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MAGNETIC FIELDS AND MORSE INEQUALITIES FOR d"-COHOMOLOGY

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0. Introduction.

Is X a variety $\mathbb{C}$-analytic compact size n , F a holomorphic vector bundle of rank r and a E hermitian holomorphic bundle in straight $\mathcal{C}^\infty$ class above X . $D = D' + D''$ is the canonical connection and E $c(E) = D^2 = D'D'' + D''D'$ the shape of curvature of this connection. Denote $X(q)$, $0 \leqslant q \leqslant n$, open points X q of index, i.e. open $x \in X$ of points which form curvature $ic(E)(x)$ exactly q values and < 0 $(n-q)$ values > 0. It also poses

$$X(\leqslant q) = X(0) \cup X(1) \cup \ldots \cup X(q).$$

We show then the following Morse inequalities, which limit the dimension of the cohomology spaces depending $H^q(X, E^k \otimes F)$ integral invariants of the curvature of E.

Theorem 0.1. – *When k tends to $+\infty$ we have for all $q = 0, 1, \ldots, n$ the following asymptotic inequalities.*

(a) *Morse inequalities* :

$$\dim H^q(X, E^k \otimes F) \leqslant r \frac{k^n}{n!} \int_{X(q)} (-1)^q \Big(\frac{i}{2\pi} c(E)\Big)^n + o(k^n).$$

(b) *Strong Morse inequalities* :

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(X, E^k \otimes F) \leqslant r \frac{k^n}{n!} \int_{X(\leqslant q)} (-1)^q \Big(\frac{i}{2\pi} c(E)\Big)^n + o(k^n).$$

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(c) *Asymptotic Riemann-Roch formula* :

$$\sum_{q=0}^{n}(-1)^q \dim H^q(X, E^k \otimes F) = r\frac{k^n}{n!}\int_X \left(\frac{i}{2\pi}c(E)\right)^n + o(k^n).$$

Estimates 0.1 (a), (b) are new to our knowledge, even in the case of projective varieties. The asymptotic equality 0.1 (C), for its part, is a weakened version of the theorem Hirzebruch-Riemann-Roch, which is itself a special case of tea Oreme index Atiyah Singer [1]. This last theorem in order to express the Euler-Poincaré

$$\chi(X, E^k \otimes F) = \sum_{q=0}^{n}(-1)^q \dim H^q(X, E^k \otimes F)$$

Under the form

$$\chi(X, E^k \otimes F) = r\frac{k^n}{n!}c_1(E)^n + P_{n-1}(k)\ ; \tag{0.2}$$

$P_{n-1}(k) \in \mathbb{Q}[k]$ means here a polynomial of degree $\leqslant n-1$ $c_1(E) \in H^2(X, \mathbb{Z})$ and is the first Chern class of E, represented in De Rham cohomology by $(1,1)$ Platform closed $\frac{i}{2\pi}c(E)$ (see e.g. [16]). It will be observed that the constant digital inequality 0.1 (a) is optimal, as shown in the example of the pullback tensor total $E = \mathcal{O}(1)^{n-q} \boxtimes \mathcal{O}(-1)^q$ above $X = (\mathbb{P}^1(\mathbb{C}))^n$. For this bundle, it was indeed $X(q) = X$ and

$$\begin{aligned} &\dim H^q(X, E^k) = (k+1)^{n-q}(k-1)^q, k \geqslant 1, \\ &\int_X \left(\frac{i}{2\pi}c(E)\right)^n = (-1)^q n!\,. \end{aligned}$$

The existence of a mark-type 0.1 (a) was conjectured by Y.T. Siu, which successively show the case especially where $ic(E)$ is > 0 in a complementary set of measure zero [16] and the case is $ic(E) \geqslant 0$ on X [17]. We also borrowed a Siu part of the technical utilisécs here, including §3 and §5. The proof of Theorem 0.1 is based on the method Analytical introduced recently by re E. Witten [18], [19]. This method allows (among others) to reprove the classic Morse inequalities on a $b_q \leqslant m_q$ variety compact differentiable M where b_q means the q th Betti number and the number of m_q critical points of index q any Morse function on M . In our situation, the role of the Morse function is held by the choice of the Hermitian metric on E. the other was fitted X hand and F arbitrary hermitian metrics, intervene only in terms $o(k^n)$ estimates finals. G iven a real $\lambda \geqslant 0$ it considers $\mathcal{H}_k^{\bullet}(\lambda)$ subcomplex of complex Dolbeault of $\mathcal{C}^{\infty}_{0,\bullet}(X, E^k \otimes F)$ $(0,q)$ class Platforms $\mathcal{C}^{\infty}$ on X values in $E^k \otimes F$ generated by the eigenfunctions of the Laplacian antiholomorphic Δ'' whose eigenvalues are $\leqslant k\lambda$. The cohomology groups of the complex are $\mathcal{H}_k^{\bullet}(\lambda)$ then isomorphic to $H^q(X, E^k \otimes F)$ groups (Proposition 4.1) so it is enough to know the size limit spaces $\mathcal{H}_k^q(\lambda)$. For this, essentially two tools are used. The first tool is a type of formula Weitzenböck

$$\frac{2}{k}\int_X \langle \Delta'' u, u\rangle = \int_X \frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu, u\rangle + \frac{1}{k}\langle \Theta u, u\rangle \tag{0.3}$$

demonstrated §3 and is derived from the identity of Bochner- Kodaira-Nakano non Kählerian [6]. ∇_k here denotes the natural Hermitian connection on $\Lambda^{0,q}T^*X \otimes E^k \otimes F$ the bundle, V is a linear potential of order related to the 0 E curvature of the bundle, and finally S Θ are operators of 0 order from the torsion of the Hermitian metric on X and curvature of F. The study of the spectrum of Δ'' is therefore reduced to the study of the spectrum the self-adjoint operator associated $\nabla_k^*\nabla_k$ the actual connection ∇_k. The second tool fundamental theorem consists precisely in a very broad spectrum of operators on Type $\nabla^*\nabla$. (M,g) be a Riemannian $\mathcal{C}^\infty$ real dimension n, E a bundle straight complex above X , provided with a connection Hermitian ∇ . If ∇_k means the connection induced ∇ on E^k, we then studied the spectrum of quadratic form

$$Q_k(u) = \int_\Omega \Big(\frac{1}{k}|\nabla_k u|^2 - V|u|^2\Big) d\sigma, \qquad u \in L^2(\Omega, E^k) \tag{0.4}$$

for the Dirichlet problem, which is an open relatively Ω compact in$M$ and where V is a continuous scalar potential of M . From a physical point of view, this amounts to study the spectrum the Schrödinger operator $\frac{1}{k}(\nabla_k^*\nabla_k - kV)$ associated with the electric field and magnetic field kV kB, $B = -i\nabla^2$ which is none other than the 2 Platform of curvature ∇ connection. It is in the presence of this magnetic field lies our main contribution from the method E. Witten [18], [19] (in the case of cohomology De Rham the magnetic field is always zero since $d^2 = 0$).

At any point $x \in X$ or $2s = 2s(x) \leqslant n$ rank $B(x)$ and $B_1(x) \geqslant \ldots \geqslant B_s(x) > 0$ modules of non-zero eigenvalues of the skew associated endomorphism. We define a function $\nu_{B(x)}(\lambda)$ couple $(x,\lambda) \in M \times \mathbb{R}$ continues to λ left, putting

$$v_B(\lambda) = \frac{2^{s-n}\pi^{-\frac{n}{2}}}{\Gamma(\frac{n}{2} - s + 1)} B_1 \ldots B_s \sum_{(p_1,\ldots,p_s)\in\mathbb{N}^s} \big[\lambda - \sum(2p_j+1)B_j\big]_+^{\frac{n}{2}-s} \tag{0.5}$$

$0^0 = 0$ with the agreement. Finally, if $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ denote the eigenvalues of Q_k (counted with multiplicity), consider the counting function $N_k(\lambda) = \mathrm{card}\{j\,;\ \lambda_j \leqslant \lambda\}$, $\lambda \in \mathbb{R}$.

Theorem 0.6. – *If $\partial\Omega$ is measure zero, there is a countable set $\mathcal{D} \subset \mathbb{R}$ as*

$$\lim_{k\to+\infty} k^{-\frac{n}{2}} N_k(\lambda) = \int_\Omega \nu_B(V+\lambda)\, d\sigma$$

for all $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$.

To prove Theorem 0.6, we first consider the case where single $M = \mathbb{R}^n$ with a constant magnetic field and B with $V = 0$. When Ω is a cube, then we know the explicit own functions by partial Fourier transformation brings the classic problem of the oscillator harmonic in a variable. The idea for this calculation was us strongly inspired by Articles [3], [4] Y. Colin Verdière. The extension of the result to the case of a field any magnetic gets an idea of [16], consisting using a paving Ω by fairly small cubes. Our method is nevertheless very different that of Siu, since we work directly on harmonic forms while Siu came down to cochains via holomorphic isomorphism Dolbeault. so much gain

in precision of estimates sought. The cubes of side should be chosen by a magnitude intermediary between $k^{-\frac{1}{2}}$ and $k^{-\frac{1}{4}}$ example $k^{-\frac{1}{3}}$: $k^{-\frac{1}{2}}$ is indeed the wavelength of the firstly res own functions, so that the action of the field magnetic B is not noticeable to a scale lower; above $k^{-\frac{1}{4}}$, B oscillation is too strong. finally is used minimax principle to compare the values on the values Ω own on the cubes. Ante'rieure in the method of [16] (as is included in [7]), the size of the cubes were made equal $k^{-\frac{1}{2}}$ to ; one can easily see that this choice was critical to allow to limit the effects of the magnetic field regardless of k , but the exact determination of the spectrum then became impossible. The last paragraph is devoted to the study of geometric characterizations of spaces Moišezon [13]. Recall that a compact analytic space X is called irreducible space Moišezon $K(X)$ body of meromorphic functions is $X \; = n = \dim_{\mathbb{C}} X$ degree of transcendence. Conjecture Grauert-Riemenschneider [10] says is X Moišezon if and only if there exists a quasi-positive beam $\mathcal{E}$ rank 1 without torsion over X . By desingularization, we reduce to the case is smooth and X $\mathcal{E}$ where is the locally free sheaf of sections of a bundle in straight E strictly positive on an open dense X. Y.T. Siu [17] recently solved the conjecture and has strengthened assuming only $ic(E)$ semi-positive and > 0 in at least one point. Using Theorem 0.1 (b) permits to find conditions Geometric even lower, which do not require the point semi-positivity $ic(E)$, but only the positivity of an integral oertaine curvature. For $q = 1$, inequality 0.1 (b) in fact implies a reduction in the number E^k of holomorphic sections, namely:

$$\dim H^0(X, E^k) \geqslant \frac{k^n}{n!} \int_{X(\leqslant 1)} \Big(\frac{i}{2\pi} c(E)\Big)^n - o(k^n). \tag{0.7}$$

secondly it can be shown, using conventional reasoning Siegel [15] formed by [16] that $\dim H^0(X, E^k) \leqslant \mathrm{cte} \cdot k^{n-1}$ X if not Moišezon (cf. theorem 5.1). Of the it follows the

Theorem 0.8. – *Let X a compact $\mathbb{C}$-analytic manifold of dimension n . For X either Moišezon, just as X has a holomorphic bundle straight hermitian checking one of the hypotheses* (a), (b), (c) *below.*

(a) $\int_{X(\leqslant 1)} (ic(E))^n > 0$.

(b) $c_1(E)^n > 0$, *and the shape of curvature $ic(E)$ has no index point* $\neq 0$ *par.*

(c) $ic(E)$ *is semi-positive at any point and X defined positive in at least one point of* X .

This work was the subject of a note [8] of the same title, Accounts published in renderings. This article is a version improved from a previous memory [7], which was closer to the initial technical Siu, who showed only inequality 0.1 (a) à constant digital close; thereby estimates 0.1 (b) and (C) remained inaccessible.

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1. spectrum of the Schrödinger operator combined with a constant magnetic field.

(M, g) be a Riemannian $\mathcal{C}^\infty$ class, n real dimension and $E \to M$ a complex line bundle above M provided with a Hermitian metric $\mathcal{C}^\infty$. note $\mathcal{C}^\infty_q(M, E)$ space of $\mathcal{C}^\infty$ class sections $\Lambda^q T^* M \otimes E$ fiber, and coupling $(?|?)$ canonical sesquilinear

$$\mathcal{C}^\infty_q(M, E) \times \mathcal{C}^\infty_q(M, E) \to \mathcal{C}^\infty_{p+q}(M, \mathbb{C}).$$

D a Hermitian connection on E we suppose given, that is to say a differential operator of order a

$$D : \mathcal{C}^\infty_q(M, E) \to \mathcal{C}^\infty_{q+1}(M, E), \qquad 0 \leqslant q < n,$$

verifying identities

$$D(f \wedge u) = df \wedge u + (-1)^m f \wedge Du, \tag{1.1}$$

$$d(u|v) = (Du|v) + (-1)^p (u|Dv), \tag{1.2}$$

for all $f \in \mathcal{C}^\infty_m(M, \mathbb{C})$ sections $u \in \mathcal{C}^\infty_p(M, E)$, $v \in \mathcal{C}^\infty_q(M, E)$. Consider a trivialization $\theta : E_{|W} \to W \times \mathbb{C}$ isometric E over a $W \subset M$ open. The Hermitian connections $E_{|W}$ are then all the data following formula:

$$Du = du + iA \wedge u,$$

and where $u \in \mathcal{C}^\infty_q(W, E) \simeq \mathcal{C}^\infty_q(W, \mathbb{C})$ where $A \in \mathcal{C}^\infty_1(W, \mathbb{R})$ is 1 Platform *real* arbitrary. The magnetic field (or shape curvature) associated with D the connection is closed 2 real $B = dA$ Platform as

$$D^2 u = iB \wedge u$$

for all $u \in \mathcal{C}^\infty_q(M, E)$. B therefore depends on the D connection, but not the trivialization θ chosen. A change $u = ve^{i\varphi}$ phase of θ led to replace by A $A + d\varphi$. The choice of a trivialization E and 1 -form A corresponding physically interpreted as the choice of a potential particularly the magnetic field vector B .

$|u|$ denote the point norm of an element $u \in \Lambda^q T^* M \otimes E$ for the metric tensor product of metrics and M E. If Ω is an open M , there $L^2(\Omega, E)$ (resp. $L^2_q(\Omega, E)$) the L^2 space sections E (resp. of $\Lambda^q T^* M \otimes E$) above of Ω , with the norm

$$\|u\|^2_\Omega = \int_\Omega |u|^2 d\sigma,$$

$d\sigma$ where is the Riemann volume density of M .

Either D_k connection induced D on th tensor power k E^k and V scalar potential M , i.e. function real continues. Given a relatively compact open $\Omega \subset M$, we propose to determine asymptotically when k approaches $+\infty$ the spectrum of the quadratic form

$$Q_{\Omega,k}(u) = \int_\Omega \Big(\frac{1}{k}|D_k u|^2 - V|u|^2\Big) d\sigma \tag{1.3}$$

where $u \in L^2(\Omega, E^k)$ with Dirichlet condition $u_{|\partial\Omega} = 0$. The domain $Q_{\Omega,k}$ is the Sobolev space $W^1_0(\Omega, E^k) =$ adhesion of 1'espace $\mathcal{D}(\Omega, E^k)$ C^∞ sections E^k compact support in Ω in $W^1(M, E^k)$ space. From a physical point of view, this amounts to study the

spectrum of the Schrödinger operator $\frac{1}{k}(D_k^* D_k - kV)$ associated with the magnetic field kB and the electric field kV when k approaches $+\infty$. We refer the reader to the classic paper [2] for a study general spectrum of the Schrödinger operator, and to work [3], [4], [5], [9], [12] to study problems asymptotic neighbors precedent.

Definition 1.4. – *will denote $N_{\Omega,k}(\lambda)$ the number of eigenvalues of $\leqslant \lambda$ the quadratic form $Q_{\Omega,k}$.*

We will first consider a simple case as a model for If the General §2. We work in the following situation : $M = \mathbb{R}^n$ with the constant metric $g = \sum_{j=1}^n dx_j^2$, is Ω cube of side r:

$$\Omega = \Big\{(x_1, \dots, x_n) \in R^n \, ; \; |x_j| < \frac{r}{2}, \; 1 \leqslant j \leqslant n\Big\},$$

$V = 0$, and finally the magnetic field B is constant, equal to the 2 Platform altered rank given by $2s$

$$B = \sum_{j=1}^{s} B_j \, dx_j \wedge dx_{j+s},$$

with $B_1 \geqslant B_2 \geqslant \cdots \geqslant B_s > 0$, $s \leqslant \frac{n}{2}$. Can then be choose a trivialization of E whose associated vector potential

$$A = \sum_{j=1}^{s} B_j x_j \, dx_{j+s}.$$

The E^k connection is thus written

$$D_k u = du + ikA \wedge u,$$

and the quadratic form is given by $Q_{\Omega,k}$

$$Q_{\Omega,k}(u) = \frac{1}{k} \int_{\Omega} \left[\sum_{1 \leqslant j \leqslant s} \Big(\Big|\frac{\partial u}{\partial x_j}\Big|^2 + \Big|\frac{\partial u}{\partial x_{j+s}} + ikB_j x_j u\Big|^2 \Big) + \sum_{j > 2s} \Big|\frac{\partial u}{\partial x_j}\Big|^2 \right] d\mu$$

$d\mu$ where denotes the Lebesgue measure on $\mathbb{R}^n$. Is carried out if $X_j = \sqrt{k}\, x_j$ the scaling, it is brought back to study the eigenvalues of the quadratic form

$$\int_{\sqrt{k}\Omega} \left[\sum_{1 \leqslant j \leqslant s} \Big(\Big|\frac{\partial u}{\partial X_j}\Big|^2 + \Big|\frac{\partial u}{\partial X_{j+s}} + iB_j X_j u\Big|^2 \Big) + \sum_{j > 2s} \Big|\frac{\partial u}{\partial X_j}\Big|^2 \right] d\mu$$

on $\sqrt{k}\Omega$ cubes $\sqrt{k}\, r$ side. B field, we combine the function of the real variable λ defined by

$$\nu_B(\lambda) = \frac{2^{s-n}\pi^{-\frac{n}{2}}}{\Gamma(\frac{n}{2} - s + 1)} B_1 \dots B_s \sum_{(p_1, \dots, p_s) \in \mathbb{N}^S} [\lambda - \sum (2p_j + 1) B_j]_{+}^{\frac{n}{2} - s} \tag{1.5}$$

where we put conventionally $\lambda_+^0 = 0$ and if $\lambda \leqslant 0$ $\lambda_+^0 = 1$ if $\lambda > 0$. ν_B the function is increasing and continuous left on$\mathbb{R}$; be observed that ν_B is actually continuous if

$s < \frac{n}{2}$. The spectrum is $Q_{\Omega,k}$ then asymptotically described by the following theorem, which the idea was suggested to us by Y. Colin Verdière [4].

Theorem 1.6. – *Let R real > 0 ,*

$$P(R) = \Big\{x \in \mathbb{R}^n\,;\ |x_j| < \frac{R}{2}\Big\}$$

the side pad R, Q_R the quadratic form

$$Q_R(u) = \int_{P(R)} \Bigg[\sum_{1 \leqslant j \leqslant s} \Big(\Big|\frac{\partial u}{\partial x_j}\Big|^2 + \Big|\frac{\partial u}{\partial x_{j+s}} + iB_j x_j u\Big|^2 \Big) + \sum_{j > 2s} \Big|\frac{\partial u}{\partial x_j}\Big|^2 \Bigg] d\mu,$$

$N_R(\lambda)$ and the number of eigenvalues of $\leqslant \lambda$ Q_R for the Dirichlet problem. So for all we $\lambda \in \mathbb{R}$

$$\lim_{R \to +\infty} R^{-n} N_R(\lambda) = \nu_B(\lambda).$$

When $s = \frac{n}{2}$, ν_B is a step function. The eigenvalues of Q_R So gather in packs around $\sum (2p_j + 1)B_j$ values with approximate multiplicity $(2\pi)^{-s} B_1 \ldots B_s R^n$. This can physically interpret as a phenomenon of quantifying own states. Returning to the original problem with the quadratic form $Q_{\Omega,k}$ we get the

Corollary 1.7. – $\lim_{k \to +\infty} k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) = r^n \nu_B(\lambda)$. □

Proof of Theorem 1.6. – First we try to increase $N_R(\lambda)$. For this purpose, being given $u \in W_0^1(P(R))$, is expressed as a series u partial Fourier compared to $x_{s+1}, \ldots, x_n$ variables:

$$u(x) = R^{-\frac{1}{2}(n-s)} \sum_{\ell \in \mathbb{Z}^{n-s}} u_\ell(x') \exp\Big(\frac{2\pi i}{R} \ell \cdot x''\Big)$$

$u_\ell \in W_0^1(\mathbb{R}^s \cap P(R))$ where, with the notation

$$x' = (x_1, \ldots, x_s), \quad x'' = (x_{s+1}, \ldots, x_n),$$
$$\ell \cdot x'' = \ell_1 x_{s+1} + \cdots + \ell_{n-s} x_n.$$

The $u \in W_0^1(P(R))$ hypothesis implies that the series

$$\sum |\ell|^2 |u_\ell(x')|^2$$

is in $L^2(\mathbb{R}^s)$. Let $\ell' = (\ell_1, \ldots, \ell_s)$, $\ell'' = (\ell_{s+1}, \ldots, \ell_{n-s})$. The standard and $\|u\|_{P(R)}$ the quadratic form Q_R are given by

$$\|u\|^2_{P(R)} = \sum_{\ell \in \mathbb{Z}^{n-s}} \int_{\mathbb{R}^s} |u_\ell(x')|^2 \, d\mu(x'),$$

$$Q_R(u) = \sum_{\ell \in \mathbb{Z}^{n-s}} \int_{\mathbb{R}^s} \Bigg[\sum_{1 \leqslant j \leqslant s} \Big(\Big|\frac{\partial u_\ell}{\partial x_j}\Big|^2 + \Big(\frac{2\pi}{R}\ell_j + B_j x_j\Big)^2 |u_\ell|^2 \Big) + \frac{4\pi^2}{R^2} |\ell''|^2 |u_\ell|^2 \Bigg] d\mu(x').$$

therefore we get a Dirichlet problem for "separated variables" on the cube $\mathbb{R}^s \cap P(R)$. By asking $t = x_j + \frac{2\pi\ell_j}{RB_j}$, it is reduced to study the shape of spectrum a quadratic variable

$$q(f) = \int_R \Big(\Big|\frac{df}{dt}\Big|^2 + B_j^2 t^2 |f|^2\Big)\, dt,$$

with $f \in W_0^1\big(]-\frac{R}{2}\frac{R}{2}[+\frac{2\pi\ell_j}{RB_j}\big)$. So we fall back on the problem classical harmonic oscillator (see e.g. [14], Vol. I, p. 142). On $\mathbb{R}$, i.e. unsupported condition for f, the sequence of values own q is $(2m+1)B_j$ subsequently $m \in \mathbb{N}$, and functions own associates are given by where $\Phi_m(\sqrt{B_j}\, t)$ Φ_0 , $\Phi_1, \ldots$ are the Hermite functions :

$$\Phi_m(t) = e^{t^2/2} \frac{d^m}{dt^m}(e^{-t^2}).$$

For $p_j \in \mathbb{N}$ include $\Psi_{p_j,\ell_j}(x_j)$ the p_j th proper function of the quadratic form

$$q(f) = \int_R \Big(\Big|\frac{df}{dx_j}\Big|^2 + \Big(\frac{2\pi}{R}\ell_j + B_j x_j\Big)^2 |f|^2\Big)\, dx_j \tag{1.8}$$

for $f \in W_0^1(]-\frac{R}{2}\frac{R}{2}[)$ and λ_{p_j,ℓ_j} the corresponding own value. We can then break down each u_ℓ function series of eigenfunctions, which leads writing as u

$$u(x) = R^{-\frac{1}{2}(n-s)} \sum_{(p,\ell)\in\mathbb{N}^s\times\mathbb{Z}^{n-s}} u_{p,\ell}\Psi_{p,\ell'}(x') \exp\Big(\frac{2\pi i}{R}\ell\cdot x''\Big) \tag{1.9}$$

with

$$u_{p,\ell} \in \mathbb{C}, \qquad \Psi_{p,\ell'}(x') = \prod_{1\leqslant j\leqslant s} \Psi_{p_j,\ell_j}(x_j).$$

Care must be taken to the fact that is $\Psi_{p,\ell'}(x') \exp(\frac{2\pi i}{R}\ell\cdot x'')$ not real clean function for the Dirichlet problem, because the term exponential takes nonzero values to the edge points $x_j = \pm\frac{R}{2}$, $j > s$. Therefore, the coefficients $(u_{p,\ell})$ are not arbitrary if $u \in W_0^1(P(R))$; they should check the cancellation policy at the edge :

$$\sum_{t_j\in\mathbb{Z}} (-1)^{\ell_j} u_{p,\ell} = 0 \tag{1.10}$$

$j = 1, \ldots, n-s$ for any and all clues other than ℓ_j attached:

$$p \in \mathbb{N}^s, \quad \ell_1, \ldots, \ell_{j-1},\ \ell_{j+1}, \ldots, \ell_{n-s} \in \mathbb{Z}.$$

With writing (1.9), the standard L^2 and the quadratic form Q_R expressed as

$$\|u\|^2_{P(R)} = \sum |u_{p,\ell}|^2, \qquad Q_R(u) = \sum \Big(\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2\Big)|u_{p,\ell}|^2,$$

where $\lambda_{p,\ell'} = \sum_{1\leqslant j\leqslant s} \lambda_{p_j,\ell_j}$. The Minimax principle of 1.20 (b) shows that further recalled

$$N_R(\lambda) \leqslant \operatorname{card}\Big\{(p,\ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}\,;\ \lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda\Big\}. \tag{1.11}$$

So just to get adequate lower bound λ_{p_j,ℓ_j} .

Lemma 1.12. — *was inequality*

$$\lambda_{p_j,\ell_j} \geqslant \max\Big((2p_j+1)B_j\ ,\ \frac{4\pi^2}{R^2}\Big[\Big(\frac{p_j+1}{2}\Big)^2 + \Big(|\ell_j| - \frac{B_jR^2}{4\pi}\Big)_+^2\Big]\Big),$$

and it is strict if $\ell_j \neq 0$ *or* $\Phi_{p_j}(R\sqrt{B_j}/2) \neq 0$.

The markdown $\lambda_{p_j,\ell_j} \geqslant (2p_j+1)B_j$ resulting effect minimax and that the eigenvalues of $q(f)$ on $\mathbb{R}$ worth $(2p_j+1)B_j$. For the other inequality, we lower bound (1.8) by the quadratic form

$$\widehat{q}(f) = \int_{x_j|<R/2}\Big(\Big|\frac{df}{dx_j}\Big|^2 + \Big(\frac{2\pi}{R}|\ell_j| - B_j\frac{R}{2}\Big)_+^2|f|^2\Big)dx_j.$$

The eigenfunctions are the functions of $\widehat{q}$

$$\sin\frac{\pi}{R}(p_j+1)\Big(x_j+\frac{R}{2}\Big), \qquad p_j \in \mathbb{N}\ ;$$

λ_{p_j,t_j} is bounded below by the corresponding own value :

$$\frac{4\pi^2}{R^2}\Big[\Big(\frac{p_j+1}{2}\Big)^2 + \Big(|t_j| - \frac{B_jR^2}{4\pi}\Big)_+^2\Big].$$

Inequality is strict because firstly $q(f) > \widehat{q}(f)$ for all $f \neq 0$, and secondly can $\Phi_{p_j}(\sqrt{B_j}t)$ q be clean depending on if $]-R/2, R/2[+2\pi\ell_j/RB_j$

$$\Phi_{p_j}\big(\pm R\sqrt{B_j}/2 + 2\pi t_j/R\sqrt{B_j}\big) = 0.$$

As Φ_{p_j} zeros are algebraic and that is π transcendent, this is only possible if

$$\ell_j = 0 \quad \text{et} \quad \Phi_{p_j}(R\sqrt{B_J}/2) = 0. \qquad \square$$

Lemma 1.13. — *Let* $\tau_n(\rho)$ *the number of points* $\mathbb{Z}^n$ *located in the closed* $\overline{B}(0,\rho) \subset \mathbb{R}^n$ *ball. So*

$$\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(\rho - \frac{\sqrt{n}}{2}\Big)_+^n \leqslant \tau_n(\rho) \leqslant \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(\rho + \frac{\sqrt{n}}{2}\Big)^n.$$

Indeed, the cubes meeting 1 side centered in points $x \in \mathbb{Z}^n$ $|x| \leqslant \rho$ such as is contained in the ball $\overline{B}(0, \rho + \frac{\sqrt{n}}{2})$ and contains ball $\overline{B}(0, \rho - \frac{\sqrt{n}}{2})$ if $\rho \geqslant \frac{\sqrt{n}}{2}$, $\frac{\sqrt{n}}{2}$ is because the half-diagonal of the cube ; integer $\tau_n(\rho)$ is framed by the volume of balls $\overline{B}(0, \rho \pm \frac{\sqrt{n}}{2})$. $\square$

We are increasing now $\limsup R^{-n}N_R(\lambda)$ using (1.11) and Lemmas 1.12, 1.13. For $p \in \mathbb{N}^s$ fixed, inequality $\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda$ involves

$$|\ell''| \leqslant \frac{R}{2\pi}\Big(\lambda - \sum(2p_j+1)B_j\Big)_+^{\frac{1}{2}}, \tag{1.14}$$

and the inequality is strict for quite a $R > R_0(p)$. When $s < n/2$ the number of multi-indices corresponding $\ell'' \in \mathbb{Z}^{n-2s}$ is therefore no

$$\begin{aligned}
&\frac{\pi^{\frac{n}{2}-s}}{\Gamma(\frac{n}{2}-s+1)}\left[\frac{R}{2\pi}\Big(\lambda-\sum(2p_j+1)B_j\Big)_+^{\frac12}+\frac{\sqrt{n}}{2}\right]^{n-2s}\\
(1.15)\qquad &\underset{R\to+\infty}{\sim}\ \frac{2^{2s-n}\pi^{s-\frac{n}{2}}}{\Gamma(\frac{n}{2}-s+1)}R^{n-2s}\Big(\lambda-\sum(2p_j+1)B_j\Big)_+^{\frac{n}{2}-s}.
\end{aligned}$$

When $s = \frac{n}{2}$, this number should be counted as valid 1 if $\lambda - \sum(2p_j+1)B_j > 0$ 0 and otherwise, which is in accordance the convention that we adopted for the λ_+^0 notation. The $\lambda_{p,\ell'} \leqslant \lambda$ inequality implies the other

$$(1.16)\qquad |\ell_j| \leqslant \frac{R}{2\pi}\sqrt{\lambda_+} + \frac{B_jR^2}{4\pi}, \qquad 1 \leqslant j \leqslant s,$$

which corresponds asymptotically to a number of multiindices $\ell' = (\ell_1, \ldots, \ell_s) \in \mathbb{Z}^s$ equivalent to

$$(1.17)\qquad \prod_{j=1}^{s}\frac{B_jR^2}{2\pi} = 2^{-s}\pi^{-s}B_1\ldots B_sR^{2s}.$$

The increase is obtained $\limsup R^{-n}N_R(\lambda) \leqslant \nu_B(\lambda)$ then by taking the product of (1.15) through (1.17), and summing for all $p \in \mathbb{N}^s$ (the sum is over). □

For convergence issues to intervene §2, we also need to know a increase independent $N_R(\lambda)$ field magnetic$B$. A Such uniform estimate is provided by the following proposition.

Proposition 1.18. – $N_R(\lambda) \leqslant (R\sqrt{\lambda_+}+1)^n$.

Proof. – On majorises j for each index number integers and p_j ℓ_j such as inequality

$$\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda$$

takes place. Lemma 1.12 implies

$$\mathrm{card}\{p_j\} \leqslant \max(p_j+1) \leqslant \min\Big(\frac{\lambda_+}{B_j}\,,\,\frac{R}{\pi}\sqrt{\lambda_+}\Big), \qquad 1 \leqslant j \leqslant s,$$

whereas (1.16) leads

$$\mathrm{card}\{l_i\} \leqslant \frac{R}{\pi}\sqrt{\lambda_+} + \frac{B_jR^2}{2\pi} + 1, \qquad 1 \leqslant j \leqslant s.$$

therefore we deduce for $1 \leqslant j \leqslant s$:

$$\mathrm{card}\{(p_j,l_j)\} \leqslant \Big(\frac{R}{\pi}\sqrt{\lambda_+}\Big)^2 + \frac{\lambda_+}{B_j}\cdot\frac{B_jR^2}{2\pi} + \frac{R}{\pi}\sqrt{\lambda_+}\cdot 1 \leqslant \big(R\sqrt{\lambda_+}+1\big)^2$$

For $s < j \leqslant n - s$, inequality (1.14) gives the other

$$|\ell_j| < \frac{R}{2\pi}\sqrt{\lambda_+},$$

where $\operatorname{card}\{l_j\} \leqslant \frac{R}{\pi}\sqrt{\lambda_+} + 1$. 1.18 The proposal follows. □

End of the proof of Theorem 1.6 (Downward adjustment $N_R(\lambda)$).

To underestimate $N_R(\lambda)$, just after 1.20 (a) to build a vector space of finite dimension on which $Q_R(u) \leqslant \lambda\|u\|^2_{P(R)}$. It is considered to the vector space $\mathcal{F}_\lambda$ linear combinations of "clean power" type (1.9), subject to Cancellation boundary conditions (1.10), and summed over the $(p,\ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}$ indices such as

$$\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda.$$

According to the reasoning of the proposal 1.18, the number of conditions (1.10) to realize is increased by

$$\sum_{j=1}^{s}\Big[\operatorname{card}\{p_j\} \times \prod_{1\leqslant i\leqslant s,\, i\neq j}\operatorname{card}\{(p_i,\ell_i)\} \times \prod_{s<i\leqslant n-s}\operatorname{card}\{\ell_i\}\Big]$$
$$+ \sum_{s<j\leqslant n-s}\Big[\prod_{1\leqslant i\leqslant s}\operatorname{card}\{(p_i,\ell_i)\} \times \prod_{s<i\neq j}\operatorname{card}\{\ell_i\}\Big] \leqslant n(R\sqrt{\lambda_+}+1)^{n-1}.$$

The entire $N_R(\lambda)$ is therefore increased by

$$\dim \mathcal{F}_\lambda \geqslant \operatorname{card}\Big\{(p,\ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}\,;\ \lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda\Big\} - O(R^{n-1}).$$

Combining the lower bound of the lemma with Lemma 1.13 below, inequality $\liminf R^{-n}N_R(\lambda) \geqslant \nu_B(\lambda)$ then results from calculations similar to those we have explained for the increase in $N_R(\lambda)$.

Lemma 1.19. – *Let $p \in \mathbb{N}^s$ a fixed multi-index. Then there exists a constant such that $C = C(p,B) \geqslant 0$*

$$\lambda_{p,\ell'} \leqslant \Big(1 + \frac{C}{R}\Big)\sum_{j=1}^{s}(2p_j+1)B_j$$

when $|\ell_j| \leqslant \frac{B_jR^2}{4\pi}(1 - R^{-\frac{1}{2}})$, $1 \leqslant j \leqslant s$.

Proof. – Again used minimax and the fact the Hermite functions are good $\Phi_p(\sqrt{B_j}t)$ approximation eigenfunctions q on any interval enough large center 0 . When $|\ell_j| \leqslant \frac{B_jR^2}{4\pi}(1-R^{-\frac{1}{2}})$ and $x_j \in \left]-\frac{R}{2},\frac{R}{2}\right[$, variable $t = x_j + \frac{2\pi\ell_j}{B_jR}$ that appears in (1.8) described indeed an interval containing $\left]-\frac{\sqrt{R}}{2},\frac{\sqrt{R}}{2}\right[$. so we $\lambda_{p_j,\ell_j} \leqslant \widetilde{\lambda}_{p_j}$ where $(\widetilde{\lambda}_m)_{m\in\mathbb{N}}$ is the sequence of eigenvalues of quadratic form

$$\widetilde{q}(f) = \int\Big[\Big|\frac{df}{dt}\Big|^2 + (B_jt)^2|f|^2\Big]dt, \qquad f \in W_0^1\Big(\Big]-\frac{\sqrt{R}}{2},\frac{\sqrt{R}}{2}\Big[\Big).$$

χ_R be a support function in tray $\big[-\frac{\sqrt{R}}{2}, \frac{\sqrt{R}}{2}\big]$, equal to about 1 $\big[-\frac{\sqrt{R}}{4}, \frac{\sqrt{R}}{4}\big]$, whose derivative is increased by $5/\sqrt{R}$. For any linear combination

$$f = \sum_{m \leqslant p_j} c_m \Phi_m(\sqrt{B_j} t),$$

the exponential decay of Φ_m functions to infinity involves large enough for R inequality

$$\|f\| \leqslant \Big(1 + C_1 \exp\Big(-\frac{R}{C_1}\Big)\Big) \|\chi_R f\|$$

where $C_1 = C_1(p_j, B_j) > 0$. therefore we deduce:

$$\begin{aligned}
\widetilde{q}(\chi_R f) &\leqslant \widetilde{q}(f) + \int_{|t| > \sqrt{R}/4} \Big(\frac{10}{\sqrt{R}}\Big|f\frac{df}{dt}\Big| + \frac{25}{R}|f|^2\Big)\,dt \\
&\leqslant \widetilde{q}(f) + \int_{|t| > \sqrt{R}/4} \Big(\frac{1}{R}\Big|\frac{df}{dt}\Big|^2 + 25\Big(1 + \frac{1}{R}\Big)|f|^2\Big)\,dt \\
&\leqslant \Big(1 + \frac{C_2}{R}\Big)\widetilde{q}(f) \leqslant \Big(1 + \frac{C_2}{R}\Big)(2p_j + 1)B_j\,\|f\|^2 \\
&\leqslant \Big(1 + \frac{C}{R}\Big)(2p_j + 1)B_j\,\|\chi_R f\|^2
\end{aligned}$$

This gives good $\lambda_{p_j,\ell_j} \leqslant \widetilde{\lambda}_{p_j} \leqslant \big(1 + \frac{C}{R}\big)(2p_j + 1)B_j$. □

For the convenience of the reader, we state now the principle of minimax as where he served us.

Proposition 1.20 (minimax principle, see [14], Vol. IV, p. 76 and 78).**—** *Let Q a quadratic form $D(Q)$ dense area in a Hilbert space $\mathcal{H}$. We assume that is Q bounded from below, i.e. $Q(f) \geqslant -C\|f\|^2$ if $f \in D(Q)$ that $D(Q)$ is complete for the $\|f\|_Q = [Q(f) + (C+1)\|f\|^2]^{\frac{1}{2}}$ standard, and finally that injection $(D(Q), \|\ \ \|_Q) \hookrightarrow (\mathcal{H}, \|\ \ \|)$ is compact. So Q has a discrete spectrum $\lambda_1 \leqslant \lambda_2 \leqslant \dots$, and we have the equalities* :

(a) $\lambda_p = \min\limits_{F \subset D(Q)} \max\limits_{f \in F, \|f\|=1} Q(f),$

F which describes the overall dimension of subspaces p of $D(Q)$;

(b) $\lambda_{p+1} = \max\limits_{F \subset D(Q)} \min\limits_{f \in F, \|f\|=1} Q(f),$

F which describes all the subspaces $\|\ \ \|_Q$ -Farms codimension p of $D(Q)$.

2. Asymptotic distribution of the spectrum (Case of a variable field).

We put ourselves back into the general framework described early §1. Our goal is to study the form of the spectrum quadratic $Q_{\Omega,k}$ (see (1.3)) in the case of a magnetic field B and an electric field any V. For any point $a \in M$, is

(2.1) $$B(a) = \sum_{j=1}^{s} B_j(a)\,dx_j \wedge dx_{j+s}$$

reduced $B(a)$ the writing in a suitable orthonormal basis $(dx_1, \ldots, dx_n)$ of T_a^*M, which is the rank $2s = 2s(a) \leqslant n$ of $B(a)$, and where are $B_1(a) \geqslant B_2(a) \geqslant \ldots \geqslant B_s(a) > 0$ the modules of non-zero eigenvalues of the endomorphism skew associated. Equal definition 1.5 lets look like $\nu_B(\lambda)$ a function of the couple $(a, \lambda) \in M \times \mathbb{R}$. We will need Also consider the $\overline{\nu}_B(\lambda)$ function right continuous in λ defined by :

$$\overline{\nu}_B(\lambda) = \lim_{0<\varepsilon\to 0} \nu_B(\lambda + \varepsilon). \tag{2.2}$$

We show then the following generalization of Corollary 1.7.

Theorem 2.3. – *tends to* $+\infty$ *When* k, $N_{\Omega,k}(\lambda)$ *the number of eigenvalues of* $\leqslant \lambda$ $Q_{\Omega,k}$ *checks the asymptotic coaching*

$$\int_\Omega \nu_B(V+\lambda)\, d\sigma \leqslant \liminf k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \limsup k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \int_\Omega \overline{\nu}_B(V+\lambda)\, d\sigma.$$

The function is $\lambda \mapsto \int_\Omega \nu_B(V+\lambda)\, d\sigma$ increasing and continuous left ; So it has more than a set $\mathcal{D}$ countable points of discontinuity. The set is $\mathcal{D}$ Besides empty if n is odd, because $\nu_B(\lambda)$ then continues. From there, once you deduct the

Corollary 2.4. – *is assumed that is* $\partial\Omega$ *measure zero. So*

$$\lim_{k\to+\infty} k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) = \int_\Omega \nu_B(V+\lambda)\, d\sigma$$

for all $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$, *and the density measurement Spectral* $k^{-\frac{n}{2}} \frac{d}{d\lambda} N_{\Omega,k}(\lambda)$ *converges* $\mathbb{R}$ *weakly to* $\frac{d}{d\lambda} \int_\Omega \nu_B(V+\lambda)\, d\sigma$. *If* n *is odd, the limit measure is diffuse.* □

The following lemma shows that the integrals of Theorem 2.3 have good one direction.

Lemma 2.5.

(a) *We inequalities*

$$\nu_B(\lambda) \leqslant \overline{\nu}_B(\lambda) \leqslant \lambda_+^{n/2}.$$

(b) $\nu_B(V)$ (*resp.* $\overline{\nu}_B(V)$) *is semi-continuous inferiorly* (*resp. superiorly*) *on* M .

(c) *In any* $x \in M$ *point where we* $s(x) < \frac{n}{2}$ $\nu_B(V)(x) = \overline{\nu}_B(V)(x)$ *and* $\nu_B(V), \overline{\nu}_B(V)$ *are continuous in* x .

(d) *If* n *is odd, is* $\nu_B(V) = \overline{\nu}_B(V)$ *continued on* M .

Proof. – (A) was still $\left(\lambda - \sum(2p_j+1)B_j\right)_+^{\frac{n}{2}-s} \leqslant \lambda_+^{\frac{n}{2}-s}$, and the number of integers p_j as $\lambda - (2p_j+1)B_j$ be $\geqslant 0$ is increased by $\frac{\lambda_+}{B_j}$. As the digital quantity contained in (1.5) is increased by 1 , inequality (a) ensues.

(B, c) The rank $s = s(x)$ is a semi-continuous function inferiorly on M , and values B_1, B_2, …, prolonged by $B_j(x) = 0$ for $j > s(x)$, are continuous over M . Since the function $t \mapsto t_+^0$ (resp. $t \mapsto (t+0)_+^0$) is semi-continuous inferiorly (resp. superiorly), the

semi-continuity of $\nu_B(V)$ and $\overline{\nu}_B(V)$ a problem only points $a \in M$ near $s(x)$ which is not locally constant. At such a point $a \in \ M$ was necessarily $s(a) < \frac{n}{2}$ so $\nu_B(V)(a) = \overline{\nu}_B(V)(a)$; we go then show that $\nu_B(V)$ are continuous and $\overline{\nu}_B(V)$ by a . B_j gives continuity to $\lim_{x\to a} B_j(x) = 0$ $j > s(a)$. If $p_1, \ldots, p_{s(a)}$ whole are fixed, the summons contained in (1.5) can be interpreted as a sum Riemann an integral over $\mathbb{R}^{s(x)-s(a)}$, and was therefore equivalent :

$$\begin{aligned}
&\sum_{(p_j;\ s(a)<j\leqslant s(x))} \Big(V(x) - \sum(2p_j+1)B_j(x)\Big)_+^{\frac{n}{2}-s(x)} \\
&\qquad \sim \int_{t\in\mathbb{R}^{s(x)-s(a)}} \left[V(a) - \sum_{j=1}^{s(a)}(2p_j+1)B_j(a) - \sum_{j=s(a)+1}^{s(x)} 2t_jB_j(x)\right]_+^{\frac{n}{2}-s(x)} dt \\
&\qquad = \frac{2^{s(a)-s(x)}\Big(V(a) - \sum(2p_j+1)B_j(a)\Big)_+^{\frac{n}{2}-s(a)}}{(\frac{n}{2}-s(x)+1)\cdots(\frac{n}{2}-s(a))B_{s(a)+1}(x)\cdots B_{s(x)}(x)}\ .
\end{aligned}$$

one obtains therefore :

$$\lim_{x\to a} \nu_B(V)(x) = \nu_B(V)(a) = \lim_{x\to a} \overline{\nu}_B(V)(x).$$

(D) is a special case of (c). $\square$

The proof of Theorem 2.3 is based primarily on two ingredients: first a principle of localization asymptotic eigenfunctions, which is obtained by applying Direct minimax (proposal 2.6) ; on the other hand, knowledge explicit spectrum of Schrödinger operator partner in a constant magnetic field (see §1). The principle of location makes it possible to reduce to the case of a constant field using a paving Ω by fairly small cubes.

Proposition 2.6.– (a) *If $\Omega_1, \cdots, \Omega_N \subset \Omega$ are open to 2 2 disjoint, then*

$$N_{\Omega,k}(\lambda) \geqslant \sum_{j=1}^{N} N_{\Omega_j,k}(\lambda).$$

(b) *Let $(\Omega'_j)_{1\leqslant j\leqslant \mathrm{N}}$ an open cover $\overline{\Omega}$ and $(\psi_j)_{1\leqslant j\leqslant \mathrm{N}}$ of a system of functions $\psi_f \in \mathcal{C}^\infty(\mathbb{R}^n)$ support in Ω'_j such on that $\sum\psi_j^2 = 1\overline{\Omega}$. we set*

$$C(\psi) = \sup_{\Omega} \sum_{j=1}^{N} |d\psi_j|^2.$$

So

$$N_{\Omega,k}(\lambda) \leqslant \sum_{j=1}^{N} N_{\Omega'_j,k}\Big(\lambda + \frac{1}{k}C(\psi)\Big).$$

Proof. – (A) Let $\mathcal{F}$ $\mathbb{C}$ the vector space generated by the collection of all the eigenfunctions quadratic forms $Q_{\Omega_j,k}$, $1 \leqslant j \leqslant N$, matching eigenvalues $\leqslant \lambda$. $\mathcal{F}$ has dimension

$$\dim \mathcal{F} = \sum_{j=1}^{N} N_{\Omega_j,k}(\lambda)$$

and for all $u \in \mathcal{F}$ was

$$Q_{\Omega,k}(u) = \sum_{j=1}^{N} Q_{\Omega_j,k}(u) \leqslant \sum_{j=1}^{N} \lambda \|u\|^2_{\Omega'_j} = \lambda \|u\|^2_{\Omega}.$$

The principle of minimax therefore shows that the eigenvalues of $Q_{\Omega,k}$ $\leqslant \dim \mathcal{F}$ of index are $\leqslant \lambda$, where the inequality (a).

(B) For $u \in W^1_0(\Omega, E^k)$ it comes

$$\sum_j |D_k(\psi_j u)|^2 = \sum_j \left|\psi_j D_k u + (d\psi_j)u\right|^2 = |D_k u|^2 + \sum_j |d\psi_j|^2 |u|^2$$

because $2\sum \psi_j d\psi_j = d(\sum \psi_j^2) = 0$. We obtain

$$\sum_{j=1}^{N} Q_{\Omega'_j,k}(\psi_j u) = Q_{\Omega,k}(u) + \int_\Omega \frac{1}{k}\sum_{j=1}^{N} |d\psi_j|^2 |u|^2 \, d\sigma \leqslant Q_{\Omega,k}(u) + \frac{1}{k} C(\psi) \|u\|^2_\Omega.$$

If each function is orthogonal $\psi_j u \in W^1_0(\Omega_j, E^k)$ Specific functions $Q_{\Omega_j,k}$ values $\leqslant \lambda + \frac{1}{k}C(\psi)$ own, we deduce successively

$$\begin{aligned} Q_{\Omega_j,k}(\psi_j u) &> \Big(\lambda + \frac{1}{k}C(\psi)\Big) \|\psi_j u\|^2_{\Omega_j}, \quad \text{si } \psi_j u \neq 0, \\ Q_{\Omega,k}(u) &> \lambda \|u\|^2_\Omega, \quad \text{si } u \neq 0. \end{aligned}$$

The principle of minimax 1.20 (b) results while $N_{\Omega,k}(\lambda)$ is increased by the number of linear equations imposed u to or at the

$$\sum_{j=1}^{N} N_{\Omega_j,k}\Big(\lambda + \frac{1}{k}C(\psi)\Big). \qquad \square$$

$W_1, \ldots, W_N$ be a recovery Ω by open card variety M . For $\varepsilon > 0$ we can find open $\Omega_i \subset \Omega'_j$ relatively compact in W_j, $1 \leqslant j \leqslant N$, such as

$$\Omega \supset \bigcup \Omega_j \ \ \text{(disjointe), et } \mathrm{Vol}(\Omega) = \sum \mathrm{Vol}(\Omega_j), \tag{2.7}$$

$$\overline{\Omega} \subset \bigcup \Omega'_j, \qquad \text{et } \sum \mathrm{Vol}(\overline{\Omega}'_j) \leqslant \mathrm{Vol}(\overline{\Omega}) + \varepsilon. \tag{2.8}$$

2.6 The proposal then brings the proof of Theorem 2.3 in case of open Ω_j and Ω'_j (be observed for it $\nu_B(V + \lambda)$ that the function is bounded and that the constant $C(\psi)$ is independent of k).

Ultimately, we can assume that $M = \mathbb{R}^n$, with a metric Riemann any g. As $M = \mathbb{R}^n$ is contractible, the E bundle is then trivial ; A is a vector potential D $B = dA$ connection and the corresponding magnetic field. We first demonstrate the following local version of Theorem 2.3.

Proposal 2.9. – *Let $a \in \mathbb{R}^n$ a fixed point, P_k and a suite of open cubic cobblestones as $P_k \ni a$. r_k Note the length of the side of P_k, and it is assumed that*

$$r_k \leqslant 1, \qquad \lim k^{\frac{1}{2}} r_k = +\infty, \qquad \lim k^{\frac{1}{4}} r_k = 0.$$

So when k approaches $+\infty$ was

$$\liminf \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) \geqslant \nu_{B(a)}(V(a) + \lambda),$$
$$\limsup \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) \leqslant \overline{\nu}_{B(a)}(V(a) + \lambda),$$

and for all $K \subset \mathbb{R}^n$ compact $N_{P_k,k}(\lambda)$ admits increase

$$N_{P_k,k}(\lambda) \leqslant C_K \Big(1 + r_k \sqrt{k\big(\lambda_+ + \max_K V_+\big)}\Big)^n$$

uniform with respect to a , since $P_k \subset K$.

Proof. – We will reduce to the theorem 1.6 in performing a dilation $\sqrt{k}$ report on P_k (This is why we had to assume $\lim k^{\frac{1}{2}} r_k = +\infty$). The following lemma how far the magnetic field deflects the field B $B(a)$ constant on each P_k .

Proof. – We will reduce to the theorem 1.6 in performing a dilation $\sqrt{k}$ report on P_k (This is why we had to assume $\lim k^{\frac{1}{2}} r_k = +\infty$). The following lemma how far the magnetic field deflects the field B $B(a)$ constant on each P_k .

Lemma 2.10. – *On each $\overline{P}_k$ pavement, we $\widetilde{A}_k$ can choose a potential of constant field such $B(a)$ that for all we have $x \in \overline{P}_k$*

$$|A_k(x) - A(x)| \leqslant C_1 r_k^2,$$

where C_1 is k (independent $\geqslant 0$ constant and independent of whether a a discloses a compact $K \subset \mathbb{R}^n$).

The $\mathcal{C}^\infty$ regularity B indeed leads to an increase

$$|B(a) - B(x)| \leqslant C_2 r_k, \qquad x \in \overline{P}_k.$$

A'_k be a potential $B(a) - B(x)$ $\overline{P}_k$ field on the cube, calculated from the usual homotopy formula for open stars. Was then

$$|A'_k(x)| \leqslant C_3 r_k^2,$$

and just ask $\widetilde{A}_k = A + A'_k$. □

Note $(x_1, \ldots, x_n)$ the standard coordinate $\mathbb{R}^n$. $(y_1, \ldots, y_n)$ is a linear coordinate system $x_1, \ldots, x_n$ in as either a base $(dy_1, \ldots, dy_n)$ orthonormal developed a g for the metric, and as in $B(a)$ this basis can be written in the diagonal form (2.1) :

$$B(a) = \sum_{j=1}^{s} B_j(a)\, dy_j \wedge dy_{j+s}.$$

Either $\widetilde{g}$ constant metric

$$\widetilde{g} \equiv g(a) = \sum_{j=1}^{n} dy_j^2.$$

Denote $D_k = d + ikA \wedge ?$, the $D_k = d + ikA_k \wedge ?$ $E^k_{|P_k}$ connections associated with potential A, $\widetilde{A}_k$, and $Q_k = Q_{P_k,k}$, $\widetilde{Q}_k$ quadratic forms respectively associated with D_k connections $\widetilde{D}_k$, the g metric, $\widetilde{g}$, and scalar potentials V, $\widetilde{V} \equiv V(a)$ (formula (1.3)).

Lemma 2.11. – *There exists a sequence ε_k tending to* 0 *(r_k dependent but independent of a if a discloses a compact $K \subset \mathbb{R}^n$) as if $\|\ \|_g$ and $\|\ \|_{\tilde{g}}$ designate L^2 global standards associated with metric g and $\widetilde{g}$, one has*

$$(1 - \varepsilon_k)\|u\|^2_{\tilde{g}} \leqslant \|u\|^2_g \leqslant (1 + \varepsilon_k)\|u\|^2_{\tilde{g}},$$
$$(1 - \varepsilon_k)\widetilde{Q}_k(u) - \varepsilon_k\|u\|^2_{\tilde{g}} \leqslant Q_k(u) \leqslant (1 + \varepsilon_k)\widetilde{Q}_k(u) + \varepsilon_k\|u\|^2_{\tilde{g}}$$

for all $u \in W^1_0(P_k)$.

On P_k, coaching was indeed :

$$(1 - C_4 r_k)\,\widetilde{g} \leqslant g \leqslant (1 + C_4 r_k)\,\widetilde{g},$$

and this gives the first double inequality 2.11. With the $A'_k = A_k - A$ notation, we deduce

$$\begin{aligned} Q_k(u) &= \int_{P_k} \Big(\frac{1}{k}|\widetilde{D}_k u - ikA'_k \wedge u|^2_g - V|u|^2\Big)\, d\sigma \\ &\leqslant (1 + C_5 r_k) \int_{P_k} \Big(\frac{1}{k}|\widetilde{D}_k u - ikA'_k \wedge u|^2_{\tilde{g}} - V(a)|u|^2\Big)\, d\widetilde{\sigma} + \eta_k\|u\|^2_{\tilde{g}} \end{aligned}$$

with $\eta_k = \sup_{P_k}|V - V(a)| + C_6 r_k$, quantity tends to 0 when k tends to $+\infty$. Using the inequality $(a+b)^2 \leqslant (1+\alpha)(a^2 + \alpha^{-1}b^2)$, Lemma 2.10 implies on the other hand

$$|\widetilde{D}_k u - ikA'_k \wedge u|^2_{\tilde{g}} \leqslant (1+\alpha)\Big[\,|\widetilde{D}_k u|^2_{\tilde{g}} + \alpha^{-1}C_1^2 k^2 r_k^4 |u|^2\Big].$$

$\alpha = \alpha_k = C_1\sqrt{k} r_k^2$ choose. Following α_k tends to 0 $\lim k^{\frac{1}{4}} r_k = 0$ according to the hypothesis, he comes

$$\frac{1}{k}|\widetilde{D}_k u - ikA'_k \wedge u|^2_{\tilde{g}} \leqslant (1+\alpha_k)\Big[\frac{1}{k}|D_k u|^2_{\tilde{g}} + \alpha_k|u|^2\Big].$$

The increase in Q_k follows. The lower bound is obtained as well with inequality $(a+b)^2 \geqslant (1-\alpha)(a^2 - \alpha^{-1}b^2)$. □

Lemma 2.11 brings the proof of Proposition 2.9 in case the mé g tric and magnetic field are constant B :

$$g = \sum_{j=1}^{n} dy_j^2, \qquad B = \sum_{j=1}^{n} B_j \, dy_j \wedge dy_{j+s}.$$

Presumably more $V \equiv 0$ performing translation $\lambda \mapsto \lambda + V(a)$. The only difficulty that remains for directly apply the theorem 1.6 is that the cubes P_k become generally parallelepipeds oblique in $(y_1, \ldots, y_n)$ coordinates; Angles between the different edges of each P_k and reports their lengths, however, remain framed by > 0 constant. To solve this problem, simply pave each cuboid P_k by $P_{k,\alpha}$ cubes with the edges are parallel to the coordinate axes $(y_1, \ldots, y_n)$. $\varepsilon \in \,]0,1[$ choose. For $\alpha \in \mathbb{Z}^n$ any, are $(P_{k,\alpha})$, $(P'_{k,\alpha})$ cubes open respective sides εr_k, $\varepsilon(1+\varepsilon)r_k$ and common center $\varepsilon r_k \alpha$. Suffice it to consider $P_{k,\alpha}$ cubes contained in P_k and cubic $P'_{k,\alpha}$ P_k meeting. Was then

$$P_k \supset \bigcup_{\alpha} P_{k,\alpha} \ \text{(disjointe)}, \quad \text{et} \quad \frac{\sum_\alpha \mathrm{Vol}(P_{k,\alpha})}{\mathrm{Vol}(P_k)} \geqslant 1 - C_7\varepsilon, \tag{2.12}$$

$$P_k \subset \bigcup_{\alpha} P'_{k,\alpha}, \quad \text{et} \quad \frac{\sum_\alpha \mathrm{Vol}(P'_{k,\alpha})}{\mathrm{Vol}(P_k)} \leqslant 1 + C_7\varepsilon, \tag{2.13}$$

where C_7 k is a constant independent of (and also a, if a discloses a compact). The number of cubic $P_{k,\alpha}$, $P'_{k,\alpha}$ contained in (2.12) or (2.13) is bounded by $C_8\varepsilon^{-n}$. As $P'_{k,\alpha}$ cubes overlap two by two on a length $\varepsilon^2 r_k$ when they are adjacent, one can construct a partition of unity $\sum \psi_{k,\alpha}^2 = 1$ on P_k with $\mathrm{Supp}\, \psi_{k,\alpha} \subset P'_{k,\alpha}$

$$\sup_{P_k} \sum_{\alpha} |d\psi_{k,\alpha}|^2 = C(\psi_k) \leqslant C_9(\varepsilon^2 r_k)^{-2}.$$

The hypothesis implies well $\lim k^{\frac{1}{2}} r_k = +\infty$ $\lim \frac{1}{k} C(\psi_k) = 0$, which allows to apply 2.6 (b). On $P_{k\alpha}$ cubes, we're $P'_{k,\alpha}$ now in the situation of Theorem 1.6 : after scaling report $\sqrt{k}$ the side of the cube homothetic $\sqrt{k}\, P_{k,\alpha}$ is well and tends $R_k = \varepsilon r_k \sqrt{k}$ $+\infty$ to hypothetically. The uniform surcharge $N_{P_k,k}(\lambda)$ follows from proposition 1.18 and the fact that all our constants were $C_1, \ldots, C_9$ uniforms. 2.9 The proposal is demonstrated. □

Proof of Theorem 2.3. – According to the note preceding the proposition 2.9, we can assume that $M = \mathbb{R}^n$ and that is a Ω open bounded $\mathbb{R}^n$. The idea of the reasoning is to combine Proposals 2.6 and 2.9 using a paving Ω by cubes $r_k = k^{-\frac{1}{3}}$ side. The actual application demands ceuvre Nevertheless, some care because of difficulties the possible non-uniformity of lim sup and lim inf.

Denote $\Pi_{k,\alpha}$, $\Pi'_{k,\alpha}$, $\alpha \in \mathbb{Z}^n$, open cubes respective sides

$$k^{-\frac{1}{3}}, \qquad k^{-\frac{1}{3}}(1 + k^{-\frac{1}{8}}) = k^{-\frac{1}{3}} + k^{-\frac{11}{24}}$$

and $k^{-\frac{1}{3}}\alpha$ common center. Is $I(k)$ (resp. $I'(k)$) the set of indices such that $\alpha \in \mathbb{Z}^n$ $\Pi_{k.\alpha} \subset \Omega$ (resp. $\overline{\Pi}'_{k,\alpha} \cap \overline{\Omega} \neq \emptyset$). As in the reasoning of the proposal 2.9, there is a partition of unity $\sum_{\alpha\in I'(k)} \psi_{k,\alpha}^2 = 1$ on Ω with and $\operatorname{Supp}\psi_{k,\alpha} \subset \Pi'_{k,\alpha}$

$$C(\psi_k) = \sup_{\Omega} \sum_{\alpha\in I'(k)} |d\psi_{k,\alpha}|^2 \leqslant C_{10} k^{\frac{11}{12}},$$

where $\lim \frac{1}{k} C(\psi_k) = 0.$ we set

$$\Omega_k = \bigcup_{\alpha\in I(k)} \Pi_{k,\alpha}, \qquad \Omega'_k = \bigcup_{\alpha\in I'(k)} \Pi'_{k,\alpha}$$

and considering all set to $\lambda \in \mathbb{R}$, functions on $\mathbb{R}^n$ defined

$$\begin{aligned} f_k &= k^{-\frac{n}{2}} \sum_{\alpha\in I(k)} N_{\Pi_{k,\alpha},k}(\lambda) \frac{1}{\operatorname{Vol}(\Pi_{k,\alpha})} \mathbb{1}_{\Pi_{k,\alpha}}, \\ f'_k &= k^{-\frac{n}{2}} \sum_{\alpha\in I'(k)} N_{\Pi'_{k,\alpha},k}\Big(\lambda + \frac{1}{k} C(\psi_k)\Big) \frac{1}{\operatorname{Vol}(\Pi_{k,\alpha})} \mathbb{1}_{\Pi_{k,\alpha}} \end{aligned}$$

where $\mathbb{1}_{\Pi_{k,\alpha}}$ denotes the characteristic function of $\Pi_{k,\alpha}$. 2.6 The proposal involves coaching

$$\int_{\mathbb{R}^n} f_k \, d\sigma \leqslant k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \int_{\mathbb{R}^n} f'_k \, d\sigma. \tag{2.14}$$

$x \in \mathbb{R}^n$ either a fixed point does not belong to all negligible

$$Z = \bigcup_{k\in\mathbb{N},\, \alpha\in\mathbb{Z}^n} \partial\Pi_{k,\alpha}.$$

Then there exists a sequence of single $\alpha(k) \in \mathbb{Z}^n$ clues as $x \in \Pi_{k,\alpha(k)}$. 2.9 The proposal applied following $P_k = \Pi_{k,\alpha(k)}$ of cubic (resp. $P'_k = \Pi'_{k,\alpha(k)}$) with $\operatorname{Vol}(P_k) \sim \operatorname{Vol} P'_k$ shows that the point suites

$$f_k(x) = \frac{k^{-\frac{n}{2}}}{\operatorname{Vol}(P_k)} N_{P_k,k}(\lambda) \mathbb{1}_{\Omega_k}(x), \qquad f'_k(x) = \frac{k^{-\frac{n}{2}}}{\operatorname{Vol}(P_k)} N_{P'_k,k}(\lambda) \mathbb{1}_{\Omega'_k}(x),$$

are as

$$\begin{cases} \liminf f_k(x) \;\geqslant \nu_{B(x)}(V(x)+\lambda)\, \mathbb{1}_{\Omega}(x) \\ \limsup f'_k(x) \leqslant \overline{\nu}_{B(x)}(V(x)+\lambda)\, \mathbb{1}_{\overline{\Omega}}(x). \end{cases} \tag{2.15}$$

The uniform mark of the proposal 2.9 causes the other the existence of C_{11} constant, independent of C_{12} k , x and λ such as

$$f_k(x) \leqslant f'_k(x) \leqslant C_{11}\big(1 + \sqrt{\lambda_+ + C_{12}}\,\big)^n.$$

Theorem 2.3 then follows from (2.14), (2.15) and Lemma Fatou. □

For applications with complex geometry, we need a slight generalization of Theorem 2.3. We give a fiber Hermitian F rank r and $\mathcal{C}^\infty$ class above of M, with a Hermitian connection ∇ and continuous sections S $\Lambda^1_R T^*X \otimes_R \mathrm{Hom}_{\mathbb{C}}(F,F)$ of the bundle and the V $\mathrm{Herm}(F)$ bundle of Hermitian endomorphisms of F. Is ∇_k the Hermitian connection on $E^k \otimes F$ induced the D and ∇ connections. To abbreviate notations, yet will designate S and V the endomorphisms $\mathrm{Id}_{E^k} \otimes S$ and operating $\mathrm{Id}_{E^k} \otimes V$ on$E^k \otimes F$. Given an open Ω relatively compact in M, consider the quadratic form

$$Q_{\Omega,k}(u) = \int_\Omega \Big(\frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu, u\rangle\Big)\, d\sigma,$$

where $u \in W^1_0(\Omega, E^k \otimes F)$. Are the eigenvalues of $V_1(x) \leqslant V_2(x) \leqslant \cdots \leqslant V_r(x)$ $V(x)$ at any point $x \in M$. We then have the following result.

Theorem 2.16. – *The counting function $N_{\Omega,k}(\lambda)$ eigenvalues of $Q_{\Omega,k}$ admits all $\lambda \in \mathbb{R}$ asymptotic estimates*

$$\liminf_{k\to+\infty} k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \geqslant \sum_{j=1}^{r} \int_\Omega \nu_B(V_j + \lambda)\, d\sigma,$$

$$\limsup_{k\to+\infty} k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \sum_{j=1}^{r} \int_\Omega \overline{\nu}_B(V_j + \lambda)\, d\sigma,$$

B where is the magnetic field associated with the connection D on$E$.

Proof. – The principle of location 2.6 is again valid in the this situation. So just to prove inequality 2.16 when Ω is quite small. Either a fixed point and $a \in M$ $(e_1, \ldots, e_r)$ a $\mathcal{C}^\infty$ orthonormal of F above a vicinity of W a as $(e_1(a), \ldots, e_r(a))$ is a clean base for $V(a)$. Write u as

$$u = \sum_{j=1}^{r} u_j \otimes e_j$$

where u_j is a section of E^k. For $\varepsilon > 0$ it is a $W'_\varepsilon \subset W$ neighborhood a on which

$$\sum_{j=1}^{r} (V_j(a) - \varepsilon)|u_j|^2 \leqslant \langle Vu, u\rangle \leqslant \sum_{j=1}^{r} (V_j(a) + \varepsilon)|u_j|^2$$

It was the other

$$\nabla_k u = \sum_{j=1}^{r} D_k u_j \otimes e_j + u_j \otimes \nabla e_j,$$

and $u_j \otimes \nabla e_j$ term can be absorbed into Su (Which brings us in fact in case the connection is flat ∇). coaching

$$(1 - k^{-\frac{1}{2}})|\nabla_k u|^2 + (1 - k^{\frac{1}{2}})|Su|^2 \leqslant |\nabla_k u + Su|^2 \leqslant (1 + k^{-\frac{1}{2}})|\nabla_k u|^2 + (1 + k^{\frac{1}{2}})|Su|^2$$

shows that the Su term modifies $Q_{\Omega,k}$ by a factor $1 \pm \varepsilon$ multiplicative and additive factor $\pm\varepsilon\|u\|^2$. For $\varepsilon > 0$, there So a W_ε neighborhood a and a whole $k_0(\varepsilon)$ such as

$$(1 - \varepsilon)\widetilde{Q}_{\Omega.k}(u) - \varepsilon\|u\|^2 \leqslant Q_{\Omega,k}(u) \leqslant (1 + \varepsilon)\widetilde{Q}_{\Omega,k}(u) + \varepsilon\|u\|^2$$

when $k \geqslant k_0(\varepsilon)$ and $\Omega \subset W_\varepsilon$, where $\widetilde{Q}_{\Omega,k}$ denotes the quadratic form

$$\widetilde{Q}_{\Omega.k}(u) = \sum_{j=1}^{r} \int_{\Omega} \left(\frac{1}{k} |D_k u_j|^2 - V_j(a)|u_j|^2 \right) d\sigma.$$

As $\widetilde{Q}_{\Omega,k}$ is a direct sum of forms r quadratic $\widetilde{Q}_{\Omega.k}$ the spectrum is the meeting (Counted with multiplicities) spectra of each term of the sum. Theorem 2.16 follows. □

3. Identity Bochner-Kodaira-Nakano in Hermitian geometry.

The purpose of the following paragraphs is to draw the consequences of the spectral distribution of Theorem 2.16 for the study of vector bundles d'' -cohomologie holomorphic Hermitian. To this end, we need to connect the Laplace antiholomorphic Δ'' to ope'rateur Schrödinger a connection adequate real. This is done by means of a formula Weitzenböck special type, known in geometry complex under the identity name of Bochner-Kodaira-Nakano.

X is a compact complex analytic variety of dimension n and F a Hermitian holomorphic vector bundle of rank above r of X . We know that there is a single connection Hermitian $D = D' + D''$ F on whose D'' component type $(0,1)$ coincides with the operator $\overline{\partial}$ of fiber (such a connection is called holomorphic). Either $c(F) = D^2 = D'D'' + D''D'$ the form of curvature F . X endow an arbitrary Hermitian me'trique type ω $(1,1)$ and $\mathcal{C}^\infty$ class. Space $\mathcal{C}^\infty_{p,q}(X,F)$ Class sections of the bundle $\mathcal{C}^\infty$ $\Lambda^{p,q}T^*X \otimes F$ is then provided with a natural prehilbertian structure. $\delta = \delta' + \delta''$ we note the formal deputy D considered differential operator $\mathcal{C}^\infty(X,F)$ and Λ Operator Assistant $L : u \mapsto \omega \wedge u$.

We will use the identity of Bochner-Kodaira-Nakano as General demonstrated in [6], although it actually might just like the fact Y.T. Siu [16], [17], the less formula accurate given by P. Griffiths. If A, B are differential operators on $\mathcal{C}^\infty(X,F)$, we define their anti- $[A,B]$ switch by the formula

$$[A,B] = AB - (-1)^{ab} BA$$

where a, b are the respective degrees of A and B . The Laplace-Beltrami operators Δ' of Δ'' and are then conventionally given by

$$\Delta' = [D', \delta'] = D'\delta' + \delta' D', \qquad \Delta'' = [D'', \delta'']$$

In the form of torsion $d'\omega$, we associate the operator to exterior multiplication on $u \mapsto d'\omega \wedge u$ $\mathcal{C}^\infty(X,F)$, type $(2,1)$ simply noted $d'\omega$, and τ operator $(1,0)$ defined type $\tau = [\Lambda, d'\omega]$. We ask finally

$$D'_\tau = D' + \tau, \qquad \delta'_\tau = (D'_\tau)^* = \delta' + \tau^*, \qquad \Delta'_\tau = [D'_\tau, \delta'_\tau].$$

then we have the following identity, for a demonstration of which the Readers should refer to [6].

Proposition 3.1. – *was* $\Delta'' = \Delta'_\tau + [ic(F),\Lambda] + T_\omega$ T_ω *which is the operator of order and type* 0 $(0,0)$ *defined by*

$$T_\omega = \Big[\Lambda, \Big[\Lambda, \frac{i}{2} d'd''\omega\Big]\Big] - [d'\omega, (d'\omega)^*].$$

According to the theory of Hodge-de Rham cohomology group $H^q(X,F)$ identifies with the space of $(0,q)$ Platforms Δ'' -harmoniques values in F . Is $u \in \mathcal{C}^\infty_{p.q}(X,F)$. 3.1 The proposal gives equal

$$(3.2)\quad \int_X |D''u|^2 + |\delta''u|^2 = \int_X \langle \Delta''u, u\rangle = \int_X |D'_\tau u|^2 + |\delta'_\tau u|^2 + \langle [ic(F),\Lambda]u,u\rangle + \langle T_\omega u, u\rangle,$$

where the integrals are calculated with respect to the volume element $d\sigma = \frac{\omega^n}{n!}$. In particular, if u is bidegree $(0,q)$ was $\delta'_\tau u = 0$ by reason of bidegree, where

$$(3.3)\qquad \int_X \langle \Delta''u, u\rangle = \int_X |D'_\tau u|^2 + \langle [ic(F),\Lambda]u,u\rangle + \langle T_\omega u, u\rangle.$$

One can also consider u as a Platform to (n,q) values in the bundle

$$\widetilde{F} := F \otimes \Lambda^n TX \ ;$$

we note $\widetilde{D} = \widetilde{D}' + \widetilde{D}''$ the Hermitian connection holomorphic $\widetilde{F}$ $\widetilde{u}$ and the canonical image of u in $\mathcal{C}^\infty_{n,q}(X,F)$.

Lemma 3.4. – *was commutative diagrams*

$$\begin{array}{ccc} \mathcal{C}^\infty_{0.q}(X,F) & \xrightarrow{D''} & \mathcal{C}^\infty_{0,q+1}(X,F) \\ \sim\downarrow & & \downarrow\sim \\ \mathcal{C}^\infty_{n,q}(X,\widetilde{F}) & \xrightarrow{\widetilde{D}''} & \mathcal{C}^\infty_{n,q+1}(X,\widetilde{F}), \end{array} \qquad \begin{array}{ccc} \mathcal{C}^\infty_{0,q}(X,F) & \xrightarrow{\Delta''} & \mathcal{C}^\infty_{0,q}(X,F) \\ \sim\downarrow & & \downarrow\sim \\ \mathcal{C}^\infty_{n,q}(X,\widetilde{F}) & \xrightarrow{\widetilde{\Delta}''} & \mathcal{C}^\infty_{n,q}(X,\widetilde{F}), \end{array}$$

where the vertical arrows are isometrics $u \mapsto \widetilde{u}$.

Proof. – The commutativity of the left diagram results from the fact that $\Lambda^n TX$ is a holomorphic bundle (We take care of the fact that the corresponding result for D' and $\widetilde{D}'$ is false). There is therefore a diagram commutative analogue for δ'' deputies $\widetilde{\delta}''$ and Δ'', $\widetilde{\Delta}''$. □

Lemma 3.4 and identity (3.2) give us

$$(3.5)\qquad \int_X \langle \Delta''u, u\rangle = \int_X \langle \widetilde{\Delta}''\widetilde{u}, \widetilde{u}\rangle = \int_X |\widetilde{\delta}'_\tau \widetilde{u}|^2 + \langle [ic(\widetilde{F}),\Lambda]\widetilde{u},\widetilde{u}\rangle + \langle T_\omega \widetilde{u}, \widetilde{u}\,\rangle.$$

We now turn slightly writing (3.3) and (3.5). The connection of hermitian holomorphic bundle $\Lambda^q T^*X$ induced on the conjugate fiber $\Lambda^{0,q}T^*X$ connection with the type of component $(1,0)$ coincides with the operatord' . We can deduce then a natural hermitian

connection ∇ Product bundle $\Lambda^{0,q}T^*X \otimes F$ tensor (it should be noted that this bundle Vector is not holomorphic in general if $q \neq 0$). Let ∇' and ∇'' components ∇ type $(1,0)$ and $(0,1)$.

Proposal 3.6. – *was*

$$\nabla' = D' : \mathcal{C}^\infty(\Lambda^{0,q}T^*X \otimes F) \to \mathcal{C}^\infty_{1,0}(\Lambda^{0,q}T^*X \otimes F),$$

and there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(X, \Lambda^{0.q}T^*X \otimes F) & \xrightarrow{\nabla''} & \mathcal{C}^\infty_{0,1}(X, \Lambda^{0.q}T^*X \otimes F) \\ \sim\downarrow & & \downarrow\Psi \\ \mathcal{C}^\infty_{n,q}(X, \widetilde{F}) & \xrightarrow{\widetilde{\delta}''} & \mathcal{C}^\infty_{n-1,q}(X, \widetilde{F}), \end{array}$$

where the vertical arrows are isometries, from that left given by $u \mapsto \widetilde{u}$.

Proof. – Equality comes from $\nabla' = D'$ $(1,0)$ that the type of component connection $\Lambda^{0,q}T^*X$ coincides with d'. In the diagram, you begin by defining the vertical arrow Ψ. Is

$$\{?|?\} : (\Lambda^{p_1,q_1}T^*X \otimes \widetilde{F}) \times (\Lambda^{p_2,q_2}T^*X \otimes \widetilde{F}) \longrightarrow \Lambda^{p_1+q_2,q_1+p_2}T^*X$$

sesquilinear canonical coupling induced by the metric on F fibers, and

$$* : \Lambda^{p,q}T^*X \otimes \widetilde{F} \longrightarrow \Lambda^{n-q,n-p}T^*X \otimes \widetilde{F}$$

the operator of Hodge-de Rham Poincaré defined

$$\{v| * w\} = \langle v, w\rangle \, d\sigma, \qquad v,\ w \in \Lambda^{p,q}T^*X \otimes \widetilde{F}.$$

We deduce composition by an isometric

$$\Psi_0 : \Lambda^{0,1}T^*X \otimes F \xrightarrow{\sim} \Lambda^{n,1}T^*X \otimes \widetilde{F} \xrightarrow{*} \Lambda^{n-1,0}T^*X \otimes \widetilde{F}$$

and Ψ arrow is obtained by definition tensoring $-i^{-n^2}\Psi_0$ by $\Lambda^{0,q}T^*X$. To demonstrate commutative, it is first assumed $q = 0$. Either $u \in \mathcal{C}^\infty(F)$. Was conventionally

$$\widetilde{\delta}'\widetilde{u} = - * \widetilde{D}'' * \widetilde{u},$$

$\widetilde{u} \in \mathcal{C}^\infty_{n,0}(X, F)$ and as it comes $*\widetilde{u} = i^{-n^2}\widetilde{u}$, where

$$\widetilde{\delta}'\widetilde{u} = -i^{-n^2} * D''\widetilde{u} = -i^{-n^2} * \sim D''u = -i^{-n^2}\Psi_0(D''u) = \Psi(\nabla''u).$$

In case any q is, simply trivialize $\Lambda^{0,q}T^*X$ adjacent a x arbitrary point in choosing an orthonormal $(e_1, \ldots, e_N)$ This bundle as $\nabla e_1(x) = \cdots = \nabla e_N(x) = 0$. □

Now consider the morphisms of bundles

$$S' : \Lambda^{0,q}T^*X \otimes F \to \Lambda^{1,0}T^*X \otimes \Lambda^{0,q}T^*X \otimes F$$
$$S'' : \Lambda^{0,q}T^*X \otimes F \to \Lambda^{0,1}T^*X \otimes \Lambda^{0,q}T^*X \otimes F$$

$S' = \tau = [\Lambda, d'\omega]$ where, and where is the statement by S'' isometrics and $\sim \Psi$ the morphism

$$\tau^* = [(d'\omega)^*, L] : \Lambda^{n,q}T^*X \otimes \widetilde{F} \to \Lambda^{n-1.q}T^*X \otimes \widetilde{F}.$$

According to Proposition 3.6, we have

$$|D'_\tau u| = |\nabla' u + S'u|, \qquad |\widetilde{\delta}'_\tau \widetilde{u}| = |\nabla'' u + S''u|.$$

If we put $S = S' \oplus S''$, identities (3.3) and (3.5) imply by addition

$$\begin{aligned} 2\int_X \langle \Delta'' u, u\rangle &= \int_x |\nabla u + Su|^2 + \int_X \langle [ic(F), \Lambda]u, u\rangle \\ &\quad + \int_X \langle [ic(\widetilde{F}), \Lambda]\widetilde{u}, \widetilde{u}\rangle + \langle T_\omega u, u\rangle + \langle T_\omega \widetilde{u}, \widetilde{u}\rangle \end{aligned} \tag{3.7}$$

for all $u \in \mathcal{C}^\infty_{0,q}(X, F)$.

Now let E a Hermitian holomorphic bundle of rank 1 above of X. For every integer k, D_k and we note the ∇_k Natural Hermitian connections on bundles $F_k = E^k \otimes F$ and $\Lambda^{0,q}T^*X \otimes F_k$, and $\Delta''_k = [D''_k, \delta''_k]$ pose. The curvature of F_k (Resp. $\widetilde{F}_k$) is given by

$$c(F_k) = c(F) + kc(E) \otimes \mathrm{Id}_F, \quad \text{resp.} \quad c(\widetilde{F}_k) = c(\widetilde{F}) + kc(E) \otimes \mathrm{Id}_{\widetilde{F}}. \tag{3.8}$$

Recall, though it is unnecessary to read that

$$c(\widetilde{F}) = c(F) + c(\Lambda^n TX) \otimes \mathrm{Id}_F = c(F) + \mathrm{Ricci}(\omega) \otimes \mathrm{Id}_F.$$

So we will need to assess $[ic(E), \Lambda]$ terms. For everything Point $x \in X$, are $\alpha_1(x)$, $\alpha_2(x), \ldots, \alpha_n(x)$ $ic(E)(x)$ eigenvalues with respect to the metric Hermitian ω on X. There is therefore a system of $(z_1, \ldots, z_n)$ local coordinates centered x $(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$ as an orthonormal basis of $T_X X$, and such that

$$\omega(x) = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j,$$

$$ic(E)(x) = \frac{i}{2}\sum_{j=1}^n \alpha_j(x)\, dz_j \wedge d\overline{z}_j.$$

$(e_1, \ldots, e_r)$ be an orthonormal fiber $E^k_x \otimes F_x$. For $v \in \Lambda^{p.q}T^*X \otimes F_k$, we can write

$$v = \sum_{|I|=p, |J|=q,\, \ell} v_{I,J,\ell}\, dz_I \wedge d\overline{z}_J \otimes e_\ell, \qquad |v|^2 = 2^{p+q}\sum_{I,J,\ell} |v_{I,J,\ell}|^2$$

An elementary calculation, explained for example in [6], gives the formula

$$\langle [ic(E), \Lambda]v, v\rangle = 2^{p+q}\sum_{I,J,\ell} (\alpha_I + \alpha_J - \sum_{j=1}^n \alpha_j)|v_{I,J,\ell}|^2 \tag{3.9}$$

with $\alpha_I = \sum_{j\in I} \alpha_j$. Is $u \in \Lambda^{0,q}T^*X \otimes F_k$. Let

$$u = \sum_{J,\ell} u_{J,\ell}\, d\overline{z}_J \otimes e_\ell.$$

By (3.9), it comes

$$\langle [ic(E),\Lambda]u,u\rangle = 2^q \sum_{J,\ell} -\alpha_{\complement J}|u_{J,\ell}|^2,$$
$$\langle [ic(E\rangle,\Lambda]\widetilde{u},\widetilde{u}\rangle = 2^q \sum_{J,\ell} \alpha_J|u_{J,\ell}|^2.$$

V is the hermitian endomorphism $\Lambda^{0,q}T^*X \otimes F_k$ defined by

$$\langle Vu,u\rangle = -\langle [ic(E),\Lambda]u,u\rangle - \langle [ic(E),\Lambda]\widetilde{u},\widetilde{u}\rangle = 2^q \sum_{J,\ell}(\alpha_{\complement J} - \alpha_J)|u_{J,\ell}|^2. \tag{3.10}$$

The eigenvalues of V are coefficients $\alpha_{\complement J} - \alpha_J$, counted with multiplicity $r = \mathrm{rank}(F)$. Or finally Θ the endomorphism defined by Hermitian

$$\langle \Theta u,u\rangle = \langle [ic(F),\Lambda]u,u\rangle + \langle [ic(\widetilde{F}),\Lambda]\widetilde{u},\widetilde{u}\rangle + \langle T_\omega u,u\rangle + \langle T_\omega \widetilde{u},\widetilde{u}\rangle. \tag{3.11}$$

Identities (3.7-11) then involve

$$\frac{2}{k}\int_X \langle \Delta''_k u,u\rangle = \int_X \frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu,u\rangle + \frac{1}{k}\langle \Theta u,u\rangle \tag{3.12}$$

where S operators V, Θ act only on the $\Lambda^{0,q}T^*X \otimes F$ component $\Lambda^{0,q}T^*X \otimes F_k$. So we will be able use Theorem 2.16 to determine the spectral distribution asymptotic Δ''_k because the term $\frac{1}{k}\langle \Theta u,u\rangle$ approaches 0 norm.

$h^q_k(\lambda)$ is the number of eigenvalues $\leqslant k\lambda$ Δ''_k of operating on $\mathcal{C}^\infty_{0,q}(E^k \otimes F)$. The magnetic field is here given by B

$$B = -ic(E) = -\sum_{j=1}^n \alpha_j\, dx_j \wedge dy_j, \qquad z_j = x_j + iy_j. \tag{3.13}$$

Given that $\dim_{\mathbb{R}} X = 2n$, Theorem 2.16 is transcribed as follows.

Theorem 3.14. – *There is a set countable $\mathcal{D}$ such that for all and $q = 0,1,\ldots,n$ all we have $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$*

$$h^q_k(\lambda) = rk^n \sum_{|J|=q} \int_X \nu_B(2\lambda + \alpha_{\complement J} - \alpha_J)\, d\sigma + o(k^n)$$

when k approaches $+\infty$.

4. Complex Witten and Morse inequalities.

E. Witten [18], [19] has recently introduced a new method analytics to demonstrate the Morse inequalities de Rham cohomology. We adapt his method here the study of d'' -cohomologie. The main difference lies in the fact that the magnetic field is always zero in the case of de Rham cohomology (we indeed $d^2 = 0$!) and it is the electric field that acts alone in this case.

With the notation of §3 or $\mathcal{H}^q_k(\lambda) \subset \mathcal{C}^\infty_{0,q}(X, E^k \otimes F)$ the direct sum of eigenspaces of Δ''_k attached to values $\leqslant k\lambda$. $\mathcal{H}^q_k(\lambda)$ is a vector space of finite dimension

$$h^q_k(\lambda) = \dim_{\mathbb{C}} \mathcal{H}^q_k(\lambda).$$

Hodge theory gives an isomorphism

$$H^q(X, E^k \otimes F) \simeq \mathcal{H}^q_k(0).$$

We ask to shorten

$$h^q_k = \dim H^q(X, E^k \otimes F) = h^q_k(0).$$

Proposition 4.1.— *is $\mathcal{H}^\bullet_k(\lambda)$ a complex of sub-Dolbeault complex*

$$D''_k : \mathcal{C}^\infty_{0,\bullet}(X, E^k \otimes F).$$

In addition, the inclusion $\mathcal{H}^\bullet_k(\lambda) \subset \mathcal{C}^\infty_{0,\bullet}(X, E^k \otimes F)$ and the orthogonal projection

$$P_\lambda : \mathcal{C}^\infty_{0,\bullet}(X, E^k \otimes F) \to \mathcal{H}^\bullet_k(\lambda)$$

induce in cohomology isomorphism inverse of one another.

Proof. – That is a $\mathcal{H}^\bullet_k(\lambda)$ sub-complex $\mathcal{C}^\infty_{0,\bullet}(X, E^k \otimes F)$ comes from D''_k operators switching property and Δ''_k . now let

$$G = \int_{\lambda>0} \frac{1}{\lambda} dP_1$$

the Green operator of the Laplace Δ''_k. As $[P_\lambda, \Delta''_k] = 0$ was the $[G, \Delta''_k] = 0$ relations

$$\Delta''_k G + P_0 = \mathrm{Id}\,.$$

Moreover, $[P_\lambda, D''_k] = [G, D''_k] = 0$. We therefore concluded

$$\begin{aligned}\mathrm{Id} - P_\lambda &= \Delta''_k G(\mathrm{Id} - P_\lambda) + P_0(\mathrm{Id} - P_\lambda) = \Delta''_k G(\mathrm{Id} - P_\lambda)\\ &= D''_k\big(\delta''_k G(\mathrm{Id} - P_\lambda)\big) + \big(\delta''_k G(\mathrm{Id} - P_\lambda)\big) D''_k,\end{aligned}$$

so that the operator is a homotopy $\delta''_k G(Id - P_\lambda)$ between Id and P_λ. □

now uses a conventional single lemma of homological algebra.

Lemma 4.2.— *Let*

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n \longrightarrow 0$$

a complex of vector spaces of finite dimensions c^0, c^1, ... , c^n on a body $\mathbb{K}$. *Either* $h^q = \dim_{\mathbb{K}} H^q(C^\bullet)$. *Then we have the following inequalities:*

(a) *Morse inequalities* : $h^q \leqslant c^q$, $0 \leqslant q \leqslant n$.

(b) *Equality of Euler-Poincaré characteristics* $\chi(H^\cdot(C^\cdot)) = \chi(C^\cdot)$:

$$h^0 - h^1 + \cdots + (-1)^n h^n = c^0 - c^1 + \cdots + (-1)^n c^n.$$

(c) *Strong Morse inequalities* : *for all* q*,* $0 \leqslant q \leqslant n$,

$$h^q - h^{q-1} + \cdots + (-1)^q h^0 \leqslant c^q - c^{q-1} + \cdots + (-1)^q c^0.$$

Proof. – If $Z^q = \operatorname{Ker} d^q$ and $B^q = \operatorname{Im} d^{q-1}$ have to z^q and b^q dimensions, equality (b) is a matter for formulas

$$c^q = z^q + b^{q+1}, \qquad h^q = z^q - b^q,$$

while (c) is the result of (b) applied to the complex

$$0 \to C^0 \to C^1 \to \cdots \to C^{q-1} \to Z^q \to 0. \qquad \square$$

If F is a holomorphic vector bundle on X, we define the Euler-Poincare

$$\chi(X, F) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, F).$$

Combining Proposition 4.1 and Lemma 4.2, we get for all $\lambda \geqslant 0$ and everything q , $0 \leqslant q \leqslant n$, inequality

$$h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant h_k^q(\lambda) - h_k^{q-1}(\lambda) + \cdots + (-1)^q h_k^0(\lambda).$$

Now evaluate $h_k^q(\lambda)$ by Theorem 3.14 and do tend to $\lambda \in \mathbb{R} \setminus \mathcal{D}$ 0 by > 0 values. he follows:

Corollary 4.3. – *was the asymptotic inequality*

(a) $h_k^q \leqslant k^n I^q + o(k^n)$,

(b) $\chi(X, E^k \otimes F) = k^n(I^0 - I^1 + \cdots + (-1)^n I^n) + o(k^n)$,

(c) $h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant k^n(I^q - I^{q-1} + \cdots + (-1)^q I^0) + o(k^n)$,

I^q *which means bending the integral*

$$I^q = r \sum_{|J|=q} \int_X \overline{\nu}_B(\alpha_{\complement J} - \alpha_J) d\sigma.$$

According to (3.13), the modules of the eigenvalues of the magnetic field B are $|\alpha_j|$, $1 \leqslant j \leqslant n$. For any point $x \in X$, Let us arranke these values so that in

$$|\alpha_1 \geqslant |\alpha_2| \geqslant \cdots \geqslant |\alpha_s| > 0 = |\alpha_{s+1}| = \cdots = |\alpha_n|, \qquad s = s(x).$$

The formula (1.5) gives

$$\overline{\nu}_B(a_{\complement J} - \alpha_J) = \frac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)}|\alpha_1 \ldots \alpha_s| \sum_{(p_1,\ldots,p_s)} \Big\{\alpha_{\complement J} - \alpha_J - \sum(2p_j+1)|\alpha_j|\Big\}_+^{n-s}$$

with the notation $\{\lambda\}_+^0 = 0$ and if $\lambda < 0$ $\{\lambda\}_+^0 = 1$ if $\lambda \geqslant 0$. As the amount

$$\alpha_{\complement J} - \alpha_J - \sum(2p_j+1)|\alpha_j|$$

always $\leqslant 0$, $\overline{\nu}_B(\alpha_{\complement J} - \alpha_J)$ can only be nonzero if $s = n$. In this last case if$\alpha_{\complement J} - \alpha_J - \sum(2p_j+1)|\alpha_j| = 0$ and only if $p_1 = \cdots = p_n = 0$ and for $\alpha_j < 0$ $j \in J$, $\alpha_j > 0$ for $j \in \complement J$. This causes the $ic(E)$ form is non-degenerate index q . For $x \in X(q)$ (cf. notations of the introduction) and $|J| = q$ was therefore

$$\overline{\nu}_B(\alpha_{\complement J} - \alpha_J) = (2\pi)^{-n}|\alpha_1 \ldots \alpha_n| > 0$$

if J is the multi-index and $J(x) = \{j\,;\ \alpha_j(x) < 0\}$ $\overline{\nu}_B(\alpha_{\complement J} - \alpha_J) = 0$ if $J \neq J(x)$. It follows

$$I^q = r\int_{X(q)} (2\pi)^{-n}(-1)^q \alpha_1 \ldots \alpha_n \, d\sigma = \frac{r}{n!}\int_{X(q)} (-1)^q \Big(\frac{i}{2\pi}c(E)\Big)^n.$$

The fundamental theorem 0.1 is then only a reformulation of Corollary 4.3. The above reasoning shows that the harmonics forms of $H^q(X, E^k \otimes F)$ focus on asymptotically $X(q)$, and that each point $X(q)$ their direction tends to align with the q -sous-space TX matching the negative part of $ic(E)$. Moreover, only the intrinsic value of minimum energy $p_1 = \cdots = p_n = 0$ of the harmonic oscillator operates for these forms. For $q = 1$, Morse inequality strong 4.3 (c) is written

$$h_k^1 - h_k^0 \leqslant k^n(I^1 - I^0) + o(k^n),$$

where in particular an asymptotic lower bound for the number of sections holomorphic bundle of $E^k \otimes F$.

Theorem 4.4. – *was*

$$\dim H^0(X, E^k \otimes F) \geqslant r\frac{k^n}{n!}\int_{X(\leqslant 1)} \Big(\frac{i}{2\pi}c(E)\Big)^n - o(k^n).$$

More generally, the addition of inequality 4.3 (c) for $q+1$ and $q-2$ clues leads

$$h_k^{q+1} - h_k^q + h_k^{q-1} \leqslant k^n(I^{q+1} - I^q + I^{q-1}) + o(k^n),$$

hence the lower bound

$$(4.5)\qquad \dim H^q(X, E^k \otimes F) \geqslant r\frac{k^n}{n!} \sum_{j=0,\pm 1} (-1)^q \int_{X(q+j)} \Big(\frac{i}{2\pi} c(E)\Big)^n - o(k^n).$$

5. Characterization of varieties of Moišezon.

Either X $\mathbb{C}$ a variety of compact connected -analytique n dimension. Called algebraic dimension X denoted $a(X)$, the transcendence degree over $\mathbb{C}$ body of $K(X)$ meromorphic functions on X . According to a well-known theorem of Siegel [15], the dimension algebraic X always satisfies the inequality $0 \leqslant a(X) \leqslant n$. $a(X) = n$ when we say that space is X Moišezon. As we shall see, the algebraic dimension X asymptotically imposes strong constraints on the size of the sections of spaces of holomorphic vector bundle.

Theorem 5.1. – *Let a size algebraic X, F a bundle Vector rank and E r holomorphic linear bundle on X. So there is a constant that depends $C_E \geqslant 0$ as E as*

$$\dim H^0(X, E^k \otimes F) \leqslant C_E r k^a + o(k^a).$$

Proof. – We essentially resuming arguments Y.T. Siu [16]. $\{W_\ell\}$ be a recovery X by open $W_\ell \subset \mathbb{C}^n$ of coordinates and $B_j = B(a_j, R_j)$, $1 \leqslant j \leqslant m$, a family of relatively compact balls in open W_ℓ such as concentric balls $B'_j = B(a_j, \frac{1}{7}R_j)$ cover X . Endow E, of F Hermitian metric, and the weight is $\exp(-\varphi_j)$ representing metric E in a trivialization of E near $\overline{B}_j$.

Let then $s \in H^0(X, E^k \otimes F)$ a holomorphic section which vanishes p to order a $x_j \in B'_j$ points. inclusions

$$B'_j \subset B(x_j, \frac{2}{7}R_j) \subset B(x_j, \frac{6}{7}R_j) \subset B_j$$

and Lemma Schwarz applied to two intermediate balls cause inequality

$$(5.2)\qquad \sup_{B'_j} |s| \leqslant \exp(Ak + C_F) 3^{-p} \sup_{B_j} |s|,$$

where $A = \max_{1 \leqslant j \leqslant m} \operatorname{diam} \varphi_j(B_j)$ depends only of E , and where C_F is a constant which depends on $\geqslant 0$ metric F .

$\rho \leqslant r = \operatorname{rank}(F)$ is the maximum for the size of $x \in X$ subspace of the fiber F_x generated by the vectors $s(x)$ when s described $\bigcup_{k\in\mathbb{N}} H^0(X, E^k \otimes F)$. If $\rho = 0$ then $H^0(X, E^k \otimes F) = 0$ for all k . Now distinguish two following cases $\rho = 1$ or $\rho > 1$.

(A) *Suppose* $\rho = 1$.

Either $h_k = \dim H^0(X, E^k \otimes F)$, supposed > 0 . Assuming $\rho = 1$, global sections of $E^k \otimes F$ define a holomorphic

$$\Phi_k : X \smallsetminus Z_k \to \mathbb{P}^{h_k - 1}(\mathbb{C})$$

where $Z_k \subset X$ is the analytical subset of their zeros common. d is the maximum rank of the differential on Φ'_k $X \smallsetminus Z_k$. We $d \leqslant a$ necessarily, otherwise the body of rational

$\mathbb{P}^{h_k-1}(\mathbb{C})$ to induce a function field meromorphic on$X \geqslant d > a$ degree of transcendence, which is absurd. Choose for any $j = 1, \ldots, m$ $x_j \in B'_j \setminus Z_k$ a point that is of maximum rank $\Phi'_k = d$ in x_j and either $s_0 \in H^0(X, E^k \otimes F)$ a section that does not cancel any Point x_j . For $s \in H^0(X, E^k \otimes F)$, the quotient s/s_0 is defined as meromorphic function on X , and more s/s_0 is a holomorphic function in the neighborhood of x_j , constant along the fibers of Φ_k . As is a Φ_k subimmersion in the vicinity of each point x_j , one can choose a submanifold M_j size d through and x_j $\Phi_k^{-1}(\Phi_k(x_j))$ transverse to the fiber. The section s will cancel the order p each point x_j, $1 \leqslant j \leqslant m$, if and only if the partial derivatives $< p$ order to s/s_0 along M_j vanish at x_j . This corresponds to a total cancellation

$$m\binom{p+d-1}{d}$$

Derived. If we choose $p = [Ak + C_F] + 1$ then the inequality (5.2) drives

$$\sup_X |s| \leqslant \left(\frac{e}{3}\right)^p \sup_X |s|,$$

where $s = 0$. As $d \leqslant a$, we get a result

$$\dim H^0(X, E^k \otimes F) \leqslant m\binom{p+a-1}{a} \leqslant C_E k^a + o(k^a)$$

$C_E = mA^a/a!$ with .

(B) *Suppose* $\rho > 1$.

There is then the $s_t \in H^0(X, E^{k_t} \otimes F)$ sections, $1 \leqslant t \leqslant \rho$, and a $x_0 \in X$ item such as vectors $s_1(x_0), \ldots, s_\rho(x_0)$ are linearly independent. By construction, for any and all $k \in \mathbb{N}$ $s \in H^0(X, E^k \otimes F)$ section, the right is $\mathbb{C} \cdot s(x)$ contained in the subspace spanned by $(s_1(x), \ldots, s_\rho(x))$, except perhaps above analytic subset $\{x \in X; s_1 \wedge \ldots \wedge s_\rho(x)\} = 0$. We therefore have an injective morphism

$$H^0(X, E^k \otimes F) \to \bigoplus_{1 \leqslant t \leqslant \rho} H^0(X, E^{k+k_{\hat{t}}} \otimes \Lambda^p F)$$

where $k_{\hat{t}} = (k_1 + \cdots + k_\rho) - k_t$, the index component t is given by the morphism $s \to s_1 \wedge \cdots \wedge \widehat{s}_t \wedge \cdots \wedge s_\rho \wedge s$. The image of $H^0(X, E^k \otimes F)$ on each component is formed collinear sections of almost any point to $s_1 \wedge \cdots \wedge s_\rho$. So we end up in a situation similar to that of (a), which is replaced F by $E^{k_{\hat{t}}} \otimes \Lambda^\rho F$; by following :

$$\dim H^0(X, E^k \otimes F) \leqslant C_E \rho k^a + o(k^a), \qquad \rho \leqslant r. \qquad \square$$

Choose especially F the trivial bundle $X \times \mathbb{C}$. Comparing Theorems 4.4 and 5.1, we obtain the characterization Next geometric varieties Moišezon.

Theorem 5.2. – *For a variety* $\mathbb{C}$ *-analytique* n *dimension* X *compact connected either Moišezon, sufficient that there exists a line bundle holomorphic Hermitian* E *above* X *as*

$$\int_{X(\leqslant 1)} (ic(E))^n > 0. \qquad \square$$

This theorem in turn leads to Theorem 0.8 as 0.8 (c) $\Rightarrow$ 0.8 (b) $\Rightarrow$ 0.8 (a). We improves results Y.T. Siu [17], [18] and thus in particular a new demonstration of the Guess Grauert-Riemenschneider. [10]

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