S) \qquad \text{for every } k \geq k_0(\delta, m). \tag{2.35}$$

Let $f \in \mathscr{I}(m(1+\delta)\varphi_k|_S)_x$. By the Ohsawa-Takegoshi extension theorem, there exists a $\widetilde{f} \in \mathscr{I}(X, m(1+\delta)\varphi_k)_x$ such that $\widetilde{f}|_S = f$. By (2.34), $\widetilde{f} \in \mathscr{I}(m\varphi)$. Thanks to (2.31), we have $\widetilde{f}|_S \in \mathscr{I}(S, m\varphi\mid_S)$. (2.35) is proved.

(iii) : First of all, the Ohsawa-Takegoshi extension theorem implies the surjectivity of the sequence. It remains to prove the exactness of the middle term, i.e., for any $f \in \mathscr{O}_x$ satisfying the conditions

$$\frac{f}{s} \in \mathscr{O}_x \qquad \text{and} \qquad \int_{U_x} \frac{|f|^2}{|s|^{2(1-\frac{\epsilon}{2})}} e^{-2(1+\epsilon)\varphi} < +\infty, \tag{2.36}$$

we should prove the existence of some $\epsilon' > 0$ such that

$$\int_{U_x} \frac{|f|^2}{|s|^2} e^{-2(1+\epsilon')\varphi} < +\infty, \tag{2.37}$$

where s is a local defining function for S. In fact, if $\frac{f}{s} \in \mathscr{O}_x$, then

$$\int_{U_x} \frac{|f|^2}{|s|^{4-\delta}} < +\infty \qquad \text{for every } \delta > 0. \tag{2.38}$$

By taking $\epsilon' = \frac{\epsilon}{4}$ in (2.37), we have

$$\int_{U_x} \frac{|f|^2}{|s|^2} e^{-2(1+\frac{\epsilon}{4})\varphi} \tag{2.39}$$

$$\leq \Big(\int_{U_x} \frac{|f|^2}{|s|^{2(1-\frac{\epsilon}{2})}} e^{-2(1+\epsilon)\varphi}\Big)^{\frac{1+\frac{\epsilon}{4}}{1+\epsilon}} \Big(\int_{U_x} \frac{|f|^2}{|s|^{\alpha}}\Big)^{\frac{\frac{3\epsilon}{4}}{1+\epsilon}}$$

by Hölder's inequality, where

$$\alpha = \Big(2 - 2\big(1 - \frac{\epsilon}{2}\big)\frac{1+\frac{\epsilon}{4}}{1+\epsilon}\Big) \cdot (1+\epsilon) \cdot \frac{4}{3\epsilon} = \frac{10\epsilon + \epsilon^2}{3\epsilon} < 4.$$

Thanks to (2.36) and (2.38), the second line of (2.39) is finite. Thus (2.37) is proved.

(iv) : By the definition of $\mathscr{I}_+(\varphi)$, we have an obvious inclusion

$$\mathrm{adj}_S^{\epsilon}(\varphi) \subset \mathscr{I}_+(\varphi).$$

In order to prove the equality, it is enough to show that for any $f \in \mathscr{I}((1+\epsilon)\varphi)_x$, we have

$$\int_{U_x} \frac{|f|^2}{|s|^{2(1-\frac{\epsilon}{2})}} e^{-2(1+\epsilon)\varphi} dV < +\infty, \tag{2.40}$$

where s is a general global section of $H^0(X, A)$ independent of the choice of f and x.

(2.40) comes from the Fubini theorem. In fact, let $\{s_0, \cdots, s_N\}$ be a basis of $H^0(X, A)$. Then

$$\sum_{i=0}^{N} |s_i(x)|^2 \neq 0 \qquad \text{for every } x \in X.$$

Taking $\{\tau_0, \cdots, \tau_N\} \in \mathbb{C}^{N+1}$, we have

$$\int_{\sum\limits_{i=0}^{N} |\tau_i|^2=1} d\tau \int_{U_x} \frac{| f |^2}{| \sum\limits_{i=0}^{N} \tau_i s_i |^{2(1-\frac{\epsilon}{2})}} e^{-2(1+\epsilon)\varphi} dV \tag{2.41}$$

$$= \int_{U_x} \frac{| f |^2}{| \sum\limits_{i=0}^{N} |s_i(x)|^2 |^{(1-\frac{\epsilon}{2})}} e^{-2(1+\epsilon)\varphi} dV \int_{\sum\limits_{i=0}^{N} |\tau_i|^2=1} \frac{1}{(\sum\limits_{i=0}^{N} \tau_i \frac{s_i}{\sum\limits_{i=0}^{N} |s_i(x)|^2})^{2(1-\frac{\epsilon}{2})}} d\tau$$

$$= \int_{U_x} \frac{| f |^2}{| \sum\limits_{i=0}^{N} |s_i(x)|^2 |^{(1-\frac{\epsilon}{2})}} e^{-2(1+\epsilon)\varphi} dV \int_{\sum\limits_{i=0}^{N} |\tau_i|^2=1} \frac{1}{|\tau_0|^{2(1-\frac{\epsilon}{2})}} d\tau < +\infty$$

For any $f \in \mathscr{I}((1+\epsilon)\varphi)_x$ fixed, by applying the Fubini theorem to (2.41), we obtain

$$\int_{U_x} \frac{| f |^2}{| s |^{2(1-\frac{\epsilon}{2})}} e^{-2(1+\epsilon)\varphi} < +\infty \tag{2.42}$$

for a general element $s \in H^0(X, A)$. Observing that $\mathscr{I}((1+\epsilon)\varphi)$ is finitely generated on X, we can thus choose a general section s such that (2.42) is true for any $f \in \mathscr{I}((1+\epsilon)\varphi)$. (2.40) is proved.

□

The next proposition confirms that our definition of the numerical dimension coincides with Tsuji's definition.

Proposition 2.4.2. *If (L, φ) is a pseudo-effective on a projective variety X of dimension n, then*

$$\nu_{\text{num}}(L, \varphi) = \text{nd}(L, \varphi).$$

Proof. We first prove

$$\nu_{\text{num}}(L, \varphi) \geq \text{nd}(L, \varphi) \tag{2.43}$$

by induction on dimension. If $\text{nd}(L, \varphi) = n$, (2.43) comes from Proposition 2.3.7. Assume that $\text{nd}(L, \varphi) < n$. Let A be a general hypersurface given by a very ample line bundle and let $\{\varphi_k\}$ be a quasi-equisingular approximation of φ. By Lemma 2.4.1, $\varphi_k|_A$ is a quasi-equisingular approximation of $\varphi|_A$. Since A is a general section and $\text{nd}(L, \varphi) < n$, we have

$$\lim_{k\to\infty} \int_A ((\frac{i}{2\pi}\Theta_{\varphi_k}(L))_{\text{ac}})^s \wedge \omega^{n-s-1} > 0$$

where $s = \text{nd}(L, \varphi)$. By Definition 2.2.2, we have

$$\text{nd}(L, \varphi|_A) \geq s = \text{nd}(L, \varphi), \tag{2.44}$$

where $\text{nd}(L, \varphi|_A)$ is the numerical dimension of $(L, \varphi|_A)$ on A. Note moreover that, by the definition of ν_{num},

$$\nu_{\text{num}}(L, \varphi) \geq \nu_{\text{num}}(L, \varphi|_A). \tag{2.45}$$

Thanks to (2.44) and (2.45), we get (2.43) by induction on dimension.

We now prove

$$\nu_{\text{num}}(L, \varphi) \leq \text{nd}(L, \varphi). \tag{2.46}$$

Assume that $\nu_{\mathrm{num}}(L,\varphi)=s$. By Definition 2.1.1, there exists a subvariety V of dimension s such that

$$\overline{\lim_{m\to\infty}}\frac{h^0(V,mL\otimes\mathscr{I}(m\varphi))}{m^s}>0. \tag{2.47}$$

Let $\{\varphi_k\}$ be a quasi-equisingular approximation of φ. To prove (2.46), by Definition 2.3.1, it is sufficient to prove that

$$\lim_{k\to+\infty}(i\Theta_{\varphi_k}(L))^s_{\mathrm{ac}}\wedge[V]>0 \tag{2.48}$$

We prove (2.48) by holomorphic Morse inequality for line bundles equipped with singular metrics (cf. [Bon98]). Let $\pi:\widetilde{X}\to X$ be a desingularization of V in X, and let $\widetilde{V}$ be the strict transform of V. Thanks to (2.47), we have

$$\overline{\lim_{m\to\infty}}\frac{h^0(\widetilde{V},m\pi^*(L)\otimes\mathscr{I}(m\varphi_k\circ\pi))}{m^s}>0\qquad\text{for every } k. \tag{2.49}$$

Let A be an ample line bundle on X and let ω be a Kähler metric such that $c_1(A)=\omega$. By Definition 2.3.1, we can find a positive sequence $\epsilon_k\to 0$ such that $(i\Theta_{\varphi_k}(L))_{\mathrm{ac}}+\epsilon_k\omega>0$. Using [Bon98, Thm 1.1], we have

$$\int_V(i\Theta_{\varphi_k}(L)+\epsilon_k\omega)^s_{\mathrm{ac}}\geq\overline{\lim_{m\to\infty}}\frac{h^0(\widetilde{V},m\pi^*(L)\otimes\mathscr{I}(m\varphi_k\circ\pi))}{m^s}.$$

Combining this with (2.49), we have

$$(i\Theta_{\varphi_k}(L)+\epsilon_k\omega)^s_{\mathrm{ac}}\wedge[V]>0.$$

By letting $k\to+\infty$, (2.48) is proved. □

Remark 2.4.3. *From the proof, it is easy to conclude that if $S_1,S_2,\ldots,S_k$ are divisors of general global sections of a very ample line bundle, then*

$$\mathrm{nd}(L,\varphi|_{S_1\cap S_2\cap\cdots\cap S_k})=\max(\mathrm{nd}(L,\varphi),n-k). \tag{2.50}$$

In fact, if $\mathrm{nd}(L,\varphi)\leq n-k$, by the same argument as above, $\varphi_m|_{S_1\cap S_2\ldots\cap S_k}$ is also a quasi-equisingular approximation of $\varphi|_{S_1\cap S_2\cap\cdots\cap S_k}$. Then (2.50) is proved by a simple calculation.

Before giving a numerical criterion to calculate the numerical dimension. we should mention the following interesting example in [Tsu07, Example 3.6]. The example tells us that we cannot expect an equality of the form :

$$\sup_A\overline{\lim_{m\to\infty}}\frac{\ln h^0(X,\mathscr{O}(A+mL)\otimes\mathscr{I}(m\varphi))}{\ln m}=\mathrm{nd}(L,\varphi), \tag{2.51}$$

where A runs over all the amples bundles on X. In fact, H.Tsuji defined a closed positive $(1,1)$-current T on $\mathbb{P}^1$:

$$T=\sum_{i=1}^{+\infty}\sum_{j=1}^{3^{i-1}}\frac{1}{4^i}P_{i,j}$$

where $\{P_{i,j}\}$ are distinct points on $\mathbb{P}^1$. There exists thus a singular metric φ on $L=\mathscr{O}(1)$ with $\frac{i}{2\pi}\Theta_\varphi(L)=T$. It is easy to construct a quasi-equisingular approximation $\{\varphi_k\}$ of φ such that

$$\frac{i}{2\pi}\Theta_{\varphi_k}(L)=\sum_{i=1}^{k}\sum_{j=1}^{3^{i-1}}\frac{1}{4^i}P_{i,j}+C^\infty.$$

Then $\mathrm{nd}(L,\varphi)=0$.

On the other hand, thanks to the construction of φ, we have

$$\varlimsup_{m\to\infty}\frac{h^0(\mathbb{P}^1,\mathscr{O}(s+m)\otimes\mathscr{I}(m\varphi))}{m}=\varlimsup_{k\to\infty}\frac{h^0(\mathbb{P}^1,\mathscr{O}(s+4^k-1)\otimes\mathscr{I}((4^k-1)\varphi))}{4^k-1}$$

for every $s\geq 1$. By construction,

$$\mathscr{I}((4^k-1)\varphi)_x=\mathscr{O}_x$$

for $x\notin\{P_{i,j}\}_{i\leq k-1}$, and $\mathscr{I}((4^k-1)\varphi)$ has multiplicity $\lfloor\frac{4^k-1}{4^i}\rfloor=4^{k-i}-1$ on 3^{i-1} points $\{P_{i,1},...,P_{i,3^{i-1}}\}$. Therefore

$$h^0(\mathbb{P}^1,\mathscr{O}(s+4^k-1)\otimes\mathscr{I}((4^k-1)\varphi))=s+4^k-\sum_{i=1}^{k-1}3^{i-1}(4^{k-i}-1)$$

$$=\frac{9}{2}3^{k-1}+s-\frac{1}{2}.$$

Then

$$\sup_A\varlimsup_{m\to\infty}\frac{\ln h^0(\mathbb{P}^1,\mathscr{O}(A+m)\otimes\mathscr{I}(m\varphi))}{\ln m}=\frac{\ln 3}{\ln 4}.$$

Therefore

$$\operatorname{nd}(L,\varphi)\neq\sup_A\varlimsup_{m\to\infty}\frac{\ln h^0(\mathbb{P}^1,\mathscr{O}(A+m)\otimes\mathscr{I}(m\varphi))}{\ln m}.$$

In view of the above example, we propose the following modified formula to calculate the numerical dimension.

Proposition 2.4.4. *Let (L,φ) be a pseudo-effective line bundle on a projectvie variety X, and let A be a very ample line bundle. Then* $\operatorname{nd}(L,\varphi)=d$ *if and only if*

$$\lim_{\epsilon\to 0}\frac{\ln(\varlimsup_{m\to\infty}\frac{h^0(X,m\epsilon A+mL\otimes\mathscr{I}(m\varphi))}{m^n})}{\ln\epsilon}=n-d.$$

Proof. First of all, the inclusion

$$H^0(X,m\epsilon A+mL\otimes\mathscr{I}(m\varphi))\supset H^0(X,m\epsilon A+mL\otimes\mathscr{I}_+(m\varphi))$$

$$\supset H^0(X,m\epsilon A+mL\otimes\mathscr{I}((m+1)\varphi)),$$

implies that $h^0(X,m\epsilon A+mL\otimes\mathscr{I}_+(m\varphi))$ has the same asymptotic comportment as $h^0(X,m\epsilon A+mL\otimes\mathscr{I}(m\varphi))$. Since we have constructed the exact sequence for $\mathscr{I}_+$ in Lemma 2.4.1, we prefer to calculate $h^0(X,m\epsilon A+mL\otimes\mathscr{I}_+(m\varphi))$ instead of $h^0(X,m\epsilon A+mL\otimes\mathscr{I}(m\varphi))$.

If $\operatorname{nd}(L,\varphi)=n$, the proposition comes directly from Proposition 2.4.2. Assume that $\operatorname{nd}(L,\varphi)=d<n$. Let $\{Y_i\}_{i=1}^n$ be the zero divisors of n very general global sections $H^0(X,A)$. By the remark of Proposition 2.4.2, there exists a uniform constant $C>0$ such that for all m,ϵ,

$$h^0(Y_1\cap\cdots\cap Y_{n-d},m\epsilon A+mL\otimes\mathscr{I}_+(m\varphi))=C(\epsilon,m)\cdot m^d. \tag{2.52}$$

and $C(\epsilon,m)\geq C$. Our aim is to prove by induction on s that

$$\frac{1}{m^{n-s}}h^0(Y_1\cap\cdots\cap Y_s,m\epsilon A+mL\otimes\mathscr{I}_+(m\varphi)) \tag{2.53}$$

$$=C(\epsilon,m)\epsilon^{n-s-d}\frac{1}{(n-d-s)!}+O(\epsilon^{n-s-d+1})+O(\frac{1}{m})$$

for $0\leq s\leq n-d$. If $s=n-d$, (2.53) comes from (2.52). Assume that (2.53) is true for $s_0\leq s\leq n-d$. We now prove (2.53) for $s=s_0-1$.

Let Y be the intersection of zero divisors of $s_0 - 1$ general global sections of $H^0(X, A)$, and let

$$e_1^{0,q}(\epsilon, m) = \binom{m\epsilon}{q} h^0(Y \cap Y_1 \cap \cdots \cap Y_q, m\epsilon A \otimes mL \otimes \mathscr{I}_+(m\varphi)). \tag{2.54}$$

We claim that :

$$\frac{1}{m^{n-s_0+1}} h^0(Y, m\epsilon A + mL \otimes \mathscr{I}_+(m\varphi)) \tag{2.55}$$

$$= -\frac{1}{m^{n-s_0+1}} (\sum_{q\geq 1} (-1)^q e_1^{0,q}(\epsilon, m)) + O(\frac{1}{m}).$$

We postpone the proof of (2.55) in Lemma 2.4.5 and conclude first the proof of (2.53). If $q > n - d - s_0 + 1$, we have by definition,

$$\lim_{m\to\infty} \frac{1}{m^{n-s_0+1}} e_1^{0,q}(\epsilon) = O(\epsilon^q) \leq O(\epsilon^{n-d-s_0+2}). \tag{2.56}$$

Then (2.55) and the induction hypothesis of (2.53) imply that

$$\frac{1}{m^{n-s_0+1}} h^0(Y, m\epsilon A + mL \otimes \mathscr{I}_+(m\varphi))$$

$$= -(\sum_{q=1}^{n-d-s_0+1} (-1)^q \frac{\epsilon^{n-d-s_0+1} C(\epsilon, m)}{q!(n-q-s_0+1-d)!}) + O(\epsilon^{n-d-s_0+2}) + O(\frac{1}{m})$$

$$= -(\sum_{q=1}^{n-d-s_0+1} (-1)^q \frac{\epsilon^{n-d-s_0+1} C(\epsilon, m)}{(n-s_0+1-d)!} \binom{n-s_0+1-d}{q}) + O(\epsilon^{n-d-s_0+2}) + O(\frac{1}{m})$$

$$= -\frac{\epsilon^{n-d-s_0+1} C(\epsilon, m)}{(n-s_0+1-d)!} (\sum_{q=1}^{n-d-s_0+1} (-1)^q \binom{n-s_0+1-d}{q}) + O(\epsilon^{n-d-s_0+2}) + O(\frac{1}{m})$$

$$= C(\epsilon, m) \epsilon^{n-d-s_0+1} \frac{1}{(n-d-s_0+1)!} + O(\epsilon^{n-d-s_0+2}) + O(\frac{1}{m}).$$

Therefore (2.53) is proved for $s = s_0 - 1$.

In particular, taking $s = 0$ in (2.53), we have

$$\lim_{\epsilon\to 0} \overline{\lim_{m\to\infty}} \frac{1}{m^n \epsilon^{n-d}} h^0(X, m\epsilon A + mL \otimes \mathscr{I}_+(m\varphi)) > 0.$$

The proposition is proved. □

We now prove formula (2.55), as promised in Proposition 2.4.4.

Lemma 2.4.5. *In the situation of Proposition 2.4.4, we have*

$$\frac{1}{m^{n-s_0+1}} h^0(Y, m\epsilon A + mL \otimes \mathscr{I}_+(m\varphi)) = \frac{1}{m^{n-s_0+1}} e_1^{0,0}(\epsilon, m)$$

$$= -\frac{1}{m^{n-s_0+1}} (\sum_{q\geq 1} (-1)^q e_1^{0,q}(\epsilon, m)) + O(\frac{1}{m}).$$

Proof. Thanks to $(iii), (iv)$ of Lemma 2.4.1 and [Kür06, Section 4], $\mathscr{O}_Y(mL\otimes\mathscr{I}_+(m\varphi))$ is resolved by a complex of sheaves

$$(*)\qquad \mathscr{O}_Y(m\epsilon A+mL\otimes\mathscr{I}_+(m\varphi))\to\oplus_{1\leq i\leq m\epsilon}\mathscr{O}_{Y\cap Y_i}(m\epsilon A+mL\otimes\mathscr{I}_+(m\varphi))$$
$$\to\oplus_{1\leq i_1<i_2\leq m\epsilon}\mathscr{O}_{Y\cap Y_{i_1}\cap Y_{i_2}}(m\epsilon A+mL\otimes\mathscr{I}_+(m\varphi))$$
$$\to\cdots$$

Then

$$H^k(Y,mL\otimes\mathscr{I}_+(m\varphi))=\mathbb{H}^k(\epsilon,m) \tag{2.57}$$

where $\mathbb{H}^k(\epsilon,m)$ represents the hypercohomology of $(*)$.

We now calculate the asymptotic behaviour on the both sides of (2.57). The Nadel vanishing theorem implies that

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}h^k(Y,mL\otimes\mathscr{I}_+(m\varphi))=0\qquad\text{for every } k\geq 1. \tag{2.58}$$

Moreover, since we assume that $\mathrm{nd}(L,h)=d<\dim Y$, we have

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}h^0(Y,mL\otimes\mathscr{I}_+(m\varphi))=0. \tag{2.59}$$

By calculating the asymptotic cohomology on both sides of (2.57), equations (2.58) and (2.59) imply in particular that

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}\sum_k(-1)^kh^k(\epsilon,m)=0, \tag{2.60}$$

where $h^k(\epsilon,m)$ denotes the dimension of $\mathbb{H}^k(\epsilon,m)$.

We now prove the lemma by using (2.60). By the Nadel vanishing theorem, we have

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}\binom{m\epsilon}{q}h^p(Y\cap Y_1\cap\cdots\cap Y_q,m\epsilon A\otimes mL\otimes\mathscr{I}_+(m\varphi))=0$$

for every $p\geq 1$. If $p=0$, we have

$$\binom{m\epsilon}{q}h^0(Y\cap Y_1\cap\cdots\cap Y_q,m\epsilon A\otimes mL\otimes\mathscr{I}_+(m\varphi))=e_1^{0,q}(\epsilon,m)$$

by (2.54). Then (2.60) implies that

$$\lim_{m\to\infty}\frac{1}{m^{n-s_0+1}}(\sum_{q\geq 0}(-1)^qe_1^{0,q}(\epsilon,m))=0\qquad\text{for every } \epsilon>0,$$

which is equivalent to say that

$$\frac{1}{m^{n-s_0+1}}h^0(Y,m\epsilon A+mL\otimes\mathscr{I}_+(m\varphi))=\frac{1}{m^{n-s_0+1}}e_1^{0,0}(\epsilon,m)$$
$$=-\frac{1}{m^{n-s_0+1}}(\sum_{q\geq 1}(-1)^qe_1^{0,q}(\epsilon,m))+O(\frac{1}{m}).$$

The lemma is proved. □

Remark 2.4.6. *On a compact Kähler manifold. S.Boucksom defined in [Bou02] a concept of numerical dimension* $\mathrm{nd}(L)$ *for a pseudo-effective line bundle* L *without any specified metric. Let* $\varphi_{\min}$ *be a positive metric of* L *with minimal singularities. Proposition 2.4.4 implies in particular that*

$$\mathrm{nd}(L) \geq \mathrm{nd}(L, \varphi_{\min}). \tag{2.61}$$

[DPS94, Example 1.7] tells us that we cannot hope for an equality

$$\mathrm{nd}(L) = \mathrm{nd}(L, \varphi_{\min}).$$

In that example, the line bundle L *is nef and* $\mathrm{nd}(L) = 1$*. On the other hand, [DPS94, Example 1.7] proved that there exists a unique singular metric* h *on* L *such that the curvature form* $\frac{i}{2\pi}\Theta_h(L)$ *is positive. Moreover,*

$$\frac{i}{2\pi}\Theta_h(L) = [C]$$

for a curve C *on* X*. Therefore* $\varphi_{\min} = h$ *and* $\mathrm{nd}(L, \varphi_{\min}) = 0$*. Therefore*

$$\mathrm{nd}(L) > \mathrm{nd}(L, \varphi_{\min})$$

in this example.

2.5 A Kawamata-Viehweg-Nadel Vanishing Theorem

The classical Nadel vanishing theorem states that

Theorem 2.5.1 (([Nad89], [Dem93]))**.** *Let* (X, ω) *be a projective manifold and let* (L, φ) *be a pseudo-effective line bundle on* X*. If* $i\Theta_\varphi(L) \geq c \cdot \omega$ *for some constant* $c > 0$*, then*

$$H^q(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}(\varphi)) = 0 \qquad \text{for every } q \geq 1.$$

One of the limitations of Theorem 2.5.1 is that the curvature $i\Theta_\varphi(L)$ should be strictly positive. Various attempts have been made to overcome this limitation. For example, the following more classical Kawamata-Viehweg vanishing theorem has found many applications in complex algebraic geometry (cf. [Dem12, Chapter 6.D])

Theorem 2.5.2 (([Dem12]))**.** *Let* X *be a projective manifold and let* F *be a line bundle over* X *such that some positive multiple* mF *can be written* $mF = L + D$ *where* L *is a nef line bundle and* D *an effective divisor. Then*

$$H^q(X, \mathscr{O}(K_X + F) \otimes \mathscr{I}(m^{-1}D)) = 0 \qquad \text{for every } q > n - \mathrm{nd}(L).$$

The classical proof of Theorem 2.5.2 uses an ample line bundle on X and a hyperplane section argument to perform an induction on dimension. Therefore the hypothesis that X is projective is crucial in Theorem 2.5.2. However, we believe that it would be useful to find a Kawamata-Viehweg type vanishing theorem for arbitrary Kähler manifolds. In this direction, [DP03b] proved

Theorem 2.5.3 (([DP03b]))**.** *Let* (L, h) *be a line bundle over a compact Kähler* n*-fold* X*. Assume that* L *is nef. Then the natural morphism*

$$H^q(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(h)) \to H^q(X, \mathscr{O}(K_X + L))$$

vanishes for $q \geq n - \mathrm{nd}(L) + 1$*.*

Following several ideas and techniques of [DP03b], we will prove in this section our Main Theorem 2.1.3, i.e., given a pseudo-effective line bundle (L, φ) over a compact Kähler manifold X of dimension n, one has

$$H^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi)) = 0 \qquad \text{for } p \geq n - \mathrm{nd}(L, \varphi) + 1.$$

By (2.61), our vanishing theorem can be view as a generalization of Theorem 2.5.3. The main advantage of this version of the Kawamata-Viehweg-Nadel vanishing theorem is that we do not need any strict positivity of the line bundle. But as a compensation, we have to use the multiplier ideal sheaf $\mathscr{I}_+(\varphi)$ instead of $\mathscr{I}(\varphi)$. When X is projective, the proof of our vanishing theorem is much easier. We first give a quick proof of Theorem 2.1.3 in the projective case by the tools developed in the previous sections. To begin with, we prove Theorem 2.1.3 in the case $\mathrm{nd}(L, \varphi) = \dim X$.

Proposition 2.5.4. *Let X be a smooth projective variety of dimension n. Let (L, φ) be a pseudo-effective line bundle over X and $\mathrm{nd}(L, \varphi) = n$. Then*

$$H^i(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi)) = 0 \qquad \textit{for every } i > 0.$$

Proof. Recall that we first fix a smooth metric h_0 on L. The quasi-psh fonction φ gives a metric $h_0 e^{-\varphi}$ on L. (L, φ) is pseudo-effective means that

$$\frac{i}{2\pi}\Theta_\varphi(L) = \frac{i}{2\pi}\Theta_{h_0}(L) + dd^c\varphi \geq 0.$$

Since $\frac{i}{2\pi}\Theta_\varphi(L)$ is not strictly positive, we cannot directly apply Theorem 2.5.1. The idea is to add a portion of the metric $\widetilde{\varphi}$ constructed in Proposition 2.3.6 to make the curvature form for the new metric becomes strictly positive. We will see that this operation preserves the multiplier ideal sheaves $\mathscr{I}_+(\varphi)$.

First of all, by the definition of $\mathscr{I}_+$ (cf. Section 2), there exists a $\delta > 0$ such that

$$\mathscr{I}_+(\varphi) = \mathscr{I}((1+\delta)\varphi). \tag{2.62}$$

Let $\widetilde{\varphi}$ be the psh function constructed in Propositon 2.3.6. Set $\varphi_1 := (1 + \sigma(\epsilon) - \epsilon)\varphi + \epsilon\widetilde{\varphi}$, where $0 < \epsilon < 1$ and $0 < \sigma(\epsilon) \ll \epsilon$. Since $dd^c\varphi \geq -c\omega$ for some constant c[4], the condition $\sigma(\epsilon) \ll \epsilon$ implies that

$$\frac{i}{2\pi}\Theta_{\varphi_1}(L) = (1 + \sigma(\epsilon) - \epsilon)\frac{i}{2\pi}\Theta_\varphi(L) + \epsilon\frac{i}{2\pi}\Theta_{\widetilde{\varphi}}(L) + \sigma(\epsilon)dd^c\varphi > 0.$$

Applying the standard Nadel vanishing theorem (cf. Theorem 2.5.1) to $(X, L, \mathscr{I}(\varphi_1))$, we get

$$H^i(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}(\varphi_1)) = 0 \qquad \text{for } i > 0. \tag{2.63}$$

On the other hand, it it not hard to prove that

$$\mathscr{I}_+(\varphi) = \mathscr{I}(\varphi_1) \qquad \text{for } \epsilon \ll 1. \tag{2.64}$$

We postpone the proof of (2.64) in Lemma 2.5.5 and conclude first the proof of Proposition 2.5.4. By taking ϵ small enough, (2.63) and (2.64) imply the proposition. □

Lemma 2.5.5. *In the situation of Proposition 2.5.4, if ϵ is small enough, then*

$$\mathscr{I}(\varphi_1) = \mathscr{I}_+(\varphi). \tag{2.65}$$

Proof. By (2.22) of Proposition 2.3.6, we have

$$(1 + \sigma(\epsilon))\varphi_m = (1 + \sigma(\epsilon) - \epsilon)\varphi_m + \epsilon\varphi_m \preccurlyeq (1 + \sigma(\epsilon) - \epsilon)\varphi + \epsilon\widetilde{\varphi}.$$

Therefore

$$\mathscr{I}(\varphi_1) \subset \mathscr{I}((1 + \sigma(\epsilon))\varphi_m). \tag{2.66}$$

4. In our context, since φ is a function on X, we have $\frac{i}{2\pi}\Theta_\varphi(L) = \frac{i}{2\pi}\Theta_{h_0}(L) + dd^c\varphi \geq 0$. Therefore $dd^c\varphi \geq -c\omega$.

Note that, by Lemma 2.3.1, we have

$$\mathscr{I}((1+\sigma(\epsilon))\varphi_m) \subset \mathscr{I}_+(\varphi) \tag{2.67}$$

for m large enough with respect to $\sigma(\epsilon)$. Combining (2.66) with (2.67), we have

$$\mathscr{I}(\varphi_1) \subset \mathscr{I}_+(\varphi).$$

As for the other side inclusion of (2.65), we need to prove that if $f \in \mathscr{I}_+(\varphi)_x$, then

$$f \in \mathscr{I}(\varphi_1)_x.$$

By (2.62), we have

$$\int_{U_x} \mid f \mid^2 e^{-2(1+\delta)\varphi} < +\infty. \tag{2.68}$$

Since $\widetilde{\varphi}$ is a quasi-psh function, by taking ϵ small enough, we have

$$\int_{U_x} e^{-2\frac{\epsilon}{\delta}\widetilde{\varphi}} < +\infty. \tag{2.69}$$

Therefore (2.68) and (2.69) imply that

$$\int_{U_x} \mid f \mid^2 e^{-2(1+\sigma(\epsilon)-\epsilon)\varphi - 2\epsilon\widetilde{\varphi}} \leq \int_{U_x} \mid f \mid^2 e^{-2(1+\delta)\varphi} \int_{U_x} e^{-2\frac{\epsilon}{\delta}\widetilde{\varphi}} < +\infty$$

by Hölder's inequality. Since $\varphi_1 = (1+\sigma(\epsilon)-\epsilon)\varphi + \epsilon\widetilde{\varphi}$ by construction, we have $f \in \mathscr{I}(\varphi_1)$. The lemma is proved. □

Using Proposition 2.5.4, we can prove the following Kawamata-Viehweg-Nadel vanishing theorem by induction on dimension.

Proposition 2.5.6. *Let (L,φ) be a pseudo-effective line bundle on a projective variety X of dimension n. Then*

$$H^p(X, \mathscr{O}(K_X+L)\otimes \mathscr{I}_+(\varphi)) = 0 \qquad \textit{for } p \geq n - \mathrm{nd}(L,\varphi)+1.$$

Proof. If $\mathrm{nd}(L,\varphi) = n$, the proposition has been proved in Proposition 2.5.4. Assume that $\mathrm{nd}(L,\varphi) < n$. Let A be an ample line bundle that is large enough with respect to L, and let S be the zero divisor of a very general global section of $H^0(X,A)$. Let $\epsilon > 0$ be small enough such that the condition (iv) of Lemma 2.4.1 is satisfied (by Lemma 2.4.1, ϵ is independent of A!) . By Lemma 2.4.1, we have an exact sequence

$$0 \to \mathscr{I}_+(\varphi)\otimes\mathscr{O}(-S) \to \mathscr{I}_+(\varphi) \to \mathscr{I}_+(S,\varphi_S) \to 0. \tag{2.70}$$

Therefore we get an exact sequence

$$H^q(S,\mathscr{O}(K_S+L)\otimes\mathscr{I}_+(\varphi|_S)) \to H^{q+1}(X,\mathscr{O}(K_X+L)\otimes\mathscr{I}_+(\varphi)) \to H^{q+1}(X,\mathscr{O}(K_X+A+L)\otimes\mathscr{I}_+(\varphi)),$$

for every $q \geq 0$. Since A is ample enough with respect ot L, we have

$$H^{q+1}(X,\mathscr{O}(K_X+A+L)\otimes\mathscr{I}_+(\varphi)) = 0$$

by the Nadel vanishing theorem. Thus the above exact sequence implies that

$$H^q(S,\mathscr{O}(K_S+L)\otimes\mathscr{I}_+(\varphi|_S)) \to H^{q+1}(X,\mathscr{O}(K_X+L)\otimes\mathscr{I}_+(\varphi))$$

is surjective for every q. The proposition is proved by induction on dimension.

□

The main goal of this section is to prove Theorem 2.1.3 for arbitrary Kähler manifolds. To achieve this, we use the methods developed in [DP03b], [Eno93] and [Mou95]. To clarify the idea of the proof, we first consider the following easy case. Assume that (X,ω) is a compact Kähler manifold and (L,φ) is a pseudo-effective line bundle with analytic singularities. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $i\Theta_\varphi(L)$ with respect to ω. Let f be a smooth (n,p)-form representing an element in $H^p(X, K_X \otimes L \otimes \mathscr{I}(\varphi))$ for some $p \geq n - \mathrm{nd}(L,\varphi) + 1$. Then $\int_X |f|^2 e^{-2\varphi}\omega^n < +\infty$. By using a L^2 estimate (cf. [DP03b] or Proposition 2.6.1 in the appendix), f can be written as

$$f = \overline{\partial} u_k + v_k \tag{2.71}$$

with the following estimate

$$\int_X |u_k|^2 e^{-2\varphi} + \frac{1}{2p\epsilon_k}\int_X |v_k|^2 e^{-2\varphi} \leq \int_X \frac{1}{2p\epsilon_k + \lambda_1 + \lambda_2 + \cdots + \lambda_p}|f|^2 e^{-2\varphi}, \tag{2.72}$$

where $\{\epsilon_k\}$ is a positive sequence tending to 0. Since $p \geq n - \mathrm{nd}(L,\varphi) + 1$, we have

$$\int_X (\sum_{i\geq p} \lambda_i(z))\omega^n > 0. \tag{2.73}$$

If $\lambda_p(z)$ is generically strictly positive, (2.72) implies that

$$\lim_{k\to+\infty}\int_X |v_k|^2 e^{-2\varphi} = 0.$$

By some standard results in functional analysis (cf. Lemma 2.5.8), we obtain

$$f = 0 \in H^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}(\varphi)).$$

The situation becomes more complicated when $\lambda_p(z)$ is not necessary generically strictly positive. In this case, thanks to the condition (2.73) and the fact that φ has analytic singularities, we can use Monge-Ampère equations to construct a sequence of new metrics $\widehat{\varphi_k}$ on L, such that $\int_X |f|^2 e^{-2\widehat{\varphi_k}}\omega^n$ can be controled by $\int_X |f|^2 e^{-2\varphi}\omega^n$, and more importantly, the place where the p-th eigenvalue of $i\Theta_{\widehat{\varphi_k}}(L)$ is strictly positive tends to cover the whole X. Letting $k \to +\infty$, we can thus prove that

$$f = 0 \in H^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}(\varphi)).$$

In the general case, since φ does not necessarily possess analytic singularities, we are in trouble when using L^2 estimates. Therefore we replace φ by a quasi-equisingular approximation $\{\varphi_k\}$ and get estimates similar to (2.71) and (2.72) with φ replaced by φ_k. We can use a Monge-Ampère equation to construct other metrics $\widehat{\varphi}_k$ for which we can control the eigenvalues. Therefore we can use L^2 estimates for every $\widehat{\varphi}_k$. By a delicate analysis, we then prove the theorem. Such ideas are already used in [DP03b], [Eno93] or [Mou95]. We will construct the key metric $\widehat{\varphi}_k$ in Lemma 2.5.10 and prove some important properties of $\widehat{\varphi}_k$ in Lemma 2.5.11 and Lemma 2.5.12. We prove finally the vanishing theorem in Theorem 2.5.13.

To begin with, we first prove that $\mathscr{I}_+$ has analytic singularities. More precisely,

Lemma 2.5.7. *Let (L,φ) be a pseudo-effective line bundle over a compact Kähler manifold X. Then there exists a quasi-equisingular approximation $\{\varphi_k\}$ of φ such that*

$$\mathscr{I}((1+\frac{2}{k})\varphi_k) = \mathscr{I}_+(\varphi) \qquad \text{for } k \gg 1. \tag{2.74}$$

Proof. By [DPS01, Thm 2.2.1], there exists a quasi-equisingular approximation $\{\varphi_k\}$ of φ. The technique of comparing integral discussed in [DPS01] implies that we can choose a subsequence $\{\varphi_{f(k)}\} \subset \{\varphi_k\}$ such that

$$\mathscr{I}((1+\frac{2}{k})\varphi_{f(k)}) \subset \mathscr{I}_+(\varphi). \tag{2.75}$$

In fact, if X is projective, we can take $s = 1 + \epsilon$ and $f(k) \gg k$ in Lemma 2.3.1. By Lemma 2.3.1, we get (2.75). If X is an arbitrary compact Kähler manifold, we can get the inclusion (2.75) on any Stein open set of X. Using standard glueing techniques, we also obtain the global inclusion (2.75) (see [DPS01, Thm 2.2.1] for details).

For the opposite inclusion, we observe that $\varphi_{f(k)}$ is less singular than φ, and the definition of $\mathscr{I}_+(\varphi)$ implies that

$$\mathscr{I}((1+\frac{2}{k})\varphi_{f(k)}) \supset \mathscr{I}_+(\varphi) \qquad \text{for } k \gg 1.$$

The lemma is proved.

□

The following lemma will be important in the proof of our Kawamata-Viehweg-Nadel vanishing theorem. The main substance of the lemma is that to prove the convergence in higher degree cohomology with multiplier ideal sheaves, we just need to check the convergence for some smooth metric. Although this technique is well known (cf. for example [DPS01, Part 2.4.2]), we will give the proof for the convenience of reader.

We first fix some notations. Let (L, φ) be a pseudo-effective line bundle over a compact Kähler manifold X and let $\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$ be a Stein covering of X. Set $U_{\alpha_0\alpha_1\cdots\alpha_q} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$. Let $\check{C}^q(\mathscr{U}, K_X \otimes L \otimes \mathscr{I}_+(\varphi))$ be the Čech q-cochain associated to $K_X \otimes L \otimes \mathscr{I}_+(\varphi)$. For an element $c \in \check{C}^q(\mathscr{U}, K_X \otimes L \otimes \mathscr{I}_+(\varphi))$, we denote its component on $U_{\alpha_0\alpha_1\cdots\alpha_q}$ by $c_{\alpha_0\alpha_1\cdots\alpha_q}$. Let

$$\delta_p : \check{C}^{p-1}(\mathscr{U}, \mathscr{I}_+(\varphi)) \to \check{C}^p(\mathscr{U}, \mathscr{I}_+(\varphi)) \tag{2.76}$$

be the Čech operator, and $\check{Z}^p(\mathscr{U}, \mathscr{I}_+(\varphi)) = \operatorname{Ker} \delta_{p+1}$.

Lemma 2.5.8. *Let L be a line bundle over a compact Kähler manifold X and let φ be a singular metric on L. Let $\{U_\alpha\}_{\alpha\in I}$ be a Stein covering of X. Let u be an element in $\check{H}^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi))$. If there exists a sequence $\{v_k\}_{k=1}^{\infty} \subset \check{C}^p(\mathscr{U}, K_X \otimes L \otimes \mathscr{I}_+(\varphi))$ in the same cohomology class as u satisfying the L^2 convergence condition :*

$$\lim_{k\to\infty} \int_{U_{\alpha_0\ldots\alpha_p}} |v_{k,\alpha_0\ldots\alpha_p}|^2 \to 0, \tag{2.77}$$

where the L^2 norm $|v|^2$ in (2.77) is taken for some fixed smooth metric on L, then $u = 0$ in $\check{H}^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi))$.

Proof. On the p-cochain space $\check{C}^p(\mathscr{U}, \mathscr{I}_+(\varphi))$, we first define a family of natural semi-norms : for $f \in \check{C}^p(\mathscr{U}, \mathscr{I}_+(\varphi))$, we define a family of semi-norms :

$$\sum_{\alpha_0\ldots\alpha_p} \int_{V_{\alpha_0\cdots\alpha_p}} |f|^2 \omega^n \qquad \text{for any open set } V_{\alpha_0\ldots\alpha_p} \Subset U_{\alpha_0\ldots\alpha_p}. \tag{2.78}$$

Claim : $\check{C}^p(\mathscr{U}, \mathscr{I}_+(\varphi))$ is a Fréchet space with respect to the family of semi-norms (2.78).

Proof of the claim : we need to prove that if $f_i \in \mathscr{I}_+(\varphi)$ and $f_i \to f_0$ with respect to the semi-norms (2.78), then $f_0 \in \mathscr{I}_+(\varphi)$. First of all, by (2.78), f_0 is holomorphic. By Lemma 2.5.7, we can choose a quasi-psh function ψ with analytic singularities such that

$$\mathscr{I}(\psi) = \mathscr{I}_+(\varphi).$$

Let $\pi : \widehat{X} \to X$ be a log resolution of ψ. Then the current $E = \lfloor dd^c(\psi \circ \pi) \rfloor$ has normal crossing singularities. Since $f_i \in \mathscr{I}_+(\varphi) = \mathscr{I}(\psi)$, we have

$$(f_i \circ \pi) \cdot J \in \mathscr{O}(-E), \tag{2.79}$$

where J is the Jacobian of π. Since $f_i \circ \pi \rightharpoonup f_0 \circ \pi$ in the sense of weak convergence and E has normal crossing singularities, (2.79) implies that

$$(f_0 \circ \pi) \cdot J \in \mathscr{O}(-E).$$

Therefore $f_0 \in \mathscr{I}_+(\varphi)$. The claim is proved.

As a consequence of the claim, the Čech operator (2.76) is continuous and its kernel $\check{Z}^{p-1}(\mathscr{U}, \mathscr{I}_+(\varphi))$ is also a Fréchet space. Therefore we have a continuous boundary morphism between Fréchet spaces :

$$\delta_p : \check{C}^{p-1}(\mathscr{U}, \mathscr{I}_+(\varphi)) \to \check{Z}^p(\mathscr{U}, \mathscr{I}_+(\varphi)). \tag{2.80}$$

Since the cokernel of δ_p in (2.80) is $\check{H}^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi))$ which is of finite dimension, by the open mapping theorem in functional analysis, the image of δ_p in (2.80) is closed. Therefore the quotient morphism

$$\mathrm{pr} : \check{Z}^p(\mathscr{U}, \mathscr{I}_+(\varphi)) \to \frac{\check{Z}^p(\mathscr{U}, \mathscr{I}_+(\varphi))}{\mathrm{Im}(\delta_p)} = \check{H}^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi)) \tag{2.81}$$

is continuous. Thanks to the claim, the condition (2.77) implies that $\{v_k\}_{k=1}^{\infty}$ tends to 0 in the Fréchet space $\check{Z}^p(\mathscr{U}, \mathscr{I}_+(\varphi))$. By the continuity of (2.81), we have

$$\lim_{k\to+\infty} \mathrm{pr}(v_k) = 0 \in \check{H}^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi)). \tag{2.82}$$

Since by construction, $\mathrm{pr}(v_k)$ are in the same class as $[u]$, we conclude by (2.82) that $u = 0$ in $\check{H}^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi))$.

□

Remark 2.5.9. *Recently, Matsumura proved in [Mat13] that the above lemma is also true for the space* $\check{H}^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}(\varphi))$.

We begin to construct the new singular metrics mentioned in the paragraphs before Lemma 2.5.7.

Lemma 2.5.10. *Let (L, φ) be a pseudo-effective line bundle over a compact kähler manifold (X, ω) of dimension n and let $p \geq n - \mathrm{nd}(L, \varphi) + 1$. Then there exists a sequence of metrics $\{\widehat{\varphi}_k\}_{k=1}^{\infty}$ with analytic singularities on L satisfying the following properties :*

(i) : $\mathscr{I}(\widehat{\varphi}_k) = \mathscr{I}_+(\varphi)$ *for all k.*

(ii) : *Let $\lambda_{1,k} \leq \lambda_{2,k} \leq \cdots \leq \lambda_{n,k}$ be the eigenvalues of $\frac{i}{2\pi}\Theta_{\widehat{\varphi}_k}(L)$ with respect to the base metric ω. Then there exist two sequences $\tau_k \to 0, \epsilon_k \to 0$ such that*

$$\epsilon_k \gg \tau_k + \frac{1}{k} \qquad \textit{and} \qquad \lambda_{1,k}(x) \geq -\epsilon_k - \frac{C}{k} - \tau_k$$

for all $x \in X$ and k, where C is a constant independent of k.

(iii) : *We can choose $\beta > 0$ and $0 < \alpha < 1$ independent of k such that for every k, there exists an open subset U_k of X satisfying*

$$\mathrm{vol}(U_k) \leq \epsilon_k^{\beta} \qquad \textit{and} \qquad \lambda_p + 2\epsilon_k \geq \epsilon_k^{\alpha} \quad \textit{on } X \setminus U_k.$$

Proof. Recall that we first fix a smooth metric h_0 on L. Taking φ as a weight, we just mean that the hermitian metric on L is $h_0 \cdot e^{-\varphi}$.

By definition, there exists $s_1 > 0$ such that

$$\mathscr{I}_+(\varphi) = \mathscr{I}((1 + s_1)\varphi). \tag{2.83}$$

Let $\{\varphi_k\}$ be the quasi-equisingular approximation of φ in Lemma 2.5.7. Then there is a positive sequence $\tau_k \to 0$ such that

$$\frac{i}{2\pi}\Theta_{\varphi_k}(L) \geq -\tau_k\omega \qquad \text{and} \qquad \mathscr{I}((1 + \frac{2}{k})\varphi_k) = \mathscr{I}_+(\varphi) \tag{2.84}$$

for every k. We can choose a positive sequence $\epsilon_k \to 0$ such that $\epsilon_k \gg \tau_k + \frac{1}{k}$.

Fix a positive sequence $\{\delta_k\}$ tending to 0. We begin to construct new metrics by solving a Monge-Ampère equation. Let $\pi : X_k \to X$ be a log resolution of φ_k. Then $dd^c(\varphi_k \circ \pi)$ is of the form $[E_k] + C^\infty$ where $[E_k]$ is a normal crossing $\mathbb{Q}$-divisor. Let $Z_k = \pi_*(E_k)$. By [Bou02], there exists a smooth metric h_k on $[E_k]$, such that for all $\delta > 0$ small enough,

$$\pi^*(\omega) + \delta \frac{i}{2\pi}\Theta_{h_k}(-E_k)$$

is a Kähler form on X_k. Then we can solve a Monge-Ampère equation on X_k :

$$((\frac{i}{2\pi}\pi^*\Theta_{\varphi_k}(L))_{\rm ac} + \epsilon_k\pi^*\omega + \delta_k\frac{i}{2\pi}\Theta_{h_k}(-E_k) + dd^c\psi_{k,\epsilon,\delta_k})^n \tag{2.85}$$

$$= C(k,\delta,\epsilon)\cdot\epsilon_k^{n-d}(\omega + \delta_k\frac{i}{2\pi}\Theta_{h_k}(-E_k))^n$$

with the normalization condition

$$\sup_{z\in X_k}(\varphi_k \circ \pi + \psi_{k,\epsilon,\delta_k} + \delta_k \ln |E_k|_{h_k})(z) = 0 \tag{2.86}$$

where $d = \mathrm{nd}(L,\varphi)$. Thanks to the definition of numerical dimension, there exists a uniform constant $C > 0$ such that $C(k,\delta,\epsilon) \geq C$. By observing moreover that

$$i\partial\overline{\partial}\ln |E_k|_{h_k} = [E_k] + \frac{i}{2\pi}\Theta_{h_k}(-E_k),$$

(2.85) implies that

$$\frac{i}{2\pi}\Theta_{\varphi_k+\psi_{k,\epsilon,\delta_k}+\delta_k\ln|E_k|_{h_k}}(\pi^*L) \geq -\epsilon_k\omega. \tag{2.87}$$

Set

$$\widehat{\varphi}_k := (1+\frac{2}{k}-s)\varphi_k\circ\pi + s(\varphi_k\circ\pi + \psi_{k,\epsilon,\delta} + \delta\ln|E_k|_{h_k}), \tag{2.88}$$

where $0 < s \ll s_1$ [5] will be made precise in Lemma 2.5.11. Now we have a new metric $\widehat{\varphi}_k$ on (X_k, π^*L) (i.e. $h_0 e^{-\widehat{\varphi}_k}$ as the actual hermitian metric on π^*L !). We prove that $\widehat{\varphi}_k$ induces a natural metric on (X, L). In fact, by (2.88), we have

$$\frac{i}{2\pi}\Theta_{\widehat{\varphi}_k}(\pi^*L) = (1-s)\frac{i}{2\pi}\Theta_{\varphi_k}(\pi^*L) + s\frac{i}{2\pi}\Theta_{\varphi_k+\psi_{k,\epsilon,\delta_k}+\delta_k\ln|E_k|_{h_k}}(\pi^*L) + \frac{2}{k}dd^c\varphi_k. \tag{2.89}$$

(2.87) gives the estimate for the second term of the right hand side of (2.89). For the last term of the right hand side of (2.89), we observe that φ_k is a function on X satisfying

$$\frac{i}{2\pi}\Theta_{\varphi_k}(L) = \frac{i}{2\pi}\Theta_{h_0}(L) + dd^c\varphi_k \geq -c\omega,$$

thus

$$dd^c\varphi_k \geq -C\omega$$

for some uniform constant C, and

$$\frac{i}{2\pi}\Theta_{\widehat{\varphi}_k}(\pi^*L) \geq -\epsilon_k\omega - \tau_k\omega - \frac{C}{k}\omega. \tag{2.90}$$

Thus $\widehat{\varphi}_k$ induces a quasi-psh function on X by extending it from $X \setminus Z_k$ to the whole X. This is the metric that we wanted to construct. We also denote it $\widehat{\varphi}_k$ for simplicity. We will prove properties (i) to (iii) in Lemma 2.5.11 and Lemma 2.5.12. □

5. Note that s_1 is the constant in (2.83) .

Lemma 2.5.11. *If we take s in (2.88) small enough with respect to s_1 in (2.83) of Lemma 2.5.10, then*

$$\int_U |f|^2 e^{-2\widehat{\varphi}_k} \leq C_{|f|_{L^\infty}} (\int_U |f|^2 e^{-2(1+s_1)\varphi})^{\frac{1}{1+s_1}} \tag{2.91}$$

for all U in X and $k \gg 1$, where $C_{|f|_{L^\infty}}$ is a constant depending only on $|f|_{L^\infty}$ (in particular, it is independent of the open subset U and k). As a consequence, we have

$$\mathscr{I}(\widehat{\varphi}_k) = \mathscr{I}_+(\varphi) \qquad \textit{for every } k. \tag{2.92}$$

Proof. Thanks to (2.87), $\varphi_k + \psi_{k,\epsilon,\delta_k} + \delta \ln |E_k|_{h_k}$ induces a quasi-psh function on X. We also denote it $\varphi_k + \psi_{k,\epsilon,\delta_k} + \delta \ln |E_k|_{h_k}$ for simplicity. Then (2.86) and (2.87) in Lemma 2.5.10 imply the existence of a constant $a > 0$ such that

$$\int_X e^{-2a(\varphi_k + \psi_{k,\epsilon,\delta} + \delta_k \ln |E_k|_{h_k})}$$

is uniformly bounded for all k.

By Hölder's inequality and the construction (2.88), we have

$$\int_U |f|^2 e^{-2\widehat{\varphi}_k} \leq (\int_U |f|^2 e^{-2(1+s_1)\varphi_k})^{\frac{1}{1+s_1}} (\int_U |f|^2 e^{-\frac{2s(1+s_1)}{s_1}(\varphi_k + \psi_{k,\epsilon,\delta_k} + \delta_k \ln |E_k|_{h_k})})^{\frac{s_1}{1+s_1}} \tag{2.93}$$

for $k \gg 1$, where U is any open subset of X. If we take a $s > 0$ satisfying $\frac{s(1+s_1)}{s_1} \leq a$, then the uniform boundedness of $\int_X e^{-2a(\varphi_k + \psi_{k,\epsilon,\delta_k} + \delta_k \ln |E_k|_{h_k})}$ implies that

$$\int_U |f|^2 e^{-\frac{2s(1+s_1)}{s_1}(\varphi_k + \psi_{k,\epsilon,\delta_k} + \delta_k \ln |E_k|_{h_k})} \leq C \cdot |f|_{L^\infty} \tag{2.94}$$

for any $U \subset X$ and $k \gg 1$. Combining (2.93) with (2.94), we have

$$\begin{aligned} \int_U |f|^2 e^{-2\widehat{\varphi}_k} &\leq C_{|f|_{L^\infty}} (\int_U |f|^2 e^{-2(1+s_1)\varphi_k})^{\frac{1}{1+s_1}} \\ &\leq C_{|f|_{L^\infty}} (\int_U |f|^2 e^{-2(1+s_1)\varphi})^{\frac{1}{1+s_1}}. \end{aligned} \tag{2.95}$$

for some constant $C_{|f|_{L^\infty}}$ independent of the open subset U and $k \gg 1$.

It remains to prove (2.92). The inclusion $\mathscr{I}(\widehat{\varphi}_k) \supset \mathscr{I}_+(\varphi)$ comes directly from (2.95). By the construction, $\widehat{\varphi}_k$ is more singular than $(1 + \frac{2}{k})\varphi_k$. Then (2.84) implies that $\mathscr{I}(\widehat{\varphi}_k) \subset \mathscr{I}_+(\varphi)$. Equality (2.92) is proved. □

The following lemma was essentially proved in [Mou95].

Lemma 2.5.12. *In the situation of Lemma 2.5.10, the new metrics $\{\widehat{\varphi}_k\}_{k=1}^\infty$ satisfy properties (ii) and (iii) in Lemma 2.5.10.*

Proof. Let $\lambda_1(z) \leq \lambda_2(z) \leq \cdots \leq \lambda_n(z)$ be the eigenvalues of $i\Theta_{\widehat{\varphi}_k}(L)$ with respect to the base metric ω. Note that λ_i is equal to $\lambda_{i,k}$ in Lemma 2.5.10. Since the proof here is for a fixed k, the simplification will not lead misunderstanding. By (2.90), we have

$$\lambda_i(z) \geq -\epsilon_k - \frac{C}{k} - \tau_k.$$

(ii) of Lemma 2.5.10 is proved.

Set $\widehat{\lambda}_i := \lambda_i + 2\epsilon_k$. Since s is a fixed positive constant, the Monge-Ampère equation (2.85) implies that

$$\prod_{i=1}^n \widehat{\lambda}_i(z) \geq C(s)\epsilon_k^{n-d} \tag{2.96}$$

where $C(s) > 0$ does not depend on k. Since $p > n - d$, we can take α such that $0 < \alpha < 1$ and $n - d < \alpha p$. Set $U_k := \{z \in X \mid \widehat{\lambda}_p(z) < \epsilon_k^{\alpha}\}$.

We now check that U_k satisfies (iii) of Lemma 2.5.10. Since $\epsilon_k \gg \tau_k + \frac{1}{k}$, we have $\widehat{\lambda}_i(z) = \lambda_i(z) + 2\epsilon_k \geq 0$ for any z and i. Thus the cohomological condition

$$\int_X (\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_n)\omega^n \leq M$$

implies that

$$\int_{U_k} (\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_n)\omega^n \leq M. \tag{2.97}$$

Observing that (2.96) and the definition of U_k imply that

$$\prod_{p+1 \leq i \leq n} \widehat{\lambda}_i(z) \geq C(s)\frac{\epsilon_k^{n-d}}{\epsilon_k^{\alpha p}} \qquad \text{for } z \in U_k,$$

we have

$$\sum_{p+1 \leq i \leq n} \widehat{\lambda}_i(z) \geq C(\frac{\epsilon_k^{n-d}}{\epsilon_k^{\alpha p}})^{\frac{1}{n-p}} \qquad \text{for } z \in U_k \tag{2.98}$$

by the inequality between arithmetic and geometric means. Applying (2.98) to (2.97), we have

$$\int_{U_k} (\frac{\epsilon_k^{n-d}}{\epsilon_k^{\alpha p}})^{\frac{1}{n-p}}\omega^n \leq M'. \tag{2.99}$$

Since $n - d < \alpha p$, (2.99) implies that

$$\mathrm{vol}(U_k) \leq \epsilon_k^{\beta}$$

for some $\beta > 0$. (iii) of Lemma 2.5.10 is proved. □

We now reach the final conclusion.

Theorem 2.5.13 ((= Theorem 2.1.3)). *Let (L, φ) be a pseudo-effective line bundle on a compact kähler manifold (X, ω). Then*

$$H^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi)) = 0 \qquad \textit{for } p \geq n - \mathrm{nd}(L, \varphi) + 1.$$

Remark 2.5.14. *One of the reason to use $\mathscr{I}_+(\varphi)$ instead of $\mathscr{I}(\varphi)$ is that it does not seem to be easy to prove*

$$H^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}(\varphi)) = 0 \qquad \textit{for } p \geq n - \mathrm{nd}(L, \varphi) + 1$$

even when X is projective (However, cf. [Mat13] for a recent progress).

Proof. We prove it in two steps.

Steps 1 : L^2 Estimates

Let $\{\widehat{\varphi}_k\}_{k=1}^{\infty}$ be the metrics constructed in Lemma 2.5.10, and let $[u]$ be an element in $H^p(X, K_X \otimes L \otimes \mathscr{I}_+(\varphi))$. Let f be a smooth (n,p)-form representing $[u]$. Then

$$\int_X |f|^2 e^{-2(1+s_1)\varphi} < +\infty,$$

where s_1 is the constant in (2.83) of Lemma 2.5.10. By Lemma 2.5.11, we have

$$\int_U |f|^2 e^{-2\widehat{\varphi}_k} \leq C(\int_U |f|^2 e^{-2(1+s_1)\varphi})^{\frac{1}{1+s_1}} \qquad \text{for every } k \gg 1 \tag{2.100}$$

for any open subset U of X, where C is a constant independent of U and k (but certainly depends on $|f|_{L^\infty}$). We now use the L^2 method in [DP03b] to get a key estimate : f can be written as

$$f = \overline{\partial} u_k + v_k \tag{2.101}$$

with the following bound

$$\int_X |u_k|^2 e^{-2\widehat{\varphi}_k} + \frac{1}{2p\epsilon_k}\int_X |v_k|^2 e^{-2\widehat{\varphi}_k} \leq \int_X \frac{1}{\widehat{\lambda}_{1,k} + \widehat{\lambda}_{2,k} + \cdots + \widehat{\lambda}_{p,k}} |f|^2 e^{-2\widehat{\varphi}_k}, \tag{2.102}$$

where $\widehat{\lambda}_{i,k} = \lambda_{i,k} + 2\epsilon_k$. Estimate (2.102) comes from the Bochner inequality :

$$\|\overline{\partial} u\|^2_{\widehat{\varphi}_k} + \|\overline{\partial}^* u\|^2_{\widehat{\varphi}_k} \geq \int_{X - Z_k} (\widehat{\lambda}_{1,k} + \widehat{\lambda}_{2,k} + \cdots + \widehat{\lambda}_{p,k} - C\epsilon_k)|u|^2_{\widehat{\varphi}_k} dV$$

where Z_k is the singular locus of φ_k in X (see [DP03b, Thm 3.3] or Proposition 2.6.1 in the appendix for details).

Using (2.102), we claim that

$$\lim_{k\to\infty} \int_X |v_k|^2 e^{-2\widehat{\varphi}_k} \to 0. \tag{2.103}$$

Proof of the claim : Properties (ii), Properties (iii) of Lemma 2.5.10 and (2.102) imply that

$$\begin{aligned} &\int_X |u_k|^2 e^{-2\widehat{\varphi}_k} + \frac{1}{2p\epsilon_k}\int_X |v_k|^2 e^{-2\widehat{\varphi}_k} \\ \leq &\int_X \frac{C_1}{\epsilon_k^{\alpha}} |f|^2 e^{-2\widehat{\varphi}_k} + \int_{U_k} \frac{1}{C_2 \epsilon_k} |f|^2 e^{-2\widehat{\varphi}_k}. \end{aligned}$$

Then

$$\int_X |v_k|^2 e^{-2\widehat{\varphi}_k} \leq C_3 \epsilon_k^{1-\alpha} \int_X |f|^2 e^{-2\widehat{\varphi}_k} + C_4 \int_{U_k} |f|^2 e^{-2\widehat{\varphi}_k}. \tag{2.104}$$

Since $\mathrm{vol}(U_k) \to 0$ by property (iii) of Lemma 2.5.10, (2.100) implies that the second term of the right hand side of (2.104) tends to 0. Since $0 < \alpha < 1$ and $\epsilon_k \to 0$ as $k \to \infty$, (2.100) implies thus that the first term of the right hand side of (2.104) also tends to 0. (2.103) is proved.

Step 2 : Final step

We use Lemma 2.5.8 to obtain the final conclusion. Let $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ be a Stein covering of X. Thanks to (2.103), we get a p-cocycle representing v_k by solving $\overline{\partial}$-equations, i.e., v_k can be written as

$$v_k = \{v_{k,\alpha_0 \ldots \alpha_p}\} \in \check{C}^p(\mathscr{U}, \mathscr{O}(K_X + L) \otimes \mathscr{I}(\widehat{\varphi}_k)),$$

where the components satisfy the L^2 conditions

$$\int_{U_{\alpha_0 \ldots \alpha_p}} |v_{k,\alpha_0 \ldots \alpha_p}|^2 e^{-2\widehat{\varphi}_k} \leq C \int_X |v_k|^2 e^{-2\widehat{\varphi}_k}, \tag{2.105}$$

and where C does not depend on k. Inequality (2.105) and property (i) in Lemma 2.5.10 imply that $\{v_k\}$ is in $\check{C}^p(\mathscr{U}, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi))$ for every k.

Since $\widehat{\varphi}_k \leq 0$ by construction, (2.103) and (2.105) imply that

$$\lim_{k\to\infty} \int_{U_{i_0 \ldots i_p}} |v_{k,i_0 \ldots i_p}|^2 = 0. \tag{2.106}$$

By (2.101), $\{v_k\}_{k=1}^{\infty}$ are in the same cohomology class as u in $H^p(X, \mathscr{O}(K_X + L) \otimes \mathscr{I}_+(\varphi))$. By Lemma 2.5.8, (2.106) implies that $[u] = 0$. The theorem is proved.

□

2.6 Appendix

For the convenience of readers, we give the proof of estimate (2.102) in Theorem 2.5.13. For the major part, the proof is just extracted from [DP03b].

Proposition 2.6.1. *Let (X, ω) be a compact Kähler manifold and let $(L, h_0 e^{-\varphi})$ be a line bundle on X where h_0 is a smooth metric on L and the quasi-psh function φ has analytic singularities and smooth outside a subvariety Z. Assume that*

$$\frac{i}{2\pi}\Theta_{\varphi}(L) \geq -\epsilon\omega$$

on $X \setminus Z$, and f is a smooth L-valued (n,p)-form satisfying

$$\int_X |f|^2 e^{-2\varphi} dV < \infty. \tag{2.107}$$

Let $\lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n$ be the eigenvalues of $\frac{i}{2\pi}\Theta_{\varphi}(L)$ and $\widehat{\lambda}_i = \lambda_i + 2\epsilon \geq \epsilon$. Then there exist u and v such that $f = \overline{\partial} u + v$ and with the following estimate

$$\int_X |u|^2 e^{-2\varphi} dV + \frac{1}{2p\epsilon}\int_X |v|^2 e^{-2\varphi} dV \leq \int_X \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p} |f|^2 e^{-2\varphi} dV.$$

Proof. Let ω_1 be a complete Kähler metric on $X \setminus Z$ and $\omega_\delta = \omega + \delta\omega_1$ for some $\delta > 0$. We now do the standard L^2 estimate on $(X \setminus Z, \omega_\delta, L, \varphi)$.

If s is a L-valued (n,p)-form in $C_c^\infty(X \setminus Z)$, then the Bochner inequality implies that :

$$\|\overline{\partial} s\|_\delta^2 + \|\overline{\partial}^* s\|_\delta^2 \geq \int_{X \setminus Z} (\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p - 2p\epsilon)|s|^2 e^{-2\varphi}\omega_\delta^n \tag{2.108}$$

where $\|s\|_\delta^2 = \int_X |s|^2 e^{-2\varphi}\omega_\delta^n$. Note that there is an abuse of notation here : we calculate the norm $|u|^2$ by the metric (or the volum form) written in the equations. For example, if the volum form is ω_δ^n, then we calculate the norm of u by means of the metrics ω_δ and h_0.

Since f is a (n,p)-form, (2.107) implies that

$$f \in L^2(X \setminus Z, L, \varphi, \omega_\delta) \qquad \text{for } \delta > 0.$$

We write every form s in the domain of the L^2 extension of $\overline{\partial}^*$ as $s = s_1 + s_2$ with

$$s_1 \in \operatorname{Ker} \overline{\partial} \qquad \text{and} \qquad s_2 \in (\operatorname{Ker} \overline{\partial})^\perp \subset \operatorname{Ker} \overline{\partial}^*.$$

Since $f \in \operatorname{Ker} \overline{\partial}$, by (2.108) we obtain

$$|\langle f, s\rangle|_{\varphi,\delta}^2 = |\langle f, s_1\rangle|_{\varphi,\delta}^2$$

$$\leq \int_{X \setminus Z} \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p} |f|^2 e^{-2\varphi} dV_\delta \int_{X\setminus Z} (\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p)|s_1|^2 e^{-2\varphi} dV_\delta$$

$$\leq \int_{X \setminus Z} \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p} |f|^2 e^{-2\varphi} dV_\delta (\|\overline{\partial}^* s_1\|_\delta^2 + 2p\epsilon\|\overline{\partial} s_1\|_\delta^2)$$

$$\leq \int_{X \setminus Z} \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p} |f|^2 e^{-2\varphi} dV_\delta (\|\overline{\partial}^* s\|_\delta^2 + 2p\epsilon\|\overline{\partial} s\|_\delta^2).$$

By the Hahn-Banach theorem, we can find v_δ, u_δ such that

$$\langle f, s\rangle_\delta = \langle u_\delta, \overline{\partial}^* s\rangle_\delta + \langle v_\delta, s\rangle_\delta \qquad \text{for every } s,$$

satisfying the estimate

$$\|u_\delta\|_\delta^2 + \frac{1}{2p\epsilon}\|v_\delta\|_\delta^2 \leq C\int_X \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p} |f|^2 e^{-2\varphi}\omega_\delta^n.$$

Therefore

$$f = \overline{\partial} u_\delta + v_\delta. \tag{2.109}$$

Since the norm $\| \cdot \|_\delta$ of (n, p)-forms is increasing when δ decreases to $\to 0$, we obtain limits

$$u = \lim_{\delta \to 0} u_\delta \qquad \text{and} \qquad v = \lim_{\delta \to 0} v_\delta \tag{2.110}$$

satisfying

$$\begin{aligned} \|u\|_\delta^2 + \frac{1}{2p\epsilon} \|v\|_\delta^2 &\leq C \int_X \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p} |f|^2 e^{-2\varphi} \omega_\delta^n \\ &\leq C \int_X \frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_p} |f|^2 e^{-2\varphi} \omega^n \end{aligned} \tag{2.111}$$

for every $\delta > 0$. Formulas (2.109) and (2.110) imply that $f = \overline{\partial} u + v$. Letting $\delta \to 0$ in (2.111), we obtain the estimate in the proposition.

□

Chapitre 3

Kawamata-Viehweg vanishing theorem and numerical dimension

3.1 Introduction

This chapter is devoted to study the Kawamata-Viehweg vanishing theorem and numerical dimension of $-K_X$ when $-K_X$ is nef. We first recall the following well known Kawamata-Viehweg vanishing theorem.

Theorem 3.1.1. *Let X be a projective manifold and let F be a nef line bundle over X. Then*

$$H^q(X, K_X + F) = 0 \qquad \textit{for all } q \geq n - \mathrm{nd}(F) + 1.$$

The main object in this chapter is to generalize this vanishing theorem to some Kähler manifolds. More precisely, we consider the manifolds admitting a fibration to a torus such that the generic fiber is projective. We will prove the following weak Kawamata-Viehweg vanishing theorem

Theorem 3.1.2. *Let X be a compact Kähler manifold of dimension n. We suppose that there exists a fibration $\pi : X \to T$ onto a torus T of dimension r. Let L be a nef, π-big line bundle on X.*

If $\mathrm{nd}(L) = n - r$, then

$$H^q(X, K_X + L) = 0 \qquad \textit{for } q > r.$$

If $\mathrm{nd}(L) \geq n - r + 1$, then

$$H^q(X, K_X + L) = 0 \qquad \textit{for } q \geq r.$$

Remark 3.1.3. *As pointed out by T.Peternell, if $q > r$, the above vanishing theorem comes from directly the fact that $R^j\pi_*(K_X + L) = 0$ for $j \geq 1$. In particular, the hypothèse that T is a torus is not necessary in this case. However, if $\mathrm{nd}(L) \geq n - r + 1$ and $q = r$, the vanishing theorem obtained here is non trivial and has the following important application.*

Theorem 3.1.4. *Let X be a compact Kähler manifold with nef anticanonical bundle of dimension n, and let $\pi : X \to T$ be a fibration onto a torus T of dimension r. If $-K_X$ is big on the generic fiber, then $\mathrm{nd}(-K_X) = n - r$.*

Remark 3.1.5. *If X is projective, this statement is well-known : in this case X is projective, so if $\mathrm{nd}(-K_X) > n - r$ we can apply Theorem 3.1.1 to see that*

$$H^r(X, \mathscr{O}_X) = H^r(X, K_X + (-K_X)) = 0,$$

which is clearly impossible.

3.2 Preparatory lemmas

It is well known that for a generic torus $T = \mathbb{C}^n/\Gamma$, we have

$$H^{q,q}(T,\mathbb{R}) \cap H^{2q}(T,\mathbb{Q}) = 0 \qquad \text{for } 1 \leq q \leq \dim T - 1.$$

Therefore a generic torus T has no strict subvariety. However, if there exists an effective divisor on T, the following lemma tells us that T is not far from an abelian variety.

Lemma 3.2.1. *Let $T = \mathbb{C}^n/\Gamma$ be a complex torus of dimension n, and $\alpha \in H^{1,1}(T,\mathbb{Z})$ an effective non trivial element. Then T possess a submersion*

$$\pi : T \to S$$

to an abelian variety S. Moreover $\alpha = \pi^* c_1(A)$ for some ample line bundle A on S.

Proof. Since T is a torus, we can suppose that α is a constant semipositive $(1,1)$-form. As α is an integral class, it defines a bilinear form

$$G_{\mathbb{Q}} : (\Gamma \otimes \mathbb{Q}) \times (\Gamma \otimes \mathbb{Q}) \to \mathbb{Q}.$$

We denote its extension to $\Gamma \otimes \mathbb{R}$ by $G_{\mathbb{R}}$. Let V be the maxium subspace of $\Gamma \otimes \mathbb{Q}$, on which $G_{\mathbb{Q}}$ is zero. Therefore $V_{\mathbb{R}} = V \otimes \mathbb{R}$ is also the kernel of $G_{\mathbb{R}}$. Moreover since α is an $(1,1)$-form, $V_{\mathbb{R}}$ is a complex subspace of $\mathbb{C}^n$. Therefore we have a natural holomorphic submersion $T \to T/V_{\mathbb{R}}$. We denote the complex torus $T/V_{\mathbb{R}}$ by S. Since $V_{\mathbb{R}}$ is the kernel of $G_{\mathbb{R}}$, α is well defined on S and is moreover strictly positive on it. The proposition is proved. □

We need a partial vanishing theorem with multiplier ideal sheaf (cf. Definition 1.2.2 for the definition of multiplier ideal sheaves and Definition 1.2.1 for analytic singularities).

Proposition 3.2.2. *Let L be a line bundle on a compact Kähler manifold (X,ω) of dimension n and let φ be a metric on L with analytic singularities. Let $\lambda_1(z) \leq \lambda_2(z) \leq \cdots \leq \lambda_n(z)$ be the eigenvalues of $\frac{i}{2\pi}\Theta_\varphi(L)$ with respect to ω. If*

$$\sum_{i=1}^{p} \lambda_i(z) \geq c \tag{3.1}$$

for some constant $c > 0$ independent of $z \in X$, then

$$H^q(X, K_X \otimes L \otimes \mathscr{I}(\varphi)) = 0 \qquad \text{for } q \geq p.$$

Proof. Since φ has analytic singularities, there exists an analytic subvariety Y such that φ is smooth on $X \setminus Y$. Moreover it is known that there exists a quasi-psh function ψ on X, smooth on $X \setminus Y$ such that $\mathscr{I}(\varphi) = \mathscr{I}(\varphi + \psi)$ and $\widetilde{\omega} = c_1\omega + i\partial\overline{\partial}\psi$ is a complete metric on $X \setminus Y$ for some fixed constant c_1 with $0 < c_1 \ll c$ (cf. [Dem, Section 5, 6 , Chapter VIII]). To prove the proposition, it is therefore equivalent to prove that

$$H^q(X, K_X \otimes L \otimes \mathscr{I}(\varphi + \psi)) = 0 \qquad \text{for } q \geq p. \tag{3.2}$$

We consider the new metric $\phi = \varphi + \psi$ on L (i.e., the new metric is $\|\cdot\|_\varphi \cdot e^{-\psi}$). Then

$$\frac{i}{2\pi}\Theta_\phi(L) = \frac{i}{2\pi}\Theta_\varphi(L) + dd^c\psi = (\frac{i}{2\pi}\Theta_\varphi(L) - c_1\omega) + \widetilde{\omega}. \tag{3.3}$$

Since φ is a quasi-psh function, there exists a constant M such that

$$\frac{i}{2\pi}\Theta_\varphi(L) - c_1\omega \geq -M\omega. \tag{3.4}$$

Combining (3.4) with (3.3), we obtain

$$\frac{i}{2\pi}\Theta_{\phi}(L) \geq -M\omega + \widetilde{\omega}. \tag{3.5}$$

Set $\omega_{\tau} := \omega + \tau\widetilde{\omega}$. We claim that, the sum of p smallest eigenvalues of $\frac{i}{2\pi}\Theta_{\phi}(L)$ with respect to ω_{τ} is larger than $\frac{c}{2}$, for any $0 < \tau \leq \frac{c_1}{1000(M+c)\cdot n\cdot(1+c_1)}$.

Proof of the claim : Let $x \in X \setminus Y$. By the minimax principle, it is sufficient to prove that for any p-dimensional subspace V of $(T_X)_x$, we have

$$\sum_{i=1}^{p} \langle \frac{i}{2\pi}\Theta_{\phi}(L)e_i, e_i\rangle \geq \frac{c}{2} \tag{3.6}$$

where $\{e_i\}_{i=1}^{p}$ is an orthonormal basis of V with respect to ω_{τ}.

We first consider the case when V contains an element e such that

$$\widetilde{\omega}(e,e) \geq \frac{c_1}{\tau} \qquad \text{and} \qquad |e|_{\omega} = 1. \tag{3.7}$$

By the choice of τ, we have

$$\widetilde{\omega}(e,e) \geq 1000n \cdot (M+c). \tag{3.8}$$

Thanks to (3.5) and (3.8), we have

$$\langle \frac{i}{2\pi}\Theta_{\phi}(L)e, e\rangle \geq -M + \widetilde{\omega}(e,e) \geq \frac{\widetilde{\omega}(e,e)}{2}.$$

Observing moreover that the construction of ω_{τ} implies

$$\langle e, e\rangle_{\omega_{\tau}} = 1 + \tau \cdot \widetilde{\omega}(e,e),$$

then

$$\frac{\langle \frac{i}{2\pi}\Theta_{\phi}(L)e, e\rangle}{\langle e, e\rangle_{\omega_{\tau}}} \geq \frac{\widetilde{\omega}(e,e)}{2 + 2\tau\widetilde{\omega}(e,e)} \geq \frac{1}{2}\min\{\frac{\widetilde{\omega}(e,e)}{2}, \frac{1}{2\tau}\} \geq n(M+c). \tag{3.9}$$

Noting that (3.5) implies that

$$\langle \frac{i}{2\pi}\Theta_{\phi}(L)e', e'\rangle \geq -M\omega(e',e') \geq -M\omega_{\tau}(e',e') \tag{3.10}$$

for any $e' \in V$, (3.9) and (3.10) imply thus the inequality (3.6).

In the case when

$$\tau \cdot \widetilde{\omega}(e,e) \leq c_1 \qquad \text{for any } e \in V \text{ with } |e|_{\omega} = 1,$$

we have

$$|\omega_{\tau} - \omega|_{\omega} \leq c_1 \qquad \text{on } V, \tag{3.11}$$

i.e., for considering the restriction on V, the difference between $\omega_{\tau}|_V$ and $\omega|_V$ is controled by $c_1\omega$. On the other hand, using again the minimax principle, (3.1) implies that

$$\sum_{i=1}^{p} \langle \frac{i}{2\pi}\Theta_{\varphi}(L)\widetilde{e}_i, \widetilde{e}_i\rangle \geq c \tag{3.12}$$

for any orthonormal basis $\{\widetilde{e}_i\}$ of V with respect to ω. By (3.3), we have

$$\frac{i}{2\pi}\Theta_{\phi}(L) \geq (\frac{i}{2\pi}\Theta_{\varphi}(L) - c_1\omega). \tag{3.13}$$

Combining (3.13) with (3.12) and the smallness assumption on c_1, we have

$$\sum_{i=1}^{p}\langle\frac{i}{2\pi}\Theta_\phi(L)\widetilde{e}_i,\widetilde{e}_i\rangle \geq \frac{3c}{4}. \tag{3.14}$$

Using again that c_1 is a fixed constant small enough with respect to c, (3.11) and (3.14) imply the inequality (3.6). The claim is proved.

The following arguement is standard. Let f be a L-valued closed (n,q)-form such that

$$\int_X |f|^2 e^{-2\phi}\omega^n < +\infty.$$

To prove (3.2), it is equivalent to find a L-valued $(n,q-1)$-form g such that

$$f=\overline{\partial}g \qquad \text{and} \qquad \int_X |g|^2 e^{-2\phi}\omega^n < +\infty.$$

Thanks to our claim, we can use the standard L^2 estimate on

$$(X\setminus Y, \omega_\tau, L, e^{-\phi}).$$

It is known that

$$\int_{X\setminus Y} |f|^2 e^{-2\phi}\omega_\tau^n \leq \int_{X\setminus Y} |f|^2 e^{-2\phi}\omega^n < +\infty. \tag{3.15}$$

Then we can find a g_τ such that $f=\overline{\partial}g_\tau$ and

$$\int_{X\setminus Y} |g|^2 e^{-2\phi}\omega_\tau^n \leq C\int_{X\setminus Y} |f|^2 e^{-2\phi}\omega_\tau^n < +\infty.$$

for a constant C depending only on c (i.e., C is independent of τ). Letting $g=\lim\limits_{\tau\to 0} g_\tau$, by (3.15), we have

$$\int_{X\setminus Y} |g|^2 e^{-2\phi}\omega^n < +\infty$$

and $f=\overline{\partial}g$ on $X\setminus Y$. [Dem12, Lemma 11.10] implies that such g can be extended to the whole space X, and $f=\overline{\partial}g$ on X. Therefore (3.2) is proved. □

Lemma 3.2.3. *Let X be a compact Kähler manifold of dimension n, and let $\pi: X\to T$ be a surjective morphism onto a compact Kähler manifold T of dimension r. Let L be a nef line bundle on X that is π-big*[1]*. If $\mathrm{nd}(L)\geq n-r+1$, then we have*

$$\int_X L^{n-r+1}\wedge(\pi^*\omega_T)^{r-1} > 0$$

for any Kähler form ω_T on T.

Proof. We suppose that $\mathrm{nd}(L)=n-r+k$ for some $k\in\mathbb{N}^*$. Since L is nef and π-big,

$$\alpha = L+\pi^*(\omega_T) \tag{3.16}$$

is a nef class, and $\int_X \alpha^n > 0$. Thanks to [DP04, Theorem 0.5] [2], there exists a $\epsilon>0$, such that $\alpha-\epsilon\omega_X$ is a pseudoeffective class. Combining this with the fact that L is nef, we have

$$\int_X L^{n-r+k}\wedge\alpha^{r-k}\geq\epsilon\int_X L^{n-r+k}\wedge\alpha^{r-k-1}\wedge\omega_X\geq ...\geq\epsilon^{r-k}\int_X L^{n-r+k}\wedge\omega_X^{r-k} > 0,$$

1. cf. Definition1.2.6
2. Their theorem is esstentially a transcendental version of holomorphic Morse inequality, cf. [Dem12, Chapter 8], [Tra11].

where the last inequality comes from Remark 1.2.2. By the definition of numerical dimension and (3.16), we obtain

$$\int_X L^{n-r+k} \wedge \pi^*(\omega_T)^{r-k} = \int_X L^{n-r+k} \wedge \alpha^{r-k} > 0. \tag{3.17}$$

On the other hand, since L is π-big, we have

$$\int_X L^{n-r} \wedge \pi^*(\omega_T)^r > 0. \tag{3.18}$$

Using the Hovanskii-Teissier inequality (cf. Appendix 6.2), (3.17) and (3.18) imply

$$\int_X L^{n-r+1} \wedge \pi^*(\omega_T)^{r-1} > 0.$$

□

3.3 A Kawamata-Viehweg vanishing theorem

As pointed out in the introduction, when X is a projective variety of dimension n and L is a nef line bundle on X with $\mathrm{nd}(L) = k$, we have the Kawamata-Viehweg vanishing theorem :

$$H^r(X, K_X + L) = 0 \qquad \text{for } r > n - k.$$

But it is probably a difficult problem to prove this vanishing theorem for a non projective compact Kähler manifold. We will prove in this section a Kawamata-Viehweg vanishing theorem for certain Kähler manifolds.

Before annoncing the main theorem in this section, we first prove a technical lemma.

Lemma 3.3.1. *Let L be a nef line bundle on a compact Kähler manifold X of dimension n . We suppose that (X, L) satisfies the following two conditions*

(i) There exists a two steps tower fibration

$$X \xrightarrow{\ \pi\ } T \xrightarrow{\ \pi_1\ } S$$

where π is a surjective to a smooth variety T of dimension r, and π_1 is a submersion to a smooth curve S.

(ii) The nef line bundle L is π-big and

$$\pi_*(L^{n-r+1}) = \pi_1^*(\mathscr{O}_S(1))$$

for some ample line bundle $\mathscr{O}_S(1)$ on S.

Then $L - c\pi^*\pi_1^*(\mathscr{O}_S(1))$ is pseudo-effective for some constant $c > 0$.

Remark 3.3.2. *We first remark that (ii) of Lemma 3.3.1 implies that*

$$\mathrm{nd}(L) > n - r.$$

Set $\mathrm{nd}(L) := n - r + t$. *Our aim in this remark is to prove that*

$$L^{n-r+t} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) = 0.$$

First of all, using the Hovanskii-Teissier inequality for arbitrary compact Kähler manifolds [Gro90], we obtain

$$\int_X L^{n-r+1} \wedge \omega_T^{r-2} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) \geq \tag{3.19}$$

$$(\int_X L^{n-r+p} \wedge \omega_T^{r-p-1} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)))^{\frac{1}{p}} (\int_X L^{n-r} \wedge \omega_T^{r-1} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)))^{\frac{p-1}{p}},$$

where ω_T is a Kähler metric on T and $p > 1$. Since $\dim S = 1$*, condition* (ii) *of Lemma 3.3.1 implies that*

$$\int_X L^{n-r+1} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) \wedge \omega_T^{r-2} = 0.$$

Moreover, the relative ampleness of L implies that

$$\int_X L^{n-r} \wedge \omega_T^{r-1} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) > 0.$$

Combining these two equations with (3.19)*, we obtain*

$$\int_X L^{n-r+p} \wedge \omega_T^{r-p-1} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) = 0 \qquad \text{for any } p \geq 1. \tag{3.20}$$

Suppose that $\mathrm{nd}(L) = n - r + t$*. If $t \geq 2$, using again the Hovanskii-Teissier inequality, we have*

$$\int_X L^{n-r+1} \wedge \omega_X^{r-2} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) \geq \tag{3.21}$$

$$(\int_X L^{n-r+t} \wedge \omega_X^{r-t-1} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)))^{\frac{1}{t}} (\int_X L^{n-r} \wedge \omega_X^{r-1} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)))^{\frac{t-1}{t}},$$

where ω_X is a Kähler metric on X. Since L is relatively ample, ω_X is controled by $L + C \cdot \omega_T$ for some $C > 0$ large enough. Then (3.20) *implies that*

$$\int_X L^{n-r+1} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) \wedge \omega_X^{r-2} = 0.$$

Moreover, the relative ampleness of L implies that

$$\int_X L^{n-r} \wedge \omega_X^{r-1} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) > 0.$$

By (3.21)*, we obtain finally*

$$L^{n-r+t} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) = 0. \tag{3.22}$$

Proof of Lemma 3.3.1. We first explain the idea of the proof. By using a Monge-Ampère equation, we can construct a sequence of metrics $\{\varphi_\epsilon\}$ on L, such that

$$\frac{i}{2\pi}\Theta_{\varphi_\epsilon}(L) \geq c\pi^*\pi_1^*(\mathscr{O}_S(1)) \qquad \text{for all small } \epsilon.$$

Then $\frac{i}{2\pi}\Theta_\varphi(L) \geq c\pi^*\pi_1^*\mathscr{O}_S(1)$, where φ is a limit of some subsequence of $\{\varphi_\epsilon\}$. In this way, the lemma would therefore be proved. This idea comes from [DP04], but the proof here is in some sense much simpler because we do not need their diagonal trick in our case.

By Remark 3.3.2, we can thus suppose that $\mathrm{nd}(L) = n - r + t$, for some $t \geq 1$ and

$$L^{n-r+t} \wedge \pi^*\pi_1^*(\mathscr{O}_S(1)) = 0. \tag{3.23}$$

For simplicity, we denote $\pi^*\pi_1^*\mathscr{O}_S(1)$ by A. Let $s \in S$, and X_s the fiber of $\pi \circ \pi_1$ over s. We first fix a smooth metric h_0 on $\mathscr{O}_S(1)$. Thanks to the semi-positivity of A, we can choose a sequence of smooth functions ψ_ϵ on X such that for the metrics $h_0 e^{-\psi_\epsilon}$ on A, the curvature forms $\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A)$ are semi-positive [3], and

$$\int_{V_\epsilon} \frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \geq C_1 \qquad \text{for } \epsilon \to 0 \tag{3.24}$$

3. Note that here ψ_ϵ are functions, but the φ's in Proposition 2.1 are metrics! Therefore in this lemma, $\frac{i}{2\pi}\Theta_{\psi_\epsilon}(\mathscr{O}_S(1)) = \frac{i}{2\pi}\Theta_{h_0}(\mathscr{O}_S(1)) + dd^c\psi_\epsilon$.

where V_ϵ is an ϵ open neighborhood of X_s, and $C_1 > 0$ is a uniform constant [4].

Let τ_1, τ_2 two constants such that $1 \gg \tau_1 \gg \tau_2 > 0$ which will be made precise later. Let h be a fixed smooth metric on L. Thanks to the nefness of L, we can solve a Monge-Ampère equation :

$$(\frac{i}{2\pi}\Theta_h(L) + \tau_1\omega + dd^c\varphi_\epsilon)^n = C_{2,\epsilon}\frac{\tau_1^{r-t}}{\tau_2^{n-1}}(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n, \tag{3.25}$$

where

$$C_{2,\epsilon} = \frac{(\frac{i}{2\pi}\Theta_h(L) + \tau_1\omega)^n \tau_2^{n-1}}{\tau_1^{r-t}(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n}.$$

Since $\mathrm{nd}(L) = n - r + t$ and $\dim S = 1$, we have $\inf\limits_{\epsilon} C_{2,\epsilon} > 0$.

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\frac{i}{2\pi}\Theta_h(L)+\tau_1\omega+dd^c\varphi_\epsilon$ with respect to $\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A)+\tau_2\omega$. Then the Monge-Ampère equation (3.25) implies that

$$\prod_{i=1}^{n}\lambda_i(z) = C_{2,\epsilon}\frac{\tau_1^{r-t}}{\tau_2^{n-1}} \qquad \text{for any } z \in X. \tag{3.26}$$

We claim that there exists a constant $\delta > 0$ independent of ϵ, τ_1, τ_2, such that

$$\int_{V_\epsilon \setminus E_{\delta,\epsilon}} \frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \geq \frac{C_1}{2} \qquad \text{for any } \epsilon, \tag{3.27}$$

where

$$E_{\delta,\epsilon} = \{z \in V_\epsilon | \prod_{i=2}^{n}\lambda_i(z) \geq C_{2,\epsilon}\frac{\tau_1^{r-t}}{\delta\tau_2^{n-1}}\}.$$

We postphone the proof of the claim in Lemma (3.3.3) and finish the proof of this lemma. Since

$$\lambda_1(z) \geq \frac{C_2\frac{\tau_1^{r-t}}{\tau_2^{n-1}}}{C_2\frac{\tau_1^{r-t}}{\delta\tau_2^{n-1}}} = \delta \qquad \text{for } z \in V_\epsilon \setminus E_{\delta,\epsilon}$$

by the definition of $E_{\delta,\epsilon}$ and (3.26), (3.27) implies hence that

$$\int_{V_\epsilon}(\frac{i}{2\pi}\Theta_h(L) + \tau_1\omega + dd^c\varphi_\epsilon) \wedge \omega^{n-1} \geq C_8\int_{V_\epsilon}\lambda_1(z)\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1}$$

$$\geq \delta C_8\int_{V_\epsilon \setminus E_{\delta,\epsilon}}\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \geq \delta \cdot C_8 \cdot \frac{C_1}{2}. \tag{3.28}$$

Letting $\epsilon \to 0$, the choice of V_ϵ and (3.28) imply that the weak limit of

$$\frac{i}{2\pi}\Theta_h(L) + \tau_1\omega + dd^c\varphi_\epsilon$$

is more positive than $C_9[X_s]$. Thus $L + \tau_1\omega - C_9[X_s]$ is pseudo-effective. Since C_9 is independent of τ_1, when $\tau_1 \to 0$, we obtain that $L - C_9\pi^*\pi_1^*(\mathscr{O}_S(1))$ is pseudo-effective. The lemma is proved.

□

Lemma 3.3.3. *We now prove the claim in Lemma 3.3.1*

4. All the constants C_i below will be uniformly strictly positive. When the uniform boundedness comes from obvious reasons, we will not make it explicit.

Proof. By construction,

$$\int_X (\prod_{i=2}^{n} \lambda_i(z))(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n \tag{3.29}$$

$$\leq C_3 \int_X (c_1(L) + \tau_1\omega + dd^c\varphi_\epsilon)^{n-1} \wedge (\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)$$

$$= C_3 \int_X (c_1(L) + \tau_1\omega)^{n-1} \wedge (c_1(A) + \tau_2\omega).$$

On the other hand, using (3.23), we have

$$\int_X (c_1(L) + \tau_1\omega)^{n-1} \wedge (c_1(A) + \tau_2\omega) \tag{3.30}$$

$$= C_4\tau_1^{r-t} c_1(L)^{n-r+t-1} \wedge c_1(A) + O(\tau_2) \leq C_5\tau_1^{r-t}.$$

where the last inequality comes from the fact that $\tau_2 \ll \tau_1$. Combining (3.29) with (3.30), we get

$$\int_X (\prod_{i=2}^{n} \lambda_i(z))(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n \leq C_6\tau_1^{r-t}. \tag{3.31}$$

For any δ fixed, (3.31) and the definition of $E_{\delta,\epsilon}$ imply that

$$\int_{E_\delta} C_{2,\epsilon}\frac{\tau_1^{r-t}}{\delta\tau_2^{n-1}}(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n \leq C_6\tau_1^{r-t}.$$

Combining this with the fact that $\inf_\epsilon C_{2,\epsilon} > 0$, we get

$$\int_{E_{\delta,\epsilon}} (\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n \leq C_7\delta\tau_2^{n-1}. \tag{3.32}$$

Since $\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A)$ is semi-positive, (3.32) implies that

$$\int_{E_{\delta,\epsilon}} \frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \leq C_7\delta. \tag{3.33}$$

By taking $\delta = \frac{C_1}{2C_7}$, (3.24) of Lemma 5.1 and (3.33) imply that

$$\int_{V_\epsilon \setminus E_{\delta,\epsilon}} \frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \geq \frac{C_1}{2}.$$

The lemma is proved.

□

Using Lemma 3.3.1, we would like to prove a Kawamata-Viehweg type vanishing theorem. Recall that T.Ohsawa proved in [Ohs84] that if $X \to T$ is a smooth fibration and (E, h) is a hermitian line bundle on X with $\frac{i}{2\pi}\Theta_h(E) \geq \pi^*\omega_T$. Then

$$H^q(T, R^0\pi_*(K_X \otimes E)) = 0$$

for $q \geq 1$. In his proof, he uses the metrics $\pi^*\omega_T + \tau\omega_X$ on X and lets $\tau \to 0$ to preserve the information on T. The idea of our proof comes from this technique.

Proposition 3.3.4. *Let (X,ω_X) be a compact Kähler manifold of dimension n, and L be a nef line bundle on X. We suppose that (X,L) satisfy the following two conditions :*

(i) X admits a two steps tower fibration

$$X \xrightarrow{\pi} T \xrightarrow{\pi_1} S$$

where π is surjective to a smooth variety T of dimension r, and π_1 is a submersion to a smooth curve S.

(ii) L is π-big and satisfies

$$\pi_*(c_1(L)^{n-r+1}) = \pi_1^*(\omega_S)$$

for a Kähler metric ω_S on S.

Then

$$H^p(X, K_X + L) = 0 \qquad \text{for } p \geq r.$$

Proof. Let ω_T be a Kähler metric on T. By Lemma 3.3.1,

$$L - d\cdot \pi^* \circ \pi_1^* \omega_S$$

is pseudoeffective for some $d > 0$. Therefore there exists a singular metric h_1 on L such that

$$i\Theta_{h_1}(L) \geq d\cdot \pi^*\pi_1^*\omega_S.$$

Since $c_1(L) + \pi^*\omega_T$ is nef and $\int_X (c_1(L) + \pi^*\omega_T)^n > 0$, [DP04, Theorem 0.5] implies the existence of a singular metric h_2 on L such that

$$i\Theta_{h_2}(L) \geq c\cdot \omega_X - \pi^*\omega_T$$

in the sense of currents for some constant $c > 0$. Thanks to Demailly's regularization theorem, we can suppose moreover that h_1, h_2 have analytic singularities. Note that L is nef. Then for any $\epsilon > 0$, there exists a smooth metric h_ϵ on L such that $i\Theta_{h_\epsilon}(L) \geq -\epsilon\omega_X$.

Now we define a new metric h on L :

$$h = \epsilon_1 h_1 + \epsilon_2 h_2 + (1 - \epsilon_1 - \epsilon_2) h_\epsilon$$

for some $1 \gg \epsilon_1 \gg \epsilon_2 \gg \epsilon > 0$. By construction, we have

$$\begin{aligned} i\Theta_h(L) &= \epsilon_1 i\Theta_{h_1}(L) + \epsilon_2 i\Theta_{h_2}(L) + (1-\epsilon_1-\epsilon_2) i\Theta_{h_\epsilon}(L) \\ &\geq d\cdot\epsilon_1\pi^*(\omega_S) - \epsilon_2\pi^*(\omega_T) + (c\cdot\epsilon_2 - \epsilon)\omega_X. \end{aligned} \tag{3.34}$$

Let $\omega_\tau = \tau\cdot\omega_X + \pi^*(\omega_T)$ for $\tau > 0$. We now check that $(i\Theta_h(L), \omega_\tau)$ satisfies the condition (3.1) in Proposition 3.2.2 for τ small enough. In fact, since $\epsilon_2 \ll \epsilon_1$, (3.34) implies that $i\Theta_h(L)$ has at most $(r-1)$-negative eigenvectors and their eigenvalues are $\geq -\epsilon_2$. Let x be any point in X. For any r dimensional subspace V of $(T_X)_x$, we have

$$\sup_{v\in V} \frac{i\Theta_h(L)(v,v)}{\langle v, v\rangle_{\omega_\tau}} \geq \frac{1}{2}\min\{\frac{c\epsilon_2 - \epsilon}{\tau}, d\cdot\epsilon_1\} \gg (r-1)\cdot\epsilon_2$$

by the choice of $\tau, \epsilon_1, \epsilon_2$. By the minimax principle, the condition (3.1) of Proposition 3.2.2 is satisfied. Thus

$$H^p(X, K_X + L\otimes \mathscr{I}(h)) = 0 \qquad \text{for } p \geq r.$$

Since ϵ_1, ϵ_2 are small enough, we have $\mathscr{I}(h) = \mathscr{O}_X$. Therefore we get

$$H^p(X, K_X + L) = 0 \qquad \text{for } p \geq r.$$

□

We now prove the main theorem in this chapter.

Theorem 3.3.5. *Let X be a compact Kähler manifold of dimension n. We suppose that there exists a surjective morphism $\pi : X \to T$ to a torus of dimension r. Let L be a nef, π-big line bundle on X.*

If $\mathrm{nd}(L) = n - r$, *then*

$$H^q(X, K_X + L) = 0 \qquad \text{for } q > r.$$

If $\mathrm{nd}(L) \geq n - r + 1$, *then*

$$H^q(X, K_X + L) = 0 \qquad \text{for } q \geq r.$$

Proof. If $\mathrm{nd}(L) = n - r$, the proof is not difficult. In fact, since $c_1(L) + \pi^*\omega_T$ is nef and $\int_X (c_1(L) + \pi^*\omega_T)^n > 0$, [DP04, Theorem 0.5] implies the existence of a singular metric h_1 on L such that

$$i\Theta_{h_1}(L) \geq c\omega_X - \pi^*\omega_T$$

in the sense of currents for some constant $c > 0$. Thanks to Demailly's regularization theorem, we can suppose moreover that h_1 have analytic singularities. Note that L is nef. Then for any $\epsilon > 0$, there exists a smooth metric h_ϵ on L such that $i\Theta_{h_\epsilon}(L) \geq -\epsilon\omega_X$.

Now we define a new metric h on L :

$$h = \epsilon_1 h_1 + (1 - \epsilon_1) h_\epsilon$$

for some $1 \gg \epsilon_1 \gg \epsilon > 0$. Let $\omega_\tau = \tau\omega_X + \pi^*(\omega_T)$ for $\tau > 0$. We apply Proposition 3.2.2 to pair (L, h, ω_τ). By the same proof of Proposition 3.3.4, we can get

$$H^q(X, K_X + L) = 0 \qquad \text{for } q > r.$$

If $\mathrm{nd}(L) \geq n - r + 1$, Lemma 3.2.3 implies that

$$\int_T \pi_*(c_1(L)^{n-r+1}) \wedge \omega_T^{r-1} > 0 \tag{3.35}$$

for any Kähler class ω_T. Since T is a torus, we can represent the cohomology class $\pi_*(c_1(L)^{n-r+1})$ by a constant $(1,1)$-form

$$\sum_{i=1}^{r} \lambda_i dz_i \wedge d\overline{z}_i$$

on T. Since (3.35) is valid for any Kähler class ω_T, an elementary computation shows that $\lambda_i \geq 0$ for any i. Thus $\pi_*(c_1(L)^{n-r+1})$ is a semipositive (non trivial) class in $H^{1,1}(T) \cap H^2(T, \mathbb{Q})$. Using Lemma 3.2.1, we get a submersion

$$\varphi : T \to S$$

where S is an abelian variety of dimension s, and

$$\pi_*(c_1(-K_X)^{n-r+1}) = \lambda\, \varphi^* A$$

for some $\lambda > 0$ and a very ample divisor A on S.

Let S_1 be a complete intersection of divisors of $(s-1)$ general elements in $H^0(S, \mathscr{O}_S(A))$ and set $X_1 := (\varphi \circ \pi)^{-1}(S_1)$ and $T_1 := \varphi^{-1}(S_1)$ is of dimension $r - s + 1$. Then we get a morphism

$$X_1 \xrightarrow{\pi|_{X_1}} T_1 \xrightarrow{\varphi|_{T_1}} S_1 \tag{1}$$

and X_1 is smooth by Bertini's theorem. Moreover, we have also the equality

$$(\pi|_{X_1})_*(c_1(L)^{n-r+1}) = \lambda \cdot (\varphi|_{T_1})^* A|_{S_1}. \tag{2}$$

Applying Proposition 3.3.4 to (X_1, L), we get

$$H^q(X_1, K_{X_1} + L) = 0 \qquad \text{for } q \geq \dim T_1.$$

The theorem is finally proved by a standard exact sequece argument. $\square$

As a consequence, we obtain :

Theorem 3.3.6. *Let X be a compact Kähler manifold of dimension n with nef anticanonical bundle. Let $\pi : X \to T$ be a surjection to a torus T of dimension r. If $-K_X$ is π-big, then* $\mathrm{nd}(-K_X) = n - r$.

Proof. We suppose by contradiction that $\mathrm{nd}(-K_X) \geq n - r + 1$. By Theorem 3.3.5, we have

$$H^r(X, \mathscr{O}_X) = H^r(X, K_X - K_X) = 0. \tag{3.36}$$

Using [Anc87, Thm.2.1], we have

$$R^j \pi_*(K_X - K_{X/T}) = 0 \qquad \text{for } j \geq 1. \tag{3.37}$$

By the Leray spectral sequence, (3.36) and (3.37) imply that

$$H^r(T, \mathscr{O}_T) = 0.$$

We thus get a contradiction. □

Chapitre 4

On the approximation of Kähler manifolds by algebraic varieties

4.1 Introduction

It is well known that the curvature of the canonical bundle controls the structure of projective varieties. C.Voisin has given a counterexample to the Kodaira conjecture , showing that one cannot always deform a compact Kähler manifold to a projective manifold. In her counterexample one can see that the canonical bundle is neither nef nor anti-nef. Therefore it is interesting to ask whether for a Kähler manifold with a nef or anti-nef canonical bundle, one can deform it to a projective variety. More precisely,

Definition 4.1.1. *Let X be a compact Kähler manifold. We say that X can be approximated by projective varieties, if there exists a deformation of $X : \mathscr{X} \to \Delta$ such that the central fiber X_0 is X, and there exists a sequence $t_i \to 0$ in Δ such that all the fibers X_{t_i} are projective.*

In this chapter, we discuss the deformation properties of Kähler manifolds in the following three cases :
(1) Compact Kähler manifolds with hermitian semipositive anticanonical bundles.
(2) Compact Kähler manifolds with real analytic metrics and nonpositive bisectional curvatures.
(3) Compact Kähler manifolds with nef tangent bundles.

The main result of chapter is

Main Theorem. *If X is a compact Kähler manifold in one of the above three classes, then X can be approximated by projective varieties.*

The proof for these three types of manifolds relies on their respective structure theorems. We first sketch the strategy of the proof when X is a compact Kähler manifold with hermitian semipositive anticanonical bundle. We first recall that a compact Kähler manifold X is said to be deformation unobstructed, if there exists a smooth deformation of X, $\pi : \mathscr{X} \to \Delta$, such that the Kodaira-Spencer map $T_\Delta \to H^1(X, T_X)$ is surjective. For this type of manifolds, we have the following proposition :

Proposition 3.3 in [Voi05]. *Assume that a deformation unobstructed compact Kähler manifold X has a Kähler class ω satisfying the following condition : the interior product*

$$\omega \wedge : H^1(X, T_X) \to H^2(X, \mathscr{O}_X)$$

is surjective. Then X can be approximated by projective varieties.

In [DPS96], it is proved that after a finite cover, a compact Kähler manifold with hermitian semipositive anticanonical bundle has a smooth fibration to a compact Kähler manifold with trivial canonical bundle and the fibers Y_t satisfy the vanishing property :

$$H^q(Y_t, \mathscr{O}_{Y_t}) = 0 \qquad \text{for } q \geq 1.$$

Therefore the Dolbeault cohomology of X is easy to calculate. One can thus construct explicitly a deformation of X satisfying the surjectivity in in [Voi05a, Proposition 3.3]. Therefore this type of manifolds can be approximated by projective varieties.

When X is a compact Kähler manifold with nef tangent bundle, the proof is more difficult. It is based on the structure theorem in [DPS94] which can be stated as follows.

Theorem 4.1.1. *Let X be a compact Kähler manifold with nef tangent bundle T_X. Let $\widetilde{X}$ be a finite étale cover of X of maximum irregularity $q = h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$. Then the Albanese map $\pi : \widetilde{X} \to T$ is a smooth fibration over a q-dimensional torus, and $-K_{\widetilde{X}}$ is relatively ample.*

Remark 4.1.2. *We will prove that after passing to some finite Galois cover $\widetilde{X} \to X$ with group G, there exists a commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ T & \longrightarrow & T/G \end{array}$$

and T/G is smooth.

In [DPS94], when X is a projective variety with nef tangent bundle, it is proved that $\pi_*(-mK_X)$ is numerically flat for all $m \geq 1$. Combining Bo Berndtsson's formula 4.7 [Ber09] and Theorem 3.3.5 in the last chapter, we can also prove that

Theorem 4.1.3. *Let X be a compact Kähler manifold of dimension n with nef tangent bundle such that the Albanese map $\pi : X \to T$ is a smooth fibration onto a torus T of dimension r, and $-K_X$ is relatively ample. Then $\mathrm{nd}(-K_X) = n - r$, and $\pi_*(-mK_X)$ is numerically flat for all $m \geq 1$.*

We combine this with a result in [Sim92] which states that any numerically flat bundle over a compact Kähler manifold is in fact a local system :

Theorem 4.1.4. *Let E be a numerically flat holomorphic vector bundle on a Galois quotient of a torus T, then E is a local system.*

Using Theorem 4.1.3 and Theorem 4.1.4, we will see that one can approximate Kähler manifolds with nef tangent bundles by projective varieties.
Acknowledgements : I would like to thank my supervisor J-P.Demailly for helpful discussions and his kindness in sharing his ideas. I would also like to thank C.Voisin who explained to me that [Voi05b, Proposition 3.3] could be used to prove certain approximation problems during a summer school in Norway, and C.Simpson who told me that the results in [Sim92] could largely simplify the original proof of Theorem 4.1.4.

4.2 Deformation of compact Kähler manifolds with hermitian semipositive anticanonical bundles or nonpositive bisectional curvatures

We first treat a special case, i.e., how to appproximate compact manifolds with numerically trivial canonical bundles by projective varieties. To prove the statement, we need the following two propositions.

Proposition 3.3 in [Voi05]. *Assume that a deformation unobstructed compact Kähler manifold X has a Kähler class ω satisfying the following condition : the interior product*

$$\omega \wedge : H^1(X, T_X) \to H^2(X, \mathscr{O}_X)$$

is surjective. Then X can be approximated by projective varieties.

Remark 4.2.1. *The proof of this proposition is based on a density criterion (cf. [Voi07, Proposition 5.20]) which will also be used in Proposition 4.2.6 and Proposition 4.2.9. We need moreover a slightly generalized version of [Voi05b, Proposition 3.3]. In fact, we can suppose ω to be a nef class in X, since the surjectivity is preserved under small perturbation. Moreover, if X is not necessarily unobstructed, we just need a deformation unobstructed subspace V of $H^1(X, T_X)$ such that*

$$\omega \wedge V \to H^2(X, \mathscr{O}_X)$$

is surjective. In summary, we have the following variant of the above proposition.

Version B of Proposition 3.3 in [Voi05]. *Let $\mathscr{X} \to \Delta$ be a deformation of a compact Kähler manifold X and V be the image of Kodaira-Spencer map of this deformation. If there exists a nef class ω in $H^{1,1}(X)$ such that*

$$\omega \wedge V \to H^2(X, \mathscr{O}_X)$$

is surjective, then there exists a sequence $t_i \to 0$ in Δ such that all the fibers X_{t_i} are projective.

In general, it is difficult to check the surjectivity in the above proposition. By a well-known observation communicated to us by J-P. Demailly, one can prove that the above morphism is surjective when $-K_X$ is hermitian semipositive by using the following Hard Lefschetz theorem.

Hard Lefschetz theorem. *(cf. [Dem12, Corollary 15.2]) Let (L, h) be a semi-positive line bundle on a compact Kähler manifold (X, ω) of dimension n i.e., h is a smooth metric on L and $i\Theta_h(L) \geq 0$. Then the wedge multiplication operator $\omega^q\wedge$ induces a surjective morphism*

$$\omega^q \wedge : H^0(X, \Omega_X^{n-q} \otimes L) \to H^q(X, \Omega_X^n \otimes L).$$

Using the above two propositions, we can reprove the following well-known fact.

Proposition 4.2.2. *Let X be a compact Kähler manifold with $c_1(X)_{\mathbb{R}} = 0$. Then it can be approximated by projective varieties.*

Proof. By a theorem due to Beauville, there exists a finite Galois cover $\widetilde{X} \to X$ such that $K_{\widetilde{X}}$ is trivial. Then K_X is a torsion line bundle. Using the Tian-Todorov theorem (cf. the torsion version in [Ran92]), X is unobstructed. To prove Proposition 4.2.2, by [Voi05b, Proposition 3.3], it is sufficient to check that

$$\omega \wedge : H^1(X, T_X) \to H^2(X, \mathscr{O}_X) \tag{4.1}$$

is surjective for some Kähler class ω.

In fact, since $c_1(K_X)_{\mathbb{R}} = 0$, there exists a smooth metric h on $-K_X$ such that $i\Theta_h(-K_X) = 0$. Thus $(-K_X, h)$ is semipositive. Then the Hard Lefschetz theorem above told us that for any Kähler metric ω, the morphism

$$\omega^2 \wedge : H^0(X, \Omega_X^{n-2} \otimes (-K_X)) \to H^2(X, K_X \otimes (-K_X)) \tag{4.2}$$

is surjective. Observing moreover that the image of (4.2) is contained in the image of

$$\omega \wedge H^1(X, \Omega_X^{n-1} \otimes (-K_X)) = \omega \wedge H^1(X, T_X),$$

i.e., the image of (4.1). Then (4.1) is surjective. Using [Voi05b, Proposition 3.3], the proposition is proved. □

We now begin to prove the main proposition in this section, i.e., one can approximate compact Kähler manifolds with hermitian semipositive anticanonical bundles by projective varieties. The main tool is the following structure theorem in [DPS96] :

Structure Theorem. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then*

(i) The universal cover $\widetilde{X}$ admits a holomorphic and isometric splitting

$$\widetilde{X} = \mathbb{C}^q \times Y_1 \times Y_2$$

with Y_1 being the product of either Calabi-Yau manifolds or symplectic manifolds, and Y_2 being projective. Moreover $H^0(Y_2, \Omega_{Y_2}^{\otimes q}) = 0$ for $q \geq 1$.

(ii) There is a normal subgroup $\Gamma_1 \subset \pi_1(X)$ of finite index such that $\widehat{X} = \widetilde{X}/\Gamma_1$ has a smooth fibration to a Ricci-flat compact manifold : $F = (\mathbb{C}^q \times Y_1)/\Gamma_1$ with fibers Y_2.

Remark 4.2.3. *Since $\Omega_{Y_2}^q \subset \Omega_{Y_2}^{\otimes q}$, the above structure theorem implies that*

$$H^0(Y_2, \Omega_{Y_2}^q) = 0.$$

Therefore $H^q(Y_2, \mathscr{O}_{Y_2}) = 0$ by duality.

Remark 4.2.4. *The Ricci semipositive metric on X induces a $\pi_1(X)$-invariant metric ω_{Y_2} on Y_2. Thanks to Remark 4.2.3, we can suppose that $\omega_{Y_2} \in H^{1,1}(Y_2, \mathbb{Q})$. Therefore ω_{Y_2} induces a rational coefficience, closed semipositive $(1,1)$-form on $\widehat{X}$, which is strictly positive on the fibers of the fibration in (ii) of the above Structure Theorem.*

We need also the following lemma.

Lemma 4.2.5. *Let X be a compact Kähler manifold with $K_X = \mathscr{O}_X$, and G a finite subgroup of the biholomorphic group $\mathrm{Aut}(X)$. Then there exists a local deformation of X : $\mathscr{X} \to \Delta$ such that the image of the Kodaira-Spencer map of this deformation is equal to $H^1(X, T_X)^{G-\mathrm{inv}}$ and $\mathscr{X}$ admits a holomorphic G-action fiberwise, where $H^1(X, T_X)^{G-\mathrm{inv}}$ is the G-invariant subspace of $H^1(X, T_X)$.*

Proof. By the Kuranishi deformation theory, it is sufficient to construct a vector valued $(0,1)$-form

$$\varphi(t) = \sum_{k_i \geq 0} \varphi_{k_1 \cdots k_m} t_1^{k_1} \cdots t_m^{k_m}$$

such that

$$\varphi(0) = 0 \qquad \text{and} \qquad \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)], \tag{4.3}$$

where $\varphi_{k_1 \cdots k_m}$ are G-invariant vector valued $(0,1)$-forms, $\{\varphi_{k_1 \cdots k_m}\}_{\sum k_i = 1}$ gives a basis of $H^1(X, T_X)^{G-\mathrm{inv}}$ and $t_1, ..., t_m$ are parameters of Δ. By [MK06], solving (4.3) is equivalent to find G-invariant vector valued $(0,1)$-forms φ_μ such that

$$\overline{\partial}\varphi_\mu = \frac{1}{2} \sum_{|\lambda| < |\mu|} [\varphi_\lambda, \varphi_{\mu - \lambda}] \tag{4.4}$$

for any μ.

Suppose that we have already found φ_μ for $|\mu| \leq N$ such that (4.4) is satisfied for all $|\mu| \leq N$. If $|\mu| = N+1$, thanks to [Tia87], there exists a vector valued $(0,1)$-form s_μ satisfying

$$\overline{\partial} s_\mu = \frac{1}{2} \sum_{|\lambda| \leq N} [\varphi_\lambda, \varphi_{\mu - \lambda}].$$

Recall that if Y_1, Y_2 are two G-invariant vector valued $(0,1)$-forms, then $[Y_1, Y_2]$ is also a G-invariant vector valued $(0,2)$-form [1]. Therefore $\overline{\partial} s_\mu$ is a G-invariant vector valued $(0,2)$-form. The finiteness of G and the above G-invariance of $\overline{\partial} s_\mu$ imply hence that $\frac{1}{|G|} \sum_{g \in G} g^* s_\mu$ is a G-invariant vector valued $(0,1)$-form satisfying (4.4). The lemma is proved. □

1. Let $\alpha \in G$, $f \in C^\infty(X)$ and $x \in X$. Using the G-invariance of Y_1 and Y_2, we have $\alpha^*(Y_1 Y_2)(f)(x) = Y_1 Y_2(f \circ \alpha)(\alpha^{-1}(x)) = Y_1(Y_2(f) \circ \alpha)(\alpha^{-1}(x)) = Y_1(Y_2(f))(x)$. Thus $[Y_1, Y_2]$ is also G-invariant.

The following proposition tells us that for a compact Kähler manifold with numerically trivial canonical bundle, if it admits "more automorphisms", then it is "more algebraic". More precisely, we have

Proposition 4.2.6. *Let $\pi : \mathscr{X} \to \Delta$ be the deformation constructed in Lemma 4.2.5. Then there exists a sequence $t_i \to 0 \in \Delta$ such that X_{t_i} are projective varieties.*

Proof. We first prove that $H^2(X,\mathbb{Q})^{G-\mathrm{inv}}$ admits a sub-Hodge structure of $H^2(X,\mathbb{Q})$. In fact, we have the equality

$$H^2(X,\mathbb{Q})^{G-\mathrm{inv}} \otimes \mathbb{R} = H^2(X,\mathbb{R})^{G-\mathrm{inv}} \tag{4.5}$$

by observing that the elements in G act continuous on $H^2(X,\mathbb{R})$. Combining (4.5) with the obvious Hodge decomposition

$$H^2(X,\mathbb{C})^{G-\mathrm{inv}} = \oplus_{p+q=2} H^{p,q}(X,\mathbb{C})^{G-\mathrm{inv}},$$

$H^2(X,\mathbb{Q})^{G-\mathrm{inv}}$ is thus a sub-Hodge structure of $H^2(X,\mathbb{Q})$. Then π induces a VHS of $H^2(X,\mathbb{Q})^{G-\mathrm{inv}}$.

Let ω_X be a G-invariant Kähler metric on X. (4.1) of Proposition 4.2.2 implies that

$$\omega_X \wedge H^1(X,T_X) \to H^2(X,\mathscr{O}_X)$$

is surjective. Thanks to the G-invariance of ω_X,

$$\omega_X \wedge H^1(X,T_X)^{G-\mathrm{inv}} \to H^2(X,\mathscr{O}_X)^{G-\mathrm{inv}}$$

is also surjective. Using the density criterion [Voi07, Proposition 5.20] and the same argument of [Voi05b, Proposition 3.3], the proposition is proved. □

We now prove the main result in this section.

Theorem 4.2.7. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then it can be approximated by projective varieties.*

Proof. We prove it in three steps.

Step 1 : We use the terminology of the Structure Theorem in this section. Let $G = \pi_1(X)/\Gamma_1$ and $\widehat{X} = \widetilde{X}/\Gamma_1$. Then G acts on $\widehat{X}$. We have $X = \widehat{X}/G$. Thanks to *(ii)* of the Structure Theorem in this section, we have a smooth fibration

$$\pi : \widehat{X} \to F \tag{4.6}$$

with the fibers Y_2. We prove in this step that

$$H^q(\widehat{X},\mathscr{O}_{\widehat{X}}) = \pi^*(H^q(F,\mathscr{O}_F)) \tag{4.7}$$

and

$$H^q(\widehat{X},\mathscr{O}_{\widehat{X}})^{G-\mathrm{inv}} = \pi^*(H^q(F,\mathscr{O}_F)^{G-\mathrm{inv}}), \tag{4.8}$$

for any q.

Using (4.6), we can calculate $H^q(\widehat{X},\mathscr{O}_{\widehat{X}})$ by the Leray spectral sequence. Then (4.7) comes directly from the fact that

$$H^q(Y_2,\mathscr{O}_{Y_2}) = 0 \qquad \text{for } q \geq 1$$

(cf. Remark 4.2.3 of the Structure Theorem in this section). To prove (4.8), we need to check that the image of the injective map

$$\pi^* : H^q(F,\mathscr{O}_F)^{G-\mathrm{inv}} \to H^q(\widehat{X},\mathscr{O}_{\widehat{X}}) \tag{4.9}$$

is $H^q(\widehat{X}, \mathscr{O}_{\widehat{X}})^{G-\mathrm{inv}}$. Let $\gamma \in G$ and α a smooth differential form on F. Since $\pi_1(X)$ acts on $\mathbb{C}^q \times Y_1$ and Y_2 separately, we have the diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\gamma} & \widehat{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ F & \xrightarrow{\gamma} & F \end{array}$$

Then the equality

$$\gamma^*(\pi^*\alpha) = \pi^*(\gamma^*\alpha)$$

implies that the image of (4.9) is contained in $H^q(\widehat{X}, \mathscr{O}_{\widehat{X}})^{G-\mathrm{inv}}$. To prove that $H^q(\widehat{X}, \mathscr{O}_{\widehat{X}})^{G-\mathrm{inv}}$ is in the image of (4.9), we first take an element $\beta \in H^q(\widehat{X}, \mathscr{O}_{\widehat{X}})^{G-\mathrm{inv}}$. Thanks to the proved equality (4.7), we can find an element $\mu \in H^q(F, \mathscr{O}_F)$ such that $\pi^*\mu = \beta$ as an element in $H^q(\widehat{X}, \mathscr{O}_{\widehat{X}})$. Since

$$\pi^*(\gamma^*\mu) = \gamma^*(\pi^*\mu) = \gamma^*(\beta) = \beta = \pi^*(\mu)$$

in $H^q(\widehat{X}, \mathscr{O}_{\widehat{X}})$, the injectivity of (4.9) implies that $\gamma^*(\mu) = \mu$ in $H^q(F, \mathscr{O}_F)$. Then μ is G-invariant. Therefore (4.9) gives an isomorphism from $H^q(F, \mathscr{O}_F)^{G-\mathrm{inv}}$ to $H^q(\widehat{X}, \mathscr{O}_{\widehat{X}})^{G-\mathrm{inv}}$. (4.8) is proved.

Step 2 : Let $\omega_F^{G-\mathrm{inv}}$ be a G-invariant Kähler metric on F. We construct in this step a deformation of F : $\mathscr{F} \to \Delta$ such that

$$\omega_F^{G-\mathrm{inv}} \wedge V_1 \to H^2(F, \mathscr{O}_F)^{G-\mathrm{inv}} \tag{4.10}$$

is surjective, where V_1 is the image of the Kodaira-Spencer map of this deformation. Moreover, $\mathscr{F}$ should admit a holomorphic G-action fiberwise.

In fact, using Lemma 4.2.5, there exists a deformation of F admitting a holomorphic G-action fiberwise. Moreover, the image of the Kodaira-Spencer map of this deformation is $H^1(F, T_F)^{G-\mathrm{inv}}$. We now check (4.10) for this deformation. Since $c_1(F)_{\mathbb{R}} = 0$ by construction, the proof of Proposition 4.2.2 implies that

$$\omega_F^{G-\mathrm{inv}} \wedge H^1(F, T_F) \to H^2(F, \mathscr{O}_F)$$

is surjective. Then

$$\omega_F^{G-\mathrm{inv}} \wedge H^1(F, T_F)^{G-\mathrm{inv}} \to H^2(F, \mathscr{O}_F)^{G-\mathrm{inv}} \tag{4.11}$$

is also surjective. Step 2 is proved.

Step 3 : Final conclusion.

Since $\widehat{X}$ is the quotient of $\Gamma_1 \curvearrowright \mathbb{C}^q \times Y_1 \times Y_2$ and Γ_1 acts on $\mathbb{C}^q \times Y_1$ and Y_2 separately, the deformation of $F = (\mathbb{C}^q \times Y_1)/\Gamma_1$ in Step 2 induces a deformation of $\widehat{X}$:

$$\widehat{\mathscr{X}} \to \Delta$$

by preserving the complex structure of Y_2. By construction, we have a natural fibration

$$\widehat{\pi} : \widehat{\mathscr{X}} \to \mathscr{F}.$$

Moreover, since G acts holomorphic on the fibers of $\mathscr{F}$ over Δ, the quotient $\mathscr{X} = \widehat{\mathscr{X}}/G$ is a smooth deformation of X. In summary, we have the following diagrams :

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{G} & X = \widehat{X}/G \\ \downarrow{\scriptstyle \pi} & & \\ F & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \widehat{\mathscr{X}} & \xrightarrow{G} & \mathscr{X} = \widehat{\mathscr{X}}/G \\ \downarrow{\scriptstyle \widehat{\pi}} & & \\ \mathscr{F} & & \end{array}.$$

Let X_t, F_t be the fibers of $\mathscr{X}$ and $\mathscr{F}$ over $t \in \Delta$. Thanks to Proposition 4.2.6, there exists a sequence $t_i \to 0 \in \Delta$ such that F_{t_i} are projective. Combining this with Remark 4.2.4 after the Structure Theorem in this section, we obtain that X_{t_i} are projective. The proposition is proved. □

Remark 4.2.8. *For the further application, we need to study the deformation $\mathscr{X}$ in detail. Let* $\mathrm{pr} : \widehat{X} \to X$ *be the quotient. Since* $\pi^*\omega_F^{G-\mathrm{inv}}$ *is a G-invariant semipositive form on* $\widehat{X}$*, we can find a nef class* α *on* X *such that* $\mathrm{pr}^*(\alpha) = \pi^*\omega_F^{G-\mathrm{inv}}$*. Let* V *be the image of Kodaira-Spencer map of the deformation* $\mathscr{X} \to \Delta$*. Our goal is prove that*

$$\alpha \wedge V \to H^2(X, \mathscr{O}_X) \tag{4.12}$$

is surjective. Thanks to the construction of $\widehat{\mathscr{X}}$ *and the surjectivity of* (4.10)*, the morphism*

$$\pi^*\omega_F^{G-\mathrm{inv}} \wedge W \to \pi^*(H^2(F, \mathscr{O}_F)^{G-\mathrm{inv}}) \tag{4.13}$$

is surjective on $\widehat{X}$*, where* W *is the image of Kodaira-Spencer map of the deformation* $\widehat{\mathscr{X}} \to \Delta$*. Combining* (4.13) *with* (4.8)*,*

$$\pi^*\omega_F^{G-\mathrm{inv}} \wedge W \to H^q(\widehat{X}, \mathscr{O}_{\widehat{X}})^{G-\mathrm{inv}}$$

is surjective. Hence (4.12) *is surjective.*

As an application, we prove [BDPP04, Conjecture 2.3 and Conjecture 10.1] for compact Kähler manifolds with hermitian semipositive anticanonical bundles.

Proposition 4.2.9. *If* X *is a compact Kähler manifold with* $-K_X$ *hermitian semipositive, then the Conjecture 2.3 and Conjecture 10.1 in [BDPP04] are all true, namely :*

(i) *: The pseudo-effective cone* $\mathscr{E} \subset H^{1,1}_{\mathbb{R}}(X)$ *and the movable cone* $\mathscr{M} \subset H^{n-1,n-1}_{\mathbb{R}}(X)$ *are dual. (cf. [BDPP04, Definition 1.2, 1.3] for the definition of* $\mathscr{E}$ *and* $\mathscr{M}$*)*

(ii) *: Let* α *be a closed,* $(1,1)$*-form on* X*. If* $\int_{X(\alpha,\leq 1)} \alpha^n > 0$ *(cf. [BDPP04, Conjecture 10.1] for the definition of* $X(\alpha, \leq 1)$*), the class* (α) *contains a Kähler current, and*

$$\mathrm{vol}(\alpha) \geq \int_{X(\alpha,\leq 1)} \alpha^n.$$

Proof. By Remark 4.2.8 after Theorem 4.2.7, there exists a local deformation of X

$$\pi : \mathscr{X} \to \Delta,$$

such that

$$\alpha \wedge V \to H^2(X, \mathscr{O}_X) \tag{4.14}$$

is surjective for some nef class $\alpha \in H^{1,1}(X, \mathbb{R})$, where V is the image of the Kodaira-Spencer map of π.

Let β be any smooth closed $(1,1)$-form on X. Thanks to the surjectivity of (4.14),

$$(\beta + s\alpha) \wedge V \to H^2(X, \mathscr{O}_X)$$

is also surjective for any $s \neq 0$ small enough. By the proof of [Voi07, Proposition 5.20], we can hence find a sequence of smooth closed 2-forms $\{\beta_t\}$ on X, such that

$$\lim_{t\to 0} \beta_t = \beta + s\alpha$$

in C^∞-topology and $\beta_t \in H^{1,1}(X_t, \mathbb{Q})$. By the same argument as in [BDPP04, Theorem 10.12], the proposition is proved. □

We now study the case when X has a real analytic metric and nonpositive bisectional curvatures. Recall first the structure theorem [WZ02, Theorem E]

Proposition 4.2.10. *Let X be a compact Kähler manifold of dimension n with real analytic metric and nonpositive bisectional curvature, and let $\widetilde{X}$ be its universal cover. Then*

(i) There exists a holomorphically isometric decomposition $\widetilde{X} = \mathbb{C}^{n-r} \times Y^r$, where Y^r is a complete manifold with nonpositve bisectional curvature and the Ricci tensor of Y^r is strictly negative somewhere.

(ii) (cf. [WZ02, Claim 2, Theorem E]) There exists a finite index sub-normal group Γ' of $\Gamma = \pi_1(X)$ such that Y^r/Γ' is a compact manifold and $\widetilde{X}/\Gamma'$ possess the smooth fibrations to Y^r/Γ' and $\mathbb{C}^{n-r}/\Gamma'$.

Remark 4.2.11. *By [WZ02, Claim 2, Theorem E], $\mathbb{C}^{n-r}/\Gamma'$ is a torus. We should notice that in contrast to the case when $-K_X$ is semipositive, Y^r is not necessary compact in this proposition. The universal covers of curves of genus $g \geq 2$ are typical exemples. The good news here is that Y^r/Γ' is a projective variety of general type thanks to (i).*

Proposition 4.2.12. *Let X be a compact Kähler manifold of dimension n with real analytic metric and nonpositive bisectional curvature. Then it can be approximated by projective varieties.*

Proof. Keeping the notation in Proposition 4.2.10, we know that $T = \mathbb{C}^{n-r}/\Gamma'$ is a torus with a finite group action $G = \Gamma/\Gamma'$. Let $\widehat{X} = \widetilde{X}/\Gamma'$. By Lemma 4.2.5, there exists a deformation of T

$$\pi : \mathscr{T} \to \Delta$$

such that G acts holomorphically fiberwise. Therefore this deformation induces the deformations of $\widehat{X}$ and X by preserving the complex structure on Y^r. We denote

$$\widehat{\mathscr{X}} \to \Delta \qquad \text{and} \qquad \mathscr{X} \to \Delta. \tag{4.15}$$

Thanks to the construction, X_t is the G-quotient of $\widetilde{X}_t/\Gamma'$, where X_t and $\widetilde{X}_t/\Gamma'$ are the fibers over $t \in \Delta$ of the above deformations.

Let $t_i \to 0$ be the sequence in Proposition 4.2.6 such that T_{t_i} are projective. By Proposition 4.2.10, we have two fibrations :

$$\widehat{X}_{t_i} \to T_{t_i} \qquad \text{and} \qquad \widehat{X}_{t_i} \to Y^r/\Gamma'.$$

Thanks to the projectivity of T_{t_i} and Remark 4.2.11 of Proposition 4.2.10, $\widehat{X}_{t_i}$ is thus projective. Therefore X_{t_i} is projective and the proposition is proved. □

4.3 A deformation lemma

The following two sections are devoted to the deformation problem of compact Kähler manifolds with nef tangent bundles. We discuss in this section how to deform varieties that are defined by certain numerically flat equations.

Proposition 4.3.1. *Let E be a numerically flat bundle on a compact Kähler manifold. Then E is a local system.*

Proof. Thanks to Thoerem 1.18 in [DPS94], all numerically flat vector bundles are successive extensions of hermitian flat bundles. By [Sim92, Corollary 3.10], all such types of bundles are local systems. The proposition is proved. □

Remark 4.3.2. *The proof use a deep result of [Sim92]. When X is just a finite étale quotient of a torus, we give a more elementary proof in the Appendix.*

Lemma 4.3.3. *Let X be a compact Kähler manifold and let E be a numerically flat vector bundle on X possessing a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E \tag{4.16}$$

such that the quotients E_i/E_{i-1} are irreducible hermitian flat vector bundles. Then E is a local system and all elements in $H^0(X, E)$ are parallel with respect to the natural local system induced by the filtration (4.16).

In particular, if there are two such filtrations, the transformation matrices between these two induced local systems should be locally constant.

Proof. Thanks to [Sim92, Corollary 3.10], the filtration (4.16) induces a natural local system on E and the natural Gauss-Manin connection on E preserves the filtration (4.16) (i.e., the connection on each successive quotient E_i/E_{i-1} induced by the Gauss-Manin connection on E is the natural hermitian flat connection on E_i/E_{i-1}). Using the recurrence process, to prove that all elements in $H^0(X,E)$ are parallel with respect to the local system, it is sufficient to prove that if E is a non trivial extension

$$0 \longrightarrow E_{m-1} \xrightarrow{\ i\ } E \longrightarrow E_m/E_{m-1} \longrightarrow 0, \tag{4.17}$$

then $H^0(X,E) = i(H^0(X,E_{m-1}))$. To prove this, we first note that (4.17) implies the exact sequence

$$H^0(X,E_{m-1}) \xrightarrow{\ i\ } H^0(X,E) \longrightarrow H^0(X,E_m/E_{m-1}) \xrightarrow{\ \delta\ } H^1(X,E_{m-1}).$$

Case 1 : $E_m/E_{m-1} \neq \mathscr{O}_X$. Since E_m/E_{m-1} is an irreducible hermitian flat bundle, we have

$$H^0(X,E_m/E_{m-1}) = 0. \tag{4.18}$$

Using the above exact sequence, we obtain $H^0(X,E) = i(H^0(X,E_{m-1}))$.

Case 2 : $E_m/E_{m-1} = \mathscr{O}_X$. Since $h^0(X,\mathscr{O}_X) = 1$ and E is a non trivial extension, we obtain that δ in the exact sequence is injective. Therefore $i(H^0(X,E_{m-1})) = H^0(X,E)$. By recurrence, all elements in $H^0(X,E)$ should be parallel with respect to the natural local system induced by (4.16).

For the second part of the lemma, if there is another filtration

$$0 = E'_0 \subset E'_1 \subset \cdots \subset E'_n = E,$$

then it induces a filtration on E^*. Using this filtration on E^* and the filtration (4.16) on E, we get a natural filtration on $\mathrm{Hom}(E,E) = E^* \otimes E$. Applying the first part of the lemma, the natural identity element $\mathrm{id} \in H^0(X,\mathrm{Hom}(E,E))$ should be parallel with respect to the filtration. In other words, the transformation matrices between these two filtrations should be locally constant. □

Remark 4.3.4. *We should remark that for a general local system on a compact Kähler manifold, the global sections may not be parallel with respect to the flat connection.*

Remark 4.3.5. *We should remark that all irreducible hermitian flat vector bundles on torus are in fact of rank one. To see this, for any irreducible hermitian flat vector bundle V of rank r on a torus T, it is defined by a representation $G : \pi_1(T) \to U_r$ by holonomy. Since $\pi_1(T)$ is abelian, then $G(\pi_1(T))$ are commutative with each other. Therefore we can diagonlise them simultaneously, and the irreducible condition implies that $r = 1$.*

Proposition 4.3.6. *Let X be a compact Kähler manifold which admits a surjective morphism $\pi : X \to T$ to a compact Kähler manifold T. If there exists a relative ample line bundle L on X such that $E_m = \pi_*(mL)$ is numerically flat for $m \gg 1$, and $S_{m,d} = \pi_*(\mathscr{I}_X \otimes \mathscr{O}_{\mathbb{P}(E_m)}(d))$ is also also numerically flat for $d \gg 1$, then the fibration π is locally trivial.*

Proof. We first explain the definition of $S_{m,d}$. Since L is relatively ample, we have the embedding

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}(E_m) \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle\pi} & \\ T & & \end{array}$$

Then $S_{m,d}$ is the direct image of the coherent sheaf $\mathscr{I}_X \otimes \mathscr{O}_{\mathbb{P}(E_m)}(d)$ on $\mathbb{P}(E_m)$. When $d \gg 1$, we have a natural inclusion

$$i : S_{m,d} \hookrightarrow \pi_*(\mathscr{O}_{\mathbb{P}(E_m)}(d)) = E_{m\cdot d} \qquad \text{on } T.$$

By assumption, $S_{m,d}$ and $E_{m\cdot d}$ are local systems on T. Let U be any small Stein open set in T, and let $e_1, \cdots, e_k$ be a local constant coordinate of $S_{m,d}$ over U. Note $\mathrm{Hom}(S_{m,d}, E_{m\cdot d})$ is also a local system on T, and $i \in H^0(T, \mathrm{Hom}(S_{m,d}, E_{m\cdot d}))$. Thanks to Lemma 4.3.3, i is parallel with respect to the local system $\mathrm{Hom}(S_{m,d}, E_{m\cdot d})$. Therefore the images of $e_1, \cdots, e_k$ in $E_{m\cdot d}$ are also locally constant, i.e. the determinant polynomials of the fibers X_t for $t \in U$ are locally constant. In particular, the fibration π is locally trivial. □

Proposition 4.3.7. *Let X be a Kähler manifold possessing a submersion $\pi : X \to T$, where T is a finite étale quotient of a torus. Assume that $-K_X$ is nef and relatively ample. If moreover $E_m = \pi_*(-mK_X)$ is numerically flat for any $m \gg 0$, then there is a smooth deformation of the fibration which can be realized as :*

$$\mathscr{X} \xrightarrow{\pi} \mathscr{T} \xrightarrow{\pi_1} \Delta$$

such that $\pi_1 : \mathscr{T} \to \Delta$ is the local universal deformation of T and the central fiber is $X \to T$.

Moreover, let T_s be the fiber of π_1 over $s \in \Delta$, and let X_s be the fiber of $\pi \circ \pi_1$ over $s \in \Delta$. Then the anticanonical bundle of X_s is also nef and relatively ample with respect to the fibration $X_s \to T_s$ for any $s \in \Delta$.

Proof. Thanks to [DPS94, Theorem 3.20], we have the embeddings $X \hookrightarrow \mathbb{P}(E_m)$ and $V_{m,p} = \pi_*(\mathscr{I}_X \otimes \mathscr{O}_{\mathbb{P}(E_m)}(p)) \subset S^p E_m$ for m, p large enough. More importantly, the numerically flatness of E_m imply that $V_{m,p}$ and $S^p E_m$ are numerically flat vector bundles. By Proposition 4.3.1, $V_{m,p}, S^p E_m$ are local systems on T. Thanks to [Ran92, Proposition 2.3], the deformation of T is unobstructed.

Let $\pi_1 : \mathscr{T} \to \Delta$ be the local universal deformation of T. Since

$$S^p E_m, V_{m,p}$$

are local systems, $S^p E_m$ and $V_{m,p}$ are holomorphic under the deformation of the complex structure on T. Therefore we get the holomorphic deformations of these vector bundles by changing the complex strucutre on T :

$$\begin{array}{ccc} \mathscr{V}_{m,p} & \hookrightarrow & S^p\mathscr{E}_m \\ & \searrow & \downarrow \\ U \times s & \hookrightarrow & \mathscr{T} \\ \downarrow & & \downarrow \pi_1 \\ s & \hookrightarrow & \Delta \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathscr{V}_{m,p} \times \mathbb{P}(\mathscr{E}_m) . \\ \downarrow \\ \mathscr{T} \\ \downarrow \pi_1 \\ \Delta \end{array}$$

By the proof of Proposition 4.3.6, on any small open neighborhood $U \subset T$, we can choose a local basis of $V_{m,p}$ over U to have constant coefficiences. By the discussion after [DPS94, Proposition 3.19], a local basis of $V_{m,p}$ gives the determinant polynomials of X in $\mathbb{P}(E_m)$ over U. Then the defining equations $V_{m,p}$ over $U \times s$ are the same as $V_{m,p}$ over $U \times \{0\}$ for $s \in \Delta$. Therefore $\mathscr{V}_{m,p}$ defines a smooth deformation of X, we denote it

$$\mathscr{X} \xrightarrow{\pi} \mathscr{T} \xrightarrow{\pi_1} \Delta.$$

As for the second part of the proposition, we first prove that $-K_{X_t}$ is ample on X_t where X_t is the fiber of $\mathscr{X} \to \mathscr{T}$ over $t \in \mathscr{T}$ and t is in a neighborhood of T in $\mathscr{T}$. Let $t_0 \in T$. Since $-K_{X_{t_0}}$ is ample, by [Yau78] there exists a Kähler metric ω_{t_0} on X_{t_0} such that $i\Theta_{\omega_{t_0}}(-K_{X_{t_0}}) > 0$. By a standard continuity argument (cf. [Sch07, Theorem 3.1] for exemple), we can construct Kähler metrics ω_t on X_t for t in a neighborhood of t_0 in $\mathscr{T}$ and by continuity the curvatures $i\Theta_{\omega_t}(-K_{X_t})$ are all positive for t in a neighborhood of t_0 in $\mathscr{T}$. Therefore $-K_{X_t}$ is ample on X_t for t near t_0 in $\mathscr{T}$. Letting t_0 run over T, then $-K_{X_t}$ is ample for all t in a neighborhood of T in $\mathscr{T}$.

We need also prove that $-K_{X_s}$ is nef on X_s, where X_s is the fiber of $\pi \circ \pi_1$ over $s \in \Delta$. Let $(E_m)_s$ be the fiber of $\mathscr{E}_m \to \Delta$ over s. By construction, $(E_m)_s$ is numerically flat on T_s, where T_s is fiber of π_1 over s. Then $\mathscr{O}_{\mathbb{P}(\mathscr{E}_m)}(1)$ is nef on $\mathbb{P}(E_m)_s$. Since X_s is embedded in $\mathbb{P}(E_m)_s$, $\mathscr{O}_{\mathbb{P}(\mathscr{E}_m)}(1)|_{X_s}$ is also nef for any $s \in \Delta$. If $s = 0$, we have

$$\mathscr{O}_{\mathbb{P}(\mathscr{E}_m)}(1)|_{X_s} = -mK_X.$$

Therefore

$$c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E}_m)}(1)|_{X_s}) = c_1(-mK_{X_s})$$

for $s \in \Delta$ by the rigidity of integral classes. Then the nefness of $\mathscr{O}_{\mathbb{P}(\mathscr{E}_m)}(1)|_{X_s}$ implies that $-mK_{X_s}$ is nef for all $s \in \Delta$.

The proposition is proved.

□

Remark 4.3.8. *In general, the nefness is not an open condition in families (cf. [Laz04, Theorem 1.2.17]). Thanks to the construction, the nefness is preserved under deformation under our special case.*

Thanks to Proposition 4.3.7, we have immediately a corollary.

Corollary 4.3.9. *Let X be a compact Kähler manifold satisfying the condition in Proposition 4.3.7. Then X can be approximated by projective varieties. Moreover,* $\mathrm{nd}(-K_X) = n - \dim T$.

Proof. We keep the notations in Proposition 4.3.7. By Proposition 4.3.7, there exists a deformation of $X \to T$:

$$\mathscr{X} \xrightarrow{\pi} \mathscr{T} \xrightarrow{\pi_1} \Delta$$

such that $\mathscr{T} \to \Delta$ is the local universal deformation of T and $X \to T$ is the central fiber of this deformation. By Proposition 4.2.2, there exists a sequence $s_i \to 0$ in Δ such that all T_{s_i} are projective. Using Proposition 4.3.7, we know that the fibers of

$$X_{s_i} \to T_{s_i}$$

are Fano manifolds. Then all X_{s_i} are projective and X can be approximated by projective manifolds.

As for the second part of the corollary, by observing that $-K_X$ is relatively ample, we have $\mathrm{nd}(-K_X) \geq n - r$. If $\mathrm{nd}(-K_X) \geq n - r + 1$, by the definition of numerical dimension we have

$$\int_X (-K_X)^{n-r+1} \wedge \omega_X^{r-1} > 0.$$

By continuity,

$$\int_{X_{s_i}} (-K_{X_{s_i}})^{n-r+1} \wedge \omega_{X_{s_i}}^{r-1} > 0 \tag{4.19}$$

for $|s_i| \ll 1$. Thanks to Proposition 4.3.7, $-K_{X_{s_i}}$ are nef. Then (4.19) implies the existence of a projective variety X_{s_i} such that $-K_{X_{s_i}}$ is nef and $\mathrm{nd}(-K_{X_{s_i}}) \geq n - r + 1$, which contradicts with the Kawamata-Viehweg vanishing theorem for projective varieties. We get a contradiction and the corollary is proved. □

4.4 Deformation of compact Kähler manifolds with nef tangent bundles

Proposition 4.4.1. *Let X be a compact Kähler manifold possessing a smooth submersion $\pi : X \to T$ to a compact Kähler manifold T. If $-K_X$ is nef on X and is relatively ample for π, then the direct image*

$$E = \pi_*(K_{X/T} - (m+1)K_X)$$

is a nef vector bundle for all $m \in \mathbb{N}$.

Proof. Let us first show that the direct image E is locally free. Let X_t be the fiber of π over $t \in T$. Thanks to the Kodaira vanishing theorem, we have

$$H^q(X_t, -mK_{X_t}) = 0 \qquad \text{for } q \geq 1.$$

By the Riemann-Roch theorem,

$$\sum_q (-1)^q h^q(X_t, -mK_{X_t})$$

is a constant independent of t. Therefore $h^0(X_t, -mK_{X_t})$ is also a constant and by a standard result of H.Grauert, the direct image

$$E = \pi_*(K_{X/T} - (m+1)K_X)$$

is locally free.

Since $-(m+1)K_X$ is also nef, for any $\epsilon > 0$ fixed, there exists a smooth metric φ on $-(m+1)K_X$ such that

$$i\Theta_\varphi(-(m+1)K_X) \geq -\epsilon\omega_T.$$

Since E is known to be locally free, we can use formula (4.8) in [Ber09]. In particular, φ gives a metric on E and we write its curvature as

$$\Theta^E = \sum_{j,k} \Theta^E_{jk,\varphi} dt_j \wedge d\bar{t}_k$$

where $\{t_i\}$ are the coordinates of T. Using the terminology in [Ber09], we assume that $\{u_i\}$ is a base of local holomorphic sections of E such that $D^{1,0}u_i = 0$ at a given point. We now calculate the curvature at this point. Let

$$T_u = \sum_{j,k} (u_j, u_k) \widehat{dt_j \wedge d\bar{t}_k}.$$

Then

$$i\partial\bar{\partial} T_u = -\sum_{j,k} (\Theta^E_{jk,\varphi} u_j, u_k) dV_t.$$

By the formula (4.8) in [Ber09], we obtain[2]

$$-i\partial\bar{\partial} T_u \geq c\pi_*(\widehat{u} \wedge \overline{\widehat{u}} \wedge i\partial\bar{\partial}\varphi e^{-\varphi})$$

where the constant c is independent of φ. Since $i\partial\bar{\partial}\varphi \geq -\epsilon\omega_T$ by the choice of φ, we have

$$-i\partial\bar{\partial} T_u \geq -c\epsilon\pi_*(\widehat{u} \wedge \overline{\widehat{u}} \wedge \omega_T e^{-\varphi})$$

$$= -c\epsilon\Big(\int_{X_t} \sum_j (u_j, u_j) e^{-\varphi}\Big) dV_t$$

$$= -c\epsilon \|u\|^2 dV_t.$$

In other words, we have

$$\sum_{j,k} (\Theta^E_{jk,\varphi} u_j, u_k) \geq -c\epsilon \|u\|^2.$$

The proposition is proved. □

As a corollary of the main theorem in [DPS94], we prove that every compact Kähler manifold with nef tangent bundle admits a smooth fibration to an étale Galois quotient of a torus.

Lemma 4.4.2. *Let X be a compact Kähler manifold with nef tangent bundle and let $\widetilde{X} \to X$ be an étale Galois cover with group G such that $\widetilde{X}$ satisfies Theorem 4.1.1 (i.e. Main theorem in [DPS94]). Then G induces a free automorphism group on $T = \mathrm{Alb}(\widetilde{X})$ and we have the following commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ T & \longrightarrow & T/G \end{array}$$

where $\widetilde{\pi} : \widetilde{X} \to T$ is the Albanese map in Theorem 4.1.1, and T/G is an étale Galois quotient of the torus T.

2. The $i\partial\bar{\partial}\varphi$ below is just $i\Theta_\varphi(-mK_X)$.

Proof. By the universal property of Albanese map, for any $g \in G$, g induces an automorphism on T, and the action of g on $\widetilde{X}$ maps fibers to fibers. We need hence only to prove that G acts on T freely.

Suppose by contradiction that $g(t_0) = t_0$ for some $t_0 \in T$ and $g \in G$. Let $\langle g \rangle$ be the subgroup generated by g. Since g acts on $\widetilde{X}$ without fixed point, g induces an automorphism on $\widetilde{X}_{t_0}$ without fixed points, where $\widetilde{X}_{t_0}$ is the fiber of $\widetilde{\pi}$ over t_0. By the same reason, any non trivial elements in $\langle g \rangle$ induces an automorphism on $\widetilde{X}_{t_0}$ without fixed points. Combining this with the fact that $\widetilde{X}_{t_0}$ is a Fano manifold, the quotient $\widetilde{X}_{t_0}/\langle g \rangle$ is hence also a Fano manifold. Thus the Nadel vanishing theorem implies that

$$\chi(\widetilde{X}_{t_0}, \mathscr{O}_{\widetilde{X}_{t_0}}) = \chi(\widetilde{X}_{t_0}/\langle g \rangle, \mathscr{O}_{\widetilde{X}_{t_0}/\langle g \rangle}) = 1. \tag{4.20}$$

(4.20) contradicts with the fact that the étale cover $\widetilde{X}_{t_0} \to \widetilde{X}_{t_0}/\langle g \rangle$ implies

$$\chi(\widetilde{X}_{t_0}, \mathscr{O}_{\widetilde{X}_{t_0}}) = |\langle g \rangle| \cdot \chi(\widetilde{X}_{t_0}/\langle g \rangle, \mathscr{O}_{\widetilde{X}_{t_0}/\langle g \rangle}).$$

Then G factorizes to an étale Galois action on T, and the lemma is proved. □

Now we can prove our main result :

Theorem 4.4.3. *Let X be a compact Kähler manifold of dimension n with nef tangent bundle. Then X can be approximated by projective varieties.*

Proof. By Lemma 4.4.2, there exists a finite étale Galois cover $\widetilde{X} \to X$ with group G such that one has a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ T & \longrightarrow & T/G \end{array}$$

where the fibers of π are Fano manifolds. We suppose that $\dim T = r$. Thanks to Theorem 3.3.6, we get $\mathrm{nd}(-K_{\widetilde{X}}) = n - r$, which is equivalent to say that $\mathrm{nd}(-K_X) = n - d$.

Let $E_m = \pi_*(-mK_X)$, for $m \geq 1$. Since $K_{T/G}$ is flat, by Proposition 4.4.1, E_m is a nef vector bundle. By the Riemann-Roch-Grothendick theorem, we have

$$\mathrm{Ch}(E_m) = \pi_*(\mathrm{Ch}(-K_X)\,\mathrm{Todd}(T_X)). \tag{4.21}$$

Since we proved that $\mathrm{nd}(-K_X) = n - r$, (4.21) implies that $c_1(E_m) = 0$ by using [DPS94, Corollary 2.6]. E_m is thus numerically flat by definition. Using Corollary 4.3.9, we conclude that X can be approximated by projective varieties.

□

Chapitre 5

Compact Kähler manifolds with nef anticanonical bundles

5.1 Introduction

Compact Kähler manifolds with semipositive anticanonical bundles have been studied in depth in [CDP12], where a rather general structure theorem for this type of manifolds has been obtained. It is a natural question to find some similar structure theorems for compact Kähler manifolds with nef anticanonical bundles. Obviously, we cannot hope the same structure theorem for this type of manifolds (cf. [CDP12, Remark 1.7]). It is conjectured that the Albanese map is a submersion and that the fibers exhibit no variation of their complex structure.

In relation with the structure of compact Kähler manifolds with nef anticanonical bundles, it is conjectured in [Pet12, Conj. 1.3] that the tangent bundles of projective manifolds with nef anticanonical bundles are generically nef. We first recall the notion of generically semipositive (resp. strictly positive) (cf. [Miy87, Section 6])

Definition 5.1.1. *Let X be a compact Kähler manifold and let E be a vector bundle on X. Let $\omega_1, \cdots, \omega_{n-1}$ be Kähler classes. Let*

$$0 \subset \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = E \qquad (\textit{resp. } \Omega_X^1)$$

be the Harder-Narasimhan semistable filtration with respect to $(\omega_1, \cdots, \omega_{n-1})$. We say that E is generically $(\omega_1, \cdots, \omega_{n-1})$-semipositive (resp. strictly positive), if

$$\int_X c_1(\mathscr{E}_{i+1}/\mathscr{E}_i) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-1} \geq 0 \qquad (\textit{ resp.} > 0) \qquad \textit{for all } i.$$

If $\omega_1 = \cdots = \omega_{n-1}$, we write the polarization as ω_1^{n-1} for simplicity.

We rephrase [Pet12, Conj. 1.3] as follows

Conjecture 5.1.1. *Let X be a projective manifold with nef anticanonical bundle. Then T_X is generically $(H_1, \cdots, H_{n-1})$-semipositive for any $(n-1)$-tuple of ample divisors $H_1, \cdots, H_{n-1}$.*

In this article, we first give a partial positive answer to this conjecture. More precisely, we prove

Theorem 5.1.2. *Let X be a compact Kähler manifold with nef anticanonical bundle (resp. nef canonical bundle). Then T_X (resp. Ω_X^1) is generically ω_X^{n-1}-semipositive for any Kähler class ω_X.*

Remark 5.1.3. *If X is projective and K_X is nef, Theorem 5.1.2 is a special case of [Miy87, Cor. 6.4]. Here we prove it for arbitrary compact Kähler manifolds with nef canonical bundles. If $-K_X$ is nef, Theorem 5.1.2 is a new result even for algebraic manifolds.*

As an application, we give a characterization of rationally connected compact Kähler manifolds with nef anticanonical bundles.

Proposition 5.1.4. *Let X be a compact Kähler manifold with nef anticanonical bundle. Then the following four conditions are equivalent*

(i) : $H^0(X,(T_X^*)^{\otimes m})=0$ *for all* $m\geq 1$.

(ii) : *X is rationally connected.*

(iii) : *T_X is generically ω_X^{n-1}-strictly positive for some Kähler class ω_X.*

(iv) : *T_X is generically ω_X^{n-1}-strictly positive for any Kähler class ω_X.*

Remark 5.1.5. *Mumford has in fact stated the following conjectured which would generalize the first part of Proposition 5.1.4 : for any compact Kähler manifold X, X is rationally connected if and only if*

$$H^0(X,(T_X^*)^{\otimes m})=0 \qquad \textit{for all } m\geq 1.$$

We thus prove the conjecture of Mumford under the assumption that $-K_X$ is nef.

As another application, we study the effectiveness of $c_2(T_X)$. It is conjectured by Kawamata that

Conjecture 5.1.6. *If X is a compact Kähler manifold with nef anticanonical bundle. Then*

$$\int_X (c_2(T_X))\wedge\omega_1\wedge\cdots\wedge\omega_{n-2}\geq 0,$$

for all nef classes $\omega_1,\cdots,\omega_{n-2}$.

When $\dim X=3$, this conjecture was solved by [Xie05]. Using Theorem 5.1.2 and an idea of A.Höring, we prove

Proposition 5.1.7. *Let (X,ω_X) be a compact Kähler manifold with nef anticanonical bundle. Then*

$$\int_X c_2(T_X)\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2}\geq 0 \tag{5.1}$$

for $\epsilon>0$ small enough. Moreover, if X is projective and the equality holds for some $\epsilon>0$ small enough, then after a finite étale cover, X is either a torus or a smooth $\mathbb{P}^1$-fibration over a torus.

As the last application, we study the Albanese map of compact Kähler manifolds with nef anticanonical bundles. It should be first mentioned that the surjectivity of the Albanese map has been studied in depth by several authors. If X is assumed to be projective, the surjectivity of the Albanese map was proved by Q.Zhang in [Zha96]. Still under the assumption that X is projective, [LTZZ10] proved that the Albanese map is equidimensional and all the fibres are reduced. Recently, M.Păun [Pău12b] proved the surjectivity for arbitrary compact Kähler manifolds with nef anticanonical bundles, as a corollary of a powerful method based on a direct image argument. Unfortunately, it is hard to get information for the singular fibers from his proof. Using Theorem 5.1.2, we give a new proof of the surjectivity for the Kähler case, and prove that the map is smooth outside a subvariety of codimension at least 2.

Proposition 5.1.8. *Let X be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map is surjective, and smooth outside a subvariety of codimension at least* 2. *In particular, the fibers of the Albanese map are connected and reduced in codimension* 1.

5.2 Preparatory lemmas

The results in this section should be well known to experts. For the convenience of readers, we give an account of the proofs here.

Lemma 5.2.1. *Let (X,ω) be a compact Kähler manifold of dimension n and let E be a torsion free coherent sheaf. Let $D_1,\cdots,D_{n-1}$ be nef classes in $H^{1,1}(X,\mathbb{Q})$ and let A be a Kähler class. Let a be a sufficiently small positive number. Then the Harder-Narasimhan semistable filtration of E with respect to $(D_1+a\cdot A,\cdots,D_{n-1}+a\cdot A)$ is independent of a.*

Remark 5.2.2. *If A has rational coefficients, Lemma 5.2.1 is proved in [KMM04]. When A is not necessarily rational, the proof turns out to be a little bit more complicated. We begin with the following easy observation.*

Lemma 5.2.3. *In the situation of Lemma 5.2.1, le we take $k\in\{0,1,2,\cdots,n\}$ arbitrary. Then we can find a basis $\{e_1,\cdots,e_s\}$ of $H^{2k}(X,\mathbb{Q})$ depending only on A^k, such that*

(i) *: $A^k=\sum\limits_{i=1}^{s}\lambda_i\cdot e_i$ for some $\lambda_i>0$.*

(ii) *: Let $\mathscr{F}$ be a torsion free coherent sheaf. Set*

$$D^t:=\sum_{i_1<i_2<\cdots<i_t} D_{i_1}\cdot D_{i_2}\cdots D_{i_t}\qquad \textit{for any } t,$$

and

$$a_i(\mathscr{F}):=c_1(\mathscr{F})\cdot D^{n-k-1}\cdot e_i. \tag{5.2}$$

Then the subset S of the set $\mathbb{Q}$ of rational numbers such that

$$S:=\{a_i(\mathscr{F})|\ \mathscr{F}\subset E, i\in\{1,\cdots s\}\}$$

is bounded from above, and the denominator (assumed positive) of all elements of S is uniformly bounded from above. Moreover, if $\{\mathscr{F}_t\}_t$ is a sequence of coherent subsheaves of E such that the set $\{c_1(\mathscr{F}_t)\cdot D^{n-k-1}\cdot A^k\}_t$ is bounded from below, then $\{c_1(\mathscr{F}_t)\cdot D^{n-k-1}\cdot A^k\}_t$ is a finite subset of $\mathbb{Q}$.

Proof. We can take a basis $\{e_i\}_{i=1}^s$ of $H^{2k}(X,\mathbb{Q})$ in a neigborhood of A^k, such that

$$A^k=\sum_{i=1}^{s}\lambda_i\cdot e_i\qquad \text{for some } \lambda_i>0$$

and $(e_i)^{k,k}$ can be represented by a smooth (k,k)-positive form on X (cf. [Dem, Chapter 3, Def 1.1] for the definition of (k,k)-positivity), where $(e)^{k,k}$ is the projection of e in $H^{k,k}(X,\mathbb{R})$. We now check that $\{e_i\}_{i=1}^s$ satisfies the lemma. By construction, (i) is satisfied. As for (ii), since e_i and D_i are fixed and $c_1(\mathscr{F})\in H^{1,1}(X,\mathbb{Z})$, the denominator of any elements in S is uniformly bounded from above. Thanks to (5.2), we know that S is bounded from above by using the same argument as in [Kob87, Lemma 7.16, Chapter 5]. For the last part of (ii), since

$$c_1(\mathscr{F}_t)\cdot D^{n-k-1}\cdot A^k=\sum_i a_i(\mathscr{F}_t)\cdot\lambda_i,$$

we obtain that $\{\sum\limits_i a_i(\mathscr{F}_t)\cdot\lambda_i\}_t$ is uniformly bounded. Since $\lambda_i>0$ and $a_i(\mathscr{F}_t)$ is uniformly upper bounded, we obtain that $a_i(\mathscr{F}_t)$ is uniformly bounded. Combining this with the fact already proved that the denominator of any elements in S is uniformly bounded, $\{c_1(\mathscr{F}_t)\cdot D^{n-k-1}\cdot A^k\}_t$ is thus finite. The lemma is proved. □

Proof of Lemma 5.2.1. Let $\{a_p\}_{p=1}^{+\infty}$ be a decreasing positive sequence converging to 0. Let $\mathscr{F}_p\subset E$ be the first piece of the Harder-Narasimhan filtration of E with respect to $(D_1+a_p\cdot A,\cdots,D_{n-1}+a_p\cdot A)$. Set $D^k:=\sum\limits_{i_1<i_2<\cdots<i_k} D_{i_1}\cdot D_{i_2}\cdots D_{i_k}$. Then

$$c_1(\mathscr{F}_p)\wedge(D_1+a_p\cdot A)\wedge\cdots\wedge(D_{n-1}+a_p\cdot A)=\sum_{k=0}^{n}(a_p)^k\cdot c_1(\mathscr{F}_p)\wedge D^{n-k-1}\wedge A^k.$$

By passing to a subsequence, we can suppose that $\mathrm{rk}\,\mathscr{F}_p$ is constant. To prove Lemma 5.2.1, it is sufficient to prove that for any k, after passing to a subsequence, the intersection number $\{c_1(\mathscr{F}_p) \wedge D^{n-k-1} \wedge A^k\}_p$ is stationary when p is large enough [1].

We prove it by induction on k. Note first that, by [Kob87, Lemma 7.16, Chapter 5], the set $\{c_1(\mathscr{F}_p) \wedge D^{n-k-1} \wedge A^k\}_{p,k}$ is upper bounded. If $k = 0$, since

$$c_1(\mathscr{F}_p) \wedge (D_1 + a_p \cdot A) \wedge \cdots \wedge (D_{n-1} + a_p \cdot A)$$

$$\geq \frac{\mathrm{rk}(\mathscr{F}_p)}{\mathrm{rk}\,E} c_1(E) \wedge (D_1 + a_p \cdot A) \wedge \cdots \wedge (D_{n-1} + a_p \cdot A),$$

and $\lim_{p \to +\infty} a_p = 0$, the upper boundedness of $\{c_1(\mathscr{F}_p) \wedge D^{n-k-1} \wedge A^k\}_{p,k}$ implies that the set $\{c_1(\mathscr{F}_p) \wedge D^{n-1}\}_{p=1}^{\infty}$ is bounded from below. Then *(ii)* of Lemma 5.2.3 implies that $\{c_1(\mathscr{F}_p) \wedge D^{n-1}\}_{p=1}^{\infty}$ is a finite set. By the pigeon hole principle, after passing to a subsequence, the set $\{c_1(\mathscr{F}_p) \wedge D^{n-1}\}_{p=1}^{\infty}$ is stationary. Now we suppose that $\{c_1(\mathscr{F}_p) \wedge D^{n-t-1} \wedge A^t\}_p$ is constant for $p \geq p_0$, where $t \in \{0, \cdots, k-1\}$. Our aim is to prove that after passing to a subsequence,

$$\{c_1(\mathscr{F}_p) \wedge D^{n-k-1} \wedge A^k\}_{p=1}^{\infty}$$

is stationary. By definition, we have

$$c_1(\mathscr{F}_p) \wedge (D_1 + a_p \cdot A) \wedge \cdots \wedge (D_{n-1} + a_p \cdot A)$$

$$\geq c_1(\mathscr{F}_{p_0}) \wedge (D_1 + a_p \cdot A) \wedge \cdots \wedge (D_{n-1} + a_p \cdot A) \qquad \text{for any } p \geq p_0.$$

Since $\{c_1(\mathscr{F}_p) \wedge D^{n-t-1} \wedge A^t\}_p$ is constant for $p \geq p_0$ when $t \in \{0, \cdots, k-1\}$, we obtain

$$c_1(\mathscr{F}_p) \wedge D^{n-k-1} \wedge A^k + \sum_{i \geq 1} (a_p)^i \cdot c_1(\mathscr{F}_p) \wedge D^{n-k-1-i} \wedge A^{k+i} \geq \tag{5.4}$$

$$c_1(\mathscr{F}_{p_0}) \wedge D^{n-k-1} \wedge A^k + \sum_{i \geq 1} (a_p)^i \cdot c_1(\mathscr{F}_{p_0}) \wedge D^{n-k-1-i} \wedge A^{k+i}$$

for any $p \geq p_0$. Therefore the upper boundedness of $\{c_1(\mathscr{F}_p) \wedge D^{n-k-1-i} \wedge A^{k+i}\}_{p,i}$ implies that $\{c_1(\mathscr{F}^p) \wedge D^{n-k-1} \wedge A^k\}_{p=1}^{+\infty}$ is lower bounded. Therefore

$$\{c_1(\mathscr{F}^p) \wedge D^{n-k-1} \wedge A^k\}_{p=1}^{+\infty}$$

is uniformly bounded. Using *(ii)* of Lemma 5.2.3, $\{c_1(\mathscr{F}^p) \wedge D^{n-k-1} \wedge A^k\}_{p=1}^{+\infty}$ is a finite set. By the pigeon hole principle, after passing to a subsequence,

$$\{c_1(\mathscr{F}_p) \wedge D^{n-k-1} \cdot A^k\}_{p=1}^{\infty}$$

is stationary. The lemma is proved. □

By the same argument as above, we can easily prove that

Lemma 5.2.4. *Let (X, ω) be a compact Kähler manifold and let E be a torsion free ω-stable coherent sheaf. Then E is also stable with respect to a small perturbation of ω.*

---

1. In fact, if $\mathscr{F} \subset E$ is always the first piece of semistable filtration with repsect to the polarization $(D_1 + a_p \cdot A, \cdots, D_{n-1} + a_p \cdot A)$ for a positive sequence $\{a_p\}_{p=0}^{+\infty}$ converging to 0, and $\mathscr{G} \subset E$ is always the first piece of semistable filtration for another sequence $\{b_p\}_{p=0}^{+\infty}$ coverging to 0, the stability condition implies that

$$\mathrm{rk}(\mathscr{G}) \cdot c_1(\mathscr{F}) \cdot D^k \cdot A^{n-k-1} = \mathrm{rk}(\mathscr{F}) \cdot c_1(\mathscr{G}) \cdot D^k \cdot A^{n-k-1} \tag{5.3}$$

for any k. Therefore $\mathscr{G}$ has the same slope as $\mathscr{F}$ with respect to $(D_1 + a \cdot A, \cdots, D_{n-1} + a \cdot A)$ for any $a > 0$. Then $\mathscr{F} = \mathscr{G}$.

Remark 5.2.5. *If the Kähler metric $\omega \in H^2(X,\mathbb{Z})$, the lemma comes directly from the fact that*

$$\{\int_X c_1(\mathscr{F})\wedge\omega^{n-1} | \mathscr{F} \text{ a coherent subsheaf of } \mathscr{E} \text{ with strictly smaller rank }\}$$

is a discrete subset.

We recall a regularization lemma proved in [Jac10, Prop. 3].

Lemma 5.2.6. *Let E be a vector bundle on a compact complex manifold X and $\mathscr{F}$ be a subsheaf of E with torsion free quotient. Then after a finite number of blowups $\pi : \widetilde{X} \to X$, there exists a holomorphic subbundle F of $\pi^*(E)$ containing $\pi^*(\mathscr{F})$ with a holomorphic quotient bundle, such that $\pi_*(F) = \mathscr{F}$ in codimension 1.*

We need another lemma which is proved in full generality in [DPS94, Prop. 1.15]. For completeness, we give the proof here in an over simplified case, but the idea is the same.

Lemma 5.2.7. *Let (X,ω) be a compact Kähler manifold. Let E be an extension of two vector bundles E_1, E_2*

$$0 \to E_1 \to E \to E_2 \to 0.$$

We suppose that there exist two smooth metrics h_1, h_2 on E_1 and E_2, such that

$$\frac{i\Theta_{h_1}(E_1)\wedge\omega^{n-1}}{\omega^n} \geq c_1\cdot \mathrm{Id}_{E_1} \qquad \text{and} \qquad \frac{i\Theta_{h_2}(E_2)\wedge\omega^{n-1}}{\omega^n} \geq c_2\cdot \mathrm{Id}_{E_2} \tag{5.5}$$

pointwise. Then for any $\epsilon > 0$, there exists a smooth metric h_ϵ on E such that

$$\frac{i\Theta_{h_\epsilon}(E)\wedge\omega^{n-1}}{\omega^n} \geq (\min(c_1,c_2)-\epsilon)\cdot \mathrm{Id}_E,$$

and

$$\|i\Theta_{h_\epsilon}(E)\|_{L^\infty} \leq C\cdot(\|i\Theta_{h_1}(E_1)\|_{L^\infty} + \|i\Theta_{h_2}(E_2)\|_{L^\infty}) \tag{5.6}$$

for some uniform constant C independent of ϵ.

Proof. Let $[E] \in H^1(X, \mathrm{Hom}(E_2,E_1))$ be the element representing E in the extension group. Let E_s be another extension of E_1 and E_2, such that $[E_s] = s\cdot[E]$, where $s \in \mathbb{C}^*$. Then there exists an isomorphism between these two vector bundles (cf. [Dem, Remark 14.10, Chapter V]). We denote the isomorphism by

$$\varphi_s : E \to E_s.$$

Thanks to (5.5), if $|s|$ is small enough with respect to ϵ, we can find a smooth metric h_s on E_s satisfying

$$\frac{i\Theta_{h_s}(E_s)\wedge\omega^{n-1}}{\omega^n} \geq (\min(c_1,c_2)-\epsilon)\cdot \mathrm{Id}_{E_s} \tag{5.7}$$

and

$$\|i\Theta_{h_s}(E_s)\|_{L^\infty} \leq C\cdot(\|i\Theta_{h_1}(E_1)\|_{L^\infty} + \|i\Theta_{h_2}(E_2)\|_{L^\infty}) \tag{5.8}$$

for some uniform constant C (cf. [Dem, Prop 14.9, Chapter V]). Let $h = \varphi_s^*(h_s)$ be the induced metric on E. Then for any $\alpha \in E$,

$$\begin{aligned} \frac{\langle i\Theta_h(E)\alpha,\alpha\rangle_h}{\langle\alpha,\alpha\rangle_h} &= \frac{\langle \varphi_s^{-1}\circ i\Theta_{h_s}(E_s)\varphi_s(\alpha),\alpha\rangle_h}{\langle\alpha,\alpha\rangle_h} \\ &= \frac{\langle i\Theta_{h_s}(E_s)\varphi_s(\alpha),\varphi_s(\alpha)\rangle_{h_s}}{\langle\varphi_s(\alpha),\varphi_s(\alpha)\rangle_{h_s}}. \end{aligned} \tag{5.9}$$

Combining this with (5.7), we get

$$\frac{\langle i\Theta_h(E)\alpha,\alpha\rangle_h\wedge\omega^{n-1}}{\langle\alpha,\alpha\rangle_h\cdot\omega^n} \geq (\min(c_1,c_2)-\epsilon)\cdot \mathrm{Id}_E\,.$$

Moreover, (5.9) implies also (5.6). The lemma is proved. □

We recall the following well-known equality in Kähler geometry.

Proposition 5.2.8. *Let (X, ω_X) be a Kähler manifold of dimension n,* R *be the curvature tensor and* Ric *be the Ricci tensor (cf. the definition of [Zhe00, Section 7.5]). Let $i\Theta_{\omega_X}(T_X)$ be the curvature of T_X induced by ω_X. We have*

$$\langle \frac{i\Theta_{\omega_X}(T_X) \wedge \omega_X^{n-1}}{\omega_X^n} u, v\rangle_{\omega_X} = \mathrm{Ric}(u, \overline{v}).$$

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of T_X with respect to ω_X. By definition, we have

$$\langle \frac{i\Theta_{\omega_X}(T_X) \wedge \omega_X^{n-1}}{\omega_X^n} u, v\rangle_{\omega_X} = \sum_{1 \leq i \leq n} \langle i\Theta_{\omega_X}(T_X) u, v\rangle(e_i, \overline{e}_i) = \sum_{1 \leq i \leq n} \mathrm{R}(e_i, \overline{e}_i, u, \overline{v}).$$

By definition of the Ricci curvature (cf. [Zhe00, Page 180]), we have

$$\mathrm{Ric}(u, \overline{v}) = \sum_{1 \leq i \leq n} \mathrm{R}(u, \overline{v}, e_i, \overline{e}_i).$$

Combining this with the First Bianchi equality

$$\sum_{1 \leq i \leq n} \mathrm{R}(e_i, \overline{e}_i, u, \overline{v}) = \sum_{1 \leq i \leq n} \mathrm{R}(u, \overline{v}, e_i, \overline{e}_i),$$

the proposition is proved. □

5.3 Main theorem

We first prove Theorem 5.1.2 in the case when $-K_X$ is nef.

Theorem 5.3.1. *Let (X, ω) be a compact n-dimensional Kähler manifold with nef anticanonical bundle. Let*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = T_X \tag{5.10}$$

be a filtration of torsion-free subsheaves such that $\mathscr{E}_{i+1}/\mathscr{E}_i$ is an ω-stable torsion-free subsheaf of $T_X/\mathscr{E}_i$ of maximal slope[2]*. Let*

$$\mu(\mathscr{E}_{i+1}/\mathscr{E}_i) = \frac{1}{\mathrm{rk}(\mathscr{E}_{i+1}/\mathscr{E}_i)} \int_X c_1(\mathscr{E}_{i+1}/\mathscr{E}_i) \wedge \omega^{n-1}$$

be the slope of $\mathscr{E}_{i+1}/\mathscr{E}_i$ with respect to ω^{n-1}. Then

$$\mu(\mathscr{E}_{i+1}/\mathscr{E}_i) \geq 0 \qquad \text{for all } i.$$

Proof. We first consider a simplified case.

Case 1 : (5.10) *is regular, i.e., all $\mathscr{E}_i$, $\mathscr{E}_{i+1}/\mathscr{E}_i$ are vector bundles.*

By the stability condition, to prove the theorem, it is sufficient to prove that

$$\int_X c_1(T_X/\mathscr{E}_i) \wedge \omega^{n-1} \geq 0 \qquad \text{for any } i. \tag{5.11}$$

Thanks to the nefness of $-K_X$, for any $\epsilon > 0$, there exists a Kähler metric ω_ϵ in the same class of ω such that (cf. the proof of [DPS93, Thm. 1.1])

$$\mathrm{Ric}_{\omega_\epsilon} \geq -\epsilon\omega_\epsilon, \tag{5.12}$$

2. Using Lemma 5.2.4, one can prove the existence of such a filtration by a standard argument [HN75].

where $\mathrm{Ric}_{\omega_\epsilon}$ is the Ricci curvature with respect to the metric ω_ϵ. Thanks to Proposition 5.2.8, we have

$$\langle \frac{i\Theta_{\omega_\epsilon}(T_X)\wedge\omega_\epsilon^{n-1}}{\omega_\epsilon^n}\alpha,\alpha\rangle_{\omega_\epsilon} = \mathrm{Ric}_{\omega_\epsilon}(\alpha,\overline{\alpha}).$$

Then (5.12) implies a pointwise estimate

$$\frac{i\Theta_{\omega_\epsilon}(T_X)\wedge\omega_\epsilon^{n-1}}{\omega_\epsilon^n} \geq -\epsilon\cdot \mathrm{Id}_{T_X}\,. \tag{5.13}$$

Taking the induced metric on $T_X/\mathscr{E}_i$ (we also denote it by ω_ϵ), we get (cf. [Dem, Chapter V])

$$\frac{i\Theta_{\omega_\epsilon}(T_X/\mathscr{E}_i)\wedge\omega_\epsilon^{n-1}}{\omega_\epsilon^n} \geq -\epsilon\cdot \mathrm{Id}_{T_X/\mathscr{E}_i}\,. \tag{5.14}$$

Therefore

$$\int_X c_1(T_X/\mathscr{E}_i)\wedge\omega_\epsilon^{n-1} \geq -\mathrm{rk}(T_X/\mathscr{E}_i)\cdot\epsilon\int_X\omega_\epsilon^n.$$

Combining this with the fact that $[\omega_\epsilon]=[\omega]$, we get

$$\int_X c_1(T_X/\mathscr{E}_i)\wedge\omega^{n-1} = \int_X c_1(T_X/\mathscr{E}_i)\wedge\omega_\epsilon^{n-1} \geq -C\epsilon, \tag{5.15}$$

for some constant C. Letting $\epsilon\to 0$, (5.11) is proved.

Case 2 : The general case

By Lemma 5.2.6, there exists a desingularization $\pi:\widetilde{X}\to X$, such that $\pi^*(T_X)$ admits a filtration :

$$0\subset E_1\subset E_2\subset\cdots\subset\pi^*(T_X), \tag{5.16}$$

where $E_i, E_i/E_{i-1}$ are vector bundles and $\pi_*(E_i)=\mathscr{E}_i$ outside an analytic subset of codimension at least 2. Let $\widetilde{\mu}$ be the slope with respect to $\pi^*(\omega)$. Then

$$\widetilde{\mu}(E_i/E_{i-1}) = \mu(\mathscr{E}_i/\mathscr{E}_{i-1}) \tag{5.17}$$

(cf. [Jac10, Lemma 2]), and E_i/E_{i-1} is a $\pi^*(\omega)$-stable subsheaf of $\pi^*(T_X)/E_{i-1}$ of maximal slope (cf. Remark 5.3.2 after the proof).

We now prove that $\widetilde{\mu}(E_i/E_{i-1})\geq 0$. Thanks to (5.13), for any $\epsilon>0$ small enough, we have

$$\frac{i\Theta_{\pi^*\omega_\epsilon}(\pi^*(T_X))\wedge(\pi^*\omega_\epsilon)^{n-1}}{(\pi^*\omega_\epsilon)^n} \geq -\epsilon\cdot\mathrm{Id}_{\pi^*(T_X)},$$

which implies that

$$\frac{i\Theta_{\pi^*\omega_\epsilon}(\pi^*(T_X)/E_i)\wedge(\pi^*\omega_\epsilon)^{n-1}}{(\pi^*\omega_\epsilon)^n} \geq -\epsilon\cdot\mathrm{Id}_{\pi^*(T_X)/E_i}\,. \tag{5.18}$$

By the same argument as in Case 1, (5.18) and the maximal slope condition of E_{i+1}/E_i in $\pi^*(T_X)/E_i$ implies that

$$\widetilde{\mu}(E_{i+1}/E_i) = \frac{1}{\mathrm{rk}(E_{i+1}/E_i)}\int_{\widetilde{X}} c_1(E_{i+1}/E_i)\wedge\pi^*\omega^{n-1} \geq -C\epsilon$$

for some constant C independent of ϵ. Letting $\epsilon\to 0$, we get $\widetilde{\mu}(E_{i+1}/E_i)\geq 0$. Combining this with (5.17), the theorem is proved. □

Remark 5.3.2. *In the situation of* (5.16) *in Theorem 5.3.1, we would like to prove that E_i/E_{i-1} is also stable for $\pi^*\omega+\epsilon\omega_{\widetilde{X}}$ for any $\epsilon>0$ small enough.*

Proof of Remark 5.3.2. Let $\mathscr{F}$ be any coherent sheaf satisfying

$$E_{i-1} \subset \mathscr{F} \subset E_i \qquad \text{and} \qquad \operatorname{rk} \mathscr{F} < \operatorname{rk} E_i. \tag{5.19}$$

It is sufficient to prove that

$$\frac{1}{\operatorname{rk} \mathscr{F}/E_{i-1}} \int_{\widetilde{X}} c_1(\mathscr{F}/E_{i-1}) \wedge (\pi^*\omega + \epsilon\omega_{\widetilde{X}})^{n-1} < \widetilde{\mu}(E_i/E_{i-1}) \tag{5.20}$$

for a uniform $\epsilon > 0$, where $\widetilde{\mu}$ is the slope with respect to $\pi^*(\omega)$ as defined in Theorem 5.3.1.

Thanks to the formula

$$\int_{\widetilde{X}} c_1(\mathscr{F}) \wedge \pi^*(\omega)^{n-1} = \int_X c_1(\pi_*(\mathscr{F})) \wedge \omega^{n-1},$$

Lemma 5.2.4 and the stability condition of $\mathscr{E}_i/\mathscr{E}_{i-1}$ imply that the upper bound of the set

$$\{\frac{1}{\operatorname{rk} \mathscr{F}/E_{i-1}} \int_{\widetilde{X}} c_1(\mathscr{F}/E_{i-1}) \wedge \pi^*\omega^{n-1} | \ \mathscr{F} \text{ satisfies (5.19)}\}$$

is strictly smaller than $\widetilde{\mu}(E_i/E_{i-1})$. Combining this with the fact that

$$\int_{\widetilde{X}} c_1(\mathscr{F}) \wedge \omega_{\widetilde{X}}^s \wedge \pi^*(\omega)^{n-s-1}$$

is uniformly bounded from above for any s, (5.20) is proved. □

We now prove Theorem 5.1.2 in the case when K_X is nef.

Theorem 5.3.3. *Let (X, ω) be a compact Kähler manifold with nef canonical bundle. Let*

$$0 \subset \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = \Omega_X^1 \tag{5.21}$$

be a filtration of torsion-free subsheaves such that $\mathscr{E}_{i+1}/\mathscr{E}_i$ is an ω-stable torsion-free subsheaf of $T_X/\mathscr{E}_i$ of maximal slope. Then

$$\int_X c_1(\mathscr{E}_{i+1}/\mathscr{E}_i) \wedge \omega^{n-1} \geq 0 \qquad \textit{for all } i.$$

Proof. The proof is almost the same as Theorem 5.3.1. First of all, since K_X is nef, for any $\epsilon > 0$, there exists a smooth fonction ψ_ϵ on X, such that

$$\operatorname{Ric}_\omega + i\partial\overline{\partial}\psi_\epsilon \leq \epsilon\omega.$$

By solving the Monge-Ampère equation

$$(\omega + i\partial\overline{\partial}\varphi_\epsilon)^n = \omega^n \cdot e^{-\psi_\epsilon - \epsilon\varphi_\epsilon}, \tag{5.22}$$

we can construct a new Kähler metric ω_ϵ in the cohomology class of ω :

$$\omega_\epsilon := \omega + i\partial\overline{\partial}\varphi_\epsilon.$$

Thanks to (5.22), we have

$$\begin{aligned} \operatorname{Ric}_{\omega_\epsilon} &= \operatorname{Ric}_\omega + i\partial\overline{\partial}\psi_\epsilon + \epsilon i\partial\overline{\partial}\varphi_\epsilon \\ &\leq \epsilon\omega + \epsilon i\partial\overline{\partial}\varphi_\epsilon = \epsilon\omega_\epsilon. \end{aligned}$$

We first suppose that (5.21) is regular, i.e., $\mathscr{E}_i$ and $\mathscr{E}_{i+1}/\mathscr{E}_i$ are free for all i. Let $\alpha \in \Omega^1_{X,x}$ for some point $x \in X$ with norm $\|\alpha\|_{\omega_\epsilon} = 1$ and let α^* be the dual of α with respect to ω_ϵ. Then we have also a pointwise estimate at x :

$$\begin{aligned} \langle \frac{i\Theta_{\omega_\epsilon}(\Omega_X) \wedge \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \alpha, \alpha \rangle &= \langle - \frac{i\Theta_{\omega_\epsilon}(T_X) \wedge \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \alpha^*, \alpha^* \rangle \\ &= -\operatorname{Ric}_{\omega_\epsilon}(\alpha^*, \alpha^*) \geq -\epsilon. \end{aligned}$$

By the same proof as in Theorem 5.3.1, $\int_X c_1(\mathscr{E}_{i+1}/\mathscr{E}_i) \wedge \omega^{n-1}$ is semi-positive for any i. For the general case, the proof follows exactly the same line as in Theorem 5.3.1. □

5.4 Applications

As an application, we give a characterization of rationally connected compact Kähler manifolds with nef anticanonical bundles.

Proposition 5.4.1. *Let X be a compact Kähler manifold with nef anticanonical bundle. Then the following four conditions are equivalent*
(i) : $H^0(X,(T_X^*)^{\otimes m})=0$ *for all* $m\geq 1$.
(ii) : *X is rationally connected.*
(iii) : *T_X is generically ω_X^{n-1} strictly positive for some Kähler class ω_X.*
(iv) : *T_X is generically ω_X^{n-1} strictly positive for any Kähler class ω_X.*

Proof. The implications $(iv)\Rightarrow(iii)$, $(ii)\Rightarrow(i)$ are obvious. For the implication $(iii)\Rightarrow(ii)$, we first note that (iii) implies (i) by Bochner technique. Therefore X is projective and any Kähler class can be approximated by rational Kähler classes. Using [BM01, Theorem 0.1], (iii) implies (ii).

We now prove that $(i)\Rightarrow(iv)$. Let ω be any Kähler class. Let

$$0\subset\mathscr{E}_0\subset\mathscr{E}_1\subset\cdots\subset\mathscr{E}_s=T_X \tag{5.23}$$

be the Harder-Narasimhan semistable filtration with respect to ω^{n-1}. To prove (iv), it is sufficient to prove

$$\int_X c_1(T_X/\mathscr{E}_{s-1})\wedge\omega^{n-1}>0.$$

Recall that Theorem 5.3.1 implies already that

$$\int_X c_1(T_X/\mathscr{E}_{s-1})\wedge\omega^{n-1}\geq 0.$$

We suppose by contradiction that

$$\int_X c_1(T_X/\mathscr{E}_{s-1})\wedge\omega^{n-1}=0. \tag{5.24}$$

Let $\alpha\in H^{1,1}(X,\mathbb{R})$. We define new Kähler metrics $\omega_\epsilon=\omega+\epsilon\alpha$ for $|\epsilon|$ small enough. Thanks to [Miy87, Cor. 2.3], the ω_ϵ^{n-1}-semistable filtration of T_X is a refinement of (5.23). Therefore, Theorem 5.3.1 implies that

$$\int_X c_1(T_X/\mathscr{E}_{s-1})\wedge(\omega+\epsilon\alpha)^{n-1}\geq 0$$

for $|\epsilon|$ small enough. Then (5.24) implies that

$$\int_X c_1(T_X/\mathscr{E}_{s-1})\wedge\omega^{n-2}\wedge\alpha=0 \qquad \text{for any } \alpha\in H^{1,1}(X,\mathbb{R}).$$

By the Hodge index theorem, we obtain that $c_1(T_X/\mathscr{E}_{s-1})=0$. By duality, there exists a subsheaf $\mathscr{F}\subset\Omega_X^1$, such that

$$c_1(\mathscr{F})=0 \qquad \text{and} \qquad \det\mathscr{F}\subset(T_X^*)^{\otimes\operatorname{rk}\mathscr{F}}. \tag{5.25}$$

Observing that $H^1(X,\mathscr{O}_X)=0$ by assumption, i.e., the group $\operatorname{Pic}^0(X)$ is trivial, hence (5.25) implies the existence of an integer m such that $(\det\mathscr{F})^{\otimes m}$ is a trivial line bundle. Observing moreover that $(\det\mathscr{F})^{\otimes m}\subset(T_X^*)^{\otimes m\cdot\operatorname{rk}\mathscr{F}}$, then

$$H^0(X,(T_X^*)^{\otimes m\cdot\operatorname{rk}\mathscr{F}})\neq 0,$$

which contradicts with (i). The implication $(i)\Rightarrow(iv)$ is proved. □

Remark 5.4.2. *One can also prove the implication $(iii)\Rightarrow(ii)$ without using the profound theorem of [BM01]. We give the proof in Appendix 6.3.*

The above results lead to the following question about rationally connected manifolds with nef anticanonical bundles.

Question 5.4.3. *Let X be a smooth compact manifold. Then X is rationally connected with nef anticanonical bundle if and only if T_X is generically ω^{n-1}-strictly positive for any Kähler metric ω.*

As a second application, we study a Conjecture of Y.Kawamata (cf. [Miy87, Thm. 1.1] for the dual case and [Xie05] for dimension 3.)

Conjecture 5.4.4. *If X is a compact Kähler manifold with nef anticanonical bundle. Then*

$$\int_X c_2(T_X)\wedge\omega_1\wedge\cdots\wedge\omega_{n-2}\geq 0$$

for all nef classes $\omega_1,\cdots,\omega_{n-1}$.

Using Theorem 5.3.1, we can prove

Proposition 5.4.5. *Let (X,ω_X) be a compact Kähler manifold with nef anticanonical bundle and let ω_X be a Kähler metric. Then*

$$\int_X c_2(T_X)\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2}\geq 0 \tag{5.26}$$

for any $\epsilon>0$ small enough.

Proof. Let nd be the numerical dimension of $-K_X$. Let

$$0=\mathscr{F}_0\subset\mathscr{F}_1\subset\ldots\subset\mathscr{F}_l=T_X \tag{5.27}$$

be a stable filtration of the semistable filtration of T_X with respect to the polarization $(c_1(-K_X)+\epsilon\omega_X)^{n-1}$ for some small $\epsilon>0$. By Lemma 5.2.1, the filtration (5.27) is independent of ϵ when $\epsilon\to 0$. By Theorem 5.3.1, we have

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1})\wedge(-K_X)^{\mathrm{nd}}\wedge(\omega_X)^{n-1-\mathrm{nd}}\geq 0\qquad\text{for any } i.$$

Since

$$\sum_i c_1(\mathscr{F}_i/\mathscr{F}_{i-1})\wedge(-K_X)^{\mathrm{nd}}\wedge(\omega_X)^{n-1-\mathrm{nd}}=(-K_X)^{\mathrm{nd}+1}\wedge(\omega_X)^{n-1-\mathrm{nd}}=0,$$

we obtain

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1})\wedge(-K_X)^{\mathrm{nd}}\wedge(\omega_X)^{n-1-\mathrm{nd}}=0\qquad\text{for any } i. \tag{5.28}$$

Combining (5.28) with Theorem 5.3.1, we obtain

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1})\wedge(-K_X)^{\mathrm{nd}-1}\wedge(\omega_X)^{n-\mathrm{nd}}\geq 0\qquad\text{for any } i. \tag{5.29}$$

Combining this with the stability condition of the filtration, we can find an integer $k\geq 1$ such that

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1})\wedge(-K_X)^{\mathrm{nd}-1}\wedge(\omega_X)^{n-\mathrm{nd}}=a_i>0\qquad\text{for } i\leq k, \tag{5.30}$$

and

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1})\wedge(-K_X)^{\mathrm{nd}-1}\wedge(\omega_X)^{n-\mathrm{nd}}=0\qquad\text{for } i>k.$$

We begin to prove (5.26). Set $r_i:=\mathrm{rk}(\mathscr{F}_i/\mathscr{F}_{i-1})$. By Lübke's inequality (cf. the proof of [Miy87, Thm. 6.1]), we have

$$c_2(T_X)\wedge(-K_X+\epsilon\omega_X)^{n-2} \tag{5.31}$$

$$\geq (c_1(-K_X)^2 - \sum_i \frac{1}{r_i} c_1(\mathscr{F}_i/\mathscr{F}_{i-1})^2)(-K_X + \epsilon\omega_X)^{n-2}.$$

There are three cases.

Case (1) : $\sum\limits_{i\leq k} r_i \geq 2$ *and* $\mathrm{nd} \geq 2$. Using the Hodge index theorem, we have[3]

$$(\alpha^2 \wedge (-K_X + \epsilon\omega_X)^{n-2})((-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}) \leq (\alpha \wedge (-K_X) \wedge (-K_X + \epsilon\omega_X)^{n-2})^2, \tag{5.32}$$

for any $\alpha \in H^{1,1}(X, \mathbb{R})$. If we take $\alpha = c_1(\mathscr{F}_i/\mathscr{F}_{i-1})$ in (5.32) and use (5.31), we obtain

$$c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2} \geq \tag{5.33}$$

$$c_1(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2} - \sum_{i\leq k} \frac{1}{r_i} \frac{(c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) \wedge (-K_X) \wedge (-K_X + \epsilon\omega_X)^{n-2})^2}{(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}}$$

Now we estimate the two terms in the right hand side of (5.33). Using (5.30), we have

$$c_1(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2} = (\sum_{i\leq k} a_i)\epsilon^{n-\mathrm{nd}} + O(\epsilon^{n-\mathrm{nd}})$$

and

$$\sum_{i\leq k} \frac{1}{r_i} \frac{(c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) \wedge (-K_X) \wedge (-K_X + \epsilon\omega_X)^{n-2})^2}{(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}} = \frac{1}{\sum_{i\leq k} a_i}(\sum_{i\leq k} \frac{a_i^2}{r_i}) \cdot \epsilon^{n-\mathrm{nd}} + O(\epsilon^{n-\mathrm{nd}}).$$

Since $\sum\limits_{i\leq k} r_i \geq 2$, we have

$$\sum_{i\leq k} a_i > \frac{1}{\sum_{i\leq k} a_i}(\sum_{i\leq k} \frac{a_i^2}{r_i}).$$

Therefore $c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2}$ is strictly positive when $\epsilon > 0$ is small enough.

Case (2) : $\sum\limits_{i\leq k} r_i = 1$ *and* $\mathrm{nd} \geq 2$. In this case, we obtain immediately that $r_1 = 1$ and $k = 1$. Moreover, (5.30) in this case means that

$$c_1(\mathscr{F}_1) \wedge (-K_X)^{\mathrm{nd}-1} \wedge (\omega_X)^{n-\mathrm{nd}} > 0,$$

and

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) \wedge (-K_X)^{\mathrm{nd}-1} \wedge (\omega_X)^{n-\mathrm{nd}} = 0 \qquad \text{for } i \geq 2. \tag{5.34}$$

Assume that s is the smallest integer such that

$$c_1(\mathscr{F}_2/\mathscr{F}_1) \wedge (-K_X)^{\mathrm{nd}-s} \wedge (\omega_X)^{n-\mathrm{nd}+s-1} > 0.$$

Taking $\alpha = c_1(\mathscr{F}_i/\mathscr{F}_{i-1})$ in (5.32) for any $i \geq 2$, we get

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1})^2 \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} \leq \frac{(c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) \wedge (-K_X) \wedge (-K_X + \epsilon\omega_X)^{n-2})^2}{(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}} \tag{5.35}$$

3. It is important that $\alpha^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}$ maybe negative.

$$\leq \frac{(\epsilon^{n+s-\mathrm{nd}-1})^2}{\epsilon^{n-\mathrm{nd}}}(1+O(1)) = \epsilon^{2s+n-\mathrm{nd}-2} + O(\epsilon^{2s+n-\mathrm{nd}-2}) \text{ for } i \geq 2.$$

Similarly, if we take $\alpha = \sum_{i\geq 2} c_1(\mathscr{F}_i/\mathscr{F}_{i-1})$ in (5.32), we obtain

$$(\sum_{i\geq 2} c_1(\mathscr{F}_i/\mathscr{F}_{i-1}))^2 \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} \leq \epsilon^{2s+n-\mathrm{nd}-2}. \tag{5.36}$$

Combining (5.35), (5.36) with (5.31), we obtain

$$c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2}$$
$$\geq (c_1(-K_X)^2 - \sum_{i\geq 2}\frac{1}{r_i}c_1(\mathscr{F}_i/\mathscr{F}_{i-1})^2 - (c_1(-K_X) - \sum_{i\geq 2} c_1(\mathscr{F}_i/\mathscr{F}_{i-1}))^2)(-K_X+\epsilon\omega_X)^{n-2}$$
$$= 2c_1(-K_X) \wedge (\sum_{i\geq 2} c_1(\mathscr{F}_i/\mathscr{F}_{i-1})) \wedge (-K_X+\epsilon\omega_X)^{n-2}$$
$$-(\sum_{i\geq 2}\frac{1}{r_i}c_1(\mathscr{F}_i/\mathscr{F}_{i-1})^2 + (\sum_{i\geq 2} c_1(\mathscr{F}_i/\mathscr{F}_{i-1}))^2) \wedge (-K_X+\epsilon\omega_X)^{n-2}$$
$$\geq \epsilon^{n-\mathrm{nd}+s-1} - \epsilon^{n-\mathrm{nd}+2s-2}.$$

Let us observe that by (5.34) we have $s \geq 2$. Therefore $c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2}$ is strictly positive for $\epsilon > 0$ small enough.

Case (3) : $\mathrm{nd} = 1$. Using (5.31), we have

$$c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2} \geq -\sum_i \frac{1}{r_i} c_1(\mathscr{F}_i/\mathscr{F}_{i-1})^2(-K_X+\epsilon\omega_X)^{n-2}.$$

By the Hodge index theorem, we obtain

$$c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2}$$
$$\geq \lim_{t\to 0^+} -\sum_i \frac{1}{r_i}\frac{(c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) \wedge (-K_X+t\omega_X) \wedge (-K_X+\epsilon\omega_X)^{n-2})^2}{(-K_X+t\omega_X)^2 \wedge (-K_X+\epsilon\omega_X)^{n-2}}.$$

Let us observe that by (5.28) we have

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) \wedge (-K_X) \wedge (\omega_X)^{n-2} = 0 \qquad \text{for any } i.$$

Then

$$c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2}$$
$$\geq \lim_{t\to 0^+} -\sum_i \frac{1}{r_i} \cdot \frac{(t\epsilon^{n-2}c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) \wedge \omega_X^{n-1})^2}{t^2\epsilon^{n-2}\omega_X^n + t\epsilon^{n-2}(-K_X)\omega_X^{n-1}} = 0.$$

□

It is interesting to study the case when the equality holds in (5.26) of Proposition 5.4.5. We will prove that in this case, X is either a torus or a smooth $\mathbb{P}^1$-fibration over a torus. Before proving this result, we first prove an auxiliary lemma.

Lemma 5.4.6. *Let (X, ω_X) be a compact Kähler manifold with nef anticanonical bundle. Let*

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}_l = T_X \tag{5.37}$$

be a stable subfiltration of the Jordan-Hölder filtration with respect to $(c_1(-K_X)+\epsilon\omega_X)^{n-1}$. If $\int_X c_2(T_X)\wedge (c_1(-K_X)+\epsilon\omega_X)^{n-2} = 0$ for some $\epsilon > 0$ small enough, we have *(i)* $\mathrm{nd}(-K_X) = 1$.

(ii) $(\mathscr{F}_i/\mathscr{F}_{i-1})^{**}$ *is projectively flat for all* i*, i.e.,* $(\mathscr{F}_i/\mathscr{F}_{i-1})^{**}$ *is locally free and there exists a smooth metric* h *on it such that* $i\Theta_h(\mathscr{F}_i/\mathscr{F}_{i-1})^{**} = \alpha\,\mathrm{Id}$*, where* α *is a* $(1,1)$*-form.*

(iii) $c_2(\mathscr{F}_i/\mathscr{F}_{i-1}) = 0$ *for all* i*, and* (5.37) *is regular outside a subvariety of codimension at least* 3.

(iv) $c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) = \mathrm{rk}(\mathscr{F}_i/\mathscr{F}_{i-1})\cdot\alpha_i$ *for some* $\alpha_i \in H^{1,1}(X,\mathbb{Z})$*. Moreover,* $c_1(\mathscr{F}_i/\mathscr{F}_{i-1})$ *is nef and proportional to* $c_1(-K_X)$*.*

Remark 5.4.7. *We first remark that for a vector bundle* V *of rank* k *supported on a subvariety* $j : Z \subset X$ *of codimension* r*, by the Grothendieck-Riemann-Roch theorem, we have*

$$c_r(j_*(V)) = (-1)^{r-1}(r-1)!k[Z].$$

Therefore for any torsion free sheaf $\mathscr{E}$*, we have* $c_2(\mathscr{E}) \geq c_2(\mathscr{E}^{**})$ *and the equality holds if and only if* $\mathscr{E} = \mathscr{E}^{**}$ *outside a subvariety of codimension at least* 3*.*

Proof. By the proof of Proposition 5.4.5, the equality

$$\int_X c_2(T_X)\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2} = 0 \tag{5.38}$$

implies that the filtration (5.37) is in the case (3), i.e., $\mathrm{nd}(-K_X) = 1$. By the proof of Proposition 5.4.5, (5.38) implies also that the filtration (5.37) satisfies the following three conditions :

$$\int_X c_1(-K_X)^2\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2} = 0. \tag{5.39}$$

$$\int_X c_2((\mathscr{F}_i/\mathscr{F}_{i-1})^{**})\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2} = \int_X c_2(\mathscr{F}_i/\mathscr{F}_{i-1})\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2} = 0. \tag{5.40}$$

$$\int_X c_1(\mathscr{F}_i/\mathscr{F}_{i-1})^2\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2} = 0. \tag{5.41}$$

By [BS94, Cor 3], (5.40) and (5.41) imply that $(\mathscr{F}_i/\mathscr{F}_{i-1})^{**}$ is locally free and projectively flat. *(ii)* is proved. *(iii)* follows by (5.40) and the Remark 5.4.7. We now check *(iv)*. *(ii)* implies that $c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) = \mathrm{rk}(\mathscr{F}_i/\mathscr{F}_{i-1})\cdot\alpha_i$ for some $\alpha_i \in H^{1,1}(X,\mathbb{Z})$. By (5.28), we have

$$\int_X c_1(-K_X)\wedge c_1(\mathscr{F}_i/\mathscr{F}_{i-1})\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2} = 0.$$

Combining this with (5.39) and (5.41), by the Hodge index theorem [4], we obtain that $c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) = a_i\cdot c_1(-K_X)$ for some $a_i \in \mathbb{Q}$. By Theorem 5.3.1, we have

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1})\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-1} \geq 0.$$

Therefore $a_i \geq 0$ and $\mathscr{F}_i/\mathscr{F}_{i-1}$ is nef. □

Using an idea of A.Höring, we finally prove that

Proposition 5.4.8. *Let* (X,ω_X) *be a projective manifold with nef anticanonical bundle. We suppose that* $\int_X c_2(X)\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2} = 0$ *for some* $\epsilon > 0$ *small enough. Then after a finite étale cover,* X *is either a torus or a smooth* $\mathbb{P}^1$*-fibration over a torus.*

4. In fact, let $Q(\alpha,\beta) = \int_X \alpha\wedge\beta\wedge(c_1(-K_X)+\epsilon\omega_X)^{n-2}$. Then Q is of index $(1,m)$. Let V be the subspace of $H^{1,1}(X,\mathbb{R})$ where Q is negative definite. If $Q(\alpha_1,\alpha_1) = Q(\alpha_2,\alpha_2) = Q(\alpha_1,\alpha_2) = 0$ for some non trivial α_1,α_2, then both α_1 and α_2 are not contained in V. Therefore we can find a $t \in \mathbb{R}$, such that $(\alpha_1 - t\alpha_2) \in V$. Since $Q(\alpha_1 - t\alpha_2, \alpha_1 - t\alpha_2) = 0$, we get $\alpha_1 - t\alpha_2 = 0$. Therefore α_1 is proportional to α_2.

Proof. Denote by $m \in \mathbb{N}$ the index of $-K_X$, that is m is the largest positive integer such that there exists a Cartier divisor L with $mL \equiv -K_X$. Let

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}_l = T_X \tag{5.42}$$

be the Jordan-Hölder filtration with respect to ω_X^{n-1}. Let Z be the locus where (5.42) is not regular. By Lemma 5.4.6, we have $\text{codim} Z \geq 3$.

1st case. $m \geq 2$. Since K_X is not nef, there exists a Mori contraction $\varphi\colon X \to Y$. Since $nd(-K_X) = 1$ and $-K_X$ is ample on all the φ-fibres, we see that all the φ-fibres have dimension at most one. By Ando's theorem [And85] we know that φ is either a blow-up along a smooth subvariety of codimension two or a conic bundle. Since $m \geq 2$ we see that the contraction has length at least two, so φ is a conic bundle without singular fibres, i.e. a $\mathbb{P}^1$-bundle. By [Miy83, 4.11] we have

$$\varphi_*(K_X^2) = -4K_Y,$$

so $K_X^2 = 0$ implies that $K_Y \equiv 0$. By the condition

$$\int_X c_2(X) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0,$$

we obtain that $c_2(Y) = 0$. Therefore Y is a torus and the proposition is proved.

2nd case. $m = 1$

By (iv) of Lemma 5.4.6, the condition $m = 1$ implies that $\text{rk}\,\mathscr{F}_1 = 1$ and $\mu(\mathscr{F}_i/\mathscr{F}_{i-1}) = 0$ for $i > 1$. By the proof of Proposition 5.4.1, we get

$$c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) = 0 \qquad \text{for } i > 1. \tag{5.43}$$

We consider a Mori contraction :

$$\varphi : X \to Y.$$

By [And85], there are two cases :

Case (1) : φ is a blow-up along a smooth subvariety $S \subset Y$ of codimension two.

Let E be the exceptional divisor. Since (5.42) is free outside Z of codimension ≥ 3, for a general fiber over $s \in S$, (5.42) is regular on the fiber X_s over s, which is $\mathbb{P}^1$. By (5.43), we know that

$$T_X|_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}^{n-1},$$

for some $a > 0$. On the other hand, over $\mathbb{P}^1$, we have a direct decomposition

$$T_X|_{\mathbb{P}^1} = T_E|_{\mathbb{P}^1} \oplus N_{X/E}|_{\mathbb{P}^1} = T_E|_{\mathbb{P}^1} \oplus [E]|_{\mathbb{P}^1}.$$

Since $[-E]|_{\mathbb{P}^1}$ is strictly positive, $T_X|_{\mathbb{P}^1}$ must contain a strictly negative part. We get a contradiction.

Case (2) : φ is a conic bundle, and Y is smooth.

We consider the reflexive subsheaf $(T_{X/Y})^{**}$ of T_X. We first prove that

$$(T_{X/Y})^{**} = \mathscr{F}_1. \tag{5.44}$$

By (5.43), we have

$$c_1(\mathscr{F}_1) = c_1(-K_X). \tag{5.45}$$

Let y be a generic point in $Y \setminus \pi(Z)$ (i.e., $\varphi_y = \mathbb{P}^1$ and (5.42) is regular on φ_y). Since (5.42) is free over φ_y, by (5.43) and (5.45) we obtain that $\mathscr{F}_1 = (T_{X/Y})^{**}$ over φ_y. Since both $\mathscr{F}_1$ and $(T_{X/Y})^{**}$ are immersed as vector subbundles in T_X outside a subvariety of codimension at least 3, combining with (5.44), we obtain that $\mathscr{F}_1 = (T_{X/Y})^{**}$ outside this subvariety. Then the reflexiveness of $\mathscr{F}_1$ and $(T_{X/Y})^{**}$ implies that $\mathscr{F}_1 = (T_{X/Y})^{**}$ on X.

We now prove that $T_X/\mathscr{F}_1 = \varphi^*(T_Y)$ outside a subvariety of codimension at least 3. Let $\widetilde{Z} \subset Y$ be the locus where the fiber is non reduced. By [And85, Thm 3.1], for any $y \in \widetilde{Z}$, we have $\varphi_y = 2C$, where $C \simeq \mathbb{P}^1$ and $N_{C/X}$ is not trivial. Then $C \cap Z \neq \emptyset$. Recall that Z is the singular set of the filtration (5.42) of codimension at least 3. Therefore the codimension $\widetilde{Z}$ in Y is at least 2. Therefore $T_X/\mathscr{F}_1 = \varphi^*(T_Y)$ outside a subvariety of codimension at least 2. Since $T_X/\mathscr{F}_1$ and $\varphi^*(T_Y)$ are locally free outside a subvariety of codimension at least 3 (thus reflexive on this open set), we obtain that $T_X/\mathscr{F}_1 = \varphi^*(T_Y)$ outside a subvariety of codimension at least 3.

As consequence, we have

$$\varphi^*(c_1(-K_Y)) = c_1(T_X/\mathscr{F}_1) = 0,$$

where the last equality comes from (5.43). Since φ is surjective and X, Y are compact Kähler, we get $c_1(-K_Y) = 0$. By Beauville's decomposition, after a finite étale cover, we can suppose that Y is a direct product $T \times Y_1$, where T is a torus and Y_1 is a product of Calabi-Yau and hyperkahler manifolds. If Y_1 is non trivial, we have $c_2(Y) > 0$. But $c_2(T_X/\mathscr{F}_1) = c_2(T_X/\mathscr{F}_1) = c_2(T_Y)$ by the above argument, we get $c_2(X) > 0$. We get a contradiction. Therefore Y is a torus. By [CH13, Thm 1.3], φ admits a smooth fibration to Y and the fibers are $\mathbb{P}^1$.

□

Remark 5.4.9. *In general, if* $\int_X c_2(X) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0$*, We cannot hope that* X *can be covered by a torus. In fact, the example [DPS94, Example 3.3] satisfies the equality* $c_2(X) = 0$ *and* X *can not be decomposed as direct product of torus with* $\mathbb{P}^1$*. Using [DHP08], we know that* X *cannot be covered by torus. Therefore we propose the following conjecture, which is a mild modification of the question of Yau :*

Conjecture 5.4.10. *Let* (X, ω_X) *be a compact Kähler manifold with nef anticanonical bundle. Then* $\int_X c_2(T_X) \wedge \omega_X^{n-2} \geq 0$*. If the equality holds for some Kähler metric, then* X *is either a torus or a smooth* $\mathbb{P}^1$*-fibration over a torus.*

Remark 5.4.11. *If one could prove that* T_X *is generically nef with respect to the polarization* $(c_1(-K_X), \omega, \cdots, \omega)$*, using the same arguement as in this section, one could prove this conjecture.*

5.5 Surjectivity of the Albanese map

As an application of Theorem 5.3.1, we give a new proof of the surjectivity of Albanese map when X is a compact Kähler manifold with nef anticanonical bundle.

Proposition 5.5.1. *Let* (X, ω) *be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map is surjective, and smooth outside a subvariety of codimension at least 2. In particular, the fibers of the Albanese map are connected and reduced in codimension 1.*

Proof. Let

$$0 \subset \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = T_X \tag{5.46}$$

be a filtration of torsion-free subsheaves such that $\mathscr{E}_{i+1}/\mathscr{E}_i$ is an ω-stable torsion-free subsheaf of $T_X/\mathscr{E}_i$ of maximal slope.

Case 1 : (5.46) is regular, i.e., all $\mathscr{E}_i$ *and* $\mathscr{E}_i/\mathscr{E}_{i-1}$ *are locally free*

In this case, we can prove that the Albanese map is submersive. Let $\tau \in H^0(X, T_X^*)$ be a nontrivial element. To prove that the Albanese map is submersive, it is sufficient to prove that τ is non vanishing everywhere. Thanks to Theorem 5.3.1 and the stability condition of $\mathscr{E}_i/\mathscr{E}_{i-1}$, we can find a smooth metric h_i on $\mathscr{E}_i/\mathscr{E}_{i-1}$ such that

$$\frac{i\Theta_{h_i}(\mathscr{E}_i/\mathscr{E}_{i-1}) \wedge \omega^{n-1}}{\omega^n} = \lambda_i \cdot \mathrm{Id}_{\mathscr{E}_i/\mathscr{E}_{i-1}}$$

for some constant $\lambda_i \geq 0$. Thanks to the construction of $\{h_i\}$ and Lemma 5.2.7, for any $\epsilon > 0$, there exists a smooth metric h_ϵ on T_X, such that

$$\frac{i\Theta_{h_\epsilon}(T_X) \wedge \omega^{n-1}}{\omega^n} \geq -\epsilon \cdot \mathrm{Id}_{T_X}, \tag{5.47}$$

and the matrix valued $(1,1)$-form $i\Theta_{h_\epsilon}(T_X)$ is uniformly bounded. Let h_ϵ^* be the dual metric on T_X^*. Then the closed $(1,1)$-current

$$T_\epsilon = \frac{i}{2\pi}\partial\overline{\partial}\ln\|\tau\|^2_{h_\epsilon^*}$$

satisfies

$$T_\epsilon \geq -\frac{\langle i\Theta_{h_\epsilon^*}(T_X^*)\tau, \tau\rangle_{h_\epsilon^*}}{\|\tau\|^2_{h_\epsilon^*}}. \tag{5.48}$$

Since $-\Theta_{h_\epsilon^*}(T_X^*) = {}^t\Theta_{h_\epsilon}(T_X)$, (5.47) and (5.48) imply a pointwise estimate

$$T_\epsilon \wedge \omega^{n-1} \geq -\epsilon\omega^n. \tag{5.49}$$

We suppose by contradiction that $\tau(x) = 0$ for some point $x \in X$, By Lemma 5.2.7, $i\Theta_{h_\epsilon}(T_X)$ is uniformly lower bounded. Therefore, there exists a constant C such that $T_\epsilon + C\omega$ is a positive current for any ϵ. After replacing by a subsequence, we can thus suppose that T_ϵ converge weakly to a current T, and $T + C\omega$ is a positive current. Since $\tau(x) = 0$, we have

$$\nu(T_\epsilon + C\omega, x) \geq 1 \qquad \text{for any } \epsilon,$$

where $\nu(T_\epsilon + C\omega, x)$ is the Lelong number of the current $T_\epsilon + C\omega$ at x. Using the main theorem in [Siu74], we obtain that $\nu(T + C\omega, x) \geq 1$. Therefore there exists a constant $C_1 > 0$ such that

$$\int_{B_x(r)} (T + C\omega) \wedge \omega^{n-1} \geq C_1 \cdot r^{2n-2} \qquad \text{for } r \text{ small enough},$$

where $B_x(r)$ is the ball of radius r centered at x. Then

$$\int_{U_x} T \wedge \omega^{n-1} > 0$$

for some neighborhood U_x of x. Therefore

$$\lim_{\epsilon \to 0} \int_{U_x} T_\epsilon \wedge \omega^{n-1} > 0. \tag{5.50}$$

Combining (5.49) with (5.50), we obtain

$$\lim_{\epsilon \to 0} \int_X T_\epsilon \wedge \omega^{n-1} > 0.$$

We get a contradiction by observing that all T_ϵ are exact forms.

Case 2 : General case

By Lemma 5.2.6, there exists a desingularization $\pi : \widetilde{X} \to X$, such that $\pi^*(T_X)$ admits a filtration :

$$0 \subset E_1 \subset E_2 \subset \cdots \subset \pi^*(T_X)$$

satisfying that $E_i, E_i/E_{i-1}$ are vector bundles and $\pi_*(E_i) = \mathscr{E}_i$ on $X \setminus Z$, where Z is an analytic subset of codimension at least 2. Let $\tau \in H^0(X, T_X^*)$ be a nontrivial element. Our aim is to prove that τ is non vanishing outside Z.

Let $x \in \widetilde{X} \setminus \pi^{-1}(Z)$. Let U_x be a small neighborhood of x such that $U_x \subset \widetilde{X} \setminus \pi^{-1}(Z)$. We suppose by contradiction that $\pi^*(\tau)(x) = 0$. By [BS94], there exists Hermitian-Einstein metrics $h_{\epsilon,i}$ on E_i/E_{i-1}

with respect to $\pi^*\omega + \epsilon\omega_{\widetilde{X}}$, and $\{i\Theta_{h_{\epsilon,i}}(E_i/E_{i-1})\}_\epsilon$ is uniformly bounded on U_x [5]. Combining this with Lemma 5.2.7, we can construct a smooth metric h_ϵ on $\pi^*(T_X)$ such that

$$\frac{i\Theta_{h_\epsilon}(\pi^*(T_X)) \wedge (\pi^*\omega + \epsilon\omega_{\widetilde{X}})^{n-1}}{(\pi^*\omega + \epsilon\omega_{\widetilde{X}})^n} \geq -2C\epsilon \cdot \mathrm{Id}_{\pi^*(T_X)}, \tag{5.51}$$

and $i\Theta_{h_\epsilon}(\pi^*(T_X))$ is uniformly bounded on U_x. Let $T_\epsilon = \frac{i}{2\pi}\partial\overline{\partial}\ln\|\pi^*(\tau)\|^2_{h_\epsilon^*}$. By the same argument as in Case 1, the uniform boundedness of $i\Theta_{h_\epsilon}(\pi^*(T_X))$ in a neighborhood of x implies the existence of a neighborhood U'_x of x and a constant $c > 0$, such that

$$\lim_{\epsilon\to 0}\int_{U'_x} T_\epsilon \wedge (\pi^*(\omega) + \epsilon\omega_{\widetilde{X}})^{n-1} \geq c.$$

Combining this with (5.51), we get

$$\lim_{\epsilon\to 0}\int_{\widetilde{X}} T_\epsilon \wedge (\pi^*(\omega) + \epsilon\omega_{\widetilde{X}})^{n-1} \geq c,$$

which contradicts with the fact that all T_ϵ are exact. Therefore τ is non vanishing outside Z. Proposition 5.5.1 is proved. □

5.6 Structure of the Albanese map

In this section, we would like to prove that

Theorem 5.6.1. *Let X be a compact Kähler manifold such that $-K_X$ is nef. Let $\pi\colon X \to T$ be the Albanese map, and let F be a general fibre. If $-K_F$ is nef and big, then π is locally trivial.*

We first prove the following lemma.

Lemma 5.6.2. *Let (X, ω_X) be a compact Kähler manifold of dimension n, and let $\pi : X \to Y$ be a smooth fibration onto a curve Y. Let E be a numerical effective vector bundle on X. Suppose that*

$$c_1(E) = M \cdot \pi^*\omega_Y$$

for some constant M and ω_Y the first Chern class of some ample divisor on Y. Let

$$0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_k = E$$

be a stable subfiltration of the Harder-Narasimhan filtration with respect to $\pi^*\omega_Y + \epsilon\omega_X$ for some $0 < \epsilon \ll 1$. Then

$$c_1(\mathscr{F}_1) = a_1 \cdot \pi^*(\omega_Y)$$

for some constant $a_1 \geq 0$.

Proof. We first fix some notations. We denote s any point in Y and X_s the fiber over s. We suppose also that $\omega_Y = c_1(\mathscr{O}_Y(1))$ is an integral Kähler class.

Thanks to the condition $c_1(E) = M \cdot \pi^*\omega_Y$, we have $c_1(E|_{X_s}) = 0$. Then $E|_{X_s}$ is numerically flat, and by [DPS94], for any reflexive subsheaf $\mathscr{F} \subset E$, we have

$$c_1(\mathscr{F}|_{X_s}) \wedge (\omega_X|_{X_s})^{n-2} \leq 0.$$

5. In fact, [BS94] proved that $h_{\epsilon,i}$ and $h_{\epsilon,i}^{-1}$ are $C^{1,\alpha}$-uniform bounded in U_x. Since U_x is in $X \setminus Z$, $\omega_\epsilon := \pi^*\omega + \epsilon\omega_{\widetilde{X}}$ is uniformly strict positive on U_x. By [Kob87, Chapter I, (14.16)] and Hermitian-Einstein condition, we obtain that $\Delta_{\omega_\epsilon}(h_{\epsilon,i})_{j,k}$ is uniformly C^α bounded on U_x, where Δ_{ω_ϵ} is the Laplacian with respect to ω_ϵ and $(h_{\epsilon,i})_{j,k} := h_{\epsilon,i}(e_j, e_k)$ for a fixed base $\{e_k\}$ of E_i/E_{i-1}. The standard elliptic estimates gives the uniform boundedness of $i\Theta_{h_{\epsilon,i}}(E_i/E_{i-1})$ on U_x.

Therefore

$$c_1(\mathscr{F}) \wedge \pi^*(\omega_Y) \wedge \omega_X^{n-2} \leq 0. \qquad \text{for any } \mathscr{F} \subset E. \tag{5.52}$$

Moreover, by the same proof of Lemma 5.2.4, we have

$$\sup\{c_1(\mathscr{F}) \wedge \omega_Y \wedge (\omega_X)^{n-2} \mid \mathscr{F} \subset E, \text{ and } c_1(\mathscr{F}) \wedge \omega_Y \wedge (\omega_X)^{n-2} < 0\} < 0. \tag{5.53}$$

Let

$$0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_k = E$$

be a stable subfiltration of the Harder-Narasimhan filtration with respect to $(\pi^*\omega_Y + \epsilon\omega_X)^{n-1}$ for some ϵ small enough. We first prove that

$$c_1(\mathscr{F}_1) \wedge \pi^*(\omega_Y) \wedge \omega_X^{n-2} = 0 \qquad \text{and} \qquad c_1(\mathscr{F}_1) \wedge (\omega_X)^{n-1} \geq 0. \tag{5.54}$$

In fact, since E is nef, by the definition of Harder-Narasimhan filtration, we have

$$c_1(\mathscr{F}_1) \wedge (\pi^*\omega_Y + \epsilon\omega_X)^{n-1} \geq 0. \tag{5.55}$$

Note that $\dim Y = 1$, then

$$(\pi^*\omega_Y + \epsilon\omega_X)^{n-1} = \epsilon^{n-2}\pi^*(\omega_Y) \wedge (\omega_X)^{n-2} + \epsilon^{n-1}(\omega_X)^{n-1}.$$

Note also that $c_1(\mathscr{F}) \wedge (\omega_X)^{n-1}$ is uniformly bounded from above for any $\mathscr{F} \subset E$ (cf. [Kob87, Lemma 7.16]). Then (5.55) implies that

$$c_1(\mathscr{F}_1) \wedge \pi^*(\omega_Y) \wedge (\omega_X)^{n-2} \geq -\epsilon \cdot M$$

for a constant M independent of ϵ. Since ϵ is sufficient small, the uniform condition (5.53) implies that

$$c_1(\mathscr{F}_1) \wedge \pi^*(\omega_Y) \wedge \omega_X^{n-2} = 0.$$

Combining this with (5.55), we obtain

$$c_1(\mathscr{F}_1) \wedge (\omega_X)^{n-1} \geq 0.$$

(5.54) is proved.

Combining (5.54) with the condition $c_1(E) = M \cdot \pi^*\omega_Y$, we get

$$c_1(E/\mathscr{F}_1) \wedge \pi^*(\omega_Y) \wedge \omega_X^{n-2} = 0.$$

Note moreover that $c_1(E/\mathscr{F}_1)$ is nef, and $\omega_Y^2 = 0$, we get

$$c_1(E/\mathscr{F}_1) = c \cdot \pi^*(\omega_Y) \tag{5.56}$$

for certain constant c by the equality condition in Hovanskii-Teissier's inequality (cf. Remark 6.2.2 of Appendix 6.2). The lemma is proved. □

We now prove

Proposition 5.6.3. *In the situation above the reflexive sheaves $\mathscr{F}_i$ are subbundles of E, in particular they and the graded pieces $\mathscr{F}_{i+1}/\mathscr{F}_i$ are locally free. Moreover each of the graded pieces $\mathscr{F}_{i+1}/\mathscr{F}_i$ is projectively flat, and there exists a smooth metric h_i on $\mathscr{F}_{i+1}/\mathscr{F}_i$, such that*

$$i\Theta_{h_i}(\mathscr{F}_{i+1}/\mathscr{F}_i) = a_i\pi^*(\omega_Y) \cdot \mathrm{Id}_{\mathscr{F}_{i+1}/\mathscr{F}_i},$$

for some constant $a_i \geq 0$.

Proof. Step 1. Proof of the first statement. We first prove the first statement for $i = 1$. By [DPS94, Lemma 1.20] it is sufficient to prove that the induced morphism

$$\det \mathscr{F}_1 \to \bigwedge^{\mathrm{rk}\mathscr{F}_1} E$$

is injective as a morphism of vector bundles. Note now that the set $Z \subset X$ where $\mathscr{F}_1 \subset E$ is not a subbundle has codimension at least two : it is contained in the union of the loci where the torsion-free sheaves $\mathscr{F}_{k+1}/\mathscr{F}_k$ are not locally free. In particular Z does not contain any fibre $X_y := \pi^{-1}(y)$ with $y \in Y$. Thus for every $y \in Y$ the restricted morphism

$$(\det \mathscr{F}_1)|_{X_y} \to (\bigwedge^{\mathrm{rk}\mathscr{F}_1} E)_{X_y} \tag{5.57}$$

is not zero. Yet by Lemma 5.6.2 the line bundle $(\det \mathscr{F}_1)|_{X_y}$ is numerically trivial and the vector bundle $(\bigwedge^{\mathrm{rk}\mathscr{F}_1} E)_{X_y}$ is numerically flat. Thus the inclusion (5.57) is injective as a morphism of vector bundles [DPS94, Prop.1.16]. Then $\mathscr{F}_1$ is a subbundle of E.

Now $E/\mathscr{F}_1$ is a nef vector bundle on X. Moreover, Lemma 5.6.2 implies that $c_1(E/\mathscr{F}_1) = M' \cdot \pi^*(\omega_Y)$ for some constant M'. Then we can argue by induction on $E/\mathscr{F}_1$, and the first statement is proved.

Step 2. The graded pieces are projectively flat. Applying Lemma 5.6.2 to $E/\mathscr{F}_i$, we obtain that $c_1(\mathscr{F}_i/\mathscr{F}_{i-1}) = a_i \cdot \pi^*(\omega_Y)$ for some constant a_i. Then $c_1^2(\mathscr{F}_i/\mathscr{F}_{i-1}) = 0$. To prove that the graded pieces are projectively flat, by [Kob87, Thm.4.7] it is sufficient to prove that

$$c_2(\mathscr{F}_i/\mathscr{F}_{i-1}) \cdot \omega_X^{n-2} = 0$$

for some Kähler form ω_X. Since $c_1(\mathscr{F}_i/\mathscr{F}_{i-1})$ is a pull-back from the curve Y for every $i \in \{1, \ldots, k\}$ it is easy to see that

$$c_2(E) = \sum_{i=1}^{k} c_2(\mathscr{F}_i/\mathscr{F}_{i-1}).$$

Since we have $c_2(\mathscr{F}_i/\mathscr{F}_{i-1}) \cdot \omega_X^{n-2} \geq 0$ for every $i \in \{1, \ldots, k\}$ by [Kob87, Thm.4.7], we are left to show that $c_2(E) \cdot \omega_X^{n-2} = 0$. Yet E is nef with $c_1(E)^2 = 0$, so this follows immediately from the Chern class inequalities for nef vector bundles [DPS94, Cor.2.6].

□

Let X be a normal compact Kähler variety with at most canonical Gorenstein singularities, and let now $\pi\colon X \to T$ be a fibration such that $-K_X$ is π-nef and π-big, that is $-K_X$ is nef on every fibre and big on the general fibre. In this case the relative base-point free theorem holds [Anc87, Thm.3.3], i.e. for every $m \gg 0$ the natural map

$$\pi^*\pi_*\mathscr{O}_X(-mK_X) \to \mathscr{O}_X(-mK_X)$$

is surjective. Thus $-mK_X$ is π-globally generated and induces a bimeromorphic morphism

$$\mu\colon X \to X' \tag{5.58}$$

onto a normal compact Kähler variety X'. Standard arguments from the MMP show that the bimeromorphic map μ is crepant [6], that is $K_{X'}$ is Cartier and we have

$$K_X \simeq \mu^* K_{X'}.$$

6. In fact, since X' is normal and μ is a bimeromorphic morphism, we can take an open set $U \subset X'$ such that μ^{-1} is well defined on U and $\mathrm{cod}_{X'}(X' \setminus U) \geq 2$. Then $\mathscr{O}_{X'}(1)|_U \cong \mathscr{O}_X(-dK_X)|_{\mu^{-1}(U)} \cong \mathscr{O}_{X'}(-dK_{X'})|_U$ for some $d \in \mathbb{N}$. Therefore $\mathscr{O}_{X'}(-dK_{X'}) \cong \mathscr{O}_{X'}(1)$ on X' and μ is crepant.

In particular X' has at most canonical Gorenstein singularities. The fibration π factors through the morphism μ, so we obtain a fibration

$$\pi' \colon X' \to T \tag{5.59}$$

such that $-K_{X'}$ is π'-ample. Therefore we call $\mu \colon X \to X'$ the relative anticanonical model of X and $\pi' \colon X' \to T$ the relative anticanonical fibration.

Our next aim is to prove

Proposition 5.6.4. *Let $E_m = \pi_*(-mK_{X/T})$ for $m \in \mathbb{N}$. Then E_m is numerically flat.*

We have divided the proof in several lemmas. We first give an important observation.

Lemma 5.6.5. *Let V be a nef vector bundle over a smooth curve C, and let $A \subset V$ be the maximal ample subbundle. Let $Z \subset \mathbb{P}(V)$ be a subvariety such that*

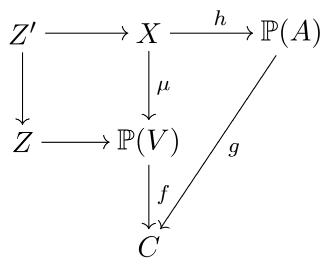
$$Z \cdot \mathscr{O}_{\mathbb{P}(V)}(1)^{\dim V} = 0.$$

Then we have an inclusion $Z \subset \mathbb{P}(V/A)$.

Proof. We will prove that if $Z \not\subset \mathbb{P}(V/A)$, then we have

$$Z \cdot \mathscr{O}_{\mathbb{P}(V)}(1)^{\dim V} > 0.$$

Let $f \colon \mathbb{P}(V) \to C$ and $g \colon \mathbb{P}(A) \to C$ be the canonical projections, and let $\mu \colon X \to \mathbb{P}(V)$ be the blow-up along the subvariety $\mathbb{P}(V/A)$. The restriction of μ to any f-fibre $f^{-1}(c)$ is the blow-up of a projective space $\mathbb{P}(V_c)$ along the linear subspace $\mathbb{P}(V_c/A_c)$, so we see that we have a fibration $h \colon X \to \mathbb{P}(A)$ which makes X into a projective bundle over $\mathbb{P}(A)$.



Since $Z \not\subset \mathbb{P}(V/A)$, the strict transform Z' is well-defined and we have

$$Z \cdot \mathscr{O}_{\mathbb{P}(V)}(1)^{\dim V} = Z' \cdot (\mu^* \mathscr{O}_{\mathbb{P}(V)}(1))^{\dim V}.$$

We claim that

$$\mu^* \mathscr{O}_{\mathbb{P}(V)}(1) \simeq h^* \mathscr{O}_{\mathbb{P}(A)}(1) + E,$$

where E is the exceptional divisor. Indeed we can write

$$\mu^* \mathscr{O}_{\mathbb{P}(V)}(1) \simeq a h^* \mathscr{O}_{\mathbb{P}(A)}(1) + bE + cF,$$

where F is a $f \circ \mu$-fibre and $a, b, c \in \mathbb{Q}$. By restricting to F one easily sees that we have $a = 1, b = 1$. In order to see that $c = 0$, note first (for example by looking at the relative Euler sequence) that we have

$$N_{\mathbb{P}(V/A)/\mathbb{P}(V)} \simeq f^* A^* \otimes \mathscr{O}_{\mathbb{P}(V/A)}(1).$$

Since the exceptional divisor E is the projectivisation of $N^*_{\mathbb{P}(V/A)/\mathbb{P}(V)}$ we deduce that

$$-E|_E \simeq \mathscr{O}_{\mathbb{P}(f^* A \otimes \mathscr{O}_{\mathbb{P}(V/A)}(-1))} \simeq (h^* \mathscr{O}_{\mathbb{P}(A)}(1))|_E + \mu|_E^* \mathscr{O}_{\mathbb{P}(V/A)}(-1).$$

Since $\mu|_E^* \mathscr{O}_{\mathbb{P}(V/A)}(-1)|_E \simeq \mu^* \mathscr{O}_{\mathbb{P}(V)}(-1)|_E$ we deduce that $c = 0$.

By induction on k one easily proves that

$$(h^*\mathscr{O}_{\mathbb{P}(A)}(1)+E)^{\dim Z}\cdot Z' = (h^*\mathscr{O}_{\mathbb{P}(A)}(1))^{\dim Z}\cdot Z'$$
$$+\sum_{j=0}^{\dim Z-1}(h^*\mathscr{O}_{\mathbb{P}(A)}(1))|_E^{\dim Z-1-j}\cdot(\mu|_E^*\mathscr{O}_{\mathbb{P}(V/A)}(1))^j\cdot(Z'\cap E).$$

Note that since A and V/A are nef, all the terms on the right hand side are non-negative. Hence if $\dim h(Z')=\dim Z$ we immediately see that $(h^*\mathscr{O}_{\mathbb{P}(A)}(1)+E)^{\dim Z}\cdot Z'>0$. If $\dim h(Z')<\dim Z$, set $j_0:=\dim Z-\dim h(Z')-1$. Since $\mu|_E^*\mathscr{O}_{\mathbb{P}(V/A)}(1)$ is ample on the fibres of $E\to\mathbb{P}(A)$ and the general fibre of $Z'\cap E\to h(Z')$ has dimension j_0-1, we see that

$$(h^*\mathscr{O}_{\mathbb{P}(A)}(1))|_E^{\dim Z-1-(j_0-1)}\cdot(\mu|_E^*\mathscr{O}_{\mathbb{P}(V/A)}(1))^{(j_0-1)}\cdot(Z'\cap E)>0.$$

□

Lemma 5.6.6. *Let X be a compact smooth Kähler manifold with nef anticanonical bundle of dimension n. Let $\pi: X\to T$ be the Albanese fibration, and suppose that $-K_F$ is nef and big for the general fibre F. Let $\pi'\colon X'\to T$ be the relative anticanonical fibration.*

Then π' is flat.

Proof. The variety X' has at most canonical singularities, so it is Cohen-Macaulay. The base T being smooth it is sufficient to prove that π' is equidimensional (cf.[Har77, III,Ex.10.9]). Let $r=\dim T$. By Theorem 3.3.6 we know that

$$(-K_X)^{n-r+1}=(-K_{X'})^{n-r+1}=0.$$

If $F\subset X'$ is an irreducible component of a π'-fibre, we have

$$(-K_{X'}|_F)^{\dim F}\neq 0,$$

since $-K_{X'}|_F$ is ample. By the preceding equation we see that $\dim F\leq n-r$. □

Now we begin to prove that E_m is numerically flat. First of all,

Lemma 5.6.7. *E_m is locally free.*

Proof. Since X' has at most canonical singularities, the relative Kawamata-Viehweg theorem applies and shows that

$$R^j(\pi')_*(-mK_{X'})=0 \qquad \forall\ j>0.$$

Since π' is flat, the statement follows. □

Lemma 5.6.8. *E_m is nef for $m\gg 1$.*

Remark 5.6.9. *If the fibration is smooth and the torus T is abelian, the nefness is proved in [DPS94, Lemma 3.21]. If the fiberation is smooth, we can also use the formula* (4.8) *in [Ber09] like the proof of Proposition 4.4.1. However, this method is difficult to genrealise in this case because of the difficulty to regularize the singular metrics on vector bundles. We use here [DP04, Theorem 0.5] and the standard regularization method (cf. [Dem12, Chapter 13], [Dem92, Section 3]) to overcome these difficulties.*

Proof. We first fix a Stein cover $\mathscr{U}=\{U_i\}$ on T as constructed in [Dem12, 13.B] [7], such that U_i are simply connected balls of radius 2δ fixed. Let

$$U_i^{'}\Subset U_i''\Subset U_i$$

be the balls constructed in [Dem12, 13.B] such that they are the balls of radius δ, $\frac{3}{2}\delta$, 2δ respectively and $\{U_i^{'}\}$ also covers T. Let θ_j be smooth partition function with support in U_j'' as constructed in

7. We keep the notations in [Dem12, 13.B], which can also be found in [Dem92, Sect. 3] .

[Dem12, Lemma 13.11]. Let $\varphi_k : T \to T$ be a 2^k-degree isogeny of the torus T, and $X_k = T \times_{\varphi_k} X$. Let $L = -(m+1)K_{X_k/T}$ and let $E_{m,k} = \pi_*(K_{X_k} + L)$. We have the commutative diagram

$$\begin{array}{ccccc} & & X_k & \xrightarrow{\varphi_k} & X \\ & & \Big\downarrow \pi & & \Big\downarrow \pi \\ \mathbb{P}(E_{m,k}) & \xrightarrow{\pi_1} & T & \xrightarrow{\varphi_k} & T \end{array}$$

Note that the cover $\mathscr{U} = \{U_i\}$, and the partition functions θ_i are independent of k. We now prove that there exists a smooth metric h on $\mathscr{O}_{\mathbb{P}(E_{m,k})}(1)$, such that

$$i\Theta_h(\mathscr{O}_{\mathbb{P}(E_{m,k})}(1)) \geq -C\pi_1^*(\omega_T)$$

for a constant C independent of k [8].

We fix a Kähler metric ω_{X_k} on X_k. Since L is nef and π-big, [DP04, Thm. 0.5] implies the existence of a singular metric $h_{\widetilde{\epsilon}_k}$ on L such that

$$i\Theta_{h_{\widetilde{\epsilon}_k}}(L) \geq \widetilde{\epsilon}_k \omega_{X_k} - C_1\omega_T,$$

for a constant C_1 independent of k, but $\widetilde{\epsilon}_k > 0$ is dependent of k. Since L is nef, for any $\epsilon > 0$, there exists a metric h_ϵ such that

$$i\Theta_{h_\epsilon}(L) \geq -\epsilon\omega_{X_k}.$$

We can thus define a new metric $h_{\epsilon_k} = h_{\widetilde{\epsilon}_k}^{r_k} \cdot h_\epsilon^{1-r_k}$ for some r_k small enough (the choice of ϵ is also depended on $\widetilde{\epsilon}_k$!), such that

$$i\Theta_{h_{\epsilon_k}}(L) \geq \epsilon_k\omega_{X_k} - 2\cdot C_1\omega_T \qquad \text{and} \qquad \mathscr{I}(h_{\epsilon_k}) = \mathscr{O}_{X_k}$$

for some $\epsilon_k > 0$. Since $\mathscr{I}(h_{\epsilon_k}) = \mathscr{O}_{X_k}$ and U_i are simply connected Stein varieties, we can suppose that L^2-bounded (with respect to h_{ϵ_k}) elements in $H^0(\pi^{-1}(U_i), K_{X_k} + L)$ generate $E_{m,k}$ over U_i.

Let $\{\widetilde{e}_{i,j}\}_j$ be an orthonormal base of $H^0(\pi^{-1}(U_i), K_{X_k}+L)$ with respect to h_{ϵ_k}, i.e., $\int_{\pi^{-1}(U_i)} \langle \widetilde{e}_{i,j}, \widetilde{e}_{i,j'} \rangle_{h_{\epsilon,k}}^2 = \delta_{j,j'}$. Then $\widetilde{e}_{i,j}$ induce an element $e_{i,j} \in H^0(\pi_1^{-1}(U_i), \mathscr{O}_{\mathbb{P}(E_{m,k})}(1))$. We now define a smooth metric h_i on $\mathscr{O}_{\mathbb{P}(E_{m,k})}(1)$ over $\pi_1^{-1}(U_i)$ by

$$\|\cdot\|_{h_i}^2 = \frac{\|\cdot\|_{h_0}^2}{\sum\limits_j \|e_{i,j}\|_{h_0}^2},$$

where h_0 is a fixed metric on $\mathscr{O}_{\mathbb{P}(E_m)}(1)$. Thanks to the construction, h_i is smooth and semi-positive on $\mathscr{O}_{\mathbb{P}(E_{m,k})}(1)(\pi_1^{-1}(U_i))$.

We claim that

$$\text{(5.60)} \qquad \frac{1}{C_2} \leq \frac{\sum\limits_j \|e_{i,j}\|_{h_0}^2(z)}{\sum\limits_j \|e_{i',j}\|_{h_0}^2(z)} \leq C_2 \qquad \text{for } z \in \pi_1^{-1}(U_i'' \cap U_{i'}'').$$

for some $C_2 > 0$ independent of z, k, i, i'. The proof is almost the same as in [Dem12, Lemma 13.10], except that we use the metric $\epsilon_k \cdot \omega_{X_k} + \pi^*\omega_T$ in stead of ω_X in the estimate. We postphone the proof of (5.60) in Appendix 5.7 and first finish the proof of Lemma 5.6.8.

We now define a global metric h on $\mathscr{O}_{\mathbb{P}(E_m)}(1)$ by

$$\|\cdot\|_h^2 = \|\cdot\|_{h_0}^2 e^{-\sum\limits_i (\pi_1^*(\theta_i'))^2 \cdot \ln(\sum\limits_j \|e_{i,j}\|_{h_0}^2)}, \qquad \text{where } (\theta_i')^2 = \frac{\theta_i^2}{\sum\limits_k \theta_k^2}.$$

8. All the constants $C, C_1, \cdots, C_i$ below are also independent of k.

Note that

$$i(\theta'_j\partial\overline{\partial}\theta'_j - \partial\theta'_j \wedge \overline{\partial}\theta'_j) \geq -C_3 \cdot \omega_T$$

by construction. Combining this with (5.60) and applying the Legendre identity in the proof of [Dem12, Lemma 13.11], [9] we obtain that

$$i\Theta_h(\mathscr{O}_{\mathbb{P}(E_{m,k})}(1)) \geq -C \cdot \pi_1^*(\omega_T)$$

for a constant C independent of k.

By [DPS94, Prop. 1.8], the metric h on $\mathscr{O}_{\mathbb{P}(E_{m,k})}(1)$ induce a smooth metric h_k on $\mathscr{O}_{\mathbb{P}(E_m)}(1)$ such that

$$i\Theta_{h_k}(\mathscr{O}_{\mathbb{P}(E_m)}(1)) \geq -\frac{C}{2^{k-1}}\omega_T.$$

The lemma is proved by letting $k \to +\infty$. □

We now prove Proposition 5.6.4

Proof of Proposition 5.6.4. Thanks to Lemma 5.6.8, we just need to prove that $c_1(E_m) = 0$. We suppose by contradiction that $c_1(E_m) \neq 0$. Then [Cao12a, Prop.2.2] implies a smooth fibration

$$\pi_1 : T \to S.$$

to an abelian variety S of dimension s, and

$$c_1(E_m) = c \cdot \pi_1^*(A),$$

for some very ample line bundle A and $c > 0$.

Let S_1 be a complete intersection of $s-1$ hypersurfaces defined by $s-1$ general elements in $H^0(S, A)$. We have thus a morphism

$$X_1 \xrightarrow{\pi|_{X_1}} T_1 \xrightarrow{\varphi|_{T_1}} S_1$$

where $X_1 := \pi^{-1}\pi_1^{-1}(S_1), T_1 := \pi_1^{-1}(S_1)$ are smooth by Bertini's theorem. Let $E'_m = E_m|_{T_1}$ for simplicity. Then E'_m is nef and $c_1(E'_m) = c\cdot\pi_1(A)$. Applying Proposition 5.6.3, we obtain a semipositive vector bundle F_1 on T_1 :

$$0 \subset F_1 \subset E'_m$$

and $c_1(F_1) = \pi_1^*(\omega_{S_1})$ for some Kähler form ω_{S_1} on S_1.

We now follow the same argument as in Lemma 5.6.5. Let $\mu : Y \to \mathbb{P}(E'_m)$ be the blow-up along the subvariety $\mathbb{P}(E'_m/F_1)$. Since X_1 is not contained in $\mathbb{P}(E'_m/F_1)$, we have thus the diagram

$$\begin{array}{ccccc}
X'_1 & \xrightarrow{i} & Y & \xrightarrow{h} & \mathbb{P}(F_1) \\
\downarrow & & \downarrow{\scriptstyle \mu} & & \\
X_1 & \longrightarrow & \mathbb{P}(E'_m) & \quad g \swarrow & \\
 & & \downarrow{\scriptstyle f} & & \\
 & & T_1 & & \\
 & & \downarrow{\scriptstyle \pi_1} & & \\
 & & S_1 & &
\end{array}$$

where X'_1 is the strict transformation of X_1. By the same argument in Lemma 5.6.5, we have

$$\mu^*\mathscr{O}_{\mathbb{P}(E'_m)}(1) \simeq h^*\mathscr{O}_{\mathbb{P}(F_1)}(1) + E,$$

9. Although in the proof of [Dem12, Lemma 13.11], θ'_i is supposed to be constant on U'_i, the uniformly strictly positive of the lower boundedness of θ'_i on U'_i is sufficient for the proof.

where E is the exceptional divisor. Since $\mathscr{O}_{\mathbb{P}(E'_m)}(1)$ is nef, we have

$$(\mu^*\mathscr{O}_{\mathbb{P}(E'_m)}(1))^{n-r+1}\cdot X'_1 \geq (\mu^*\mathscr{O}_{\mathbb{P}(E'_m)}(1))^{n-r}\cdot h^*\mathscr{O}_{\mathbb{P}(F_1)}(1)\cdot X'_1. \tag{5.61}$$

By Proposition 5.6.3, there is a smooth metric h on F_1 such that $i\Theta_h(F_1)=\pi_1^*\omega_{S1}\cdot \mathrm{Id}_{F_1}$. Then

$$i\Theta_h(g^*F_1)=g^*\pi_1^*\omega_{S1}\cdot \mathrm{Id}_{g^*F_1} \qquad \text{on } \mathbb{P}(F_1).$$

The metric h induces a natural metric h' on $\mathscr{O}_{\mathbb{P}(F_1)}(1)$, and by [DPS94, Proposition 1.11], we obtain

$$i\Theta_{h'}(\mathscr{O}_{\mathbb{P}(F_1)}(1))\geq g^*\pi_1^*\omega_{S1}$$

Since $h\circ g=\mu\circ f$, we get $h^*\mathscr{O}_{\mathbb{P}(F_1)}(1)\geq \mu^*f^*\omega_{S_1}$. Combining this with the fact that $f\circ\mu(X'_1)=T$ by construction, we obtain

$$h^*\mathscr{O}_{\mathbb{P}(F_1)}(1)\cdot X'_1\geq C\cdot X'_{1,s},$$

where $X'_{1,s}$ is the general fiber of $i\circ\mu\circ f\circ\pi_1$, and $C>0$. Combining with the fact that $\mathscr{O}_{\mathbb{P}(E'_m)}(1)$ is f-relative ample, we get

$$(\mu^*\mathscr{O}_{\mathbb{P}(E'_m)}(1))^{n-r}\cdot h^*\mathscr{O}_{\mathbb{P}(F_1)}(1)\cdot X'_1\neq 0.$$

Combining this with (5.61), we obtain $(-K_{X_1/T_1})^{n-r+1}\cdot X_1=(\mu^*\mathscr{O}_{E'_m}(1))^{n-r+1}\cdot X'_1\neq 0$. Therefore $(-K_{X/T})^{n-r+1}\cdot X\neq 0$ which contradicts Theorem 3.3.6. □

We now prove the main theorem in this section.

Proof of Theorem 5.6.1. Step 1. The relative anticanonical fibration is locally trivial. Let $E_m=\pi_*(-mK_X)$ and $j: X'\hookrightarrow \mathbb{P}(E_m)$. Then $\pi'_*(j_*(\mathscr{O}_{X'})\otimes\mathscr{O}_{\mathbb{P}(V)}(p))$ is extensible since X' is normal. By [DPS94], we have

$$\pi'_*(j_*(\mathscr{O}_{X'})\otimes\mathscr{O}_{\mathbb{P}(V)}(p))=E_{mp} \qquad \text{on } T\setminus\Delta.$$

Therefore

$$\pi'_*(j_*(\mathscr{O}_{X'})\otimes\mathscr{O}_{\mathbb{P}(V)}(p))=E_{mp} \qquad \text{on } T.$$

Recall that E_{mp} is numerically flat by Proposition 5.6.4. Then E_{mp} is a local system by Proposition 4.3.1. Therefore the natural restriction : $S^pE_m\to E_{mp}$ induces a holomorphic section of $H^0(T,\mathrm{Hom}(S^pE_m,E_{mp}))$, which is parallel with respect to the local system by Lemma 4.3.3. Since the restriction $S^pE_m\to E_{mp}$ is surjective on the generic point, then it is surjective on T, and the kernel is also a numerical flat bundle.

Step 2. The Albanese map π is locally trivial.

We prove the case when $-K_F$ is ample. In the general case, we should use MMP method. In the case $-K_F$ is ample, $\mu\colon X\to X'$ is an isomorphism on the general fibre F, in particular the general π'-fibre F' is isomorphic to F. Since π' is locally trivial by the first step, we see that X' is smooth, in particular X' has terminal singularities. The birational map μ being crepant we see that μ does not contract any divisor. However X is smooth, so $\mathbb{Q}$-factorial, hence the μ-exceptional locus is empty or of pure codimension one. This implies two things : the birational map μ is divisorial, since X' is $\mathbb{Q}$-factorial. Thus we see that $X\simeq X'$.

□

5.7 Appendix

We now prove the claim (5.60) in Lemma 5.6.8, which is in some sense a relative gluing estimate.

Lemma 5.7.1. *We have*

$$\frac{1}{C_2}\leq \frac{\sum\limits_j \|e_{i,j}\|^2_{h_0}(z)}{\sum\limits_j \|e_{i',j}\|^2_{h_0}(z)}\leq C_2 \qquad \textit{for } z\in\pi_1^{-1}(U''_i\cap U''_{i'}). \tag{5.62}$$

(i.e., (5.60) *in Lemma 5.6.8.)*

Proof. Recall that $U_i' \Subset U_i'' \Subset U_i$ are the balls of radius δ, $\frac{3}{2}\delta$, 2δ respectively as constructed in [Dem12, 13.B]. Let z be a fixed point in $\pi_1^{-1}(U_i'' \cap U_{i'}'')$. Since $e_{i,j}$ is a section of a line bundle, we have

$$\sum_j \|e_{i,j}\|^2_{h_0}(z) = \sup_{\sum_j |a_j|^2=1} \|\sum_j a_j e_{i,j}\|^2_{h_0}(z).$$

Therefore, there exists a $\widetilde{e}_i \in H^0(\pi^{-1}(U_i), K_{X_k} + L)$ such that

$$\int_{\pi^{-1}(U_i)} \|\widetilde{e}_i\|^2_{h_{\epsilon_k}} = 1 \qquad \text{and} \qquad \|e_i\|^2_{h_0}(z) = \sum_j \|e_{i,j}\|^2_{h_0}(z),$$

where $e_i \in H^0(\pi_1^{-1}(U_i), \mathscr{O}_{\mathbb{P}(E_{m,k})}(1))$ is induced by $\widetilde{e}_i$. Let θ be a cut-off function with support in the ball of radius $\frac{\delta}{4}$ centered at $\pi_1(z)$ (thus is supported in $U_i \cap U_{i'}$), and equal to 1 on the ball of radius $\frac{\delta}{8}$ centered at $\pi_1(z)$.

By construction, $(\pi^*(\theta)\cdot\widetilde{e}_i)$ is supported $\pi^{-1}(U_i \cap U_{i'})$, thus it is well defined on $\pi^{-1}(U_{i'})$. Therefore we can solve the $\overline{\partial}$-equation for $\overline{\partial}(\pi^*(\theta)\cdot\widetilde{e}_i)$ on $\pi^{-1}(U_{i'})$ with respect to the metric

$$\omega_{X_k,\epsilon_k} = \epsilon_k \cdot \omega_{X_k} + \pi^*\omega_T$$

by choosing a good metric on L. The choice of the good metric on L will be give later. We first give some estimates.

Since θ is defined on T, we have

$$\|\overline{\partial}\pi^*(\theta)\|_{\omega_{X_k,\epsilon_k}} \leq C_4$$

for some constant C_4 independent of k, ϵ_k.[10] Therefore

$$\int_{\pi^{-1}(U_{i'})} \|\overline{\partial}(\pi^*(\theta)\cdot\widetilde{e}_i)\|^2_{h_{\epsilon_k},\omega_{X_k,\epsilon_k}} = \int_{\pi^{-1}(U_i)} \|\overline{\partial}(\pi^*(\theta)\cdot\widetilde{e}_i)\|^2_{h_{\epsilon_k},\omega_{X_k,\epsilon_k}} \tag{5.63}$$

$$\leq C_4 \int_{\pi^{-1}(U_i)} \|\widetilde{e}_i\|^2_{h_{\epsilon_k}} = C_4,$$

where the first equality comes from the fact that $(\pi^*(\theta)\cdot\widetilde{e}_i)$ is supported $\pi^{-1}(U_i \cap U_{i'})$. Notice for the metric h_{ϵ_k} on L, we have

$$i\Theta_{h_{\epsilon_k}}(L) \geq \epsilon_k\omega_{X_k} - 2\cdot C_1\pi^*(\omega_T) \geq \omega_{X_k,\epsilon_k} - (2\cdot C_1 + 1)\pi^*(\omega_T). \tag{5.64}$$

We now define a metric $\widetilde{h}_{\epsilon_k} = h_{\epsilon_k}\cdot e^{-(n+1)\pi^*(\ln|t-\pi_1(z)|)-\pi^*\psi_{i'}(t)}$ on L over $\pi^{-1}(U_{i'})$, where $\psi_{i'}(t)$ is a uniformly bounded function on $U_{i'}$ satisfying

$$dd^c\psi_{i'}(t) \geq (2C_1+1)\omega_T.$$

Then (5.64) implies that

$$i\Theta_{\widetilde{h}_{\epsilon_k}}(L) \geq \omega_{X_k,\epsilon_k} \qquad \text{on } \pi^{-1}(U_{i'}).$$

By solving the $\overline{\partial}$-equation for $\overline{\partial}(\pi^*(\theta)\cdot\widetilde{e}_i)$ with respect to $(\widetilde{h}_{\epsilon_k}, \omega_{X_k,\epsilon_k})$ on $\pi^{-1}(U_{i'})$, we find a $g_{i'} \in L^2(\pi^{-1}(U_{i'}), K_{X_k} + L)$ such that $\overline{\partial}g_{i'} = \overline{\partial}(\pi^*(\theta)\widetilde{e}_i)$ and

$$\int_{\pi^{-1}(U_{i'})} \|g_{i'}\|^2_{\widetilde{h}_{\epsilon_k}} \leq C_5 \int_{\pi^{-1}(U_{i'})} \|\overline{\partial}(\pi^*(\theta)\cdot\widetilde{e}_i)\|^2_{\widetilde{h}_{\epsilon_k},\omega_{X,\epsilon_k}} \leq C_6, \tag{5.65}$$

where the last inequality comes from the inequality (5.63) and the fact that[11]

$$\overline{\partial}(\pi^*(\theta)\cdot\widetilde{e}_i)(z) = 0 \qquad \text{for } z \in \pi^{-1}(B_{\frac{\delta}{8}}(\pi_1(z))).$$

10. C_4 depends on δ. But by construction, the radius δ is independent of k !
11. $B_{\frac{\delta}{8}}(\pi_1(z))$ below is the ball radius of $\frac{\delta}{8}$ centered at $\pi_1(z)$.

Now we obtain a holomorphic section $(\pi^*(\theta)\cdot\widetilde{e}_i - g_{i'}) \in H^0(\pi^{-1}(U_{i'}), K_{X_k}+L)$. By the definition of the metric $\widetilde{h}_{\epsilon_k}$ and (5.65), we have $g_{i'} = 0$ on $\pi^{-1}(\pi_1(z))$. Therefore $\pi^*(\theta)\cdot\widetilde{e}_i - g_{i'} = \widetilde{e}_i$ on $\pi^{-1}(\pi_1(z))$. Moreover, (5.65) implies

$$\int_{\pi^{-1}(U_{i'})} \|\pi^*(\theta)\cdot\widetilde{e}_i - g_{i'}\|^2_{h_{\epsilon_k}} \leq C$$

for a constant C independent of k. By the extremal property of Bergman kernel, (5.62) is proved.

□

Chapitre 6

Appendix

6.1 Numerically flat vector bundles and local systems

In this section, we would give an elementary proof of Proposition 4.3.1 under the assumption that the variety X is an étale quotient of a torus. Assume first that E is a numerical flat bundle on a complex torus T, we would like to prove that E is in fact a local system. By [DPS94, Theorem 1.18], the numerical flat bundle E admits a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E \tag{6.1}$$

such that the quotients E_i/E_{i-1} are irreducible hermitian flat vector bundles.

Definition 6.1.1. *If $E_i/E_{i-1} = \mathscr{O}_T$ for all i in the filtration* (6.1)*, we say that E is a unipotent numerical flat bundle.*

We first prove the following lemma which helps us to simplify the situation.

Lemma 6.1.1. *Let E be a numerical flat vector bundle on a torus T. Then we have an orthogonal decomposition*

$$E = E_{\mathrm{uni}} \oplus E'$$

where E_{uni} is a unipotent numerical flat bundle and E' admits a filtration

$$\{0\} = E'_0 \subset E'_1 \subset ... \subset E'_k = E',$$

such that E'_s/E'_{s-1} are all non trivial irreducible hermitian flat bundles.

Moreover, we have $H^1(T, E') = 0$.

Proof. Let

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_p = E$$

be a filtration of E such that all E_k/E_{k-1} are irreducible hermitian flat bundes. We prove the lemma by induction on E_i.

If $i = 1$, since E_1 is irreducible, the lemma is evident in this case. Assume now that the lemma is true for E_k. Then we have

$$E_k = E_{k,\mathrm{uni}} \oplus E'_k \qquad \text{and} \qquad H^1(T, E'_k) = 0. \tag{6.2}$$

We now prove the lemma for E_{k+1}. Since E_{k+1} fits into the exact sequence

$$0 \to E_k \to E_{k+1} \to E_{k+1}/E_k \to 0, \tag{6.3}$$

applying (6.2) to (6.3), we get

$$0 \to E_{k,\mathrm{uni}} \oplus E'_k \to E_{k+1} \to E_{k+1}/E_k \to 0 \tag{6.4}$$

and

$$H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_k)) \tag{6.5}$$

$$= H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_{k,\mathrm{uni}})) \oplus H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E'_k)).$$

There are two cases.

Case 1 : $E_{k+1}/E_k \neq \mathscr{O}_T$.

We claim that

$$H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_{k,\mathrm{uni}})) = 0. \tag{6.6}$$

Proof of the claim : If $E_{k,\mathrm{uni}}$ is irreducible, then the claim is obvious. If not, by the definition of $E_{k,\mathrm{uni}}$, $E_{k,\mathrm{uni}}$ admits a filtration

$$0 \to \mathscr{O}_T \to E_{k,\mathrm{uni}} \to E_{k,\mathrm{uni}}/\mathscr{O}_T \to 0$$

such that $E_{k,\mathrm{uni}}/\mathscr{O}_T$ is also a unipotent numerical flat bundle. Then we have an exact sequence

$$H^1(T, \mathrm{Hom}(E_{k+1}/E_k, \mathscr{O}_T)) \to H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_{k,\mathrm{uni}}))$$

$$\to H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_{k,\mathrm{uni}}/\mathscr{O}_T)).$$

Combining with the fact that $H^1(T, \mathrm{Hom}(E_{k+1}/E_k, \mathscr{O})) = 0$ if E_{k+1}/E_k is non trivial, we obtain that

$$H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_{k,\mathrm{uni}})) \to H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_{k,\mathrm{uni}}/\mathscr{O}_T)) \tag{6.7}$$

is injective. Observing that $E_{k,\mathrm{uni}}/\mathscr{O}_T$ in (6.7) is also a unipotent bundle with smaller rank than $E_{k,\mathrm{uni}}$, the injectivity of (6.7) implies the claim (6.6) by induction on the rank of $E_{k,\mathrm{uni}}$.

Applying the claim (6.6) to (6.4) and (6.5), we obtain a decomposition

$$E_{k+1} = E_{k,\mathrm{uni}} \oplus E'_{k+1}$$

where E'_{k+1} is an extension of E'_k and E_{k+1}/E_k. Moreover, thanks to (6.2) and the fact that $H^1(T, E_{k+1}/E_k) = 0$, we get $H^1(T, E'_{k+1}) = 0$. The lemma is thus proved for E_{k+1}.

Case 2 : $E_{k+1}/E_k = \mathscr{O}_T$.

(6.2) implies

$$H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E'_k)) = 0. \tag{6.8}$$

Applying (6.8) to (6.3) and (6.4), we obtain

$$E_{k+1} = E_{k+1,\mathrm{uni}} \oplus E'_k$$

where $E_{k+1,\mathrm{uni}}$ is an extension of $E_{k,\mathrm{uni}}$ and E_{k+1}/E_k. Since $E_{k+1}/E_k = \mathscr{O}_T$ in this case, $E_{k+1,\mathrm{uni}}$ is also a unipotent numerical flat bundle. The lemma is proved. □

We now study the unipotent numerical flat bundle. We first prove the following lemmas.

Lemma 6.1.2. *Let f be a function on $\mathbb{C}^n$ such that*

$$\overline{\partial} f(z+\Lambda) = \overline{\partial} f(z) + \sum_{i=1}^{n} P_i(\Lambda) d\overline{z}_i, \tag{6.9}$$

for all $\Lambda \in \Gamma$, where Γ is a lattice of $\mathbb{C}^n$ and $P_i(\Lambda)$ are anti-holomorphic polynomials on Λ (independent on z). Then there exists an anti-holomorphic polynomial $g(z)$ of pure degree 2 such that

$$\overline{\partial} g(z+\Lambda) - \overline{\partial} g(z) = \sum_{i=1}^{n} P_i(\Lambda) d\overline{z}_i.$$

Proof. Since (6.9) is true for all $\Lambda, \Lambda' \in \Gamma$, we have

$$P_i(\Lambda + \Lambda') = P_i(\Lambda) + P_i(\Lambda').$$

Therefore all P_i are linear polynomials. In particular we have

$$\overline{\partial} f(z+\Lambda) - \sum_{i=1}^{n} P_i(z+\Lambda) d(\overline{z}_i + \overline{\Lambda}_i) = \overline{\partial} f(z) - \sum_{i=1}^{n} P_i(z) d\overline{z}_i.$$

Then

$$\overline{\partial}(\overline{\partial} f(z) - \sum_{i=1}^{n} P_i(z) \wedge d\overline{z}_i) = 0 \in H^2(T, \mathscr{O}).$$

Then

$$\sum_{i=1}^{n} \overline{\partial} P_i(z) \wedge d\overline{z}_i = 0 \text{ in } H^2(T, \mathscr{O}).$$

Therefore we can rewrite $\sum_{i=1}^{n} P_i(\Lambda) d\overline{z}_i$ as the form

$$\sum_{i=1}^{n} a_i \overline{\Lambda}_i d\overline{z}_i + \sum_{i\neq j} b_{i,j}(\overline{\Lambda}_i d\overline{z}_j + \overline{\Lambda}_j d\overline{z}_i).$$

where $a_i, b_{i,j}$ are constants independent of Λ, z. Then

$$g(z) = \sum_{i=1}^{n} \frac{a_i}{2} \overline{z}_i^2 + \sum_{i\neq j} b_{i,j} \overline{z}_i \overline{z}_j$$

satisfies equation (6.9). □

We now generalise Lemma 6.1.2 in higer degrees.

Lemma 6.1.3. *Let f be a function on $\mathbb{C}^n$ satisfying the equation*

$$\overline{\partial} f(z+\Lambda) = \overline{\partial} f(z) + \sum_{I} P_I(\Lambda) \overline{\partial} g_I(z) \tag{6.10}$$

for all $\Lambda \in \Gamma$, where $g_I(z)$ are monomial anti-holomorphic polynomials of index I and $P_I(\Lambda)$ are anti-holomorphic polynomials on Λ. We suppose that

$$m = \max\{|I|, g_I(z) \neq 0\}.$$

Then there is an anti-holomorphic polynomial $g(z)$ of pure degree $m+1$ such that

$$\overline{\partial} g(z+\Lambda) - \overline{\partial} g(z) - \sum_{|I|=m} P_I(\Lambda) \overline{\partial} g_I(z)$$

is $\overline{\partial}$ of a polynomial of degree $\leq m-1$.

Remark. *Lemma 6.1.2 is a special case of this lemma for $m = 1$. We need also remark that the existence of the function f satisfying* (6.10) *has already given a lot of restrictions on $P_I(\Lambda)$ and $g_I(z)$. We have already seen these restrictions explicitly for $m = 1$.*

Proof. Assume that the lemma is true when the maximum index is $\leq m-1$. We now prove that it is also true for m. First of all, we can rewrite (6.10) as

$$\overline{\partial} f(z+\Lambda) = \overline{\partial} f(z) + \overline{\partial}(\overline{z}_1^{\alpha_1} P_{\alpha_1}(\Lambda, z_2, ..., z_n)) \tag{6.11}$$

$$+\sum_{i>0}\overline{\partial}(\overline{z}_1^{\alpha_1-i}P_{\alpha_1-i}(\Lambda,z_2,...,z_n))+\overline{\partial}S_0$$

where P_{α_1-i} are polynomials independent of z_1 of degree $m-\alpha_1+i$ and $\deg S_0\leq m-1$.

Case 1 : $\alpha_1=m$.

Then (6.11) becomes

$$\overline{\partial}f(z+\Lambda)=\overline{\partial}f(z)+\overline{\partial}(\overline{z}_1^{m}P_m(\Lambda)) \tag{6.12}$$
$$+\sum_{i>0}\overline{\partial}(\overline{z}_1^{m-i}P_{m-i}(\Lambda,z_2,...,z_n))+\overline{\partial}S_0.$$

By differentiating (6.12) $m-1$ times of $\overline{z}_1$, we get

$$\overline{\partial}\widetilde{f}(z+\Lambda)=\overline{\partial}\widetilde{f}(z)+m\cdot P_m(\Lambda)d\overline{z}_1+\overline{\partial}P_{m-1}(\Lambda,z_2,...,z_n).$$

for some function $\widetilde{f}(z)$. Thanks to Lemma 6.1.2, $P_m(\Lambda)$ is thus a linear anti-holomorphic function on Λ. Let $f'(z)=f(z)-\overline{z}_1^mP_m(z)$. Then

$$\overline{\partial}f'(z+\Lambda)=\overline{\partial}f'(z)+c\overline{\partial}(\overline{z}_1^m\overline{\Lambda}_1)$$
$$+\sum_{i>0}\overline{\partial}(\overline{z}_1^{m-i}P'_{m-i}(\Lambda,z_2,...,z_n))+\overline{\partial}S'_0.$$

for some constant c independent of Λ. Replacing f' by $f''(z)=f'(z)-\frac{c}{m+1}\overline{z}_1^{m+1}$, we reduce therefore the Case 1 to the following Case 2.

Case 2 : $\alpha_1<m$.

By differentiating (6.11) α_1 times of $\overline{z}_1$, we get

$$\overline{\partial}\widetilde{f}(z+\Lambda)=\overline{\partial}\widetilde{f}(z)+\overline{\partial}(P_{\alpha_1}(\Lambda,z_2,...,z_n))+\overline{\partial}S_2(\Lambda,z)$$

for all $\Lambda\in\Gamma$, where $\widetilde{f}(z)=\frac{1}{\alpha_1!}\overline{\partial}_{\overline{z}_1}^{(\alpha_1)}f(z)$ and $S_2(\Lambda,z)$ is of degree $\leq m-\alpha_1-1$. By induction, there is an anti-holomorphic polynomial $Q(z)$ of pure degree $m-\alpha_1+1$ such that

$$\overline{\partial}(Q(z+\Lambda)-Q(z)-P_{\alpha_1}(\Lambda,z_2,...,z_n)) \tag{6.13}$$

is $\overline{\partial}$ of a polynomial of degree $\leq m-\alpha_1-1$ for all $\Lambda\in\Gamma$ (here we use the hypothesis $\alpha_1<m$).

Let $f'=f-(\overline{z}_1^{\alpha_1}Q(z))$. Then

$$\overline{\partial}f'(z+\Lambda)-\overline{\partial}f'(z) \tag{6.14}$$
$$=\overline{\partial}f(z+\Lambda)-\overline{\partial}f(z)-\overline{\partial}((\overline{z}_1+\Lambda_1)^{\alpha_1}Q(z+\Lambda)-\overline{z}_1^{\alpha_1}Q(z))$$
$$=\overline{\partial}(\alpha_1\cdot\overline{z}_1^{\alpha_1-1}Q(z+\Lambda))+\sum_{i>0}\overline{\partial}(\overline{z}_1^{\alpha_1-i}P'_{\alpha_1-i}(\Lambda,z_2,...,z_n))+\overline{\partial}S_3(\Lambda,z)$$

where $S_3(\Lambda,z)$ is of degree $\leq m-1$. We will prove in Lemma 6.1.4 that $Q(z)$ can be choosen to be independent of z_1. We postphone the proof in Lemma 6.1.4 and finish first the proof of Lemma 6.1.3. Then the maximal degree of $\overline{z}_1$ in the first two terms of the last line of (6.14) is α_1-1. We repeat the process and finally find polynomial g such that

$$\overline{\partial}(f-g)(z+\Lambda)=\overline{\partial}(f-g)(z)+\sum_{|I|\leq m}P'_I(\Lambda)\overline{\partial}g'_I(z) \tag{6.15}$$

where $g'_I(z)$ is independent of z_1 if $|I|=m$.

Repeating the same process for z_2 in (6.15), thanks again to Lemma 6.1.4, we can find a polynomial g_1 independent of z_1 such that

$$\overline{\partial}(f-g-g_1)(z+\Lambda)=\overline{\partial}(f-g-g_1)(z)+\sum_{I}P''_I(\Lambda)\overline{\partial}g''_I(z)$$

where $g''_I(z)$ is independent of z_2 for $|I|=m$. Since g_1 is independent of z_1, (6.15) implies that $g''_I(z)$ is independent of both z_1 and z_2 for $|I|=m$. Repeating the same argument for $z_3,\cdots,z_n$, the lemma is proved. □

Lemma 6.1.4. *If all $\{g_I(z),\ \text{where } |I| = m\}$ in (6.10) of Lemma 6.1.3 depends only on variables $z_k, z_{k+1}, ...z_n$, then the polynomial g found in that lemma can be asked to depend also only on $z_k, z_{k+1}, ...z_n$.*

Proof. Assume that $k > 1$. We prove that $g(z)$ does not depend on z_1. Other cases follow from the same argument.

In fact, if $g(z)$ depends on z_1, then $g_{\overline{z}_1}(z)$ is of pure degree m. We differentiate

$$\overline{\partial} g(z+\Lambda) - \overline{\partial} g(z) - \sum_{|I|=m} P_I(\Lambda)\overline{\partial} g_I(z)$$

in Lemma 6.1.3 by $\overline{z}_1$. Since $g_I(z)$ is supposed to be independent of z_1 for $|I| = m$, we get that

$$\overline{\partial} g_{\overline{z}_1}(z+\Lambda) - \overline{\partial} g_{\overline{z}_1}(z)$$

is $\overline{\partial}$ of a polynomial of degree $\leq m-2$. Then $g_{\overline{z}_1}(z+\Lambda) - g_{\overline{z}_1}(z)$ should be a polynomial of degree $\leq m-2$ for all Λ. On the other hand, since $g_{\overline{z}_1}(z)$ is an anti-holomorphic polynomial of pure degree m, using Taylor development, we get

$$g_{\overline{z}_1}(z+\Lambda) - g_{\overline{z}_1}(z) - \sum_{i=1}^{n} \overline{\Lambda}_i g_{\overline{z}_1\overline{z}_i}(z)$$

is a polynomial of degree $\leq m-2$ for all $\Lambda \in \Gamma$, where $(\Lambda_1, \Lambda_2, \cdots, \Lambda_n)$ is the coordinate of Λ. Then

$$\sum_{i=1}^{n} \overline{\Lambda}_i g_{\overline{z}_1\overline{z}_i}(z)$$

is also of degree $\leq m-2$ for all $\Lambda \in \Gamma$. Observing that Γ is a lattice of $\mathbb{C}^n$ and at least one of $g_{\overline{z}_1\overline{z}_i}(z)$ is of pure degree $m-1$, we get a contradiction. □

Now we can prove the following proposition.

Proposition 6.1.5. *Let E_m be a unipotent numerical flat bundle of rank m on a torus T. Then E_m is a local system.*

Moreover, let $T = \mathbb{C}^n/\Gamma$ and $\Lambda = (\Lambda_1, ..., \Lambda_n) \in \Gamma$, We have the following two propositions :

***A**$_m$: The transformation matrices of E_m can be choosen as*

$$g_\Lambda(x, z) = (x + \Lambda, M_m(\Lambda)z)$$

where

$$M_m(\Lambda) = \begin{pmatrix} Id & c_{1,2}(\Lambda) & c_{1,3}(\Lambda) & c_{1,m}(\Lambda) \\ 0 & Id & c_{2,3}(\Lambda) & ... \\ 0 & 0 & ... & .. \\ 0 & 0 & 0 & Id \end{pmatrix}$$

and $c_{i,j}(\Lambda)$ are anti-holomorphic polynomials on Λ (independent on z).

***B**$_m$: If A_m is true, using the above transformation matrices, any element in $H^1(T, E_m)$ can be represented by*

$$\begin{pmatrix} \overline{\partial} s_1(z) \\ \overline{\partial} s_2(z) \\ ... \\ \overline{\partial} s_m(z) \end{pmatrix} \in H^1(T, E_m),$$

where s_i are anti-holomorphic polynomials on $\mathbb{C}^n$. [1]

1. We identify the vector bundle (T, E_m) with the local system $(\mathbb{C}^n, V = \mathbb{C}^m, M_m(\Lambda))$ in A_m.

Proof. We prove these two statements by induction on the rank of the unipotent numerical flat bundle. We will prove that

$$A_{k-1} + B_{k-1} \Rightarrow A_k$$

and

$$A_{k-1} + B_{k-2} \Rightarrow B_{k-1}.$$

If the above two implications are proved, then by induction A_m, B_m are true for all $m \in \mathbb{N}$.

Proof of $A_{k-1} + B_{k-1} \Rightarrow A_k$:

By defintion, E_k is an extension of a unipotent numerical flat bundle E_{k-1} and $\mathscr{O}_T$:

$$0 \to E_{k-1} \to E_k \to \mathscr{O}_T \to 0.$$

Using A_{k-1}, we can suppose that E_{k-1} is a local system given by the transformation matrices $M_{k-1}(\Lambda)$. Using B_{k-1}, E_k is induced by an element of the form

$$\alpha = \begin{pmatrix} \overline{\partial} s_1(z) \\ \overline{\partial} s_2(z) \\ \cdots \\ \overline{\partial} s_{k-1}(z) \end{pmatrix} \in H^1(T, E_{k-1}),$$

where all s_i are anti-holomorphic polynomials on $\mathbb{C}^n$. Then the transformation matrices of E_k are given by

$$M_k(\Lambda) = \begin{pmatrix} M_{k-1}(\Lambda) & c_k(\Lambda, z) \\ 0 & Id \end{pmatrix}$$

where

$$c_k(\Lambda, z) = \begin{pmatrix} s_1(z+\Lambda) \\ s_2(z+\Lambda) \\ \cdots \\ s_{k-1}(z+\Lambda) \end{pmatrix} - M_{k-1}(\Lambda) \cdot \begin{pmatrix} s_1(z) \\ s_2(z) \\ \cdots \\ s_{k-1}(z) \end{pmatrix}$$

is holomorphic on z. Since all s_i are anti-holomorphic, $c_k(z, \Lambda)$ is also anti-holomorphic on z. Therefore $c_k(z, \Lambda)$ is constant on z and depends only on Λ. Moreover, since $M_{k-1}(\Lambda)$ and $s_i(z+\Lambda)$ are anti-holomorphic polynomials on Λ, all $c_k(\Lambda, z)$ are also anti-holomorphic polynomials on Λ. A_k is proved.

Proof of $A_k + B_{k-1} \Rightarrow B_k$:

By A_k, E_k is given by a local system $M_k(\Lambda)$. Taken an element $\alpha \in H^1(T, E_k)$, since $\mathbb{C}^n$ in Stein, α can be represented by

$$\begin{pmatrix} \overline{\partial} f_1(z) \\ \overline{\partial} f_2(z) \\ \cdots \\ \overline{\partial} f_k(z) \end{pmatrix} \in H^1(T, E_k),$$

for some smooth functions $f_i(z)$ on $\mathbb{C}^n$ and

$$\begin{pmatrix} \overline{\partial} f_1(z+\Lambda) \\ \overline{\partial} f_2(z+\Lambda) \\ \cdots \\ \overline{\partial} f_k(z+\Lambda) \end{pmatrix} = M_k(\Lambda) \cdot \begin{pmatrix} \overline{\partial} f_1(z) \\ \overline{\partial} f_2(z) \\ \cdots \\ \overline{\partial} f_k(z) \end{pmatrix}. \tag{6.16}$$

We need to prove that α can be also represented by $\begin{pmatrix} \overline{\partial} s_1(z) \\ \overline{\partial} s_2(z) \\ \cdots \\ \overline{\partial} s_k(z) \end{pmatrix}$, where all s_i are anti-holomorphic polynomials on $\mathbb{C}^n$.

Thanks to the exact sequence

$$0 \to \mathscr{O}_T = E_1 \to E_k \to E_k/E_1 \to 0, \tag{6.17}$$

we have

$$\begin{pmatrix} \overline{\partial} f_2(z) \\ \overline{\partial} f_3(z) \\ \cdots \\ \overline{\partial} f_k(z) \end{pmatrix} \in H^1(T, E_k/E_1).$$

By A_{k-1} (which is implied by A_k), E_k/E_1 is a local system. Then B_{k-1} implies the existence of $h(z) \in C^\infty(T, E_k/E_1)$ and anti-holomorphic polynomials $s_i(z)$ on $\mathbb{C}^n$, such that

$$\begin{pmatrix} \overline{\partial} f_2(z) \\ \overline{\partial} f_3(z) \\ \cdots \\ \overline{\partial} f_k(z) \end{pmatrix} = \begin{pmatrix} \overline{\partial} s_2(z) \\ \overline{\partial} s_3(z) \\ \cdots \\ \overline{\partial} s_k(z) \end{pmatrix} + \overline{\partial} h(z). \tag{6.18}$$

Case 1 : $h = 0$.

Applying (6.18) to (6.16), we have implies that :

$$\overline{\partial} f_1(z+\Lambda) = \overline{\partial} f_1(z) + \sum_I P_I(\Lambda) \overline{\partial} g_I(z),$$

where $g_I(z)$ are monomial anti-holomorphic polynomials with index I and $P_I(\Lambda)$ are anti-holomorphic polynomials on Λ. Let

$$M = \max\{|I|, g_I(z) \neq 0\}.$$

By Lemma 6.1.3, there exists an anti-holomorphic polynomial g of pure degree $m+1$ such that

$$\overline{\partial} g(z+\Lambda) - \overline{\partial} g(z) - \sum_{|I|=M} P_I(\Lambda) \overline{\partial} g_I(z)$$

is of the form $\sum_{|I|<M} \widetilde{P}_I(\Lambda) \overline{\partial} \widetilde{g}_I(z)$. Then $f_1(z) - g(z)$ satisfies the equation

$$\overline{\partial}(f_1 - g)(z+\Lambda) = \overline{\partial}(f_1 - g)(z) + \sum_{|I|<M} P'_I(\Lambda) \overline{\partial} g'_I(z)$$

for some new anti-holomorphic polynomials P'_I and $g'_I(z)$ with degree smaller than M. We repeat the process and get finally an anti-holomorphic polynomial $\widetilde{f}_1(z)$ such that

$$\overline{\partial}(f_1 - \widetilde{f}_1)(z+\Lambda) = \overline{\partial}(f_1 - \widetilde{f}_1)(z).$$

Then

$$\overline{\partial}(f_1 - \widetilde{f}_1)(z) \in H^1(T, \mathscr{O}).$$

Thus there is a linear anti-holomorphic function $l(z)$ such that

$$\overline{\partial} f_1(z) = \overline{\partial}(\widetilde{f}_1(z) + l(z)).$$

Then

$$\alpha = \begin{pmatrix} \overline{\partial} f_1(z) \\ \overline{\partial} f_2(z) \\ \cdots \\ \overline{\partial} f_k(z) \end{pmatrix} = \begin{pmatrix} \overline{\partial}(\widetilde{f}_1(z) + l(z)) \\ \overline{\partial} s_2(z) \\ \cdots \\ \overline{\partial} s_k(z) \end{pmatrix}.$$

B_k is proved.

Case 2 : h is not 0.

We would like to reduce Case 2 to Case 1. Since $\overline{\partial} h(z) = 0 \in H^1(T, E_k/E_{k-1})$, using (6.17), we can find a $(0,1)$-form $h_0(z)$ on $\mathbb{C}^n$ such that $\begin{pmatrix} h_0(z) \\ \overline{\partial} h(z) \end{pmatrix}$ is a E_k-valued $(0,1)$-form and

$$\overline{\partial} h_0(z) = 0 \in H^2(T, \mathscr{O}_T).$$

Therefore there exists a Γ-periodic $(0,1)$-form $h_1(z)$ such that

$$\overline{\partial} h_0(z) = \overline{\partial} h_1(z).$$

Then

$$\begin{pmatrix} h_0(z) - h_1(z) \\ \overline{\partial} h(z) \end{pmatrix} \in H^1(T, E_k).$$

Using the exact sequence

$$H^1(T, \mathscr{O}_T) \to H^1(T, E_k) \to H^1(T, E_k/E_1),$$

the image of $\begin{pmatrix} h_0 - h_1 \\ \overline{\partial} h \end{pmatrix}$ in $H^1(T, E_k/E_1)$ is $\overline{\partial} h(z)$ which is $[0] \in H^1(T, E_k/E_1)$ since $h(z)$ takes values in E_k/E_{k-1}. Therefore $\begin{pmatrix} h_0 - h_1 \\ \overline{\partial} h \end{pmatrix}$ comes from an element in $H^1(T, \mathscr{O}_T)$, i.e.

$$\begin{pmatrix} h_0(z) - h_1(z) \\ \overline{\partial} h(z) \end{pmatrix} = \begin{pmatrix} \overline{\partial} h_2(z) \\ 0 \end{pmatrix} + \overline{\partial} H(z)$$

for some $H(z) \in C^\infty(T, E_k)$ and an anti-holomorphic polynomial $h_2(z)$. Then

$$\begin{pmatrix} \overline{\partial} f_1(z) \\ \overline{\partial} f_2(z) \\ \dots \\ \overline{\partial} f_k(z) \end{pmatrix} = \begin{pmatrix} \overline{\partial} f_1(z) - h_0(z) + h_1(z) \\ \overline{\partial} s_2(z) \\ \dots \\ \overline{\partial} s_k(z) \end{pmatrix} + \begin{pmatrix} h_0(z) - h_1(z) \\ \overline{\partial} h(z) \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\partial} f_1(z) - h_0(z) + h_1(z) \\ \overline{\partial} s_2(z) \\ \dots \\ \overline{\partial} s_k(z) \end{pmatrix} + \begin{pmatrix} \overline{\partial} h_2(z) \\ 0 \end{pmatrix} \in H^1(T, E_k).$$

By Case 1, we can choose $\overline{\partial} f_1(z) - h_0(z) + h_1(z)$ to be an $\overline{\partial}$ of an anti-holomorphic polynomial. B_k is proved. □

Remark. *Thanks to the fact that all $c_{i,j}(\Lambda)$ are anti-holomorphic polynomials on Λ, we get that all element in $H^0(T, E_m)$ should be parallel with respect to the Gauss-Manin connection induced by A_m.*

Proposition 6.1.6. *Let E be a numerically flat holomorphic vector bundle on a torus T. Then E is induced by locally constant transformation*

$$g_\Lambda(x, z) = (x + \Lambda, M(\Lambda) z)$$

where

$$M(\Lambda) = \begin{pmatrix} U_1(\Lambda) & c_2(\Lambda) & \dots & c_k(\Lambda) \\ 0 & U_2(\Lambda) & \dots & \dots \\ 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & U_k(\Lambda) \end{pmatrix}$$

are constant supertriangle matrices by blocks for $\Lambda \in \Gamma$ and U_i are irreducible hermitian flat vector bundle, where Γ is the lattice of T.

Proof. Using [DPS94, Theorem 1.18], we know that E admits a filtration

$$0 = E_0 \subset E_1 \subset \dots \subset E_m = E$$

such that the quotients E_i/E_{i-1} are irreducible hermitian flat vector bundles. We prove the proposition by induction on the number m.

If $m = 1$, the proposition comes from the definition of hermitian flat vector bundles.

Assume now that the proposition is true for $m = k$. If $m = k + 1$, thanks to the exact sequence

$$0 \to E_k \to E_{k+1} \to E_{k+1}/E_k \to 0,$$

we have $E_{k+1} \in H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_k))$. To prove the proposition, it is sufficient to prove that all the elements in $H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_k))$ induce constant tranformation matrices.

Since the bundle $\mathrm{Hom}(E_{k+1}/E_k, E_k)$ is also numerically flat, Lemma 6.1.1 implies that we can write $\mathrm{Hom}(E_{k+1}/E_k, E_k)$ as

$$\mathrm{Hom}(E_{k+1}/E_k, E_k) = \mathrm{Hom}(E_{k+1}/E_k, E_k)_{\mathrm{uni}} \oplus E'$$

where $H^1(T, E') = 0$ and $\mathrm{Hom}(E_{k+1}/E_k, E_k)_{\mathrm{uni}}$ is a unipotent numerical flat bundle. Then

$$H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_k)) = H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_k)_{\mathrm{uni}}). \tag{6.19}$$

By Proposition 6.1.5, all elements in $H^1(T, \mathrm{Hom}(E_{k+1}/E_k, E_k))_{\mathrm{uni}}$ induce constant tranformation matrices. The proposition is proved. □

Finally, we have

Corollary 6.1.7. *Let E be a numerically flat holomorphic vector bundle on an étale quotient of a torus X. Then E is a local system.*

Proof. By definition, we have an étale quotient $\pi : T \to X$ for a torus T and a finite group G. Applying Proposition 6.1.5 to the vector bundle $\pi^*(E)$. Thanks to the finiteness of G, we can suppose that all $s_i(z)$ in Proposition 6.1.5 are G-invariant. E is thus a local system. □

6.2 A Hovanskii-Teissier inequality

In this appendix, we give the proof of the Hovanskii-Teissier concavity inequality in the Kähler case, which is a direct consequence of [DN06, Thm A, C].

Proposition 6.2.1. *Let (X, ω_X) be a compact Kähler manifold of dimension n, and let α, β be two nef class. Then we have*

$$\int_X (\alpha^i \wedge \beta^j \wedge \omega_X^{n-i-j}) \geq \tag{6.20}$$

$$(\int_X \alpha^{i-k} \wedge \beta^{j+k} \wedge \omega_X^{n-i-j})^{\frac{s}{k+s}} \cdot (\int_X \alpha^{i+s} \wedge \beta^{j-s} \wedge \omega_X^{n-i-j})^{\frac{k}{k+s}}.$$

Proof. Let α, β be two nef class, and let $\omega_1, \cdots, \omega_{n-2}$ be $n-2$ arbitrary Kähler classes. Thanks to [DN06, Thm.A], the bilinear form on $H^{1,1}(X)$

$$Q([\lambda], [\mu]) = \int_X \lambda \wedge \mu \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \qquad \lambda, \mu \in H^{1,1}(X)$$

is of signature $(1, h^{1,1} - 1)$. Since α, β are all nef, the function $f(t) = Q(\alpha + t\beta, \alpha + t\beta)$ is indefinite on $\mathbb{R}$ if and only if α and β are linearly independent. Therefore

$$\int_X (\alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2}) \geq \tag{6.21}$$

$$(\int_X \alpha^2 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2})^{\frac{1}{2}} \cdot (\int_X \beta^2 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2})^{\frac{1}{2}},$$

and the equality holds if and only if α, β are linearly dependent.

If we let $\omega_1, \cdots, \omega_{i-1}$ tend to α, let $\omega_i, \cdots, \omega_{i+j-2}$ tend to β and let $\omega_{i+j-1} = \cdots = \omega_{n-2} = \omega_X$ in inequality (6.21), we get

$$\int_X (\alpha^i \wedge \beta^j \wedge \omega_X^{n-i-j}) \geq (\int_X \alpha^{i-1} \wedge \beta^{j+1} \wedge \omega_X^{n-i-j})^{\frac{1}{2}} \cdot (\int_X \alpha^{i+1} \wedge \beta^{j-1} \wedge \omega_X^{n-i-j})^{\frac{1}{2}}.$$

Then (6.20) is an easy consequence of the above inequality. □

Remark 6.2.2. *It is easy to see that the equality holds in* (6.21) *if and only if* α *and* β *are colinear.*

6.3 A Bochner technique proof

We would like to give a proof of the implication $(iii) \Rightarrow (ii)$ in Proposition 5.4.1 without using [BM01, Theorem 0.1].

Proof. By [CDP12, Criterion 1.1], to prove the implication, it is sufficient to prove that for some ample line bundle F on X, there exists a constant $C_F > 0$, such that

$$H^0(X, (T_X^*)^{\otimes m} \otimes F^{\otimes k}) = 0 \qquad \text{for all } m, k \text{ with } m \geq C_F \cdot k. \tag{6.22}$$

Thanks to the condition (iii), there exists a Kähler class A, such that

$$\mu_A(\mathscr{F}_i / \mathscr{F}_{i-1}) \geq c \qquad \text{for all } i,$$

for some constant $c > 0$. Moreover, for the Harder-Narasimhan filtration of $(T_X)^{\otimes m}$ with respect to A, $m \cdot c$ is also a lower bound of the minimal slope with respect to the filtration.

We now prove (6.22) by a basic Bochner technique. After replacing by a more refined filtration, we can suppose that

$$0 \subset \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = (T_X)^{\otimes m} \tag{6.23}$$

is a filtration of torsion-free subsheaves such that $\mathscr{E}_{i+1}/\mathscr{E}_i$ is an ω-stable torsion-free subsheaf of $T_X/\mathscr{E}_i$ of maximal slope for simplicity. Let ω be a positive $(1,1)$-form representing $c_1(A)$.

If all the quotients of the filtration (6.23) are free, then there exists a Hermitian-Einstein metric on every quotient. Since $\mu_A(\mathscr{E}_i/\mathscr{E}_{i-1}) \geq c \cdot m$, thanks to Lemma 5.2.7, we can construct a smooth metric h on $(T_X)^{\otimes m}$, such that

$$\frac{i\Theta_h(T_X^{\otimes m}) \wedge \omega^{n-1}}{\omega^n} \geq \frac{m \cdot c}{2} \operatorname{Id}. \tag{6.24}$$

Let $\tau \in H^0(X, (T_X^*)^{\otimes m} \otimes F^{\otimes k})$. We have

$$\Delta_\omega(\|\tau\|_{h^*}^2) = \|D_h'\tau\|^2 - \frac{\langle i\Theta_{h^*}((T_X^*)^{\otimes m} \otimes F^{\otimes k})\tau, \tau\rangle \wedge \omega^{n-1}}{\omega^n}. \tag{6.25}$$

If $m \geq C_F \cdot k$ for some constant C_F big enough with respect to c, (6.24) implies that

$$\int_X \|D_h'\tau\|^2 \omega^n - \langle i\Theta_{h^*}((T_X^*)^{\otimes m} \otimes F^{\otimes k})\tau, \tau\rangle \wedge \omega^{n-1} \geq c_1 \|\tau\|_{h^*}^2$$

for some constant $c_1 > 0$. Observing moreover that

$$\int_X \Delta_\omega(\|\tau\|_{h^*}^2)\omega^n = 0,$$

then $\tau = 0$.

If the quotients $\mathscr{E}_i/\mathscr{E}_{i-1}$ of (6.23) are not necessary free, by Lemma 5.2.6, we can find a resolution $\pi : \widetilde{X} \to X$ such that there exists a filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset \pi^*(T_X)$$

where $E_i, E_i/E_{i-1}$ are vector bundles and

$$\mu_{\pi^*(A)}(E_i/E_{i-1}) = \mu_A(\mathscr{E}_i/\mathscr{E}_{i-1}) \geq c \cdot m.$$

Thanks to the strict positivity of c, for ϵ small enough,

$$\mu_\epsilon(E_i/E_{i-1}) \geq \frac{c \cdot m}{2} \qquad \text{for any } i, \tag{6.26}$$

where μ_ϵ is the slope with respect to $\pi^*(A)+\epsilon\omega_{\widetilde{X}}$. Thanks to the remark of Theorem 5.3.1, E_i/E_{i-1} are also stable for $\pi^*(A) + \epsilon\omega_{\widetilde{X}}$ when ϵ small enough. Therefore there exists a smooth Hermitian-Einstein metric on every quotient E_i/E_{i-1}. Using Lemma 5.2.7, (6.26) implies that we can thus construct a smooth metric h_ϵ on $\pi^*(T_X)^{\otimes m}$, such that

$$\frac{i\Theta_{h_\epsilon}(\pi^* T_X^{\otimes m}) \wedge (\pi^*(\omega) + \epsilon\omega_{\widetilde{X}})^{n-1}}{(\pi^*(\omega) + \epsilon\omega_{\widetilde{X}})^n} \geq \frac{m \cdot c}{4} \operatorname{Id} \tag{6.27}$$

for ϵ small enough. Using the same Bochner technique on $\pi^*(T_X)$ with respect to $\pi^*(A) + \epsilon\omega_{\widetilde{X}}$ as in (6.24) and (6.25), we get

$$H^0(\widetilde{X}, \pi^*((T_X^*)^{\otimes m} \otimes F^{\otimes k})) = 0 \qquad \text{for all } m, k \text{ with } m \geq C_F \cdot k.$$

(6.22) is thus proved. □

Résumé

L'objet principal de cette thèse est de généraliser un certain nombre de résultats bien connus de la géométrie algébrique au cas kählerien non nécessairement projectif. On généralise d'abord le théorème d'annulation de Nadel au cas kählerien arbitraire. On obtient aussi un cas particulier du théorème d'annulation de Kawamata-Viehweg pour les variétés qui admettent une fibration vers un tore dont la fibre générique est projective. En utilisant ce résultat, on étudie le problème de déformation pour les variétés kählériennes compactes sous une hypothèse portant sur leurs fibrés canoniques. On étudie enfin les variétés à fibré anticonique nef. On montre que si le fibré anticanonique est nef, alors le fibré tangent est à pentes semi-positif relative à la filtration de Harder-Narasimhan pour la polarization ω_X^{n-1}. Comme application, on donne une preuve simple de la surjectivité de l'application d'Albanese, et on étudie aussi la trivialité locale de l'application d'Albanese.

Resume

The aim of this thesis is to generalize a certain number of results of algebraic geometry to Kähler geometry. We first generalize the Nadel vanishing theorem to arbitrary compact Kähler manifolds. We prove also a particular version of the Kawamata-Viehweg vanishing theorem for manifolds admitting a fibration to a torus such that the generic fiber is projective. Using this result, we study the theory of deformations of compact Kähler manifolds under certain assumptions on their canonical bundles. Finally, we study varieties with nef anticanonical bundles. We prove that the slopes of the Harder-Narasimhan filtration of the tangent bundles with respect to a polarization of the form ω_X^{n-1} are semi-positive. As an application, we give a simple proof of the surjectivity of the Albanese map, and we investigate also its local triviality.

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