Séminaire P. LELONG, P. DOLBEAULT, H. SKODA
(Analyse) 24e année, 1983/84
Lecture Notes in Math. **1198**, Springer, pages 98–104

An example of non Stein holomorphic bundle with fiber $\mathbb{C}^2$ over the disc or the plane

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Nous construisons un exemple simple d'espace fibré holomorphe à fibre $\mathbb{C}^2$ au-dessus du disque ou du plan, dont les automorphismes de transition sont de type exponentiel. Nous montrons en fait que toutes les fonctions holomorphes ou plurisousharmoniques de ce fibré proviennent de fonctions sur la base.

We construct a simple example of a non-Stein holomorphic fiber bundle over the disk or the plane, with fiber $\mathbb{C}^2$ and with structural automorphisms of exponential type. We show in fact that all holomorphic or plurisubharmonic functions on the bundle arise from functions on the base.

0. Introduction.

The goal of the present note is to give an example, as simple as possible, of a non Stein holomorphic bundle with fiber $\mathbb{C}^2$ over the disc or the plane as the base, and whose transition automorphisms are of exponential type.

In ([6], 1977), H. Skoda have the first example of a non Stein fiber bundle with fiber of Stein, thereby answering by the negative a problem raised by J.-P. Serre [5] in 1953. We later improved H. Skoda's construction and produced a counterexample whose base was simply connected [1], [2], but the proof remained a bit obscure as a consequence of many unneeded technical artifacts. Hope Prpers have here awfully clarifié that example.

The principle of the building rests on an owed inequality at P. Lelong [4], who imposes of the severe restrictions at the growth of the functions plurisousharmoniques (psh at abbreviated) along the fibers, cf. lemme 1.1. That inequality trains a strong distorsion of the growth tracking the different fibers for an election adéquat of the automorphismes of transition. Thanks to a calculation of enveloppe pseudoconvexe using the principle of the disc, deduces whereas the functions psh of the bundle are constantes on the fibers, cf. theorem 4.6. On our example the bundle is besides topologiquement trivial.

1. A convexity inequality due to P. Lelong.

Be Ω a variety complex complex analytique of dimension p and V a function psh on $\Omega \times \mathbb{C}^n$. Having given an open $\omega \Subset \Omega$ relatively compact, states on

$$M(V, \omega, r) = \sup_{\omega \times D(r)} V,$$

where $D(r)$ designates the polydisque of centre 0 and of ray r on $\mathbb{C}^n$. As P. Lelong [4], $M(V, \omega, r)$ is function convexe increasing of $\log r$; outrer that function is no conaunt if V is no constante on at least a fiber $\{x\} \times \mathbb{C}^n$, $x \in \Omega$.

Prpers redémontrons here the inequality of P. Lelong On the particular case where the open considered on the base are of the polydisques concentriques of $\mathbb{C}^n$ (the general inequality deduce se of elsewhere easily of that particular case).

Lemma 1.1. – *Be V a function psh $\geqslant 0$ on $\Omega \times \mathbb{C}^n$, where Ω is an open of $\mathbb{C}^p$, and $D(\alpha) \Subset D(\beta) \Subset D(\gamma) \Subset \Omega$. Then for all $r > 0$ has on the inequality*

$$M(v, D(\alpha), r) \leqslant M(V, D(\beta), r^\sigma) + M(V, D(\gamma), 1)$$

where $\sigma = \log(\gamma/\alpha)/\log(\gamma/\beta) > 1$.

Démonstration. At effect, as P. Lelong [4], the function $M(V, D(\rho), r)$ is function convexe of the couple $(\log \rho, \log r)$. Has On so

$$\begin{aligned} M(V, D(\beta), r) &\leqslant \frac{1}{\sigma} M(V, D(\alpha), r^\sigma) + \Big(1 - \frac{1}{\sigma}\Big) M(V, D(\gamma), 1) \\ &\leqslant M(V, D(\alpha), r^\sigma) + M(V, D(\gamma), 1) \end{aligned}$$

if chooses on σ as

$$\log \beta = \frac{1}{\sigma} \log \alpha + \Big(1 - \frac{1}{\sigma}\Big) \log \gamma. \qquad \square$$

2. Construction of the bundle X.

The base of the bundle will be an open $\Omega \subset \mathbb{C}$ containing the disc $D(0, 3)$. States On then

$$\begin{aligned} &\Omega_1 = \Omega \smallsetminus \{-1\}, \qquad \Omega_2 = \Omega \smallsetminus \{1\}, \\ &\Omega_0 = \Omega_1 \cap \Omega_2 = \Omega \smallsetminus \{-1, 1\}, \end{aligned}$$

defines On a bundle X at fiber $\mathbb{C}^2$ at the-above of Ω recollant both cards trivialisantes $\Omega_1 \times \mathbb{C}^2$ and $\Omega_2 \times \mathbb{C}^2$ at the half of the automorphisme of transition

$$\tau_{12} : \Omega_0 \times \mathbb{C}^2 \longrightarrow \Omega_0 \times \mathbb{C}^2$$

defined by the formula $\tau_{12} = \tau_{01}^{-1} \circ \tau_{02}$ with

$$\left\{ \begin{aligned} &\tau_{01}(x\,;\, z_1, z_2) = \big(x\,;\, z_1\,,\, z_2\, \exp(z_1 u(x))\big) \\ &\tau_{02}(x\,;\, z_1, z_2) = \big(x\,;\, z_1\, \exp(z_2 u(x))\,,\, z_2\big) \end{aligned} \right. \tag{2.1}$$

where $x \in \Omega_0$, $(z_1, z_2) \in \mathbb{C}^2$ and $u(x) = \exp(1/(x^2-1))$. The card $\Omega_0 \times \mathbb{C}^2$ and the automorphismes τ_{01} , τ_{02} corresponding have be entered here at alone end of simplify the writing of τ_{12} , although are at unnecessary principle for define the bundle X .

A function psh V on X is so represented by a triplet $(V_j)_{j=0,1,2}$ of functions psh on $\Omega_j \times \mathbb{C}^2$ tied by the relatlons of transition

$$V_k = V_j \circ \tau_{jk}, \qquad 0 \leqslant j, k \leqslant 2. \tag{2.2}$$

Observes 2.3. Is easy of see that the bundle X is trivial at the sense C^∞ -différentiable, relatively at the structural group of the automorphismes analytique of the fiber. Be at effect f_1 , f_2 , of the functions C^∞ at compact support on of the voisinage disjoints of 1 and -1 respectively, equal at 1 on of the best small voisinage. Obtains On then a trivialisation global $\gamma : X \to \Omega \times \mathbb{C}^2$ recollant the morphismes $\gamma_j : \Omega_j \times \mathbb{C}^2 \to \Omega_j \times \mathbb{C}^2$ of class C^∞ defined by

$$\begin{cases} \gamma_0(x\,;\, z_1, z_2) = \big(x\,;\, z_1 \exp(-z_2 f_2(x) u(x))\,,\, z_2\, \exp(-z_1 f_1(x) u(x))\big) \\ \gamma_1(x\,;\, z_1, z_2) = \big(x\,;\, z_1\,,\, z_2 \exp(z_1(1-f_1(x))u(x))\big) \\ \gamma_2(x\,;\, z_1, z_2) = \big(x\,;\, z_1\, \exp(z_2(1-f_2(x))u(x))\,,\, z_2\big). \end{cases} \tag{2.4}$$

The reader will check that that morphismes satisfy very the accounts of want to transition $\gamma_j \circ \tau_{jk} = \gamma_k$.

3. Restrictions on the growth of the psh functions.

Notes On $\Delta = D(0,1)$ the disc unity on $\mathbb{C}$, $\omega = D(0, \frac{1}{2}) \Subset \Omega_0$, and considers on the automorphismes of the defined discs by

$$h_a(x) = \frac{x+a}{1+\overline{a}x}, \qquad a \in \Delta.$$

The following inequalities display that the growth of the functions psh along the fibers of X has subjected at of the very strong restrictions.

Proposal 3.1. – *Be V a function psh on X . Then il there is a constante $C = C(V) > 0$ tel that for all $j = 1,\, 2$ and $r > 1$ have on*

$$M(V_j, \omega, r) \leqslant M\big(V_0, \omega, \exp\big((\log r)^3\big)\big) + C.$$

Démonstration. Comme the application $(x\,;\, z_1, z_2) \mapsto (-x\,;\, z_2, z_1)$ defined on $\Omega_0 \times \mathbb{C}^2$ stretches se at an automorphisme of X who exchanges the cards $\Omega_1 \times \mathbb{C}^2$ and $\Omega_2 \times \mathbb{C}^2$, prpers suffice of reason for $j = 1$. Quitte à replace V by $V = \max(V, 0)$, pouvoir on equally assume $V \geqslant 0$. Consider a real $a \in [0,1]$ who have fixed ultérieurement. The idea consister observe that V is “presque” equal at V_0 on $h_a(\omega) \times D(r)$ if a is enough nearby of 1 , because the function $u(x)$ is very small on $h_a(\omega)$. The lemme 1.1 allows

then of connect the growth of the functions V_0 and V_1 on ω at leur growth on $h_a(\omega)$, and so of compare V_0 and V_1 on ω .

The open $h_a(\omega)$ is the determined disc by the dots diamétralement opposer $h_a(\pm\frac{1}{2})$, having respectively for centre the dot x_a and for ray the real α tel that

$$x_a = \frac{3a}{4-a^2} \in \,]0,1[, \qquad \alpha = \frac{2(1-a^2)}{4-a^2} \in \Big]0, \frac{1}{2}\Big[.$$

Consider both discs concentriques $D(x_a,\beta) \Subset D(x_a,\gamma)$, eux-same concentriques at the disc $h_a(\omega) = D(x_a,\alpha)$, of respective rays $\beta = 1/2 + x_a$, $\gamma = 3/4 + x_a$. Has On clearly

$$\begin{gathered}\log\gamma/\beta > \log(7/4)/(3/2) = \log 7/6 > 1/7,\\ \omega \subset D(x_a,\beta), \qquad D(x_a,\gamma) \Subset \Omega_1.\end{gathered}$$

As the lemme 1.1, prpers comes so

$$M(V_1,\omega,r) \leqslant M(V_1,D(x_a,\beta),r) \leqslant M(V_1,h_a(\omega),r^\sigma) + M(V_1,D(1,\tfrac{7}{4}),1) \tag{3.2}$$

with

$$\sigma = \frac{\log\gamma/\alpha}{\log\gamma/\beta} \leqslant 7\,\log 4/(1-a). \tag{3.3}$$

The image of $h_a(\omega)$ by the homographie $x \mapsto \frac{1}{x-1}$ is the defined disc by the dots diamétralement opposer $1/(h_a(\pm\frac{1}{2})-1)$, of where

$$\begin{gathered}\sup_{x\in h_a(\omega)} \operatorname{Re}\frac{1}{x-1} = \frac{1}{h_a(-\frac{1}{2})-1} = \frac{1}{3(a-1)},\\ \sup_{x\in h_a(\omega)} \log|u(x)| = \sup \frac{1}{2}\Big(\operatorname{Re}\frac{1}{x-1} - \operatorname{Re}\frac{1}{x+1}\Big) < \frac{1}{6(a-1)}.\end{gathered}$$

The election of a as

$$\frac{1}{1-a} = 48\,\log r\cdot\log\log r \tag{3.4}$$

give for r enough big $\sigma \leqslant 8\log\log r$, of where :

$$\sup_{x\in h_a(\omega)} |u(x)| \leqslant r^{-8\log\log r} \leqslant r^{-\sigma}.$$

The equality of definition (2.1) watch whereas

$$\tau_{01}(\{x\}\times D(r^\sigma)) \subset \{x\}\times D(e\,r^\sigma), \qquad \forall x \in h_a(\omega),$$

of where

$$M(V_1,h_a(\omega),r^\sigma) \leqslant M(V_0,h_a(\omega),e\,r^\sigma). \tag{3.5}$$

Apply now the lemme 1.1 at the function V_0 and at the discs concentriques

$$D(0,\tfrac{1}{2})=\omega,\qquad D(0,h_a(\tfrac{1}{2}))\supset h_a(\omega),\qquad D(0,1)=\Delta.$$

Prpers comes

$$M(V_0,h_a(\omega),r)\leqslant M(V_0,D(0,h_a(\tfrac{1}{2}),r)\leqslant M(V_0,\omega,r^\tau)+M(V_0,\Delta,1) \tag{3.6}$$

with

$$\tau=\frac{\log 2}{\log 1/h_a(\frac{1}{2})}<\frac{\log 2}{1-h_a(\frac{1}{2})}<\frac{3\log 2}{1-a}.$$

The constante $M(V_0,\Delta,1)$ has ended, because $u(x)$ has limited (by 1) on Δ , and pouvoir on type

$$M(V_0,\Delta,1)=\max\Big(\sup_{\tau_{10}(\Delta_+\times D(1))}V_1\,,\ \sup_{\tau_{20}(\Delta_-\times D(1))}V_2\Big)$$

with $\Delta_+=\Delta\cap\{\operatorname{Re}x\geqslant 0\}\Subset\Omega_1$, $\Delta_-=\Delta\cap\{\operatorname{Re}x\leqslant 0\}\Subset\Omega_2$. Obtains On finally for r enough big

$$\sigma\leqslant 8\,\log\log r,\qquad \tau\leqslant 144\,\log 2\cdot\log r\cdot\log\log r,$$

and combining (3.2), (3.5) and (3.6) prpers comes

$$\begin{aligned}M(V,\omega,r)&\leqslant M(V_0,\omega,e^\tau r^{\sigma\tau})+C\\&\leqslant M\big(V_0,\omega,\exp\big(800\,(\log r\ \log\log r)^2\big)\big)+C.\end{aligned}$$ □

4. Distorsion Induced by the automorphismes of transition.

Observes On maintenant que by definition of the functions V_j . Has on

$$\max_{j=1,\,2}M(V_j,\omega,r)=\sup_{x\in\omega}\ \sup_{z\in K(x,r)}V_0(x,z) \tag{4.1}$$

where $K(x,r)=\tau_{01}(\{x\}\times\overline{D(r)})\cup\tau_{02}(\{x\}\times\overline{D(r)})$. Since V_0 is psh, has on the equality

$$\sup_{z\in K(x,r)}V_0(x,z)=\sup_{z\in\widehat{K}(x,r)}V_0(x,z) \tag{4.2}$$

where $\widehat{K}(x,r)$ designates the enveloppe holomorphic convexe of $K(x,r)$. For conclude, goes on now price the size of $\widehat{K}(x,r)$ using the principle of the disc (cf. By example L. Hörmander [3], th. 2.4.3). That "principle" trains that for all $0<\alpha\leqslant\beta$ has on

$$\Big(\overline{D(\alpha)}\times\overline{D(\beta)}\cup\overline{D(\beta)}\times\overline{D(\alpha)}\Big)^{\widehat{}}=\big\{(z_1,z_2)\in\mathbb{C}^2\,;\ |z_1|\leqslant\beta,\ |z_2|\leqslant\beta,\ |z_1z_2|\leqslant\alpha\beta\big\}.$$

Lemme 4.3. – *For r enough big $\widehat{K}(x,r)$ contains the polydisque $D(\hat r)$ of ray*

$$\hat r=\exp(r/32).$$

Démonstration. Has On $\inf_{x\in\omega}|u(x)| = u(\frac{1}{2}) = \exp(-4/3)$. Having given $x \in \omega$, note θ the argument of $u(x)$. On the disc

$$\left\{|z_1 - \tfrac{r}{2}\, e^{-i\theta}| < \tfrac{r}{4}\right\} \subset \{|z_1 < r\}$$

has on trivialement $\mathrm{Re}(z_1 e^{i\theta}) \geqslant \frac{r}{4}$, so obtains on

$$|\exp(z_1 u(x))| \geqslant \exp\left(\tfrac{r}{4}\exp(-4/3)\right) \geqslant \exp\left(\tfrac{r}{16}\right)$$

and en conséquence

$$\tau_{01}(\{x\} \times D(r)) \supset \{x\} \times \left\{|z_1 - \tfrac{r}{2}\, e^{-i\theta}| < \tfrac{r}{4}\,,\ |z_2| < r\,\exp(\tfrac{r}{16})\right\}.$$

By continuation $\tau_{0j}(\{x\} \times D(r))$ contains the bidisque of centre $\zeta = (\frac{r}{2}e^{-i\theta}, \frac{r}{2}e^{-i\theta})$ and of birayon

$$(r_1, r_2) = \left(\tfrac{r}{4}\,,\ r\,\exp(\tfrac{r}{16}) - \tfrac{r}{2}\right) \quad \text{si } j = 1 \qquad \left[\text{resp. } (r_2, r_1) \text{ si } j = 2\right].$$

As the principle of the disc, $\widehat{K}(x, r)$ contains then the bidisque of centre ζ and of geometrical half ray $\sqrt{r_1 r_2}$. Prpers results that $\widehat{K}(x, r) \supset D(\hat{r})$ with

$$\hat{r} = \sqrt{r_1 r_2} - \tfrac{r}{2} = \tfrac{r}{2}\left(\sqrt{\exp(\tfrac{r}{16}) - \tfrac{1}{2}} - 1\right),$$

so $\hat{r} > \exp(r/32)$ if r is enough big. □

The proposal 3.1, the lemme 4.3 and the equalities (4.1), (4.2) give

$$M\left(V_0, \omega, \exp(\tfrac{r}{32})\right) \leqslant M\left(V_0, \omega, \exp\left((\log r)^3\right)\right) + C. \tag{4.4}$$

If V is no constante on at least a fiber of X , the function $M(V_0, \omega, r)$ is, for r enough big, strictly increasing convexe at the variable $\log r$, those that trains

$$M\left(V_0, \omega, \exp(\tfrac{r}{32})\right) - M\left(V_0, \omega, \exp\left((\log r)^3\right)\right) \geqslant c\left(\tfrac{r}{32} - (\log r)^3\right) \tag{4.5}$$

with $c > 0$. The member of left of (4.5) extends so to $+\infty$ when r extends to $+\infty$, those that contradict (4.4). Prpers deduce therefore the following outcome.

Theorem 4.6. – *All psh functions V (resp. all holomorphic functions F) on X are constant on the fibers. In particular, X is not Stein.*

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(June 1984)