Elliptic Functions sn, cn, dn, as Trigonometry

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Background: Jacobi discovered that rather than studying elliptic integrals themselves, it is simpler to think of them as inverses for some functions like trig functions. For instance, recall that

$$\sin^{-1}(x) = \int_0^x \frac{dx}{\sqrt{1-x^2}},$$

but that it is easier to study $\sin(x)$ than the inverse sine. The resulting elliptic functions satisfy non-linear DEs that arise in many applications.

Here we develop the Jacobi elliptic functions as a form of trigonometric functions, but using an ellipse rather than a circle. These notes evolved from a lecture by William M. Kinnersley, circa 1975. The approach ought to be in some classic text, but I have not found it.

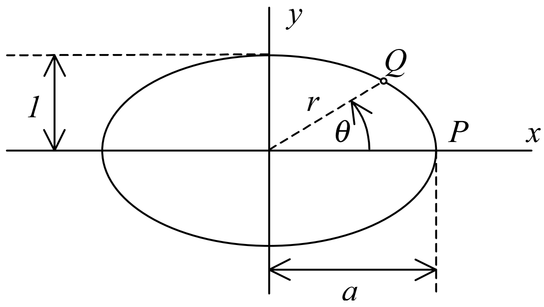

Figure 1: ellipse featured in construction.

Trigonometry of the ellipse: The ellipse equation is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

but we normalize the ellipse by choosing $b = 1$ so that,

$$\left(\frac{x}{a}\right)^2 + y^2 = 1. \tag{1}$$

Also of course,

$$x^2 + y^2 = r^2. \tag{2}$$

The eccentricity of an ellipse with general a, b is

$$\frac{b^2}{a^2} = 1 - \epsilon^2, \qquad \text{or} \qquad \epsilon = \sqrt{1 - \frac{b^2}{a^2}},$$

so that $\epsilon = 0$ for a circle, $\epsilon = 1$ for a parabola. Since $b = 1$, the eccentricity is

$$\epsilon \equiv k = \sqrt{1 - \frac{1}{a^2}},$$

which is the *modulus* of the corresponding elliptic functions. Thus $0 \le k \le 1$, and $k = 1$ should give ordinary trigonometry.

The next and very important thing to define is the *argument* u of the elliptic functions. The u is the thing the elliptic functions are functions <u>of</u>. In the case of trig functions, the argument would be the angle θ, but here u is a bit more complicated.

$$u \equiv \int_P^Q r\, d\theta, \tag{3}$$

where P and Q are as shown in Fig. 1. Notice that u is not an angle. It is not arc length and it is not area either. However, u becomes the angle θ or arc length in the limit $a \to 1$, or $k \to 0$ when the ellipse becomes a circle.

With the argument and modulus of the elliptic functions defined, the functions themselves are just ratios, just as in the case of trigonometry.

$$\begin{aligned} \mathrm{sn}(u,k) &= y, & (4)\\ \mathrm{cn}(u,k) &= x/a, & (5)\\ \mathrm{dn}(u,k) &= r/a. & (6) \end{aligned}$$

The first two generalize the sine and cosine, and the third comes about because the radius is not constant on an ellipse. When $k \to 0$, so that $a = 1$, these become just y, x, and $+1$, since $r \to 1$ also. This connects the elliptic functions to $\sin\theta$, $\cos\theta$ and $+1$.

There are several notational points to mention here. First, one often omits the modulus k in writing the elliptic functions and just writes

$$\mathrm{sn}\, u = \mathrm{sn}(u,k), \qquad \text{and so on.}$$

Corresponding to a given modulus k there is a *complementary modulus* k' such that

$$k' = \sqrt{1-k^2}.$$

There are also other notations. For example, a modern invention is to use $m = k^2$ so that fewer square roots appear. Then one defines

$$\mathrm{sn}(u|m) \equiv \mathrm{sn}(u,k), \qquad \text{where } m = k^2.$$

In fact there are twelve Jacobi elliptic functions, defined using a simple convention

$$\begin{array}{lll}
\mathrm{ns}\,u = \frac{1}{\mathrm{sn}\,u} & \mathrm{nc}\,u = \frac{1}{\mathrm{cn}\,u} & \mathrm{nd}\,u = \frac{1}{\mathrm{dn}\,u} \\
\mathrm{sc}\,u = \frac{\mathrm{sn}\,u}{\mathrm{cn}\,u} & \mathrm{dc}\,u = \frac{\mathrm{dn}\,u}{\mathrm{cn}\,u} & \mathrm{cs}\,u = \frac{\mathrm{cn}\,u}{\mathrm{sn}\,u} \\
\mathrm{ds}\,u = \frac{\mathrm{dn}\,u}{\mathrm{sn}\,u} & \mathrm{sd}\,u = \frac{\mathrm{sn}\,u}{\mathrm{dn}\,u} & \mathrm{cd}\,u = \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u}
\end{array}$$

and these all satisfy certain nonlinear differential equations, as we shall see.

From Eq.(1) we have

$$\mathrm{cn}^2\,u + \mathrm{sn}^2\,u = 1, \tag{7}$$

which generalizes $\cos^2\theta + \sin^2\theta = 1$. An then from Eq.(2),

$$\mathrm{dn}^2\,u + k^2\,\mathrm{sn}^2\,u = 1. \tag{8}$$

The differential relations now follow essentially from Eqs(1) and (2), just as the differentials of the sine and cosine follow from the Pytagorean formula. From

$$\theta = \tan^{-1}\left(\frac{y}{x}\right),$$

one has

$$d\,\theta = \frac{1}{r^2}(x\,dy - y\,dx).$$

But

$$du = r\,d\theta = \frac{1}{r}(x\,dy - y\,dx).$$

Also, from Eq.(1),

$$\frac{x\,dx}{a^2} + y\,dy = 0,$$

so one can replace either

$$dy = -\frac{x}{a^2\,y}\,dx,$$

or

$$dx = -\frac{a^2\,y}{x}\,dy.$$

The corresponding substitutions for du are therefore

$$du = \frac{1}{r}\left(-\frac{x^2}{a^2\,y} - y\right)dx,$$

or

$$du = \frac{1}{r}\left(x + \frac{a^2\,y^2}{x}\right)dy.$$

With these substitutions we get the following formulas for differentiating elliptic functions (with respect to the argument u, not k),

$$\frac{d}{du}\,\mathrm{sn}\,u = \mathrm{cn}\,u\,\mathrm{dn}\,u, \tag{9}$$

$$\frac{d}{du}\,\mathrm{cn}\,u = -\,\mathrm{sn}\,u\,\mathrm{dn}\,u, \tag{10}$$

$$\frac{d}{du}\,\mathrm{dn}\,u = -\,k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u. \tag{11}$$

Equations (9) and (10) relate in obvious ways to the trigonometric limit, while Eq.(11) is new. It reduces to an identity when $k \to 0$.

The elliptic functions satisfy differential equations that we find by starting with a solution and working backward. Apparently the modulus k should enter the DE as a parameter.

$$\frac{d}{du}\,\mathrm{sn}\,u = \mathrm{cn}\,u\,\mathrm{dn}\,u = \sqrt{1-\mathrm{sn}^2\,u}\sqrt{1-k^2\,\mathrm{sn}^2\,u},$$

so if $y(u) = \mathrm{sn}\,u$, then

$$\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2\,y^2). \tag{12}$$

If I solve for $u(y)$,

$$u = c + \int \frac{dy}{\sqrt{1-y^2}\sqrt{1-k^2\,y^2}},$$

which I recognize as an elliptic integral of the first kind, $F(y, k)$. Thus, as I mentioned earlier, the elliptic functions are the inverse functions for the elliptic integrals. On the other hand, If I differentiate Eq.(12) again with respect to u I get

$$y'' + (1 + k^2)\, y - 2\, k^2\, y^3 = 0. \tag{13}$$

This relates to a nonlinear duffing-type oscillator. In fact, all twelve of the Jacobi elliptic functions satisfy nonlinear first order DEs like Eq.(12), and also nonlinear second order DEs like Eq.(13). Moreover, you will find that the squares of the elliptic functions satisfy equations of the form

$$(y')^2 + \alpha\, y^2 + \beta\, y^3 = 0,$$

and of the form

$$y'' + \gamma\, y + \delta\, y^2 = 0.$$

One can thus solve all such equations exactly, in closed form, in terms of elliptic functions. Different functions cover different parameter ranges.

Elliptic functions open up a window of solvable nonlinear (polynomial) DEs, all of which relate to physical problems and physical phenomena. I do not know of other types of solutions of this quality for any nonlinear dynamical problems.

Homework: Perform the same construction starting from a hyperbola,

$$\frac{x^2}{a^2} - y^2 = 1$$

rather than from the ellipse in Fig.(1). Thus define the "Jacobi hyperbolic functions," $\mathrm{sn}(u, k), = y$, $\mathrm{ch}(u, k) = x/a$ and $\mathrm{dh}(u, k) = r/a$ and derive their properties. You should find that,

$$\mathrm{ch}^2\, u - \mathrm{sh}^2\, u = 1$$

and

$$\frac{d}{du}\, \mathrm{sh}\, u = \mathrm{ch}\, u\, \mathrm{dh}\, u$$

and then compute all the other properties, including the first and second order DEs these functions satisfy. (By the way, these functions are not discussed in the literature, since they are related to elliptic functions with complex arguments, just as hyperbolic sines and cosines relate to sines and cosines of complex argument. Using the DEs, can you show this relationship?)