On the Hyperbolic Plane and Chinese Checkers

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Abstract

Chinese checkers is a game played on a hexagonal grid. This regular hexagonal tesselation is an artifact of Euclidean geometry that provides a fair playing field only for games of two, three, four or six players. Hyperbolic geometry allows tessallations of the plane by regular polygons with any number of sides. Hyperbolic versions of the chinese checkers board permit fair games with five, seven and even more players.

1 Introduction

Chinese checkers is a game for two to six players. Each player has ten marbles arranged in a triangle. The goal of the game is to move these marbles to the opposite triangle. In each turn, the player may move one marble to any adjacent position or jump the marble over any adjacent marble into an empty space, and that marble may continue jumping as permitted.

Chinese checkers is played on a hexagonal grid arranges as in Figure 1. This board provides a fair playing field for games of two, three, four and six players. In five player games, the goal triangles of the four players are filled whereas whereas the goal triangle of the fifth player is empty. We have called this the "tainted win[1]" since this position provides an unfair advantage over the other.

This paper attempts to create a fair playing grid for five players, and new fair playing grids for seven or more players. Triangulating a pentagon can produce such a fair grid, though sacrifices the straightness of edges across vertices, as shown in Figure 2.

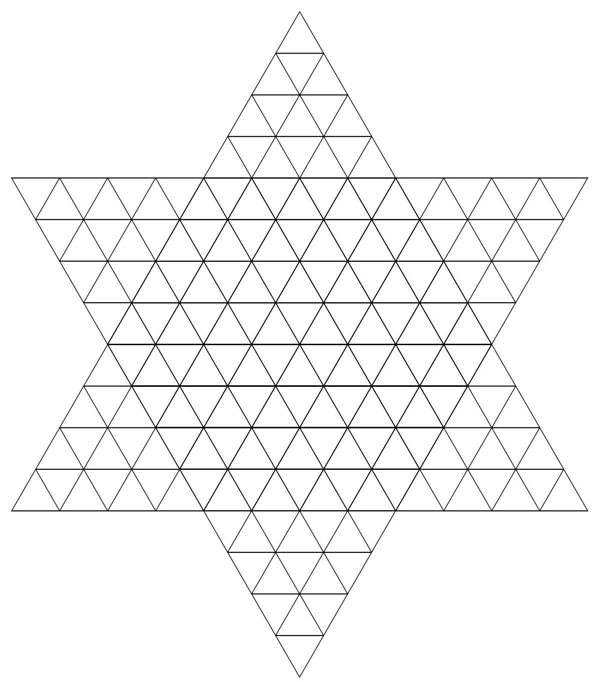

Figure 1: The chinese checkers playing board.

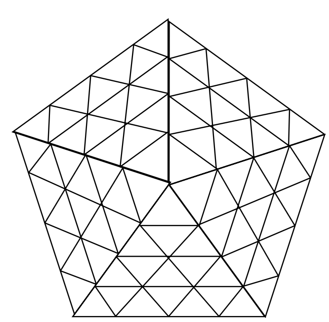

Figure 2: Pentagon triangulation.

2 Hyperbolic Geometry

Euclid defined the geometry that now shares his name with five axioms. The fifth axiom states that given

[1]Wayne Cochran, personal communication, lunchtime 1995.

a line and a point not on the line, then there exists only one line passing through the point that does not intersect the original line. This fifth axiom was long believed to be a consequence of the first four axioms, until *hyperbolic geometry* was devised, which follows the first four of Euclid's axioms but not the fifth.

There are several Euclidean models of hyperbolic geometry that make it easier to visualize the relationship between hyperbolic points and lines. Typically, the hyperbolic plane is represented by the unit open disk D centered at the origin. The unit-radius circle that forms the boundary of this disk is called the *circle at infinity* C_∞.

The projective *Klein* model represents hyperbolic lines with Euclidean lines. This model is popular for computer rendering of 3-D hyperbolic structures, such as those in the animated educational short "Not Knot" [Gunn, 1993], because it can be incorporated into the standard homogeneous 4×4 transformation matrix implemented in computer graphics hardware [Phillips & Gunn, 1992].

An alternative representation of hyperbolic geometry is the conformal *Poincare* model. This wass the model used by M.C. Escher [Dunham *et al.*, 1981]. The Poincare model represents hyperbolic lines with Euclidean circles such that the hyperbolic line passing through any two points is uniquely representing by the Euclidean circle piercing the two points that intersects the infinity disk orthogonally. The angle formed by the intersection of two circular arcs is the angle formed by their tangents at the intersection. Given two points $\mathbf{a}, \mathbf{b} \in D$ the origin o of the circle passing through $\mathbf{a}$ and $\mathbf{b}$, orthogonal to C_∞ is given by

$$\begin{aligned} o_x &= \frac{a_y(1+\mathbf{b}\cdot\mathbf{b}) - b_y(1+\mathbf{a}\cdot\mathbf{a})}{2(a_y b_x - a_x b_y)} \\ o_y &= \frac{b_x(1+\mathbf{a}\cdot\mathbf{a}) - a_x(1+\mathbf{b}\cdot\mathbf{b})}{2(a_y b_x - a_x b_y)} \end{aligned} \quad (1)$$

The radius r of this circle is then the Euclidean distance between a and o. If the hyperbolic line passes through the origin, then its representation is a Euclidean line (a diameter of the circle at infinity). An example is shown in Figure 3

The property of hyperbolic space that makes it so appealing for chinese checkers is that the hyperbolic plane can be tessellated by regular polygons with any number of sides. Tessellations of the hyperbolic plane are denoted $\{p, q\}$ which indicates a tesselation with p-gons, and q of these p-gons meet at each vertex. The only restriction is that

$$(p-2)(q-2) > 4. \quad (2)$$

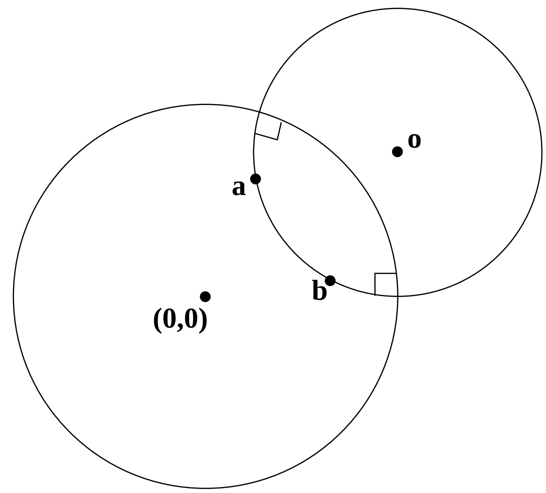

Figure 3: Geometry of the hyperbolic line piercing points $\mathbf{a}$ and $\mathbf{b}$.

The angles of hyperbolic triangles sum to less than π. Regular hyperbolic tesselations are described by a single right triangle with angles of π/p and π/q. The π/p vertex is placed at the origin, which causes two of the triangle's three hyperbolic sides to be straight Euclidean lines in the Poincare model. The only remaining degree of freedom is the scale of the triangle. The length of the edge from the π/p vertex to the π/q vertex is given by d as

$$r = \operatorname{acosh} \frac{\cos\frac{\pi}{p}\cos\frac{\pi}{q} + 1}{\sin\frac{\pi}{p}\sin\frac{\pi}{q}} \quad (3)$$

$$d = \frac{e^r - 1}{e^r + 1}. \quad (4)$$

Successive reflections of this triangle about its "straight" (non-circular) edges form an initial p-sided polygon. Successive reflections of the p-gon about its circular edge tessellate space. The reflection of a point about a Euclidean line is straightforward. The reflection of a point $\mathbf{x}$ about a circle of radius r centered at the point $\mathbf{o}$ is the point $\mathbf{y}$ that lies on a Euclidean ray extending from $\mathbf{o}$ through $\mathbf{x}$ and satisfies

$$d(\mathbf{o}, \mathbf{x}) d(\mathbf{o}, \mathbf{y}) = r^2 \quad (5)$$

where d returns the Euclidean distance.

3 Some Results

We can apply the regular hyperbolic polygon tesselation to solve the problem of providing a fair playing field for 5, 7 or more players. Figure 4 shows a fair playing field for five players whereas Figure 5 shows a fair playing field for seven players. These tessellations contain the polygons that fit within a Euclidean circle of radius .97.

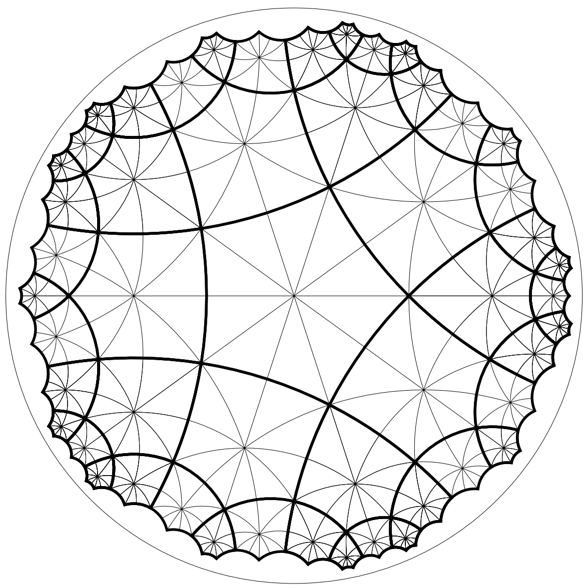

Figure 4: A fair playing field for five players generated by the tessellation {5,4}.

Small "start" and "goal" triangles need to be placed around the perimiter of the playing field. For the pentagonal tessellation, ten triangles would be placed, facing the ends of the five diameters (straight Euclidean lines). Likewise, for the septagonal tessellation, fourteen triangles would appear, placed at the ends of the seven diameters.

Unlike the pentagon triangulation in Figure 2, the conformal property of the Poincare model of hyperbolic geometry clearly defines the direction of each jump. Each hyperbolic polygon contains an extra vertex at its center. Line segments connect its vertices and edges (at their midpoints) to its center. For odd-sided polygons, jumping over a marble at the polygon's center takes a marble from the polygon's corner to its opposing edge.

4 Future Work

We've never actually played a chinese checkers game on one of these boards. The next obvious step is implementation, either in software or cardboard.

The standard chinese checkers board (Figure 1) organizes marbles into four distinct phase spaces[2]. A phase space is the set of all positions a marble could

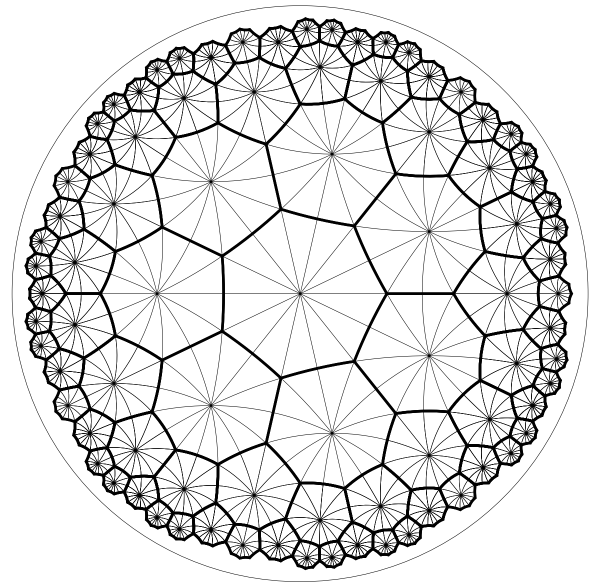

Figure 5: A fair playing field for seven players generated by the tessellation {7,3}.

ever jump into. Such analysis of hyperbolic chinese checkers boards remains open.

[2]Bart Stander, personal communication, lunchtime, 1995.

References

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