Towards The Green-Griffiths-Lang Conjecture

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In memory of M. Salah Baouendi

Abstract The Green-Griffiths-Lang conjecture stipulates that for every projective variety X of general type over $\mathbb{C}$, there exists a proper algebraic subvariety of X containing all non constant entire curves $f : \mathbb{C} \to X$. Using the formalism of directed varieties, we prove here that this assertion holds true in case X satisfies a strong general type condition that is related to a ceain jet-semistitoperty of the tangent bundle T_X. We then give a sufficient criterion for the Kobayashi hyperbolicity of an arbitrary directed variety (X, V).

Keywords Projective algebraic variety · Variety of general type · Entire curve · Jet bundle · Semple tower · Green-griffiths-lang conjecture · Holomorphic morse inequality · Semistable vector bundle · Kobayashi hyperbolic

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1 Introduction

The goal of this paper is to study the Green-Griffiths-Lang conjecture, as stated in [7, 10]. It is useful to work in a more general context and consider the category of directed projective manifolds (or varieties). Since the basic problems we deal with are birationally invariant, the varieties under consideration can always be replaced by nonsingular models. A directed projective manifold is a pair (X, V) where X is a projective manifold equipped with an analytic linear subspace $V \subset T_X$, i.e. a closed irreducible complex analytic subset V of the total space of T_X, such that each fiber

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$V_x = V \cap T_{X,x}$ is a complex vector space [If X is not irreducible, V should rather be assumed to be irreducible merely over each component of X, but we will hereafter assume that our varieties are irreducible]. A morphism $\Phi : (X, V) \to (Y, W)$ in the category of directed manifolds is an analytic map $\Phi : X \to Y$ such that $\Phi_* V \subset W$. We refer to the case $V = T_X$ as being the *absolute case*, and to the case $V = T_{X/S} = \mathrm{Ker}\, d\pi$ for a fibration $\pi : X \to S$, as being the *relative case*; V may also be taken to be the tangent space to the leaves of a singular analytic foliation on X, or maybe even a non integrable linear subspace of T_X.

We are especially interested in *entire curves* that are tangent to V, namely non constant holomorphic morphisms $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ of directed manifolds. In the absolute case, these are just arbitrary entire curves $f : \mathbb{C} \to X$. The Green-Griffiths-Lang conjecture, in its strong form, stipulates

1.1 GGL conjecture Let X be a projective variety of general type. Then there exists a proper algebraic variety $Y \subsetneq X$ such that every entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$.

[The weaker form would state that entire curves are algebraically degenerate, so that $f(\mathbb{C}) \subset Y_f \subsetneq X$ where Y_f might depend on f]. The smallest admissible algebraic set $Y \subset X$ is by definition the *entire curve locus* of X, defined as the Zariski closure

$$\mathrm{ECL}(X) = \overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}. \tag{1.1}$$

If $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined over a number field $\mathbb{K}_0$ (i.e. by polynomial equations with equations with coefficients in $\mathbb{K}_0$) and $Y = \mathrm{ECL}(X)$, it is expected that for every number field $\mathbb{K} \supset \mathbb{K}_0$ the set of $\mathbb{K}$-points in $X(\mathbb{K}) \smallsetminus Y$ is finite, and that this property characterizes $\mathrm{ECL}(X)$ as the smallest algebraic subset Y of X that has the above property for all $\mathbb{K}$ [10]. This conjectural arithmetical statement would be a vast generalization of the Mordell-Faltings theorem, and is one of the strong motivations to study the geometric GGL conjecture as a first step.

1.2 Problem (*generalized GGL conjecture*) Let (X, V) be a projective directed manifold. Find geometric conditions on V ensuring that all entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ are contained in a proper algebraic subvariety $Y \subsetneq X$. Does this hold when (X, V) is of general type, in the sense that the canonical sheaf K_V is big ?

As above, we define the entire curve locus set of a pair (X, V) to be the smallest admissible algebraic set $Y \subset X$ in the above problem, i.e.

$$\mathrm{ECL}(X, V) = \overline{\bigcup_{f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)} f(\mathbb{C})}^{\mathrm{Zar}}. \tag{1.2}$$

We say that (X, V) is *Brody hyperbolic* if $\mathrm{ECL}(X, V) = \emptyset$; as is well-known, this is equivalent to Kobayashi hyperbolicity whenever X is compact.

In case V has no singularities, the *canonical sheaf* K_V is defined to be $(\det \mathcal{O}(V))^*$ where $\mathcal{O}(V)$ is the sheaf of holomorphic sections of V, but in general this naive definition would not work. Take for instance a generic pencil of elliptic curves $\lambda P(z) + \mu Q(z) = 0$ of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$, and the linear space V consisting of the tangents to the fibers of the rational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ defined by $z \mapsto Q(z)/P(z)$. Then V is given by

$$0 \longrightarrow \mathcal{O}(V) \longrightarrow \mathcal{O}(T_{\mathbb{P}^2_{\mathbb{C}}}) \xrightarrow{PdQ-QdP} \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6) \otimes \mathcal{J}_S \longrightarrow 0$$

where $S = \mathrm{Sing}(V)$ consists of the 9 points $\{P(z) = 0\} \cap \{Q(z) = 0\}$, and $\mathcal{J}_S$ is the corresponding ideal sheaf of S. Since $\det \mathcal{O}(T_{\mathbb{P}^2}) = \mathcal{O}(3)$, we see that $(\det(\mathcal{O}(V))^* = \mathcal{O}(3)$ is ample, thus Problem 1.2 would not have a positive answer (all leaves are elliptic or singular rational curves and thus covered by entire curves). An even more "degenerate" example is obtained with a generic pencil of conics, in which case $(\det(\mathcal{O}(V))^* = \mathcal{O}(1)$ and $\#S = 4$.

If we want to get a positive answer to Problem 1.2, it is therefore indispensable to give a definition of K_V that incorporates in a suitable way the singularities of V; this will be done in Definition 2.1 (see also Proposition 2.2). The goal is then to give a positive answer to Problem 1.2 under some possibly more restrictive conditions for the pair (X, V). These conditions will be expressed in terms of the tower of Semple jet bundles

$$(X_k, V_k) \to (X_{k-1}, V_{k-1}) \to \cdots \to (X_1, V_1) \to (X_0, V_0) := (X, V) \tag{1.3}$$

which we define more precisely in Sect. 2, following [1]. It is constructed inductively by setting $X_k = P(V_{k-1})$ (projective bundle of *lines* of V_{k-1}), and all V_k have the same rank $r = \mathrm{rank}\, V$, so that $\dim X_k = n + k(r-1)$ where $n = \dim X$. Entire curve loci have their counterparts for all stages of the Semple tower, namely, one can define

$$\mathrm{ECL}_k(X, V) = \overline{\bigcup_{f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)} f_{[k]}(\mathbb{C})}^{\mathrm{Zar}}. \tag{1.4}$$

where $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the k-jet of f. These are by definition algebraic subvarieties of X_k, and if we denote by $\pi_{k,\ell} : X_k \to X_\ell$ the natural projection from X_k to X_ℓ, $0 \le \ell \le k$, we get immediately

$$\pi_{k,\ell}(\mathrm{ECL}_k(X, V)) = \mathrm{ECL}_\ell(X, V), \qquad \mathrm{ECL}_0(X, V) = \mathrm{ECL}(X, V). \tag{1.5}$$

Let $\mathcal{O}_{X_k}(1)$ be the tautological line bundle over X_k associated with the projective structure. We define the k-stage Green-Griffiths locus of (X, V) to be

$$\mathrm{GG}_k(X, V) = \overline{(X_k \smallsetminus \Delta_k) \cap \bigcap_{m\in\mathbb{N}} \Big(\text{base locus of } \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* A^{-1}\Big)} \tag{1.6}$$

where A is any ample line bundle on X and $\Delta_k = \bigcup_{2\le\ell\le k} \pi_{k,\ell}^{-1}(D_\ell)$ is the union of "vertical divisors" (see Sect. 2; the vertical divisors play no role and have to be removed in this context). Clearly, $\mathrm{GG}_k(X, V)$ does not depend on the choice of A. The basic vanishing theorem for entire curves (cf. [1, 7, 16]) asserts that every entire curve $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ satisfies all differential equations $P(f) = 0$ arising from sections $P \in H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* A^{-1})$, hence

$$\mathrm{ECL}_k(X, V) \subset \mathrm{GG}_k(X, V). \tag{1.7}$$

(For this, one uses the fact that $f_{[k]}(\mathbb{C})$ is not contained in any component of Δ_k, cf. [1]). It is therefore natural to define the global Green-Griffiths locus of (X, V) to be

$$\mathrm{GG}(X, V) = \bigcap_{k\in\mathbb{N}} \pi_{k,0}\left(\mathrm{GG}_k(X, V)\right). \tag{1.8}$$

By (1.5) and (1.7) we infer that

$$\mathrm{ECL}(X, V) \subset \mathrm{GG}(X, V). \tag{1.9}$$

The main result of [4] (Theorem 2.37 and Corollary 4.4) implies the following useful information:

1.3 Theorem *Assume that (X, V) is of "general type", i.e. that the canonical sheaf K_V is big on X. Then there exists an integer k_0 such that $\mathrm{GG}_k(X, V)$ is a proper algebraic subset of X_k for $k \ge k_0$ [though $\pi_{k,0}(\mathrm{GG}_k(X, V))$ might still be equal to X for all k].*

In fact, if F is an invertible sheaf on X such that $K_V \otimes F$ is big, the probabilistic estimates of [4, Corollarys 2.38 and 4.4] produce sections of

$$\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big) F\Big) \tag{1.10}$$

for $m \gg k \gg 1$. The (long and involved) proof uses a curvature computation and singular holomorphic Morse inequalities to show that the line bundles involved in (0.11) are big on X_k for $k \gg 1$. One applies this to $F = A^{-1}$ with A ample on X to produce sections and conclude that $\mathrm{GG}_k(X, V) \subsetneq X_k$.

Thanks to (1.9), the GGL conjecture is satisfied whenever $\mathrm{GG}(X, V) \subsetneq X$. By [5], this happens for instance in the absolute case when X is a generic hypersurface of degree $d \ge 2^{n^5}$ in $\mathbb{P}^{n+1}$ (see also [13] for better bounds in low dimensions, and [14, 15]). However, as already mentioned in [10], very simple examples show that one can have $\mathrm{GG}(X, V) = X$ even when (X, V) is of general type, and this already occurs in the absolute case as soon as $\dim X \ge 2$. A typical example is a product of directed manifolds

$$(X, V) = (X', V') \times (X'', V''), \qquad V = \mathrm{pr}'^* V' \oplus \mathrm{pr}''^* V''. \tag{1.11}$$

The absolute case $V = T_X$, $V' = T_{X'}$, $V'' = T_{X''}$ on a product of curves is the simplest instance. It is then easy to check that $\mathrm{GG}(X, V) = X$, cf. (3.2). Diverio and Rousseau [6] have given many more such examples, including the case of indecomposable varieties (X, T_X), e.g. Hilbert modular surfaces, or more generally compact quotients of bounded symmetric domains of rank ≥ 2. The problem here is the failure of some sort of stability condition that is introduced in Sect. 4. This leads to a somewhat technical concept of more manageable directed pairs (X, V) that we call *strongly of general type*, see Definition 4.1. Our main result can be stated

1.4 Theorem (partial solution to the generalized GGL conjecture) *Let (X, V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X, V), namely* $\mathrm{ECL}(X, V)$ *is a proper algebraic subvariety of X.*

The proof proceeds through a complicated induction on $n = \dim X$ and $k = \operatorname{rank} V$, which is the main reason why we have to introduce directed varieties, even in the absolute case. An interesting feature of this result is that the conclusion on $\mathrm{ECL}(X, V)$ is reached without having to know anything about the Green-Griffiths locus $\mathrm{GG}(X, V)$, even a posteriori. Nevetheless, this is not yet enough to confirm the GGL conjecture. Our hope is that pairs (X, V) that are of general type without being strongly of general type—and thus exhibit some sort of "jet-instability"—can be investigated by different methods, e.g. by the diophantine approximation techniques of McQuillan [11]. However, Theorem 1.4 provides a sufficient criterion for Kobayashi hyperbolicity [8, 9], thanks to the following concept of algebraic jet-hyperbolicity.

1.5 Definition A directed variety (X, V) will be said to be algebraically jet-hyperbolic if the induced directed variety structure (Z, W) on every irreducible algebraic variety Z of X such that $\operatorname{rank} W \geq 1$ has a desingularization that is strongly of general type [see Sects. 3 and 5 for the definition of induced directed structures and further details]. We also say that a projective manifold X is algebraically jet-hyperbolic if (X, T_X) is.

In this context, Theorem 1.4 yields the following connection between algebraic jet-hyperbolicity and the analytic concept of Kobayashi hyperbolicity.

1.6 Theorem *Let (X, V) be a directed variety structure on a projective manifold X. Assume that (X, V) is algebraically jet-hyperbolic. Then (X, V) is Kobayashi hyperbolic.*

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2 Semple Jet Bundles and Associated Canonical Sheaves

Let (X, V) be a directed projective manifold and $r = \operatorname{rank} V$, that is, the dimension of generic fibers. Then V is actually a holomorphic subbundle of T_X on the complement $X \smallsetminus \operatorname{Sing}(V)$ of a certain minimal analytic set $\operatorname{Sing}(V) \subsetneq X$ of codimension ≥ 2, called hereafter the singular set of V. If $\mu : \widehat{X} \to X$ is a proper modification (a composition of blow-ups with smooth centers, say), we get a directed manifold $(\widehat{X}, \widehat{V})$ by taking $\widehat{V}$ to be the closure of $\mu_*^{-1}(V')$, where $V' = V_{|X'}$ is the restriction of V over a Zariski open set $X' \subset X \smallsetminus \operatorname{Sing}(V)$ such that $\mu : \mu^{-1}(X') \to X'$ is a biholomorphism. We will be interested in taking modifications realized by iterated blow-ups of certain nonsingular subvarieties of the singular set $\operatorname{Sing}(V)$, so as to eventually "improve" the singularities of V ; outside of $\operatorname{Sing}(V)$ the effect of blowing-up will be irrelevant, as one can see easily. Following [4], the canonical sheaf K_V is defined as follows.

2.1 Definition For any directed pair (X, V) with X nonsingular, we define K_V to be the rank 1 analytic sheaf such that

$$K_V(U) = \text{sheaf of locally bounded sections of } \mathcal{O}_X(\Lambda^r V'^*)(U \cap X')$$

where $r = \operatorname{rank}(V)$, $X' = X \smallsetminus \operatorname{Sing}(V)$, $V' = V_{|X'}$, and "bounded" means bounded with respect to a smooth hermitian metric h on T_X.

For $r = 0$, one can set $K_V = \mathcal{O}_X$, but this case is trivial: clearly $\operatorname{ECL}(X, V) = \emptyset$. The above definition of K_V may look like an analytic one, but it can easily be turned into an equivalent algebraic definition:

2.2 Proposition *Consider the natural morphism $\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*)$ where $r = \operatorname{rank} V$ [$\mathcal{O}(\Lambda^r V^*)$ being defined here as the quotient of $\mathcal{O}(\Lambda^r T_X^*)$ by r-forms that have zero restrictions to $\mathcal{O}(\Lambda^r V^*)$ on $X \smallsetminus \operatorname{Sing}(V)$]. The bidual $\mathcal{L}_V = \mathcal{O}_X(\Lambda^r V^*)^{**}$ is an invertible sheaf, and our natural morphism can be written*

$$\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*) = \mathcal{L}_V \otimes \mathcal{J}_V \subset \mathcal{L}_V \tag{2.1}$$

where $\mathcal{J}_V$ is a certain ideal sheaf of $\mathcal{O}_X$ whose zero set is contained in $\operatorname{Sing}(V)$ *and the arrow on the left is surjective by definition. Then*

$$K_V = \mathcal{L}_V \otimes \overline{\mathcal{J}}_V \tag{2.2}$$

where $\overline{\mathcal{J}}_V$ is the integral closure of $\mathcal{J}_V$ in $\mathcal{O}_X$. In particular, K_V is always a coherent sheaf.

Proof Let (u_k) be a set of generators of $\mathcal{O}(\Lambda^r V^*)$ obtained (say) as the images of a basis $(dz_I)_{|I|=r}$ of $\Lambda^r T_X^*$ in some local coordinates near a point $x \in X$. Write $u_k = g_k \ell$ where ℓ is a local generator of $\mathcal{L}_V$ at x. Then $\mathcal{J}_V = (g_k)$ by definition. The boundedness condition expressed in Definition 2.1 means that we take sections of the form $f\ell$ where f is a holomorphic function on $U \cap X'$ (and U a neighborhood of x), such that

$$|f| \leq C \sum |g_k| \tag{2.3}$$

for some constant $C > 0$. But then f extends holomorphically to U into a function that lies in the integral closure $\overline{\mathcal{J}}_V$, and the latter is actually characterized analytically by condition (2.3). This proves Proposition 2.2. □

By blowing-up $\mathcal{J}_V$ and taking a desingularization $\widehat{X}$, one can always find a *log-resolution* of $\mathcal{J}_V$ (or K_V), i.e. a modification $\mu : \widehat{X} \to X$ such that $\mu^*\mathcal{J}_V \subset \mathcal{O}_{\widehat{X}}$ is an invertible ideal sheaf (hence integrally closed); it follows that $\mu^*\overline{\mathcal{J}}_V = \mu^*\mathcal{J}_V$ and $\mu^*K_V = \mu^*\mathcal{L}_V \otimes \mu^*\mathcal{J}_V$ are invertible sheaves on $\widehat{X}$. Notice that for any modification $\mu' : (X', V') \to (X, V)$, there is always a well defined natural morphism

$$\mu'^* K_V \to K_{V'} \tag{2.4}$$

(though it need not be an isomorphism, and $K_{V'}$ is possibly non invertible even when μ' is taken to be a log-resolution of K_V). Indeed $(\mu')_* = d\mu' : V' \to \mu^*V$ is continuous with respect to ambient hermitian metrics on X and X', and going to the duals reverses the arrows while preserving boundedness with respect to the metrics. If $\mu'' : X'' \to X'$ provides a simultaneous log-resolution of $K_{V'}$ and μ'^*K_V, we get a non trivial morphism of invertible sheaves

$$(\mu' \circ \mu'')^* K_V = \mu''^* \mu'^* K_V \longrightarrow \mu''^* K_{V'}, \tag{2.5}$$

hence the bigness of μ'^*K_V with imply that of $\mu''^*K_{V'}$. This is a general principle that we would like to refer to as the "monotonicity principle" for canonical sheaves: one always get more sections by going to a higher level through a (holomorphic) modification.

2.3 Definition We say that the rank 1 sheaf K_V is "big" if the invertible sheaf μ^*K_V is big in the usual sense for any log resolution $\mu : \widehat{X} \to X$ of K_V. Finally, we say that (X, V) is of *general type* if there exists a modification $\mu' : (X', V') \to (X, V)$ such that $K_{V'}$ is big; any higher blow-up $\mu'' : (X'', V'') \to (X', V')$ then also yields a big canonical sheaf by (2.4).

Clearly, "general type" is a birationally (or bimeromorphically) invariant concept, by the very definition. When $\dim X = n$ and $V \subset T_X$ is a subbundle of rank $r \geq 1$, one constructs a tower of "Semple k-jet bundles" $\pi_{k,k-1} : (X_k, V_k) \to (X_{k-1}, V_{k-1})$ that are $\mathbb{P}^{r-1}$-bundles, with $\dim X_k = n + k(r-1)$ and $\operatorname{rank}(V_k) = r$. For this, we take $(X_0, V_0) = (X, V)$, and for every $k \geq 1$, we set inductively $X_k := P(V_{k-1})$ and

$$V_k := (\pi_{k,k-1})_*^{-1} \mathcal{O}_{X_k}(-1) \subset T_{X_k},$$

where $\mathcal{O}_{X_k}(1)$ is the tautological line bundle on X_k, $\pi_{k,k-1} : X_k = P(V_{k-1}) \to X_{k-1}$ the natural projection and $(\pi_{k,k-1})_* = d\pi_{k,k-1} : T_{X_k} \to \pi_{k,k-1}^* T_{X_{k-1}}$ its differential (cf. [1]). In other terms, we have exact sequences

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_{k,k-1})_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0, \tag{2.6}$$

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow (\pi_{k,k-1})^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0, \tag{2.7}$$

where the last line is the Euler exact sequence associated with the relative tangent bundle of $P(V_{k-1}) \to X_{k-1}$. Notice that we by definition of the tautological line bundle we have

$$\mathcal{O}_{X_k}(-1) \subset \pi_{k,k-1}^* V_{k-1} \subset \pi_{k,k-1}^* T_{X_{k-1}},$$

and also $\mathrm{rank}(V_k) = r$. Let us recall also that for $k \geq 2$, there are "vertical divisors" $D_k = P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k$, and that D_k is the zero divisor of the section of $\mathcal{O}_{X_k}(1) \otimes \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(-1)$ induced by the second arrow of the first exact sequence (2.6), when k is replaced by $k-1$. This yields in particular

$$\mathcal{O}_{X_k}(1) = \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k). \tag{2.8}$$

By composing the projections we get for all pairs of indices $0 \leq j \leq k$ natural morphisms

$$\pi_{k,j} : X_k \to X_j, \quad (\pi_{k,j})_* = (d\pi_{k,j})_{|V_k} : V_k \to (\pi_{k,j})^* V_j,$$

and for every k-tuple $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ we define

$$\mathcal{O}_{X_k}(\mathbf{a}) = \bigotimes_{1 \leq j \leq k} \pi_{k,j}^* \mathcal{O}_{X_j}(a_j), \quad \pi_{k,j} : X_k \to X_j.$$

We extend this definition to all weights $\mathbf{a} \in \mathbb{Q}^k$ to get a $\mathbb{Q}$-line bundle in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Now, Formula (2.8) yields

$$\mathcal{O}_{X_k}(\mathbf{a}) = \mathcal{O}_{X_k}(m) \otimes \mathcal{O}(-\mathbf{b} \cdot D) \quad \text{where } m = |\mathbf{a}| = \sum a_j, \mathbf{b} = (0, b_2, \ldots, b_k) \tag{2.9}$$

and $b_j = a_1 + \cdots + a_{j-1}$, $2 \leq j \leq k$.

When $\mathrm{Sing}(V) \neq \emptyset$, one can always define X_k and V_k to be the respective closures of X'_k, V'_k associated with $X' = X \smallsetminus \mathrm{Sing}(V)$ and $V' = V_{|X'}$, where the closure is taken in the nonsingular "absolute" Semple tower (X^a_k, V^a_k) obtained from $(X^a_0, V^a_0) = (X, T_X)$. We leave the reader check the following easy (but important) observation.

2.4 Fonctoriality *If* $\Phi : (X, V) \to (Y, W)$ *is a morphism of directed varieties such that* $\Phi_* : T_X \to \Phi^* T_Y$ *is injective (i.e.* Φ *is an immersion), then there is a corresponding natural morphism* $\Phi_{[k]} : (X_k, V_k) \to (Y_k, W_k)$ *at the level of Semple bundles. If one merely assumes that the differential* $\Phi_* : V \to \Phi^* W$ *is non zero,* *there is still a well defined meromorphic map* $\Phi_{[k]} : (X_k, V_k) \dashrightarrow (Y_k, W_k)$ *for all* $k \geq 0$.

In case V is singular, the k-th Semple bundle X_k will also be singular, but we can still replace (X_k, V_k) by a suitable modification $(\widehat{X}_k, \widehat{V}_k)$ if we want to work with a nonsingular model $\widehat{X}_k$ of X_k. The exceptional set of $\widehat{X}_k$ over X_k can be chosen to lie above $\mathrm{Sing}(V) \subset X$, and proceeding inductively with respect to k, we can also arrange the modifications in such a way that we get a tower structure $(\widehat{X}_{k+1}, \widehat{V}_{k+1}) \to (\widehat{X}_k, \widehat{V}_k)$; however, in general, it will not be possible to achieve that $\widehat{V}_k$ is a subbundle of $T_{\widehat{X}_k}$.

It is not true that $K_{\widehat{V}_k}$ is big in case (X, V) is of general type (especially since the fibers of $X_k \to X$ are towers of $\mathbb{P}^{r-1}$ bundles, and the canonical bundles of projective spaces are always negative !). However, a twisted version holds true, that can be seen as another instance of the "monotonicity principle" when going to higher stages in the Semple tower.

2.5 Lemma *If* (X, V) *is of general type, then there is a modification* $(\widehat{X}, \widehat{V})$ *such that all pairs* $(\widehat{X}_k, \widehat{V}_k)$ *of the associated Semple tower have a twisted canonical bundle* $K_{\widehat{V}_k} \otimes \mathcal{O}_{\widehat{X}_k}(p)$ *that is still big when one multiplies* $K_{\widehat{V}_k}$ *by a suitable* $\mathbb{Q}$*-line bundle* $\mathcal{O}_{\widehat{X}_k}(p)$, $p \in \mathbb{Q}_+$.

Proof. First assume that V has no singularities. The exact sequences (2.6) and (2.7) provide

$$K_{V_k} := \det V_k^* = \det(T^*_{X_k/X_{k-1}}) \otimes \mathcal{O}_{X_k}(1) = \pi^*_{k,k-1} K_{V_{k-1}} \otimes \mathcal{O}_{X_k}(-(r-1))$$

where $r = \mathrm{rank}(V)$. Inductively we get

$$K_{V_k} = \pi^*_{k,0} K_V \otimes \mathcal{O}_{X_k}(-(r-1)\mathbf{1}), \qquad \mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^k. \tag{2.10}$$

We know by [1] that $\mathcal{O}_{X_k}(\mathbf{c})$ is relatively ample over X when we take the special weight $\mathbf{c} = (2\,3^{k-2}, \ldots, 2\,3^{k-j-1}, \ldots, 6, 2, 1)$, hence

$$K_{V_k} \otimes \mathcal{O}_{X_k}((r-1)\mathbf{1} + \varepsilon\mathbf{c}) = \pi^*_{k,0} K_V \otimes \mathcal{O}_{X_k}(\varepsilon\mathbf{c})$$

is big over X_k for any sufficiently small positive rational number $\varepsilon \in \mathbb{Q}^*_+$. Thanks to Formula (2.9), we can in fact replace the weight $(r-1)\mathbf{1} + \varepsilon\mathbf{c}$ by its total degree $p = (r-1)k + \varepsilon|\mathbf{c}| \in \mathbb{Q}_+$. The general case of a singular linear space follows by considering suitable "sufficiently high" modifications $\widehat{X}$ of X, the related directed structure $\widehat{V}$ on $\widehat{X}$, and embedding $(\widehat{X}_k, \widehat{V}_k)$ in the absolute Semple tower $(\widehat{X}^a_k, \widehat{V}^a_k)$ of $\widehat{X}$. We still have a well defined morphism of rank 1 sheaves

$$\pi^*_{k,0} K_{\widehat{V}} \otimes \mathcal{O}_{\widehat{X}_k}(-(r-1)\mathbf{1}) \to K_{\widehat{V}_k} \tag{2.11}$$

because the multiplier ideal sheaves involved at each stage behave according to the monotonicity principle applied to the projections $\pi^a_{k,k-1} : \widehat{X}^a_k \to \widehat{X}^a_{k-1}$ and their differentials $(\pi^a_{k,k-1})_*$, which yield well-defined transposed morphisms from the $(k-1)$-st stage to the k-th stage at the level of exterior differential forms. Our contention follows. □

3 Induced Directed Structure on a Subvariety of a Jet Space

Let Z be an irreducible algebraic subset of some k-jet bundle X_k over X, $k \geq 0$. We define the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure

$$W := \overline{T_{Z'} \cap V_k} \tag{3.1}$$

taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection $T_{Z'} \cap V_k$ has constant rank and is a subbundle of $T_{Z'}$. Alternatively, we could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage (X^a_k, V^a_k) of the absolute Semple tower, which has the advantage of being nonsingular. We say that (Z, W) is the *induced* directed variety structure; this concept of induced structure already applies of course in the case $k = 0$. If $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ is such that $f_{[k]}(\mathbb{C}) \subset Z$, then

$$\text{either} \quad f_{[k]}(\mathbb{C}) \subset Z_\alpha \quad \text{or} \quad f'_{[k]}(\mathbb{C}) \subset W, \tag{3.2}$$

where Z_α is one of the connected components of $Z \smallsetminus Z'$ and Z' is chosen as in (3.1); especially, if $W = 0$, we conclude that $f_{[k]}(\mathbb{C})$ must be contained in one of the Z_α's. In the sequel, we always consider such a subvariety Z of X_k as a directed pair (Z, W) by taking the induced structure described above. By (3.2), if we proceed by induction on $\dim Z$, the study of curves tangent to V that have a k-lift $f_{[k]}(\mathbb{C}) \subset Z$ is reduced to the study of curves tangent to (Z, W). Let us first quote the following easy observation.

3.1 Observation *For $k \geq 1$, let $Z \subsetneq X_k$ be an irreducible algebraic subset that projects onto X_{k-1}, i.e. $\pi_{k,k-1}(Z) = X_{k-1}$. Then the induced directed variety $(Z, W) \subset (X_k, V_k)$, satisfies*

$$1 \leq \operatorname{rank} W < r := \operatorname{rank}(V_k).$$

Proof. Take a Zariski open subset $Z' \subset Z_{\mathrm{reg}}$ such that $W' = T_{Z'} \cap V_k$ is a vector bundle over Z'. Since $X_k \to X_{k-1}$ is a $\mathbb{P}^{r-1}$-bundle, Z has codimension at most $r-1$ in X_k. Therefore $\operatorname{rank} W \geq \operatorname{rank} V_k - (r-1) \geq 1$. On the other hand, if we had $\operatorname{rank} W = \operatorname{rank} V_k$ generically, then $T_{Z'}$ would contain $V_{k|Z'}$, in particular it would contain all vertical directions $T_{X_k/X_{k-1}} \subset V_k$ that are tangent to the fibers of $X_k \to X_{k-1}$. By taking the flow along vertical vector fields, we would conclude that Z' is a union of fibers of $X_k \to X_{k-1}$ up to an algebraic set of smaller dimension, but this is excluded since Z projects onto X_{k-1} and $Z \subsetneq X_k$. □

3.2 Definition For $k \geq 1$, let $Z \subset X_k$ be an irreducible algebraic subset of X_k. We assume moreover that $Z \not\subset D_k = P(T_{X_{k-1}/X_{k-2}})$ (and put here $D_1 = \emptyset$ in what follows to avoid to have to single out the case $k = 1$). In this situation we say that (Z, W) is of general type modulo $X_k \to X$ if either $W = 0$, or $\operatorname{rank} W \geq 1$ and there exists $p \in \mathbb{Q}_+$ such that $K_W \otimes \mathcal{O}_{X_k}(p)_{|Z}$ is big over Z, possibly after replacing Z by a suitable nonsingular model $\widehat{Z}$ (and pulling-back W and $\mathcal{O}_{X_k}(p)_{|Z}$ to the nonsingular variety $\widehat{Z}$).

The main result of [4] mentioned in the introduction as Theorem 1.3 implies the following important "induction step".

3.3 Proposition *Let (X, V) be a directed pair where X is projective algebraic. Take an irreducible algebraic subset $Z \not\subset D_k$ of the associated k-jet Semple bundle X_k that projects onto X_{k-1}, $k \geq 1$, and assume that the induced directed space $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, $\operatorname{rank} W \geq 1$. Then there exists a divisor $\Sigma \subset Z_\ell$ in a sufficiently high stage of the Semple tower (Z_ℓ, W_ℓ) associated with (Z, W), such that every non constant holomorphic map $f : \mathbb{C} \to X$ tangent to V that satisfies $f_{[k]}(\mathbb{C}) \subset Z$ also satisfies $f_{[k+\ell]}(\mathbb{C}) \subset \Sigma$.*

Proof Let $E \subset Z$ be a divisor containing $Z_{\text{sing}} \cup (Z \cap \pi_{k,0}^{-1}(\operatorname{Sing}(V)))$, chosen so that on the nonsingular Zariski open set $Z' = Z \smallsetminus E$ all linear spaces $T_{Z'}$, $V_{k|Z'}$ and $W' = T_{Z'} \cap V_k$ are subbundles of $T_{X_k|Z'}$, the first two having a transverse intersection on Z'. By taking closures over Z' in the absolute Semple tower of X, we get (singular) directed pairs $(Z_\ell, W_\ell) \subset (X_{k+\ell}, V_{k+\ell})$, which we eventually resolve into $(\widehat{Z}_\ell, \widehat{W}_\ell) \subset (\widehat{X}_{k+\ell}, \widehat{V}_{k+\ell})$ over nonsingular bases. By construction, locally bounded sections of $\mathcal{O}_{\widehat{X}_{k+\ell}}(m)$ restrict to locally bounded sections of $\mathcal{O}_{\widehat{Z}_\ell}(m)$ over $\widehat{Z}_\ell$.

Since Theorem 1.3 and the related estimate (1.10) are universal in the category of directed varieties, we can apply them by replacing X with $\widehat{Z} \subset \widehat{X}_k$, the order k by a new index ℓ, and F by

$$F_k = \mu^*\Big(\big(\mathcal{O}_{X_k}(p) \otimes \pi_{k,0}^*\mathcal{O}_X(-\varepsilon A)\big)_{|Z}\Big)$$

where $\mu : \widehat{Z} \to Z$ is the desingularization, $p \in \mathbb{Q}_+$ is chosen such that $K_W \otimes \mathcal{O}_{x_k}(p)_{|Z}$ is big, A is an ample bundle on X and $\varepsilon \in \mathbb{Q}_+^*$ is small enough. The assumptions show that $K_{\widehat{W}} \otimes F_k$ is big on $\widehat{Z}$, therefore, by applying our theorem and taking $m \gg \ell \gg 1$, we get in fine a large number of (metric bounded) sections of

$$\begin{aligned}\mathcal{O}_{\widehat{Z}_\ell}(m) &\otimes \widehat{\pi}_{k+\ell,k}^*\mathcal{O}\Big(\frac{m}{\ell r'}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{\ell}\Big)F_k\Big)\\ &= \mathcal{O}_{\widehat{X}_{k+\ell}}(m\mathbf{a}') \otimes \widehat{\pi}_{k+\ell,0}^*\mathcal{O}\Big(-\frac{m\varepsilon}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)A\Big)_{|\widehat{Z}_\ell}\end{aligned}$$

where $\mathbf{a}' \in \mathbb{Q}_+^{k+\ell}$ is a positive weight (of the form $(0, \ldots, \lambda, \ldots, 0, 1)$ with some non zero component $\lambda \in \mathbb{Q}_+$ at index k). These sections descend to metric bounded sections of

$$\mathcal{O}_{X_{k+\ell}}((1+\lambda)m) \otimes \widehat{\pi}^*_{k+\ell,0}\mathcal{O}\Big(-\frac{m\varepsilon}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)A\Big)_{|Z_\ell}.$$

Since A is ample on X, we can apply the fundamental vanishing theorem (see e.g. [2] or [4], Statement 8.15), or rather an "embedded" version for curves satisfying $f_{[k]}(\mathbb{C}) \subset Z$, proved exactly by the same arguments. The vanishing theorem implies that the divisor Σ of any such section satisfies the conclusions of Proposition 3.3, possibly modulo exceptional divisors of $\widehat{Z} \to Z$; to take care of these, it is enough to add to Σ the inverse image of the divisor $E = Z \smallsetminus Z'$ initially selected. □

4 Strong General Type Condition for Directed Manifolds

Our main result is the following partial solution to the Green-Griffiths-Lang conjecture, providing a sufficient algebraic condition for the analytic conclusion to hold true. We first give an ad hoc definition.

4.1 Definition Let (X, V) be a directed pair where X is projective algebraic. We say that that (X, V) is "strongly of general type" if it is of general type and for every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, that projects onto X, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$.

4.2 Example The situation of a product $(X, V) = (X', V') \times (X'', V'')$ described in (1.11) shows that (X, V) can be of general type without being strongly of general type. In fact, if (X', V') and (X'', V'') are of general type, then $K_V = \mathrm{pr}'^* K_{V'} \otimes \mathrm{pr}''^* K_{V''}$ is big, so (X, V) is again of general type. However

$$Z = P(\mathrm{pr}'^* V') = X'_1 \times X'' \subset X_1$$

has a directed structure $W = \mathrm{pr}'^* V'_1$ which does not possess a big canonical bundle over Z, since the restriction of K_W to any fiber $\{x'\} \times X''$ is trivial. The higher stages (Z_k, W_k) of the Semple tower of (Z, W) are given by $Z_k = X'_{k+1} \times X''$ and $W_k = \mathrm{pr}'^* V'_{k+1}$, so it is easy to see that $\mathrm{GG}_k(X, V)$ contains Z_{k-1}. Since Z_k projects onto X, we have here $\mathrm{GG}(X, V) = X$ (see [6] for more sophisticated indecomposable examples).

4.3 Remark It follows from Definition 3.2 that $(Z, W) \subset (X_k, V_k)$ is automatically of general type modulo $X_k \to X$ if $\mathcal{O}_{X_k}(1)_{|Z}$ is big. Notice further that

$$\mathcal{O}_{X_k}(1+\varepsilon)_{|Z} = \big(\mathcal{O}_{X_k}(\varepsilon) \otimes \pi^*_{k,k-1}\mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k)\big)_{|Z}$$

where $\mathcal{O}(D_k)_{|Z}$ is effective and $\mathcal{O}_{X_k}(1)$ is relatively ample with respect to the projection $X_k \to X_{k-1}$. Therefore the bigness of $\mathcal{O}_{X_{k-1}}(1)$ on X_{k-1} also implies that every directed subvariety $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$. If (X, V) is of general type, we know by the main result of [4] that $\mathcal{O}_{X_k}(1)$ is big for $k \geq k_0$ large enough, and actually the precise estimates obtained therein give explicit bounds for such a k_0. The above observations show that we need to check the condition of Definition 4.1 only for $Z \subset X_k$, $k \leq k_0$. Moreover, at least in the case where V, Z, and $W = T_Z \cap V_k$ are nonsingular, we have

$$K_W \simeq K_Z \otimes \det(T_Z/W) \simeq K_Z \otimes \det(T_{X_k}/V_k)_{|Z} \simeq K_{Z/X_{k-1}} \otimes \mathcal{O}_{X_k}(1)_{|Z}.$$

Thus we see that, in some sense, it is only needed to check the bigness of K_W modulo $X_k \to X$ for "rather special subvarieties" $Z \subset X_k$ over X_{k-1}, such that $K_{Z/X_{k-1}}$ is not relatively big over X_{k-1}. □

4.4 Hypersurface case Assume that $Z \neq D_k$ is an irreducible hypersurface of X_k that projects onto X_{k-1}. To simplify things further, also assume that V is nonsingular. Since the Semple jet-bundles X_k form a tower of $\mathbb{P}^{r-1}$-bundles, their Picard groups satisfy $\mathrm{Pic}(X_k) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}^k$ and we have $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{k,0}^* B$ for some $\mathbf{a} \in \mathbb{Z}^k$ and $B \in \mathrm{Pic}(X)$, where $a_k = d > 0$ is the relative degree of the hypersurface over X_{k-1}. Let $\sigma \in H^0(X_k, \mathcal{O}_{X_k}(Z))$ be the section defining Z in X_k. The induced directed variety (Z, W) has rank $W = r - 1 = \mathrm{rank}\, V - 1$ and formula (2.11) yields $K_{V_k} = \mathcal{O}_{X_k}(-(r-1)\mathbf{1}) \otimes \pi_{k,0}^*(K_V)$. We claim that

$$K_W \supset \big(K_{V_k} \otimes \mathcal{O}_{X_k}(Z)\big)_{|Z} \otimes \mathcal{J}_S = \big(\mathcal{O}_{X_k}(\mathbf{a}-(r-1)\mathbf{1}) \otimes \pi_{k,0}^*(B \otimes K_V)\big)_{|Z} \otimes \mathcal{J}_S \quad (4.1)$$

where $S \subsetneq Z$ is the set (containing Z_{sing}) where σ and $d\sigma_{|V_k}$ both vanish, and $\mathcal{J}_S$ is the ideal locally generated by the coefficients of $d\sigma_{|V_k}$ along $Z = \sigma^{-1}(0)$. In fact, the intersection $W = T_Z \cap V_k$ is transverse on $Z \smallsetminus S$; then (4.1) can be seen by looking at the morphism

$$V_{k|Z} \xrightarrow{d\sigma_{|V_k}} \mathcal{O}_{X_k}(Z)_{|Z},$$

and observing that the contraction by $K_{V_k} = \Lambda^r V_k^*$ provides a metric bounded section of the canonical sheaf K_W. In order to investigate the positivity properties of K_W, one has to show that B cannot be too negative, and in addition to control the singularity set S. The second point is a priori very challenging, but we get useful information for the first point by observing that σ provides a morphism $\pi_{k,0}^*\mathcal{O}_X(-B) \to \mathcal{O}_{X_k}(\mathbf{a})$, hence a nontrivial morphism

$$\mathcal{O}_X(-B) \to E_{\mathbf{a}} := (\pi_{k,0})_* \mathcal{O}_{X_k}(\mathbf{a})$$

By [1, ,Section 12] , there exists a filtration on $E_{\mathbf{a}}$ such that the graded pieces are irreducible representations of $\mathrm{GL}(V)$ contained in $(V^*)^{\otimes \ell}$, $\ell \leq |\mathbf{a}|$. Therefore we get a nontrivial morphism

$$\mathcal{O}_X(-B) \to (V^*)^{\otimes \ell}, \qquad \ell \leq |\mathbf{a}|. \tag{4.2}$$

If we know about certain (semi-)stability properties of V, this can be used to control the negativity of B. □

We further need the following useful concept that slightly generalizes entire curve loci.

4.5 Definition If Z is an algebraic set contained in some stage X_k of the Semple tower of (X, V), we define its "induced entire curve locus" $\mathrm{IEL}_{X,V}(Z) \subset Z$ to be the Zariski closure of the union $\bigcup f_{[k]}(\mathbb{C})$ of all jets of entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ such that $f_{[k]}(\mathbb{C}) \subset Z$.

We have of course $\mathrm{IEL}_{X,V}(\mathrm{IEL}_{X,V}(Z)) = \mathrm{IEL}_{X,V}(Z)$ by definition. It is not hard to check that modulo certain "vertical divisors " of X_k, the $\mathrm{IEL}_{X,V}(Z)$ locus is essentially the same as the entire curve locus $\mathrm{ECL}(Z, W)$ of the induced directed variety, but we will not use this fact here. Notice that if $Z = \bigcup Z_\alpha$ is a decomposition of Z into irreducible divisors, then

$$\mathrm{IEL}_{X,V}(Z) = \bigcup_\alpha \mathrm{IEL}_{X,V}(Z_\alpha).$$

Since $\mathrm{IEL}_{X,V}(X_k) = \mathrm{ECL}_k(X, V)$, proving the Green-Griffiths-Lang property amounts to showing that $\mathrm{IEL}_{X,V}(X) \subsetneq X$ in the stage $k = 0$ of the tower. The basic step of our approach is expressed in the following statement.

4.6 Proposition *Let (X, V) be a directed variety and $p_0 \leq n = \dim X$, $p_0 \geq 1$. Assume that there is an integer $k_0 \geq 0$ such that for every $k \geq k_0$ and every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, such that $\dim \pi_{k,k_0}(Z) \geq p_0$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$. Then $\dim \mathrm{ECL}_{k_0}(X, V) < p_0$.*

Proof We argue here by contradiction, assuming that $\dim \mathrm{ECL}_{k_0}(X, V) \geq p_0$. If

$$p_0' := \dim \mathrm{ECL}_{k_0}(X, V) > p_0$$

and if we can prove the result for p_0', we will already get a contradiction, hence we can assume without loss of generality that $\dim \mathrm{ECL}_{k_0}(X, V) = p_0$. The main argument consists of producing inductively an increasing sequence of integers

$$k_0 < k_1 < \cdots < k_j < \cdots$$

and directed varieties $(Z^j, W^j) \subset (X_{k_j}, V_{k_j})$ satisfying the following properties :

(3.6.1) Z^0 is one of the irreducible components of $\mathrm{ECL}_{k_0}(X, V)$ and $\dim Z^0 = p_0$.

(3.6.2) Z^j is one of the irreducible components of $\mathrm{ECL}_{k_j}(X, V)$ and $\pi_{k_j,k_0}(Z^j) = Z^0$.

(3.6.3) For all $j \ge 0$, $\mathrm{IEL}_{X,V}(Z^j) = Z^j$ and $\mathrm{rank}\, W_j \ge 1$.

(3.6.4) For all $j \ge 0$, the directed variety (Z^{j+1}, W^{j+1}) is contained in some stage (of order $\ell_j = k_{j+1} - k_j$) of the Semple tower of (Z^j, W^j), namely

$$(Z^{j+1}, W^{j+1}) \subsetneq (Z^j_{\ell_j}, W^j_{\ell_j}) \subset (X_{k_{j+1}}, V_{k_{j+1}})$$

and

$$W^{j+1} = \overline{T_{Z^{j+1\prime}} \cap W^j_{\ell_j}} = \overline{T_{Z^{j+1\prime}} \cap V_{k_j}} \tag{4.3}$$

is the induced directed structure; moreover $\pi_{k_{j+1},k_j}(Z^{j+1}) = Z^j$.

(3.6.5) For all $j \ge 0$, we have $Z^{j+1} \subsetneq Z^j_{\ell_j}$ but $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j-1}$.

For $j = 0$, we simply take Z^0 to be one of the irreducible components S_α of $\mathrm{ECL}_{k_0}(X, V)$ such that $\dim S_\alpha = p_0$, which exists by our hypothesis that $\dim \mathrm{ECL}_{k_0}(X, V) = p_0$. Clearly, $\mathrm{ECL}_{k_0}(X, V)$ is the union of the $\mathrm{IEL}_{X,V}(S_\alpha)$ and we have $\mathrm{IEL}_{X,V}(S_\alpha) = S_\alpha$ for all those components, thus $\mathrm{IEL}_{X,V}(Z^0) = Z^0$ and $\dim Z^0 = p_0$. Assume that (Z^j, W^j) has been constructed. The subvariety Z^j cannot be contained in the vertical divisor D_{k_j}. In fact no irreducible algebraic set Z such that $\mathrm{IEL}_{X,V}(Z) = Z$ can be contained in a vertical divisor D_k, because $\pi_{k,k-2}(D_k)$ corresponds to stationary jets in X_{k-2} ; as every non constant curve f has non stationary points, its k-jet $f_{[k]}$ cannot be entirely contained in D_k ; also the induced directed structure (Z, W) must satisfy $\mathrm{rank}\, W \ge 1$ otherwise $\mathrm{IEL}_{X,V}(Z) \subsetneq Z$. Condition (3.6.2) implies that $\dim \pi_{k_j,k_0}(Z^j) \ge p_0$, thus (Z^j, W^j) is of general type modulo $X_{k_j} \to X$ by the assumptions of the proposition. Thanks to Proposition 3.3, we get an algebraic subset $\Sigma \subsetneq Z^j_\ell$ in some stage of the Semple tower (Z^j_ℓ) of Z^j such that every entire curve $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ satisfying $f_{[k_j]}(\mathbb{C}) \subset Z^j$ also satisfies $f_{[k_j+\ell]}(\mathbb{C}) \subset \Sigma$. By definition, this implies the first inclusion in the sequence

$$Z^j = \mathrm{IEL}_{X,V}(Z^j) \subset \pi_{k_j+\ell,k_j}(\mathrm{IEL}_{X,V}(\Sigma)) \subset \pi_{k_j+\ell,k_j}(\Sigma) \subset Z^j$$

(the other ones being obvious), so we have in fact an equality throughout. Let (S'_α) be the irreducible components of $\mathrm{IEL}_{X,V}(\Sigma)$. We have $\mathrm{IEL}_{X,V}(S'_\alpha) = S'_\alpha$ and one of the components S'_α must satisfy

$$\pi_{k_j+\ell,k_j}(S'_\alpha) = Z^j = Z^j_0.$$

We take $\ell_j \in [1, \ell]$ to be the smallest order such that $Z^{j+1} := \pi_{k_j+\ell,k_j+\ell_j}(S'_\alpha) \subsetneq Z^j_{\ell_j}$, and set $k_{j+1} = k_j + \ell_j > k_j$. By definition of ℓ_j, we have $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j-1}$, otherwise ℓ_j would not be minimal. Then $\pi_{k_{j+1},k_j}(Z^{j+1}) = Z^j$, hence $\pi_{k_{j+1},k_0}(Z^{j+1}) = Z^0$ by induction, and all properties (3.6.1–3.6.5) follow easily. Now, by Observation 3.1, we have

$$\mathrm{rank}\, W^j < \mathrm{rank}\, W^{j-1} < \cdots < \mathrm{rank}\, W^1 < \mathrm{rank}\, W^0 = \mathrm{rank}\, V.$$

This is a contradiction because we cannot have such an infinite sequence. Proposition 4.6 is proved. □

The special case $k_0 = 0$, $p_0 = n$ of Proposition 4.6 yields the following consequence.

4.7 Partial solution to the generalized GGL conjecture *Let (X, V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X, V), namely* $\mathrm{ECL}(X, V) \subsetneq X$, *in other words there exists a proper algebraic variety $Y \subsetneq X$ such that every non constant holomorphic curve $f : \mathbb{C} \to X$ tangent to V satisfies $f(\mathbb{C}) \subset Y$.*

4.8 Remark The proof is not very constructive, but it is however theoretically effective. By this we mean that if (X, V) is strongly of general type and is taken in a bounded family of directed varieties, i.e. X is embedded in some projective space $\mathbb{P}^N$ with some bound δ on the degree, and $P(V)$ also has bounded degree $\le \delta'$ when viewed as a subvariety of $P(T_{\mathbb{P}^N})$, then one could theoretically derive bounds $d_Y(n, \delta, \delta')$ for the degree of the locus Y. Also, there would exist bounds $k_0(n, \delta, \delta')$ for the orders k and bounds $d_k(n, \delta, \delta')$ for the degrees of subvarieties $Z \subset X_k$ that have to be checked in the definition of a pair of strong general type. In fact, [4] produces more or less explicit bounds for the order k such that Proposition 3.3 holds true. The degree of the divisor Σ is given by a section of a certain twisted line bundle $\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}_X(-A)$ that we know to be big by an application of holomorphic Morse inequalities – and the bounds for the degrees of (X_k, V_k) then provide bounds for m. □

4.9 Remark The condition that (X, V) is strongly of general type seems to be related to some sort of stability condition. We are unsure what is the most appropriate definition, but here is one that makes sense. Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \ge 1$, and $Z = X = X_0$ for $k = 0$, we define the slope $\mu_A(Z, W)$ of the corresponding directed variety (Z, W) to be

$$\mu_A(Z, W) = \frac{\inf \lambda}{\mathrm{rank}\, W},$$

where λ runs over all rational numbers such that there exists $m \in \mathbb{Q}_+$ for which

$$K_W \otimes \big(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}(\lambda A)\big)_{|Z} \quad \text{is big on } Z$$

(again, we assume here that $Z \not\subset D_k$ for $k \ge 2$). Notice that (X, V) is of general type if and only if $\mu_A(X, V) < 0$, and that $\mu_A(Z, W) = -\infty$ if $\mathcal{O}_{X_k}(1)_{|A}$ is big. Also, the proof of Lemma 2.5 shows that

$$\mu_A(X_k, V_k) \le \mu_A(X_{k-1}, V_{k-1}) \le \ldots \le \mu_A(X, V) \quad \text{for all } k$$

(with $\mu_A(X_k, V_k) = -\infty$ for $k \geq k_0 \gg 1$ if (X, V) is of general type). We say that (X, V) is *A-jet-stable* (resp. *A-jet-semi-stable*) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above. It is then clear that if (X, V) is of general type and A-jet-semi-stable, then it is strongly of general type in the sense of Definition 4.1. It would be useful to have a better understanding of this condition of stability (or any other one that would have better properties). □

4.10 Example (**case of surfaces**) Assume that X is a minimal complex surface of general type and $V = T_X$ (absolute case). Then K_X is nef and big and the Chern classes of X satisfy $c_1 \leq 0$ ($-c_1$ is big and nef) and $c_2 \geq 0$. The Semple jet-bundles X_k form here a tower of $\mathbb{P}^1$-bundles and $\dim X_k = k + 2$. Since $\det V^* = K_X$ is big, the strong general type assumption of 4.6 and 4.8 need only be checked for irreducible hypersurfaces $Z \subset X_k$ distinct from D_k that project onto X_{k-1}, of relative degree m. The projection $\pi_{k,k-1} : Z \to X_{k-1}$ is a ramified $m : 1$ cover. Putting $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{k,0}(B)$, $B \in \mathrm{Pic}(X)$, we can apply (4.1) to get an inclusion

$$K_W \supset \big(\mathcal{O}_{X_k}(\mathbf{a} - \mathbf{1}) \otimes \pi^*_{k,0}(B \otimes K_X)\big)_{|Z} \otimes \mathcal{J}_S, \qquad \mathbf{a} \in \mathbb{Z}^k,\ \ a_k = m.$$

Let us assume $k = 1$ and $S = \emptyset$ to make things even simpler, and let us perform numerical calculations in the cohomology ring

$$H^\bullet(X_1, \mathbb{Z}) = H^\bullet(X)[u]/(u^2 + c_1 u + c_2), \qquad u = c_1(O_{X_1}(1))$$

(cf. [3, Section 2] for similar calculations and more details). We have

$$Z \equiv mu + b \quad \text{where} \quad b = c_1(B) \quad \text{and} \quad K_W \equiv (m-1)u + b - c_1.$$

We are allowed here to add to K_W an arbitrary multiple $\mathcal{O}_{X_1}(p)$, $p \geq 0$, which we rather write $p = mt + 1 - m$, $t \geq 1 - 1/m$. An evaluation of the Euler-Poincaré characteristic of $K_W + \mathcal{O}_{X_1}(p)_{|Z}$ requires computing the intersection number

$$\begin{aligned}\big(K_W + \mathcal{O}_{X_1}(p)_{|Z}\big)^2 \cdot Z &= \big(mt\,u + b - c_1\big)^2 (mu + b)\\ &= m^2 t^2 \big(m(c_1^2 - c_2) - bc_1\big) + 2mt(b - mc_1)(b - c_1)\\ &\quad + m(b - c_1)^2,\end{aligned} \tag{4.4}$$

taking into account that $u^3 \cdot X_1 = c_1^2 - c_2$. In case $S \neq \emptyset$, there is an additional (negative) contribution from the ideal $\mathcal{J}_S$ which is $O(t)$ since S is at most a curve. In any case, for $t \gg 1$, the leading term in the expansion is $m^2 t^2 (m(c_1^2 - c_2) - bc_1)$ and the other terms are negligible with respect to t^2, including the one coming from S. We know that T_X is semistable with respect to $c_1(K_X) = -c_1 \geq 0$. Multiplication by the section σ yields a morphism $\pi^*_{1,0}\mathcal{O}_X(-B) \to \mathcal{O}_{X_1}(m)$, hence by direct image, a morphism $\mathcal{O}_X(-B) \to S^m T^*_X$. Evaluating slopes against K_X (a big nef class), the semistability condition implies $bc_1 \leq \frac{m}{2} c_1^2$, and our leading term is bigger that

$m^3t^2(\frac{1}{2}c_1^2 - c_2)$. We get a positive answer in the well-known case where $c_1^2 > 2c_2$, corresponding to T_X being almost ample. Analyzing positivity for the full range of values (k, m, t) and of singular sets S seems an unsurmountable task at this point; in general, calculations made in [3, 12] indicate that the Chern class and semistability conditions become less demanding for higher order jets (e.g. $c_1^2 > c_2$ is enough for $Z \subset X_2$, and $c_1^2 > \frac{9}{13}c_2$ suffices for $Z \subset X_3$). When $\operatorname{rank} V = 1$, major gains come from the use of Ahlfors currents in combination with McQuillan's tautological inequalities [11]. We therefore hope for a substantial strengthening of the above sufficient conditions, and a better understanding of the stability issues, possibly in combination with a use of Ahlfors currents and tautological inequalities. In the case of surfaces, an application of Proposition 4.6 for $k_0 = 1$ and an analysis of the behaviour of rank 1 (multi-)foliations on the surface X (with the crucial use of [11]) was the main argument used in [3] to prove the hyperbolicity of very general surfaces of degree $d \geq 21$ in $\mathbb{P}^3$. For these surfaces, one has $c_1^2 < c_2$ and $c_1^2/c_2 \to 1$ as $d \to +\infty$. Applying Proposition 4.6 for higher values $k_0 \geq 2$ might allow to enlarge the range of tractable surfaces, if the behavior of rank 1 (multi)-foliations on X_{k_0-1} can be analyzed independently.

5 Algebraic Jet-Hyperbolicity Implies Kobayashi Hyperbolicity

Let (X, V) be a directed variety, where X is an irreducible projective variety; the concept still makes sense when X is singular, by embedding (X, V) in a projective space $(\mathbb{P}^N, T_{\mathbb{P}^N})$ and taking the linear space V to be an irreducible algebraic subset of $T_{\mathbb{P}^n}$ that is contained in T_X at regular points of X.

5.1 Definition Let (X, V) be a directed variety. We say that (X, V) is algebraically jet-hyperbolic if for every $k \geq 0$ and every irreducible algebraic subvariety $Z \subset X_k$ that is not contained in the union Δ_k of vertical divisors, the induced directed structure (Z, W) either satisfies $W = 0$, or is of general type modulo $X_k \to X$, i.e. has a desingularization $(\widehat{Z}, \widehat{W})$, $\mu : \widehat{Z} \to Z$, such that some twisted canonical sheaf $K_{\widehat{W}} \otimes \mu^*(\mathcal{O}_{X_k}(\mathbf{a})_{|Z})$, $\mathbf{a} \in \mathbb{N}^k$, is big.

Proposition 4.6 then gives

5.2 Theorem *Let (X, V) be an irreducible projective directed variety that is algebraically jet-hyperbolic in the sense of the above definition. Then (X, V) is Brody (or Kobayashi) hyperbolic, i.e.* $\mathrm{ECL}(X, V) = \emptyset$.

Proof Here we apply Proposition 4.6 with $k_0 = 0$ and $p_0 = 1$. It is enough to deal with subvarieties $Z \subset X_k$ such that $\dim \pi_{k,0}(Z) \geq 1$, otherwise $W = 0$ and can reduce Z to a smaller subvariety by (3.2). Then we conclude that $\dim \mathrm{ECL}(X, V) < 1$. All entire curves tangent to V have to be constant, and we conclude in fact that $\mathrm{ECL}(X, V) = \emptyset$. □

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