A HIGH-SCHOOL ALGEBRA[1] , WALLET-SIZED PROOF, OF THE BIEBERBACH CONJECTURE [After L. Weinstein]

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Dedicated to Leonard Carlitz[3] *, master of formal mathematics*

Weinstein's[2] brilliant short proof of de Branges'[1] theorem can be made yet much shorter(modulo routine calculations), completely elementary (modulo Löwner theory), self contained(no need for the esoteric Legendre polynomials' addition theorem), and motivated(ditto), as follows. Replace the text between p. 62, line 7 and p. 63, line 7, by Fact 1 below, and the text between the last line of p.63 and p.64, line 7, by Fact 2 below.

FACT 1: Let $f_t(z) = e^t z \exp(\sum_{k=0}^{\infty} c_k(t) z^k)$ where $c_k(t)$ are *formal* functions of t. Let z and w be related by $z/(1-z)^2 = e^t w/(1-w)^2$. The following formal identity holds. (For any formal Laurent series $f(z)$, $CT_z f(z)$ denotes the *Constant Term* of $f(z)$.)

$$(1+w)\frac{d}{dt}\{\sum_{k=1}^{\infty}(4/k - kc_k(t)\overline{c_k(t)})w^k\} =$$

$$(1-w)\sum_{k=1}^{\infty} \mathrm{Re}\, CT_z \left\{ \frac{\frac{\partial f_t(z)}{\partial t}}{\frac{z\partial f_t(z)}{\partial z}} \cdot (2(1+\ldots+kc_k(t)z^k) - kc_k(t)z^k)\cdot(2(1+\ldots+k\overline{c_k(t)}z^{-k}) - k\overline{c_k(t)}z^{-k}) \right\} w^k$$

Proof: Routine. (Obviously computer-implementable.) □

FACT 2: The polynomials $A_{k,n}(c)$, defined in terms of the formal power series (Laurent in w) expansion $(1-z(2c+(1-c)(w+1/w))+z^2)^{-1} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,n}(c)(w^k+w^{-k})z^n$ are non-negative.

Proof: This follows immediately from the stronger fact that the polynomials$B_{k,n}(c)$, defined by the expansion $(1-z(2c+(1-c)(w+1/w))+z^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{k,n}(c)(w^k+w^{-k})z^n$ are perfect squares. To prove that

$$B_{k,n}(c) := CT_{z,w} F(z,w,c,k,n) := CT_{z,w}[\frac{(1-z(2c+(1-c)(w+1/w))+z^2)^{-1/2}}{z^n w^k}]$$

are indeed perfect squares, the reader[4] can easily find polynomials in (n,k,c), p_0, p_1, p_2, p_3, and polynomials in (n,k,c,z,w), G_1, and G_2, both of degree 2 in both z and w, such that

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[1] and high-school (purely formal) calculus.

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[3] L. Carlitz was, for many years, editor of the Duke Journal, until he was relieved from his duties by the proponents of so-called "modern math", who proceeded to reject anything that smacked, even faintly, of Carlitz-style mathematics. But those who have drowned Carlitz will soon be drowned themselves, as post-modern, computer-assisted and computer-generated mathematics, that by its very nature is purely formal, will soon take over and make so-called "modern" mathematics a relic of the past.

[4] Human readers: Find a computer friend to help you with this. All you have to do is express G_1, G_2, qua polynomials in z, w (of degree 2 in each), generically, with "indeterminate coefficients", then divide (WZ) by F, simplify clear denominators, and equate all the coefficients of the monomials $z^i w^j$ in the resulting identity to 0, getting a linear system of equations, with 2[(2+1)·(2+1)]+4 unknowns, that your computer friend can easily solve. This method is called the WZ method, and the fact that such a recurrence *always* exists follows from the WZ theory (Wilf and Zeilberger, Invent. Math.108(1992), 575-633.), but at any particular instance, like in this case, no explicit reference to WZ theory need be made.

$$p_0 F(z,w,c,k,n) + p_1 F(z,w,c,k,n+1) + p_2 F(z,w,c,k,n+2) + p_3 F(z,w,c,k,n+3) = z\frac{d}{dz}\left(\frac{G_1 F}{z^3 w}\right) + w\frac{d}{dw}\left(\frac{G_2 F}{z w^3}\right) \quad . \tag{WZ}$$

Applying CT to both sides of (WZ), remembering the obvious fact that for *any* formal Laurent series $f(z)$, $CT(z(d/dz)f(z)) = 0$, we get that the $B_{k,n}$ satisfy the linear recurrence, in n:

$$p_0 B_{k,n}(c) + p_1 B_{k,n+1}(c) + p_2 B_{k,n+2}(c) + p_3 B_{k,n+3}(c) = 0 \qquad . \tag{Rec^2}$$

The recurrence (Rec^2) can be used to generate many $B_{k,n}(c)$, and it turns out, empirically for now, that they are all perfect squares $B_{k,n}(c) = L_{k,n}(c)^2$, for some double-sequence of polynomials $L_{k,n}(c)$. These empirically-generated polynomials can be used to find a (conjectured) linear recurrence

$$q_0 L_{k,n}(c) + q_1 L_{k,n+1}(c) + q_2 L_{k,n+2}(c) = 0 \quad , \tag{Rec}$$

where q_0, q_1, q_2 are polynomials of (n,k,c). Let's now *define* $L'_{k,n}(c)$ to be the solution of (Rec) under the appropriate initial conditions $L_{k,0}$, $L_{k,1}$, and define $B'_{k,n}(c) := L'_{k,n}(c)^2$. Using high-school linear algebra (which is implemented in the *gfun* Maple package developed by Salvy and Zimmerman) one can easily find a (third order) recurrence satisfied by $B'_{k,n}(c)$, that turns out to be identical with (Rec^2). Matching the three initial values $n = 0, 1, 2$ completes the proof. $\square$

References

1. L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154**(1985), 137-152.
2. L. Weinstein, *The Bieberbach Conjecture*, Duke Math. J. **59**(1991) 61-64.