Compact Kähler 3-manifolds without non-trivial subvarieties

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Abstract
We prove that any compact Kähler 3-dimensional manifold which has no non-trivial complex subvarieties is a torus. This is a very special case of a general conjecture on the structure of 'simple manifolds', central in the bimeromorphic classification of compact Kähler manifolds. The proof follows from the Brunella pseudo-effectivity theorem, combined with fundamental results of Siu and ~~Demailly~~ of the second author on the Lelong numbers of closed positive $(1,1)$-currents, and ~~on the Takegoshi and Demailly-Peternell-Schneider~~ with a version of the hard Lefschetz theorem for pseudo-effective line bundles, ~~together with special~~ due to Takegoshi and Demailly-Peternell-Schneider. Special features of the Riemann-Roch formula in dimension 3 are also used.

1 Introduction

We consider here connected compact Kähler manifolds X of (complex) dimension $n > 1$. An irreducible compact analytic subset Z of X will be said to be a 'subvariety' of X. It is said to be 'non-trivial' if its (complex) dimension is neither 0, nor n.

The bimeromorphic classification of compact Kähler manifolds can be reduced, by means of suitable functorial[1] fibrations[2], to the following two extreme particular cases: either X is projective, or X is 'simple', which means that its 'general'[3]) point x of X is not contained in any non-trivial subvariety of X.

The present text is concerned with the bimeromorphic classification of such 'simple' X. Its difficulty, in contrast to the projective case, is not due to the abundance and complexity of the examples, but to their (expected) scarceness and 'simple structure', which makes essentially all usual invariants of the classification vanish.

For example, if X is 'simple', its algebraic dimension[4] $a(X)$ vanishes, which implies, by [?], theorem 9.3, that its Albanese map is surjective and has connected fibres. Because the fibres must then have dimension either 0 or n, we

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[1]In the category of connected compact Kähler manifolds, morphisms being the dominant rational maps with connected fibres.

[2]Relative algebraic reductions, and relative Albanese maps, see [?], [?], [?].

[3]'general' means: in the intersection of countably many dense Zariski open subsets.

[4]Recall that $a(X) \in \{0, ..., n\}$ is the transcendence degree of the field of global meromorphic functions on X, $a(X) = n$ if and only if X is projective.

get: either $q = 0$, or X is bimeromorphic to its Albanese torus. Thus only the case where $q = 0$ needs to be considered.

The known[5] examples of 'simple' compact Kähler manifolds are (up to bimeromorphic equivalence) 'general' complex tori, $K3$ surfaces, or the 'general' member of the deformation families of hyperkähler manifolds[6]. 'Simple' surfaces are thus classified: either Tori ($q > 0$), or $K3$ (q = 0). But when $n \geqslant 3$, the classification is open. The situation is however expected to be similar to the surface case in higher dimensions (see Conjecture 1.1), where the only known examples are constructed out of surfaces which are either $K3$, or tori.

The following was formulated in [**?**], as question 1.4.

Conjecture 1.1: Let X be a 'simple' compact Kähler manifold. Then:

1. either X has a finite étale cover bimeromorphic to a complex torus, or $H^0(X, \Omega_X^2)$ is generated by some σ which is 'generically symplectic' (i.e. $n = 2m$ is even, and $\wedge^m \sigma \neq 0$).

We should thus have: $\kappa(X) = 0$. If $\dim(X)$ is odd, then X should be bimeromorphic to a complex torus, possibly after some finite étale cover. Such a cover will be needed, as shown by the Kummer quotient by the -1 involution.

2. When X does not contain any nontrivial subvariety, X should be either a complex torus, or an irreducible hyperkähler manifold[7].

Thus K_X should be trivial in this case, the two cases being told by $q > 0$, or $q = 0$.

3. If X is 'generically symplectic', then $\pi_1(X)$ should be finite, of cardinality at most 2^m, where $2m = n$.

The assertion 3 follows from [**?**], if $\chi(\mathcal{O}_X) \neq 0$ for X 'simple' and 'generically symplectic'.

In the sequel we prove ~~that~~ the second assertion of the conjecture for $n = 3$.

This conjecture can be motivated by the conjectural existence of minimal models in the bimeromorphic category of connected compact Kähler manifolds.

[5]It is shown in [**?**] that a stable bundle which does not degenerate to a direct sum of stable bundles on the generic deformation of a Hilbert scheme of $K3$ has a compact moduli space; this may lead to new examples of hyperkähler manifolds. The 'rigidity' of the category of coherent sheaves on the general members of these families is established in [**?**], [**?**].

[6]Recall that X is 'hyperkähler' it is Kähler and admits a holomorphic symplectic form. It is said to be 'irreducible' if, moreover, $\pi_1(X)$ is finite, and $h^{2,0}(X) = 1$.

[7]Notice that a general deformation of a Hilbert scheme of a $K3$ surface has no subvarieties [**?**], while a general deformation of a generalized Kummer variety has some subvarieties, partially classified in [**?**].

Indeed, if such a theory exists, and if X is simple, we have $a(X) = 0$, which brings $\kappa(X) \leqslant 0$. The possibility $\kappa(X) = -\infty$ is excluded, since X should then be uniruled. This implies $\kappa(X) = 0$.

In this case, a Kähler minimal model is a map $X \dashrightarrow X'$ such that X' has terminal singularities and $K_{X'} \equiv 0$). A (conjectural) Bogomolov-type decomposition ([?], [?]) for manifolds with terminal singularities and $K_{X'} \equiv 0$ could be used to represent a finite cover of X' as a product. However X is simple, hence either X' is covered by a simple torus (with finite locus of ramification), or X' carries a symplectic holomorphic 2-form, establishing the conjecture above.

Theorem 1.2: Let X be a compact Kähler 3-dimensional manifold. Assume that X has no non-trivial subvariety. Then X is a torus.

The proof will be given as ~~corollary~~ ??. Although it is an easy combination of known results, it seems worth to present it, because it indicates that the above conjecture might be accessible in dimension 3 with the actually existing techniques (see ~~Remark~~ ??). Let us start with an easy remark:

Lemma 1.3: Let X be a 'simple' compact Kähler threefold. If $\chi(\mathcal{O}_X) = 0$, then X is bimeromorphic to its Albanese torus.

Proof: By the above remarks, it is sufficient to show that $q > 0$. But ~~$0 = \chi(\mathcal{O}_X) = 1 - q + h^{2,0} - h^{3,0}$.~~ $0 = \chi(*0\mathcal{O}_X) = 1 - q + h^{2,0} - h^{3,0}$. By Kodaira's theorem, $h^{2,0} > 0$

$_X) = 0 \chi(\mathcal{O}_X) = 0$ *, which is the objective of the following sections.

Finally, let us mention that compact (connected) complex manifolds, not necessarily Kähler, without non-trivial subvarieties are also quite important in model theory, giving examples of so-called 'trivial' Zariski geometries ([?], [?] [?]). Our arguments below show that compact three-dimensional complex manifolds without non-trivial subvarieties are tori, provided their canonical bundle are pseudo-effective. Brunella's theorem is indeed the single step of our proof requiring the Kähler hypothesis.

2 Pseudoeffective currents and line bundles

We refer to [?] for the basic notions and properties of currents on complex manifolds, and only recall briefly some of the facts needed here.

Let X be a compact, n-dimensional complex manifold, and L a holomorphic line bundle on X. Assume that L is equipped with a singular hermitian metric h with local weights $e^{-\varphi}$ which are locally integrable. Its curvature current is then $\Theta(L,h) := i\partial\overline{\partial}\varphi$, and its first Chern class $c_1(L)$ is represented by the cohomology class $\frac{1}{2\pi}.[\Theta(L,h)] \in H^{1,1}(X) \subset H^2(X,\mathbb{R})$ $\frac{1}{2\pi}[\Theta(L,h)] \in H^{1,1}(X) \subset H^2(X,\mathbb{R})$.

Definition 2.1: A positive, closed (1,1)-current Θ is **nef** (resp. pseudo-effective) if it is a limit of positive, closed $(1,1)$-forms (resp. if it is positive, or equivalently, if the functions φ are plurisubharmonic). If Θ is nef, and $\int_M [\Theta]^n > 0$, Θ is said to be **big**[1]. A line bundle L on X is said to be nef (resp. pseudo-effective, resp. big) if it admits a singular metric h such that $\Theta(L,h)$ has the same property.

Theorem 2.2: ([?]) Let X be a compact complex manifold. Assume that X carries a nef and big line bundle L. Then X is bimeromorphic to a projective manifold (using sections of $L^{\otimes m}$, for $m > 0$ sufficiently large). In particular, X is not simple.

This result has been generalized by Demailly by means of his in [?], using holomorphic Morse inequalities(see) . This improved version is however not used here.

Definition 2.3: The **Lelong number** $\nu_x(\Theta(L,h)) = \nu(\varphi, x)$ of the $(1,1)$-current $\Theta(L,h)$ at $x \in X$, is defined as $lim\ inf_{z\to x} \frac{\varphi(z)}{|z-x|}$ $\liminf_{z\to x} \frac{\varphi(z)}{|z-x|}$, for a local metric on X near x.

Definition 2.4: For a positive real number $c > 0$, **the Lelong set** F_c of a (1,1)-current η is the set of points $x \in M$ with $\nu(\eta, x) \geqslant c$. By Siu's theorem a well-known theorem of Siu ([?], with a considerably simplified proof using Demailly's regularization of currents and the Ohsawa-Takegoshi extension theorem) any Lelong set of a positive, closed current is a complex analytic subvariety of M . (the proof of this difficult result has been considerably simplified, using regularization of currents and the Ohsawa-Takegoshi extension theorem, see [?].)

Theorem 2.5: ([?]) Let X be a compact complex manifold. Let L be a pseudo-effective holomorphic line bundle on X, with singular hermitian metric h with positive curvature current $\Theta(L,h)$. Assume that the Lelong sets of $\Theta(L,h)$ are all zero-dimensional. Then L is nef, and big unless all Lelong numbers vanish as soon as there is at least one nonzero Lelong number .

Proof: The first assertion is ([?], corollary 6.4). The second results from theorem applied to follows from ([?], corollary 7.6) . See also [?], theorem 3.12. ■

Corollary 2.6: If X is a compact Kähler manifold without nontrivial subvarieties, and with pseudo-effective canonical bundle, then K_X is nef, and for any singular hermitian metric h on $K_X, m > 0$ with positive curvature current, the Lelong number vanish at any point. In particular, the associated multiplier

[1] A more general notion exists, assuming Θ pseudo-effective only.

ideal sheaves on $K_X^{\otimes m}$ are all trivial for any $m > 0$ (i.e: $e^{-m.\varphi}$ $e^{-m\varphi}$ is integrable for any $m > 0$).

3 Hard Lefschetz theorem for the cohomology of pseudo-effective line bundles

We recall the version of the Hard Lefschetz theorem which is going to be used here:

Theorem 3.1: ([?], [?]) Let (X, ω) be a compact Kähler manifold, of dimension n with Kähler form ω, let K_X be its canonical bundle, and L be a pseudo-effective holomorphic line bundle on X equipped with a singular Hermitian metric h. Assume that the curvature Θ of (L, h) is a positive current on X, and denote by $\mathcal{I}(h)$ the corresponding multiplier ideal sheaf. Then the wedge multiplication operator $\eta \longrightarrow \omega^i \wedge \eta$ induces a surjective map

$$H^0(X, \Omega_X^{n-i} \otimes L \otimes \mathcal{I}(h)) \xrightarrow{\omega^i \wedge} H^i(X, K \otimes L \otimes \mathcal{I}(h)).$$

Here ω is considered as an element in $H^1(X, \Omega_X^1)$, and multiplication by ω maps $H^k(X, \Omega_X^{n-l} \otimes F)$ to $H^{k+1}(X, \Omega_X^{n-l+1} \otimes F)$. ■

This theorem was obtained under various hypothesis during the decade 1990: see [?], and [?]. The most general form given here is due to [?], Theorem 2.1.1. It was proved in [?] when L nef.

Corollary 3.2: Let X be a compact Kähler manifold of dimension $n > 1$ without non-trivial subvariety. Assume that K_X is pseudo-effective. Then $h^i(X, m.K_X) \leqslant \binom{n}{i}$, for any $i \geqslant 0$, and the polynomial $P(m) := \chi(X, m.K_X)$ is constant, equal to $\chi(\mathcal{O}_\mathcal{X})$ $\chi(\mathcal{O}_X)$.

Proof: Since $n > 1$ and X has no non-trivial subavrieties, the algebraic imension of X vanishes: $a(X) = 0$. Therefore $h^0(X, E) \leqslant \mathrm{rank}(E)$, for any holomorphic vector bundle E on X. Combining with the Hard Lefschetz theorem (Theorem **??**), this gives the first claim of **??**. The second claim is clear because a polynomial function $P(m)$ which remains bounded when $m \to +\infty$ is necessarily constant. ■

Corollary 3.3: Let X be a compact Kähler manifold of dimension 3 without non-trivial subvariety. Assume that K_X is pseudo-effective. The polynomial $P(m) := \chi(X, m.K_X)$ is constant, equal to $\chi(\mathcal{O}_X) = 0$.

Proof: The intersection number K_X^3 vanishes (either by **??**, or because it is the leading term of $P(m)$, up to the factor 3!). The Riemann-Roch formula then gives: $P(m) := \frac{(1-2n)}{24}.c_1(X).c_2(X)$. The boundedness of $P(m)$ then implies that $24.\chi(\mathcal{O}_X) = c_1(X).c_2(X) = 0$ $24.\chi(\mathcal{O}_X) = c_1(X).c_2(X) = 0$. ■

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Remark 3.4: Arguments close to some of the ones presented here have been already used in the proof [?], theorem 2.7.3: if X is a compact Kähler manifold with pseudo-effective canonical bundle admitting a metric with weights φ having analytic singularities and positive curvature current, then either $H^0(X, \Omega^i_X \otimes (m.K_X)) \neq 0$ for infinitely many $m > 0$ and some $i \geqslant 0$, or $\chi(X, \mathcal{O}_{\mathcal{X}}) = '$ $\chi(X, \mathcal{O}_X) = 0$.

4 Brunella's pseudo-effectivity criterion.

The rest of our arguments is based on the following strong (and very difficult) theorem by Brunella.

Theorem 4.1: ([?]) Let X be a compact Kähler manifold with a 1-dimensional holomorphic foliation F given by a nonzero morphism of vector bundle $L \to T_X$, where L is a line bundle on X, and T_X is its holomorphic tangent bundle. If L^{-1} is not pseudo-effective, the closures of the leaves of F are rational curves (and X is thus uniruled).

The following corollary had already been observed in [?], proposition 4.2.

Corollary 4.2: If X is an n-dimensional compact Kähler manifold with $H^0(X, \Omega^{n-1}_X) \neq 0$. Then K_X is pseudo effective pseudo-effective .

Proof: Ω^{n-1}_X is canonically isomorphic to $K_X \otimes T_X$. Any nonzero section thus provides a nonzero map $K_X^{-1} \to T_X$, and an associated foliation ■

Corollary 4.3: If X is a 3-dimensional non-projective compact Kähler manifold. Then: either K_X is pseudo effective pseudo-effective , or X is uniruled. If X is 'simple', K_X is pseudoeffective pseudo-effective .

Proof: Kodaira's theorem implies that any compact Kähler manifold with $H^{2,0}(X) = 0$ is projective. Thus $H^{2,0}(X) \neq 0$, and the preceding corollary applies (since $2 = (n-1)$, here) and gives the first claim. If X is uniruled, it is not simple, hence the second assertion. ■

By combining **??**, **??**, and **??**, we get our main result:

Corollary 4.4: If X is a 3-dimensional compact Kähler manifold without non-trivial subvariety, then X is a complex torus.

In higher dimensions, we can replace the assumption that K_X was pseudo-effective from our preceding corollary **??** by the existence of a holomorphic $(n-1)$-form:

Corollary 4.5: Let X be a compact Kähler manifold of dimension $n > 1$ without non-trivial subvariety. Assume that $H^0(X, \Omega_X^{n-1}) \neq 0$. Then $h^i(X, m.K_X) \leqslant \binom{n}{i}$, for any $i \geqslant 0$, and the polynomial $P(m) := \chi(X, m.K_X)$ is constant, equal to $\chi(\mathcal{O}_X)$[1].

Remark 4.6: In order to show that if X is compact Kähler and 'simple', of dimension 3, then X is bimeromorphic to a torus, it were sufficient either:

A. show that $\chi(\mathcal{O}_X) = 0$ $\chi(\mathcal{O}_X) = 0$, by a suitable control of the multiplier ideal sheaves on $K_X^{\otimes m}, m > 0$, or proceed in the following two steps:

Assume first that K_X is nef: X should then contain no divisor, but possibly curves. One might then show that $\chi(\mathcal{O}_X) = 0$ $\chi(\mathcal{O}_X) = 0$ through a suitable control of the multiplier ideal sheaves of $K_X^{\otimes m}, m > 0$, which are concentrated on curves and points.

In the remaining case where K_X is pseudo-effective, but not nef, by [**?**], theorem 5.4, and [**?**], [**?**], one has a Mori contraction. Showing that its image can be chosen to be smooth and Kähler would permit to conclude by induction on the rank of $H^2(X, \mathbb{R})$.

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