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MAGNETIC FIELDS AND MORSE INEQUALITIES FOR d"-COHOMOLOGY

by Jean-Pierre DEMAILLY

0. Introduction.

Let X be a compact complex analytic manifold of dimension n, F a holomorphic fiber bundle of rank r, and E a holomorphic line bundle with a hermitian structure of class $\mathcal{C}^\infty$ on X. Let $D = D' + D''$ be the canonical connection of E and $c(E) = D^2 = D'D'' + D''D'$ be the curvature tensor of this connection. Let us denote by $X(q)$, $0 \leqslant q \leqslant n$, the open set of points of $x \in X$ that are of index q, i.e. the points at which the $(1,1)$ curvature form $ic(E)(x)$ has exactly q negative eigenvalues and $n - q$ positive ones. We also set

$$X(\leqslant q) = X(0) \cup X(1) \cup \ldots \cup X(q).$$

We then prove the following Morse inequalities, which bound the dimension of the cohomology groups $H^q(X, E^k \otimes F)$ in terms of integral invariants of the curvature of E.

Theorem 0.1. – *When k tends towards $+\infty$, one has for all $q = 0, 1, \ldots, n$ the following asymptotic inequalities.*

(a) *Morse Inequalities* :

$$\dim H^q(X, E^k \otimes F) \leqslant r\frac{k^n}{n!}\int_{X(q)} (-1)^q \Big(\frac{i}{2\pi}c(E)\Big)^n + o(k^n).$$

(b) *Strong Morse inequalities* :

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(X, E^k \otimes F) \leqslant r\frac{k^n}{n!}\int_{X(\leqslant q)} (-1)^q \Big(\frac{i}{2\pi}c(E)\Big)^n + o(k^n).$$

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(c) *Asymptotic Riemann-Roch Formula* :

$$\sum_{q=0}^{n}(-1)^q \dim H^q(X, E^k \otimes F) = r\frac{k^n}{n!}\int_X \left(\frac{i}{2\pi}c(E)\right)^n + o(k^n).$$

Estimates 0.1 (a), (b) are new to our knowledge, even in the case of projective varieties. Asymptotic equality 0.1 (c), quand à elle, est une version affaiblie of the theorem of Hirzebruch-Riemann-Roch, which is itself a special case of the Atiyah-Singer index theorem [1]. The latter theorem allows indeed to express the characteristic of Euler-Poincaré

$$\chi(X, E^k \otimes F) = \sum_{q=0}^{n}(-1)^q \dim H^q(X, E^k \otimes F)$$

in the form

$$\chi(X, E^k \otimes F) = r\frac{k^n}{n!}c_1(E)^n + P_{n-1}(k) \; ; \tag{0.2}$$

$P_{n-1}(k) \in \mathbb{Q}[k]$ here designates a polynomial of degree $\leqslant n-1$ and $c_1(E) \in H^2(X, \mathbb{Z})$ is the first class of Chern de E, represented in De Rham's cohomology by the closed $(1,1)$-form $\frac{i}{2\pi}c(E)$ (cf. for example [16]). It can be observed that the numerical constant of the inequality 0.1 (a) is optimal, as shown by the example of the total tensor product fiber Xq above $X = (\mathbb{P}^1(\mathbb{C}))^n$. For this fiber, we have indeed $X(q) = X$ and

$$\dim H^q(X, E^k) = (k+1)^{n-q}(k-1)^q, k \geqslant 1,$$
$$\int_X \left(\frac{i}{2\pi}c(E)\right)^n = (-1)^q n!\,.$$

The existence of a 0.1(a) surcharge was speculated by Y. T. Siu, who has successively demonstrated the case where $ic(E)$ is > 0 in the complement of a particular set of measure zero [16], then the case where $ic(E) \geqslant 0$ over X [17]. We have borrowed from Siu a part of the techniques used here, especially at§3 and§5. The proof of theorem 0.1 is based on the method analytical approach recently introduced by E. Witten [18], [19]. This method allows (among other things) to redemonstrate the classical Morse inequalities $b_q \leqslant m_q$ on a compact differentiable variety M , where b_q designates the q-th number of Betti and m_q the number of critical points of index q of any Morse function on M . In our situation, the role of the Morse function is held by the choice of the Hermitian metric on E . We provide other part X and F of arbitrary Hermitian metrics, which will intervene only in the $o(k^n)$ terms of the estimates finals. Given a real $\lambda \geqslant 0$, one can considers the $\mathcal{H}^\bullet_k(\lambda)$ sub-complex of the Dolbeault $\mathcal{C}^\infty_{0,\bullet}(X, E^k \otimes F)$ des $(0,q)$ -forms of class $\mathcal{C}^\infty$ on X with values in $E^k \otimes F$, generated by the Laplacian's own functions antiholomorph Δ'' whose eigenvalues are $\leqslant k\lambda$. The cohomology groups of the $\mathcal{H}^\bullet_k(\lambda)$ complex are then isomorphic to the $H^q(X, E^k \otimes F)$ groups (proposal 4.1), so that it is enough to know how to limit the dimension of the spaces $\mathcal{H}^q_k(\lambda)$. For this, two tools are essentially used. The first tool consists of a Weitzenböck type formula

$$\frac{2}{k}\int_X \langle \Delta'' u, u\rangle = \int_X \frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu, u\rangle + \frac{1}{k}\langle \Theta u, u\rangle \tag{0.3}$$

demonstrated at§3, and derived from the identity of Bochner- Kodaira-Nakano non-Kählerian [6]. ∇_k designates here the natural hermitian connection on the $\Lambda^{0,q}T^*X \otimes E^k \otimes F$ fiber, $\mathbb{C}$y is a linear potential of order 0 related to the curvature of the fiber na , finally S and Θ are operators of order 0 from the twist of the Hermitian metric on nf and curvature of F . The study of the spectrum of ng is is thus brought back to the study of the spectrum of the associated self-adhesive operator $\nabla_k^*\nabla_k$ to the actual connection ∇_k . The second tool fundamental consists precisely of a fundamental theorem very general spectrum relative to the operators of the type nj . Let nk be a riemannian variety nl of real dimension nm , nn a fibrous in complex straight lines above X , provided with a connection hermitienne ∇ . If nq designates the connection induced by ∇ on E^k , we then study the spectrum of the quadratic form

$$Q_k(u) = \int_\Omega \Big(\frac{1}{k}|\nabla_k u|^2 - V|u|^2\Big) d\sigma, \qquad u \in L^2(\Omega, E^k) \tag{0.4}$$

for the Dirichlet problem, where Ω is a relatively open-ended compact in M , and where V is a continuous scalar potential on M . From a physical point of view, this is equivalent to studying the spectrum of the operator of Schrödinger $\frac{1}{k}(\nabla_k^*\nabla_k - kV)$ associated with the electric field kV and the magnetic field kB , where $B = -i\nabla^2$ is none other than the 2 -form of curvature of the connection Fd . It is in the presence of this magnetic field that our main contribution to the method of E. Witten [18], [19] (in the case of the cohomology of De Rham the magnetic field is always zero since Fe).

At any point $x \in X$, that is $2s = 2s(x) \leqslant n$ the rank of $B(x)$ and $B_1(x) \geqslant \ldots \geqslant B_s(x) > 0$ the non-zero eigenvalue modules of the associated antisymmetric endomorphism. We define a function Fj of couple Fk , continues at left in λ , by posing

$$v_B(\lambda) = \frac{2^{s-n}\pi^{-\frac{n}{2}}}{\Gamma(\frac{n}{2}-s+1)} B_1 \ldots B_s \sum_{(p_1,\ldots,p_s)\in\mathbb{N}^s} \big[\lambda - \sum(2p_j+1)B_j\big]_+^{\frac{n}{2}-s} \tag{0.5}$$

with the convention Fn . Finally, if Fo denote the eigenvalues of Q_k (counted with multiplicity), we consider the enumeration function Fq , Fr .

Theorem 0.6. – *If $\partial\Omega$ is of zero measurement, there is a countable set $0 \leqslant q \leqslant nd$ such as*

$$\lim_{k\to+\infty} k^{-\frac{n}{2}} N_k(\lambda) = \int_\Omega \nu_B(V+\lambda)\, d\sigma$$

for all $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$.

To demonstrate the 0.6 theorem, we start by studying the case of simple where Fw with a constant magnetic field Fx and with Fy . When Fz is a cube, then we know how to explain the own functions by a partial Fourier transformation that brings the problem back to the classic oscillator problem harmonic in one variable. The idea of this calculation was strongly inspired by articles [3], [4] of Y. Colin from Verdière. The extension of the result to the case of a field any kind of magnetic takes up an idea from [16], consisting of to use a paving of Ω by rather small cubes. Our method is nevertheless very different the one of Siu, since we work directly on the harmonic forms while Siu was reduced to the holomorphic cochains via the isomorphism of Dolbeault. We thus

gain a lot of precision on the estimates sought. The side of the cubes must be chosen here with a order of magnitude intermediate between $k^{-\frac{1}{2}}$ and $k^{-\frac{1}{4}}$, for example $k^{-\frac{1}{3}}$: $k^{-\frac{1}{2}}$ is indeed the wavelength of the the first proper functions, so that the action of the field magnetic rf is not perceptible at a scale lower ; above $k^{-\frac{1}{4}}$, the oscillation of B is on the contrary too strong. We finally use the principle of minimax to compare the eigenvalues on Ω to the values clean on the cubes. In the ante'rieure method of [16] (as it was used in the is repeated in [7]), the size of the cubes was chosen equal to to $k^{-\frac{1}{2}}$; one can easily see that this choice was critical to allow the effects of the magnetic field to be limited. independently of k , but the exact determination of the spectrum became impossible. The last paragraph is devoted to the study of geometrical characterizations of spaces of Mešezon [13]. Recall that a compact analytical space irreducible X is called ego spacešezon if the body $K(X)$ of the meromorphic functions on X is degree of transcendence $= n = \dim_{\mathbb{C}} X$. The conjecture of Grauert-Riemenschneider [10] states that X is Mešezon if and only if there is a quasi-positive beam $\mathcal{E}$ Row 1 without torsion above X . By de-ingularization, we come back to the case where X is smooth and where $\mathcal{E}$ is the locally free bundle of sections of a fibrous material straight E strictly positive on a dense open X . Y.T. Siu [17] has recently resolved the conjecture and strengthened it assuming only $ic(E)$ semi-positive and > 0 in at least one point. The use of theorem 0.1 (b) makes it possible to find the following conditions geometrically even weaker, which do not require the ry's semi-positive point activity, but only the positivity of an integral curvature oertain. For $q = 1$, the inequality 0.1 (b) indeed implies a minusing of the number holomorphic sections of E^k , namely:

$$\dim H^0(X, E^k) \geqslant \frac{k^n}{n!} \int_{X(\leqslant 1)} \Big(\frac{i}{2\pi} c(E)\Big)^n - o(k^n). \tag{0.7}$$

On the other hand, we can show, using classical Siegel reasoning 15] formatted by [16] that $\dim H^0(X, E^k) \leqslant \text{cte} \cdot k^{n-1}$ if X is not of Mešezon (cf. theorem 5.1). From there it results in the

Theorem 0.8. – *Let X a variety $\mathbb{C}$ -analytical compact related dimension n . For Eh to be of Mešezon, it is enough that X has a holomorphic fiber in Hermitian straight lines verifying one of the hypotheses*(a), (b), (c) *below.*

(a) $\int_{X(\leqslant 1)} (ic(E))^n > 0.$

(b) $c_1(E)^n > 0$, *and the shape of curvature El has no even index pointEm .*

(c) $ic(E)$ *is semi-positive at any point of X and defined positive in at least one point of X .*

This work was the subject of a note [8] of the same title, published in the Comptes Rendus. This article is a version improved from a previous memory [7], which was closer to Siu's initial techniques, and which demonstrated only the inequality 0.1 (a) to the constant to the nearest numerical value; therefore, the estimates 0.1 (b) and (c) remained inaccessible.

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1. Spectrum of Schrödinger operator associated with a constant magnetic field.

Let Eq a riemannian variety of class Er , of real dimension n , and $E \to M$ one fiber in complex straight lines above M , provided with a hermitian metric Ev . Note $\mathcal{C}_q^\infty(M,E)$ the space of the class sections $\mathcal{C}^\infty$ of the fibered $\Lambda^q T^*M \otimes E$, and Ez the coupling canonical sesquilinear

$$\mathcal{C}_q^\infty(M,E) \times \mathcal{C}_q^\infty(M,E) \to \mathcal{C}_{p+q}^\infty(M,\mathbb{C}).$$

It is assumed to give a hermitian connection $\mathcal{C}^\infty$b on $\mathcal{C}^\infty$c, c'est-à-dire un opérateur difféorder frame of order one

$$D : \mathcal{C}_q^\infty(M,E) \to \mathcal{C}_{q+1}^\infty(M,E), \qquad 0 \leqslant q < n,$$

verifying identities

$$D(f \wedge u) = df \wedge u + (-1)^m f \wedge Du, \tag{1.1}$$

$$d(u|v) = (Du|v) + (-1)^p(u|Dv), \tag{1.2}$$

for all sections $\mathcal{C}^\infty$f , $\mathcal{C}^\infty$g , $v \in \mathcal{C}_q^\infty(M,E)$. Let's consider a trivialization isometric $\theta : E_{|W} \to W \times \mathbb{C}$ of E over a open $W \subset M$. The hermitian connections of $E_{|W}$ are then all given by the following formula :

$$Du = du + iA \wedge u,$$

where $u \in \mathcal{C}_q^\infty(W,E) \simeq \mathcal{C}_q^\infty(W,\mathbb{C})$ and where $A \in \mathcal{C}_1^\infty(W,\mathbb{R})$ is a 1 -form*real* arbitrary. The*magnetic field* (or shape of curvature) associated with the connection D is the 2 -real closed form $B = dA$ such that

$$D^2 u = iB \wedge u$$

for all $u \in \mathcal{C}_q^\infty(M,E)$. $\mathcal{C}^\infty$v therefore depends only on the connection $\mathcal{C}^\infty$w , but not of the selected trivialisation $\mathcal{C}^\infty$x. A change phase $u = ve^{i\varphi}$ in θ leads to replace Xa by $A + d\varphi$. The choice of a trivialization of E and 1-form A is interpreted physically as the choice of a potential particular vector of the magnetic field B .

Let us designate by $|u|$ the point norm of an element $u \in \Lambda^q T^*M \otimes E$ for the tensor product metrics of the metrics of M and E . If Ω is an open of M , we note $L^2(\Omega,E)$ (resp. Xn) the space L^2 of the sections of E (resp. of $\Lambda^q T^*M \otimes E$) above of Ω, with the norm

$$\|u\|_\Omega^2 = \int_\Omega |u|^2 d\sigma,$$

where $d\sigma$ is the Riemannian volume density on M .

Let D_k be the connection induced by D on the tensor power k-th E^k , and V a scalar potential on $D = D' + D''$a , i.e. a function actual continues. Given a relatively compact

open $\Omega \subset M$, we propose to determine asymptotically when k tends towards $+\infty$ the spectrum of the quadratic form

$$Q_{\Omega,k}(u) = \int_{\Omega} \Big(\frac{1}{k}|D_k u|^2 - V|u|^2\Big) d\sigma \tag{1.3}$$

where $D = D' + D''$f , with Dirichlet condition $u_{|\partial\Omega} = 0$. The domain of $Q_{\Omega,k}$ is therefore the Sobolev space $W_0^1(\Omega, E^k)$ = closure of the space $\mathcal{D}(\Omega, E^k)$ of C^∞ sections of E^k with compact support in Ω in the space $W^1(M, E^k)$. From a physical point of view, this amounts to study the spectrum of the Schrödinger operator $\frac{1}{k}(D_k^* D_k - kV)$ associated with the magnetic field kB and the electric field kV , when k tends towards $+\infty$. We let us refer the reader to the classic article [2] for a study general spectrum of the Schrödinger operator, and to works [3], [4], [5], [9], [12] for the study of problems asymptotic neighbors of the previous one.

Definition 1.4. – *This is designated by $N_{\Omega,k}(\lambda)$ the number of eigenvalues $\leqslant \lambda$ of the quadratic form $Q_{\Omega,k}$.*

We will first study a simple case that will serve as a model for the general case at§2. We are in the following situation : $M = \mathbb{R}^n$ with the constant metric $g = \sum_{j=1}^n dx_j^2$, Ω is the side cube r :

$$\Omega = \Big\{(x_1, \dots, x_n) \in R^n \,;\ |x_j| < \frac{r}{2},\ 1 \leqslant j \leqslant n\Big\},$$

$V = 0$, and finally the magnetic field B is constant, equal to the 2 - alternate form of rank $2s$ given by

$$B = \sum_{j=1}^{s} B_j \, dx_j \wedge dx_{j+s},$$

with $B_1 \geqslant B_2 \geqslant \cdots \geqslant B_s > 0$, $s \leqslant \frac{n}{2}$. You can then choose a trivialization of E whose associated vector potential is

$$A = \sum_{j=1}^{s} B_j x_j \, dx_{j+s}.$$

The connection of Ek is written as follows

$$D_k u = du + ikA \wedge u,$$

and the quadratic form Em is given by

$$Q_{\Omega,k}(u) = \frac{1}{k} \int_{\Omega} \Bigg[\sum_{1 \leqslant j \leqslant s} \Big(\Big|\frac{\partial u}{\partial x_j}\Big|^2 + \Big|\frac{\partial u}{\partial x_{j+s}} + ikB_j x_j u\Big|^2 \Big) + \sum_{j>2s} \Big|\frac{\partial u}{\partial x_j}\Big|^2 \Bigg] d\mu$$

where Eo is the measure of Lebesgue on Ep . If one performs the homothety $X_j = \sqrt{k}\, x_j$, we are reduced to study the eigenvalues of the quadratic form

$$\int_{\sqrt{k}\Omega} \Bigg[\sum_{1 \leqslant j \leqslant s} \Big(\Big|\frac{\partial u}{\partial X_j}\Big|^2 + \Big|\frac{\partial u}{\partial X_{j+s}} + iB_j X_j u\Big|^2 \Big) + \sum_{j>2s} \Big|\frac{\partial u}{\partial X_j}\Big|^2 \Bigg] d\mu$$

on the side cubes Et . on the field Eu , we associate the function of the real variable λ defined by

$$(1.5) \qquad \nu_B(\lambda) = \frac{2^{s-n}\pi^{-\frac{n}{2}}}{\Gamma(\frac{n}{2}-s+1)} B_1 \ldots B_s \sum_{(p_1,\ldots,p_s)\in\mathbb{N}^S} \Big[\lambda - \sum(2p_j+1)B_j\Big]_+^{\frac{n}{2}-s}$$

where one poses by convention Ex if Ey and $\lambda_+^0 = 1$ si $c(E) = D^2 = D'D'' + D''D'$a . The function $c(E) = D^2 = D'D'' + D''D'$b is then increasing and continuous to the left on $\mathbb{R}$; one will observe that ν_B is actually continuous if $s < \frac{n}{2}$. The spectrum of $Q_{\Omega,k}$ is then described asymptotically by the following theorem, of which the idea was suggested to us by Y. Colin of Verdière [4].

Theorem 1.6. – *Let R be a real$c(E) = D^2 = D'D'' + D''D'h$,*

$$P(R) = \Big\{x \in \mathbb{R}^n\,;\ |x_j| < \frac{R}{2}\Big\}$$

the side paving stone R , Q_R the quadratic form

$$Q_R(u) = \int_{P(R)} \Bigg[\sum_{1\leqslant j\leqslant s} \Big(\Big|\frac{\partial u}{\partial x_j}\Big|^2 + \Big|\frac{\partial u}{\partial x_{j+s}} + iB_j x_j u\Big|^2\Big) + \sum_{j>2s} \Big|\frac{\partial u}{\partial x_j}\Big|^2\Bigg] d\mu,$$

and $\leqslant \lambda$ the number of eigenvalues $\leqslant \lambda$ of Q_R for the Dirichlet problem. So for all $\lambda \in \mathbb{R}$ we have

$$\lim_{R\to+\infty} R^{-n} N_R(\lambda) = \nu_B(\lambda).$$

When $s = \frac{n}{2}$, ν_B is a step function. The eigenvalues of Q_R are thus grouped in packets around the values $\sum(2p_j+1)B_j$, with approximate multiplicity $(2\pi)^{-s}B_1\ldots B_s R^n$. This can be interpreted physically as a phenomenon of eigen-state quantification. Returning to the initial problem of the quadratic form $c(E) = D^2 = D'D'' + D''D'$w , we get the

Corollary 1.7. – $\lim_{k\to+\infty} k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) = r^n \nu_B(\lambda)$. □

Demonstration of theorem 1.6. – First we try to increase $N_R(\lambda)$. To this end, being given $X(q)$a , one expresses $X(q)$b as a series of partial Fourier with respect to the variables $x_{s+1}, \ldots, x_n$:

$$u(x) = R^{-\frac{1}{2}(n-s)} \sum_{\ell\in\mathbb{Z}^{n-s}} u_\ell(x') \exp\Big(\frac{2\pi i}{R}\ell\cdot x''\Big)$$

where $u_\ell \in W_0^1(\mathbb{R}^s \cap P(R))$, with the notations

$$x' = (x_1, \ldots, x_s), \quad x'' = (x_{s+1}, \ldots, x_n),$$
$$\ell\cdot x'' = \ell_1 x_{s+1} + \cdots + \ell_{n-s} x_n.$$

The $u \in W_0^1(P(R))$ hypothesis implies that the series

$$\sum |\ell|^2 |u_\ell(x')|^2$$

is in $L^2(\mathbb{R}^s)$. Let's put $\ell' = (\ell_1, \dots, \ell_s)$, $\ell'' = (\ell_{s+1}, \dots, \ell_{n-s})$. $\|u\|_{P(R)}$ and the quadratic form Q_R are given by

$$\|u\|^2_{P(R)} = \sum_{\ell\in\mathbb{Z}^{n-s}} \int_{\mathbb{R}^s} |u_\ell(x')|^2\, d\mu(x'),$$

$$Q_R(u) = \sum_{\ell\in\mathbb{Z}^{n-s}} \int_{\mathbb{R}^s} \Big[\sum_{1\leqslant j\leqslant s} \Big(\Big|\frac{\partial u_\ell}{\partial x_j}\Big|^2 + \Big(\frac{2\pi}{R}\ell_j + B_j x_j\Big)^2 |u_\ell|^2 \Big) + \frac{4\pi^2}{R^2} |\ell''|^2 |u_\ell|^2 \Big]\, d\mu(x').$$

We therefore get a Dirichlet problem at "separate variables" on the cube $\mathbb{R}^s \cap P(R)$. By posing $X(q)$p , we are brought back to study the spectrum of the form quadratic of a variable

$$q(f) = \int_R \Big(\Big|\frac{df}{dt}\Big|^2 + B_j^2 t^2 |f|^2 \Big)\, dt,$$

with $f \in W_0^1\big(]-\frac{R}{2}\frac{R}{2}[+\frac{2\pi\ell_j}{RB_j}\big)$. So we come back to the problem classic harmonic oscillator (see for example [14], Vol. I, p. 142). On $\mathbb{R}$, i.e. without support condition for f , the sequence of values of q is the suite $(2m+1)B_j$, $m \in \mathbb{N}$, and the functions of q. are given by $\Phi_m(\sqrt{B_j}\, t)$ where Φ_0 , $\Phi_1, \dots$ are the functions of Hermite :

$$\Phi_m(t) = e^{t^2/2} \frac{d^m}{dt^m}(e^{-t^2}).$$

For all $p_j \in \mathbb{N}$, note $\Psi_{p_j,\ell_j}(x_j)$ the p_j the p_j -third. proper function of the quadratic form

$$q(f) = \int_R \Big(\Big|\frac{df}{dx_j}\Big|^2 + \Big(\frac{2\pi}{R}\ell_j + B_j x_j\Big)^2 |f|^2 \Big)\, dx_j \tag{1.8}$$

for $f \in W_0^1(]-\frac{R}{2}\frac{R}{2}[)$, and λ_{p_j,ℓ_j} the corresponding eigenvalue. We can then decompose each function u_ℓ as a series of own functions, which leads to write u in the form

$$u(x) = R^{-\frac{1}{2}(n-s)} \sum_{(p,\ell)\in\mathbb{N}^s\times\mathbb{Z}^{n-s}} u_{p,\ell} \Psi_{p,\ell'}(x') \exp\Big(\frac{2\pi i}{R}\ell\cdot x''\Big) \tag{1.9}$$

with

$$u_{p,\ell} \in \mathbb{C}, \qquad \Psi_{p,\ell'}(x') = \prod_{1\leqslant j\leqslant s} \Psi_{p_j,\ell_j}(x_j).$$

We will take care of the fact that $\Psi_{p,\ell'}(x')\exp(\frac{2\pi i}{R}\ell\cdot x'')$ is not not a real proper function for the Dirichlet problem, because the term exponential takes non-zero values at the edge points $x_j = \pm\frac{R}{2}$, $0 \leqslant q \leqslant n$n . Therefore, the coefficients $(u_{p,\ell})$ are not arbitrary if $u \in W_0^1(P(R))$; they must check the cancellation conditions at the edge:

$$\sum_{t_j\in\mathbb{Z}} (-1)^{\ell_j} u_{p,\ell} = 0 \tag{1.10}$$

for all $j = 1, \dots, n-s$ and all indices other than ℓ_j fixed :

$$p \in \mathbb{N}^s, \quad \ell_1, \dots, \ell_{j-1},\ \ell_{j+1}, \dots, \ell_{n-s} \in \mathbb{Z}.$$

With the writing (1.9), the norm L^2 and the quadratic form Q_R are expressed in the form of

$$\|u\|^2_{P(R)} = \sum |u_{p,\ell}|^2, \qquad Q_R(u) = \sum \Big(\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2\Big)|u_{p,\ell}|^2,$$

where $\lambda_{p,\ell'} = \sum_{1\leqslant j\leqslant s} \lambda_{p_j,\ell_j}$. The principle of the minimax 1.20 (b) recalled below shows that

$$N_R(\lambda) \leqslant \operatorname{card}\Big\{(p,\ell)\in \mathbb{N}^s\times\mathbb{Z}^{n-s}\,;\ \lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda\Big\}. \tag{1.11}$$

It is therefore sufficient to obtain an adequate reduction of λ_{p_j,ℓ_j} .

Lemma 1.12. – *We have inequality*

$$\lambda_{p_j,\ell_j} \geqslant \max\bigg((2p_j+1)B_j\ ,\ \frac{4\pi^2}{R^2}\Big[\Big(\frac{p_j+1}{2}\Big)^2 + \Big(|\ell_j| - \frac{B_jR^2}{4\pi}\Big)^2_+\Big]\bigg),$$

and this one is strict if $\ell_j \neq 0$ or if Xc .

The $\lambda_{p_j,\ell_j} \geqslant (2p_j+1)B_j$ reduction results in fact of the minimax and the fact that the eigenvalues of $q(f)$ on $\mathbb{R}$ are worth Xg . To obtain the other inequality, we minimize (1.8) by the quadratic form

$$\widehat{q}(f) = \int_{x_j|<R/2}\bigg(\Big|\frac{df}{dx_j}\Big|^2 + \Big(\frac{2\pi}{R}|\ell_j| - B_j\frac{R}{2}\Big)^2_+|f|^2\bigg)dx_j.$$

The specific functions of $\widehat{q}$ are the functions

$$\sin\frac{\pi}{R}(p_j+1)\Big(x_j+\frac{R}{2}\Big), \qquad p_j\in\mathbb{N}\ ;$$

λ_{p_j,t_j} is therefore reduced by the corresponding eigenvalue :

$$\frac{4\pi^2}{R^2}\Big[\Big(\frac{p_j+1}{2}\Big)^2 + \Big(|t_j| - \frac{B_jR^2}{4\pi}\Big)^2_+\Big].$$

The inequalities are strict because on the one hand $q(f) > \widehat{q}(f)$ for any $\circ$on , and on the other hand Xo cannot be proper function of q on $]-R/2, R/2[+2\pi\ell_j/RB_j$ only if

$$\Phi_{p_j}\big(\pm R\sqrt{B_j}/2 + 2\pi t_j/R\sqrt{B_j}\big) = 0.$$

Since the zeros of Φ_{p_j} are algebraic and π is transcendent, this is only possible if

$$\ell_j = 0 \quad \text{et} \quad \Phi_{p_j}(R\sqrt{B_J}/2) = 0. \qquad \square$$

Lemma 1.13. – *Let Xv the number of points of $\mathbb{Z}^n$ located in the closed ball $\overline{B}(0,\rho)\subset\mathbb{R}^n$. Then*

$$\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(\rho - \frac{\sqrt{n}}{2}\Big)^n_+ \leqslant \tau_n(\rho) \leqslant \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(\rho + \frac{\sqrt{n}}{2}\Big)^n.$$

Indeed, the meeting of the side cubes 1 centered at the points $x \in \mathbb{Z}^n$ such that $|x| \leqslant \rho$ is contained in the ball $\overline{B}(0, \rho + \frac{\sqrt{n}}{2})$, and contains the ball $\overline{B}(0, \rho - \frac{\sqrt{n}}{2})$ si qe , because qf is the half-diagonal of the cube ; the integer $\tau_n(\rho)$ is thus framed by the volume of the balls qh . □

We now enhance $\limsup R^{-n} N_R(\lambda)$ using (1.11) and the lemmas 1.12, 1.13. For qj fixed, inequality $\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda$ implies

$$|\ell''| \leqslant \frac{R}{2\pi}\Big(\lambda - \sum(2p_j+1)B_j\Big)_+^{\frac{1}{2}}, \tag{1.14}$$

and the inequality is strict for qm big enough. When $s < n/2$ the number of corresponding multi-indices qo is so at most qp When $s = \frac{n}{2}$, this number must be counted as qr. if $\lambda - \sum(2p_j+1)B_j > 0$ and 0 if not, which is well in conformity to the convention we adopted for the rating λ_+^0 . Inequality $\lambda_{p,\ell'} \leqslant \lambda$ implies on the other hand

$$|\ell_j| \leqslant \frac{R}{2\pi}\sqrt{\lambda_+} + \frac{B_j R^2}{4\pi}, \qquad 1 \leqslant j \leqslant s, \tag{1.16}$$

which asymptotically corresponds to a number of multi-indices qx equivalent to

$$\prod_{j=1}^{s} \frac{B_j R^2}{2\pi} = 2^{-s}\pi^{-s} B_1 \dots B_s R^{2s}. \tag{1.17}$$

The qz increase is obtained by then by performing the product of (1.15) by (1.17), and summing for all $p \in \mathbb{N}^s$ (the sum is finite). □

For questions of convergence that will take place at the§2, we will also need to know a field-independent $N_R(\lambda)$ increase magnetic B . One Such a uniform estimate is provided by the following proposal.

Proposition 1.18. – $N_R(\lambda) \leqslant (R\sqrt{\lambda_+} + 1)^n$.

Demonstration. – For each index j is increased the number of integers $x \in X$f and $x \in X$g such as inequality

$$\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda$$

takes place. The lemma 1.12 implies

$$\operatorname{card}\{p_j\} \leqslant \max(p_j+1) \leqslant \min\Big(\frac{\lambda_+}{B_j}, \frac{R}{\pi}\sqrt{\lambda_+}\Big), \qquad 1 \leqslant j \leqslant s,$$

while (1.16) results in

$$\operatorname{card}\{l_i\} \leqslant \frac{R}{\pi}\sqrt{\lambda_+} + \frac{B_j R^2}{2\pi} + 1, \qquad 1 \leqslant j \leqslant s.$$

We deduce therefore for $1 \leqslant j \leqslant s$:

$$\operatorname{card}\{(p_j, l_j)\} \leqslant \Big(\frac{R}{\pi}\sqrt{\lambda_+}\Big)^2 + \frac{\lambda_+}{B_j} \cdot \frac{B_j R^2}{2\pi} + \frac{R}{\pi}\sqrt{\lambda_+} \cdot 1 \leqslant \big(R\sqrt{\lambda_+} + 1\big)^2$$

For $x \in X\mathrm{m}$, the inequality (1.14) gives on the other hand

$$|\ell_j| < \frac{R}{2\pi}\sqrt{\lambda_+},$$

hence $\operatorname{card}\{l_j\} \leqslant \frac{R}{\pi}\sqrt{\lambda_+} + 1$. Proposition 1.18 follows. □

End of the demonstration of the theorem 1.6 (minus $x \in X\mathrm{p}$).

To minimize $x \in X\mathrm{q}$, it is sufficient to construct a vector space of finite dimension on which $Q_R(u) \leqslant \lambda\|u\|^2_{P(R)}$. We consider for this vector space $\mathcal{F}_\lambda$ of linear combinations of "own functions" of the type (1.9), subject to the conditions of cancellation on board (1.10), and summoned on the $(p,\ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}$ indices such as

$$\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda.$$

Based on the reasoning in Proposition 1.18, the number of conditions (1.10) to be carried out is increased by

$$\sum_{j=1}^{s}\Big[\operatorname{card}\{p_j\} \times \prod_{1\leqslant i\leqslant s,\, i\neq j}\operatorname{card}\{(p_i,\ell_i)\} \times \prod_{s<i\leqslant n-s}\operatorname{card}\{\ell_i\}\Big] \\ + \sum_{s<j\leqslant n-s}\Big[\prod_{1\leqslant i\leqslant s}\operatorname{card}\{(p_i,\ell_i)\} \times \prod_{s<i\neq j}\operatorname{card}\{\ell_i\}\Big] \leqslant n(R\sqrt{\lambda_+}+1)^{n-1}.$$

The integer $x \in X\mathrm{w}$ is therefore increased by

$$\dim \mathcal{F}_\lambda \geqslant \operatorname{card}\Big\{(p,\ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}\,;\ \lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda\Big\} - O(R^{n-1}).$$

By combining the minusing of lemma 1.13 with the lemma below, inequality $\liminf R^{-n}N_R(\lambda) \geqslant \nu_B(\lambda)$ then results from calculations similar to those we have explained to obtain the increase of $N_R(\lambda)$.

Lemma 1.19. – *Let* $p \in \mathbb{N}^s$ *a fixed multi-indice. Then there is a constant* $C = C(p,B) \geqslant 0$ *such that*

$$\lambda_{p,\ell'} \leqslant \Big(1 + \frac{C}{R}\Big)\sum_{j=1}^{s}(2p_j+1)B_j$$

when $|\ell_j| \leqslant \frac{B_j R^2}{4\pi}(1 - R^{-\frac{1}{2}})$ *, ic(E)(x)e .*

Demonstration. – We use the minimax again and do that the functions of Hermite $\Phi_p(\sqrt{B_j}t)$ are a good approximation of the eigenfunctions of q over any interval enough

large center $ic(E)(x)$h . When $|\ell_j| \leqslant \frac{B_j R^2}{4\pi}(1 - R^{-\frac{1}{2}})$ and $x_j \in \left]-\frac{R}{2}, \frac{R}{2}\right[$, the variable $ic(E)(x)$k that appears in (1.8) described in (1.8) indeed an interval containing $ic(E)(x)$l . So we have $\lambda_{p_j,\ell_j} \leqslant \widetilde{\lambda}_{p_j}$ where $(\widetilde{\lambda}_m)_{m\in\mathbb{N}}$ is the continuation of the eigenvalues of the quadratic form

$$\widetilde{q}(f) = \int \Big[\Big|\frac{df}{dt}\Big|^2 + (B_j t)^2 |f|^2\Big] dt, \qquad f \in W_0^1\Big(\Big] - \frac{\sqrt{R}}{2}, \frac{\sqrt{R}}{2}\Big[\Big).$$

Let χ_R a function tray to support in $\left[-\frac{\sqrt{R}}{2}, \frac{\sqrt{R}}{2}\right]$, equal to $ic(E)(x)$r on $\left[-\frac{\sqrt{R}}{4}, \frac{\sqrt{R}}{4}\right]$, which is derived from is increased by $5/\sqrt{R}$. For any linear combination

$$f = \sum_{m \leqslant p_j} c_m \Phi_m(\sqrt{B_j} t),$$

the exponential decay of the Φ_m functions to infinity implies for $ic(E)(x)$w quite big inequality

$$\|f\| \leqslant \Big(1 + C_1 \exp\Big(-\frac{R}{C_1}\Big)\Big) \|\chi_R f\|$$

where $C_1 = C_1(p_j, B_j) > 0$. We deduce therefore:

$$\begin{aligned}
\widetilde{q}(\chi_R f) &\leqslant \widetilde{q}(f) + \int_{|t|>\sqrt{R}/4} \Big(\frac{10}{\sqrt{R}}\Big|f\frac{df}{dt}\Big| + \frac{25}{R}|f|^2\Big) dt \\
&\leqslant \widetilde{q}(f) + \int_{|t|>\sqrt{R}/4} \Big(\frac{1}{R}\Big|\frac{df}{dt}\Big|^2 + 25\Big(1 + \frac{1}{R}\Big)|f|^2\Big) dt \\
&\leqslant \Big(1 + \frac{C_2}{R}\Big)\widetilde{q}(f) \leqslant \Big(1 + \frac{C_2}{R}\Big)(2p_j + 1)B_j \|f\|^2 \\
&\leqslant \Big(1 + \frac{C}{R}\Big)(2p_j + 1)B_j \|\chi_R f\|^2
\end{aligned}$$

This gives qa . □

For the reader's convenience, we now state the principle of minimax in the form in which it has served us.

Proposition 1.20 (minimax principle, see [14], Vol. IV, p. 76 and 78)**. –** *Let Q be a quadratic form with dense domain $D(Q)$ in a space of Hilbert $\mathcal{H}$. We assume that Q is bounded from below, i.e. $Q(f) \geqslant -C\|f\|^2$ if $f \in D(Q)$, that $D(Q)$ is complete for the $\|f\|_Q = [Q(f) + (C+1)\|f\|^2]^{\frac{1}{2}}$ norm, and finally that injection $(D(Q), \|\ \|_Q) \hookrightarrow (\mathcal{H}, \|\ \|)$ is compact. Then Q has a discrete spectrum $\lambda_1 \leqslant \lambda_2 \leqslant \dots$, and we have the equalities* :

(a) $\lambda_p = \min\limits_{F\subset D(Q)} \max\limits_{f\in F,\, \|f\|=1} Q(f),$

where F describes the set of subspaces of dimension p of $D(Q)$;

(b) $\lambda_{p+1} = \max\limits_{F\subset D(Q)} \min\limits_{f\in F,\, \|f\|=1} Q(f),$

where F describes all the Q-closed subspaces of codimension p of $D(Q)$.

2. Asymptotic distribution of the spectrum (case of a variable field).

Again, we place ourselves within the general framework described above. at the beginning of the§1. Our objective is to study the spectrum of the form quadratic $Q_{\Omega,k}$ (see (1.3)) in the case of a magnetic field B and any V electric field. For any point qz , or

$$B(a) = \sum_{j=1}^{s} B_j(a)\, dx_j \wedge dx_{j+s} \tag{2.1}$$

reduced writing of $B(a)$ in a suitable orthonormal base $(dx_1, \ldots, dx_n)$ of $< 0d$, where $< 0e$ is the row of $B(a)$, and where $< 0g$ are the non-zero eigenvalue modules of endomorphism associated antisymmetric. Equality of definition 1.5 allows you to view $\nu_B(\lambda)$ as a function of the torque $(a, \lambda) \in M \times \mathbb{R}$. We will need also to consider the function $< 0j$, continuous right in $< 0k$, defined by :

$$\overline{\nu}_B(\lambda) = \lim_{0<\varepsilon\to 0} \nu_B(\lambda+\varepsilon). \tag{2.2}$$

We then demonstrate the following generalization of the corollary 1.7.

Theorem 2.3.
When k tends towards $+\infty$, the number $N_{\Omega,k}(\lambda)$ of eigenvalues $\leqslant \lambda$ of $Q_{\Omega,k}$ checks the asymptotic framing

$$\int_\Omega \nu_B(V+\lambda)\, d\sigma \leqslant \liminf k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \limsup k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \int_\Omega \overline{\nu}_B(V+\lambda)\, d\sigma.$$

The $\lambda \mapsto \int_\Omega \nu_B(V+\lambda)\, d\sigma$ function is increasing and continuous to the left ; therefore it has at most one set $\mathcal{D}$ countable points of discontinuity. The $< 0u$ set is also empty if $< 0v$ is odd, because $< 0w$ is then continuous. From this, we immediately deduce the

Corollary 2.4.
It is assumed that $\partial\Omega$ is zero measurement. So

$$\lim_{k\to+\infty} k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) = \int_\Omega \nu_B(V+\lambda)\, d\sigma$$

for all $< 0z$, and density measurement spectral $k^{-\frac{n}{2}} \frac{d}{d\lambda} N_{\Omega,k}(\lambda)$ converge weakly on $\mathbb{R}$ to $\frac{d}{d\lambda} \int_\Omega \nu_B(V+\lambda)\, d\sigma$. If n is odd, the limit measurement is diffuse. □

The following lemma shows that the integrals of theorem 2.3 have well a sense.

Lemma 2.5.

(a) *We have the inequalities*

$$\nu_B(\lambda) \leqslant \overline{\nu}_B(\lambda) \leqslant \lambda_+^{n/2}.$$

(b) $\nu_B(V)$ *(resp. $\overline{\nu}_B(V)$) is semi-continuous lower (resp. upper) on M .*

(c) *At any point $(n-q)l$ where $s(x) < \frac{n}{2}$ we have $(n-q)n$ and $\nu_B(V), \overline{\nu}_B(V)$ are continuous in x .*

(d) *If n is odd, $\nu_B(V) = \overline{\nu}_B(V)$ is continues on M .*

Demonstration. – (a) We always have $\big(\lambda - \sum(2p_j+1)B_j\big)_+^{\frac{n}{2}-s} \leqslant \lambda_+^{\frac{n}{2}-s}$, and the number of integers p_j such that $\lambda - (2p_j+1)B_j$ or $\geqslant 0$ is increased by $\frac{\lambda_+}{B_j}$. As the numerical quantity shown in (1.5) is increased by 1 , the inequality (a) follows.

(b, c) Rank $s = s(x)$ is a semi-continuous function below on $\circ va$, and the eigenvalues $> 0b$, prolonged by $B_j(x) = 0$ for $j > s(x)$, are continuous on $> 0e$. As the function $t \mapsto t_+^0$ (resp. $t \mapsto (t+0)_+^0$) is semi-continuous lower (resp. upper), the semi-continuity of $\nu_B(V)$ and $\overline{\nu}_B(V)$ is a problem only at points $a \in M$ in the vicinity of which $s(x)$ is not local constant. At such a point $a \in M$, we necessarily have $s(a) < \frac{n}{2}$, so $> 0n$; we are going to then show that $\nu_B(V)$ and $\overline{\nu}_B(V)$ are continuous in $> 0q$. The continuity of B_j gives $> 0s$ for $j > s(a)$. If the integers $p_1, \ldots, p_{s\langle a)}$ are fixed, the summation in (1.5) may be construed as a sum of of Riemann's complete works on $\mathbb{R}^{s(x)-s(a)}$, and we have therefore the equivalent :

$$\sum_{(p_j;\ s(a)<j\leqslant s(x))} \Big(V(x) - \sum(2p_j+1)B_j(x)\Big)_+^{\frac{n}{2}-s(x)}$$
$$\sim \int_{t\in\mathbb{R}^{s(x)-s(a)}} \Big[V(a) - \sum_{j=1}^{s(a)}(2p_j+1)B_j(a) - \sum_{j=s(a)+1}^{s(x)} 2t_jB_j(x)\Big]_+^{\frac{n}{2}-s(x)} dt$$
$$= \frac{2^{s(a)-s(x)}\Big(V(a) - \sum(2p_j+1)B_j(a)\Big)_+^{\frac{n}{2}-s(a)}}{(\frac{n}{2}-s(x)+1)\cdots(\frac{n}{2}-s(a))B_{s(a)+1}(x)\cdots B_{s(x)}(x)} .$$

Therefore, we get :

$$\lim_{x\to a} \nu_B(V)(x) = \nu_B(V)(a) = \lim_{x\to a} \overline{\nu}_B(V)(x).$$

(d) Is a special case of (c). □

The demonstration of theorem 2.3 is essentially based on on two ingredients : first of all a principle of localization asymptotic of the eigenfunctions, which is obtained by applying direct from the minimax (proposal 2.6) ; on the other hand, knowledge of the minimax is a explicit spectrum of the associated Schrödinger operator spectrum to a constant magnetic field (see §1). The principle of Indeed, the localization allows to get back to the case of a constant field. by using a paving of Ω by quite small cubes.

Proposition 2.6.
(a)
it If $\Omega_1, \cdots, \Omega_N \subset \Omega$ are open 2 to 2 disjointed, then

$$N_{\Omega,k}(\lambda) \geqslant \sum_{j=1}^{N} N_{\Omega_j,k}(\lambda).$$

(b) *Let* $(\Omega'_j)_{1\leqslant j\leqslant \mathbb{N}}$ *an open overlap of* $\overline{\Omega}$ *and*

$$X(\leqslant q) = X(0) \cup X(1) \cup \ldots \cup X(q).$$

f a system of functions $\psi_f \in \mathcal{C}^\infty(\mathbb{R}^n)$ *to support in* Ω'_j *, such as than* $\sum \psi_j^2 = 1$ *on* $\overline{\Omega}$ *. We pose*

$$C(\psi) = \sup_{\Omega} \sum_{j=1}^{N} |d\psi_j|^2.$$

So

$$N_{\Omega,k}(\lambda) \leqslant \sum_{j=1}^{N} N_{\Omega'_j,k}\Big(\lambda + \frac{1}{k} C(\psi)\Big).$$

Demonstration. – (a) Let $\mathcal{F}$ *be* $\mathbb{C}$ *- vector space generated by the collection of all the functions proper to the quadratic forms*

$$X(\leqslant q) = X(0) \cup X(1) \cup \ldots \cup X(q).$$

o , $1 \leqslant j \leqslant N$ *, corresponding to eigenvalues* $\leqslant \lambda$ *.* $\mathcal{F}$ *is of dimension*

$$\dim \mathcal{F} = \sum_{j=1}^{N} N_{\Omega_j,k}(\lambda)$$

and for all

$$X(\leqslant q) = X(0) \cup X(1) \cup \ldots \cup X(q).$$

t , we have

$$Q_{\Omega,k}(u) = \sum_{j=1}^{N} Q_{\Omega_j,k}(u) \leqslant \sum_{j=1}^{N} \lambda \|u\|^2_{\Omega'_j} = \lambda \|u\|^2_{\Omega}.$$

The minimax principle therefore shows that the eigenvalues of $Q_{\Omega,k}$ *of index* $\leqslant \dim \mathcal{F}$ *are*

$$X(\leqslant q) = X(0) \cup X(1) \cup \ldots \cup X(q).$$

x , from where inequality (a).

(b) For every $u \in W^1_0(\Omega, E^k)$ *it comes*

$$\sum_j |D_k(\psi_j u)|^2 = \sum_j \big|\psi_j D_k u + (d\psi_j)u\big|^2 = |D_k u|^2 + \sum_j |d\psi_j|^2 |u|^2$$

because $H^q(X, E^k \otimes F)a$ *. So we get*

$$\sum_{j=1}^{N} Q_{\Omega'_j,k}(\psi_j u) = Q_{\Omega,k}(u) + \int_\Omega \frac{1}{k} \sum_{j=1}^{N} |d\psi_j|^2 |u|^2 \, d\sigma \leqslant Q_{\Omega,k}(u) + \frac{1}{k} C(\psi) \|u\|^2_{\Omega}.$$

If each function $H^q(X, E^k \otimes F)c$ is orthogonal to the own functions of $Q_{\Omega_j,k}$ of values own

$$\sum_{j=1}^{N} N_{\Omega_j,k}\Big(\lambda + \frac{1}{k}C(\psi)\Big). \qquad \square$$

s , we deduce successively $H^q(X, E^k \otimes F)f$ The principle of the minimax 1.20 (b) then leads to $N_{\Omega,k}(\lambda)$ is increased by the number of imposed linear equations to $H^q(X, E^k \otimes F)h$, or at most

$$\sum_{j=1}^{N} N_{\Omega_j,k}\Big(\lambda + \frac{1}{k}C(\psi)\Big). \qquad \square$$

Either $W_1, \ldots, W_N$ an overlap of $H^q(X, E^k \otimes F)k$ by openings of card of the variety M . For any $H^q(X, E^k \otimes F)m$, we can find openings $\Omega_i \subset \Omega'_j$, relatively compact in W_j , $H^q(X, E^k \otimes F)p$, such as

$$\text{(2.7)} \qquad \Omega \supset \bigcup \Omega_j \quad \textit{(disjointe)}, \quad \textit{et } \mathrm{Vol}(\Omega) = \sum \mathrm{Vol}(\Omega_j),$$

$$\text{(2.8)} \qquad \overline{\Omega} \subset \bigcup \Omega'_j, \qquad \textit{et } \sum \mathrm{Vol}(\overline{\Omega}'_j) \leqslant \mathrm{Vol}(\overline{\Omega}) + \varepsilon.$$

Proposition 2.6 then brings back the proof of theorem 2.3. in the case of open Ω_j and Ω'_j (we will observe for this purpose that the function $H^q(X, E^k \otimes F)t$ is bounded and that the constant $C(\psi)$ is independent of k).

In the end, we can assume that $H^q(X, E^k \otimes F)w$, with a metric riemannian $H^q(X, E^k \otimes F)x$ whatever. Since $H^q(X, E^k \otimes F)y$ is contractile, the $H^q(X, E^k \otimes F)z$ is then trivial ; that is, $E a$ a vector potential of the connection D and $B = dA$ the corresponding magnetic field. We first demonstrate the following local version of theorem 2.3.

Proposition 2.9. – *Let $a \in \mathbb{R}^n$ be a fixed point, and P_k a suite of open cubic pavers such as $P_k \ni a$. We note r_k the length of the side of P_k , and it is assumed that*

$$r_k \leqslant 1, \qquad \lim k^{\frac{1}{2}} r_k = +\infty, \qquad \lim k^{\frac{1}{4}} r_k = 0.$$

So when k tends towards $+\infty$, we have

$$\liminf \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) \geqslant \nu_{B(a)}(V(a) + \lambda),$$
$$\limsup \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) \leqslant \overline{\nu}_{B(a)}(V(a) + \lambda),$$

and for any compact $K \subset \mathbb{R}^n$, $N_{P_k,k}(\lambda)$ admits the increase

$$N_{P_k,k}(\lambda) \leqslant C_K \Big(1 + r_k \sqrt{k\big(\lambda_+ + \max_K V_+\big)}\Big)^n$$

uniform with respect to a , since $P_k \subset K$.

Demonstration. – We will return to theorem 1.6 in performing a homothety of ratio $\sqrt{k}$ *on* P_k *(that's why we had to assume* $\lim k^{\frac{1}{2}} r_k = +\infty$ *). The following lemma measures how much the magnetic field* B *deviates from the field constant* $B(a)$ *on each* P_k *.*

Lemma 2.10. – *On each paving stone* $\overline{P}_k$*, one can choose a potential* $\widetilde{A}_k$ *of the constant field* $B(a)$ *tel that for all* $x \in \overline{P}_k$ *we have*

$$|A_k(x) - A(x)| \leqslant C_1 r_k^2,$$

where C_1 *is a constant* $\geqslant 0$ *independent of* k *(and independent of* a *if* a *describes a compact* $K \subset \mathbb{R}^n$*) .*

The regularity $\mathcal{C}^\infty$ *of* B *indeed leads to a surcharge*

$$|B(a) - B(x)| \leqslant C_2 r_k, \qquad x \in \overline{P}_k.$$

Let A'_k *be a potential of the field* $B(a) - B(x)$ *on the cube* $\overline{P}_k$ *, calculated using the usual homotopic formula for open starred. We have then*

$$|A'_k(x)| \leqslant C_3 r_k^2,$$

and just put $\widetilde{A}_k = A + A'_k$ *.* □

Note $(x_1, \ldots, x_n)$ *the norm coordinates of* $\mathbb{R}^n$ *. Let* $(y_1, \ldots, y_n)$ *a linear coordinate system in* $x_1, \ldots, x_n$ *such that* $(dy_1, \ldots, dy_n)$ *is a base orthonormal to the point* a *for the metric* g *, and as in this base* $B(a)$ *is written in the diagonal form (2.1) :*

$$B(a) = \sum_{j=1}^{s} B_j(a)\, dy_j \wedge dy_{j+s}.$$

Let $\widetilde{g}$ *be the constant metric*

$$\widetilde{g} \equiv g(a) = \sum_{j=1}^{n} dy_j^2.$$

Let us designate by $D_k = d + ikA \wedge ?$ *,* $D_k = d + ikA_k \wedge ?$ *the connections on* $E^k_{|P_k}$ *associated with aabaf ,* $\widetilde{A}_k$ *, and by* $Q_k = Q_{P_k,k}$ *,* $\widetilde{Q}_k$ *quadratic forms*

metric g *,* $\widetilde{g}$ *, and scalar potentials* V *,* $\widetilde{V} \equiv V(a)$ *(formula (1.3)).*

Lemma 2.11. – *There is a suite* ε_k *tendant vers* 0 *(dépendant des* r_k *, mais indépendante of* a *if* a *describes a compact* $K \subset \mathbb{R}^n$*) as if* $\|\ \ \|_g$ *and* $\|\ \ \|_{\widetilde{g}}$ *refer to the global* L^2 *norms associated with metrics* g *and* $\widetilde{g}$ *, we have*

$$(1 - \varepsilon_k)\|u\|^2_{\widetilde{g}} \leqslant \|u\|^2_g \leqslant (1 + \varepsilon_k)\|u\|^2_{\widetilde{g}},$$
$$(1 - \varepsilon_k)\widetilde{Q}_k(u) - \varepsilon_k\|u\|^2_{\widetilde{g}} \leqslant Q_k(u) \leqslant (1 + \varepsilon_k)\widetilde{Q}_k(u) + \varepsilon_k\|u\|^2_{\widetilde{g}}$$

for all $u \in W^1_0(P_k)$ *.*

On P_k , we have indeed a frame:

$$(1 - C_4 r_k)\,\widetilde{g} \leqslant g \leqslant (1 + C_4 r_k)\,\widetilde{g},$$

and this gives the first double inequality in 2.11. With the notation $A'_k = A_k - A$, we deduce from it

$$\begin{aligned} Q_k(u) &= \int_{P_k} \Big(\frac{1}{k}|\widetilde{D}_k u - ikA'_k \wedge u|_g^2 - V|u|^2\Big)\, d\sigma \\ &\leqslant (1 + C_5 r_k) \int_{P_k} \Big(\frac{1}{k}|\widetilde{D}_k u - ikA'_k \wedge u|_{\tilde{g}}^2 - V(a)|u|^2\Big)\, d\widetilde{\sigma} + \eta_k \|u\|_{\tilde{g}}^2 \end{aligned}$$

with $\eta_k = \sup_{P_k} |V - V(a)| + C_6 r_k$, quantity that tends towards 0 when k tends towards $+\infty$. Using the inequality $(a+b)^2 \leqslant (1+\alpha)(a^2 + \alpha^{-1} b^2)$, the lemma 2.10 implies on the other hand

$$|\widetilde{D}_k u - ikA'_k \wedge u|_{\tilde{g}}^2 \leqslant (1+\alpha)\Big[\, |\widetilde{D}_k u|_{\tilde{g}}^2 + \alpha^{-1} C_1^2 k^2 r_k^4 |u|^2 \Big].$$

Let's choose $\alpha = \alpha_k = C_1 \sqrt{k} r_k^2$. The suite α_k tends towards 0 according to the $\lim k^{\frac{1}{4}} r_k = 0$ *hypothesis, and it comes*

$$\frac{1}{k}|\widetilde{D}_k u - ikA'_k \wedge u|_{\tilde{g}}^2 \leqslant (1+\alpha_k)\Big[\frac{1}{k}|D_k u|_{\tilde{g}}^2 + \alpha_k |u|^2\Big].$$

The Q_k surcharge follows. The minoration is obtained in the same way thanks to inequality $(a+b)^2 \geqslant (1-\alpha)(a^2 - \alpha^{-1} b^2)$. □

Lemma 2.11 brings back the evidence of proposal 2.9 in case the metamorphosis is not completed. brick g and the magnetic field B are constant :

$$g = \sum_{j=1}^{n} dy_j^2, \qquad B = \sum_{j=1}^{n} B_j\, dy_j \wedge dy_{j+s}.$$

We can assume moreover $V \equiv 0$ by carrying out the translation $\lambda \mapsto \lambda + V(a)$. The only remaining difficulty for applying directly the theorem 1.6 comes from the fact that the cubes P_k usually become parallelepipeds oblique in the coordinates $(y_1, \ldots, y_n)$; the angles between the different edges of each P_k and the ratios of their lengths remain however framed by constants > 0 . To solve this difficulty, you just have to pave every parallelepiped P_k by cubes $P_{k,\alpha}$ of which the edges are parallel to the coordinate axes $(y_1, \ldots, y_n)$. let's choose $\varepsilon \in \,]0,1[$. For all $\alpha \in \mathbb{Z}^n$, are $(P_{k,\alpha})$, $(P'_{k,\alpha})$ the cubes open sides εr_k , $\varepsilon(1+\varepsilon) r_k$, and of common center $\varepsilon r_k \alpha$. We will limit ourselves to considering the $P_{k,\alpha}$ cubes contained in P_k and $P'_{k,\alpha}$ cubes meeting P_k . Then we have

$$\text{(2.12)} \qquad P_k \supset \bigcup_{\alpha} P_{k,\alpha} \ \textit{(disjointe)}, \quad \textit{et} \quad \frac{\sum_\alpha \mathrm{Vol}(P_{k,\alpha})}{\mathrm{Vol}(P_k)} \geqslant 1 - C_7 \varepsilon,$$

$$\text{(2.13)} \qquad P_k \subset \bigcup_{\alpha} P'_{k,\alpha}, \quad \textit{et} \quad \frac{\sum_\alpha \mathrm{Vol}(P'_{k,\alpha})}{\mathrm{Vol}(P_k)} \leqslant 1 + C_7 \varepsilon,$$

where C_7 *is a constant independent of* k *(and also of* a *, if* a *describes a compact). The number of cubes* $P_{k,\alpha}$ *,* $P'_{k,\alpha}$ *which are listed in (2.12) or (2.13) is increased by* $C_8\varepsilon^{-n}$ *. As the* $P'_{k,\alpha}$ *cubes overlap two by two on a length* $\varepsilon^2 r_k$ *when they are contiguous, a partition of the unit can be constructed* $\sum\psi^2_{k,\alpha}=1$ *on* P_k *, with Supp* $\psi_{k,\alpha}\subset P'_{k,\alpha}$ *and*

$$\sup_{P_k}\sum_{\alpha}|d\psi_{k,\alpha}|^2=C(\psi_k)\leqslant C_9(\varepsilon^2 r_k)^{-2}.$$

The $\lim k^{\frac{1}{2}}r_k=+\infty$ *hypothesis leads well to* $\lim\frac{1}{k}C(\psi_k)=0$ *, which allows to apply 2.6 (b). On the cubes* $P_{k\alpha}$ *,* $P'_{k,\alpha}$ *we are maintenant dans la situation du théorth 1.6 : after ratio homothety* $\sqrt{k}$ *, the side of the cube homothetic* $\sqrt{k}\,P_{k,\alpha}$ *is worth* $R_k=\varepsilon r_k\sqrt{k}$ *and tends well to* $+\infty$ *by hypothesis. The uniform mark-up of* $N_{P_k,k}(\lambda)$ *results from Proposition 1.18 and the fact that all our* $C_1,\dots,C_9$ *constants were uniforms. Proposition 2.9 is demonstrated.* □

Demonstration of theorem 2.3. – According to the remark preceding the proposition 2.9, we can assume that $M=\mathbb{R}^n$ *and* Ω *is a open bounded of* $\mathbb{R}^n$ *. The idea of the reasoning is to combine the proposals 2.6 and 2.9 using a paving of* Ω *by cubes of side* $r_k=k^{-\frac{1}{3}}$ *. The actual implementation requires nevertheless a little care because of the difficulties related to the possible non-uniformity of the* lim sup *and* lim inf *.*

Designate by $\Pi_{k,\alpha}$ *,* $\Pi'_{k,\alpha}$ *,* $\alpha\in\mathbb{Z}^n$ *, the cubes open on their respective sides*

$$k^{-\frac{1}{3}},\qquad k^{-\frac{1}{3}}(1+k^{-\frac{1}{8}})=k^{-\frac{1}{3}}+k^{-\frac{11}{24}}$$

and common center $k^{-\frac{1}{3}}\alpha$ *. Either* $I(k)$ *(resp.* $I'(k)$ *) the set of* $\alpha\in\mathbb{Z}^n$ *indices such as* $\Pi_{k.\alpha}\subset\Omega$ *(resp.* $\overline{\Pi}'_{k,\alpha}\cap\overline{\Omega}\neq\emptyset$ *). As in the reasoning of proposal 2.9, there is a partition of the unit* $\sum_{\alpha\in I'(k)}\psi^2_{k,\alpha}=1$ *on* Ω *, with Supp* $\psi_{k,\alpha}\subset\Pi'_{k,\alpha}$ *and*

$$C(\psi_k)=\sup_{\Omega}\sum_{\alpha\in I'(k)}|d\psi_{k,\alpha}|^2\leqslant C_{10}k^{\frac{11}{12}},$$

hence $\lim\frac{1}{k}C(\psi_k)=0$ *. We pose*

$$\Omega_k=\bigcup_{\alpha\in I(k)}\Pi_{k,\alpha},\qquad \Omega'_k=\bigcup_{\alpha\in I'(k)}\Pi'_{k,\alpha}$$

and one considers for every $\lambda\in\mathbb{R}$ *fixed, the functions on* $\mathbb{R}^n$ *defined by*

$$f_k=k^{-\frac{n}{2}}\sum_{\alpha\in I(k)}N_{\Pi_{k,\alpha},k}(\lambda)\frac{1}{\mathrm{Vol}(\Pi_{k,\alpha})}1\!\!1_{\Pi_{k,\alpha}},$$

$$f'_k=k^{-\frac{n}{2}}\sum_{\alpha\in I'(k)}N_{\Pi'_{k,\alpha},k}\Big(\lambda+\frac{1}{k}C(\psi_k)\Big)\frac{1}{\mathrm{Vol}(\Pi_{k,\alpha})}1\!\!1_{\Pi_{k,\alpha}}$$

where $1\!\!1_{\Pi_{k,\alpha}}$ *designates the characteristic function of* $\Pi_{k,\alpha}$ *. La proposition 2.6 implique l'encahard*

$$\int_{\mathbb{R}^n}f_k\,d\sigma\leqslant k^{-\frac{n}{2}}N_{\Omega,k}(\lambda)\leqslant\int_{\mathbb{R}^n}f'_k\,d\sigma. \tag{2.14}$$

Either $x \in \mathbb{R}^n$ a fixed point not belonging to the set negligible

$$Z = \bigcup_{k\in\mathbb{N},\, \alpha\in\mathbb{Z}^n} \partial\Pi_{k,\alpha}.$$

There is then a unique $\alpha(k) \in \mathbb{Z}^n$ index sequence such as $x \in \Pi_{k,\alpha(k)}$. Proposition 2.9 applied to $P_k = \Pi_{k,\alpha(k)}$ cubes (resp. $P'_k = \Pi'_{k,\alpha(k)}$) with $\mathrm{Vol}(P_k) \sim \mathrm{Vol}\,P'_k$ shows that the punctual sequences

$$f_k(x) = \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) \mathbb{1}_{\Omega_k}(x), \qquad f'_k(x) = \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P'_k,k}(\lambda) \mathbb{1}_{\Omega'_k}(x),$$

are such that

$$\left\{\begin{array}{l} \liminf f_k(x) \;\geqslant \nu_{B(x)}(V(x)+\lambda)\, \mathbb{1}_\Omega(x) \\ \limsup f'_k(x) \leqslant \overline{\nu}_{B(x)}(V(x)+\lambda)\, \mathbb{1}_{\overline{\Omega}}(x). \end{array}\right. \tag{2.15}$$

The uniform increase in proposal 2.9 also results in the existence of constants C_{11} , C_{12} independent of k , x and λ such as

$$f_k(x) \leqslant f'_k(x) \leqslant C_{11}\big(1+\sqrt{\lambda_+ + C_{12}}\,\big)^n.$$

The theorem 2.3 then results from (2.14), (2.15) and the lemma of Fatou. □

For applications with complex geometry, we will require d'une légère généralisation of theorem 2.3. We give ourselves a F hermitian fiber of rank r and class $\mathcal{C}^\infty$ above of M , provided with a hermitian connection ∇ , and continuous sections S of the fiber $\Lambda^1_R T^*X \otimes_R \mathrm{Hom}_{\mathbb{C}}(F,F)$ and V of the fibered $\mathrm{Herm}(F)$ of F hermitian endomorphisms. Either ∇_k the hermitian connection on $E^k \otimes F$ induced by the connections D and ∇ . To shorten the notations, one will still designate by S and V the endomorphisms $\mathrm{Id}_{E^k} \otimes S$ and $\mathrm{Id}_{E^k} \otimes V$ operating on $E^k \otimes F$. Given an open Ω relatively compact in M , we consider the quadratic form

$$Q_{\Omega,k}(u) = \int_\Omega \Big(\frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu,u\rangle\Big)\, d\sigma,$$

where $u \in W^1_0(\Omega, E^k \otimes F)$. Let $V_1(x) \leqslant V_2(x) \leqslant \cdots \leqslant V_r(x)$ be the eigenvalues of $V(x)$ at any point $x \in M$. We then have the following result.

Theorem 2.16. – *The counting function $N_{\Omega,k}(\lambda)$ of the eigenvalues of $Q_{\Omega,k}$ admits for all $\lambda \in \mathbb{R}$ asymptotic estimates*

$$\liminf_{k\to+\infty}\ k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \geqslant \sum_{j=1}^r \int_\Omega \nu_B(V_j+\lambda)\, d\sigma,$$

$$\limsup_{k\to+\infty}\ k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \sum_{j=1}^r \int_\Omega \overline{\nu}_B(V_j+\lambda)\, d\sigma,$$

where B is the magnetic field associated with the connection D on E .

Demonstration. – The localization principle 2.6 is still valid in the present situation. It is therefore sufficient to demonstrate the inequalities 2.16 when Ω is small enough. Let $a \in M$ a fixed point and $(e_1, \ldots, e_r)$ an orthonormal marker $\mathcal{C}^\infty$ of F above a neighborhood W of a , such as $(e_1(a), \ldots, e_r(a))$ or a clean base for $V(a)$. Let's write u as

$$u = \sum_{j=1}^{r} u_j \otimes e_j$$

where u_j is a section of E^k . For any $\varepsilon > 0$, there is exists a neighborhood $W'_\varepsilon \subset W$ of a on which

$$\sum_{j=1}^{r}(V_j(a) - \varepsilon)|u_j|^2 \leqslant \langle Vu, u\rangle \leqslant \sum_{j=1}^{r}(V_j(a) + \varepsilon)|u_j|^2$$

On the other hand, we have

$$\nabla_k u = \sum_{j=1}^{r} D_k u_j \otimes e_j + u_j \otimes \nabla e_j,$$

and the term $u_j \otimes \nabla e_j$ can be absorbed into Su (which actually brings us back to the case where the ∇ connection is flat). Coaching

$$(1 - k^{-\frac{1}{2}})|\nabla_k u|^2 + (1 - k^{\frac{1}{2}})|Su|^2 \leqslant |\nabla_k u + Su|^2 \leqslant (1 + k^{-\frac{1}{2}})|\nabla_k u|^2 + (1 + k^{\frac{1}{2}})|Su|^2$$

shows that the term Su only changes $Q_{\Omega,k}$ by a factor of multiplicative $1 \pm \varepsilon$ and by an additive factor $\pm\varepsilon\|u\|^2$. For all $\varepsilon > 0$, there are so a neighborhood W_ε of a and an integer $k_0(\varepsilon)$ such as

$$(1 - \varepsilon)\widetilde{Q}_{\Omega.k}(u) - \varepsilon\|u\|^2 \leqslant Q_{\Omega,k}(u) \leqslant (1 + \varepsilon)\widetilde{Q}_{\Omega,k}(u) + \varepsilon\|u\|^2$$

as soon as $k \geqslant k_0(\varepsilon)$ and $\Omega \subset W_\varepsilon$, where $\widetilde{Q}_{\Omega,k}$ designates the quadratic form

$$\widetilde{Q}_{\Omega.k}(u) = \sum_{j=1}^{r} \int_\Omega \Big(\frac{1}{k}|D_k u_j|^2 - V_j(a)|u_j|^2\Big)\, d\sigma.$$

As $\widetilde{Q}_{\Omega,k}$ is a direct sum of r forms quadratic, the spectrum of $\widetilde{Q}_{\Omega.k}$ is the meeting (counted with multiplicities) of the spectra of each of the terms of the sum. The theorem 2.16 follows. □

3. Identity of Bochner-Kodaira-Nakano in hermitian geometry.

The purpose of the following paragraphs is to draw the consequences of the théorème de réparspectral analysis 2.16 for the study of the d'' -cohomology of holomorphic vectorial fibrils hermitians. For this purpose, we will need to connect the laplacian antiholomorph

Δ'' to Schrödinger's ope'rateur of a connection adequate real one. This is done by means of a formula Weitzenböck type, known in geometry and design for the complex under the identity of Bochner-Kodaira-Nakano.

Either X a compact complex analytical variety of dimension n and F a hermitian holomorphic vectorial fiber of rank r above of X . It is known that there is a unique hermitian connection $D = D' + D''$ on F whose D'' component of type $(0,1)$ coincides with the operator $\overline{\partial}$ of the fibrous (such a connection is called holomorphic). Let $c(F) = D^2 = D'D'' + D''D'$ the bending shape of F . Let's provide X with an arbitrary hermitian me'trique ω of type $(1,1)$ and class $\mathcal{C}^\infty$. $\mathcal{C}^\infty_{p,q}(X,F)$ space sections of class $\Lambda^{p,q}T^*X \otimes F$ fiber $\Lambda^{p,q}T^*X \otimes F$ se trouve alors muni d'une structure préhilbernatural hold. We note $\delta = \delta' + \delta''$ the formal assistant to D considered as a differential operator on $\mathcal{C}^\infty(X,F)$, and Λ the operator's assistant $L : u \mapsto \omega \wedge u$.

We will use the identity of Bochner-Kodaira-Nakano in the form of demonstrated in [6], although one can in fact make a general statement in be content, as Y.T. does. Siu [16], [17], of the formula minus precise given by P. Griffiths. If A , B are differential operators on $\mathcal{C}^\infty(X,F)$, one defines their anti-switch $[A,B]$ by the formula

$$[A,B] = AB - (-1)^{ab}BA$$

where a , b are the respective degrees of A and B . The Laplace-Beltrami operators $\Delta' j$ and Δ'' are then classically given by

$$\Delta' = [D', \delta'] = D'\delta' + \delta' D', \qquad \Delta'' = [D'', \delta'']$$

To the torsion shape $d'\omega$, we associate the operator of the outward multiplication $u \mapsto d'\omega \wedge u$ on $\mathcal{C}^\infty(X,F)$, type $(2,1)$, simply noted $d'\omega$, and the operator τ of type $(1,0)$ defined by $\tau = [\Lambda, d'\omega]$. We finally pose

$$D'_\tau = D' + \tau, \qquad \delta'_\tau = (D'_\tau)^* = \delta' + \tau^*, \qquad \Delta'_\tau = [D'_\tau, \delta'_\tau].$$

We then have the following identity, for a demonstration of which the reader will refer to [6].

Proposition 3.1.
We have $\Delta'' = \Delta'_\tau + [ic(F), \Lambda] + T_\omega$ where T_ω is the operator of order 0 and of type $(0,0)$ defined by

$$T_\omega = \Big[\Lambda, \Big[\Lambda, \frac{i}{2} d'd''\omega\Big]\Big] - [d'\omega, (d'\omega)^*].$$

According to Hodge-De Rham's theory, the cohomology group $H^q(X,F)$ identifies itself with the space of $(0,q)$ -forms Δ'' -harmonics with values in F . Or $u \in \mathcal{C}^\infty_{p.q}(X,F)$. Proposition 3.1 gives us equality

$$(3.2)\quad \int_X |D''u|^2 + |\delta''u|^2 = \int_X \langle \Delta''u, u\rangle = \int_X |D'_\tau u|^2 + |\delta'_\tau u|^2 + \langle [ic(F), \Lambda]u, u\rangle + \langle T_\omega u, u\rangle,$$

where integrals are calculated relative to l'élément de volume $d\sigma = \frac{\omega^n}{n!}$. En particulier, if u is of bidegré $(0,q)$, we have $\delta'_\tau u = 0$ by reason of bidegré, from where

$$(3.3)\qquad \int_X \langle \Delta''u, u\rangle = \int_X |D'_\tau u|^2 + \langle [ic(F), \Lambda]u, u\rangle + \langle T_\omega u, u\rangle.$$

One can also consider u *as a* (n,q) *-form to values in fiber*

$$\widetilde{F} := F \otimes \Lambda^n TX \ ;$$

we will note $\widetilde{D} = \widetilde{D}' + \widetilde{D}''$ *the hermitian connection holomorph of* $\widetilde{F}$ *and* $\widetilde{u}$ *the canonical image of* u *in* $\mathcal{C}^\infty_{n,q}(X, F)$ *.*

Lemma 3.4.
We have switching diagrams

$$\begin{tikzcd} \mathcal{C}^\infty_{0.q}(X,F) \arrow[r, "D''"] \arrow[d, "\sim"'] & \mathcal{C}^\infty_{0,q+1}(X,F) \arrow[d, "\sim"] \\ \mathcal{C}^\infty_{n,q}(X,\widetilde{F}) \arrow[r, "\widetilde{D}''"] & \mathcal{C}^\infty_{n,q+1}(X,\widetilde{F}), \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^\infty_{0,q}(X,F) \arrow[r, "\Delta''"] \arrow[d, "\sim"'] & \mathcal{C}^\infty_{0,q}(X,F) \arrow[d, "\sim"] \\ \mathcal{C}^\infty_{n,q}(X,\widetilde{F}) \arrow[r, "\widetilde{\Delta}''"] & \mathcal{C}^\infty_{n,q}(X,\widetilde{F}), \end{tikzcd}$$

where the vertical arrows are the isometrics $u \mapsto \widetilde{u}$ *.*

Demonstration. – Switching of the left diagram results from the fact that $\Lambda^n TX$ *is a holomorphic fiber (be careful that the corresponding result for* D' *and* $\widetilde{D}'$ *is wrong). So we have a diagram analogue switch for assistants* δ'' *,* $\widetilde{\delta}''$ *and for* Δ'' *,* $\widetilde{\Delta}''$ *.* □

The lemma 3.4 and the identity (3.2) give us

$$(3.5) \qquad \int_X \langle \Delta'' u, u\rangle = \int_X \langle \widetilde{\Delta}''\widetilde{u}, \widetilde{u}\rangle = \int_X |\widetilde{\delta}'_\tau \widetilde{u}|^2 + \langle [ic(\widetilde{F}), \Lambda]\widetilde{u}, \widetilde{u}\rangle + \langle T_\omega \widetilde{u}, \widetilde{u}\rangle.$$

We will now slightly transform the writing of (3.3) and (3.5). The holomorphic hermitian connection of the fibroid $\Lambda^q T^*X$ *induced on conjugated fiber* $\Lambda^{0,q}T^*X$ *a connection with a component of type* $(1,0)$ *coincides with the operator* d' *. From this we deduce then a natural hermitian connection* ∇ *on the fibered product produces tensorial* $\Lambda^{0,q}T^*X \otimes F$ *(we will observe that this fibrous vector is not generally holomorphic if* $q \neq 0$ *). Let* ∇' *et* ∇'' *the components of* ∇ *of type* $(1,0)$ *and* $(0,1)$ *.*

Proposition 3.6.
We have

$$\nabla' = D' : \mathcal{C}^\infty(\Lambda^{0,q}T^*X \otimes F) \to \mathcal{C}^\infty_{1,0}(\Lambda^{0,q}T^*X \otimes F),$$

and there is a commutative diagram

$$\begin{tikzcd} \mathcal{C}^\infty(X, \Lambda^{0.q}T^*X \otimes F) \arrow[r, "\nabla''"] \arrow[d, "\sim"'] & \mathcal{C}^\infty_{0,1}(X, \Lambda^{0.q}T^*X \otimes F) \arrow[d, "\Psi"] \\ \mathcal{C}^\infty_{n,q}(X,\widetilde{F}) \arrow[r, "\widetilde{\delta}''"] & \mathcal{C}^\infty_{n-1,q}(X,\widetilde{F}), \end{tikzcd}$$

where the vertical arrows are isometries, that of left being given by $u \mapsto \widetilde{u}$ *.*

Demonstration. – The equality $\nabla' = D'$ *comes from the makes the* $(1,0)$ *component of the connection from* $\Lambda^{0,q}T^*X$ *coincides with* d' *. For the diagram, we start by defining the vertical arrow* Ψ *. Either*

$$\{?|?\} : (\Lambda^{p_1,q_1}T^*X \otimes \widetilde{F}) \times (\Lambda^{p_2,q_2}T^*X \otimes \widetilde{F}) \longrightarrow \Lambda^{p_1+q_2,q_1+p_2}T^*X$$

the canonical sesquilinear coupling induced by the metric on the F *fibers, and*

$$*:\Lambda^{p,q}T^*X\otimes\widetilde{F}\longrightarrow\Lambda^{n-q,n-p}T^*X\otimes\widetilde{F}$$

the operator of Hodge-De Rham-Poincaré defined by

$$\{v|*w\}=\langle v,w\rangle\,d\sigma,\qquad v,\ w\in\Lambda^{p,q}T^*X\otimes\widetilde{F}.$$

We deduce from this by composition an isometry

$$\Psi_0:\Lambda^{0,1}T^*X\otimes F\xrightarrow{\sim}\Lambda^{n,1}T^*X\otimes\widetilde{F}\xrightarrow{*}\Lambda^{n-1,0}T^*X\otimes\widetilde{F}$$

and the arrow $E^k\otimes F$ *is obtained by definition by tensorizing* $-i^{-n^2}\Psi_0$ *by* $\Lambda^{0,q}T^*X$. *To demonstrate the commutativity, we first assume* $q=0$. *Let* $u\in\mathcal{C}^\infty(F)$. *We have classically*

$$\widetilde{\delta}'\widetilde{u}=-*\widetilde{D}''*\widetilde{u},$$

and like $\widetilde{u}\in\mathcal{C}^\infty_{n,0}(X,F)$, *he comes* $*\widetilde{u}=i^{-n^2}\widetilde{u}$, *hence*

$$\widetilde{\delta}'\widetilde{u}=-i^{-n^2}*D''\widetilde{u}=-i^{-n^2}*\sim D''u=-i^{-n^2}\Psi_0(D''u)=\Psi(\nabla''u).$$

In case q *is arbitrary, you just have to trivialize* $\Lambda^{0,q}T^*X$ *in the vicinity of an arbitrary point* x*, in choosing an orthonormal marker* $(e_1,\ldots,e_N)$ *of this fiber, such as*$\nabla e_1(x)=\cdots=\nabla e_N(x)=$
□

We now consider the morphisms of fibrils

$$S':\Lambda^{0,q}T^*X\otimes F\to\Lambda^{1,0}T^*X\otimes\Lambda^{0,q}T^*X\otimes F$$
$$S'':\Lambda^{0,q}T^*X\otimes F\to\Lambda^{0,1}T^*X\otimes\Lambda^{0,q}T^*X\otimes F$$

where $S'=\tau=[\Lambda,d'\omega]$, *and where* S'' *is the reading by the* $\sim$ *and* Ψ *isometries of the morphism*

$$\tau^*=[(d'\omega)^*,L]:\Lambda^{n,q}T^*X\otimes\widetilde{F}\to\Lambda^{n-1.q}T^*X\otimes\widetilde{F}.$$

According to proposal 3.6, we have

$$|D'_\tau u|=|\nabla'u+S'u|,\qquad|\widetilde{\delta}'_\tau\widetilde{u}|=|\nabla''u+S''u|.$$

If $S=S'\oplus S''$ *is set, identities (3.3) and (3.5) imply by addition*

$$\begin{aligned}2\int_X\langle\Delta''u,u\rangle=\int_x|\nabla u+Su|^2+\int_X\langle[ic(F),\Lambda]u,u\rangle\\+\int_X\langle[ic(\widetilde{F}),\Lambda]\widetilde{u},\widetilde{u}\rangle+\langle T_\omega u,u\rangle+\langle T_\omega\widetilde{u},\widetilde{u}\rangle\end{aligned}\tag{3.7}$$

for all $u\in\mathcal{C}^\infty_{0,q}(X,F)$.

Let now be E a hermitian holomorphic fiber of rank 1 above of X . For the whole k , we note D_k and ∇_k the natural hermitian connections on the fibrils $F_k = E^k \otimes F$ and $\Lambda^{0,q}T^*X \otimes F_k$, and on pose $\Delta''_k = [D''_k, \delta''_k]$. The curvature of F_k (resp. $\widetilde{F}_k$) is given by

$$c(F_k) = c(F) + kc(E) \otimes \mathrm{Id}_F, \quad \textit{resp.} \quad c(\widetilde{F}_k) = c(\widetilde{F}) + kc(E) \otimes \mathrm{Id}_{\widetilde{F}}. \tag{3.8}$$

Let us recall, although it is useless for the rest, that

$$c(\widetilde{F}) = c(F) + c(\Lambda^n TX) \otimes \mathrm{Id}_F = c(F) + \mathrm{Ricci}(\omega) \otimes \mathrm{Id}_F .$$

We will therefore need to evaluate the terms $[ic(E), \Lambda]$. For all point $x \in X$, are $\alpha_1(x)$, $\alpha_2(x), \ldots, \alpha_n(x)$ the eigenvalues of $ic(E)(x)$ relative to the metrics hermitienne ω on X . So there is a system of local coordinates $(z_1, \ldots, z_n)$ centered in x such that $(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$ is an orthonormal base of $T_X X$, and such as

$$\omega(x) = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j,$$

$$ic(E)(x) = \frac{i}{2} \sum_{j=1}^{n} \alpha_j(x)\, dz_j \wedge d\overline{z}_j.$$

Either $(e_1, \ldots, e_r)$ an orthonormal marker of the fiber $E^k_x \otimes F_x$. For $v \in \Lambda^{p.q}T^*X \otimes F_k$, you can write

$$v = \sum_{|I|=p, |J|=q,\, \ell} v_{I,J,\ell}\, dz_I \wedge d\overline{z}_J \otimes e_\ell, \qquad |v|^2 = 2^{p+q} \sum_{I,J,\ell} |v_{I,J,\ell}|^2$$

An elementary calculation, as explained for example in [6], gives the formula

$$\langle [ic(E), \Lambda]v, v\rangle = 2^{p+q} \sum_{I,J,\ell} (\alpha_I + \alpha_J - \sum_{j=1}^{n} \alpha_j) |v_{I,J,\ell}|^2 \tag{3.9}$$

with $\alpha_I = \sum_{j \in I} \alpha_j$. Either $u \in \Lambda^{0,q}T^*X \otimes F_k$. Let's post

$$u = \sum_{J,\ell} u_{J,\ell}\, d\overline{z}_J \otimes e_\ell.$$

According to (3.9), it comes from

$$\langle [ic(E), \Lambda]u, u\rangle = 2^q \sum_{J,\ell} -\alpha_{\complement J} |u_{J,\ell}|^2,$$

$$\langle [ic(E\rangle, \Lambda]\widetilde{u}, \widetilde{u}\rangle = 2^q \sum_{J,\ell} \alpha_J |u_{J,\ell}|^2.$$

Either V the hermitian endomorphism of $\Lambda^{0,q}T^*X \otimes F_k$ defined by

$$\langle Vu, u\rangle = -\langle [ic(E), \Lambda]u, u\rangle - \langle [ic(E), \Lambda]\widetilde{u}, \widetilde{u}\rangle = 2^q \sum_{J,\ell} (\alpha_{\complement J} - \alpha_J) |u_{J,\ell}|^2. \tag{3.10}$$

The eigenvalues of V are therefore the coefficients $\alpha_{\complement J} - \alpha_J$, counted with multiplicity $r = \mathrm{rank}(F)$. Or finally Θ the hermitian endomorphism defined by

$$\langle \Theta u, u\rangle = \langle [ic(F), \Lambda]u, u\rangle + \langle [ic(\widetilde{F}), \Lambda]\widetilde{u}, \widetilde{u}\rangle + \langle T_\omega u, u\rangle + \langle T_\omega \widetilde{u}, \widetilde{u}\rangle. \tag{3.11}$$

The identities (3.7-11) then imply

$$\frac{2}{k}\int_X \langle \Delta''_k u, u\rangle = \int_X \frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu, u\rangle + \frac{1}{k}\langle \Theta u, u\rangle \tag{3.12}$$

where the operators S , V , Θ act only on the component $\Lambda^{0,q}T^*X\otimes F$ of $\Lambda^{0,q}T^*X\otimes F_k$. So we're going to be able to use theorem 2.16 to determine the spectral distribution asymptotic of Δ''_k , because the term $\frac{1}{k}\langle\Theta u, u\rangle$ tends towards 0 as norm.

Let $h^q_k(\lambda)$ be the number of eigenvalues $\leqslant k\lambda$ of Δ''_k operating on $\mathcal{C}^\infty_{0,q}(E^k\otimes F)$. The magnetic field B is here given by

$$B = -ic(E) = -\sum_{j=1}^n \alpha_j\, dx_j\wedge dy_j, \qquad z_j = x_j + iy_j. \tag{3.13}$$

Given that $\dim_{\mathbb{R}} X = 2n$, the theorem 2.16 is transcribed as follows.

Theorem 3.14.
There is a set of countable $\mathcal{D}$ such as for any $q = 0, 1, \ldots, n$ and all $\lambda \in \mathbb{R}\setminus\mathcal{D}$ we have

$$h^q_k(\lambda) = rk^n \sum_{|J|=q}\int_X \nu_B(2\lambda + \alpha_{\complement J} - \alpha_J)\, d\sigma + o(k^n)$$

when k tends towards $+\infty$.

4. Witten's complex and Morse inequalities.

E. Witten [18], [19] has recently introduced a new method analytique pour démonto overcome the inequalities of Morse in de Rham's cohomology. We adapt here his method to the study of d'' -cohomology. The main difference lies in the fact that the magnetic field is always null. in the case of de Rham's cohomology (we have indeed $d^2 = 0$!), and it is the electric field which intervenes alone in this case.

With the§3 notations, that is $\mathcal{H}^q_k(\lambda) \subset \mathcal{C}^\infty_{0,q}(X, E^k\otimes F)$ the direct sum of the clean subspaces of Δ''_k attached to eigenvalues $\leqslant k\lambda$. $\mathcal{H}^q_k(\lambda)$ is therefore a vector space of finite dimension.

$$h^q_k(\lambda) = \dim_{\mathbb{C}} \mathcal{H}^q_k(\lambda).$$

Hodge's theory gives an isomorphism

$$H^q(X, E^k\otimes F) \simeq \mathcal{H}^q_k(0).$$

We will pose to shorten

$$h^q_k = \dim H^q(X, E^k\otimes F) = h^q_k(0).$$

Proposition 4.1. – *$\mathcal{H}_k^\bullet(\lambda)$ is a sub-complex of the Dolbeault complex*

$$D_k'' : \mathcal{C}_{0,\bullet}^\infty(X, E^k \otimes F).$$

In addition, the $\mathcal{H}_k^\bullet(\lambda) \subset \mathcal{C}_{0,\bullet}^\infty(X, E^k \otimes F)$ inclusion and the orthogonal projection

$$P_\lambda : \mathcal{C}_{0,\bullet}^\infty(X, E^k \otimes F) \to \mathcal{H}_k^\bullet(\lambda)$$

induce in cohomology inverse isomorphisms of each other.

Demonstration. – The fact that $\mathcal{H}_k^\bullet(\lambda)$ is a sub-complex of $\mathcal{C}_{0,\bullet}^\infty(X, E^k \otimes F)$ comes from the switching property of the operators D_k'' and Δ_k'' . Either now

$$G = \int_{\lambda > 0} \frac{1}{\lambda} dP_1$$

the Green operator of the laplacian Δ_k'' . Like $[P_\lambda, \Delta_k''] = 0$, we have the relations $[G, \Delta_k''] = 0$ and

$$\Delta_k'' G + P_0 = \mathrm{Id}\,.$$

In addition, $[P_\lambda, D_k''] = [G, D_k''] = 0$. We can therefore deduce

$$\begin{aligned} \mathrm{Id} - P_\lambda &= \Delta_k'' G(\mathrm{Id} - P_\lambda) + P_0(\mathrm{Id} - P_\lambda) = \Delta_k'' G(\mathrm{Id} - P_\lambda) \\ &= D_k''\big(\delta_k'' G(\mathrm{Id} - P_\lambda)\big) + \big(\delta_k'' G(\mathrm{Id} - P_\lambda)\big) D_k'', \end{aligned}$$

so that the operator $\delta_k'' G(Id - P_\lambda)$ is a homotopy between Id *and P_λ .* □

We now use a simple classical lemma of homological algebra.

Lemma 4.2. – *Either*

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n \longrightarrow 0$$

a complex of vector spaces of finite dimensions c^0 , c^1 , $\ldots q$, c^n on a body $\mathbb{K}$. Let $h^q = \dim_{\mathbb{K}} H^q(C^\bullet)$. So we have the following inequalities:

(a) *Morse code Inequalities : $h^q \leqslant c^q$, $0 \leqslant q \leqslant n$.*

(b) *Equality of characteristics of Euler-Poincaré $\chi(H^\bullet(C^\bullet)) = \chi(C^\bullet)$:*

$$h^0 - h^1 + \cdots + (-1)^n h^n = c^0 - c^1 + \cdots + (-1)^n c^n.$$

(c) *Strong Morse code inequalities : for all q , $0 \leqslant q \leqslant n$,*

$$h^q - h^{q-1} + \cdots + (-1)^q h^0 \leqslant c^q - c^{q-1} + \cdots + (-1)^q c^0.$$

Demonstration. – If $Z^q = \mathrm{Ker}\, d^q$ and $B^q = \mathrm{Im}\, d^{q-1}$ are for dimensions z^q and b^q , the equality (b) results from the formulas

$$c^q = z^q + b^{q+1}, \qquad h^q = z^q - b^q,$$

while (c) results from (b) applied to the complex

$$0 \to C^0 \to C^1 \to \cdots \to C^{q-1} \to Z^q \to 0. \qquad \square$$

If F *is a holomorphic vectorial fiber on* X *, we define its Euler-Poincaré characteristic by*

$$\chi(X,F) = \sum_{q=0}^{n} (-1)^q \dim H^q(X,F).$$

Combining proposition 4.1 and lemma 4.2, we get for everything $\lambda \geqslant 0$ *and all* q *,* $0 \leqslant q \leqslant n$ *, inequality*

$$h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant h_k^q(\lambda) - h_k^{q-1}(\lambda) + \cdots + (-1)^q h_k^0(\lambda).$$

Let us now evaluate $h_k^q(\lambda)$ *using the 3.14 theorem and let's stretch* $\lambda \in \mathbb{R} \setminus \mathcal{D}$ *to* 0 *by values* > 0 *. It follows :*

Corollary 4.3. – *We have asymptotic inequalities*

(a) $h_k^q \leqslant k^n I^q + o(k^n)$,

(b) $\chi(X, E^k \otimes F) = k^n(I^0 - I^1 + \cdots + (-1)^n I^n) + o(k^n)$,

(c) $h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant k^n(I^q - I^{q-1} + \cdots + (-1)^q I^0) + o(k^n)$,

where I^q *is the integral of curvature*

$$I^q = r \sum_{|J|=q} \int_X \overline{\nu}_B(\alpha_{\complement J} - \alpha_J) d\sigma.$$

According to (3.13), the eigenvalue modules of the magnetic field B *are the* $|\alpha_j|$ *,* $1 \leqslant j \leqslant n$ *. For any point* $x \in X$ *, let us arranke these eigenvalues so that*

$$|\alpha_1 \geqslant |\alpha_2| \geqslant \cdots \geqslant |\alpha_s| > 0 = |\alpha_{s+1}| = \cdots = |\alpha_n|, \qquad s = s(x).$$

Formula (1.5) gives

$$\overline{\nu}_B(a_{\complement J} - \alpha_J) = \frac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)} |\alpha_1 \dots \alpha_s| \sum_{(p_1,\dots,p_s)} \left\{ \alpha_{\complement J} - \alpha_J - \sum (2p_j+1)|\alpha_j| \right\}_+^{n-s}$$

with the notation $\{\lambda\}_+^0 = 0$ *if* $\lambda < 0$ *and* $\{\lambda\}_+^0 = 1$ *if* $\lambda \geqslant 0$ *. As the quantity*

$$\alpha_{\complement J} - \alpha_J - \sum (2p_j+1)|\alpha_j|$$

is always $\leqslant 0$ *,* $\overline{\nu}_B(\alpha_{\complement J} - \alpha_J)$ *can be non-null only if* $s = n$ *. In the latter case* $\alpha_{\complement J} - \alpha_J - \sum(2p_j+1)|\alpha_j| = 0$ *if and only if* $p_1 = \cdots = p_n = 0$ *and* $\alpha_j < 0$ *for* $j \in J$ *,*

$\alpha_j > 0$ for $j \in \complement J$. This results in the form $ic(E)$ is non-degenerated q . For $x \in X(q)$ (see notations of the introduction) and $|J| = q$, so we have

$$\overline{\nu}_B(\alpha_{\complement J} - \alpha_J) = (2\pi)^{-n}|\alpha_1 \ldots \alpha_n| > 0$$

if J is the multi-index $J(x) = \{j\,;\ \alpha_j(x) < 0\}$ and $\overline{\nu}_B(\alpha_{\complement J} - \alpha_J) = 0$ si $J \neq J(x)$. This results in

$$I^q = r\int_{X(q)} (2\pi)^{-n}(-1)^q\alpha_1 \ldots \alpha_n \, d\sigma = \frac{r}{n!}\int_{X(q)} (-1)^q\Big(\frac{i}{2\pi}c(E)\Big)^n.$$

The fundamental theorem 0.1 is then only a reformulation of the corollary 4.3. The above reasoning shows that the harmonic forms of $H^q(X, E^k \otimes F)$ focus asymptotically on $X(q)$, and that by each point of $X(q)$ their direction tends to align with the q -sub-space of TX corresponding to the negative part of $ic(E)$. Moreover, only the minimum eigenvalue of energy $p_1 = \cdots = p_n = 0$ of the harmonic oscillator intervenes for these forms. For $q = 1$, the inequality of Morse code 4.3 (c) is written as follows

$$h^1_k - h^0_k \leqslant k^n(I^1 - I^0) + o(k^n),$$

hence in particular an asymptotic reduction of the number of sections. holomorphs of the fiber $E^k \otimes F$.

Theorem 4.4. – *On a*

$$\dim H^0(X, E^k \otimes F) \geqslant r\frac{k^n}{n!}\int_{X(\leqslant 1)} \Big(\frac{i}{2\pi}c(E)\Big)^n - o(k^n).$$

More generally, the addition of inequality 4.3 (c) for the indexes $q + 1$ and $q - 2$ leads to

$$h^{q+1}_k - h^q_k + h^{q-1}_k \leqslant k^n(I^{q+1} - I^q + I^{q-1}) + o(k^n),$$

hence the reduction

$$\dim H^q(X, E^k \otimes F) \geqslant r\frac{k^n}{n!}\sum_{j=0,\pm 1} (-1)^q \int_{X(q+j)} \Big(\frac{i}{2\pi}c(E)\Big)^n - o(k^n). \tag{4.5}$$

5. Characterization of the varieties of Egošezon.

Let X a variety $\mathbb{C}$ -analytical compact related of dimension n . We call algebraic dimension of X , noted $a(X)$, the degree of transcendence on $\mathbb{C}$ of the body $K(X)$ of the meromorphic functions on X . According to a well-known theorem by Siegel [15], the dimension algebraic of X always checks for inequality $0 \leqslant a(X) \leqslant n$. When $a(X) = n$, it is said that X is a space of Mešezon. As we will see, the algebraic dimension of X asymptotically imposes strong constraints on the dimension of the section spaces of a holomorphic vectorial fiber.

Theorem 5.1. – *Let a the dimension algebraic of X , F a fibrous one holomorphic vector of rank r and E a linear fiber on X . Then, there is a constant $C_E \geqslant 0$ that does not depend on than E such as*

$$\dim H^0(X, E^k \otimes F) \leqslant C_E r k^a + o(k^a).$$

Demonstration. – We essentially take the arguments of Y.T. Siu [16]. Let $\{W_\ell\}$ be an overlay of X by open of coordinates $W_\ell \subset \mathbb{C}^n$, and $B_j = B(a_j, R_j)$, $1 \leqslant j \leqslant m$, a family of relatively compact balls in the open W_ℓ , such as concentric balls $B'_j = B(a_j, \frac{1}{7}R_j)$ cover X . Provide E , F of hermitian metrics, and either $\exp(-\varphi_j)$ the weight representing the metric of E in a trivialization of E in the vicinity of $\overline{B}_j$.

Let $s \in H^0(X, E^k \otimes F)$ be a holomorphic section that cancels itself out to the order p in one point $x_j \in B'_j$. Inclusions

$$B'_j \subset B(x_j, \frac{2}{7}R_j) \subset B(x_j, \frac{6}{7}R_j) \subset B_j$$

and the Schwarz lemma applied to the two intermediate balls lead to inequality

$$\sup_{B'_j} |s| \leqslant \exp(Ak + C_F) 3^{-p} \sup_{B_j} |s|, \tag{5.2}$$

where $A = \max_{1\leqslant j\leqslant m} \operatorname{diam} \varphi_j(B_j)$ depends only on of E , and where C_F is a constant $\geqslant 0$ which depends on the metric of F .

Let $\rho \leqslant r = \operatorname{rank}(F)$ be the maximum for $x \in X$ of the dimension of the subspace of the F_x fiber generated by the $s(x)$ vectors when s describes $\bigcup_{k\in\mathbb{N}} H^0(X, E^k \otimes F)$. If $\rho = 0$, then $H^0(X, E^k \otimes F) = 0$ for all k . Let us now distinguish two cases according to whether $\rho = 1$ or $\rho > 1$.

(a)Assume $\rho = 1$.

Let $h_k = \dim H^0(X, E^k \otimes F)$, assumed > 0 . Under the hypothesis $\rho = 1$, the global sections of $E^k \otimes F$ define a holomorphic application

$$\Phi_k : X \smallsetminus Z_k \to \mathbb{P}^{h_k - 1}(\mathbb{C})$$

where $Z_k \subset X$ is the analytical subset of their zeros common. Let d the maximum rank of the differential Φ'_k on $X \smallsetminus Z_k$. We necessarily have $d \leqslant a$, otherwise the body of the rational fractions of $\mathbb{P}^{h_k-1}(\mathbb{C})$ would induce a body of functions metamorphs on X of degree of transcendence $\geqslant d > a$, which is absurd. Let's choose for all $j = 1, \ldots, m$ a point $x_j \in B'_j \smallsetminus Z_k$ such that Φ'_k is of maximum rank $= d$ in x_j , and either $s_0 \in H^0(X, E^k \otimes F)$ a section that does not cancel in any way point x_j . For all $s \in H^0(X, E^k \otimes F)$, the quotient s/s_0 is well defined as a meromorphic function on X , and moreover s/s_0 is a holomorphic function in the neighborhood of x_j , constant along the fibers of Φ_k . As Φ_k is a submersion in the vicinity of each point x_j , one can choose a submergence sub-variety M_j of dimension d passing through x_j and transverse to the fiber $\Phi_k^{-1}(\Phi_k(x_j))$. The section s will cancel in the order p at each point x_j , $1 \leqslant j \leqslant m$

, if and only if partial derivatives of order $< p$ *of* s/s_0 *along* M_j *cancel each other in* x_j *. This corresponds to the total cancellation of*

$$m\binom{p+d-1}{d}$$

derivatives. If we choose $p = [Ak + C_F] + 1$ *, then inequality (5.2) leads to*

$$\sup_X |s| \leqslant \left(\frac{e}{3}\right)^p \sup_X |s|,$$

hence $s = 0$ *. As* $d \leqslant a$ *, we therefore get*

$$\dim H^0(X, E^k \otimes F) \leqslant m\binom{p+a-1}{a} \leqslant C_E k^a + o(k^a)$$

with $C_E = mA^a/a!$ *.*

(b)Assume $\rho > 1$*.*

There are then sections $s_t \in H^0(X, E^{k_t} \otimes F)$ *,* $1 \leqslant t \leqslant \rho$ *, and a point* $x_0 \in X$ *such as the vectors* $s_1(x_0), \ldots, s_\rho(x_0)$ *are linear independent. By construction, for any* $k \in \mathbb{N}$ *and any section* $s \in H^0(X, E^k \otimes F)$ *, the right* $\mathbb{C} \cdot s(x)$ *is contained in the subspace generated by* $(s_1(x), \ldots, s_\rho(x))$ *, except perhaps above the analytical subset* $\{x \in X; s_1 \wedge \ldots \wedge s_\rho(x)\} = 0$ *. So we have an injectable morphism*

$$H^0(X, E^k \otimes F) \to \bigoplus_{1 \leqslant t \leqslant \rho} H^0(X, E^{k + k_{\hat{t}}} \otimes \Lambda^p F)$$

where $k_{\hat{t}} = (k_1 + \cdots + k_\rho) - k_t$ *, whose index component* t *is given by the morphism* $s \to s_1 \wedge \cdots \wedge \widehat{s}_t \wedge \cdots \wedge s_\rho \wedge s$ *. The image of* $H^0(X, E^k \otimes F)$ *on each component is formed collinear sections at almost any point in* $s_1 \wedge \cdots \wedge s_\rho$ *. So we meet again in a situation analogous to (a), where* F *is replaced by by* $E^{k_{\hat{t}}} \otimes \Lambda^\rho F$ *; thereafter :*

$$\dim H^0(X, E^k \otimes F) \leqslant C_E \rho k^a + o(k^a), \qquad \rho \leqslant r. \qquad \square$$

Let's choose in particular for F *the trivial fiber* $X \times \mathbb{C}$ *. By comparing the theorems 4.4 and 5.1, we obtain the characterization geometrically following varieties of egošezon.*

Theorem 5.2. – *For a variety* $\mathbb{C}$ *-analytical compact related* X *of dimension* n *either of Mešezon, it is enough that there is a fiber in straight lines hermitian holomorph* E *above* X *such as*

$$\int_{X(\leqslant 1)} (ic(E))^n > 0. \qquad \square$$

This theorem in turn leads to the 0.8 theorem. since 0.8 (c) $\Rightarrow$ *0.8(b)* $\Rightarrow$ *0.8(a). On improves Y.T.'s results. Siu [17], [18], and one can thus finds in particular a new demonstration of the conjecture of Grauert-Riemenschneider [10].*

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Jean-Pierre Demailly
Université de Grenoble I
Institut Fourier
Laboratoire de Mathématiques, B.P. 74
38402 St-Martin d'Hères Cedex