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MAGNETIC FIELDS AND MORSE INEQUALITIES FOR d"-COHOMOLOGY

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0. Introduction.

Be X a variety $\mathbb{C}$ -compact analytique of dimension n , F a bundle vectoriel holomorphic of rank r and E a bundle holomorphic at right hermitian of class $\mathcal{C}^\infty$ at the-above of X . Be $D = D' + D''$ the connection canonique of E and $c(E) = D^2 = D'D'' + D''D'$ the form of curvature of that connection. Designate by $X(q)$, $0 \leqslant q \leqslant n$, the open of the dots of X of rate q , i.e. The open of the dots $x \in X$ at which the form of curvature $ic(E)(x)$ has exactly q eigenvalues < 0 and $(n-q)$ eigenvalues > 0 . States On equally

$$X(\leqslant q) = X(0) \cup X(1) \cup \ldots \cup X(q).$$

show then the inequalities of Morse following, who limit the dimension of the spaces of cohomologie $H^q(X, E^k \otimes F)$ en fonction of invariants integral of the curvature of E .

Theorème 0.1. – *When k extends to $+\infty$ has on for all $q = 0, 1, \ldots, n$ the inequalities asymptotics following.*

(a) *Inequalities of Morse* :

$$\dim H^q(X, E^k \otimes F) \leqslant r \frac{k^n}{n!} \int_{X(q)} (-1)^q \Big(\frac{i}{2\pi} c(E)\Big)^n + o(k^n).$$

(b) *Inequalities of Morse strong* :

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(X, E^k \otimes F) \leqslant r \frac{k^n}{n!} \int_{X(\leqslant q)} (-1)^q \Big(\frac{i}{2\pi} c(E)\Big)^n + o(k^n).$$

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(c) *Formula of Riemann-Roch asymptotic* :

$$\sum_{q=0}^{n}(-1)^q \dim H^q(X, E^k \otimes F) = r\frac{k^n}{n!}\int_X \Big(\frac{i}{2\pi}c(E)\Big)^n + o(k^n).$$

The assessments 0.1 (a), (b) are new at our knowledge, even on the case of the projectif varieties. The equality asymptotic 0.1 (c) , when hers, is a weakened version of the theorem of Hirzebruch-Riemann-Roch, who has luire-even a particular case of the tea orème of the rate of Atiyah- Imitate [1]. That latter theorem allows at effect of express the characteristic of Euler-Poincaré

$$\chi(X, E^k \otimes F) = \sum_{q=0}^{n}(-1)^q \dim H^q(X, E^k \otimes F)$$

under the form

$$\chi(X, E^k \otimes F) = r\frac{k^n}{n!}c_1(E)^n + P_{n-1}(k) \ ; \tag{0.2}$$

$P_{n-1}(k) \in \mathbb{Q}[k]$ designates here a polynôme of grade $\leqslant n-1$ and $c_1(E) \in H^2(X, \mathbb{Z})$ is the first class of Chern of E , represented at cohomologie of Of Rham by the $(1,1)$ -fermé form $\frac{i}{2\pi}c(E)$ (cf. By example [16]). Observe On that the constante numerical of the inequality 0.1 (a) is optimum, comme display it the example of the bundle produced tensoriel total $E = \mathcal{O}(1)^{n-q} \boxtimes \mathcal{O}(-1)^q$ at the-above of $X = (\mathbb{P}^1(\mathbb{C}))^n$. For that bundle, has on at effect $X(q) = X$ and

$$\dim H^q(X, E^k) = (k+1)^{n-q}(k-1)^q, k \geqslant 1,$$
$$\int_X \Big(\frac{i}{2\pi}c(E)\Big)^n = (-1)^q n!\,.$$

The existence of a majoration of the type 0.1(a) be conjecturée by . T. Siu, who has showed successively the particular case where $ic(E)$ is > 0 on the complementary of an ensemble of measure any [16], afterwards the case where $ic(E)$ is $\geqslant 0$ on X [17]. We have of elsewhere emprunté at Siu a part of the technical utilisécs here, especially at the §3 and §5. The test of the theorem 0.1 rests on the method entered analytique recently by E. Witten [18], [19]. That method allows (amid autres) of reprove the inequalities of Morse classical $b_q \leqslant m_q$ on a variety différentiable compact M , where b_q designates the q -ième number of Betti and m_q the number of dots critics of rate q of a function of Morse quelconque on M . On our situation, the role of the function of Morse has kept by the election of the métrique hermitian on E . Caters On on the other hand X and F of métrique hermitians arbitrary, who will take part alone on the terms $o(k^n)$ of the final assessments. É So much given a real $\lambda \geqslant 0$, considers on the under-complex $\mathcal{H}_k^\bullet(\lambda)$ of the complex of Dolbeault $\mathcal{C}^\infty_{0,\bullet}(X, E^k \otimes F)$ of the $(0,q)$ -forms of class $\mathcal{C}^\infty$ on X at courages on $E^k \otimes F$, engendré by the own functions of the Laplacien antiholomorphic Δ'' whose eigenvalues are $\leqslant k\lambda$. The groups of cohomologie of the complex $\mathcal{H}_k^\bullet(\lambda)$ are then isomorphes at the groups $H^q(X, E^k \otimes F)$ (proposal 4.1), so that it suffice of know

limit the dimension of the spaces $\mathcal{H}_k^q(\lambda)$. For that, uses on essentially deux tools. The first tool consists at a formula of type Weitzenböck

$$\frac{2}{k}\int_X \langle \Delta'' u, u\rangle = \int_X \frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu, u\rangle + \frac{1}{k}\langle \Theta u, u\rangle \tag{0.3}$$

showed at the §3, and derived of the identity of Bochner- Kodaira-Nakano no kählérienne [6]. ∇_k Designates here the connection hermitian natural on the bundle $\Lambda^{0,q}T^*X \otimes E^k \otimes F$, V is a potential linear of order 0 tied at the curvature of the bundle E , lastly S and Θ are of the operators of order 0 pertinent of the torsion of the métrique hermitian on X and of the curvature of F . The studio of the spectrum of Δ'' finds se so brought at the studio of the spectrum of the operator autoadjoint $\nabla_k^*\nabla_k$ associé at the real connection ∇_k . The deuxième tool fundamental consists precisely at a theorem spectral very general relative at the operators of the type $\nabla^*\nabla$. Be (M, g) a variety riemannienne $\mathcal{C}^\infty$ of real dimension n , E a bundle at right complex at the- above of X , equipped of a connection hermitian ∇ . If ∇_k designates the induced connection by ∇ on E^k , studies on then the spectrum of the form quadratique

$$Q_k(u) = \int_\Omega \Big(\frac{1}{k}|\nabla_k u|^2 - V|u|^2\Big)d\sigma, \qquad u \in L^2(\Omega, E^k) \tag{0.4}$$

for the problem of Dirichlet, where Ω is an open relatively compact on M , and where V is a potential scalaire continuous on M . Of a dot of physical view, this goes back at study the spectrum of the operator of Schrödinger $\frac{1}{k}(\nabla_k^*\nabla_k - kV)$ associé at the electrical field kV and at the magnetic field kB , where $B = -i\nabla^2$ no is autre that the 2 -form of curvature of the connection ∇ . Is on the presence of that magnetic field what résider our principal contribution by report at the method of E. Witten [18], [19] (on the case of the cohomologie of Of Rham the magnetic field is always any since $d^2 = 0$).

Entirely $x \in X$, be $2s = 2s(x) \leqslant n$ the rank of $B(x)$ and $B_1(x) \geqslant \ldots \geqslant B_s(x) > 0$ the modules of the eigenvalues no nulles of the endomorphisme antisymétrique associé. Defines On a function $\nu_{B(x)}(\lambda)$ of the couple $(x, \lambda) \in M \times \mathbb{R}$, continues at left at λ , stating

$$v_B(\lambda) = \frac{2^{s-n}\pi^{-\frac{n}{2}}}{\Gamma(\frac{n}{2} - s + 1)} B_1 \ldots B_s \sum_{(p_1,\ldots,p_s)\in\mathbb{N}^s} \big[\lambda - \sum(2p_j + 1)B_j\big]_+^{\frac{n}{2}-s} \tag{0.5}$$

with the convention $0^0 = 0$. Lastly, if $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ designate the eigenvalues of Q_k (counted with multiplicity), considers on the function of headcount $N_k(\lambda) = \mathrm{card}\{j\,;\ \lambda_j \leqslant \lambda\}$, $\lambda \in \mathbb{R}$.

Théorème 0.6. – *If $\partial\Omega$ is of measure any, il there is an ensemble dénombrable $\mathcal{D} \subset \mathbb{R}$ as*

$$\lim_{k\to+\infty} k^{-\frac{n}{2}} N_k(\lambda) = \int_\Omega \nu_B(V + \lambda)\,d\sigma$$

for all $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$.

For show the theorem 0.6, begins on by study the simple case where $M = \mathbb{R}^n$ with a magnetic field constant B and with $V = 0$. When Ω is a cube, knows on then expliciter

the own functions by a transformation of Fourier partial who brings the problem at that classical of the oscillateur harmonique at a variable. The idea of that calculation it has be strongly inspired by the articles [3], [4] of Y. Colin de Verdière. The extension of the outcome at the case of a magnetic field quelconque restarts an idea of [16], consister use a pavage of Ω by of the cubes enough small. Our method is néanmoins very different of that of Siu, since work it straight on the forms harmoniques whereas Siu bring se at the cochaînes holomorphics via the isomorphisme of Dolbeault. Earns On thus awfully at precision on the searched assessments. The side of the cubes owes stand here chosen of an order of intermediate greatness go in $k^{-\frac{1}{2}}$ and $k^{-\frac{1}{4}}$, by example $k^{-\frac{1}{3}}$: $k^{-\frac{1}{2}}$ is at effect the length of wave of the premiè res own functions, so that the action of the magnetic field B no is perceivable at an inferior stair ; at the-above of $k^{-\frac{1}{4}}$, the swing of B is at the contrary too much strong. Uses On finally the principle of the minimax for compare the eigenvalues on Ω at the own courages on the cubes. On the method ante'rieure of [16] (such what it has restarted on [7]), the size of the cubes have chosen equalises at $k^{-\frac{1}{2}}$; pouvoir on see easily that that election be critical for allow limit the effects of the magnetic field independently of k , but the exact determination of the spectrum become then impossible. The latter paragraph has consacrer the studio of geometrical characterisations of the spaces of Moišezon [13]. Recall who a space compact analytique irréductible X has urged space of Moišezon if the body $K(X)$ of the functions méromorphes on X is of grade of transcendence $= n = \dim_{\mathbb{C}} X$. The conjecture of Grauert-Riemenschneider [10] affirms that X is of Moišezon if and alone if il there is a quasi positive-bunch $\mathcal{E}$ of rank 1 without torsion at the-above of X . By désingularisation, on se brings au cas où X is lisse and where $\mathcal{E}$ is the locally free bunch of the sections of a bundle at right E strictly positive on an open dense of X . .T. Siu [17] has resolved recently the conjecture and it has reinforced assuming alone $ic(E)$ semi-positive and > 0 at at least a dot. The utilisation of the theorem 0.1 (b) allows find of the geometrical conditions best feeble still, who no demand the semi-positivité punctual of $ic(E)$, but alone the positivité of an oertaine integral of curvature. For $q = 1$, the inequality 0.1 (b) involves at effect a minoration of the number of sections holomorphics of E^k , at know:

$$\dim H^0(X, E^k) \geqslant \frac{k^n}{n!} \int_{X(\leqslant 1)} \Big(\frac{i}{2\pi} c(E)\Big)^n - o(k^n). \tag{0.7}$$

pouvoir On display on the other hand, using a classical reasoning of Siegel [15] laid at form by [16] that $\dim H^0(X, E^k) \leqslant \mathrm{cte} \cdot k^{n-1}$ if X no is of Moišezon (cf. theorem 5.1). De là it results the

Théorème 0.8. – *Be X a variety $\mathbb{C}$ -compact analytique connexe of dimension n . So that X be of Moišezon, it suffice that X owns a bundle holomorphic at right hermitian checking the an of the hypothesis* (a), (b), (c) *here- below.*

(a) $\int_{X(\leqslant 1)} (ic(E))^n > 0$.

(b) $c_1(E)^n > 0$ *, and the form of curvature $ic(E)$ no owns any dot of rate pair $\neq 0$.*

(c) *$ic(E)$ is semi-positive entirely of X and defined positive at at least a dot ofX .*

That work has taken the object of a note [8] of the same title, issued at the Summaries. The present article is an improved version of a former memory [7], who be best nearby

of the technical initial of Siu, and who show alone the inequality 0.1 (a) at the constante numerical near; of that fact, the assessments 0.1 (b) and (c) remain inaccessible.

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1. Spectrum of the operator of Schrödinger associé at a magnetic field constant.

Be (M,g) a variety riemannienne of class $\mathcal{C}^\infty$, of real dimension n , and $E \to M$ a bundle at right complex at the-above of M , equipped of a métrique hermitian $\mathcal{C}^\infty$. Note $\mathcal{C}^\infty_q(M,E)$ the space of the sections of class $\mathcal{C}^\infty$ of the bundle $\Lambda^q T^*M \otimes E$, and $(?|?)$ the accouplement sesquilinéaire canonique

$$\mathcal{C}^\infty_q(M,E) \times \mathcal{C}^\infty_q(M,E) \to \mathcal{C}^\infty_{p+q}(M,\mathbb{C}).$$

Assumes On given a connection hermitian D on E , c'est-à-dire an operator différentiel of order a

$$D : \mathcal{C}^\infty_q(M,E) \to \mathcal{C}^\infty_{q+1}(M,E), \qquad 0 \leqslant q < n,$$

checking the identities

$$D(f \wedge u) = df \wedge u + (-1)^m f \wedge Du, \tag{1.1}$$

$$d(u|v) = (Du|v) + (-1)^p (u|Dv), \tag{1.2}$$

for all sections $f \in \mathcal{C}^\infty_m(M,\mathbb{C})$, $u \in \mathcal{C}^\infty_p(M,E)$, $v \in \mathcal{C}^\infty_q(M,E)$. Consider a trivialisation isométrique $\theta : E_{|W} \to W \times \mathbb{C}$ of E at the-above of an open $W \subset M$. The connections hermitians of $E_{|W}$ are then all data by the following formula :

$$Du = du + iA \wedge u,$$

where $u \in \mathcal{C}^\infty_q(W,E) \simeq \mathcal{C}^\infty_q(W,\mathbb{C})$ and where $A \in \mathcal{C}^\infty_1(W,\mathbb{R})$ is a 1 -form *real* arbitrary. The *magnetic field* (or form of curvature) associé at the connection D is the 2 -fermé real form $B = dA$ such that

$$D^2 u = iB \wedge u$$

for all $u \in \mathcal{C}^\infty_q(M,E)$. B No depends so that of the connection D , but of the trivialisation θ chosen. A change of phase $u = ve^{i\varphi}$ on θ conducted at replace A by $A + d\varphi$. The election of a trivialisation of E and of the 1 -form A corresponding interprets se physically comme the election of a potential vecteur particular of the magnetic field B .

Designate by $|u|$ the punctual norm of an element $u \in \Lambda^q T^*M \otimes E$ for the produced métrique tensoriel of the métrique of M and E . If Ω is an open of M , notes on $L^2(\Omega,E)$ (resp. $L^2_q(\Omega,E)$) The space L^2 of the sections of E (resp. Of $\Lambda^q T^*M \otimes E$) at the-above of Ω , equipped of the norm

$$\|u\|^2_\Omega = \int_\Omega |u|^2 d\sigma,$$

where $d\sigma$ is the density of volume riemannien on M .

Be D_k the induced connection by D on the puissance tensorielle k -ième E^k , and V a potential scalaire on M , i.e. A real function continuous. Having given an open relatively compact $\Omega \subset M$, it propose it of determine asymptoticment when k extends to $+\infty$ the spectrum of the form quadratique

$$Q_{\Omega,k}(u) = \int_{\Omega} \Big(\frac{1}{k}|D_k u|^2 - V|u|^2\Big) d\sigma \tag{1.3}$$

where $u \in L^2(\Omega, E^k)$, with condition of Dirichlet $u_{|\partial\Omega} = 0$. The property of $Q_{\Omega,k}$ is so the space of Sobolev $W_0^1(\Omega, E^k) =$ adhérence of 1'space $\mathcal{D}(\Omega, E^k)$ of the sections C^∞ of E^k at compact support on Ω on the space $W^1(M, E^k)$. Of a dot of physical view, this goes back at study the spectrum of the operator of Schrödinger $\frac{1}{k}(D_k^* D_k - kV)$ associé at the magnetic field kB and at the electrical field kV , when k extends to $+\infty$. We renvoyons the reader at the classical article [2] for a general studio of the spectrum of the operator of Schrödinger, and at the works [3], [4], [5], [9], [12] for the studio of problems asymptotics neighbouring of the précédent.

Definition 1.4. – *Designate On by $N_{\Omega,k}(\lambda)$ the number of eigenvalues $\leqslant \lambda$ of the form quadratique $Q_{\Omega,k}$.*

We Go firstly study a simple case who will serve of modèle for the general case at the §2. On se places on the following situation : $M = \mathbb{R}^n$ with the métrique constante $g = \sum_{j=1}^n dx_j^2$, Ω is the cube of side r :

$$\Omega = \Big\{(x_1, \ldots, x_n) \in R^n \,;\; |x_j| < \frac{r}{2},\; 1 \leqslant j \leqslant n\Big\},$$

$V = 0$, and lastly the magnetic field B is constant, equal at the 2 -form alterée of rank $2s$ data by

$$B = \sum_{j=1}^{s} B_j \, dx_j \wedge dx_{j+s},$$

with $B_1 \geqslant B_2 \geqslant \cdots \geqslant B_s > 0$, $s \leqslant \frac{n}{2}$. Pouvoir On then choose a trivialisation of E whose the potential vecteur associé is

$$A = \sum_{j=1}^{s} B_j x_j \, dx_{j+s}.$$

The connection of E^k types se so

$$D_k u = du + ikA \wedge u,$$

and the form quadratique $Q_{\Omega,k}$ has given by

$$Q_{\Omega,k}(u) = \frac{1}{k} \int_{\Omega} \Bigg[\sum_{1 \leqslant j \leqslant s} \Big(\Big|\frac{\partial u}{\partial x_j}\Big|^2 + \Big|\frac{\partial u}{\partial x_{j+s}} + ikB_j x_j u\Big|^2 \Big) + \sum_{j > 2s} \Big|\frac{\partial u}{\partial x_j}\Big|^2 \Bigg] d\mu$$

where $d\mu$ designates the measure of Lebesgue on $\mathbb{R}^n$. If effects on the homothétie $X_j = \sqrt{k}\, x_j$, on has brought at study the eigenvalues of the form quadratique

$$\int_{\sqrt{k}\Omega} \Big[\sum_{1\leqslant j\leqslant s} \Big(\Big|\frac{\partial u}{\partial X_j}\Big|^2 + \Big|\frac{\partial u}{\partial X_{j+s}} + iB_jX_ju\Big|^2 \Big) + \sum_{j>2s} \Big|\frac{\partial u}{\partial X_j}\Big|^2 \Big] d\mu$$

on the cubes $\sqrt{k}\Omega$ of side $\sqrt{k}\, r$. At the field B , associate it the function of the real variable λ defined by

$$\nu_B(\lambda) = \frac{2^{s-n}\pi^{-\frac{n}{2}}}{\Gamma(\frac{n}{2}-s+1)} B_1 \dots B_s \sum_{(p_1,\dots,p_s)\in\mathbb{N}^S} [\lambda - \sum (2p_j+1)B_j]_{2+}^{\frac{n}{2+}-s} \tag{1.5}$$

where states on by convention $\lambda_+^0 = 0$ if $\lambda \leqslant 0$ and $\lambda_+^0 = 1$ if $\lambda > 0$. The function ν_B is so increasing and continues at left on $\mathbb{R}$; observe on that ν_B is in fact continues if $s < \frac{n}{2}$. The spectrum of $Q_{\Omega,k}$ is then depicted asymptoticment by the theorem following, whose ideas it has be suggested by . Hake of Verdière [4].

Théorème 1.6. – *Be R a real > 0 ,*

$$P(R) = \Big\{ x \in \mathbb{R}^n \,;\ |x_j| < \frac{R}{2} \Big\}$$

the pavé of side R , Q_R the form quadratique

$$Q_R(u) = \int_{P(R)} \Big[\sum_{1\leqslant j\leqslant s} \Big(\Big|\frac{\partial u}{\partial x_j}\Big|^2 + \Big|\frac{\partial u}{\partial x_{j+s}} + iB_jx_ju\Big|^2 \Big) + \sum_{j>2s} \Big|\frac{\partial u}{\partial x_j}\Big|^2 \Big] d\mu,$$

and $N_R(\lambda)$ the number of eigenvalues $\leqslant \lambda$ of Q_R for the problem of Dirichlet. Then for all $\lambda \in \mathbb{R}$ has on

$$\lim_{R\to+\infty} R^{-n} N_R(\lambda) = \nu_B(\lambda).$$

When $s = \frac{n}{2}$, ν_B is a function at stairs. The eigenvalues of Q_R regrouper se so by bundles autour de the courages $\sum(2p_j+1)B_j$, with multiplicity approximative $(2\pi)^{-s}B_1 \dots B_s R^n$. This pouvoir se interpre'ter physically comme a phenomenon of quantification of the own states. Going back at the initial problem relative at the form quadratique $Q_{\Omega,k}$, obtain it the

Corollaire 1.7. – $\lim_{k\to+\infty} k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) = r^n \nu_B(\lambda)$. □

Démonstration of the theorem 1.6. – Searches On firstly at majorer $N_R(\lambda)$. On that aim, having given $u \in W_0^1(P(R))$, expresses on u under form of series of Fourier partial by report at the variable $x_{s+1}, \dots, x_n$:

$$u(x) = R^{-\frac{1}{2}(n-s)} \sum_{\ell\in\mathbb{Z}^{n-s}} u_\ell(x') \exp\Big(\frac{2\pi i}{R}\ell\cdot x''\Big)$$

where $u_\ell \in W_0^1(\mathbb{R}^s \cap P(R))$, with the notations

$$x' = (x_1, \ldots, x_s), \quad x'' = (x_{s+1}, \ldots, x_n),$$
$$\ell \cdot x'' = \ell_1 x_{s+1} + \cdots + \ell_{n-s} x_n.$$

The hypotheses $u \in W_0^1(P(R))$ trains that the series

$$\sum |\ell|^2 |u_\ell(x')|^2$$

is on $L^2(\mathbb{R}^s)$. State $\ell' = (\ell_1, \ldots, \ell_s)$, $\ell'' = (\ell_{s+1}, \ldots, \ell_{n-s})$. The norm $\|u\|_{P(R)}$ and the form quadratique Q_R have given by

$$\|u\|^2_{P(R)} = \sum_{\ell \in \mathbb{Z}^{n-s}} \int_{\mathbb{R}^s} |u_\ell(x')|^2 \, d\mu(x'),$$
$$Q_R(u) = \sum_{\ell \in \mathbb{Z}^{n-s}} \int_{\mathbb{R}^s} \left[\sum_{1 \leqslant j \leqslant s} \left(\left| \frac{\partial u_\ell}{\partial x_j} \right|^2 + \left(\frac{2\pi}{R} \ell_j + B_j x_j \right)^2 |u_\ell|^2 \right) + \frac{4\pi^2}{R^2} |\ell''|^2 |u_\ell|^2 \right] d\mu(x').$$

obtains On therefore a problem of Dirichlet at «parted variables» on the cube $\mathbb{R}^s \cap P(R)$. Stating $t = x_j + \frac{2\pi \ell_j}{R B_j}$, on has brought at study the spectrum of the form quadratique of a variable

$$q(f) = \int_R \left(\left| \frac{df}{dt} \right|^2 + B_j^2 t^2 |f|^2 \right) dt,$$

with $f \in W_0^1 \big( ] - \frac{R}{2} \frac{R}{2} [+ \frac{2\pi \ell_j}{R B_j} \big)$. Retomber On so on the classical problem of the oscillateur harmonique (cf. By example [14], Theft. I, p. 142). On $\mathbb{R}$, i.e. Without condition of support for f , the continuation of the own courages of q is the continuation $(2m+1)B_j$, $m \in \mathbb{N}$, and the own functions partners have given by $\Phi_m(\sqrt{B_j}\, t)$ where Φ_0 , $\Phi_1, \ldots$ are the functions of Hermite :

$$\Phi_m(t) = e^{t^2/2} \frac{d^m}{dt^m} (e^{-t^2}).$$

For all $p_j \in \mathbb{N}$, note $\Psi_{p_j, \ell_j}(x_j)$ the p_j -ième own function of the form quadratique

$$q(f) = \int_R \left(\left| \frac{df}{dx_j} \right|^2 + \left(\frac{2\pi}{R} \ell_j + B_j x_j \right)^2 |f|^2 \right) dx_j \tag{1.8}$$

for $f \in W_0^1(] - \frac{R}{2} \frac{R}{2} [)$, and λ_{p_j, ℓ_j} the eigenvalue corresponding. Pouvoir On then décomposer each function u_ℓ sérier of own functions, those that conducts at type u under the form

$$u(x) = R^{-\frac{1}{2}(n-s)} \sum_{(p,\ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}} u_{p,\ell} \Psi_{p,\ell'}(x') \exp\left(\frac{2\pi i}{R} \ell \cdot x'' \right) \tag{1.9}$$

with

$$u_{p,\ell} \in \mathbb{C}, \qquad \Psi_{p,\ell'}(x') = \prod_{1 \leqslant j \leqslant s} \Psi_{p_j, \ell_j}(x_j).$$

seize On custody at the fact that $\Psi_{p,\ell'}(x')\exp(\frac{2\pi i}{R}\ell\cdot x'')$ no is a true function own for the problem of Dirichlet, because the term exponentiel seizes of the courages no nulles at the dots of the edge $x_j=\pm\frac{R}{2}$, $j>s$. Therefore, the coefficients $(u_{p,\ell})$ no are arbitrary if $u\in W_0^1(P(R))$; it have to check the conditions of cancellation at the edge :

$$\sum_{t_j\in\mathbb{Z}}(-1)^{\ell_j}u_{p,\ell}=0 \tag{1.10}$$

for all $j=1,\ldots,n-s$ and all the rates autres that ℓ_j fixed :

$$p\in\mathbb{N}^s,\quad \ell_1,\ldots,\ell_{j-1},\ \ell_{j+1},\ldots,\ell_{n-s}\in\mathbb{Z}.$$

With the writing (1.9), the norm L^2 and the form quadratique Q_R express se under the form

$$\|u\|^2_{P(R)}=\sum|u_{p,\ell}|^2,\qquad Q_R(u)=\sum\Big(\lambda_{p,\ell'}+\frac{4\pi^2}{R^2}|\ell''|^2\Big)|u_{p,\ell}|^2,$$

where $\lambda_{p,\ell'}=\sum_{1\leqslant j\leqslant s}\lambda_{p_j,\ell_j}$. The principle of the minimax 1.20 (b) recalled best far displays that

$$N_R(\lambda)\leqslant\operatorname{card}\Big\{(p,\ell)\in\mathbb{N}^s\times\mathbb{Z}^{n-s}\,;\ \lambda_{p,\ell'}+\frac{4\pi^2}{R^2}|\ell''|^2\leqslant\lambda\Big\}. \tag{1.11}$$

It suffices so of obtain a minoration adéquate of λ_{p_j,ℓ_j} .

Lemme 1.12. — *Has On the inequality*

$$\lambda_{p_j,\ell_j}\geqslant\max\bigg((2p_j+1)B_j\ ,\ \frac{4\pi^2}{R^2}\Big[\Big(\frac{p_j+1}{2}\Big)^2+\Big(|\ell_j|-\frac{B_jR^2}{4\pi}\Big)_+^2\Big]\bigg),$$

and this is strict if $\ell_j\neq0$ *or if* $\Phi_{p_j}(R\sqrt{B_j}/2)\neq0$.

The minoration $\lambda_{p_j,\ell_j}\geqslant(2p_j+1)B_j$ result at effect of the minimax and since the eigenvalues of $q(f)$ on $\mathbb{R}$ cost $(2p_j+1)B_j$. For obtain the autre inequality, on minore (1.8) by the form quadratique

$$\widehat{q}(f)=\int_{x_j|<R/2}\bigg(\Big|\frac{df}{dx_j}\Big|^2+\Big(\frac{2\pi}{R}|\ell_j|-B_j\frac{R}{2}\Big)_+^2|f|^2\bigg)dx_j.$$

The own functions of $\widehat{q}$ are the functions

$$\sin\frac{\pi}{R}(p_j+1)\Big(x_j+\frac{R}{2}\Big),\qquad p_j\in\mathbb{N}\,;$$

λ_{p_j,t_j} is so minorée by the eigenvalue corresponding :

$$\frac{4\pi^2}{R^2}\Big[\Big(\frac{p_j+1}{2}\Big)^2+\Big(|t_j|-\frac{B_jR^2}{4\pi}\Big)_+^2\Big].$$

The inequalities are strict because on the one hand $q(f) > \widehat{q}(f)$ for all $f \neq 0$, and on the other hand $\Phi_{p_j}(\sqrt{B_j}t)$ no peut être own function of q on $]-R/2, R/2[+2\pi\ell_j/RB_j$ that if

$$\Phi_{p_j}\big(\pm R\sqrt{B_j}/2 + 2\pi t_j/R\sqrt{B_j}\big) = 0.$$

Comme the zero of Φ_{p_j} are algébriques and that π is transcendant, this no is possible that if

$$\ell_j = 0 \quad \text{et} \quad \Phi_{p_j}(R\sqrt{B_J}/2) = 0. \qquad \square$$

Lemme 1.13. – *Be $\tau_n(\rho)$ the number of dots of$\mathbb{Z}^n$ situated on the fermé bowl $\overline{B}(0,\rho) \subset \mathbb{R}^n$. Then*

$$\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(\rho - \frac{\sqrt{n}}{2}\Big)_+^n \leqslant \tau_n(\rho) \leqslant \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(\rho + \frac{\sqrt{n}}{2}\Big)^n.$$

At effect, the meeting of the cubes of side 1 centred at the dots $x \in \mathbb{Z}^n$ such that $|x| \leqslant \rho$ has contained on the bowl $\overline{B}(0, \rho + \frac{\sqrt{n}}{2})$, and contains the bowl $\overline{B}(0, \rho - \frac{\sqrt{n}}{2})$ if $\rho \geqslant \frac{\sqrt{n}}{2}$, because $\frac{\sqrt{n}}{2}$ is the half-diagonale of the cube ; the entire $\tau_n(\rho)$ is so framed by the volume of the bowls $\overline{B}(0, \rho \pm \frac{\sqrt{n}}{2})$. $\square$

We majorons now $\limsup R^{-n} N_R(\lambda)$ using (1.11) and the lemmes 1.12, 1.13. For $p \in \mathbb{N}^s$ fixed, the inequality $\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda$ involve

$$|\ell''| \leqslant \frac{R}{2\pi}\Big(\lambda - \sum(2p_j+1)B_j\Big)_+^{\frac{1}{2}}, \tag{1.14}$$

and the inequality is strict for $R > R_0(p)$ enough big. When $s < n/2$ the number of multi-rates $\ell'' \in \mathbb{Z}^{n-2s}$ corresponding is so at the best

$$\begin{aligned} &\frac{\pi^{\frac{n}{2}-s}}{\Gamma(\frac{n}{2}-s+1)}\left[\frac{R}{2\pi}\Big(\lambda - \sum(2p_j+1)B_j\Big)_+^{\frac{1}{2}} + \frac{\sqrt{n}}{2}\right]^{n-2s} \\ &\underset{R\to+\infty}{\sim} \frac{2^{2s-n}\pi^{s-\frac{n}{2}}}{\Gamma(\frac{n}{2}-s+1)} R^{n-2s}\Big(\lambda - \sum(2p_j+1)B_j\Big)_+^{\frac{n}{2}-s}. \end{aligned} \tag{1.15}$$

When $s = \frac{n}{2}$, that number owes stand counted comme costing 1 if $\lambda - \sum(2p_j+1)B_j > 0$ and 0 sinon, those that is very conforme at the convention that it have adopted for the notation λ_+^0 . The inequality $\lambda_{p,\ell'} \leqslant \lambda$ involves on the other hand

$$|\ell_j| \leqslant \frac{R}{2\pi}\sqrt{\lambda_+} + \frac{B_j R^2}{4\pi}, \qquad 1 \leqslant j \leqslant s, \tag{1.16}$$

those that corresponds asymptoticment at a number of multi-rates $\ell' = (\ell_1, \ldots, \ell_s) \in \mathbb{Z}^s$ equivalent at

$$\prod_{j=1}^{s} \frac{B_j R^2}{2\pi} = 2^{-s}\pi^{-s} B_1 \ldots B_s R^{2s}. \tag{1.17}$$

The majoration $\limsup R^{-n} N_R(\lambda) \leqslant \nu_B(\lambda)$ obtains se then effecting the produce of (1.15) by (1.17), and ordering for all $p \in \mathbb{N}^s$ (the sum have ended). $\square$

For of the questions of convergence who will take part at the §2, have it need equally of know a majoration of $N_R(\lambda)$ independent of the magnetic field B . A such assessment uniform has afforded by the following proposal.

Proposal 1.18. – $N_R(\lambda) \leqslant (R\sqrt{\lambda_+} + 1)^n$.

Démonstration. – On majore for each rate j the number of entire p_j and ℓ_j such that the inequality

$$\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda$$

have place. The lemme 1.12 involves

$$\mathrm{card}\{p_j\} \leqslant \max(p_j + 1) \leqslant \min\Big(\frac{\lambda_+}{B_j}\,,\ \frac{R}{\pi}\sqrt{\lambda_+}\Big), \qquad 1 \leqslant j \leqslant s,$$

while (1.16) trains

$$\mathrm{card}\{l_i\} \leqslant \frac{R}{\pi}\sqrt{\lambda_+} + \frac{B_j R^2}{2\pi} + 1, \qquad 1 \leqslant j \leqslant s.$$

Deduces therefore for $1 \leqslant j \leqslant s$:

$$\mathrm{card}\{(p_j, l_j)\} \leqslant \Big(\frac{R}{\pi}\sqrt{\lambda_+}\Big)^2 + \frac{\lambda_+}{B_j}\cdot\frac{B_j R^2}{2\pi} + \frac{R}{\pi}\sqrt{\lambda_+}\cdot 1 \leqslant \big(R\sqrt{\lambda_+} + 1\big)^2$$

For $s < j \leqslant n - s$, the inequality (1.14) gives on the other hand

$$|\ell_j| < \frac{R}{2\pi}\sqrt{\lambda_+},$$

of where $\mathrm{card}\{l_j\} \leqslant \frac{R}{\pi}\sqrt{\lambda_+} + 1$. The proposal 1.18 se ensuit. $\square$

End of the proof of the theorem 1.6 (minoration of $N_R(\lambda)$).

For minorer $N_R(\lambda)$, it suffice as 1.20 (a) of build a space vectoriel of ended dimension on which $Q_R(u) \leqslant \lambda\|u\|^2_{P(R)}$. Considers On for that spaces it vectoriel $\mathcal{F}_\lambda$ of the linear combinations of «own functions» of the type (1.9), assujetties at the conditions of cancellation at the edge (1.10), and ordered on the rates $(p, \ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}$ such that

$$\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda.$$

As the reasoning of the proposal 1.18, the number of conditions (1.10) at realise is majoré by

$$\sum_{j=1}^{s}\Big[\mathrm{card}\{p_j\} \times \prod_{1\leqslant i\leqslant s,\, i\neq j} \mathrm{card}\{(p_i,\ell_i)\} \times \prod_{s<i\leqslant n-s} \mathrm{card}\{\ell_i\}\Big]$$
$$+ \sum_{s<j\leqslant n-s}\Big[\prod_{1\leqslant i\leqslant s} \mathrm{card}\{(p_i,\ell_i)\} \times \prod_{s<i\neq j} \mathrm{card}\{\ell_i\}\Big] \leqslant n(R\sqrt{\lambda_+} + 1)^{n-1}.$$

The entire $N_R(\lambda)$ is so majoré by

$$\dim \mathcal{F}_\lambda \geqslant \operatorname{card}\Big\{(p,\ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}\,;\ \lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda\Big\} - O(R^{n-1}).$$

Combining the minoration of the lemme 1.13 with the lemme here-below, the inequality $\liminf R^{-n} N_R(\lambda) \geqslant \nu_B(\lambda)$ result then of analogous calculations at those that have it explicité for obtain the majoration of $N_R(\lambda)$.

Lemme 1.19. – *Be $p \in \mathbb{N}^s$ a multi-fixed rate. Then il there is a constante $C = C(p,B) \geqslant 0$ such that*

$$\lambda_{p,\ell'} \leqslant \Big(1 + \frac{C}{R}\Big) \sum_{j=1}^{s} (2p_j+1) B_j$$

when $|\ell_j| \leqslant \frac{B_j R^2}{4\pi}(1 - R^{-\frac{1}{2}})$, $1 \leqslant j \leqslant s$.

Démonstration. – Uses On again the minimax and takes it that the functions of Hermite $\Phi_p(\sqrt{B_j}t)$ are a good approximation of the own functions of q on all enough big interval of centre 0 . When $|\ell_j| \leqslant \frac{B_j R^2}{4\pi}(1 - R^{-\frac{1}{2}})$ and $x_j \in \,]-\frac{R}{2}, \frac{R}{2}[$, the variable $t = x_j + \frac{2\pi \ell_j}{B_j R}$ who appears on (1.8) depicted at effect an interval containing $]-\frac{\sqrt{R}}{2}, \frac{\sqrt{R}}{2}[$. Has On so $\lambda_{p_j,\ell_j} \leqslant \widetilde{\lambda}_{p_j}$ where $(\widetilde{\lambda}_m)_{m\in\mathbb{N}}$ is the continuation of the eigenvalues of the form quadratique

$$\widetilde{q}(f) = \int \Big[\Big|\frac{df}{dt}\Big|^2 + (B_j t)^2 |f|^2\Big] dt, \qquad f \in W_0^1\Big(\Big] - \frac{\sqrt{R}}{2}, \frac{\sqrt{R}}{2}\Big[\Big).$$

Be χ_R a function stage at support on $\big[-\frac{\sqrt{R}}{2}, \frac{\sqrt{R}}{2}\big]$, equalises at 1 on $\big[-\frac{\sqrt{R}}{4}, \frac{\sqrt{R}}{4}\big]$, whose the derive is majorée by $5/\sqrt{R}$. For all linear combination

$$f = \sum_{m \leqslant p_j} c_m \Phi_m(\sqrt{B_j}t),$$

the décroissance exponentielle of the functions Φ_m at the infinite involves for R enough big the inequality

$$\|f\| \leqslant \Big(1 + C_1 \exp\Big(-\frac{R}{C_1}\Big)\Big)\|\chi_R f\|$$

where $C_1 = C_1(p_j, B_j) > 0$. Deduces therefore:

$$\begin{aligned}
\widetilde{q}(\chi_R f) &\leqslant \widetilde{q}(f) + \int_{|t|>\sqrt{R}/4} \Big(\frac{10}{\sqrt{R}}\Big|f\frac{df}{dt}\Big| + \frac{25}{R}|f|^2\Big)\, dt \\
&\leqslant \widetilde{q}(f) + \int_{|t|>\sqrt{R}/4} \Big(\frac{1}{R}\Big|\frac{df}{dt}\Big|^2 + 25\Big(1+\frac{1}{R}\Big)|f|^2\Big)\, dt \\
&\leqslant \Big(1 + \frac{C_2}{R}\Big)\widetilde{q}(f) \leqslant \Big(1 + \frac{C_2}{R}\Big)(2p_j+1)B_j\, \|f\|^2 \\
&\leqslant \Big(1 + \frac{C}{R}\Big)(2p_j+1)B_j\, \|\chi_R f\|^2
\end{aligned}$$

This gives very $\lambda_{p_j,\ell_j} \leqslant \widetilde{\lambda}_{p_j} \leqslant \big(1+\frac{C}{R}\big)(2p_j+1)B_j$. □

For facilitate the task of the reader, it énonçons now the principle of the minimax under the form where it it has served.

Proposal 1.20 (principle of the minimax, cf. [14], Theft. IV, p. 76 and 78)**. –** *Be Q a form quadratique at dense property $D(Q)$ on a space of Hilbert $\mathcal{H}$. Assumes On that Q has limited inférieurement, i.e. $Q(f) \geqslant -C\|f\|^2$ If $f \in D(Q)$, that $D(Q)$ is complete for the norm $\|f\|_Q = [Q(f)+(C+1)\|f\|^2]^{\frac{1}{2}}$, and lastly that the injection $(D(Q),\|\ \|_Q) \hookrightarrow (\mathcal{H},\|\ \|)$ is compact. Then Q has a discreet spectrum $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$, and has on the equalities* :

(a) $\lambda_p = \min\limits_{F\subset D(Q)} \ \max\limits_{f\in F,\,\|f\|=1} Q(f),$

where F depicted the ensemble of the under-spaces of dimension p of $D(Q)$;

(b) $\lambda_{p+1} = \max\limits_{F\subset D(Q)} \ \min\limits_{f\in F,\,\|f\|=1} Q(f),$

where F depicted the ensemble of the under-spaces $\|\ \|_Q$ -closed of codimension p of $D(Q)$.

2. Distribution asymptotic of the spectrum (case of a variable field).

Place it again on the general frame depicts at first of the §1. Our objective is of study the spectrum of the form quadratique $Q_{\Omega,k}$ (cf. (1.3)) on the case of a magnetic field B and of an electrical field V quelconques. For all dot $a \in M$, be

$$B(a) = \sum_{j=1}^{s} B_j(a)\,dx_j \wedge dx_{j+s} \tag{2.1}$$

the reduced writing of $B(a)$ on a base orthonormée convenient $(dx_1,\ldots,dx_n)$ of T_a^*M , where $2s = 2s(a) \leqslant n$ is the rank of $B(a)$, and where $B_1(a) \geqslant B_2(a) \geqslant \ldots \geqslant B_s(a) > 0$ are the modules of the eigenvalues no nulles of the endomorphisme antisymétrique associé. The equality of definition 1.5 allows watch $\nu_B(\lambda)$ comme a function of the couple $(a,\lambda) \in M\times\mathbb{R}$. We need equally of consider the function $\overline{\nu}_B(\lambda)$, continues at right at λ , defined by :

$$\overline{\nu}_B(\lambda) = \lim_{0<\varepsilon\to 0} \nu_B(\lambda+\varepsilon). \tag{2.2}$$

We show then the following generalisation of the corollaire 1.7.

Théorème 2.3. – *When k extends to $+\infty$, the number $N_{\Omega,k}(\lambda)$ of eigenvalues $\leqslant \lambda$ of $Q_{\Omega,k}$ checks the encadrement asymptotic*

$$\int_\Omega \nu_B(V+\lambda)\,d\sigma \leqslant \liminf k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \limsup k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \int_\Omega \overline{\nu}_B(V+\lambda)\,d\sigma.$$

The function $\lambda \mapsto \int_\Omega \nu_B(V+\lambda)\,d\sigma$ is increasing and continues at left ; no has so at the plus who an ensemble $\mathcal{D}$ dénombrable of dots of discontinuité. The ensemble $\mathcal{D}$ is of elsewhere empty if n is impair, because $\nu_B(\lambda)$ is then continuous. De là, deduces on forthwith the

Corollaire 2.4. – *Assumes On that $\partial\Omega$ is of measure any. Then*

$$\lim_{k\to+\infty} k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) = \int_\Omega \nu_B(V+\lambda)\,d\sigma$$

for all $\lambda \in \mathbb{R} \setminus \mathcal{D}$, and the measure of density spectrale $k^{-\frac{n}{2}} \frac{d}{d\lambda} N_{\Omega,k}(\lambda)$ converge faiblement on $\mathbb{R}$ to $\frac{d}{d\lambda}\int_\Omega \nu_B(V+\lambda)\,d\sigma$. If n is impair, the measure limit is diffuser. □

The lemme following displays that the integral of the theorem 2.3 have very a sense.

Lemme 2.5.

(a) *has On the inequalities*

$$\nu_B(\lambda) \leqslant \overline{\nu}_B(\lambda) \leqslant \lambda_+^{n/2}.$$

(b) $\nu_B(V)$ *(resp. $\overline{\nu}_B(V)$) Is semi-continuous inférieurement (resp. supérieurement) On M .*

(c) *Entirely $x \in M$ where $s(x) < \frac{n}{2}$ has on $\nu_B(V)(x) = \overline{\nu}_B(V)(x)$ and $\nu_B(V), \overline{\nu}_B(V)$ are continuous at x .*

(d) *If n is impair, $\nu_B(V) = \overline{\nu}_B(V)$ is continuous on M .*

Démonstration. – (Has) has On always $\left(\lambda - \sum(2p_j+1)B_j\right)_+^{\frac{n}{2}-s} \leqslant \lambda_+^{\frac{n}{2}-s}$, and the number of entire p_j such that $\lambda - (2p_j+1)B_j$ be $\geqslant 0$ is majoré by $\frac{\lambda_+}{B_j}$. Comme the numerical amount featuring on (1.5) is majorée by 1 , the inequality (a) se ensuit.

(b, c) The rank $s = s(x)$ is a function semi-continuous inférieurement on M , and the eigenvalues $B_1, B_2, \ldots$, prolonger by $B_j(x) = 0$ for $j > s(x)$, are continuous on M . Comme the function $t \mapsto t_+^0$ (resp. $t \mapsto (t+0)_+^0$) Is semi-continuous inférieurement (resp. supérieurement), the semi-continuity of $\nu_B(V)$ and $\overline{\nu}_B(V)$ states a problem only at the dots $a \in M$ at the voisinage desquels $s(x)$ no is locally constant. At a such dot $a \in M$, has on necessarily $s(a) < \frac{n}{2}$, so $\nu_B(V)(a) = \overline{\nu}_B(V)(a)$; goes on then display that $\nu_B(V)$ and $\overline{\nu}_B(V)$ are continuous at a . The continuity of the B_j give $\lim_{x\to a} B_j(x) = 0$ for $j > s(a)$. If the entire $p_1, \ldots, p_{s(a)}$ have fixed, the sommation featuring on (1.5) pouvoir interpret comme a sum of Riemann of an integral on $\mathbb{R}^{s(x)-s(a)}$, and has on so the equivalent :

$$\begin{aligned}
&\sum_{(p_j;\ s(a)<j\leqslant s(x))} \Big(V(x) - \sum(2p_j+1)B_j(x)\Big)_+^{\frac{n}{2}-s(x)} \\
&\quad\sim \int_{t\in\mathbb{R}^{s(x)-s(a)}} \Bigg[V(a) - \sum_{j=1}^{s(a)}(2p_j+1)B_j(a) - \sum_{j=s(a)+1}^{s(x)} 2t_jB_j(x)\Bigg]_+^{\frac{n}{2}-s(x)} dt \\
&\quad= \frac{2^{s(a)-s(x)}\Big(V(a) - \sum(2p_j+1)B_j(a)\Big)_+^{\frac{n}{2}-s(a)}}{(\frac{n}{2}-s(x)+1)\cdots(\frac{n}{2}-s(a))B_{s(a)+1}(x)\cdots B_{s(x)}(x)} \,.
\end{aligned}$$

obtains On very therefore :

$$\lim_{x \to a} \nu_B(V)(x) = \nu_B(V)(a) = \lim_{x \to a} \overline{\nu}_B(V)(x).$$

(d) Is a particular case of (c). □

The proof of the theorem 2.3 rests essentially on deux ingredients : firstly a principle of localisation asymptotic of the own functions, who obtain se by direct application of the minimax (proposal 2.6) ; on the other hand, the explicit knowledge of the spectrum of the operator of Schrödinger associé at a magnetic field constant (cf. §1). The principle of localisation allows at effect of bring at the case of a field constant using a pavage of Ω by of the cubes enough small.

Proposal 2.6. – (Has) *If* $\Omega_1, \cdots, \Omega_N \subset \Omega$ *are of the open* 2 *at* 2 *disjoints, then*

$$N_{\Omega,k}(\lambda) \geqslant \sum_{j=1}^{N} N_{\Omega_j,k}(\lambda).$$

(b) *Be* $(\Omega'_j)_{1 \leqslant j \leqslant \mathbb{N}}$ *a recouvrement opened of* $\overline{\Omega}$ *and* $(\psi_j)_{1 \leqslant j \leqslant \mathbb{N}}$ *a system of functions* $\psi_f \in \mathcal{C}^\infty(\mathbb{R}^n)$ *at support on* Ω'_j *, such that* $\sum \psi_j^2 = 1$ *on* $\overline{\Omega}$ *. States On*

$$C(\psi) = \sup_{\Omega} \sum_{j=1}^{N} |d\psi_j|^2.$$

Then

$$N_{\Omega,k}(\lambda) \leqslant \sum_{j=1}^{N} N_{\Omega'_j,k}\Big(\lambda + \frac{1}{k} C(\psi)\Big).$$

Démonstration. – (Has) Be $\mathcal{F}$ the $\mathbb{C}$ -space vectoriel engendré by the collection of all the functions own of the forms quadratiques $Q_{\Omega_j,k}$, $1 \leqslant j \leqslant N$, corresponding at of the eigenvalues $\leqslant \lambda$. $\mathcal{F}$ Is of dimension

$$\dim \mathcal{F} = \sum_{j=1}^{N} N_{\Omega_j,k}(\lambda)$$

and for all $u \in \mathcal{F}$, has on

$$Q_{\Omega,k}(u) = \sum_{j=1}^{N} Q_{\Omega_j,k}(u) \leqslant \sum_{j=1}^{N} \lambda \|u\|^2_{\Omega'_j} = \lambda \|u\|^2_{\Omega}.$$

The principle of the minimax display so that the eigenvalues of $Q_{\Omega,k}$ of rate $\leqslant \dim \mathcal{F}$ are $\leqslant \lambda$, of where the inequality (a).

(b) For all $u \in W^1_0(\Omega, E^k)$ it comes

$$\sum_j |D_k(\psi_j u)|^2 = \sum_j \big|\psi_j D_k u + (d\psi_j)u\big|^2 = |D_k u|^2 + \sum_j |d\psi_j|^2 |u|^2$$

because $2\sum \psi_j d\psi_j = d(\sum \psi_j^2) = 0$. Obtains On so

$$\sum_{j=1}^{N} Q_{\Omega_j',k}(\psi_j u) = Q_{\Omega,k}(u) + \int_\Omega \frac{1}{k}\sum_{j=1}^{N} |d\psi_j|^2 |u|^2\, d\sigma \leqslant Q_{\Omega,k}(u) + \frac{1}{k}C(\psi)\|u\|_\Omega^2.$$

If each function $\psi_j u \in W_0^1(\Omega_j, E^k)$ is orthogonale at the own functions of $Q_{\Omega_j,k}$ of own courages $\leqslant \lambda + \frac{1}{k}C(\psi)$, deduces successively

$$\begin{aligned} Q_{\Omega_j,k}(\psi_j u) &> \Big(\lambda + \frac{1}{k}C(\psi)\Big)\|\psi_j u\|_{\Omega_j}^2, \quad \text{si } \psi_j u \neq 0,\\ Q_{\Omega,k}(u) &> \lambda\|u\|_\Omega^2, \quad \text{si } u \neq 0. \end{aligned}$$

The principle of the minimax 1.20 (b) trains whereas $N_{\Omega,k}(\lambda)$ is majoré by the number of impose linear equation at u , be at the best

$$\sum_{j=1}^{N} N_{\Omega_j,k}\Big(\lambda + \frac{1}{k}C(\psi)\Big). \qquad \square$$

Be $W_1, \ldots, W_N$ a recouvrement of Ω by of the open of card of the variety M . For all $\varepsilon > 0$, pouvoir on find of the open $\Omega_i \subset \Omega_j'$, relatively compact on W_j , $1 \leqslant j \leqslant N$, such that

$$\Omega \supset \bigcup \Omega_j \ \text{(disjointe)}, \quad \text{et } \mathrm{Vol}(\Omega) = \sum \mathrm{Vol}(\Omega_j), \tag{2.7}$$

$$\overline{\Omega} \subset \bigcup \Omega_j', \quad \text{et } \sum \mathrm{Vol}(\overline{\Omega}_j') \leqslant \mathrm{Vol}(\overline{\Omega}) + \varepsilon. \tag{2.8}$$

The proposal 2.6 brings then the test of the theorem 2.3 at the case of the open Ω_j and Ω_j' (observe on for that that the function $\nu_B(V+\lambda)$ has limited and that the constante $C(\psi)$ is independent of k).

At definite, pouvoir on assume that $M = \mathbb{R}^n$, with a métrique riemannienne g quelconque. Comme $M = \mathbb{R}^n$ is contractile, the bundle E is then trivial ; be A a potential vecteur of the connection D and $B = dA$ the magnetic field corresponding. We show firstly the local version following of the theorem 2.3.

Proposal 2.9. – *Be $a \in \mathbb{R}^n$ a fixed dot, and P_k a continuation of pavés cubiques opened such that $P_k \ni a$. Notes On r_k the length of the side of P_k , and assumes on that*

$$r_k \leqslant 1, \qquad \lim k^{\frac{1}{2}} r_k = +\infty, \qquad \lim k^{\frac{1}{4}} r_k = 0.$$

Then when k extends to $+\infty$, has on

$$\begin{aligned} \liminf \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) &\geqslant \nu_{B(a)}(V(a)+\lambda),\\ \limsup \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) &\leqslant \overline{\nu}_{B(a)}(V(a)+\lambda), \end{aligned}$$

and for all compact $K \subset \mathbb{R}^n$, $N_{P_k,k}(\lambda)$ admits the majoration

$$N_{P_k,k}(\lambda) \leqslant C_K\Big(1 + r_k\sqrt{k\big(\lambda_+ + \max_K V_+\big)}\Big)^n$$

uniform by report at a , dès lors que $P_k \subset K$.

Démonstration. – Goes On bring at the theorem 1.6 effecting a homothétie of report $\sqrt{k}$ on P_k (therefore it have had to assume $\lim k^{\frac{1}{2}} r_k = +\infty$). The lemme following measures what the magnetic field B divert of the field constant $B(a)$ on each P_k .

Lemme 2.10. – *On each pavé $\overline{P}_k$, pouvoir on choose a potential $\widetilde{A}_k$ of the field constant $B(a)$ as for all $x \in \overline{P}_k$ have on*

$$|A_k(x) - A(x)| \leqslant C_1 r_k^2,$$

where C_1 is a constante $\geqslant 0$ independent of k (and independent of a if a depicted a compact $K \subset \mathbb{R}^n$) .

The regularity $\mathcal{C}^\infty$ of B train at effect a majoration

$$|B(a) - B(x)| \leqslant C_2 r_k, \qquad x \in \overline{P}_k.$$

Be A'_k a potential of the field $B(a) - B(x)$ on the cube $\overline{P}_k$, reckoned at the half of the formula of homotopie usuelle for the open étoiler. Has On then

$$|A'_k(x)| \leqslant C_3 r_k^2,$$

and it suffice of state $\widetilde{A}_k = A + A'_k$. □

Note $(x_1, \ldots, x_n)$ the coordinate standard of $\mathbb{R}^n$. Be $(y_1, \ldots, y_n)$ a system of coordinated linear at $x_1, \ldots, x_n$ as $(dy_1, \ldots, dy_n)$ be a base orthonormée at the dot a for the métrique g , and as on that base $B(a)$ type se under the form diagonale (2.1) :

$$B(a) = \sum_{j=1}^{s} B_j(a)\, dy_j \wedge dy_{j+s}.$$

Be $\widetilde{g}$ the métrique constante

$$\widetilde{g} \equiv g(a) = \sum_{j=1}^{n} dy_j^2.$$

Designate by $D_k = d + ikA \wedge ?$, $D_k = d + ikA_k \wedge ?$ the connections on $E^k_{|P_k}$ associated at the potential A , $\widetilde{A}_k$, and by $Q_k = Q_{P_k,k}$, $\widetilde{Q}_k$ the forms quadratiques associated respectively at the connections D_k , $\widetilde{D}_k$, at the métrique g , $\widetilde{g}$, and at the potential scalaires V , $\widetilde{V} \equiv V(a)$ (formula (1.3)).

Lemme 2.11. – *Il there is a continuation ε_k extending to* 0 (*dépendant of the r_k , but indépendante of a if a depicted a compact $K \subset \mathbb{R}^n$) such that if $\| \ \|_g$ and $\| \ \|_{\tilde g}$ designate the norms L^2 global partners at the métrique g and $\widetilde{g}$, have on*

$$(1-\varepsilon_k)\|u\|_{\tilde g}^2 \leqslant \|u\|_g^2 \leqslant (1+\varepsilon_k)\|u\|_{\tilde g}^2,$$
$$(1-\varepsilon_k)\widetilde{Q}_k(u) - \varepsilon_k\|u\|_{\tilde g}^2 \leqslant Q_k(u) \leqslant (1+\varepsilon_k)\widetilde{Q}_k(u) + \varepsilon_k\|u\|_{\tilde g}^2$$

for all $u \in W_0^1(P_k)$.

On P_k , has on at effect an encadrement :

$$(1 - C_4 r_k)\,\widetilde{g} \leqslant g \leqslant (1 + C_4 r_k)\,\widetilde{g},$$

and this gives the first inequality twofold on 2.11. With the notation $A_k' = A_k - A$, deduces

$$\begin{aligned} Q_k(u) &= \int_{P_k} \Big(\frac{1}{k}|\widetilde{D}_k u - ikA_k' \wedge u|_g^2 - V|u|^2\Big)\,d\sigma \\ &\leqslant (1 + C_5 r_k)\int_{P_k} \Big(\frac{1}{k}|\widetilde{D}_k u - ikA_k' \wedge u|_{\tilde g}^2 - V(a)|u|^2\Big)\,d\widetilde{\sigma} + \eta_k\|u\|_{\tilde g}^2 \end{aligned}$$

with $\eta_k = \sup_{P_k}|V - V(a)| + C_6 r_k$, amount who extends to 0 when k extends to $+\infty$. Using the inequality $(a+b)^2 \leqslant (1+\alpha)(a^2 + \alpha^{-1}b^2)$, the lemme 2.10 involves on the other hand

$$|\widetilde{D}_k u - ikA_k' \wedge u|_{\tilde g}^2 \leqslant (1+\alpha)\Big[\,|\widetilde{D}_k u|_{\tilde g}^2 + \alpha^{-1}C_1^2 k^2 r_k^4 |u|^2\Big].$$

Choose $\alpha = \alpha_k = C_1\sqrt{k}r_k^2$. The continuation α_k extends to 0 as the hypothesis $\lim k^{\frac14} r_k = 0$, and it comes

$$\frac{1}{k}|\widetilde{D}_k u - ikA_k' \wedge u|_{\tilde g}^2 \leqslant (1+\alpha_k)\Big[\frac{1}{k}|D_k u|_{\tilde g}^2 + \alpha_k|u|^2\Big].$$

The majoration of Q_k se ensuit. The minoration obtains se likewise thanks to the inequality $(a+b)^2 \geqslant (1-\alpha)(a^2 - \alpha^{-1}b^2)$. □

The lemme 2.11 brings the test of the proposal 2.9 au cas où the mé trique g and the magnetic field B are constants :

$$g = \sum_{j=1}^n dy_j^2, \qquad B = \sum_{j=1}^n B_j\,dy_j \wedge dy_{j+s}.$$

pouvoir On assume besides $V \equiv 0$ effecting the translation $\lambda \mapsto \lambda + V(a)$. The alone difficulty who subsister for apply straight the theorem 1.6 comes since the cubes P_k become at general of the parallélépipèdes obliques on the coordinate $(y_1, \ldots, y_n)$; the angles amid the different arêtes of each P_k and the reports of leur lengths remain however framed by of the constantes > 0 . For resolve that difficulty, it suffice of paver each parallélépipède P_k by of the cubes $P_{k,\alpha}$ whose the arêtes are parallel at the axes of the coordinate $(y_1, \ldots, y_n)$. Choose $\varepsilon \in]0,1[$. For all $\alpha \in \mathbb{Z}^n$, are $(P_{k,\alpha})$, $(P'_{k,\alpha})$ the cubes

opened of respective sides εr_k , $\varepsilon(1+\varepsilon)r_k$, and of common centre $\varepsilon r_k\alpha$. On se limit at consider the cubes $P_{k,\alpha}$ contenu on P_k and the cubes $P'_{k,\alpha}$ finding P_k . Has On then

$$(2.12)\qquad P_k \supset \bigcup_\alpha P_{k,\alpha}\ \text{(disjointe)},\quad \text{et}\quad \frac{\sum_\alpha \operatorname{Vol}(P_{k,\alpha})}{\operatorname{Vol}(P_k)} \geqslant 1 - C_7\varepsilon,$$

$$(2.13)\qquad P_k \subset \bigcup_\alpha P'_{k,\alpha},\quad \text{et}\quad \frac{\sum_\alpha \operatorname{Vol}(P'_{k,\alpha})}{\operatorname{Vol}(P_k)} \leqslant 1 + C_7\varepsilon,$$

where C_7 is a constante independent of k (and also of a , if a depicted a compact). The number of cubes $P_{k,\alpha}$, $P'_{k,\alpha}$ who feature on (2.12) or (2.13) is majoré by $C_8\varepsilon^{-n}$. Comme the cubes $P'_{k,\alpha}$ recouvrer se deux at deux on a length $\varepsilon^2 r_k$ when it are contigus, pouvoir on build a partition of the unity $\sum \psi_{k,\alpha}^2 = 1$ on P_k , with $\operatorname{Supp}\psi_{k,\alpha} \subset P'_{k,\alpha}$ and

$$\sup_{P_k} \sum_\alpha |d\psi_{k,\alpha}|^2 = C(\psi_k) \leqslant C_9(\varepsilon^2 r_k)^{-2}.$$

The hypothesis $\lim k^{\frac{1}{2}} r_k = +\infty$ trains very $\lim \frac{1}{k} C(\psi_k) = 0$, those that allows apply 2.6 (b). On the cubes $P_{k\alpha}$, $P'_{k,\alpha}$ stand it now on the situation of the théorème 1.6 : apres homothétie of report $\sqrt{k}$, the side of the cube homothétique $\sqrt{k}\, P_{k,\alpha}$ costs $R_k = \varepsilon r_k \sqrt{k}$ and extends very to $+\infty$ by hypothesis. The majoration uniform of $N_{P_k,k}(\lambda)$ result of the proposal 1.18 and since all our constantes $C_1, \ldots, C_9$ étayer uniform. The proposal 2.9 has showed. ☐

Demonstration Of the theorem 2.3. – As observes it preceding the proposal 2.9, pouvoir it assume that $M = \mathbb{R}^n$ and that Ω is an open limited of $\mathbb{R}^n$. The idea of the reasoning is of combine the proposals 2.6 and 2.9 using a pavage of Ω by of the cubes of side $r_k = k^{-\frac{1}{3}}$. Bets it at ceuvre effectif claims néanmoins a peu de cure owing to the lié difficulties at the no-uniformité eventual of the $\limsup$ and $\liminf$.

Designate by $\Pi_{k,\alpha}$, $\Pi'_{k,\alpha}$, $\alpha \in \mathbb{Z}^n$, the cubes opened of respective sides

$$k^{-\frac{1}{3}}, \qquad k^{-\frac{1}{3}}(1 + k^{-\frac{1}{8}}) = k^{-\frac{1}{3}} + k^{-\frac{11}{24}}$$

and of common centre $k^{-\frac{1}{3}}\alpha$. Be $I(k)$ (resp. $I'(k)$) The ensemble of the rates $\alpha \in \mathbb{Z}^n$ such that $\Pi_{k.\alpha} \subset \Omega$ (resp. $\overline{\Pi}'_{k,\alpha} \cap \overline{\Omega} \neq \emptyset$). Comme on the reasoning of the proposal 2.9, il there is a partition of the unity $\sum_{\alpha \in I'(k)} \psi_{k,\alpha}^2 = 1$ on Ω , with $\operatorname{Supp}\psi_{k,\alpha} \subset \Pi'_{k,\alpha}$ and

$$C(\psi_k) = \sup_\Omega \sum_{\alpha \in I'(k)} |d\psi_{k,\alpha}|^2 \leqslant C_{10} k^{\frac{11}{12}},$$

of where $\lim \frac{1}{k} C(\psi_k) = 0$. States On

$$\Omega_k = \bigcup_{\alpha \in I(k)} \Pi_{k,\alpha}, \qquad \Omega'_k = \bigcup_{\alpha \in I'(k)} \Pi'_{k,\alpha}$$

and considers on for all $\lambda \in \mathbb{R}$ fixed, the functions on $\mathbb{R}^n$ defined by

$$f_k = k^{-\frac{n}{2}} \sum_{\alpha \in I(k)} N_{\Pi_{k,\alpha},k}(\lambda) \frac{1}{\mathrm{Vol}(\Pi_{k,\alpha})} 1\!\!1_{\Pi_{k,\alpha}},$$
$$f'_k = k^{-\frac{n}{2}} \sum_{\alpha \in I'(k)} N_{\Pi'_{k,\alpha},k}\Big(\lambda + \frac{1}{k} C(\psi_k)\Big) \frac{1}{\mathrm{Vol}(\Pi_{k,\alpha})} 1\!\!1_{\Pi_{k,\alpha}}$$

where $1\!\!1_{\Pi_{k,\alpha}}$ designates the characteristic function of $\Pi_{k,\alpha}$. The proposal 2.6 involves the encadrement

$$\int_{\mathbb{R}^n} f_k \, d\sigma \leqslant k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \int_{\mathbb{R}^n} f'_k \, d\sigma. \tag{2.14}$$

Be $x \in \mathbb{R}^n$ a fixed dot no pertaining at the ensemble négligeable

$$Z = \bigcup_{k \in \mathbb{N},\, \alpha \in \mathbb{Z}^n} \partial \Pi_{k,\alpha}.$$

Il there is then a continuation of rates $\alpha(k) \in \mathbb{Z}^n$ unique such that $x \in \Pi_{k,\alpha(k)}$. The proposal 2.9 applied as a result of the cubes $P_k = \Pi_{k,\alpha(k)}$ (resp. $P'_k = \Pi'_{k,\alpha(k)}$) With $\mathrm{Vol}(P_k) \sim \mathrm{Vol}\, P'_k$ displays that the punctual continuations

$$f_k(x) = \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) 1\!\!1_{\Omega_k}(x), \qquad f'_k(x) = \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P'_k,k}(\lambda) 1\!\!1_{\Omega'_k}(x),$$

are such that

$$\begin{cases} \liminf f_k(x) \ \geqslant \nu_{B(x)}(V(x) + \lambda)\, 1\!\!1_{\Omega}(x) \\ \limsup f'_k(x) \leqslant \overline{\nu}_{B(x)}(V(x) + \lambda)\, 1\!\!1_{\overline{\Omega}}(x). \end{cases} \tag{2.15}$$

The majoration uniform of the proposal 2.9 trains on the other hand the existence of constantes C_{11} , C_{12} independent of k , x and λ such that

$$f_k(x) \leqslant f'_k(x) \leqslant C_{11}\big(1 + \sqrt{\lambda_+ + C_{12}}\,\big)^n.$$

The theorem 2.3 results then of (2.14), (2.15) and of the lemme of Fatou. □

En vue de the applications at the géométrie complex, have it need of a slight generate-lisation of the theorem 2.3. On se gives a bundle hermitian F of rank r and of class $\mathcal{C}^\infty$ at the-above of M , equipped of a connection hermitian ∇ , and of the continuous sections S of the bundle $\Lambda^1_R T^* X \otimes_R \mathrm{Hom}_{\mathbb{C}}(F, F)$ and V of the bundle $\mathrm{Herm}(F)$ of the endomorphismes hermitiens of F . Be ∇_k the connection hermitian on $E^k \otimes F$ induced by the connections D and ∇ . For abbreviate the notations, designate on still by S and V the endomorphismes $\mathrm{Id}_{E^k} \otimes S$ and $\mathrm{Id}_{E^k} \otimes V$ operating on $E^k \otimes F$. Having given an open Ω relatively compact on M , considers on the form quadratique

$$Q_{\Omega,k}(u) = \int_\Omega \Big(\frac{1}{k} |\nabla_k u + Su|^2 - \langle Vu, u\rangle\Big)\, d\sigma,$$

where $u \in W_0^1(\Omega, E^k \otimes F)$. Are $V_1(x) \leqslant V_2(x) \leqslant \cdots \leqslant V_r(x)$ the eigenvalues of $V(x)$ entirely $x \in M$. Has On then the following outcome.

Théorème 2.16. – *The function of headcount $N_{\Omega,k}(\lambda)$ of the eigenvalues of $Q_{\Omega,k}$ admits for all $\lambda \in \mathbb{R}$ the assessments asymptotics*

$$\liminf_{k\to+\infty} \ k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \geqslant \sum_{j=1}^{r} \int_\Omega \nu_B(V_j+\lambda)\, d\sigma,$$
$$\limsup_{k\to+\infty} \ k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \sum_{j=1}^{r} \int_\Omega \overline{\nu}_B(V_j+\lambda)\, d\sigma,$$

where B is the magnetic field associé at the connection D on E .

Démonstration. – The principle of localisation 2.6 is still valid on the present situation. It suffice so of show the inequalities 2.16 when Ω is enough small. Be $a \in M$ a fixed dot and $(e_1, \ldots, e_r)$ a repérer orthonormé $\mathcal{C}^\infty$ of F at the-above of a voisinage W of a , as $(e_1(a), \ldots, e_r(a))$ be an own base for $V(a)$. Type u under the form

$$u = \sum_{j=1}^{r} u_j \otimes e_j$$

where u_j is a section of E^k . For all $\varepsilon > 0$, il there is a voisinage $W'_\varepsilon \subset W$ of a on which

$$\sum_{j=1}^{r} (V_j(a)-\varepsilon)|u_j|^2 \leqslant \langle Vu, u\rangle \leqslant \sum_{j=1}^{r} (V_j(a)+\varepsilon)|u_j|^2$$

has On on the other hand

$$\nabla_k u = \sum_{j=1}^{r} D_k u_j \otimes e_j + u_j \otimes \nabla e_j,$$

and the term $u_j \otimes \nabla e_j$ peut être absorbed on Su (those that bring it in fact au cas où the connection ∇ is flat). The encadrement

$$(1-k^{-\frac{1}{2}})|\nabla_k u|^2 + (1-k^{\frac{1}{2}})|Su|^2 \leqslant |\nabla_k u + Su|^2 \leqslant (1+k^{-\frac{1}{2}})|\nabla_k u|^2 + (1+k^{\frac{1}{2}})|Su|^2$$

displays that the term Su no modifies $Q_{\Omega,k}$ that by a factor multiplicatif $1 \pm \varepsilon$ and by an additif factor $\pm\varepsilon\|u\|^2$. For all $\varepsilon > 0$, il there is so a voisinage W_ε of a and an entire $k_0(\varepsilon)$ such that

$$(1-\varepsilon)\widetilde{Q}_{\Omega.k}(u) - \varepsilon\|u\|^2 \leqslant Q_{\Omega,k}(u) \leqslant (1+\varepsilon)\widetilde{Q}_{\Omega,k}(u) + \varepsilon\|u\|^2$$

dès que $k \geqslant k_0(\varepsilon)$ and $\Omega \subset W_\varepsilon$, where $\widetilde{Q}_{\Omega,k}$ designates the form quadratique

$$\widetilde{Q}_{\Omega.k}(u) = \sum_{j=1}^{r} \int_\Omega \Big(\frac{1}{k}|D_k u_j|^2 - V_j(a)|u_j|^2\Big)\, d\sigma.$$

Comme $\widetilde{Q}_{\Omega,k}$ is a direct sum of r form quadratiques, the spectrum of $\widetilde{Q}_{\Omega.k}$ is the meeting (counted with multiplicities) of the spectrums of each of the terms of the sum. The theorem 2.16 se ensuit. □

3. Identity of Bochner-Kodaira-Nakano at géométrie hermitian.

The object of the paragraphs who track is of tirer the consequences of the theorem of répartition spectrale 2.16 for the studio of the d'' -cohomologie of the bundles vectoriels holomorphics hermitiens. On that aim, have it need of connect the laplacien antiholomorphic Δ'' at the ope'rateur of Schrödinger of a real connection adéquate. This take se at the half of a particular formula of type Weitzenböck, known at géométrie complex under the name of identity of Bochner-Kodaira-Nakano.

Be X a variety compact complex analytique of dimension n and F a bundle vectoriel holomorphic hermitian of rank r at the-above of X . Knows On who il there is a unique connection hermitian $D = D' + D''$ on F whose the composante D'' of type $(0,1)$ coincide with the operator $\overline{\partial}$ of the bundle (a such connection have said holomorphic). Be $c(F) = D^2 = D'D'' + D''D'$ the form of curvature of F . Cater X of one me'trique hermitian arbitrary ω of type $(1,1)$ and of class $\mathcal{C}^\infty$. The space $\mathcal{C}^\infty_{p,q}(X,F)$ of the sections of class $\mathcal{C}^\infty$ of the bundle $\Lambda^{p,q}T^*X \otimes F$ finds se then equipped of a structure prehilbertian natural. Notes On $\delta = \delta' + \delta''$ the adjoint formal of D considered comme operator differential on $\mathcal{C}^\infty(X,F)$, and Λ the adjoint of the operator $L : u \mapsto \omega \wedge u$.

We use the identity of Bochner-Kodaira-Nakano under the general form showed on [6], although pouvoir on in fact cheer, comme the fact .T. Siu [16], [17], of the less precise formula data by P. Griffiths. If A , B are of the operators differential on $\mathcal{C}^\infty(X,F)$, defines on leur anti-commutateur $[A,B]$ by the formula

$$[A,B] = AB - (-1)^{ab}BA$$

where a , b are the respective grades of A and B . The operators of Laplace-Beltrami Δ' and Δ'' are then given classiquement by

$$\Delta' = [D',\delta'] = D'\delta' + \delta'D', \qquad \Delta'' = [D'',\delta'']$$

At the form of torsion $d'\omega$, associate it the operator of external multiplication $u \mapsto d'\omega \wedge u$ on $\mathcal{C}^\infty(X,F)$, of type $(2,1)$, noted simply $d'\omega$, and the operator τ of type $(1,0)$ defined by $\tau = [\Lambda, d'\omega]$. We state lastly

$$D'_\tau = D' + \tau, \qquad \delta'_\tau = (D'_\tau)^* = \delta' + \tau^*, \qquad \Delta'_\tau = [D'_\tau, \delta'_\tau].$$

has On then the following identity, for a proof of which the reader postpone se at [6].

Proposal 3.1. – *Has On* $\Delta'' = \Delta'_\tau + [ic(F),\Lambda] + T_\omega$ *where* T_ω *is the operator of order* 0 *and of type* $(0,0)$ *defined by*

$$T_\omega = \left[\Lambda, \left[\Lambda, \frac{i}{2}d'd''\omega\right]\right] - [d'\omega, (d'\omega)^*].$$

As the theory of Hodge-Of Rham, the group of cohomologie $H^q(X, F)$ identifies se at the space of the $(0, q)$ -form Δ'' -harmoniques at courages on F . Be $u \in \mathcal{C}^\infty_{p.q}(X, F)$. The proposal 3.1 gives it the equality

$$(3.2)\quad \int_X |D''u|^2+|\delta''u|^2 = \int_X \langle\Delta''u, u\rangle = \int_X |D'_\tau u|^2+|\delta'_\tau u|^2+\langle[ic(F), \Lambda]u, u\rangle+\langle T_\omega u, u\rangle,$$

where the integral have reckoned relatively at the element of volume $d\sigma = \frac{\omega^n}{n!}$. At partyculier, if u is of bidegré $(0, q)$, has on $\delta'_\tau u = 0$ by reason of bidegré, of where

$$(3.3)\qquad \int_X \langle\Delta''u, u\rangle = \int_X |D'_\tau u|^2 + \langle[ic(F), \Lambda]u, u\rangle + \langle T_\omega u, u\rangle.$$

pouvoir On equally consider u comme a (n, q) -form at courages on the bundle

$$\widetilde{F} := F \otimes \Lambda^n TX \ ;$$

note on $\widetilde{D} = \widetilde{D}' + \widetilde{D}''$ the connection hermitian holomorphic of $\widetilde{F}$ and $\widetilde{u}$ the image canonique of u on $\mathcal{C}^\infty_{n,q}(X, F)$.

Lemme 3.4. – *Has On of the diagrammes commutatifs*

$$\begin{array}{ccc} \mathcal{C}^\infty_{0.q}(X, F) & \stackrel{D''}{\longrightarrow} & \mathcal{C}^\infty_{0,q+1}(X, F) \\ \sim\downarrow & & \downarrow\sim \\ \mathcal{C}^\infty_{n,q}(X, \widetilde{F}) & \stackrel{\widetilde{D}''}{\longrightarrow} & \mathcal{C}^\infty_{n,q+1}(X, \widetilde{F}), \end{array} \qquad \begin{array}{ccc} \mathcal{C}^\infty_{0,q}(X, F) & \stackrel{\Delta''}{\longrightarrow} & \mathcal{C}^\infty_{0,q}(X, F) \\ \sim\downarrow & & \downarrow\sim \\ \mathcal{C}^\infty_{n,q}(X, \widetilde{F}) & \stackrel{\widetilde{\Delta}''}{\longrightarrow} & \mathcal{C}^\infty_{n,q}(X, \widetilde{F}), \end{array}$$

where the vertical arrows are the isométries $u \mapsto \widetilde{u}$.

Démonstration. – The commutativité of the diagramme of left results since $\Lambda^n TX$ is a bundle holomorphic (seize on custody at the fact that the corresponding outcome for D' and $\widetilde{D}'$ is false). Has On so a diagramme commutatif analogous for the adjoint δ'' , $\widetilde{\delta}''$ and for Δ'' , $\widetilde{\Delta}''$. □

The lemme 3.4 and the identity (3.2) give it

$$(3.5)\qquad \int_X \langle\Delta''u, u\rangle = \int_X \langle\widetilde{\Delta}''\widetilde{u}, \widetilde{u}\rangle = \int_X |\widetilde{\delta}'_\tau \widetilde{u}|^2 + \langle[ic(\widetilde{F}), \Lambda]\widetilde{u}, \widetilde{u}\rangle + \langle T_\omega \widetilde{u}, \widetilde{u}\,\rangle.$$

We now transform slightly the writing of (3.3) and (3.5). The connection hermitian holomorphic of the bundle $\Lambda^q T^*X$ induced on the bundle conjugué $\Lambda^{0,q}T^*X$ a connection whose the composante of type $(1, 0)$ coincide with the operator d' . Deduces then a connection hermitian natural ∇ on the bundle produced tensoriel $\Lambda^{0,q}T^*X \otimes F$ (observe on that that bundle vectoriel no is holomorphic at general if $q \neq 0$). Are ∇' and ∇'' the composantes of ∇ of type $(1, 0)$ and $(0, 1)$.

Proposal 3.6. – *Has On*

$$\nabla' = D' : \mathcal{C}^\infty(\Lambda^{0,q}T^*X \otimes F) \to \mathcal{C}^\infty_{1,0}(\Lambda^{0,q}T^*X \otimes F),$$

and il there is a diagramme commutatif

$$\begin{array}{ccc}
\mathcal{C}^\infty(X, \Lambda^{0.q}T^*X \otimes F) & \xrightarrow{\nabla''} & \mathcal{C}^\infty_{0,1}(X, \Lambda^{0.q}T^*X \otimes F) \\
\sim \downarrow & & \downarrow \Psi \\
\mathcal{C}^\infty_{n,q}(X, \widetilde{F}) & \xrightarrow{\widetilde{\delta}''} & \mathcal{C}^\infty_{n-1,q}(X, \widetilde{F}),
\end{array}$$

where the vertical arrows are of the isométries, that of left having given by $u \mapsto \widetilde{u}$ *.*

Démonstration. – The equality $\nabla' = D'$ provenir since the composante of type $(1,0)$ of the connection of $\Lambda^{0,q}T^*X$ coincide with d' . For the diagramme, begins on by define the vertical arrow Ψ . Be

$$\{?|?\} : (\Lambda^{p_1,q_1}T^*X \otimes \widetilde{F}) \times (\Lambda^{p_2,q_2}T^*X \otimes \widetilde{F}) \longrightarrow \Lambda^{p_1+q_2,q_1+p_2}T^*X$$

the accouplement sesquilinéaire canonique induced by the métrique on the fibres of F , and

$$* : \Lambda^{p,q}T^*X \otimes \widetilde{F} \longrightarrow \Lambda^{n-q,n-p}T^*X \otimes \widetilde{F}$$

the operator of Hodge-Of Rham-Poincaré defined by

$$\{v| * w\} = \langle v, w\rangle \, d\sigma, \qquad v,\ w \in \Lambda^{p,q}T^*X \otimes \widetilde{F}.$$

Deduces by composition an isométrie

$$\Psi_0 : \Lambda^{0,1}T^*X \otimes F \xrightarrow{\sim} \Lambda^{n,1}T^*X \otimes \widetilde{F} \xrightarrow{*} \Lambda^{n-1,0}T^*X \otimes \widetilde{F}$$

and the arrow Ψ obtains se by definition at tensorisant $-i^{-n^2}\Psi_0$ by $\Lambda^{0,q}T^*X$. For show the commutativité, assumes on firstly $q = 0$. Be $u \in \mathcal{C}^\infty(F)$. Has On classiquement

$$\widetilde{\delta}'\widetilde{u} = - * \widetilde{D}'' * \widetilde{u},$$

and comme $\widetilde{u} \in \mathcal{C}^\infty_{n,0}(X, F)$, it comes $*\widetilde{u} = i^{-n^2}\widetilde{u}$, of where

$$\widetilde{\delta}'\widetilde{u} = -i^{-n^2} * D''\widetilde{u} = -i^{-n^2} * \sim D''u = -i^{-n^2}\Psi_0(D''u) = \Psi(\nabla''u).$$

Dans le cas où q is quelconque, it suffice of trivialiser $\Lambda^{0,q}T^*X$ at the voisinage of a dot x arbitrary, choosing a repérer orthonormé $(e_1, \ldots, e_N)$ of that bundle, as $\nabla e_1(x) = \cdots = \nabla e_N(x) = 0$. □

Considers On now the morphismes of bundles

$$S' : \Lambda^{0,q}T^*X \otimes F \to \Lambda^{1,0}T^*X \otimes \Lambda^{0,q}T^*X \otimes F$$
$$S'' : \Lambda^{0,q}T^*X \otimes F \to \Lambda^{0,1}T^*X \otimes \Lambda^{0,q}T^*X \otimes F$$

where $S' = \tau = [\Lambda, d'\omega]$, and where S'' is the extract by the isométries $\sim$ and Ψ of the morphisme

$$\tau^* = [(d'\omega)^*, L] : \Lambda^{n,q}T^*X \otimes \widetilde{F} \to \Lambda^{n-1.q}T^*X \otimes \widetilde{F}.$$

As the proposal 3.6, has on

$$|D'_\tau u| = |\nabla' u + S'u|, \qquad |\widetilde{\delta}'_\tau \widetilde{u}| = |\nabla'' u + S''u|.$$

If states on $S = S' \oplus S''$, the identities (3.3) and (3.5) involve by addition

$$\begin{aligned}2\int_X \langle \Delta'' u, u\rangle = \int_x |\nabla u + Su|^2 + \int_X \langle [ic(F),\Lambda]u,u\rangle \\ + \int_X \langle [ic(\widetilde{F}),\Lambda]\widetilde{u},\widetilde{u}\rangle + \langle T_\omega u,u\rangle + \langle T_\omega \widetilde{u},\widetilde{u}\rangle\end{aligned} \tag{3.7}$$

for all $u \in \mathcal{C}^\infty_{0,q}(X,F)$.

Be now E a bundle holomorphic hermitian of rank 1 at the-above of X . For all entire k , notes on D_k and ∇_k the connections hermitians natural on the bundles $F_k = E^k \otimes F$ and $\Lambda^{0,q}T^*X \otimes F_k$, and states on $\Delta''_k = [D''_k, \delta''_k]$. The curvature of F_k (resp. $\widetilde{F}_k$) Has given by

$$c(F_k) = c(F) + kc(E) \otimes \mathrm{Id}_F, \quad \text{resp.} \quad c(\widetilde{F}_k) = c(\widetilde{F}) + kc(E) \otimes \mathrm{Id}_{\widetilde{F}}. \tag{3.8}$$

Recall, although that be useless for the continuation, that

$$c(\widetilde{F}) = c(F) + c(\Lambda^n TX) \otimes \mathrm{Id}_F = c(F) + \mathrm{Ricci}(\omega) \otimes \mathrm{Id}_F.$$

We have so need of score the terms $[ic(E),\Lambda]$. For all dot $x \in X$, are $\alpha_1(x)$, $\alpha_2(x),\dots,\alpha_n(x)$ the eigenvalues of $ic(E)(x)$ relatively at the métrique hermitian ω on X . Il there is so a system of coordinated local $(z_1,\dots,z_n)$ centred at x as $(\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n})$ be a base orthonormée of $T_X X$, and as

$$\omega(x) = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j,$$

$$ic(E)(x) = \frac{i}{2}\sum_{j=1}^n \alpha_j(x)\, dz_j \wedge d\overline{z}_j.$$

Be $(e_1,\dots,e_r)$ a repérer orthonormé of the fibre $E^k_x \otimes F_x$. For $v \in \Lambda^{p.q}T^*X \otimes F_k$, pouvoir on type

$$v = \sum_{|I|=p,|J|=q,\,\ell} v_{I,J,\ell}\, dz_I \wedge d\overline{z}_J \otimes e_\ell, \qquad |v|^2 = 2^{p+q}\sum_{I,J,\ell} |v_{I,J,\ell}|^2$$

An elementary calculation, explicité by example on [6], gives the formula

$$\langle [ic(E),\Lambda]v,v\rangle = 2^{p+q}\sum_{I,J,\ell}(\alpha_I + \alpha_J - \sum_{j=1}^n \alpha_j)|v_{I,J,\ell}|^2 \tag{3.9}$$

with $\alpha_I = \sum_{j\in I}\alpha_j$. Be $u \in \Lambda^{0,q}T^*X \otimes F_k$. State

$$u = \sum_{J,\ell} u_{J,\ell}\, d\overline{z}_J \otimes e_\ell.$$

As (3.9), it comes

$$\langle [ic(E),\Lambda]u,u\rangle = 2^q \sum_{J,\ell} -\alpha_{\complement J}|u_{J,\ell}|^2,$$
$$\langle [ic(E\rangle,\Lambda]\widetilde{u},\widetilde{u}\rangle = 2^q \sum_{J,\ell} \alpha_J|u_{J,\ell}|^2.$$

Be V the endomorphisme hermitian of $\Lambda^{0,q}T^*X\otimes F_k$ defined by

$$(3.10)\qquad \langle Vu,u\rangle = -\langle [ic(E),\Lambda]u,u\rangle - \langle [ic(E),\Lambda]\widetilde{u},\widetilde{u}\rangle = 2^q\sum_{J,\ell}(\alpha_{\complement J}-\alpha_J)|u_{J,\ell}|^2.$$

The eigenvalues of V are so the coefficients $\alpha_{\complement J}-\alpha_J$, counted with multiplicity $r =$ rang(F) . Be lastly Θ the endomorphisme hermitian defined by

$$(3.11)\qquad \langle \Theta u,u\rangle = \langle [ic(F),\Lambda]u,u\rangle + \langle [ic(\widetilde{F}),\Lambda]\widetilde{u},\widetilde{u}\rangle + \langle T_\omega u,u\rangle + \langle T_\omega\widetilde{u},\widetilde{u}\rangle.$$

The identities (3.7-11) involve then

$$(3.12)\qquad \frac{2}{k}\int_X \langle \Delta''_k u,u\rangle = \int_X \frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu,u\rangle + \frac{1}{k}\langle \Theta u,u\rangle$$

where the operators S , V , Θ no act that on the composante $\Lambda^{0,q}T^*X\otimes F$ of $\Lambda^{0,q}T^*X\otimes F_k$. Goes On so pouvoir use the theorem 2.16 for determine the distribution spectrale asymptotic of Δ''_k , because the term $\frac{1}{k}\langle\Theta u,u\rangle$ extends to 0 at norm.

Be $h^q_k(\lambda)$ the number of eigenvalues $\leqslant k\lambda$ of Δ''_k operating on $\mathcal{C}^\infty_{0,q}(E^k\otimes F)$. The magnetic field B has given here by

$$(3.13)\qquad B = -ic(E) = -\sum_{j=1}^n \alpha_j\, dx_j\wedge dy_j,\qquad z_j = x_j+iy_j.$$

kept-Account that $\dim_{\mathbb{R}} X = 2n$, the theorem 2.16 se transcrit comme tracks.

Théorème 3.14. – *Il there is an ensemble dénombrable $\mathcal{D}$ as for all $q = 0,1,\ldots,n$ and all $\lambda\in\mathbb{R}\smallsetminus\mathcal{D}$ have on*

$$h^q_k(\lambda) = rk^n\sum_{|J|=q}\int_X \nu_B(2\lambda+\alpha_{\complement J}-\alpha_J)\,d\sigma + o(k^n)$$

when k extends to $+\infty$.

4. Complex of Witten and inequalities of Morse.

E. Witten[18], [19] has entered recently a new method analytique for demontrer the inequalities of Morse cohomologie of of Rham. We adapt here his method for the studio of the d'' -cohomologie. The principal difference résider on the fact that the magnetic field is always any on the case of the cohomologie of of Rham (has on at effect $d^2 = 0$!), and is the electrical field who takes part alone on that case.

With the notations of the §3, be $\mathcal{H}_k^q(\lambda) \subset \mathcal{C}_{0,q}^\infty(X, E^k \otimes F)$ the direct sum of the under-own spaces of Δ_k'' bound at the eigenvalues $\leqslant k\lambda$. $\mathcal{H}_k^q(\lambda)$ Is so a space vectoriel of ended dimension

$$h_k^q(\lambda) = \dim_{\mathbb{C}} \mathcal{H}_k^q(\lambda).$$

The theory of Hodge gives an isomorphisme

$$H^q(X, E^k \otimes F) \simeq \mathcal{H}_k^q(0).$$

state On for abbreviate

$$h_k^q = \dim H^q(X, E^k \otimes F) = h_k^q(0).$$

Proposal 4.1. – $\mathcal{H}_k^\bullet(\lambda)$ *Is an under-complex of the complex of Dolbeault*

$$D_k'' : \mathcal{C}_{0,\bullet}^\infty(X, E^k \otimes F).$$

Besides, the inclusion $\mathcal{H}_k^\bullet(\lambda) \subset \mathcal{C}_{0,\bullet}^\infty(X, E^k \otimes F)$ *and the projection orthogonale*

$$P_\lambda : \mathcal{C}_{0,\bullet}^\infty(X, E^k \otimes F) \to \mathcal{H}_k^\bullet(\lambda)$$

induce at cohomologie of the isomorphismes reverse l'un de l'autre.

Démonstration. – Takes it that $\mathcal{H}_k^\bullet(\lambda)$ be an under-complex of $\mathcal{C}_{0,\bullet}^\infty(X, E^k \otimes F)$ provenir of the ownership of commutation of the operators D_k'' and Δ_k'' . Be now

$$G = \int_{\lambda > 0} \frac{1}{\lambda} dP_1$$

the operator of Green of the laplacien Δ_k'' . Comme $[P_\lambda, \Delta_k''] = 0$, has on the accounts $[G, \Delta_k''] = 0$ and

$$\Delta_k'' G + P_0 = \mathrm{Id}\,.$$

Besides, $[P_\lambda, D_k''] = [G, D_k''] = 0$. Deduces so

$$\begin{aligned} \mathrm{Id} - P_\lambda &= \Delta_k'' G(\mathrm{Id} - P_\lambda) + P_0(\mathrm{Id} - P_\lambda) = \Delta_k'' G(\mathrm{Id} - P_\lambda) \\ &= D_k''\big(\delta_k'' G(\mathrm{Id} - P_\lambda)\big) + \big(\delta_k'' G(\mathrm{Id} - P_\lambda)\big) D_k'', \end{aligned}$$

so that the operator $\delta_k'' G(Id - P_\lambda)$ is a homotopie go in Id and P_λ . □

Uses On now a lemme classical simple of algèbre homologique.

Lemme 4.2. – *Be*

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n \longrightarrow 0$$

a complex of spaces vectoriels of ended dimensions c^0 , c^1 , ... , c^n *on a body* $\mathbb{K}$. *Be* $h^q = \dim_{\mathbb{K}} H^q(C^\bullet)$. *Then, has on the following inequalities:*

(a) *Inequalities of Morse :* $h^q \leqslant c^q$, $0 \leqslant q \leqslant n$.

(b) *Equality of the characteristic of Euler-Poincaré* $\chi(H^\bullet(C^\bullet)) = \chi(C^\bullet)$:

$$h^0 - h^1 + \cdots + (-1)^n h^n = c^0 - c^1 + \cdots + (-1)^n c^n.$$

(c) *Inequalities of Morse strong* : *for all* q , $0 \leqslant q \leqslant n$,

$$h^q - h^{q-1} + \cdots + (-1)^q h^0 \leqslant c^q - c^{q-1} + \cdots + (-1)^q c^0.$$

Démonstration. – If $Z^q = \operatorname{Ker} d^q$ and $B^q = \operatorname{Im} d^{q-1}$ have for dimensions z^q and b^q , the equality (b) results at effect of the formulas

$$c^q = z^q + b^{q+1}, \qquad h^q = z^q - b^q,$$

while (c) results of (b) applied at the complex

$$0 \to C^0 \to C^1 \to \cdots \to C^{q-1} \to Z^q \to 0. \qquad \square$$

If F is a bundle vectoriel holomorphic on X , defines on his characteristic of Euler-Poincaré by

$$\chi(X, F) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, F).$$

Combining the proposal 4.1 and the lemme 4.2, obtain it for all $\lambda \geqslant 0$ and all q , $0 \leqslant q \leqslant n$, the inequality

$$h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant h_k^q(\lambda) - h_k^{q-1}(\lambda) + \cdots + (-1)^q h_k^0(\lambda).$$

Score now $h_k^q(\lambda)$ at the half of the theorem 3.14 and take extend $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$ to 0 by courages > 0 . It se ensuit :

Corollaire 4.3. – *Has On the inequalities asymptotics*

(a) $h_k^q \leqslant k^n I^q + o(k^n)$,

(b) $\chi(X, E^k \otimes F) = k^n(I^0 - I^1 + \cdots + (-1)^n I^n) + o(k^n)$,

(c) $h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant k^n(I^q - I^{q-1} + \cdots + (-1)^q I^0) + o(k^n)$,

where I^q *designates the integral of curvature*

$$I^q = r \sum_{|J|=q} \int_X \overline{\nu}_B(\alpha_{\complement J} - \alpha_J) d\sigma.$$

As (3.13), the modules of the eigenvalues of the magnetic field B are the $|\alpha_j|$, $1 \leqslant j \leqslant n$. For all dot $x \in X$, order that eigenvalues at type that

$$|\alpha_1 \geqslant |\alpha_2| \geqslant \cdots \geqslant |\alpha_s| > 0 = |\alpha_{s+1}| = \cdots = |\alpha_n|, \qquad s = s(x).$$

The formula (1.5) gives

$$\overline{\nu}_B(a_{\complement J}-\alpha_J)=\frac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)}|\alpha_1\ldots\alpha_s|\sum_{(p_1,\ldots,p_s)}\Big\{\alpha_{\complement J}-\alpha_J-\sum(2p_j+1)|\alpha_j|\Big\}_+^{n-s}$$

with the notation $\{\lambda\}_+^0=0$ if $\lambda<0$ and $\{\lambda\}_+^0=1$ if $\lambda\geqslant 0$. Comme the amount

$$\alpha_{\complement J}-\alpha_J-\sum(2p_j+1)|\alpha_j|$$

is always $\leqslant 0$, $\overline{\nu}_B(\alpha_{\complement J}-\alpha_J)$ no peut être no any that if $s=n$. On that latter case $\alpha_{\complement J}-\alpha_J-\sum(2p_j+1)|\alpha_j|=0$ if and alone if $p_1=\cdots=p_n=0$ and $\alpha_j<0$ for $j\in J$, $\alpha_j>0$ for $j\in\complement J$. This trains that the form $ic(E)$ is no dégénérée of rate q . For $x\in X(q)$ (cf. Notations of the introduction) and $|J|=q$, has on so

$$\overline{\nu}_B(\alpha_{\complement J}-\alpha_J)=(2\pi)^{-n}|\alpha_1\ldots\alpha_n|>0$$

if J is the multi-rate $J(x)=\{j\,;\ \alpha_j(x)<0\}$ and $\overline{\nu}_B(\alpha_{\complement J}-\alpha_J)=0$ if $J\neq J(x)$. It se ensuit

$$I^q=r\int_{X(q)}(2\pi)^{-n}(-1)^q\alpha_1\ldots\alpha_n\,d\sigma=\frac{r}{n!}\int_{X(q)}(-1)^q\Big(\frac{i}{2\pi}c(E)\Big)^n.$$

The theorem fundamental 0.1 no is whereas a reformulation of the corollaire 4.3. The reasoning here-above displays that the forms harmoniques of $H^q(X,E^k\otimes F)$ mass se asymptoticment on $X(q)$, and what at each dot of $X(q)$ leur address extends at range on the$q$ -under-space of TX corresponding at the negative part of $ic(E)$. Besides, alone the eigenvalue of minimum energy $p_1=\cdots=p_n=0$ of the oscillateur harmonique takes part for that forms. For $q=1$, the inequality of Morse strong 4.3 (c) types se

$$h_k^1-h_k^0\leqslant k^n(I^1-I^0)+o(k^n),$$

of where at particular a minoration asymptotic of the number of sections holomorphics of the bundle $E^k\otimes F$.

Théorème 4.4. – *Has On*

$$\dim H^0(X,E^k\otimes F)\geqslant r\frac{k^n}{n!}\int_{X(\leqslant 1)}\Big(\frac{i}{2\pi}c(E)\Big)^n-o(k^n).$$

Best generally, the addition of the inequalities 4.3 (c) for the rates $q+1$ and $q-2$ train

$$h_k^{q+1}-h_k^q+h_k^{q-1}\leqslant k^n(I^{q+1}-I^q+I^{q-1})+o(k^n),$$

of where the minoration

$$\dim H^q(X,E^k\otimes F)\geqslant r\frac{k^n}{n!}\sum_{j=0,\pm1}(-1)^q\int_{X(q+j)}\Big(\frac{i}{2\pi}c(E)\Big)^n-o(k^n). \tag{4.5}$$

5. Characterisation of the varieties of Moišezon.

Be X a variety $\mathbb{C}$ -compact analytique connexe of dimension n . Urges On dimension algébrique of X , noted $a(X)$, the grade of transcendence on $\mathbb{C}$ of the body $K(X)$ of the functions méromorphes on X . As a theorem very known of Siegel [15], the dimension algébrique of X checks always the inequality $0 \leqslant a(X) \leqslant n$. When $a(X) = n$, say on that X is a space of Moišezon. Comme go on see, the dimension algébrique of X impose asymptoticment of strong curtailments on the dimension of the spaces of sections of a bundle vectoriel holomorphic.

Théorème 5.1. – *Be a the dimension algébrique of X , F a bundle vectoriel holomorphic of rank r and E a bundle linear on X . Then, il there is a constante $C_E \geqslant 0$ no dépendant that of E such that*

$$\dim H^0(X, E^k \otimes F) \leqslant C_E r k^a + o(k^a).$$

Démonstration. – We restart for the essential the arguments of .T. Siu [16]. Be $\{W_\ell\}$ a recouvrement of X by of the open of coordinated $W_\ell \subset \mathbb{C}^n$, and $B_j = B(a_j, R_j)$, $1 \leqslant j \leqslant m$, a family of bowls relatively compact on the open W_ℓ , such that the bowls concentriques $B'_j = B(a_j, \frac{1}{7}R_j)$ recouvrer X . Cater E , F of métrique hermitians, and be $\exp(-\varphi_j)$ the weight representing the métrique of E on a trivialisation of E at the voisinage of $\overline{B}_j$.

Be then $s \in H^0(X, E^k \otimes F)$ a section holomorphic who cancel se at the order p at a dot $x_j \in B'_j$. The inclusions

$$B'_j \subset B(x_j, \frac{2}{7}R_j) \subset B(x_j, \frac{6}{7}R_j) \subset B_j$$

and the lemme of Schwarz applied at the deux bowls intermediate train the inequality

$$\sup_{B'_j} |s| \leqslant \exp(Ak + C_F) 3^{-p} \sup_{B_j} |s|, \tag{5.2}$$

where $A = \max_{1 \leqslant j \leqslant m} \operatorname{diam} \varphi_j(B_j)$ no depends that of E , and where C_F is a constante $\geqslant 0$ who depends of the métrique of F .

Be $\rho \leqslant r = \operatorname{rang}(F)$ the maximum for $x \in X$ of the dimension of the under-space of the fibre F_x engendré by the vecteurs $s(x)$ when s depicted $\bigcup_{k \in \mathbb{N}} H^0(X, E^k \otimes F)$. If $\rho = 0$, then $H^0(X, E^k \otimes F) = 0$ for all k . Distinguish now deux cases as $\rho = 1$ or $\rho > 1$.

(Has) *Assume* $\rho = 1$.

Be $h_k = \dim H^0(X, E^k \otimes F)$, assumed > 0 . Under the hypothesis $\rho = 1$, the global sections of $E^k \otimes F$ define an application holomorphic

$$\Phi_k : X \smallsetminus Z_k \to \mathbb{P}^{h_k - 1}(\mathbb{C})$$

where $Z_k \subset X$ is the under-ensemble analytique of leur zero common. Be d the rank maximum of the differential Φ'_k on $X \smallsetminus Z_k$. Has On necessarily $d \leqslant a$, sinon the body of the rational fractions of $\mathbb{P}^{h_k - 1}(\mathbb{C})$ would induce a body of functions méromorphes on X of grade of transcendence $\geqslant d > a$, those that is absurd. Choose for all $j = 1, \ldots, m$ a dot $x_j \in B'_j \smallsetminus Z_k$ as Φ'_k be of rank maximum $= d$ x_j , and be $s_0 \in H^0(X, E^k \otimes F)$ a section who no cancels se at any dot x_j . For all $s \in H^0(X, E^k \otimes F)$, the quotient s/s_0 is

very defined as function méromorphe on X , and besides s/s_0 is a function holomorphic at the voisinage of x_j , constante along the fibres of Φ_k . Comme Φ_k is a subimmersion at the voisinage of each dot x_j , pouvoir on choose an under-variety M_j of dimension d spending by x_j and transverse at the fibre $\Phi_k^{-1}(\Phi_k(x_j))$. The section s cancel se at the order p at each dot x_j , $1 \leqslant j \leqslant m$, if and alone if the derive partial of order $< p$ of s/s_0 along M_j cancel se at x_j . This corresponds at the total at the cancellation of

$$m\binom{p+d-1}{d}$$

derived. If choose it $p = [Ak + C_F] + 1$, then the inequality (5.2) trains

$$\sup_X |s| \leqslant \left(\frac{e}{3}\right)^p \sup_X |s|,$$

of where $s = 0$. Comme $d \leqslant a$, obtain it therefore

$$\dim H^0(X, E^k \otimes F) \leqslant m\binom{p+a-1}{a} \leqslant C_E k^a + o(k^a)$$

with $C_E = mA^a/a!$.

(b) *Assume* $\rho > 1$.

Il there is then of the sections $s_t \in H^0(X, E^{k_t} \otimes F)$, $1 \leqslant t \leqslant \rho$, and a dot $x_0 \in X$ such that the vecteurs $s_1(x_0), \ldots, s_\rho(x_0)$ are linéairement independent. By building, for all $k \in \mathbb{N}$ and all section $s \in H^0(X, E^k \otimes F)$, the right $\mathbb{C} \cdot s(x)$ has contained on the under-space engendré by $(s_1(x), \ldots, s_\rho(x))$, sauf perhaps at the-above of the under-ensemble analytique $\{x \in X; s_1 \wedge \ldots \wedge s_\rho(x)\} = 0$. Has On so a morphisme injectif

$$H^0(X, E^k \otimes F) \to \bigoplus_{1 \leqslant t \leqslant \rho} H^0(X, E^{k+k_{\hat{t}}} \otimes \Lambda^p F)$$

where $k_{\hat{t}} = (k_1 + \cdots + k_\rho) - k_t$, whose the composante of rate t has given by the morphisme $s \to s_1 \wedge \cdots \wedge \widehat{s}_t \wedge \cdots \wedge s_\rho \wedge s$. The image of $H^0(X, E^k \otimes F)$ on each composante has formed of sections colinéaires at presque all dot at $s_1 \wedge \cdots \wedge s_\rho$. On se finds so on an analogous situation at that of the (a), where F has replaced by $E^{k_{\hat{t}}} \otimes \Lambda^\rho F$; by continuation :

$$\dim H^0(X, E^k \otimes F) \leqslant C_E \rho k^a + o(k^a), \qquad \rho \leqslant r. \qquad \square$$

Choose at particular for F the bundle trivial $X \times \mathbb{C}$. Comparing the theorems 4.4 and 5.1, obtain it the geometrical characterisation following of the varieties of Moišezon.

Théorème 5.2. – *So that a variety* $\mathbb{C}$ *-compact analytique connexe* X *of dimension* n *be of Moišezon, it suffice what il there is a bundle at right holomorphic hermitian* E *at the-above of* X *as*

$$\int_{X(\leqslant 1)} (ic(E))^n > 0. \qquad \square$$

That theorem trains à son tour the theorem 0.8 since 0.8 (c) $\Rightarrow$ 0.8 (b) $\Rightarrow$ 0.8 (a). Improves On thus the outcomes of .T. Siu [17], [18], and finds on so at particular a new proof of the conjecture of Grauert-Riemenschneider [10].

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