Mémoire de la Société Mathématique de France, n°19
Supplément au Bulletin de la S.M.F.
Tome 113, 1985, fascicule 2

Monge-Ampère measures and geometric characterization of affine algebraic varieties

by Jean-Pierre Demailly

Université de Grenoble I, Institut Fourier
Laboratoire de Mathématiques associé au C.N.R.S. n°188
BP 74, F-38402 Saint-Martin d'Hères, France

Résumé. À toute fonction d'exhaustion plurisousharmonique continue φ sur un espace de Stein, nous associons une collection de mesures positives portées par les surfaces de niveau de φ, et définies à l'aide des opérateurs de Monge-Ampère au sens de Bedford et Taylor. Nous montrons que ces mesures jouent un rôle fondamental dans l'étude des propriétés de croissance et de convexité des fonctions plurisousharmoniques ou holomorphes. Lorsque le volume de Monge-Ampère de la variété est fini, un théorème d'algébricité de type Siegel s'applique aux fonctions holomorphes à croissance φ-polynomiale. Nous en déduisons que la finitude du volume de Monge-Ampère, associée à une minoration convenable de la courbure de Ricci, est une condition géométrique nécessaire et suffisante caractérisant les variétés algébriques affines.

Abstract. To every continuous plurisubharmonic exhaustion function φ on a Stein space, we associate a collection of positive measures with support in the level sets of φ, defined by means of the Monge-Ampère operators in the sense of Bedford and Taylor. We show that these measures play a prominent part in the study of growth and convexity properties of plurisubharmonic or holomorphic functions. When the variety has finite Monge-Ampère volume, an algebraicity theorem of Siegel type holds for holomorphic functions with φ-polynomial growth. From this result, we deduce that the finiteness of Monge-Ampère volume, together with a suitable lower bound of the Ricci curvature, is a necessary and sufficient geometric condition characterizing affine algebraic varieties.

0037-9484/85 02 1 \$ 14.50

Contents

0. Introduction.

This study takes place in the framework of the spaces complex analytical. The first section is devoted to a definition of differential forms, positive currents and functions plurisubharmonic on a complex space possibly X singular. Given a local X dipping in an open $\Omega \subset \mathbb{C}^N$, we define the differential forms on X as restrictions X forms "ambient" on Ω ; spaces currents are deduced by duality as in the smooth case.

Definition 0.1. – *Either a $V : X \to [-\infty, +\infty[$ function.*

(a) *V be called plurisubharmonic (psh abbreviated /) on X if V is locally restriction X of psh functions on space ambient $\mathbb{C}^N$.*

(b) *V will be called weakly psh if V is locally integrable and increased on X and if $dd^cV \geqslant 0$.*

Notation : *raised here*

$$d^c = i(\overline{\partial} - \partial), \quad \textit{de sorte que} \quad dd^c = 2i\,\partial\overline{\partial}.$$

Any psh psh function is low, but in general a low psh function does not necessarily identify almost everywhere a psh function. However, we show that both concepts coincide when X space is locally irreducible. The proof of this result uses two ingredients: firstly characterization psh functions due to Fornaess and Narasimhan [FN], secondly an extension theorem psh functions of bounded through the singular place X (which uses the resolution of singularities). We also study the transformation of closed positive currents and psh functions own direct image.

In§2 we essentially resuming the method developed by Bedford and Taylor [BT2] to give meaning to the positive current $dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$ when φ_j psh functions are locally bounded, and we generalize to the case where one of the functions (either by φ_i example) is not locally bounded. Conventional inequalities Chern-Levine-Nirenberg can then be stated as follows :

Theorem 0.2. – *For open $\omega \Subset X$ and any compact $K \subset \omega$ there are constant C_1, C_2 depending only on ω K and we have such mark-ups following mass* :

(a) $\displaystyle\int_K \|dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k\| \leqslant C_1 \|\varphi_1\|_{L^1(\omega)} \|\varphi_2\|_{L^\infty(\omega)} \cdots \|\varphi_k\|_{L^\infty(\omega)},$

(b) $\displaystyle\int_K \|\varphi_1\, dd^c\varphi_2 \wedge \ldots \wedge dd^c\varphi_k\| \leqslant C_2 \|\varphi_1\|_{L^1(\omega)} \|\varphi_2\|_{L^\infty(\omega)} \cdots \|\varphi_k\|_{L^\infty(\omega)}.$

Finally, we show in this continuity Monge-Ampère of sequential operators

$$(\varphi_1, \ldots, \varphi_k) \longmapsto dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k \quad \text{et} \quad \varphi_1\, dd^c\varphi_2 \wedge \ldots \wedge dd^c\varphi_k$$

for decreasing sequences of $\varphi_1^\nu, \ldots, \varphi_k^\nu$ psh functions.

Assume now that X is a Stein space and is X with a comprehensive continuous psh function $\varphi : X \to [-\infty, R[$. Then we denote

$$B(r) = \big\{z \in X\,;\ \varphi(z) < r\big\}, \qquad S(r) = \big\{z \in X\,;\ \varphi(z) = r\big\}, \qquad r \in [-\infty, R[$$

the "pseudoboules" and "Pseudosphères" associated with φ. With these data, we show that we can naturally associate a collection of positive measures μ_r carried by spheres $S(r)$, we call for measures Monge-Ampère associated φ. These are defined simply by

$$\mu_r(h) = \int_{S(r)} h\,(dd^c\varphi)^{n-1} \wedge d^c\varphi, \qquad n = \dim X,$$

when φ is $\mathcal{C}^2$ class and when r is regular value ofφ. In the case where only φ is continuous, it is necessary to use the definition of Bedford-Taylor and ask for $(dd^c)^n$

$$\mu_r = \big(dd^c \max(\varphi, r)\big)^n - \mathbb{1}_{X \smallsetminus B(r)}(dd^c\varphi)^n.$$

We then have a formula type Lelong-Jensen, whose proof is an immediate consequence of the theorems of Stokes and Fubini (cf. §3).

Theorem 0.3. – *Any function psh V is on X μ_r integrable regardless $r < R$, and has the formula*

$$\int_{-\infty}^{r} dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1} = \mu_r(V) - \int_{B(r)} V(dd^c\varphi)^n.$$

It further shows that μ_r measures depend continuously on φ respect to decreasing sequences. This allows you to see as in the case $\mathcal{C}^\infty$ (μ_r) that the family is the family of weakly continuous measures left that disintegrates the current $(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi$ positive on the spheres $S(r)$.

The measures thus constructed μ_r enjoy a number of important natural properties for the study of growth and the convexity of psh functions.

Section 4 examines the extent "Residual" $\mu_{-\infty} = \mathbb{1}_{S(-\infty)}(dd^c\varphi)^n$, driven by in the polar -sem-ble $S(-\infty)$. From (0.3), the measurement $\mu_{-\infty}$ can also be defined as the weak limit ofμ_r when r approaches $-\infty$. Drawing inspiration from our Previous work [De4, De5], we show that the measure $\mu_{-\infty}$ essentially depends behavior asymptotic φ near $S(-\infty)$. This result follows the classical inequality

$$(dd^\varphi)^n \geqslant 2^n \sum_{x \in X} \nu(\varphi, x)^n\, \delta_x, \tag{0.4}$$

where denotes the number of $\nu(\varphi, x)$ Lelong in any of φ Point $x \in X$ and δ_x the Dirac measure in x (at a point singular x, this measure must be counted with multiplicity equal the multiplicity of X in x).

In§5, we show that the measures satisfy the principle μ_r the maximum with respect to psh functions, namely that for all psh function V we have the equality:

$$\sup_{B(r)} V = \text{sup essentiel de } V \text{ relativement à } \mu_r. \tag{0.5}$$

The remarkable fact is that equality holds that the support μ_r can be very incomplete in $S(r)$, such as in If the $B(r)$ pseudoboules are analytic polyhedra.

Paragraph 6 generalizes to the present situation of the properties classic convexity due to P. Lelong, on averages psh functions on the balls, spheres, polydisques .... We show that the natural geometrical assumption that underlies the validity of the convexity properties is the fact that the function φ either homogeneous solution of Monge-Ampère equation $(dd^c\varphi)^n \equiv 0$. Specifically :

Theorem 0.6. – *Assume that $(dd^c\varphi)^n \equiv 0$ on the open $\{\varphi > A\}$. V be a psh function on X. Then the V sup on $B(r)$, average $\mu_r(V)$ and more generally averages in standard L^p, $r \mapsto [\mu_r(V_+^p)]^{1/p}$ are increasing convex functions of $r \in]A, R]$.*

Verification of this result is obtained by elementary calculations of second derivatives, involving the formula of Jensen and 0.3 theorems of Stokes and Fubini. More generally, we demonstrate a version with "parameter" theorem 0.6, on $\mu_{y,r}$ measurements on fiber $\pi^{-1}(y)$ an holomorphic fibration $\pi : X \to Y$. The psh function φ Data on X is assumed exhaustive on the fibers and such that $(dd^c\varphi)^n \equiv 0$ on open $\{\varphi > A\}$ where is n fiber size. Then the average $\mu_{y,r}(V)$ and averages L^p standard functions are weakly psh couple (y, z) on $Y \times \mathbb{C}$, if we set $r = \operatorname{Re} z$. One draws easily the following extension of Theorem 0.6 product areas.

Theorem 0.7. – *Let $X_1, \ldots, X_k$ Stein spaces, provided comprehensive continuous psh functions $\varphi_j : X_j \to [-\infty, R_j[$ such as the open $(dd^c\varphi_j)^{n_j} \equiv 0$ $\{\varphi_j > A_j\}$, $n_j = \dim X_j$. So if V is psh on $X_1 \times \cdots \times X_k$, the average standard L^p*

$$M_V^p(r_1, \ldots, r_k) = \Big[\mu_{r_1} \otimes \cdots \otimes \mu_{r_k}(V_+^p)\Big]^{1/p}$$

is convex in the same $(r_1, \ldots, r_k) \in \prod_{1 \leqslant j \leqslant k}]A_j, R_j[$ variables.

In paragraphs 7 and 8, we make the additional assumption the volume of X has moderate growth to infinity (the "radius" R is here assumed to $+\infty$). In a way precise, we assume that

$$\lim_{r \to +\infty} \frac{1}{r} \|\mu_r\| = 0. \tag{0.8}$$

Under this assumption, the formula of Jensen 0.3 implies inequality following fundamental :

$$\int_X dd^c V \wedge (dd^c\varphi)^{n-1} \leqslant \liminf_{r \to +\infty} \frac{1}{r}\, \mu_r(V_+), \tag{0.9}$$

which follows a number of results concerning the growth of psh functions or distribution of values holomorphic functions (as suggested in Article

N. Sibony M.W. and Wong [SW]). In particular, any function psh or holomorphic bounded on X is constant.

Given a holomorphic function on f X we define secondly the "degree" of f relatively to φ by

$$\delta_\varphi(f) = \limsup_{r \to +\infty} \frac{1}{r}\, \mu_r(\log_+ |f|), \tag{0.10}$$

and we say that is f φ -polynomiale is if $\delta_\varphi(f)$ finished. Inequality (0.9) results while the cancellation order f a regular point $a \in X$ verifies the estimate:

$$\mathrm{ord}_a(f) \leqslant C(a)\, \delta_\varphi(f).$$

By elementary reasoning linear algebra due to Siegel, he algebraicity result the following theorem (X assumed irreducible).

Theorem 0.11. – *Either $K_\varphi(X)$ body meromorphic functions of the form where f/g f and are g φ -polynomiales. So* :

(a) $0 \leqslant \deg \mathrm{tr}_{\mathbb{C}} K_\varphi(X) \leqslant \dim_{\mathbb{C}} X$;

(b) If $\deg \mathrm{tr}_{\mathbb{C}} K_\varphi(X) = \dim_{\mathbb{C}} X$, *then the body $K_\varphi(X)$ is finitely.*

As a special case of this theorem, we find the result W. Stoll [St1] characterizing algebraic varieties by $\mathbb{C}^N$ the property that the growth area is minimal.

The second part B of this work is devoted to a characterization of affine algebraic varieties with an intrinsic geometric criterion, involving the finite volume Monge-Ampère and a reduction of the Ricci curvature. In a way precisely, we prove the following result:

Theorem 0.12. – *Either X analytical variety complex, smooth, connected, dimension n. So is X analytically isomorphic to an affine algebraic variety if X_{alg} and only if X satisfies condition*(c) *below and if X has a function of exhaustion of φ strictly psh $\mathcal{C}^\infty$ class such as:*

(a) $\mathrm{Vol}(X) = \displaystyle\int_X (dd^c\varphi)^n < +\infty$;

(b) *The Ricci curvature of the metric $\beta = dd^c(e^\varphi)$ admits a reduction of the form*

$$\mathrm{Ricci}(\beta) \geqslant -\frac{1}{2} dd^c\psi,$$

with $\psi \in \mathcal{C}^\infty(X, \mathbb{R})$ where $\psi \leqslant A\varphi + B$ and A, are B constants $\geqslant 0$;

(c) *even degree cohomology spaces $H^{2q}(X, \mathbb{R})$ are finite dimensional.*

The ring of regular functions of the algebraic structure is X_{alg} then given by the $K_\varphi(X) \cap \mathcal{O}(X)$ intersection.

Following the work of W. Stoll on varieties strictly parabolic (see [St2] and [Bu]), D. Burns posed the problem of the characterization of affine algebraic varieties in terms of functions

of exhaustion with specific properties, checking for example the condition of homogeneity in $(dd^c\varphi)^n \equiv 0$ outside a compact. 0.12 The theorem provides a partial response this issue. This second in line conditions sufficient obtained by Mok Siu and Yau [SY], [MI], [Mok1,2,3], although that our assumptions are significantly different from those of abovementioned work. Our argument is, moreover similar in outline to the approach taken by [Mok1,2,3].

Section 10 demonstrates the necessity of the conditions 0.12 (a, b, c) for any algebraic set $X \subset \mathbb{C}^N$. The function φ is then given by $\varphi(z) = \log(1 + |z|^2)$, so that the $dd^c\varphi$ metric coincides with the metric of Fubini-Study projective space $\mathbb{P}^N$. Through an explicit calculation of the curvature Ricci we check the curvature of inequality (b) takes place with $\psi = \log\sum_{K,L} |J_{K,L}|^2$ where $J_{K,L}$ designate Jacobian determinants associated with a system of polynomial equations of$X$. We show more against by-example that the condition curvature (b) is indispensable.

Proof of the sufficiency of conditions (a, b, c) is the subject of§11,12,13. General demonstration scheme is as follows, With L^2 estimates Hörmander-Nakano-Skoda for the operator $\overline{\partial}$ and through hypothesis (b), a system built $F = (f_1, \ldots, f_N)$ functions holomorphic φ -polynomiales that, outside an analytic set $S \subset X$ defines an embedding of $X \smallsetminus S$ in $\mathbb{C}^N$. Assuming (a) finite volume, the algebraicity 0.11 theorem implies that the degree of transcendence $f_1, \ldots, f_N$ of functions equals $n = \dim X$. The morphism F send therefore X in an algebraic variety $M \subset \mathbb{C}^N$ dimension n.

The main challenge that remains is then to prove that the dip is almost surjective, that is to say the open $\Omega = F(X \smallsetminus S)$ is an open Zariski M. We get this result showing first that the $(1,1)$ Platform extends $F_*(dd^c\varphi)$ a closed positive current T finite mass M as $T = 0$ on $M \smallsetminus \Omega$; given the mass resulting estimates construction, it is mainly through the integration method by parts developed by H. Skoda [SK 5] and H. El Mir [EM]. To give an overview of the result of reasoning, look at the already significant case where $N = n$, i.e. $M = \mathbb{C}^n$ the case. There is then a function psh V on $\mathbb{C}^n$ to minimal growth, i.e. $V(z) \leqslant C_1 \log_+ |z| + C_0$, as $dd^cV = T$. By construction, the function $\tau = V - \varphi \circ F^{-1}$ is pluriharmonique on Ω to more τ approaches $-\infty$ at any point $\partial\Omega$. he The result is that the closed assembly $M \smallsetminus \Omega$ is multipolar. For $M \smallsetminus \Omega$ show that is actually an algebraic set, our method is to verify, using the theorem again algebraicity of 0.11, the 1 Platform is holomorphic $h = \partial\tau$ extends in a rational meromorphic form on $M = \mathbb{C}^n$.

Where M is an affine algebraic variety any, the link in $\mathbb{C}^n$ between -courants positive $(1,1)$ closed finite projective mass T psh functions and growth Minimum no longer applies. However, we can show (see Appendix§15) that $dd^cV \geqslant T$ differential inequality is always resolves on M with psh V solution such that the current $dd^cV - T$ either $\mathcal{C}^\infty$ class and growth polynomial. The end of the demonstration is then almost identical. The hypothesis (c), in turn, serves to demonstrate that the "Zariski topology" on X is quasi-compact and So X that can be covered by a finite number of open the $X \smallsetminus S$ form (cf. S13). We do not know in fact if the hypothesis (C) is really essential.

Finally, note that the 0.12 theorem can be extended to areas complexes isolated Singularities (see
S9), but the extension to the case any raises difficulties which will be studied§14.

A. Monge-Ampère measures and croissance of plurisubharmonic functions.

1. Courants and plurisubharmonic functions on complex spaces.

The purpose of this section is to provide a definition of current and psh functions of a complex analytic space eventually singular. The reader who wishes to consider in the following that the smooth case can jump directly to§2.

Let X be a complex surface of reduced sheer size n, X_{reg} (resp. X_{sing}) all of its regular points (resp. singular). As the definitions that we will consider are local, we can unrestricted X assume that identifies with an analytic subset closed an open $\Omega \subset \mathbb{C}^N$ by means of an embedding $j : X \to \Omega$.

We define the space $\mathcal{C}^k_{p,q}(X)$ class Platforms (p,q) $\mathcal{C}^k$ on X, $k \in \mathbb{N} \cup \{\infty\}$, as the image of the morphism restriction

$$j* : \mathcal{C}^k_{p,q}(\Omega) \to \mathcal{C}^k_{p,q}(X_{\text{reg}}),$$

equipped with the quotient topology. If $j_1 : X \to \Omega_1 \subset \mathbb{C}^{N_1}$ Another dip, there are (locally) applications holomorphic $f : \Omega \to \mathbb{C}^{N_1}$ and $g : \Omega_1 \to \mathbb{C}^N$ such as $j_1 = f \circ j$ and $j = g \circ j_1$. The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ j\ } & f^{-1}(\Omega_1) \subset \Omega \\ {\scriptstyle j_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}\times f} \\ \Omega_1 \supset g^{-1}(\Omega) & \xrightarrow[g \times \mathrm{Id}]{} & \Omega \times \Omega_1, \end{array}$$

watch while the morphisms j and j_1 induce much the same image space $\mathcal{C}^k_{p,q}(X)$ because $\mathrm{Id} \times f$ and are $g \times \mathrm{Id}$ Closed smooth dips.

Definition 1.1. – *by $\mathcal{D}_{p,q}(X)$ (resp designates. $\mathcal{D}^k_{p,q}(X)$) space on the Platforms (p,q) class X $\mathcal{C}^\infty$ (resp. $\mathcal{C}^k$) and compact support, endowed with the topology induetive limit. The dual space $\mathcal{D}'_{p,q}(X)$ is by definition space currents bidimension of (p,q) and bidegree $(n-p, n-q)$ on X. The currents belonging to the subspace will be called $[\mathcal{D}^k_{p,q}(X)]'$ Current order k.*

If $T \in [\mathcal{D}^k_{p,q}(X)]'$ the current $j_*T \in [\mathcal{D}^k_{p,q}(\Omega)]'$ defined by

$$\langle j_*T, v\rangle = \langle T, j * v\rangle$$

$v \in \mathcal{D}^k_{p,q}(\Omega)$ for any form, has support in $j(\Omega)$. However, for $k \geqslant 1$ a current $\theta \in [\mathcal{D}^k_{p,q}(\Omega)]'$ support in $j(\Omega)$ not necessarily from a current T set to X although X is smooth.

Differential operators d, ∂, $\overline{\partial}$ usual and the operator of exterior multiplication by a form $\mathcal{C}^\infty$ other hand are extended by the current duality, just as in the smooth case. It would be particularly interesting to know generally calculate the cohomology groups d of local and $\overline{\partial}$ operators; we do not even know if not done in these groups are always zero in the case of whatever singularities.

Definition 1.2. – *A current $T \in \mathcal{D}'_{p,p}(X)$ be said () weakly positive if the current bidegree (n, n)*

$$T \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p$$

$\geqslant 0$ *is a measurement for any system of $(1, 0)$ Platforms $(\alpha_1, \ldots, \alpha_p)$ of $\mathcal{C}^\infty$ class on X.*

This amounts to saying that the current is $j_*T \geqslant 0$ on Ω ; in particular, on current $T \geqslant 0$ X is necessarily order 0.

Now let $F : X \to Y$ a morphism of analytic spaces X, Y respective dimensions n, m. To ensure that the morphism inverse image

$$F^* : \mathcal{C}^k_{p,q}(Y) \to \mathcal{C}^k_{p,q}(X)$$

is well defined, just check the following lemma :

Lemma 1.3. – *Either a dip and $j : Y \to \Omega \subset \mathbb{C}^N$ $\alpha \in \mathcal{C}^k_{p,q}(\Omega)$ a shape such that $\alpha_{|Y_{\text{reg}}} = 0$. So $F^*\alpha_{|x_{\text{reg}}} = 0$.*

Demonstration. Presumably smooth and related X. If $F(X) \not\subset Y_{\text{sing}}$ then $F^{-1}(Y_{\text{reg}})$ is dense in X and the result follows by continuity. The only problem is if $F(X) \subset Y_{\text{sing}}$. By induction on the dimension of Y and decomposing F as

$$X \xrightarrow{F} Y_{\text{sing}} \hookrightarrow Y$$

we see that it suffices to consider instead of F the case of the morphism $Y_{\text{sing}} \hookrightarrow Y$ of inclusion. Lemma 1.3 then follows from α and continuity of the following lemma :

Lemma 1.4. – *Either a a regular on Y_{sing}. So he $\{a_\nu\} \subset Y_{\text{reg}}$ exists a sequence of points converging a, as in the grasmannienne of m -planes of $\mathbb{C}^N$ space tangent $T_{a_\nu}Y_{\text{reg}}$ converges to a plane containing T_aY_{sing}.*

Demonstration. This is a consequence of the existence of stratifications whitney Y, see [Wh1] and [Wh2]. □

Suppose that the morphism $F : X \to Y$ own. We define the direct image enforcement

$$F_* : [\mathcal{D}^k_{p,q}(X)]' \longrightarrow [\mathcal{D}^k_{p,q}(Y)]'$$

by duality, asking for any current $T \in [\mathcal{D}^k_{p,q}(X)]'$ and any form $\alpha \in \mathcal{D}^k_{p,q}(Y)$

$$\langle F_*T, \alpha\rangle = \langle T, F^*\alpha\rangle.$$

If $T \geqslant 0$ is, it is clearly the case for F_*T. In addition, the direct image morphism commutes with F_* d operators d^c, ∂, $\overline{\partial}$. If $T \geqslant 0$ is closed, is F_*T therefore also $\geqslant 0$ closed.

Now to the definition of psh functions.

Definition 1.5. – *Either $V : X \to [-\infty, +\infty[$ a function which is not identically $-\infty$ on any open X. We say that is plurisubharmonic on V X (psh for short) if for all local dip $j : X \hookrightarrow \Omega \subset \mathbb{C}^N$, is V locally restricting a psh function on Ω.*

J. E. and R. Fornaess Narasimhan gave basic characterization Next of psh functions on a complex space.

Theorem 1.6([FN], Th. 5.3.1) **–** *A function $V : X \to [-\infty, +\infty[$ is psh on X iff :*

(a) *V is upper semi-continuous ;*

any holomorphic *$f : \Delta \to X$ disc X unit, $V \circ f$ is sub-harmonic or identically equal to$-\infty$ on Δ.*

With this result, it can easily generalize theorem Brelot extension to the case of complex spaces.

Theorem 1.7. – *Let X a complex space $Y \subset X$ locally irreducible analytic subset empty interior in X. V be a psh function on $X \setminus Y$, locally increased near Y. Then there is a function psh V^* on X extending V, unique, given by*

$$V * (y) = \limsup_{x \in X \setminus Y,\ x \to y} V(x), \qquad y \in Y.$$

Demonstration.

(A) UniquenessV^*. As V^* is upper semi-continuous, we have for all $y \in Y$

$$V^*(y) = \limsup_{x \in X \setminus Y,\ x \to y} V^*(x) \geqslant \limsup_{x \in X \setminus Y,\ x \to y} V(x).$$

Conversely, choose a holomorphic such $f : \Delta \to X$ and that $f(0) = y$ $f(\Delta) \not\subset Y$. So 0 is isolated in $f^{-1}(Y)$ and as $V^* \circ f$ is psh on Δ he comes

$$V^(y) = V^*(f(0)) = \limsup_{t \neq 0,\ t \to 0} V(f(t)) \leqslant \limsup_{x \in X \setminus Y,\ x \to y} V(x).$$

(B)*Plurisousharmonicité of* V^*. The result is local to X. According desingularization theorem of Hironaka [Hi], there X' a smooth space and a proper change $\sigma : X' \to X$; by definition, is proper and σ induced outside a analytic set $Z \subset X$ an isomorphism

$$\sigma : X' \setminus \sigma^{-1}(Z) \xrightarrow{\sim} X \setminus Z.$$

For $x \in X$ the $\sigma^{-1}(x)$ fiber is compact and connected. While $\sigma^{-1}(x)$ was not connected, the point would be a x open neighborhood irreducible U (X is assumed locally irreducible) as $\sigma^{-1}(U)$ is not related ; but then would $U \setminus Z$ related and not related $\sigma^{-1}(U) \setminus$

$\sigma^{-1}(Z)$, which is absurd. The $V \circ \sigma$ psh function on$X' \setminus \sigma^{-1}(Y)$ and locally increased near $\sigma^{-1}(Y)$. By Theorem of Brelot on the smooth case, $V \circ \sigma$ extends to a function psh V' on X'. The V' function is necessarily constant on $\sigma^{-1}(x)$ fiber, so V' induced transition to a quotient function V^* upper semi-continuous on X. show now that is V^* psh on X using 1.7 theorem. Given a germ of holomorphic $f : (\Delta, 0) \to (X, x)$, there X' in a curve germ (Γ', x') above image $\Gamma = f(\Delta)$, so there exists an integer $k \in \mathbb{N}^*$ and a germ $f' : (\Delta, 0) \to (X', x')$ such as $f(t^k) = \sigma(f'(t))$. By Following is $V^*(f(t^k)) = V'(f'(t))$ psh on $(\Delta, 0)$, which implies that $V^* \circ f$ also psh. □

Proposition 1.8. – *Any function psh V is on X locally integrable for area measuring X (on a $j : X \to \Omega \subset \mathbb{C}^N$) any dip.*

Demonstration. V being locally increased by definition, we can assume $V \leqslant 0$. Left to rotate the coordinates, it existe at any point X a local dip $j : X \hookrightarrow P$ a polydisk of $\mathbb{C}^N$ as it has $\pi^I : X \to P^I$ own projections of finite fibers on n -planes of z_j coordinates $j \in I$, $I \subset \{1, \ldots, N\}$, $|I| = n$. Thus, for any I, there analytical $S_I \subset P^I$ together such that the restriction $\pi^I : X \setminus (\pi^I)^{-1}(S_I) \to P^I \setminus S_I$ is a finished coating. The $\pi^I_* V$ defined function

$$\pi^I_* V(y)(y) = \sum_{x \in (\pi^I)^{-1}(y)} V(x)$$

psh is $\leqslant 0$ on $P^I \setminus S_I$, thus extends into a psh function V_I on P^I whole. As the area measurement is given by X

$$d\sigma_X = \sum_{|I|=n} (\pi^I)^* d\lambda_{\mathbb{C}^n},$$

where $d\lambda$ is Lebesgue measure, the conclusion then follows from that the V_I are locally integrable on P^I. □

The proposed 1.8 shows that we can consider any function psh on X as a current bidegree $(0, 0)$. For regularization a local extension to V $\mathbb{C}^N$ and passage to the limit decreasing, it is easily verified that the $(1, 1)$ -current $dd^c V = 2i\partial\overline{\partial} V$ is positive on X.

Definition 1.9. – *A function locally integrated V on X will be called weakly psh is if V plus locally and if $dd^c V \geqslant 0$ the direction of the currents.*

Unlike the smooth case, the assumption that V either locally plus is fundamental. Consider, for example. curve $(z_1, z_2) = (t^2, t^3)$ set in $\mathbb{C}^2$; function $V(t) = \mathrm{Re}(l/t)$ is not locally increased in 0, however we $dd^c V = 0$ can verify that the current direction (see definition 1.1). Observe the other hand a low psh function does not identify not necessarily almost everywhere to a psh function, as shown Example of the function defined on the curve $z_1 z_2 = 0$ $\mathbb{C}^2$ by $V(z_1, 0) = 1$, $V(0, z_2) = 0$ if $z_2 \neq 0$. However, it was the following result:

Theorem 1.10. – *Given a function $V : X \to [-\infty + \infty[$, there is equivalence between properties* (a), (b), (c) *below:*

(a) is weakly V *psh on X.*

(b) *V coincides almost everywhere with a function V_{reg} psh on X_{reg} locally increased near X_{sing}.*

(c) *There is a psh function $\tilde{V}$ on standardization $\tilde{X}$ of X, as $\tilde{V} = V \circ \pi$ almost everywhere where $\pi : \tilde{X} \to X$ is the natural morphism.*

For V is equal almost everywhere to a psh function on X, must and only if the following condition is achieved : for any $a \in X$, is designated by the $V^*(a)$ critical upper limit when $V(x)$ $x \in X$ approaches a and (X_j, a) the irreducible components then the germ (X, a) ;

$$\limsup_{x \in X_j \;;\; x \to a} \mathrm{ess}\; V(x) = V^*(a), \quad \forall j.$$

Under this assumption is V^* psh on X and $V = V^*$ almost everywhere.

Demonstration. (A) $\Rightarrow$ (b). this involvement follows immediately from the definition 1.9 and the well-known case X is smooth.

(B) $\Rightarrow$ (a). Is $h_1 = \ldots = h_m = 0$ local equations of X_{sing} in X. Then for every function $\varepsilon > 0$

$$V_\varepsilon = \begin{cases} V_{\mathrm{reg}} + \varepsilon \log\left(|h_1|^2 + \cdots + |h_m|^2\right) & \text{sur } X_{\mathrm{reg}} \\ -\infty & \text{sur } X_{\mathrm{sing}} \end{cases}$$

psh is on X by Theorem 1.6. so we $dd^c V_\varepsilon \geqslant 0$. As $dd^c V_\varepsilon$ converges weakly to when $dd^c V$ ε tends to 0, it follows that $dd^c V \geqslant 0$.

(B) $\Rightarrow$ (c). The $V_{\mathrm{reg}} \circ \pi$ psh function on $\tilde{X} \smallsetminus \pi^{-1}(X_{\mathrm{sing}}) = \pi^{-1}(X_{\mathrm{reg}})$, plus locally near $\pi^{-1}(X_{\mathrm{sing}})$, and is locally $\tilde{X}$ irreducible. Theorem 1.7 shows that $V_{\mathrm{reg}} \circ \pi$ extends a psh function on V X.

(C) $\Rightarrow$ (b) arises because $\pi : \tilde{X} \smallsetminus \pi^{-1}(X_{\mathrm{sing}}) \to X_{\mathrm{reg}}$ is an isomorphism.

Regarding the latter claim, provided we have given to the plurisousharmonicité of V^* is obviously necessary. To see that it is sufficient, it is observed all irreducible components is (X_j, a) one correspondence with all the points of a_j $\tilde{X}$ located above a (this results e.g. Narasimhan [Nar], prop. VI.2) and

$$\tilde{V}(a_j) = \limsup_{x \in X_j \;,\; x \to a} \mathrm{ess}\; V(x).$$

It was therefore by $V^* \circ \pi = \tilde{V}$ hypothesis at any point X ; like any holomorphic $f : \Delta \to X$ rises in a Application $\tilde{f} : \Delta \to \tilde{X}$, the plurisousharmonicité of $\tilde{V}$ involves that of V^*. □

Corollary 1.11. – *If X is locally irreducible if V is weakly psh on X, then the function defined by*

$$V^*(a) = \limsup_{x \to a} \mathrm{ess}\; V(x), \qquad a \in X,$$

psh is on X and $V = V^*$ almost everywhere.

Corollary 1.12. – *If $V : X \to [-\infty, +\infty[$ is continuous and weakly psh, psh is then V.*

To conclude this section, we examine the transformation of psh functions by direct image.

Proposition 1.13. – *Either $F : X \to Y$ a morphism surjective own finite fibers.*

(a) If V *is a weakly psh function on* X, *the* F_*V *defined function*

$$F_*V(y) = \sum_{x \in F^{-1}(y)} V(x)$$

psh is low on Y *and more*

$$dd^c(F_*V) = F_*(dd^cV).$$

(b) Y *One more guess is locally irreducible. If* V *is psh and if the sum is counted* $\sum_{x \in F^{-1}(y)} V(x)$ *with multiplicities, then* F_*V *is psh on* Y.

Demonstration.

(A) It is known that F is a branched covering, i.e. there is a analytic set Y as $Z \subset$

$$F : X \smallsetminus F^{-1}(Z) \to Y \smallsetminus Z$$

or a coating of smooth varieties. We see that the definition of F_*V coincides with that given for the direct images of streams. Clearly F_*V is locally bounded on Y, and property $dd^c(F_*V) = F_*dd^cV \geqslant 0$ results from the fact that commutes with F_* the d and d^c operators.

(B) Assuming locally irreducible X, Cardinal of $F^{-1}(y)$ fiber, $y \in Y \smallsetminus Z$, is locally constant neighborhood of a point of Z. If more V is continuous, is F_*V extended by continuity on Y through Z, and Corollary 1.12 shows that F_*V is psh. In the general case, there exists for any fiber $F^{-1}(y) = \{x_1, \ldots x_m\}$ neighborhoods arbitrarily small O_j of x_j, $1 \leqslant j \leqslant m$, and U neighborhood y such as$F^{-1}(U) = O_1 \cup \ldots \cup O_m$. Write as V decreasing limit of continuous psh functions on such V_k $F^{-1}(V)$ neighborhood. He comes

$$F_*V = \lim_{k \to +\infty} F_*V_k \quad \text{sur } U,$$

F_*V result is psh. □

2. Opérateurs $(dd^c)^k$ Unequal Chern-Levine-Nirenberg.

In this section, we recall the definition of operators Monge-Ampère of $(dd^c)^k$ introduced by Bedford and Taylor [BT1], [BT2]. This definition allows to make sense of the current $dd^cV_1 \wedge \ldots \wedge dd^cV_k$ when V_j are functions psh bounded. We need here to consider the case a little more General V_j where one of the functions may not be limited, and we will restore in this framework demonstrated the inequalities Chern-Levine-Nirenberg [CLN]. Finally, we study as in [BT2] the continuity of the operator relating to boundaries $(dd^c)^k$ decreasing of psh functions.

Let φ_1, φ_2, ..., φ_k functions psh locally bounded on X and V any psh function. According Bedford-Taylor [BT2] we can define the current closed $\geqslant 0$ $dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$ by induction on k posing

$$dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k = dd^c\big(\varphi_k\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}\big). \tag{2.1}$$

Positivity side is obvious by induction hypothesis if $\varphi_k \in \mathcal{C}^\infty(X)$; the general case follows by regularization φ_k and transition to the low limit space currents.

$\Omega = \{\rho < 0\}$ be a relatively compact open in X, ρ defined by a function strictly psh in $\mathcal{C}^\infty$ Ω' neighborhood of Ω and as $d\rho \neq 0$ on $\partial\Omega$. For any real $a > 0$ and whole $0 \leqslant k \leqslant n$ is placed

$$\beta_k = |\rho|^{k+a}(dd^c\rho)^{n-k} + (k+a)|\rho|^{k-1+a}d\rho \wedge d^c\rho \wedge (dd^c\rho)^{n-k-1}$$

and is designated by the $\|v\|_p$ L^p a standard feature on v Ω for Measuring β_0, $p \in [1, +\infty]$. The mass of the current (2.1) then admits the following estimates (see [CLN]).

Theorem 2.2. *Let V, $V_1, \ldots, V_k$ functions psh on X such as $V \leqslant 0$ and $V_1 \geqslant 0$, ..., $V_k \geqslant 0$ on Ω. Then there are constants $C_j = C_j(k,a) \geqslant 0$, $j = 1, 2, 3$ such as*

(a) $\displaystyle\int_\Omega \beta_{k+1} \wedge dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k \leqslant C_1\,\|V\|_1\,\|\varphi_1\|_\infty \cdots \|\varphi_k\|_\infty.$

(b) $\displaystyle\int_\Omega \beta_k \wedge |V|\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k \leqslant C_2\,\|V\|_1\,\|\varphi_1\|_\infty \cdots \|\varphi_k\|_\infty.$

(c) $\displaystyle\int_\Omega \beta_k \wedge dd^cV_1 \wedge \ldots \wedge dd^cV_k \leqslant C_3\,\|V_1\|_k\,\|V_2\|_k \cdots \|V_k\|_k.$

Demonstration. Through the approximation Lemma 2.4 below, it Presumably the V_j and φ_j are $\mathcal{C}^\infty$ class. An immediate calculation gives

$$\begin{aligned} d^c\beta_k &= -2(k+a)\,|\rho|^{k-1+a}\,d^c\rho \wedge (dd^c\rho)^{n-k},\\ dd^c\beta_k &= 2(k+a)\Big[-|\rho|^{k-1+a}\,(dd^c\rho)^{n-k+1} + (k-1+a)\,d\rho \wedge d^c\rho \wedge (dd^c\rho)^{n-k}\Big], \end{aligned}$$

hence the inequality forms

$$|dd^c\beta_k| \leqslant 2(k+a)\beta_{k-1}.$$

Note I_k, J_k integrals (a), (b) and respectively $\psi_k = dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$. According to the formula of integration by parts Lemma 2.5 and taking into account that

$$\beta_{k+1|\partial\Omega} = d^c\beta_{k+1|\partial\Omega} = 0,$$

he comes

$$\begin{aligned} I_k &= \int_\Omega dd^c\beta_{k+1} \wedge \varphi_k\, dd^cV \wedge \psi_{k-1}\\ &\leqslant 2(k+1+a)\,\|\varphi_k\|_\infty \int_\Omega \beta_k \wedge dd^cV \wedge \psi_{k-1} = 2(k+1+a)\,\|\varphi_k\|_\infty I_{k-1},\\ I_0 &= \int_\Omega V\, dd^c\beta_1\\ &\leqslant 2(1+a)\int_\Omega |V|\,|\rho|^a(dd^c\rho)^n \leqslant 2(1+a)\,\|V\|_1. \end{aligned}$$

This demonstrates (a) by induction with $C_1(k.a) = 2^{k+1}(1+a)\dots(k+1+a)$, inequality is also satisfied even if $a = 0$. On the other hand if $k \geqslant 1$ obtained

$$\begin{aligned} J_k &= \int_\Omega -\varphi_k dd^c(V\beta_k) \wedge \psi_{k-1} \\ &= \int_\Omega -\varphi_k \big(V\, dd^c\beta_k + \beta_k \wedge dd^c V + 2\, dV \wedge d^c\beta_k\big) \wedge \psi_{k-1} \\ &= \int_\Omega \varphi_k \big(V\, dd^c\beta_k - \beta_k \wedge dd^c V\big) \wedge \psi_{k-1} + 2V\, d\varphi_k \wedge d^c\beta_k \wedge \psi_{k-1} \end{aligned}$$

by integrating the parts $-2\varphi_k\, dV \wedge d^c\beta_k \wedge \psi_{k-1}$ end to the second line. Assume now and $\varphi_k > 0$ the Cauchy-Schwarz inequality is applied to the component of bidegree $(n-k+1, n-k+1)$ current

$$2d\varphi_k \wedge d^c\beta_k = -4(k+a)\,|\rho|^{k-1+a} d\varphi_k \wedge d^c\rho \wedge (dd^c\rho)^{n-k},$$

which gives the upper bound

$$\Big[4(k+a)^2\varphi_k|\rho|^{k-2+a} d\rho \wedge d^c\rho + |\rho|^{k+a}\frac{d\varphi_k \wedge d^c\varphi_k}{\varphi_k}\Big] \wedge (dd^c\rho)^{n-k}.$$

The result

$$\begin{aligned} J_k \leqslant \int_\Omega \varphi_k|V|\Big[-dd^c\beta_k + 4(k+a)^2|\rho|^{k-2+a} d\rho \wedge d^c\rho \wedge (dd^c\rho)^{n-k}\Big] \wedge \psi_{k-1} \\ + \int_\Omega |V|\,|\rho|^{k+a}(dd^c\rho)^{n-k} \wedge \frac{d\varphi_k \wedge d^c\varphi_k}{\varphi_k} \wedge \psi_{k-1}. \end{aligned}$$

The shape brackets in the first integral equals

$$2(k+a)\Big[|\rho|^{k-1+a}(dd^c\rho)^{n-k+1} + (k+1+a)|\rho|^{k-2+a} d\rho \wedge d^c\rho \wedge (dd^c\rho)^{n-k}\Big] \leqslant C_4\beta_{k-1}$$

with $C_4 = C_4(k,a) = \frac{2(k+a)(k+1+a)}{k-1+a}$. We obtain finally

$$J_k \leqslant C_4\|\varphi_k\|_\infty J_{k-1} + \int_\Omega |V|\,\beta_k \wedge \psi_{k-1} \wedge \frac{d\varphi_k \wedge d^c\varphi_k}{\varphi_k}.$$

Note J_k' the integral obtained by replacing φ_k $\varphi_k' = \exp(B\varphi_k)$ in and ask J_k $M = \|\varphi_k\|_\infty$. He comes

$$dd^c\varphi' = e^{B\varphi_k}\big(B\, dd^c\varphi_k + B^2\, d\varphi_k \wedge d^c\varphi_k\big) \geqslant B\, e^{-BM} dd^c\varphi_k + \frac{d\varphi_k' \wedge d^c\varphi_k'}{\varphi_k'},$$

$$B\, e^{-BM} J_k \leqslant J_k' - \int_\Omega |V|\,\beta_k \wedge \psi_{k-1} \wedge \frac{d\varphi_k' \wedge d^c\varphi_k'}{\varphi_k'} \leqslant C_4(k,a)\, e^{BM} J_{k-1}.$$

As $\inf_{B>0} \frac{1}{B} e^{2BM} = 2eM = 2e\,\|\varphi_k\|_\infty$ This completes the proof of (b) by induction on k, with constant

$$C_2(k,a) = (4e)^k\, \frac{k+a}{a}\,(2+a)\cdots(k+1+a).$$

To prove (c), assume first that $V_1 = V_2 = \ldots = V_k = v \geqslant 0$. As

$$\Big(dd^c v^{\frac{k}{k-1}}\Big)^{k-1} \geqslant v\Big(\frac{k}{k-1}\,dd^c v\Big)^{k-1},$$

integration by parts gives

$$\int_\Omega \beta_k \wedge (dd^c v)^k = \int_\Omega dd^c\beta_k \wedge v\,(dd^c v)^{k-1} \leqslant 2(k+a)\Big(\frac{k-1}{k}\Big)^{k-1}\int_\Omega \beta_{k-1}\wedge\Big(dd^c v^{\frac{k}{k-1}}\Big)^{k-1}.$$

By induction on k this in turn leads to inequality

$$\int_\Omega \beta_k \wedge (dd^c v)^k \leqslant 2^k(1+a)...(k+a)\frac{(k-1)!}{k^{k-1}}\int_\Omega \beta_0 v^k.$$

Now replace by v

$$v = \frac{V_1}{\|V_1\|_k} + \cdots + \frac{V_k}{\|V_k\|_k}.$$

He comes

$$\frac{k!}{\|V_1\|_k\cdots\|V_k\|_k}\int_\Omega \beta_k \wedge dd^c V_1 \wedge \ldots \wedge dd^c V_k \leqslant 2^k(1+a)...(k+a)\frac{(k-1)!}{k^{k-1}}\int_\Omega \beta_0 v^k$$

while $\|v\|_k \leqslant k$. Inequality (c) follows with

$$C_3(k,a) = 2^k(1+a)\ldots(k+a). \qquad \square$$

An immediate consequence of Theorem 2.2 (b) is that the current $V\,dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$ is mass locally finite on X ; especially found the result next due in Bedford-Taylor [BT2].

Corollary 2.3. – *Coefficients measures $dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$ do not charge sets multipolar $\{V = -\infty\}$.* $\square$

The X space being Stein, there are after J.E. Fornaess and R. Narasimhan [FN] decreasing V_m suites, of $\varphi_{j,m}$ psh functions such as $\mathcal{C}^\infty$ on X

$$V_m \to V, \qquad \varphi_{j,m} \to \varphi_j \qquad \text{pour } 1 \leqslant j \leqslant k.$$

Lemma 2.4. – *There are suites strictly increasing integers $m(\nu)$, $m_1(\nu)$, ... , $m_k(\nu)$, $\nu \in \mathbb{N}$ such that the sense of weak convergence of measures we is chosen from one or other of the convergence properties below* :

(a) $dd^c V_{m(\nu)} \wedge dd^c\varphi_{1,m_1(\nu)} \wedge \ldots \wedge dd^c\varphi_{k,m_k(\nu)} \longrightarrow dd^c V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$

(b) $V_{m(\nu)} \wedge dd^c\varphi_{1,m_1(\nu)} \wedge \ldots \wedge dd^c\varphi_{k,m_k(\nu)} \longrightarrow V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$.

Demonstration. According to the Theorem 2.2 already demonstrated in the For psh functions $\mathcal{C}^\infty$ suites (a), (b) are locally bounded by mass, and bounded subsets of the order currents of space are 0 metrizable in the weak topology. In case (a) that is observed

$$\varphi_{k,m} dd^c V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \longrightarrow \varphi_k dd^c V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}$$

by monotone convergence $\varphi_{k,m}$ when $m \to +\infty$; passing the dd^c was therefore

$$dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge dd^c\varphi_{k,m} \longrightarrow dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge dd^c\varphi_k.$$

The topology is metrizable, we can choose successively $m_k(\nu) = \nu$ then $m_{k-1}(\nu)$, ... , $m_1(\nu)$, $m(\nu)$ by induction on ν (resp. $m(\nu) = \nu$ then $m_1(\nu)$, ... , in $m_1(\nu)$ case (b)) to obtain the desired convergence. □

Let us state here for future reference by the integration lemma parts that we used.

Lemma 2.5. – *If u and v are forms of $\mathcal{C}^2$ respective class bidegrés (p,q) and with $(n - p - 1, n - q - 1)$ $p + q$ even, then*

$$\int_\Omega u \wedge dd^cv = \int_{\partial\Omega} u \wedge d^cv - d^cu \wedge v.$$

It suffices to apply the Stokes theorem to the form

$$d\big(u \wedge d^cv - d^cu \wedge v\big) = u \wedge dd^cv - dd^cu \wedge v + du \wedge d^cv + d^cu \wedge dv$$

and observe that

$$du \wedge d^cv = i(\partial u \wedge \overline{\partial} v - \overline{\partial} u \wedge \partial v) = -d^cu \wedge dv.$$ □

By adapting techniques [BT2] to the present situation, we now show the continuity of the operator $(dd^c)^k$ compared to decreasing limits of psh functions.

Theorem 2.6. – *Let $\{\varphi_j^\nu\}_{\nu\in\mathbb{N}} \subset L^\infty_{\mathrm{loc}}(X)$ and $\{V^\nu\}_{\nu\in\mathbb{N}}$ decreasing suites of psh functions such as*

$$\varphi_j = \lim_{\nu\to+\infty} \varphi_j^\nu \in L^\infty_{\mathrm{loc}}(X), \qquad V = \lim_{\nu\to+\infty} V^\nu \not\equiv -\infty.$$

In the sense of weak convergence of measures, we then

(a) $dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \longrightarrow dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$,

(b) $V^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \longrightarrow V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$,

(c) $\varphi_k^\nu dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_{k-1}^\nu \longrightarrow \varphi_k \wedge dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$.

We will prove the theorem 2.6 simultaneously with the property which is a corollary.

Corollary 2.7. –

(a) $dd^c\big(V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k\big) = dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$.

(b) Common $V \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$ *and* $dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$ *are symmetrical in* $\varphi_1, \ldots, \varphi_k$.

Demonstration. Using Lemma 2.4 and a clear process diagonal result, we reduce to the case V^ν, φ_1^ν, ... , φ_k^ν are $\mathcal{C}^\infty$ class. Like the properties 2.6 (a, b, c) are local, we can without loss of generality be placed in an open $\Omega = \{\rho < 0\} \Subset X$. We go now back to

the situation where φ_j, φ_j^ν $\mathcal{C}^\infty$ class are near zero and $\partial\Omega$ $\partial\Omega$ on, so as to apply the Stokes formula without boundary terms. either $\mathbb{R}^2 \ni (u,v) \mapsto \lambda(u,v)$ a convex increasing function $\mathcal{C}^\infty$ u and v, coinciding with $\max(u,v)$ for $|u-v|>1$. so $\tilde\varphi_j^\nu = \lambda\big(\varphi_j^\nu - \frac{2}{\varepsilon}, \varepsilon^{-2}\rho\big)$ psh is $\mathcal{C}^\infty$, more for $\varepsilon>0$ was quite small

$$\begin{cases} \tilde\varphi_j^\nu = \varphi_j^\nu - \frac{2}{\varepsilon} & \text{sur} \quad \Omega_{3\varepsilon} = \{\rho < -3\varepsilon\}, \\ \tilde\varphi_j^\nu = \varepsilon^{-2}\rho & \text{sur} \quad \overline{\Omega} \smallsetminus \Omega_\varepsilon = \{-\varepsilon \leqslant \rho \leqslant 0\}. \end{cases}$$

We can finally assume that $\varphi_j^\nu = \varphi_j = \varepsilon^{-2}\rho$ on "crown" $\overline{\Omega}\smallsetminus\Omega_\varepsilon$ (and this regardless j, ν).

Proof 2.6 (a). We proceed by induction on k. From (2.1) it suffices to prove that

$$\begin{aligned} &\lim_{\nu\to+\infty} \int_\Omega \varphi_k^\nu \, dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_{k-1}^\nu \wedge dd^c\psi \\ &\qquad\qquad = \int_\Omega \varphi_k \, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \end{aligned} \tag{2.8}$$

$\psi \in \mathcal{C}^\infty_{n-k-1,n-k-1}(\overline{\Omega})$ for any form such as $\psi_{|\partial\Omega} = 0$ (note that by hypothesis $\varphi^\nu_{k|\partial\Omega} = 0$). Even replace ψ successively $\rho(dd^c\rho)^{n-k-1}$ and $\psi + A\rho(dd^c\rho)^{n-k-1}$, $A \gg 0$, can assume $dd^c\psi \geqslant 0$ on $\overline{\Omega}$. The $\leqslant$ inequality (2.8) then results simply the induction hypothesis

$$dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_{k-1}^\nu \longrightarrow dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}$$

and the monotone convergence theorem. To prove inequality Conversely$\geqslant$ is carried out by successive integrations parties means of Lemma 2.5 :

$$\begin{aligned} &\int_\Omega \varphi_k \, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge dd^c\psi \\ &\qquad \leqslant \int_\Omega \varphi_k^\nu \, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge dd^c\psi \\ &\qquad = \int_\Omega \varphi_{k-1} \, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-2} \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\ &\qquad \leqslant \int_\Omega \varphi_{k-1}^\nu \, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-2} \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\ &\qquad = \cdots \leqslant \int_\Omega \varphi_1^\nu \, dd^cV \wedge dd^c\varphi_2^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\ &\qquad = \int_\Omega V \, dd^c\varphi_1^\nu \wedge dd^c\varphi_2^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\ &\qquad\qquad\qquad - \int_{\partial\Omega} V \, d^c\varphi_1^\nu \wedge dd^c\varphi_2^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\ &\qquad \leqslant \int_\Omega V^\nu \, dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \wedge dd^c\psi - \varepsilon^{-2k} \int_{\partial\Omega} V \, d^c\rho \wedge (dd^c\rho)^{k-1} \wedge dd^c\psi \\ &\qquad = \int_\Omega \varphi_k^\nu \, dd^cV^\nu \wedge dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu \wedge dd^c\psi \\ &\qquad\qquad\qquad - + \varepsilon^{-2k} \int_{\partial\Omega} (V^\nu - V) \, d^c\rho \wedge (dd^c\rho)^{k-1} \wedge dd^c\psi. \end{aligned}$$

The last integral tends to 0 by monotone convergence, which completes the proof of 2.6 (a).

Proof 2.7 (a). immediate consequence of 2.4 (b) and 2.6 (a).

Proof 2.6 (b). Inequality 2.2 (b) implies that the sequence $V^\nu\, dd^c\varphi_1^\nu \wedge \ldots \wedge dd^c\varphi_k^\nu$ mass is locally uniformly bounded on X. More worthless T adhesion of this sequence is such that

$$T \leqslant V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k,$$

with equality $\Omega \smallsetminus \Omega_\varepsilon$ (where $\varphi_j^\nu = \varphi_j = \varepsilon^{-2}\rho$). According 2.6 (a) and 2.7 (a) was the other

$$dd^cT = dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k = dd^c\big(V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k\big).$$

Now distinguish two cases according to the value of the integer k.

If $k \leqslant n-1$, Lemma 2.5 applied $v = \rho(dd^c\rho)^{n-k-1}$ causes the positive current

$$u = V\, dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k - T$$

sucks on Ω.

If $k = n$, we can consider V, $\varphi_1, \ldots, \varphi_k$ as functions on $X \times \mathbb{C}$ not dependent on the last variable, and apply the results 2.6 (b) already known $X \times \mathbb{C}$. Demonstrating 2.6 (c) is identical. □

Note 2.9. – If $\varphi_j \geqslant 0$ is, we can write

$$d\varphi_j \wedge d^c\varphi_j = \frac{1}{2} dd^c\varphi_j^2 - \varphi_j\, dd^c\varphi_j\ ;$$

therefore the low 2.6 convergence theorem remains valid for any product or $V\ dd^cV$ by $(1,1)$ Platforms of $dd^c\varphi_j$ type $d\varphi_j \wedge d^c\varphi_j$, or (polarization) $d\varphi_i \wedge d^c\varphi_j + d\varphi_j \wedge d^c\varphi_i$.

Note 2.10. – The reader will find a interesting discussion on the problem of definition and continuity of the operator Monge-Ampère in [Ki] and [it]. In particular, it is possible to extend some of the results in case the previous φ_j functions are no longer necessarily bounded, provided they make an assumption compactness on the poles of φ_j. We assume that exists a compact $K \subset X$ as $\varphi_1, \ldots, \varphi_k$ are locally bounded on $X \smallsetminus K$. Then the definition (2.1) Lemma 2.4 (a) and Theorem 2.6 (a) remain valid.

To see this, we observe that the problem arises only near K. ρ is a function $\mathcal{C}^\infty$ strictly psh and ω open as $K \subset \omega \Subset \Omega = \{\rho < 0\} \Subset X$. Even replaced by φ_j

$$\begin{cases} \varphi_j - \frac{2}{\varepsilon} & \text{sur } \omega, \\ \varepsilon^{-2}\rho & \text{sur } X \smallsetminus \Omega, \\ \max\left(\varphi_j - \frac{2}{\varepsilon}, \varepsilon^{-2}\rho\right) & \text{sur } \Omega \smallsetminus \omega, \end{cases}$$

it can be assumed in the vicinity $\varphi_j = \varepsilon^{-2}\rho$ of$\partial\Omega$. First demonstrate by induction that $\varphi_k\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}$ (And thus also $dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_k$) is locally finite mass if $k \leqslant n-1$. For $a < 0$ was in Indeed, with the notation $\varphi_{k,a} = \max(\varphi_k, a)$:

$$\begin{aligned}\int_\Omega |\varphi_{k,a}|\, dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1} \wedge (dd^c\rho)^{n-k} \\ &= \int_\Omega |\rho|\, dd^c\big(\varphi_{k,a} dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}\big) \wedge (dd^c\rho)^{n-k-1} \\ &\leqslant C \int_\Omega dd^c\big(\varphi_{k,a} dd^cV \wedge dd^c\varphi_1 \wedge \ldots \wedge dd^c\varphi_{k-1}\big) \wedge (dd^c\rho)^{n-k-1} \\ &= C\varepsilon^{-2k} \int_\Omega dd^cV \wedge (dd^c\rho)^{n-1} < +\infty,\end{aligned}$$

the last equality from the Stokes theorem. The demo 2.4 (a) and 2.6 (a) is then done without any modification.

3. Mesures and Monge-Ampère formula of Jensen.

At any φ psh function on continuous and comprehensive space Stein, we will associate canonically a family positive measures carried by sets of level φ. These measures appear naturally when seeking expand the formula of Jensen in several variables. The main ideas of this paragraph are based on calculations made by P. Lelong [Le1] to show the existence of Lelong numbers of a current closed positive. We take essentially the notation [De4] [DE5] (see also the article by H. Skoda [Sk1]).

Consider a Stein space X pure n dimension reduced, with a psh function continues $\varphi : X \to [-\infty, R[$ where $R \in\,]-\infty, +\infty]$. For there is $r < R$

$$B(r) = \big\{z \in X\,;\ \varphi(z) < r\big\}, \qquad B(r) = \big\{z \in X\,;\ \varphi(z) \leqslant r\big\}$$

the “pseudoboules” associated with open and closed φ (care will be taken to the fact that is not $\overline{B}(r)$ necessarily adhesion $B(r)$!). We assume that is φ exhaustive, that is to say that the pseudoboules $\overline{B}(r)$, $r < R$ are compact. Finally we set for all $r \in [-\infty, R[$

$$\begin{aligned}&S(r) = \big\{z \in X\,;\ \varphi(z) = r\big\} = \overline{B}(r) \smallsetminus B(r),\\ &\varphi_r = \max(\varphi, r), \qquad \alpha = dd^c\varphi = 2i\,\partial\overline{\partial}\varphi.\end{aligned}$$

The current $(dd^c\varphi_r)^n$ is well defined by (2.1) if $r > -\infty$ and if $r = -\infty$, $(dd^c\varphi_r)^n = (dd^c\varphi)^n$ exists by Remark 2.10.

Lemma 3.1. – *The implementation $r \mapsto (dd^c\varphi_r)^n$ is continuous in space $[-\infty, R[$ measurements on X provided with the weak topology.*

Demonstration. The right continuity follows from Theorem 2.6 (a) while the left continuity is obtained by writing

$$(dd^c\varphi_r)^n = \big(dd^c \max(\varphi - r, 0)\big)^n. \qquad \square$$

As $(dd^c\varphi_r)^n$ sucks on $B(r)$ and coincides with $(dd^c\varphi)^n$ on $X \smallsetminus \overline{B}(r)$, continuity left leads

$$(dd^c\varphi_r)^n \geqslant \mathbb{1}_{X \smallsetminus B(r)}(dd^c\varphi)^n,$$

$\mathbb{1}_A$ where denotes the characteristic function of a part $A \subset X$. The results below immediately follow from these remarks and justified the following definition.

Theorem and Definition 3.2. – *be called Monge-Ampère measures associated with φ families (μ_r) positive steps, worn by $(\overline{\mu}_r)$ $S(r)$, $r \in [-\infty, R[$ defined by*

$$\mu_r = (dd^c\varphi_r)^n - \mathbb{1}_{X \smallsetminus B(r)}(dd^c\varphi)^n,$$
$$\overline{\mu}_r = (dd^c\varphi_r)^n - \mathbb{1}_{X \smallsetminus \overline{B}(r)}(dd^c\varphi)^n.$$

Family μ_r (resp. $\overline{\mu}_r$) is weakly continuous (left resp. right), and one has the relationships

$$\overline{\mu}_r = \lim_{\rho \to r+0} \mu_\rho, \qquad \mu_r = \lim_{\rho \to r-0} \overline{\mu}_\rho,$$
$$\overline{\mu}_r = \mathbb{1}_{S(r)}(dd^c\varphi_r)^n = \mu_r + \mathbb{1}_{S(r)}(dd^c\varphi)^n.$$

$D_\varphi \subset [-\infty, R[$ be all countable real r as $S(r)$ be negli-gible for $(dd^c\varphi)^n$ custom X. So $\overline{\mu}_r = \mu_r$ $r \notin D_\varphi$ for all, and $r \mapsto \mu_r$ applications are continuous $r \mapsto \overline{\mu}_r$ at any point $r \notin D_\varphi$. □

We thank E. Bedford for having suggested this definition, which simplifies the one we used in earlier version of this work. At any point where is φ regular, μ_r $\overline{\mu}_r$ and can be described by a simple differential form on hypersurface $S(r)$.

Proposition 3.3. – *$x \in X$ either a regular point φ the vicinity of which is $\mathcal{C}^2$ class and as $d\varphi(x) \neq 0$. Orienting $S(r)$ as ships $B(r)$. then μ_r $\overline{\mu}_r$ measures and are defined by adjacent x the volume $(dd^c\varphi)^{n-1} \wedge \varphi_{|S(r)}$ $(2n-1)$ Platform.*

Demonstration. Ω is a neighborhood where x $d\varphi \neq 0$ and h a compact support $\mathcal{C}^\infty$ function in Ω. write

$$\max(r,t) = \lim_{\nu \to +\infty} \chi_\nu(t)$$

χ_ν which is a sequence of regularized by convolution $t \mapsto \max(r,t)$. Can ensure that a result is (χ_ν) decreasing convex functions $\mathcal{C}^\infty$, as $0 \leqslant \chi'_\nu \leqslant 1$ with $\lim \chi'_\nu(t)$ equal to 0 for $t < r$ and equal to 1 for $t > r$. Theorem 2.6 (a) therefore leads

$$\begin{aligned}
\int_\Omega h\,(dd^c\varphi_r)^n &= \lim_{\nu \to +\infty} \int_\Omega h\,(dd^c\chi_\nu \circ \varphi)^n \\
&= \lim_{\nu \to +\infty} -\int_\Omega dh \wedge (dd^c\chi_\nu \circ \varphi)^{n-1} \wedge d^c(\chi_\nu \circ \varphi) \\
&= \lim_{\nu \to +\infty} -\int_\Omega \chi'_\nu(\varphi)^n\,dh \wedge (dd^c\varphi)^{n-1} \wedge d^c\varphi \\
&= -\int_{\Omega \smallsetminus B(r)} dh \wedge (dd^c\varphi)^{n-1} \wedge d^c\varphi \\
&= \int_{\Omega \cap S(r)} h\,(dd^c\varphi)^{n-1} \wedge d^c\varphi + \int_{\Omega \smallsetminus B(r)} h\,(dd^c\varphi)^n
\end{aligned}$$

according to Stokes' formula. We therefore deduce Ω on equality measures

$$(dd^c\varphi_r)^n = (dd^c\varphi)^{n-1} \wedge d^c\varphi_{|S(r)} + \mathbb{1}_{\Omega \smallsetminus B(r)}(dd^c\varphi)^n.$$ □

We can now prove the formula of Jensen-Lelong we aircraft in sight.

Theorem 3.4. – *Either V a psh function on X. So V μ_r is integrable for all $r \in]-\infty, R[$. Furthermore*

$$\int_{-\infty}^{r} dt \int_{B(t)} dd^c V \wedge \alpha^{n-1} = \mu_r(V) - \int_{B(r)} V\,\alpha^n,$$

where $\alpha = dd^c\varphi$. The two members are finished if $\inf_X \varphi > -\infty$ or $\inf_{B(r)} V > -\infty$.

Demonstration. Integrability of V for μ_r (And $\overline{\mu}_r$) results from the fact that V is integrable $(dd^c\varphi_r)^n$ by Theorem 2.2 (b). Note also that the integrals and $\int_{B(t)} dd^c V \wedge (dd^c\varphi)^{n-1} \geqslant 0$ $\int_{B(r)} V\,(dd^c\varphi)^n$ well make sense under the remark 2.10, the first being also always convergent. The second converges if $\inf_X \varphi > -\infty$ through 2.2 (b), or if $\inf_{B(r)} V > -\infty$ through 2.10. To prove the formula 3.4, it is first assumed

$$\inf_X \varphi > -\infty \quad \text{et} \quad \inf_{B(r)} V > -\infty,$$

and we give $c > r$. Fubini's theorem implies

$$\begin{aligned}\int_{-\infty}^{c} dt \int_{B(t)} dd^c V \wedge (dd^c\varphi)^{n-1} &= \int_{B(c)} \left[\int_{\{\varphi<t<c\}} dt\right] dd^c V \wedge (dd^c\varphi)^{n-1}\\ &= \int_{B(c)} (c-\varphi)\, dd^c V \wedge (dd^c\varphi)^{n-1}.\end{aligned}$$

According to Stokes formula, we have the equality

$$\begin{aligned}&\int_{B(c)} d\Big[(c-\varphi)\, d^c V \wedge (dd^c\varphi)^{n-1} + V\,(dd^c\varphi)^{n-1} \wedge d^c\varphi\Big]\\ &\qquad = \int_{B(c)} d\Big[(c-\varphi_r)\, d^c V \wedge (dd^c\varphi_r)^{n-1} + V\,(dd^c\varphi_r)^{n-1} \wedge d^c\varphi\Big]\end{aligned}$$

since the currents to integrate coincide on the crown $B(c) \smallsetminus \overline{B}(r)$. If we develop the first complete, it comes

$$\int_{B(c)} (c-\varphi)\, dd^c V \wedge (dd^c\varphi)^{n-1} + V\,(dd^c\varphi)^n + \int_{B(c)} (dV \wedge d^c\varphi - d\varphi \wedge d^c V) \wedge (dd^c\varphi)^{n-1}$$

and as the $(1,1)$ type $dV \wedge d^c\varphi - d\varphi \wedge d^c V$ component is zero, the second sum is zero. therefore

$$\int_{B(c)} (c-\varphi)\, dd^c V \wedge (dd^c\varphi)^{n-1} + V\,(dd^c\varphi)^n = \int_{B(c)} (c-\varphi_r)\, dd^c V \wedge (dd^c\varphi_r)^{n-1} + V\,(dd^c\varphi_r)^n.$$

Let us now tend to c r right. As $0 \leqslant c - \varphi_r \leqslant c - r$ it comes to the limit

$$\int_{\overline{B}(r)} (r-\varphi)\,(dd^c\varphi)^{n-1} + V\,(dd^c\varphi)^n = \int_{\overline{B}(r)} V\,(dd^c\varphi_r)^n = \overline{\mu}_r(V).$$

Given (3.5) and equal $\overline{\mu}_r = \mu_r + \mathbb{1}_{S(r)}(dd^c\varphi)^n$ This demonstrates the formula under 3.4 the restrictive assumption that φ and V are undervalued. In the case Generally, we can write $V = \lim_{\nu\to+\infty} V_\nu$ which is V_ν a decreasing sequence of psh functions on $\mathcal{C}^\infty$ X (cf. [FN]). $a < r$ be fixed. According to the above, by replacing φ φ_a the locally bounded function, equality is achieved

$$\mu_r(V_\nu) = \int_a^r dt \int_{B(t)} dd^cV_\nu \wedge (dd^c\varphi_a)^{n-1} + \int_{B(r)} V_\nu\,(dd^c\varphi_a)^n.$$

A passage to the limit when ν tends to give $+\infty$

$$\mu_r(V) = \int_a^r dt \int_{B(t)} dd^cV \wedge (dd^c\varphi_a)^{n-1} + \int_{B(r)} V\,(dd^c\varphi_a)^n\ ;$$

in fact, the measurement converges $dd^cV_\nu \wedge (dd^c\varphi_a)^{n-1}$ weakly to $dd^cV \wedge (dd^c\varphi_a)^{n-1}$ through theorem 2.6 (a), and this measure is assessed on the continuous function $\mathbb{1}_{B(r)}(r-\varphi_a)$ according to (3.5). The Stokes theorem shows that the measure

$$dd^cV \wedge \left[(dd^c\varphi_a)^{n-1} - (dd^c\varphi)^{n-1}\right]$$

is zero integral over $B(t)$ for all $t > a$, therefore obtained

$$\int_a^r dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1} = \mu_r(V) - \int_{B(r)} V\,(dd^c\varphi_a)^n.$$

This formula implies that the continuous functions $a \mapsto \int_{B(r)} V_\nu\,(dd^c\varphi_a)^n$ are increasing on $[-\infty, r[$. Decreasing their limits $a \mapsto \int_{B(r)} V\,(dd^c\varphi_a)^n$ is continuous right, which allows to pass to the limit in $a = -\infty$. □

The formula 3.4 we deduce immediately the similar formula for $\overline{\mu}_r$ measures:

$$\int_{-\infty}^r dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1} = \overline{\mu}_r(V) - \int_{\overline{B}(r)} V\,(dd^c\varphi)^n. \tag{3.6}$$

Especially $V = 1$ it comes :

Corollary 3.7. – *Total masses and μ_r $\overline{\mu}_r$ are given by*

$$\|\mu_r\| = \int_{B(r)} (dd^c\varphi)^n = \int_{B(r)} \alpha^n, \qquad \|\overline{\mu}_r\| = \int_{\overline{B}(r)} (dd^c\varphi)^n = \int_{\overline{B}(r)} \alpha^n.$$

In the following, we will leave it to the reader to translate the results obtained in the case of measures $\overline{\mu}_r$. We now studying the continuity of action based on μ_r the exhaustion φ.

Proposition 3.8. – *Either $(\varphi^\nu)_{\nu\in\mathbb{N}}$ a decreasing sequence of continuous functions converging psh φ on X and Monge-Ampère μ_r^ν associated measures to φ^ν. So μ_r^ν converges weakly to μ_r for all $r \in\,]-\infty, R[\smallsetminus D_\varphi$.*

Demonstration. Just apply the definition 3.2, which provides

$$\mu_r^\nu = (dd^c\varphi_r^\nu)^n - \mathbb{1}_{X\smallsetminus B(r)}(dd^c\varphi^\nu)^n$$

with

$$\varphi_r^\nu = \max(\varphi^\nu, r), \qquad B_r^\nu = \{|z \in X\,;\ \varphi^\nu(z) < r\},$$

and observe that $B(r) = \bigcup B^\nu(r)$. Theorem 2.6 (a) then implies

$$(dd^c\varphi^\nu)^n \to (dd^c\varphi)^n, \qquad (dd^c\varphi_r^\nu)^n \to (dd^c\varphi_r)^n. \qquad \square$$

The following proposition shows that μ_r measures are essentially the current decay $(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi$ measures on the family pseudosphères $S(r)$.

Proposition 3.9. – *Either h function Borel bounded with compact support in the open$X \smallsetminus S(-\infty)$. So*

(a) $\displaystyle\int_{-\infty}^{R} \mu_r(h)\,dr = \int_X h\,\alpha^{n-1} \wedge d\varphi \wedge d^c\varphi.$

(b) If more *h is $\mathcal{C}^1$ class was*

$$\mu_r(h)\,dr = \int_{B(r)} h\,\alpha^n + dh \wedge d^c\varphi \wedge \alpha^{n-1}.$$

Demonstration. (A) two members identify measures Positive operating on h. So just to prove equality when h is continuous with compact support. By proposition 3.3 and Fubini, the formula is true when φ is Class $\mathcal{C}^\infty$: Sard's theorem shows that all critical values φ is negli-gible. The case General is then obtained by applying Proposition 3.8 to a suite φ^ν of regularized φ.

(B) According to 3.3, the formula is true if and φ is $\mathcal{C}^\infty$ if r is not critical value φ. 3.8 The proposal extends the result should φ only continue as long as $r \notin D_\varphi$. Just then observed that the two members are continuous functions left r.

$\square$

$\chi :]-\infty, R[\to \mathbb{R}$ is increasing convex function not constant. The μ_r^* measures associated with exhaustion $\varphi^* = \chi\circ\varphi$ are then connected to μ_r measures by following variable change formula :

Proposition 3.10. – *For $r \in]-\infty, R[$, we have the formulas*

$$\mu_{\chi(r)}^* = \chi'_-(r)^n\,\mu_r, \qquad \overline{\mu}_{\chi(r)}^* = \chi'_+(r)^n\,\overline{\mu}_r$$

where χ'_+, χ'_- are derived right and left of χ.

Demonstration. Equalities result from Proposition 3.3 when φ, χ are $\mathcal{C}^\infty$ class and when is r regular value of φ. 3.8 The proposal involves the general case if $r \notin D_\varphi$ after passing the decreasing limit on φ and χ. The result follows by continuity $r \in D_\varphi$. $\square$

4. o

f residualMesure $(dd^c\varphi)^n$ on $S(-\infty)$.

If V is a psh function $\geqslant 0$, Theorem 3.4 shows that $r \mapsto \mu_r(V)$ the function is increasing $\geqslant 0$. In addition, as dual integrated

$$\int_{-\infty}^{r} dt \int_{B(t)} dd^c V \wedge \alpha^{n-1} \leqslant \mu_r(V)$$

converges, it comes

$$\lim_{\rho\to-\infty} \mu_r(V) = \lim_{\rho\to-\infty} \int_{B(r)} V\,\alpha^n = \int_{S(-\infty)} V\,\alpha^n.$$

Theorem and Definition 4.1. – *Measurement $\overline{\mu}_{-\infty} = \mathbb{1}_{S(-\infty)}\alpha^n$ carried by the compact $S(-\infty)$ will be called residual measurement associated with φ. For $V \geqslant 0$ psh function on wasX*

$$\overline{\mu}_{-\infty}(V) = \lim_{r\to-\infty} \mu_r(V),$$

and μ_r tends weakly to when $\overline{\mu}_{-\infty}$ $r \to -\infty$.

The last assertion follows from Theorem 3.2, or because we can write any h depending on the class $\mathcal{C}^2$ Stein space under X $h = h_1 - h_2$ the form with h_1, $h_2 \geqslant 0$ psh class $\mathcal{C}^2$.

The purpose of this section is to set out some general properties Residual $\overline{\mu}_{-\infty}$ measures. To evaluate $\overline{\mu}_{-\infty}$ on concrete examples, one has the theorem following comparison, based on the results of [De4], [DE5] on Lelong numbers.

Theorem 4.2. – Let$\varphi_j : X \to [-\infty, R_j[$, $j = 1, 2$, *two psh functions continuous and comprehensive measures $\mu_{r,j}$ Associated respective $S_j(r) = \{\varphi_j = r\}$ on. we set*

$$\ell = \liminf_{\varphi_1(z)\to-\infty} \frac{\varphi_2(z)}{\varphi_1(z)}.$$

Then for every function $V \geqslant 0$ psh, we have the inequality

$$\overline{\mu}_{-\infty,2}(V) \geqslant \ell^n\, \overline{\mu}_{-\infty,1}(V).$$

In particular, if when $\varphi_2 \sim \varphi_1$ $\varphi_1(z) \to -\infty$, we have

$$\overline{\mu}_{-\infty,2}(V) = \ell^n\, \overline{\mu}_{-\infty,1}(V).$$

Demonstration. It suffices to show that $\overline{\mu}_{-\infty,2}(V) \geqslant \ell^n\,\overline{\mu}_{-\infty,1}(V)$ assuming $\liminf \varphi_2/\varphi_1 > 1$. fix $r < R_2$ and ask where $\varphi = \max(\varphi_1 - A, \varphi_2)$ A is chosen large enough that coincides with φ φ_2 near $S_2(r)$. are μ_r measures associated φ. The

$\liminf \varphi_2/\varphi_1 > 1$ hypothesis implies that there is such $t < r$ φ that coincides with $\varphi_1 - A$ on $B_1(t) = \{\varphi_1 < t\}$. We obtain

$$\overline{\mu}_{-\infty,1}(V) = \lim_{t\to-\infty} \overline{\mu}_{t,1}(V) = \lim_{t\to-\infty} \overline{\mu}_t(V) \leqslant \mu_r(V) = \mu_{r,2}(V),$$

from where

$$\overline{\mu}_{-\infty,1}(V) \leqslant \lim_{r\to-\infty} \overline{\mu}_{r,2}(V) = \overline{\mu}_{-\infty,2}(V). \qquad \square$$

Under the above assumptions, we may conjecture that the inequality between measures $\overline{\mu}_{-\infty,2} \geqslant \ell^n\, \overline{\mu}_{-\infty,1}$ always occurs, but the conclusions of Theorem 4.2 we have not demonstrated it. However, we have the result especially following:

Corollary 4.3. – *With the notation of Theorem* 4.2 *or $A \subset S_1(-\infty)$ a Borel party is Meeting and related components $S_1(-\infty)$ $\mathbb{1}_A$ function feature A. Then for every function psh $V \geqslant 0$*

$$\overline{\mu}_{-\infty,2}(\mathbb{1}_A V) \geqslant \ell^n\, \overline{\mu}_{-\infty,1}(\mathbb{1}_A V).$$

In particular, if $S_1(-\infty)$ is totally disconnected, it was $\overline{\mu}_{-\infty,2} \geqslant \ell^n\, \overline{\mu}_{-\infty,1}$.

Demonstration. There is an increasing sequence of compact $K_\nu \subset A$ such that there is $\overline{\mu}_{-\infty,j}(A \smallsetminus K_\nu) < 2^{-\nu}$, $j = 1, 2$. The equivalence relation whose classes are Related components $S_1(-\infty)$ is closed graph ($S_1(-\infty)$ being compact). The saturated $\tilde{K}_\nu$ of K_ν is a compact subset of A ; more $\tilde{K}_\nu$ is intersection of a decreasing sequence of parts that are open and closed in $S_1(-\infty)$ (cf. Bourbaki [Bo], chap. II,§4, n° 4). This suggests that A is opened and closed in $S_1(-\infty)$. There is then an open $U \Subset X$ as $A = U \cap S_1(-\infty)$, $\partial U \cap S_1(-\infty) = \emptyset$. And either $r_0 = \inf_{\partial U} \varphi_1 > -\infty$

$$\Omega = \big\{z \in U\,;\ \varphi_1(z) < r_0\big\}.$$

The open meeting is Ω related components $B_1(r_0)$ so Ω of Stein ; more $\varphi_1 : \Omega \to [-\infty, r_0[$ is exhaustive. Let

$$\varphi_\nu = \max\big(\varphi_2, \nu(\varphi_1 - r_0 + 1)\big).$$

For $\nu > \sup_\Omega \varphi_2$, $\varphi_\nu : \Omega \to [-\infty, \nu[$ the application is complete, while for $\nu \geqslant \ell$ was

$$\liminf_{\varphi_1(z)\to-\infty} \frac{\varphi_\nu(z)}{\varphi_1(z)} = \liminf \frac{\varphi_2}{\varphi_1} = \ell.$$

According to Theorem 4.2 applied to φ_1 and φ_ν on Ω he comes

$$\overline{\mu}_{-\infty,\nu}(\mathbb{1}_\Omega V) \geqslant \ell^n\, \overline{\mu}_{-\infty,1}(\mathbb{1}_\Omega V).$$

If ℓ is > 0 (only interesting case to consider) was $S_2(-\infty) \supset S_1(-\infty)$ so $S_\nu(-\infty) = S_1(-\infty)$ and $\Omega \cap S_1(-\infty) = U \cap S_1(-\infty) = A$. Therefore

$$(dd^c\varphi_\nu)^n(\mathbb{1}_A V) = \overline{\mu}_{-\infty,\nu}(\mathbb{1}_A V) \geqslant \ell^n\, \overline{\mu}_{-\infty,1}(\mathbb{1}_A V).$$

Now we observe that the φ_ν subsequently decreases to φ_2 on the open $B_1(r_0 - 1)$ when $\nu \to +\infty$ so $(dd^c\varphi_\nu)^n$ weakly tends to $(dd^c\varphi_2)^n$ on $B_1(r_0 - 1)$ from 2.6 (a) and 2.10.

As A is compact and $A \subset S_1(-\infty) \subset B_1(r_0 - 1)$, as is $\mathbb{1}_A V$ upper semi-continuous, we deduce the limit

$$\overline{\mu}_{-\infty,2}(\mathbb{1}_A V) = (dd^c\varphi_2)^n(\mathbb{1}_A V) \geqslant \ell^n\,\overline{\mu}_{-\infty,1}(\mathbb{1}_A V). \qquad \square$$

In the conventional calculation that follows, we will need to assess $\|\mu_{-\infty}\|$ the mass from the function $\varphi^* = e^{\varphi}$.À this purpose, according to the observed 3.10 $\mu_r^* = \mu_{\log r}$ proposal that, where

$$\overline{\mu}_{-\infty}(1) = \lim_{r\to 0} r^{-n}\mu_r^*(1) = \lim_{r\to 0} r^{-n}\int_{\{\varphi^*<r\}}(dd^c\varphi^*)^n. \tag{4.4}$$

Proposition 4.5. – *Either $\varphi = \log\varphi^*$ psh a continuous function in $\mathbb{C}^n$ where φ^* is homogeneous degree $\ell > 0$ and $(\varphi^*)^{-1}(0) = 0$. So*

$$(dd^c\varphi)^n = (2\pi\ell)^n\delta_0$$

δ_0 where is the Dirac measure in 0.

Demonstration. The homogeneity of φ^* involves $(dd^c\varphi)^n = 0$ on $\mathbb{C}^n \smallsetminus \{0\}$. In the particular case $\varphi^*(z) = |z|^2$, we located $(dd^c\varphi^*)^n = 4^n\,n!\,d\lambda$ $(d\lambda =$ measurement Lebesgue) whereby $\overline{\mu}_{-\infty}(1) = (4\pi)^n$ and $(dd^c\varphi)^n = \overline{\mu}_{-\infty} = (4\pi)^n\delta_0$. The general case follows from the theo-rth$\tilde{4}$.2. $\square$

Listing 4.6. – way of illustration of the foregoing, look if $\varphi(z) = \log\max(|z_1|,\dots,|z_n|)$ in $\mathbb{C}^n$. As φ depends only $n-1$ variables neighborhood of each point the complement of the "diagonal" $\Delta = \{|z_1| = \cdots = |z_n|\}$, we deduce by homogeneity e^{φ} that $(dd^c\varphi)^{n-1} = 0$ on $\mathbb{C}^n \smallsetminus \Delta$. The proposition 3.7 then shows that the measure is μ_r support in the distinguished board

$$\Gamma(r) = \big\{|z_1| = \cdots = |z_n| = e^r\big\} = S(r)\cap\Delta$$

the polydisk $B(r)$. As μ_r is invariant under rotations preserving $B(r)$ and as $\|\mu_r\| = (2\pi)^n$ according 4.5, it follows that $\mu_r = d\theta_1\wedge\ldots\wedge d\theta_n$ with $z_j = e^{r+i\theta_j}$, $1 \leqslant j \leqslant n$. more we get:

$$\begin{aligned}&(dd^c\varphi)^n = (2\pi)^n\delta_0\,,\\ &(dd^c\varphi)^{n-1}\wedge d\varphi\wedge d^c\varphi = dr\wedge d\theta_1\wedge\ldots\wedge d\theta_n \quad \text{sur } \Delta\smallsetminus\{0\}.\end{aligned} \qquad \square$$

Returning now to the general case. $x \in X$ be any point and $w = (w_1, w_2, \dots, w_N)$ the N coordinates functions related an embedding of a neighborhood $U \subset X$ x in $\mathbb{C}^N$ as $w(x) = 0$. There is a $\Omega \Subset U$ neighborhood and x $r_0 < 0$ such as $\varphi_1(z) = \log|w(z)|^2 : \Omega \to\,]\infty, r_0[$ function is comprehensive. The formula then gives 4.4 (see [DE5]):

$$\overline{\mu}_{-\infty,1}(1) = (4\pi)^n\,\nu([X], x)$$

$\nu([X], x)$ which is the number of x Lelong in the integration stream X in $\mathbb{C}^N$ equal after P. Thie [Th] to the algebraic multiplicity $m(X, x)$ of X developed x. It is concluded that

$$\overline{\mu}_{-\infty,1} = (4\pi)^n \, m(X, x) \, \delta_x \, . \tag{4.7}$$

For some φ function yields the following result, which is well known at least in the case where X is smooth.

Corollary 4.8. – *denote $\nu(\varphi, x)$ the Lelong number of φ any point $x \in X$. So*

$$(dd^c\varphi)^n \geqslant \overline{\mu}_{-\infty} \geqslant (2\pi)^n \sum_{x \in X} m(X, x) \, \nu(\varphi, x)^n \, \delta_x \, .$$

Demonstration. With the previous notations, one of the definitions equivalent numbers Lelong is:

$$\nu(\varphi, x) = \liminf_{z \to x} \frac{\varphi(z)}{\log |w(z)|} .$$

$\varphi_1(z) = \chi(z) \log |w(z)|^2 + A\psi(z)$ then ask where is χ $\mathcal{C}^\infty$ compact support in Ω, near $\chi \equiv 1$ of$x$, ψ strictly psh of $\mathcal{C}^2$ class on X and $A > 0$ large enough. By Corollary 4.3 and formula (4.7) it comes :

$$\liminf_{z \to x} \frac{\varphi(z)}{\varphi_1(z)} = \frac{1}{2} \nu(\varphi, x),$$

from where

$$\overline{\mu}_{-\infty} \geqslant \Big(\frac{1}{2} \nu(\varphi, x) \Big)^n \overline{\mu}_{-\infty,1} \geqslant (2\pi)^n \, m(X, x) \, \nu(\varphi, x)^n \, \delta_x . \qquad \square$$

5. Principe the maximum.

φ is a function of exhaustion psh continuous space X complex. We will see that the plurisubharmonic functions X to satisfy the maximum principle in relation to measures Monge-Ampère associated φ.

Theorem 5.1. – *If $B(r) = \{\varphi < r\} \neq \emptyset$ then $\|\mu_r\| > 0$ and for any function V psh on X was :*

$$\sup_{B(r)} V = \textit{sup essentiel de } V \textit{ relativement à } \mu_r .$$

Example 4.6 shows that the hypothesis of plurisousharmonicité in V 5.1 theorem is relevant.

Demonstration. It is not restrictive to assume $V \leqslant 0$. We will then show that $\sup_{B(r)} V = \|V\|_{L^\infty(\mu_r)}$ Jensen applying the formula to a function exhaustion φ' well chosen.

ψ is strictly psh function on X $\mathcal{C}^2$ class, $z_0 \in B(r) \cap X_{\mathrm{reg}}$ a regular point and $U \Subset B(r) \cap X_{\mathrm{reg}}$ a neighborhood of z_0. For $\varepsilon > 0$ small enough, the function

$$\varphi'(z) = \max\big(\varphi(z), \varphi(z_0), r - \sqrt{\varepsilon} + \varepsilon\psi(z)\big)$$

equals $\varepsilon\psi(z) + \mathrm{Cte}$ on U and coincides with φ near $S(r)$. μ_r the measure may therefore both be defined by φ', giving

$$\begin{aligned}\mu_r(V) &= \int_{-\infty} r dt \int_{B(r)\cap\{\varphi'<t\}} dd^c V \wedge (dd^c\varphi')^{n-1} + \int_{B(r)} V\,(dd^c\varphi')^n \\ &\geqslant \varepsilon^n \int_U V\,(dd^c\psi)^n.\end{aligned}$$

In particular $\|\mu_r\| = \mu_r(1) > 0$. replace now V by V^p and do tend to p $+\infty$. It comes :

$$V(z_0) \leqslant \lim_{p\to+\infty} \left[\int_U V^p\,(dd^c\psi)^n\right]^{1/p} \leqslant \lim_{p\to+\infty} \left[\varepsilon^{-n}\mu_r(V^p)\right]^{1/p} = \|V\|_{L^\infty(\mu_r)}.$$

therefore we obtain

$$\sup_{B(r)} V = \sup_{B(r)\cap X_{\mathrm{reg}}} V \leqslant \|V\|_{L^\infty(\mu_r)}.$$

In the other direction, inequality

$$\|V\|_{L^\infty(\mu_r)} \leqslant \sup_{S(r)} V$$

is obvious. If we prove the continuity of the function left $r \mapsto \|V\|_{L^\infty(\mu_r)}$ we will have

$$\|V\|_{L^\infty(\mu_r)} \leqslant \lim_{t<r,\, t\to r} L < nr) \sup_{S(t)} V \leqslant \sup_{S(r)} V.$$

Lemma 5.2. – *For psh function $V \geqslant 0$, $r \mapsto \|V\|_{L^\infty(\mu_r)}$ the application is growing and still left.*

Demonstration. 3.4 The formula shows that the function $r \mapsto \mu_r(V)$ is growing and continues to the left. On every interval $]-\infty, r_0]$, $r_0 < R$, functions

$$r \mapsto \left[\|\mu_{r_0}\|^{-1}\mu_r(V^p)\right]^{1/p}$$

are increasing and continuous on the left, and form a family increasing compared to p under unequal H "older ($\|\mu_{r_0}\|^{-1}\mu_r$ the measure is $\leqslant 1$ mass. The limit when $p \to +\infty$, namely $r \mapsto \|V\|_{L^\infty(\mu_r)}$, is growing and continues to left on $]-\infty, r_0]$. □

6. Propriétés convexity of psh functions.

A well-known result of P .Lelong (cf. [Le1]) argues that the greater the average and generally the average L^p a psh function on Euclidean sphere of radius r in $\mathbb{C}^n$ are convex functions $\log r$. We intend to extend these properties to a situation much more general.

X be a Stein space of pure dimension n, $\varphi : X \to [-\infty, R[$ psh a continuous function exhaustive. We assume that is φ Monge-Ampère homogeneous, i.e. there as $A \in]-\infty, R[$

$$(6.1) \qquad (dd^c\varphi)^n = 0 \quad \text{sur l'ouvert } \{\varphi > A\}.$$

For psh function on V X and all $r > A$ Theorem 3.4 shows while the left derivative

$$(6.2) \qquad \frac{d}{dr_-}\,\mu_r(V) = \int_{B(r)} dd^cV \wedge \alpha^{n-1}$$

is increasing positively in r, hence the

Theorem 6.3. – *The function average* $r \mapsto M_V(r) = \mu_r(V)$ *is convex on growing* $]A, R[$.

The classic case mentioned at the beginning is the ball of radius in e^R $\mathbb{C}^n$ with$\varphi(z) = \log|z|$, $A = -\infty$. More generally, was a result of convexity to the average standard defined in L^p

$$M_V^p(r) = \Big[\mu_r(V_+^p)\Big]^{1/p}, \qquad p \in [1, +\infty[\,.$$

Theorem 6.4. – *The function* $r \mapsto M_V^p(r)$ *is convex on growing* $]A, R[$.

Demonstration. Accrued we reduce to the case is V psh class > 0 $\mathcal{C}^\infty$. Since $\varepsilon > 0$, consider the function

$$h_\varepsilon(r) = \int_{r-\varepsilon}^{r} \mu_t(V^p)\,dt = \int_{B(r)\smallsetminus B(r-\varepsilon)} V^p\,\alpha^{n-1} d\varphi \wedge d^c\varphi\,dt, \qquad r \in]A+\varepsilon, R[$$

(The last equality follows from Proposition 3.9 (a)). As $\mu_r(V^p) = \lim_{\varepsilon\to 0} h_\varepsilon(r)$, it suffices to prove $h_\varepsilon^{1/p}$ that is convex for all $\varepsilon > 0$. We should therefore verify inequality

$$h_\varepsilon h_\varepsilon'' - \Big(1 - \frac{1}{p}\Big) h_\varepsilon'^2 \geqslant 0$$

where the second derivative h_ε'' is, say, calculated left. According to the proposal 3.9 (b) and the assumption (6.1) it comes

$$\begin{aligned} h_\varepsilon'(r) &= \mu_r(V^p) - \mu_{r-\varepsilon}(V^p) \\ &= \int_{B(r)\smallsetminus B(r-\varepsilon)} d\big[V^p\,\alpha^{n-1} \wedge d^c\varphi\big] \\ &= \int_{B(r)\smallsetminus B(r-\varepsilon)} p\,V^{p-1}\,dV \wedge \alpha^{n-1} \wedge d^c\varphi. \end{aligned}$$

The formula (6.2) also implies

$$h''_\varepsilon(r) = \int_{B(r)\smallsetminus B(r-\varepsilon)} dd^c(V^p) \wedge \alpha^{n-1}.$$

Thanks to the Cauchy-Schwarz inequality we obtain

$$h'_\varepsilon(r)^2 \leqslant \int_{B(r)\smallsetminus B(r-\varepsilon)} V^p\, \alpha^{n-1} \wedge d\varphi \wedge d^c\varphi \;\cdot \int_{B(r)\smallsetminus B(r-\varepsilon)} p^2 V^{p-2}\, dV \wedge d^cV \wedge \alpha^{n-1},$$

and equation (6.5) sought follows from inequality

$$dd^c(V^p) \geqslant 2p(p-1)\, V^{p-2} dV \wedge d^cV. \qquad \square$$

Corollary 6.6. – *defined functions*

(a) $M_V^{\exp}(r) = \log \mu_r(e^V)$,

(b) $M_V^{\infty}(r) = \sup_{B(r)} V$,

are increasing convex on $]A, R[$.

Proof. Property (a) follows from Theorem 6.4 and equality

$$\log \mu_r(e^V) = \lim_{p\to+\infty} p\left\{\left[\mu_r\Big(1+\frac{V}{p}\Big)_+^p\right]^{1/p} - 1\right\}.$$

The maximum principle (Theorem 5.1) causes the other

$$\sup_{B(r)} V = \lim_{\lambda\to+\infty} \frac{1}{\lambda} \log \mu_r(e^{\lambda V})$$

by sequence (b) is a result of (a). $\square$

For applications in the study of fiber spaces, we demonstrate Now a version with parameter of Theorem 6.4. We are given a $\pi : X \to Y$ morphism of analytic spaces of pure dimensions $\dim X = m+n$, $\dim Y = m$ and functions $\varphi : X \to [-\infty, +\infty[$ psh continues $R : Y \to]-\infty, +\infty]$) (resp. $A : Y \to [-\infty, +\infty[$) lower semicontinuous (Resp. superiorly) satisfying the following properties.

Assumptions 6.7. –

(a) *π is onto, and $\pi^{-1}(y)$ fiber $y \in Y$ are purely n dimension.*

b) is a morphism *π Stein, i.e. has a Y open cover $(\Omega_j)_{j\in J}$ as $\pi^{-1}(\Omega_j)$ Stein or for any $j \in J$.*

(c) *$\varphi(x) < R(\pi(x))$ and $A(y) < R(y)$ whatever $x \in X$, $y \in Y$.*

(d) *For $y \in Y$ and all $r < R(y)$, there exists a neighborhood U of y in Y as $\pi^{-1}(U) \cap B(r) \Subset X$.*

(e) $(dd^c\varphi)^n \equiv 0$ *on open* $\{x \in X\,;\ \varphi(x) > A(\pi(x))\}$.

It is noted here again $B(r) = \{\varphi < r\}$, and $S(r) = \{\varphi = r\}$ $\alpha = dd^c\varphi$. Under the hypotheses (c) and (d) allows the§3 to associate each fiber $\pi^{-1}(y)$ family measures $\mu_{y,r}$ carried by $\pi^{-1}(y)\cap S(r)$ for $\in\ ]-\infty, R(y)[$. Étant given a psh function on V X introducing the mean values

$$M_V(y,r) = \mu_{y,r}(V),$$

$$M_V^p(y,r) = \big[\mu_{y,r}(V_+^p)\big]^{1/p} \quad \text{si } p \in [1,+\infty[,$$

$$M_V^{\exp}(y,r) = \log \mu_{y,r}(e^V),$$

$$M_V^\infty(y,r) = \sup_{\pi^{-1}(y)\cap B(r)} V.$$

Proposition 6.8. – *For fixed* r, $y \mapsto \mu_{y,r}(V)$ *the applications and* $y \mapsto M_V^p(y,r)$ *psh are low within the meaning of the definition of the open* 1.9 $\{y \in Y\,;\ A(y) < r < R(y)\}$.

Demonstration. As the result is local Y on the assumption 6.7 (b) to suppose X, Y Stein. A passage to the decreasing limit then brings us back to where V is psh of $\mathcal{C}^\infty$ class; if $p > 1$ we can assume more $V > 0$. Is arbitrary and $\varepsilon > 0$ $\chi :]r-\varepsilon, r[\to \mathbb{R}$ a function not $\mathcal{C}^\infty \geqslant 0$ no compact support. In analogy to the 6.4 theorem, introduce auxiliary function

$$h(y) = \int_{r-\varepsilon}^{r} \mu_{y,t}(V^p)\,\chi(t)\,dt = \int_{\pi^{-1}(y)} V^p\,\chi(\varphi)\,\alpha^{n-1}\wedge d\varphi \wedge d^c\varphi$$

set to open

$$U_\varepsilon = \big\{y \in Y\,;\ A(y) + \varepsilon < r < R(y)\big\}.$$

To conclude, it suffices to show that $h^{1/p}$ is weakly psh on U_ε. If $p > 1$, so this is to show that

$$h\,dd^c h - \Big(1 - \frac{1}{p}\Big) dh \wedge d^c h \geqslant 0.$$

Let u, v, w real forms $\mathcal{C}^\infty$ class on Y compact support in U_ε, respective bidegrés (m,m), $(m, m-1)\oplus(m-1, m)$ and $(m-1, m-1)$. According to Fubini, we first applied assuming class φ $\mathcal{C}^\infty$, he comes

$$\int_Y hu = \int_X V^p\,\chi(\varphi)\,\alpha^{n-1}\wedge d\varphi\wedge d^c\varphi \wedge \pi^* u\ ;$$

If φ only continues is deduced by limit decreasing (2.6 theorem). Now we observe that the integrand has support in

$$\pi^{-1}(\mathrm{Supp}\,u) \cap \big(B(r) \smallsetminus B(r-\varepsilon)\big) \Subset X$$

(Hypothesis 6.7 (d)) and the current is $\chi(\varphi)\,\alpha^{n-1}\wedge d\varphi\wedge d^c\varphi$ d -closed (hypothesis 6.7 (e)). Thanks to an integration by parts on Y and another on opposite X therefore obtained successively

$$\int_Y dh \wedge v = \int_X d(V^p)\wedge \chi(\varphi)\,\alpha^{n-1}\wedge d\varphi\wedge d^c\varphi\wedge \pi^* v, \tag{6.9}$$

$$\int_Y dd^c h \wedge w = \int_X dd^c(V^p)\wedge \chi(\varphi)\,\alpha^{n-1}\wedge d\varphi\wedge d^c\varphi\wedge \pi^* w. \tag{6.10}$$

Suppose the $(m-1, m-1)$ Platform w be $\geqslant 0$. Equality (6.10) GAINST $\int_Y dd^c h \wedge w \geqslant 0$ that already, so $dd^c h \geqslant 0$ U_ε on, which solves the case $p = 1$. In the general case $p > 1$ or γ a real 1 Platform $\mathcal{C}^\infty$ on and Y $\gamma^c = i(\gamma^{0,1} - \gamma^{1,0})$. Equality (6.9) combined inequality Cauchy-Schwarz leads

$$\begin{aligned}
\int_Y dh \wedge \gamma^c \wedge w &= \int_X p\, V^{p-1} \wedge dV \wedge \pi^* \gamma^c \wedge \chi(\varphi)\, \alpha^{n-1} \wedge d\varphi \wedge d^c\varphi \wedge \pi^* w \\
&\leqslant \frac{1}{2} \int_X \big(V^p\, \pi^*(\gamma \wedge \gamma^c) + p^2\, V^{p-2} dV \wedge d^c V \big) \wedge \chi(\varphi)\, \alpha^{n-1} \wedge d\varphi \wedge d^c\varphi \wedge \pi^* w \\
&\leqslant \frac{1}{2} \int_Y h\, \gamma \wedge \gamma^c + \frac{p}{p-1} dd^c h \wedge w,
\end{aligned}$$

given that $dd^c V^p \geqslant p(p-1)\, dV \wedge d^c V$. As is true for any form $w \geqslant 0$, we deduce the direction of the currents inequality

$$dh \wedge \gamma^c + \gamma \wedge d^c h \leqslant h\, \gamma \wedge \gamma^c + \frac{p}{p-1}\, dd^c h.$$

Observe that h is everywhere on > 0 U_ε from 4.1 ; if we now tend to γ dh/h he comes the expected inequality

$$\frac{1}{h}\, dh \wedge d^c h \leqslant \frac{p}{p-1} dd^c h.$$

To see that h is locally bounded on Y, just look at the If $V \equiv 1$. Equality (6.9) then shows that $dh = 0$ so h is locally constant on X_{reg}. □

6.8 The proposal actually contains the following more general result, which was our main objective.

Theorem 6.11. – *functions on $Y \times \mathbb{C}$ defined by*

$$(y, z) \mapsto M_V(y, \operatorname{Re} z), \quad M_V^p(y, \operatorname{Re} z), \quad M_V^{\exp}(V, \operatorname{Re} z), \quad M_V^\infty(y, \operatorname{Re} z)$$

psh are slightly on the open

$$\big\{(y, z) \in Y \times \mathbb{C}\, ;\ A(y) < \operatorname{Re} z < R(y)\big\}.$$

Demonstration. We consider the morphism

$$\tilde{\pi} = \pi \times \mathrm{Id} : X \times \mathbb{C} \to Y \times \mathbb{C}$$

and we equip $X \times \mathbb{C}$, $Y \times \mathbb{C}$ functions

$$\tilde{\varphi}(x, z) = \varphi(x) - \operatorname{Re} z, \quad \tilde{R}(y, z) = R(y) - \operatorname{Re} z, \quad \tilde{A}(y, z) = A(y) - \operatorname{Re} z,$$

so that the assumptions 6.7 (a-e) are satisfied with respect to these data. If $\tilde{V}(x, z) = V(x)$ was built by

$$\tilde{\mu}_{(y,z),0}(\tilde{V}) = \mu_{y, \operatorname{Re} z}(V),$$

Theorem 6.11 and thus derives from the proposition 6.8. □

Corollary 6.12. – *Let $(X_j)_{1\leqslant j\leqslant k}$ Stein spaces of pure n_j $\varphi_j : X_j \to [-\infty, R_j[$ dimension and comprehensive continuous psh functions such as $(dd^c\varphi_j)^{n_j} \equiv 0$ on open $\{x \in X_j\,;\ \varphi_j(x) > A_j\}$. If V is on psh $X_1 \times \cdots \times X_k$, functions*

$$M_V(r_1, \ldots, r_k) = \mu_{1,r_1} \otimes \cdots \otimes \mu_{k,r_k}(V),$$

$$M_V^p(r_1, \ldots, r_k) = M_{V_+^p}(r_1, \ldots, r_k)^{1/p},$$

$$M_V^{\exp}(r_1, \ldots, r_k) = \log M_{e^V}(r_1, \ldots, r_k),$$

$$M_V^{\infty}(r_1, \ldots, r_k) = \sup_{B(r_1)\times\cdots\times B(r_k)} V$$

are convex in the $(r_1, \ldots, r_k) \in \prod_{1\leqslant j\leqslant k}]A_j, R_j[$ variables simultaneously, and increasing relative to each r_j.

More generally, if X_0 is an analytical space of pure dimension n and if V is psh on $X_0 \times X_1 \times \cdots \times X_k$, function

$$M_V^p(x_0, \operatorname{Re} z_1, \ldots, \operatorname{Re} z_k) = M_{V(x_0,\bullet)}^p(\operatorname{Re} z_1, \ldots, \operatorname{Re} z_k)$$

(resp. $p = \emptyset$, $\exp$, ∞) is psh on open

$$X_0 \times \prod_{1\leqslant j\leqslant k} \big\{A_j < \operatorname{Re} z_j < R_j\big\} \subset X_0 \times \mathbb{C}^k.$$

Demonstration. Just to prove the last statement. We proceed by induction on k. For $k = 1$, Theorem 6.11 applied to $\pi : X = X_0 \times X_1 \to X_0 = Y$ and $\varphi = \varphi_1$ shows that the function

$$(x_0, z_1) \mapsto M_V^p(x_0, \operatorname{Re} z_1)$$

psh is low. If more V is continuous, this function is separately x continuous and convex $\operatorname{Re} z$, so continuous $(x_0, \operatorname{Re} z_1)$. With the corollary 1.12, $M_V^p(x_0, \operatorname{Re} z)$ is psh. If any is V be obtained by writing V as limit decreasing continuous psh functions.

$k > 1$ to the property results in k about its validity to 1 $k-1$ orders and posing

$$W(x_0, x_1, \ldots, x_{k-1}, z_k) = M_{V(x_0,\ldots,x_{k-1},\bullet)}^p(\operatorname{Re} z_k)$$

and observing that

$$M_V^p(x_0, \operatorname{Re} z_1, \ldots, \operatorname{Re} z_k) = M_{W(x_0,\bullet,z_k)}^p(\operatorname{Re} z_1, \ldots, \operatorname{Re} z_{k-1}). \qquad \square$$

We end this section by reviewing in the light of results preceding the convexity inequality P. Lelong, a measure of how specific changes of growing a psh function on a space bundle along different fibers. This inequality was used H. Skoda [Sk2] to build a first-instance against the problem posed by J.-P. Serre in 1953, whether a basic bundle and fiber Stein

himself Stein ; see also [De1], [De2] for other examples and-against [De3] for a simple construction and fast.

Ω be an irreducible Stein space dimension m which will act as a base of fiber and X space Stein n sheer size, which will be the fiber. It is assumed that there are $\psi : \Omega \to [-\infty, R[$ functions $\varphi : X \to [-\infty, +\infty[$ continuous psh exhaustive as

$$(dd^c\psi)^m = 0 \quad \text{sur} \quad \{\psi > A\}, \qquad (dd^c\varphi)^n = 0 \quad \text{sur } \{\varphi > 0\}.$$

For example if X is an affine algebraic variety of dimension n, there is a finite morphism $F : X \to \mathbb{C}^n$ (normalization theorem Noether), and just take $\varphi(z) = \log \|F(z)\| \ \| \ \ \|$ which is a standard $\mathbb{C}^n$; the same reasoning applies locally on Ω for the existence of ψ.

V be a psh function on $\Omega \times X$ and real a, b, c, r such as $A < a < b < c < R$ and $r > 0$. The convexity property corollary 6.12 show that

$$M_V^\infty(b, r) \leqslant M_V^\infty(a, \sigma r) + \left(1 - \frac{1}{\sigma}\right) \left[M_V^\infty(c, 0) - M_V^\infty(a, \sigma r)\right]$$

with $\sigma = \frac{c-a}{c-b}$. It follows from Theorem 7.5 demonstrated following paragraph which was $M_V^\infty(a, r) \to +\infty$ when $r \to +\infty$, when V is not constant over at least one fiber $\{z\} \times X$, $z \in B(a)$. Then there exists a constant dependent r_0 a, b, c, V as

$$M_V^\infty(b, r) < M_V^\infty(a, \sigma r) \quad \text{pour } r > r_0, \text{ où } \ \sigma = \frac{c-a}{c-b}. \tag{6.13}$$

If ω is an open relatively compact in Ω, we set now

$$M_V^\infty(\omega\,;\, r) = \sup_{\omega \times B(r)} V.$$

With an elementary reasoning compactness and connectedness (cf. [Le2], Theorem 6.5.4) then it follows from (6.13) the following result :

Corollary 6.14(inequality P. Lelong)**.** – Ω *be an irreducible complex space* ω_1, ω_2 *two relatively compact open in* Ω *and* V *function psh on* $\Omega \times X$*, assumed not constant over at least one fiber* $\{z\} \times X$*. So there is a constant that depends* $\sigma > 1$ *as* $\omega_1, \omega_2, \Omega$*, and constantly dependent* r_0 V *off, such that for all we have* $r > r_0$

$$M_V^\infty(\omega_2\,;\, r) < M_V^\infty(\omega_1\,;\, \sigma r).$$

In practical applications there is the problem of calculating Constant explicit σ. The inequality (6.13) brings complete theoretical answer to this problem. Yes $\omega_1, \omega_2 \Subset \Omega$ are open to $\mathbb{C}$ it seeks a harmonic function on ψ $\Omega \smallsetminus \overline{\omega}_1$ which tends to 0 on $\partial\omega_1$ and to 1 on $\partial\Omega$ (Resp. to $+\infty$ if $\partial\Omega$ capacity is 0) ; is extended by ψ 0 on ω_1 and pose $b = \sup_{\omega_2} \psi$. Any constant $\sigma > \frac{1}{1-b}$ (resp. $\sigma > 1$) then meets the question. If the base is Ω $m > 1$ dimension returns to solve a similar Dirichlet problem for the Monge-Ampère equation $(dd^c\psi)^m = 0$ on $\Omega \smallsetminus \overline{\omega}_1$. It est easy to see that the constant is obtained σ best. Aele mentary calculation shows that the function

$$\chi(t, r) = \exp\left(\frac{r}{1-t} + \frac{1}{(1-t)^2}\right)$$

growing is convex on $[0,1[\times[0,+\infty[$. The psh function $V=\chi(\psi_+,\varphi_+)$ then contradicts (6.13) for all $\sigma\leqslant\frac{1}{1-b}$.

7. Croissance to infinity psh functions.

X be a Stein space of irreducible n $\varphi : X\to[-\infty,+\infty[$ dimension and a comprehensive continuous psh function (so we here $R=+\infty$). We notice

$$\tau(r)=\|\mu_r\|=\int_{B(r)}\alpha^n$$

where $\alpha=dd^c\varphi$, the volume of the pseudoboule $B(r)=\{\varphi<r\}$. Jensen formula 3.4 is then used to connect growth counant dd^cV growth of V. Specifically :

Proposition 7.1. – *Let V a psh function on X, $r_0\in\mathbb{R}$ and $\varepsilon\in\,]0,1[$. There is a constant $C>0$ dependent V,ε,r_0 such that for all $r\geqslant r_0$ we are*

$$(l-\varepsilon)r\int_{B(r_0)}dd^cV\wedge\alpha^{n-1}\leqslant\mu_r(V_+)+C\,\tau(r).$$

Demonstration. Let $V_\nu=\max(V,-\nu)$, $\nu\in\mathbb{N}$. Measures $dd^cV_\nu\wedge\alpha^{n-1}$ weakly converge $dd^cV\wedge\alpha^{n-1}$ $\nu\to+\infty$ when, following

$$\liminf_{\nu\to+\infty}\int_{B(r_0)}dd^cV_\nu\wedge\alpha^{n-1}\geqslant\int_{B(r_0)}dd^cV\wedge\alpha^{n-1}.$$

There is therefore $\nu\in\mathbb{N}$ (depending V,ε,r_0) as

$$\int_{B(r_0)}dd^cV_\nu\wedge\alpha^{n-1}\geqslant(1-\varepsilon)\int_{B(r_0)}dd^cV\wedge\alpha^{n-1}.$$

The formula applied to V_ν 3.4 gives the other

$$(r-r_0)\int_{B(r_0)}dd^cV_\nu\wedge\alpha^{n-1}\geqslant\mu_r(V_\nu)-\int_{B(r)}V_\nu\,\alpha^n\leqslant\mu_r(V_+)+\nu\,\tau(r).$$

These two combined inequalities entail the proposal with 7.1

$$C=\nu+\frac{(1-\varepsilon)r_0}{\tau(r_0)}\int_{B(r_0)}dd^cV\wedge\alpha^{n-1}.\qquad\square$$

In the remainder of this paragraph, we will do a moderation hypothesis on volume growth of X.

Hypothesis 7.2. – $\displaystyle\lim_{r\to+\infty}\frac{\tau(r)}{r}=0.$

We then get the immediate consequence of the proposal 7.1 the following basic inequality:

Corollary 7.3. – (7.2) *Under the hypothesis, for any function V psh* :

$$\int_X dd^c V \wedge \alpha^{n-1} \leqslant \liminf_{r\to+\infty} \frac{1}{r}\mu_r(V_+).$$

This inequality will operate mainly through the following lemma:

Lemma 7.4. – *Let ψ strictly a function Class psh $\mathcal{C}^2$ on X and $r_1 < r_2$ with $B(r_2) \neq \emptyset$. Then there exists a constant $C(r_1, r_2) > 0$ such that for every function psh V we have* :

$$\int_{B(r_1)} dd^c V \wedge (dd^c\psi)^{n-1} \leqslant C(r_1, r_2) \int_{B(r_2)} dd^c V \wedge \alpha^{n-1}.$$

Demonstration. Let $\varphi' = \max(\varphi, r_1 + \varepsilon\psi + \sqrt{\varepsilon})$ where $\varepsilon > 0$ is chosen small enough that $\varphi' = \varphi$ near $S(r_2)$ and $\varphi' = r_1 + \varepsilon\psi + \sqrt{\varepsilon}$ on $B(r_1)$. By Theorem Stokes it comes

$$\int_{B(r_2)} dd^c V \wedge (dd^c\varphi)^{n-1} = \int_{B(r_2)} dd^c V \wedge (dd^c\varphi')^{n-1} \geqslant \varepsilon^{n-1} \int_{B(r_2)} dd^c V \wedge (dd^c\psi)^{n-1}. \quad \square$$

Theorem 7.5. – *Any function psh V on X checking a growth assumptions below is constant* :

(a) $\liminf_{r\to+\infty} \frac{1}{r}\mu_r(V_+) = 0.$

(b) $\sup_{B(r)} V = o\Big(\frac{r}{\tau(r)}\Big)$ *when* $r \to +\infty$.

Demonstration. 5.1 theorem gives

$$\mu_r(V_+) \leqslant \|\mu_r\| \sup_{B(r)} V_+ = \tau(r) \sup_{B(r)} V_+,$$

therefore the assumption 7.5 (b) involves 7.5 (a). Assuming 7.5 (a), Corollary 7.3 and Lemma 7.4 show that $dd^c V = 0$, i.e. V is pluriharmonique. For the $x \in X$ $z \mapsto \tilde{V}(z) = \max(V(z), V(x))$ function still checks 7.5 (a), it is pluriharmonique. According to the maximum principle is constant $-\tilde{V}$ on X (X being assumed irreducible), i.e. $V \leqslant V(x)$; V therefore constant est. $\square$

In the usual situation of the Hermitian space and $\mathbb{C}^n$ $\varphi(z) = \log|z|$ function of exhaustion, Theorem 7.5 gives (With a slightly simpler proof) a result due to N. Sibony and P.-M. Wong. Define the logarithmic function of order $\rho(V)$ psh V in $\mathbb{C}^n$ (resp. an entire function F) by

$$1 + \rho(V) = \limsup_{r\to+\infty} \frac{\log \sup_{|z|<r} V(z)}{\log\log r}, \qquad (\text{resp. } \rho(F) = \rho(\log|F|).$$

In other words V and F are logarithmic order to $\leqslant \rho$ if all $\varepsilon > 0$ was

$$V_+(z) \leqslant \varphi(z)^{1+\rho+\varepsilon}, \qquad (\text{resp. } |F(z)| \leqslant \exp\big(\varphi(z)^{1+\rho+\varepsilon}\big))$$

when $|z|$ is large enough. In particular, any polynomial is of order logarithmic zero and entire function of finite logarithmic order is no order in the usual sense.

Corollary 7.6([SW]). **–** *Either a F non-constant entire function of logarithmic order $\rho < 1$ and X an irreducible component of the hypersurface $F^{-1}(0)$. then all psh function on V X logarithmic order $< 1-\rho$ is constant. In particular, psh functions holomorphic and bounded on X are constant.*

Demonstration. Theorem 7.5 leads us to estimate the volume X φ of respect, which is a common problem. The flow Integration on X is indeed increased by $\frac{1}{2\pi}dd^c \log|F|$ in Under Lelong-Poincaré equation (see [Le1]) ; was therefore

$$\tau(r) = \int_{X\cap\{\varphi<r\}} (dd^c\varphi)^{n-1} \leqslant \int_{\{\varphi<r\}} \frac{1}{2\pi} dd^c \log|F| \wedge \alpha^{n-1}$$

After any translation we can assume $F(0) \neq 0$. The 4.5 proposal and the formula applied to 3.4 $V = \frac{1}{2\pi}\log|F|$ in $\mathbb{C}^n$ then give

$$\int_{-\infty}^{r} \tau(t)\,dt \leqslant \mu_r\Big(\frac{1}{2\pi}\log|F|\Big) - (2\pi)^n \frac{1}{2\pi}\log|F(0)| \leqslant C\, r^{1+\rho+\varepsilon}.$$

for $r \geqslant r_0(\varepsilon)$. τ as the function is increasing, it deduces

$$\tau(r) \leqslant \frac{1}{r}\int_{r}^{2r} \tau(t)\,dt \leqslant C'\, r^{\rho+\varepsilon}.$$

The conclusion then follows from 7.5 (b). □

8. Fonctions holomorphicφ-polynomial.

We keep here the notations and assumptions§7 : X means a Stein space irreducible dimension n provided with a function of exhaustion psh φ the condition (7.2) of the volume growth.

Definition 8.1. **–** *If F is a function holomorphic on X, the term degree of respect F φ number*

$$\delta_\varphi(F) = \limsup_{r\to+\infty}\ \mu_r(\log_+|F|) \in [0,+\infty].$$

The obvious inequalities and $\log_+|FG| \leqslant \log_+|F| + \log_+|G|$

$$\log_+|\lambda F + \mu G| \leqslant \log_+|F| + \log_+|G| + \log_+|\lambda + \mu|,$$

attached to (7.2), leads to all scalar $\lambda, \mu \in \mathbb{C}$:

$$\delta_\varphi(FG) \leqslant \delta_\varphi(F) + \delta_\varphi(G), \qquad \delta_\varphi(\lambda F + \mu G) \leqslant \delta_\varphi(F) + \delta_\varphi(G).$$

The set of holomorphic functions of finite degree is a $\mathbb{C}$ integrates algebra.

Notations 8.2. – *is noted* :

(a) $A_\varphi(X)$ *algebra of holomorphic functions of finite degree, who will say* φ *-polynomiales functions.*

(b) $K_\varphi(X)$ *the body of quotients with* F/G $F, G \in A_\varphi(X)$ *said body functions* φ *ra-tion-tional.*

The terminology is justified by Theorem 8.5 below. In all the examples we know, equality $A_\varphi(X) = K_\varphi(X) \cap \mathcal{O}(X)$ appropriate, but we do not know if this property is general. On the other hand, if X is normal, $A_\varphi(X)$ is fully closed subalgebra $K_\varphi(X)$ (Immediate verification).

With these definitions, we have the following fundamental inequality, which stems 7.3 of the proposal applied to $V = \log|F|$.

Proposition 8.3. – $[Z_F] = \frac{1}{2\pi} dd^c \log|F|$ *Let the divisor of zeros of a function* $F \in A_\varphi(X)$ *not identically zero. So*

$$2\pi \int_X [Z_F] \wedge \alpha^{n-1} \leqslant \delta_\varphi(F). \qquad \square$$

Corollary 8.4. – *a either a regular point of* X. $\operatorname{ord}_a(F)$ *is designated by the order of cancellation of a holomorphic function* F *in* a. *There is a constant such that for any* $C(a) > 0$ $F \in A_\varphi(X)$ *nonzero function we have* :

$$\operatorname{ord}_a(F) \leqslant C(a)\, \delta_\varphi(F).$$

Demonstration. Is a system of $(z_1, z_2, \ldots, z_n)$ local coordinates on X centered in a as the ball $|z| \leqslant \varepsilon$ is relatively compact in X. The Lemma 7.4 implies the existence of a constant $C_1 > 0$ as

$$\int_{|z|\leqslant\varepsilon} [Z_F] \wedge (dd^c|z|^2)^{n-1} \leqslant C_1 \int_X [Z_F] \wedge \alpha^{n-1}.$$

Corollary 8.4 then follows from Proposition 8.3 and inequality classic P. Lelong [Le1] :

$$\frac{1}{(4\pi\varepsilon^2)^{n-1}} \int_{|z|\leqslant\varepsilon} [Z_F] \wedge (dd^c|z|^2)^{n-1} \geqslant \operatorname{ord}_a(F). \qquad \square$$

Using conventional reasoning back to Poincaré and developed Siegel [Si1], [Si2], we now derive a theorem algebraicity of very general.

Theorem 8.5. – *The transcendence degree of* $\mathbb{C}$ *body* $K_\varphi(X)$ *functions* φ *ra-tion-tional is as* :

(a) $0 \leqslant \deg \operatorname{tr}_{\mathbb{C}} K_\varphi(X) \leqslant n = \dim X$.

(b) If $\deg\operatorname{tr}_{\mathbb{C}} K_\varphi(X) = n$ *then is* $K_\varphi(X)$ *a finitely generated extension of* $\mathbb{C}$.

Demonstration. Let $F_1, \ldots, F_N$ functions φ -polynomiales, $(k_1, \ldots, k_N)$ aN tuple $\geqslant 0$ of integers $P \in \mathbb{C}[X_1, \ldots, X_N]$ and a polynomial of variables such as N $\deg_{X_j} P \leqslant k_j$ and $P(F_1, \ldots, F_N) \not\equiv 0$. Was then

$$\log_+ |P(F_1, \ldots, F_N)| \leqslant \sum_{1\leqslant j\leqslant N} k_j \log_+ |F_j| + \mathrm{Cte},$$

$$\delta_\varphi(P(F_1, \ldots, F_N)) \leqslant \sum_{1\leqslant j\leqslant N} k_j\, \delta_\varphi(F_j).$$

Corollary 8.4 therefore gives the inequality

$$\operatorname{ord}_a P(F_1, \ldots, F_N) \leqslant C(a) \sum_{1\leqslant j\leqslant N} k_j\, \delta_\varphi(F_j). \tag{8.6}$$

Suppose $F_1, \ldots, F_N$ algebraically independent. Then the dimension of the vector space of polynomials equals $P(F_1, \ldots, F_N)$ $(k_1+1)\ldots(k_N+1)$. For whole $s \geqslant 0$ and any point $a \in X_{\mathrm{reg}}$ given the homogeneous linear system

$$\frac{\partial^\nu}{\partial z^\nu} P(F_1, \ldots, F_N)_{|z=a} = 0, \qquad \nu \in \mathbb{N}^n, \quad |\nu| \leqslant s,$$

admits a nonzero solution as soon as this size exceeds the number equations, equal to $\binom{n+s}{n} \leqslant \frac{1}{n!}(n+s)^n$. The $P(F_1, \ldots, F_N)$ function then cancels at least to order $s+1$ developed a, and the choice of s as $s \leqslant C(a)\sum k_j\, \delta_\varphi(F_j) < s+1$ contradicts the inequality (8.6) unless

$$(k_1+1)\ldots(k_N+1) \leqslant \binom{n+s}{s} \leqslant \frac{1}{n!}\Big[n + C(a) \sum_{1\leqslant j\leqslant N} k_j\, \delta_\varphi(F_j)\Big]^n.$$

Take $k_1 = \cdots = k_N = k$ and then do tend to k $+\infty$. The above inequality shows that $(k+1)^N \leqslant \mathrm{Cte}(k+1)^n$, due $N \leqslant n$ and property (a) is demonstrated.

Suppose $\deg\operatorname{tr}_{\mathbb{C}} K_\varphi(X) = n$ and are $F_1, \ldots, F_n$ n algebraically independent functions of $A_\varphi(X)$. To demonstrate (B) simply to increase the degree of the algebraic extension $[K_\varphi(X) : \mathbb{C}(F_1, \ldots, F_n)$. If $F_{n+1} \in A_\varphi(X)$ is d degree algebraic over $\mathbb{C}(F_1, \ldots, F_n)$, the monomials $F_1^{\ell_1} \ldots F_n^{\ell_n} F_{n+1}^{\ell_{n+1}}$ are linearly Independent once $\ell_{n+1} < d$. The above reasoning applied with $k_{n-1} = d-1$ therefore gives

$$(k_1+1)\ldots(k_n+1)d \leqslant \frac{1}{n!}\Big[n + C(a) \sum_{1\leqslant j\leqslant n} k_j\, \delta_\varphi(F_j) + (d-1)\,\delta_\varphi(F_{n+1})\Big]^n.$$

Take $k_1 \sim q_1 k, \ldots, k_n \sim q_n k$ where are $q_1, \ldots, q_n$ real > 0 and $k \to +\infty$. He comes to the limit

$$q_1 \ldots q_n d \leqslant \frac{1}{n!}\Big[C(a) \sum_{1\leqslant j\leqslant n} q_j\, \delta_\varphi(F_j) + (d-1)\,\delta_\varphi(F_{n+1})\Big]^n,$$

and choice $q_j = 1/\delta_\varphi(F_j)$ gives explicit increase expected degree :

$$d \leqslant \frac{(nC(a))^n}{n!}\ \delta_\varphi(F_1)\cdots\delta_\varphi(F_n). \qquad \square$$

Note that Theorem 8.5 (b) is silent regarding algebra $A_\varphi(X)$ itself ; as we shall see in§10 it may well be that algebra $A_\varphi(X)$*not soit pas* of finite type.

As an application, consider the special case where is X an analytic subset (closed) of pure n dimension $\mathbb{C}^N$, provided with the function of conventional exhaustion$\varphi(z) = \log(1+|z|^2)$. The $\alpha = dd^c\varphi$ associated metric identifies with the metric Fubini-Study of the projective space $\mathbb{P}^N$, while the metric $\beta = dd^c e^\varphi$ coincides with the Hermitian metric flat of$\mathbb{C}^N$. 3.10 The proposal involves relationships

$$\int_{X\cap\{|z|<r\}} \beta^n = (1+r^2)^n \int_{X\cap\{\varphi<\log(1+r^2\}} \alpha^n,$$
$$\mathrm{Vol}_\alpha(X) = \int_X \alpha^n = \lim_{r\to+\infty} \int_{X\cap\{|z|<r\}} \beta^n.$$

Theorem 8.5 then gives an elementary way the classical result following due to W. Stoll [St1].

Corollary 8.7. – *Either X a subset closed analytic pure n dimension $\mathbb{C}^N$, the volume of projective* $\mathrm{Vol}_\alpha(X)$ *is finished, i.e. volume Euclidean verifies the estimate*

$$\mathrm{Vol}_\beta(X\cap\{|z|<r\}) \leqslant C\cdot r^{2n}, \qquad C \geqslant 0.$$

So X is algebraic.

Demonstration. Each irreducible component of X is volume at least equal to the volume of a -plan n (see [Le1]), so these components is finite, and presumably irreducible X.

Now we observe that the polynomials induce $P \in \mathbb{C}[z_1,\ldots,z_N]$ X on the φ -polynomiales functions under the definitions 8.1 and 8.2. Indeed, the obvious result estimate $\log_+|P| \leqslant \frac{1}{2}\deg(P)\,\varphi + \mathrm{Cte}$

$$\delta_\varphi(P) = \limsup_{r\to+\infty} \frac{1}{r}\,\mu_r(\log_+|P|) \leqslant \frac{1}{2}\,\mathrm{Vol}_\alpha(X)\cdot\deg(P).$$

Consider then the restriction morphism

$$\mathbb{C}[z_1,\ldots,z_N] \to A_\varphi(X)$$

and the ideal I, core of this morphism. Since $A_\varphi(X)$ is intact, I is an ideal first ; more irreducible algebraic variety of zeros $V(I)$ X contains by definition. Theorem 8.5 (a) shows $\mathbb{C}[z_1,\ldots,z_N]/I \subset A_\varphi(X)$ that has a degree of transcendence at most equal to $n = \dim X$; therefore $\dim V(I) \leqslant n$ and $X = V(I)$. $\square$

8.8 Note. – In the situation of the corollary was an isomorphism

$$\mathbb{C}[z_1,\ldots,z_N]/I \xrightarrow{\simeq} A_\varphi(X),$$

especially $A_\varphi(X)$ is finitely. Otherwise, there would be a B algebra finitely as $\mathbb{C}[z_1,\ldots,z_N]/I \subsetneq B \subset A_\varphi(X)$. Either $M = \operatorname{Spm} B$ algebraic variety refines associated B (see [Sun], Volume 2, chap. I for formalism based on algebraic varieties) ; previous inclusions then respectively induce an algebraic morphism and $M \to V(I)$ an analytical morphism $V(I) = X \to M$, inverses of one another. Following the morphism $V(I) \to M$ would algebraic, and it would $\mathbb{C}[z_1,\ldots,z_N]/I = B$ contrary to the hypothesis. □

We will now see how these results are transposed to the case sections polynomial of a linear bundle. Either a L fiber linear Hermitian above X, D la Hermitian connection canonical$L$ and $c(L) = D^2$ the $(1,1)$ Platform curvature L.

If σ is a nonzero holomorphic section of L for is obtained all $\varepsilon > 0$:

$$i\partial\overline{\partial}\log(\varepsilon + |\sigma|^2) = i\partial\left[\frac{\langle\sigma, D\sigma\rangle}{\varepsilon + |\sigma|^2}\right] = \frac{\varepsilon\langle D\sigma, D\sigma\rangle}{(\varepsilon + |\sigma|^2)^2} - \frac{|\sigma|^2}{\varepsilon + |\sigma|^2}\, ic(L).$$

Formula applied to the Jensen 3.4 $V = \frac{1}{2}\log(\varepsilon + |\sigma|^2)$ function then gives, given that $V \geqslant \log \varepsilon^{1/2}$:

$$\begin{aligned}
&\int_0^r dt \int_{B(t)} \frac{\varepsilon\langle D\sigma, D\sigma\rangle}{(\varepsilon + |\sigma|^2)^2} \wedge \alpha^{n-1}\\
&\qquad\leqslant \int_0^r dt \int_{B(t)} \frac{|\sigma|^2}{\varepsilon + |\sigma|^2}\, ic(L) \wedge \alpha^{n-1} + \mu_r(V) - \mu_0(V) - \int_{B(r)\smallsetminus B(0)} V\,\alpha^n\\
&\qquad\leqslant r\int_X \left[ic(L) \wedge \alpha^{n-1}\right]_+ + \mu_r(V) + \tau(r)\,\log \varepsilon^{-\frac{1}{2}},
\end{aligned}$$

where $[ic(L) \wedge \alpha^{n-1}]_+$ denotes the positive part of the measurement $c(L) \wedge \alpha^{n-1}$. Divide this inequality by r and make tender r to $+\infty$. As $V \leqslant \log_+ |\sigma| + \frac{1}{2}\log(1+\varepsilon)$, he comes

$$\int_X \frac{\varepsilon\langle D\sigma, D\sigma\rangle}{(\varepsilon + |\sigma|^2)^2} \wedge \alpha^{n-1} \leqslant \limsup_{r\to+\infty}\ \frac{1}{r}\,\mu_r(\log_+ |\sigma|) + \int_X \left[ic(L) \wedge \alpha^{n-1}\right]_+ .$$

When ε approaches 0 the term $\frac{\varepsilon\langle D\sigma, D\sigma\rangle}{(\varepsilon+|\sigma|^2)^2}$ converges weakly to the associated integration of current $2\pi[Z_\sigma]$ the divisor of zeros σ. therefore obtained the widespread Next the 8.3 inequality.

Proposition 8.9. – *any holomorphic section $\sigma \not\equiv 0$ of L was*

$$2\pi \int_X [Z_\sigma] \wedge \alpha^{n-1} \leqslant \delta_\varphi(\sigma) + \delta_\varphi(L)$$

where $\delta_\varphi(\sigma)$, $\delta_\varphi(L)$ designate "degrees" respective of σ and L :

$$\delta_\varphi(\sigma) = \limsup_{r\to+\infty}\ \frac{1}{r}\mu_r(\log_+ |\sigma|),$$

$$\delta_\varphi(L) = \int_X \left[ic(L) \wedge \alpha^{n-1}\right]_+ .$$

The theorem algebraicity now reads.

Theorem 8.10. – *is designated by $K_\varphi(X,L)$ the field of meromorphic functions on X shape σ_1/σ_2 with $\sigma_j \in H^0(X, \mathcal{O}(L^m))$, $m \in \mathbb{N}$, $\delta_\varphi(\sigma_j) < +\infty$, $j = 1, 2$. If the bundle is L $\delta_\varphi(L)$ degree finished, then :*

(a) $0 \leqslant \deg\operatorname{tr} K_\varphi(X,L) \leqslant n = \dim X$;

(b) If $\deg\operatorname{tr} K_\varphi(X,L) = n$, *body $K_\varphi(X,L)$ is finitely.*

Demonstration. Are $F_1 = \sigma_1'/\sigma_1, \ldots, F_N = \sigma_N'/\sigma_N$ of $K_\varphi(X,L)$ elements with $\sigma_j, \sigma_j' \in \mathcal{O}(L^{m_j})$ and $P \in \mathbb{C}[X_1, \ldots, X_N]$ polynomial that $\deg_{X_j} P \leqslant k_j$. Let

$$\sigma = P\Big(\frac{\sigma_1'}{\sigma_1}, \ldots, \frac{\sigma_N'}{\sigma_N}\Big)\sigma_1^{k_1} \cdots \sigma_N^{k_N} \in H^0(X, \mathcal{O}(L^m)), \qquad m = \sum k_j m_j.$$

Inequality (8.6) then generalized as follows:

$$\operatorname{ord}_a(\sigma) \leqslant C(a) \sum_{1 \leqslant j \leqslant N} k_j \Big[\max(\delta_\varphi(\sigma_j), \delta_\varphi(\sigma_j')) + m_j \delta_\varphi(L)\Big], \tag{8.11}$$

and the rest of the proof is the same as 8.5. □

B. Characterization geometric des affine algebraic varieties.

9. Énoncé criterion of algebraicity.

The purpose of the following paragraphs is to show that the varieties Affine Algebraic are characterized among spaces by Stein simple geometric conditions, namely the finite volume Monge-Ampère and a suitable lower bound of Ricci curvature.

Recall that affine algebraic variety is by definition a closed algebraic subvariety of a $\mathbb{C}^N$ space. In the case of a space X to isolated singularities, we obtain the characterization necessary and sufficient below.

Theorem 9.1. – *Either X an analytic space complex of dimension n, having at most a finite number of points Singular. So X is analytically isomorphic to a variety affine algebraic X iff X has a function Class φ exhaustion strictly psh having $\mathcal{C}^\infty$ properties*(a), (b), (c) *below.*

(a) $\operatorname{Vol}(X) = \displaystyle\int_X (dd^c\varphi)^n < +\infty$;

(b) *The Ricci curvature of the metric* $\beta = dd^c(e^\varphi)$ *admits a reduction of the form*

$$\mathrm{Ricci}(\beta) \geqslant -\frac{1}{2}dd^c\psi,$$

and with $\psi \in L^1_{\mathrm{loc}}(X, \mathbb{R}) \cap \mathcal{C}^0(X_{\mathrm{reg}}, \mathbb{R})$ $\psi \leqslant A\varphi + B$ *where* A, B *are constants* $\geqslant 0$;

(c) φ *has a finite number of critical points on* X_{reg}.

If these conditions are satisfied, the ring $R_\varphi(X) = K_\varphi(X) \cap \mathcal{O}(X)$ (*cf.* 8.2) *definition is an algebra of finite type and* $\mathbb{C}$ *of transcendence degree* n. *The algebraic structure is* X_{alg} *then defined as the unique algebraic structure on which* X *the ring of regular functions is* $R_\varphi(X)$.

The extension of this characterization the case of analytic spaces with whatever singularities presents challenges that will §14 examined.

The role of different assumptions of Theorem 9.1 is divided grosso modo as follows. The existence of a function of exhaustion φ psh strictly ensures that X is a Stein manifold, according to the solution of the problem given by Levi H. Grauert [Group].

Under the hypothesis (a), Theorem 8.5 implies that the other the body of φ -rationnelles functions of transcendence degree finished. The hypothesis (b), in turn, ensures the existence of a sufficient number of functions φ -polynomiales thanks to L^2 estimates Hörmander-Nakano-Bombieri-Skoda for $\overline{\partial}$ operator. Note here that we can replace condition (b) by a provided on the curvature of the metric $dd^c\varphi$ (cf. note 10.2). by obtained against an equivalent condition β replaced by a metric as any γ

$$\exp(-A_1\varphi - B_1) \leqslant \gamma \leqslant \exp(A_2\varphi + B_2,$$

eg metric or $\gamma = dd^c \log(1 + e^\varphi)$ $\gamma = dd^c(\varphi^2)$.

Finally hypothesis (c) results from Morse theory that has X same type of homo-topia a finite cell complex, and therefore the cohomology X is finitely. We do not know in fact if the assumption (c) is really essential, therefore supposed X irreducible. Without hypothesis (c), we can already show that X is meeting an increasing sequence of algebraic varieties X Almost affine (= Zariski open affine varieties) cf. proposition 13.1. This result follows the following improvements Theorem 9.1.

Theorem 9.1′. – *The theorem remains true if* 9.1 *assumptions*(a, b, c) are weakened *as follows* :

(a') = (a) : $\mathrm{Vol}(X) = \int_X (dd^c\varphi)^n < +\infty$;

(b ') $\mathrm{Ricci}(\beta) \geqslant -\frac{1}{2}dd^c\psi$, *where* $\psi \in L^1_{\mathrm{loc}}(X, \mathbb{R}) \cap Cc^0(X_{\mathrm{reg}}, \mathbb{R})$ *admits estimate the form*

$$\int_X \exp(c\psi - A\varphi)\,\beta^n < +\infty, \qquad c > 0, \quad A > 0 \ ;$$

(that) the *even degree cohomology spaces* $H^{2q}(X_{\mathrm{reg}}; \mathbb{R})$ *are finite dimensional.*

Assumptions (a '), (b') further imply that $X = \bigcup_{k\in\mathbb{N}} X_k$ X_k with quasi-affine $X_k \subset X_{k+1}$, and assuming (that) implies that Following X_k is necessarily stationary ; therefore, is X algebraic. Observe that the hypothesis (that) is always checked if $n = 1$; when $n = 2$ or $n = 3$, it is equivalent to assuming only $\dim H^2(X;\mathbb{R}) < +\infty$ because $H^q(X;\mathbb{R})$ groups are always zero for $q > n$ when X of Stein.

The likelihood of Theorems 9.1, 9.1 'was suggested to us in part by the work of W. and D. Stoll Burns on varieties parabolic. Let us recall the fundamental result of W. Stoll (1980), which characterizes the strictly parabolic varieties radius any.

Theorem 9.2(cf. [St2] and [Bu]). **–** *Either M a connected complex analytic variety of dimension n. Suppose that there exists a real and $R \in]0,+\infty]$ $\tau : M \to]0,R^2[$ function strictly exhaustive psh $\mathcal{C}^\infty$ class, as is $\log\tau$ psh and checks $(dd^c\tau)^n \equiv 0$ on $M \smallsetminus \tau^{-1}(0)$. Then there exists a biholomorphic $F : B(R) \to M$ implementation of the ball radius of R in M as $F^*\tau(z) = |z|^2$.*

If you lift the strict assumption of plurisousharmonicité τ it easily seen that any affine algebraic variety M still checks the condition of Theorem 9.2 (with $R = +\infty$) : just $\tau(z) = \log|\pi(z)|^2$ choose where $\pi : M \to \mathbb{C}^n$ is a morphism proper finish. This remark led D. Burns to the problem of characterization of such varieties in function of exhaustion with special properties. Note in particular the following open problems.

Problem 9.3. – *Consider a variety of Stein M size n, having a function of exhaustion psh Class$\tau : M \to [0,+\infty[$ $\mathcal{C}^\infty$ as $\log\tau$ psh be $(dd^c\log\tau)^n \equiv 0$ and checks on $M \smallsetminus \tau^{-1}(0)$. Is the M affine algebraic variety* ?

Problem 9.4. – *characterize varieties M Assuming a function of exhaustion strictly $\tau : M \to [0,+\infty[$ psh psh either as $\log\tau$ and checks $(dd^c\log\tau) \equiv 0$ outside a compact.*

D. Burns has shown that there are not affine algebraic varieties not satisfying the condition 9.4 Such is the case of $M = (\mathbb{C}^*)^n$, $n \geqslant 2$. However, the condition 9.4 is checked by a variety refines generic, i.e. a variety diving in $\mathbb{C}^N$ whose projective completion is smooth and transverse to the hyperplane at infinity.

Shortly after proving Theorem 9.1, we learned of another N. Mok hand that had previously obtained a geometric condition sufficient (not necessary in general) for a variety is affine algebraic.

Theorem 9.5([Mok 1,2,3]). **–** *Either a X complete Kähler manifold of dimension n, curvature bisectionnelle positive, as*

(a) *volume*$(B(x_0,r)) \geqslant c\,r^{2n}$,

(b) $0 <$ *courbure scalaire* $\leqslant C/d(x_0,x)^2$,

where $B(x_0,r)$ and $d(x_0,x)$ designate balls respectively the geodesic distance and $c, C > 0$. So X is biholomorphically isomorphic to an affine algebraic variety.

N. Mok deduced from this theorem that any X surface curvature Riemannian positive checking assumptions 9.5 (a), (b) is is isomorphic to $\mathbb{C}^2$. The analogous result in

dimension $n > 2$ remains a conjecture. The theorem is based primarily on 9.5 work [MSY] Mok Siu and Yau on resolution the Poincare-Lelong equation on Kähler varieties bisectionnelle > 0 curvature. This result aside, the Demonstration N. Mok follows in outline an approach substantially parallel to ours.

The assumption that the curvature is positive bisectionnelle appears however rather restrictive and does not cover the general For affine algebraic varieties (Euclidean curvature of a Such variety is always negative, cf. §10). however quote some known results in the case of not necessarily curvatures positive. Siu and Yau [SY] demonstrated that a variety Kählerian complete simply connected X whose sectional curvature checks

$$-\frac{C}{d(x_0,x)^{2+\varepsilon}} < \text{courbure sectionnelle} < 0$$

is biholomorphic to $\mathbb{C}^n$. It is the same if the curvature checks

$$\text{courbure sectionnelle} \leqslant \frac{C_\varepsilon}{d(x_0,x)^{2+\varepsilon}}$$

with a fairly small C_ε constant (cf. [MI]).

10. Nécessité conditions on the volume and curvature.

We will demonstrate here that conditions (a), (b), (c) Theorem 9.1 are checked for any algebraic subvariety irreducible $X \subset \mathbb{C}^N$ size n.

Is chosen in this case $\varphi(z) = \log(1+|z|^2)$, so that the $\alpha = dd^c\varphi$ metric coincides with the metric Fubini-Study of the projective space $\mathbb{P}^N$. As $\overline{X}$ adhesion X $\mathbb{P}^N$ in a compact algebraic submanifold is obtained

$$\int_X (dd^c\varphi)^n = \int_{\overline{X}} \alpha^n < +\infty,$$

therefore the condition (a) is satisfied.

By Theorem of Bertini-Sard, the set of critical values of φ on X_{reg} is finished. Following the critical set of φ is compact. Even slightly disturbing in φ $\mathcal{C}^\infty(X,\mathbb{R})$ the vicinity of this compact ([Mil], cor. 6.8) constructing a function whose φ' critical points are non-degenerate. The critical points of φ' are then finite number [hypothesis (c)].

It now remains to show that satisfies the condition X curvature 9.1 (b) with respect to the metric

$$\beta = dd^c(e^\varphi) = dd^c|z|^2 = 2i\sum_{j=1}^N dz_j \wedge d\overline{z}_j\,,$$

what we will verify by an explicit calculation of $\mathrm{Ricci}(\beta_{|X})$ and ψ.

$(P_1,\ldots,P_m)$ either a generator polynomials system for ideal $I(X)$ sub-variety X in $\mathbb{C}[z_1,\ldots,z_N]$ and either $s = \operatorname{codim} X = N-n$. For every pair of multi-indices

$$K = \{k_1 < \ldots < k_s\} \subset \{1,\ldots,m\}, \qquad L = \{\ell_1 < \ldots < \ell_s\} \subset \{1,\ldots,N\}$$

s length, it is considered part jacobien

$$J_{K,L}(z) = \det\left(\partial P_{k_i}/\partial z_{\ell_j}\right)_{1\leqslant i,j\leqslant s}\ ,$$

and we set

$$\psi(z) = \log\left(\sum_{|K|=|L|=s} |J_{K,L}|^2\right).$$

The $J_{K,L}$ polynomial functions, particularly there are $A, B \geqslant 0$ constants such as $\psi \leqslant A\varphi + B$. Proposal Next shows φ, ψ meet more unequal 9.1 curvature (b).

Proposition 10.1. – *is noted* $U_K = U_{k_1,\ldots,k_s}$ *open to* X *on which the differential* $dP_{k_1}, \ldots, dP_{k_s}$ *are linearly independent. So :*

(a) $\mathrm{Ricci}(\beta_{|X}) = -\dfrac{1}{2}\, dd^c \log\left(\displaystyle\sum_{|L|=s} |J_{K,L}|^2\right)$ *on* U_K ,

(a) $\mathrm{Ricci}(\beta_{|X}) \geqslant -\dfrac{1}{2}\, dd^c \log\left(\displaystyle\sum_{|K|=|L|=s} |J_{K,L}|^2\right)$ *on* X_{reg} .

Demonstration of (a). Either $x \subset U_K$. Suppose the coordinates $\mathbb{C}^N$ of rows so that $(z_1, \ldots, z_n)$ or a system of local coordinate X developed x. Ricci curvature of X is in Kähler the case opposite to the curvature of the canonical bundle $\Lambda^n T^*X$. It was therefore in the vicinity of the relationship x

$$\mathrm{Ricci}(\beta_{|X}) = dd^c \log g^2,$$

g which is the norm with respect to the β $(n, 0)$ Platform holomorph $dz_1 \wedge \ldots \wedge dz_n$; g is given by

$$i\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i\, dz_n \wedge d\overline{z}_{n\,|X} = g^2\ \frac{1}{n!}\ \beta^n_{|X}.$$

Let $L_0 = \{n+1, \ldots, N\}$, L any multi-index of length $s = N - n$ and $\complement L$ complementary son in $\{1, \ldots, N\}$. If we set

$$dP_K = dP_{k_1} \wedge \ldots \wedge dP_{k_s},$$

it comes by definition $J_{K,L}$:

$$\begin{aligned}
dP_K \wedge dz_{\complement L} &= \pm J_{K,L}\, dz_1 \wedge \ldots \wedge dz_N,\\
i^{n^2+s^2} dP_K \wedge d\overline{P}_K \wedge dz_{\complement L} \wedge d\overline{z}_{\complement L} &= |J_{K,L}|^2 i\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i\, dz_N \wedge d\overline{z}_N\\
i^{s^2} dP_K \wedge d\overline{P}_K \wedge \frac{1}{n!}\,\beta^n &= 2^n \sum_{|L|=s} |J_{K,L}|^2\, i\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i\, dz_N \wedge d\overline{z}_N\\
&= 2^n |J_{K,L_0}|^{-2} \sum_{|L|=s} |J_{K,L}|^2\, i^{s^2}\, dP_K \wedge d\overline{P}_K \wedge i\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i\, dz_n \wedge d\overline{z}_n.
\end{aligned}$$

In "Simplifying" by $dP_K \wedge d\overline{P}_K$, therefore we find

$$g^2 = 2^{-n}\big|J_{K,L_0}\big|^2\Bigg(\sum_{|L|=s}|J_{K,L}|^2\Bigg)^{-1}$$

and as J_{K,L_0} is an invertible holomorphic function in x, the formula (a) ensues.

Demonstration of (b). The result (a) shows that the function

$$\log\Bigg(\sum_{|L|=s}|J_{K,L}|^2\Bigg/\sum_{|L|=s}|J_{K_0,L}|^2\Bigg)$$

pluriharmonique is on the open $U_K \cap U_{K_0}$. It is more locally increased, so psh on U_{K_0}. As a result, the function

$$\log\Bigg(\sum_{K,L}|J_{K,L}|^2\Bigg/\sum_{L}|J_{K_0,L}|^2\Bigg)$$

psh is on U_{K_0}, and this opened so we :

$$dd^c\psi \geqslant dd^c\log\sum_L|J_{K_0,L}|^2 = -2\ \mathrm{Ricci}(\beta_{|X}). \qquad \square$$

Note 10.2. – When X is a submanifold closed analytics $\mathbb{C}^N$ and $\varphi(z) = \log(1+|z|^2)$, the provided 9.1 (a) of finite volume is in itself a sufficient condition for algebraicity of X (theorem W. Stoll, cf. corollaire 8.6). We'll see however by example we can generally dispense with the condition 9.1 curvature (b), even though 9.1 ' (c') is satisfied.

$X = \mathbb{C} \smallsetminus E$ choose where $E = \{z_j\ ;\ j \in \mathbb{N}\}$ is a set closed countable infinity, and put

$$\varphi(z) = \log(1+|z|^2) - \sum_{j=0}^{+\infty}\varepsilon_j\log\frac{|z-z_j|}{1+|z_j|}$$

$\sum_{j=0}^{+\infty}\varepsilon_j = 1$ which is a series > 0 terms and converged fast enough to be in φ $\mathcal{C}^\infty(\mathbb{C}\smallsetminus E)$. As

$$\log\frac{|z-z_j|}{1+|z_j|} \leqslant \log(1+|z|),$$

he comes

$$\varphi(z) \geqslant \log\frac{1+|z|^2}{1+|z|}\ ;$$

more $\lim_{z\to z_j}\varphi(z) = +\infty$ for all $z_j \in E$. φ the function is comprehensive on X. Otherwise, $dd^c\varphi = dd^c\log(1+|z|^2)$, so $\int_X dd^c\varphi = 4\pi < +\infty$. However X is not algebraic. $\square$

This example shows incidentally that we can not replace condition 9.1 (b) by a condition on the curvature metric $dd^c\varphi$.

It is easy to see the other as algebra and $A_\varphi(X)$ $R_\varphi(X) = K_\varphi(X) \cap \mathcal{O}(X)$ coincide with algebra $\mathcal{A}_E$ of -rationnelles φ $\mathbb{C}[z]$ fractions whose poles Appar- like to E. For every element clearly admits $f \in \mathcal{A}_E$ an increase $\log|f| \leqslant C_1\varphi + C_2$ so $\mathcal{A}_E \subset A_\varphi(X) \subset R_\varphi(X)$; conversely, any element of $K_\varphi(X)$ is algebraic over $\mathbb{C}(z)$ by 8.5 theorem so

$$R_\varphi(X) \subset \mathbb{C}(z) \cap \mathcal{O}(X) = \mathcal{A}_E.$$

Clearly the $\mathcal{A}_E$ algebra is not finitely.

11. Existence a dip in an open a variety algebraic.

Paragraphs 11-14 that follow are devoted to demonstration adequacy criterion algebraicity 9.1 '. We assume here that X is a smooth connected manifold, and data functions (φ, ψ) satisfy conditions 9.1 '(a', b '). We do the other non-restrictive supplementary hypothesis $\varphi \geqslant 0$. We ask as previously $\alpha = dd^c\varphi \geqslant 0$, $\beta = dd^c(e^\varphi)$; μ_r the measures are defined as the§3.

Definition 11.1. – *Either $p \in]0, +\infty]$. We notice*

(a) L^p_φ *(X) the vector space of measurable applications $f : X \to \mathbb{C}$ such that there exists a constant such $C \geqslant 0$ that* :

$$\int_X |f|^p e^{-C\varphi}\,\beta^n < +\infty, \qquad si\ 0 < p < +\infty\ ,$$

$$|f| \leqslant e^{C(1+\varphi)}\quad presque\ partout, \qquad si\ p = +\infty\ ;$$

(b) $L^0_\varphi(X) = \bigcup_{p>0} L^p_\varphi(X)$;

(c) $A^p_\varphi(X) = L^p_\varphi(X) \cap \mathcal{O}(X), \qquad si \in p[0, +\infty]$.

The advantage of this definition appears in both technical results below, which will be used repeatedly in the sequel.

Lemma 11.2. – *It has the following properties* :

(a) $1 \in A^p_\varphi(X)$ *for all* $p > 0$;

(b) and $L^p_\varphi(X) \subset L^q_\varphi(X)$ $A^p_\varphi(X) \subset A^q_\varphi(X)$ *for all* $p \geqslant q > 0$;

(c) $L^0_\varphi(X)$ *is* $\mathbb{C}$ *algebra* ;

(d) $A^0_\varphi(X)$ *is fully closed subalgebra* $L^0_\varphi(X)$.

Lemma 11.3. – *was the inclusion $A^0_\varphi(X) \subset A_\varphi(X)$.Étant given $f \in A^0_\varphi(X)$ such as*

$$\int_X |f|^p \exp(-C\varphi)\, \beta^n < +\infty,$$

then :

(a) $\delta_\varphi(f) \leqslant \dfrac{C-n}{p} \operatorname{Vol}(X)$;

(b) if $\displaystyle\int_X |df|^p_\beta \exp\Big[\Big(\frac{p}{2} - C\Big)\varphi\Big]\, \beta^n < +\infty \qquad p \in]0,2]$,

where $|df|_\beta$ standard is calculated with respect to the metric β.

Demonstration 11.2. The proposal entails successively 3.10

$$v(r) := \int_{\{\varphi<r\}} \beta^n = e^{nr} \int_{\{\varphi<r\}} \alpha^n = e^{nr} \operatorname{Vol}(X),$$
$$\int_X e^{-(n+1)\varphi}\, \beta^n = \int_0^{+\infty} e^{-(n+1)r}\, dv(r) = (n+1) \int_0^{+\infty} e^{-(n+1)r}\, v(r)\, dr < +\infty,$$

demonstrating (a). The property (b) then follows from inequality Hölder. The same implies inequality

$$\int_X |fg|^{\frac{pq}{p+q}} e^{-C\varphi}\, \beta^n \leqslant \Big[\int_X |f|^p\, e^{-C\varphi}\, \beta^n\Big]^{\frac{q}{p+q}} \Big[\int_X |g|^q\, e^{-C\varphi}\, \beta^n\Big]^{\frac{p}{p+q}}$$

for all $p, q > 0$. thus obtained 1'inclusion

$$L^p_\varphi(X) \cdot L^q_\varphi(X) \subset L^{\frac{pq}{p+q}}_\varphi(X), \tag{11.4}$$

and property (c) follows. Now check the assertion (d). Is f a meromorphic function on X checking an entire equation $A^0_\varphi(X)$ shape

$$f^d + a_1 f^{d-1} + \cdots + a_{d-1} f + a_d = 0, \qquad a_j \in A^0_\varphi(X).$$

From this equation, we deduce the increase

$$|f| \leqslant 2 \max_{1\leqslant j\leqslant d} |a_j|^{1/j},$$

if not equal

$$-1 = a_1 f^{-1} + \cdots + a_d f^{-d}$$

absolute lead to the absurd inequality

$$1 \leqslant 2^{-1} + \cdots + 2^{-d}.$$

Therefore, as X is smooth, f extends to a function holomorphic on X, and it is clear that $f \in A^0_\varphi(X)$. $\square$

Demonstration of 11.3.

(A) was $\beta^n \geqslant e^{n\varphi}(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi$ therefore (Proposition 3.8) the integral

$$\int_{-\infty}^{+\infty} e^{(n-C)r}\,\mu_r(|f|^p)\,dr = \int_X |f|^p e^{(n-C)\varphi}\,(dd^c\varphi)^{n-1} \wedge d\varphi \wedge d^c\varphi \leqslant \int_X |f|^p e^{-C\varphi}\,\beta^n$$

is over. As $r \mapsto \mu_r(|f|^p)$ application is growing, we can deduce

$$\mu_r(|f|^p) \leqslant \exp\big((C-n)(r+1)\big) \int_r^{r+1} e^{(n-C)t}\,\mu_t(|f|^p)\,dt \leqslant C_1 e^{(C-n)r}$$

with constant $C_1 \geqslant 0$. According inequality Convexity Jensen and $\|\mu_r\| \leqslant \mathrm{Vol}(X)$ inequality, it comes

$$\frac{\mu_r\big(\log(1+|f|^p)\big)}{\|\mu\|_r} \leqslant \log\left[\frac{\mu_r\big(1+|f|^p\big)}{\|\mu\|_r}\right] \leqslant (C-n)r + C_2\,,$$

$$\delta_\varphi(f) = \limsup_{r\to+\infty}\ \frac{1}{r}\ \mu_r(\log_+|f|) \leqslant \frac{C-n}{p}\ \mathrm{Vol}(X).$$

(B) To increase $|dF|_\beta$, it is observed

$$dd^c(|F|^p) \wedge \beta^{n-1} = \frac{p^2}{2n}\,|F|^{p-2}|dF|^2_\beta\,\beta^n.$$

With the Stokes theorem, that equality leads to all $r > 0$:

$$\int_{B(r)} |F|^{p-2}|dF|^2_\beta\Big(1-\frac{\varphi}{r}\Big)^2 e^{(1-C)\varphi}\,\beta^n \leqslant \frac{2n}{p^2}\int_{B(r)} |F|^p\,dd^c\Big[\Big(1-\frac{\varphi}{r}\Big)^2 e^{(1-C)\varphi}\Big] \wedge \beta^{n-1}.$$

Easy calculation gives the other on the $B(r) = \{\varphi < r\}$ uniform increase in r :

$$dd^c\Big[\Big(1-\frac{\varphi}{r}\Big)^2 e^{(1-C)\varphi}\Big] \leqslant C_3\Big(1+\frac{1}{r}\Big)^2 e^{-C\varphi}\beta, \qquad r > 0.$$

After passing the limit when $r \to +\infty$, we obtain

$$\int_X |F|^{p-2}|dF|^2_\beta\,e^{(1-C)\varphi}\,\beta^n \leqslant C_3\int_X |F|^p e^{-C\varphi}\beta^n.$$

Property 11.3 (b) now results from Hölder inequality applied to the measurement $|F|^p e^{-C\varphi}\beta^n$, the pair of functions $(|F|^{-p}|dF|^p_\beta \exp(\frac{p}{2}\varphi)\ ;\ 1)$ and exhibitors Conjufords $(\frac{2}{p}, \frac{2}{2-p})$. $\square$

The existence of non-constant holomorphic functions in $L^p_\varphi(X)$ will result conventional estimates L. Hörmander [HO1] for the operator $\overline{\partial}$.

Proposition 11.5. – *The following properties are met.*

(a) Either $\tau \in L^1_{\mathrm{loc}}(X)$ *a function such that*

$$i\,\partial\overline{\partial}\tau + \mathrm{Ricci}(\beta) \geqslant \lambda\beta$$

λ *which is a continuous function on* > 0 X. *either* u $(0,1)$ *a Platform to* L^2_{loc} *coefficients as* X *and*$\overline{\partial}u = 0$

$$\int_X \lambda^{-1}|u|^2 e^{-\tau}\,\beta^n < +\infty.$$

Then there exists a function such as $g \in L^2_{\mathrm{loc}}(X)$ *and*$\overline{\partial}g = u$

$$\int_X |g|^2 e^{-\tau}\,\beta^n \leqslant \int_X \lambda^{-1}|u|^2 e^{-\tau}\,\beta^n < +\infty.$$

(b) Let ψ, c, A *data*9.1 '(b'). *If* ρ *psh* X *is on and if* u *checks and* $\overline{\partial}u = 0$

$$\int_X |u|^2\, e^{-\rho-\psi}\,\beta^n < +\infty,$$

then there $g \in L^2_{\mathrm{loc}}(X)$ *as* $\overline{\partial}g = u$ *and*

$$\int_X |g|^2 e^{-\rho-\psi-2\varphi}\,\beta^n \leqslant 4\int_X |u|^2 e^{-\rho-\psi}\,\beta^n.$$

(c) *Given a finite set and* $\{x_1, x_2, \ldots, x_m\} \subset X$ ρ *a psh function on* X *as* $e^{-\rho}$ *be integrable in* $x_1, x_2, \ldots, x_m$ *neighborhood. Then there is a function holomorphic* f *having an order* s *jet at each point and* x_j *such as*

$$\int_X |f|^2 e^{-\rho-\psi-C_1\varphi}\,\beta^n < +\infty, \qquad \text{où } C_1 \geqslant 0.$$

In particular, if $\rho \equiv 0$ *it* $f \in A^b_\varphi(X)$ *obtained with* $b = \frac{2c}{1+c}$ *and* c *as in* 9.1 '(b').

(d) $A^b_\varphi(X)$ *is dense in the topology* $\mathcal{O}(X)$ *of uniform convergence on compact.*

Demonstration.

(A) is classical, see for example H. Skoda [Sk3].

(B) We apply (a) with $\tau = \rho + \psi + 2\log(1+e^\varphi)$. As $\varphi \geqslant 0$ was $\tau \leqslant \rho + \psi + 2\varphi + \log 4$ and Assuming 9.1 '(b') results in

$$i\,\partial\overline{\partial}\tau + \mathrm{Ricci}(\beta) \geqslant \frac{e^\varphi\, dd^c\varphi}{1+e^\varphi} + \frac{e^\varphi\, d\varphi\wedge d^c\varphi}{(1+e^\varphi)^2} \geqslant \lambda\beta$$

with $\lambda = (1+e^\varphi)^{-2}$. The estimate (b) result.

(C) is a consequence of (b) through a classical reasoning due to Bombieri and Skoda [Sk1]. $U_1, \ldots, U_m$ are open neighborhoods disjoint from $x_1, \ldots, x_m$ on which is $e^{-\rho}$ locally integrable. U_j Assume provided with a coordinate system Local $z^{(j)} = (z_1^j, z_2^j, \ldots, z_n^j)$ centered x_j and pose

$$\rho_1 = \rho + (n+s)\left[\sum_{j=1}^{m} \chi_j \log |z^{(j)}|^2 + C_j \varphi\right]$$

χ_j which is a class function $\geqslant 0$ $\mathcal{C}^\infty$ support compact in U_j equal to 1 near x_j and a $C_j \geqslant 0$ constant large enough for either ρ_1 psh on X. The constant $n+s$ is selected here so that the jet order s g a function $\mathcal{C}^\infty$ class, locally integrable for $e^{-\rho_1}\,\beta^n$ far, is necessarily to zero x_j point. Be now $P_j(z^{(j)})$ a polynomial of degree $\leqslant s$ with the jet imposed x_j. we set

$$h = \sum_{j=1}^{m} \chi_j\, P_j(z^{(j)}).$$

The $(0,1)$ Platform

$$u := \overline{\partial} h = \sum_{j=1}^{m} \overline{\partial}\chi_j\ P_j(z^{(j)}).$$

is $\mathcal{C}^\infty$ class, no near $x_1, \ldots, x_m$ and by building

$$\int_X |u|^2 e^{-\rho_1 - \psi}\,\beta^n < +\infty\ ;$$

was used here that ψ either locally bounded. According to (b), are $g \in \mathcal{C}^\infty(X)$ as $\overline{\partial} g = u$ and

$$\int_X |g|^2 e^{-\rho_1 - \psi - 2\varphi}\,\beta^n < +\infty\ .$$

The $f = h - g$ function then answers the question. If $\rho \equiv 0$, we can write

$$|f| = \left[|f| \exp\left(-\frac{1}{2}\psi\right)\right]\ \exp\left(\frac{1}{2}\psi\right)$$

and where $|f| \exp(-\frac{1}{2}\psi) \in L^2_\varphi(X)$ $\exp(\frac{1}{2}\psi) \in L^{2c}_\varphi(X)$. It follows through (11.4) that $f \in L^b_\varphi(X)$.

(D) is proved from (b) exactly as Lemma 4.3.1 of [HO2]. □

Now we use the proposal to build many 11.5 holomorphic functions on X, and thus obtain a partial dipping X in $\mathbb{C}^N$. $x_0 \in X$ is a fixed point. According 11.5 (c) $\rho \equiv 0$ applied, there are functions $f_1, \ldots, f_n \in A^b_\varphi(X)$ such as

$$df_1 \wedge \ldots \wedge df_n(x_0) \neq 0.$$

In particular, $f_1, \ldots, f_n$ are algebraically independent in $A_\varphi(X)$. Theorem 8.5 (b) therefore applies, giving :

Proposition 11.6. – *The body of $K_\varphi(X)$ φ functions -rationnelles is a finitely generated extension of $\mathbb{C}$, of transcendence degree n.* □

As we shall see, it is easy to deduce results the existence of a previous $F : X \to M$ morphism in X algebraic variety M dimension n, that is, outside a algebraic hypersurface $S \subset X$ an isomorphism on $X \smallsetminus S$ an open M. The main difficulty will be overcome to prove that F is almost surjective, i.e. that is $F(X \smallsetminus S)$ an open Zariski M.

Proposition 11.7. – *The existence of properties following are met.*

(a) There is a $f_{n+1} \in A^0_\varphi(X)$ *function such that*

$$f_{n+1}(x_0) = 1 \quad et \quad \int_X |f_{n+1}|^2 \, |df_1 \wedge \ldots \wedge df_n|^{-2}_\beta \, \exp(-2\psi - C\varphi) \, \beta^n < +\infty.$$

In particular $\{x \in X \,;\, df_1 \wedge \ldots \wedge df_n(x) = 0\} \subset f^{-1}_{n+1}(0)$.

(b) *There are functions $f_{n+2}, \ldots, f_N \in A^0_\varphi(X)$ and an irreducible algebraic subvariety $M \subset \mathbb{C}^N$ size n such as $F = (f_1, \ldots, f_N)$ morphism sends X in M and either an analytic isomorphism $X \smallsetminus f^{-1}_{n+1}(0)$ on an open plain ofM.*

Demonstration.

(A) Just apply 11.5 (c) to psh function

$$\rho = \psi + \log |df_1 \wedge \ldots \wedge df_n|^2.$$

If Z is the divisor of the zeros of holomorphic n Platform $df_1 \wedge \ldots \wedge df_n$ considered section $\bigwedge^n T^*X$ was well indeed on the assumption 9.1 '(b') :

$$dd^c\rho = dd^c\psi + 2\,\mathrm{Ricci}(\beta) + 4\pi\,[Z] \geqslant 0,$$

ρ and is continuous in the vicinity of x_0. There is therefore $f_{n+1} \in \mathcal{O}(X)$ as $f_{n+1}(x_0) = 1$, checking L^2 annoncée estimate. This estimate implies that f_{n+1} vanishes on the support Z, and by Lemma 11.3 (b) was $|df_j| \in L^b_\varphi(X)$, where

$$|f_0| \leqslant \Big(|f_0| \; |df_1 \wedge \ldots \wedge df_n|^{-1} \, e^{-\psi} \Big) \; |df_1| \cdots |df_n| \; e^{\psi} \in L^0_\varphi(X).$$

(B) is first constructed by induction on the j $f_1, \ldots, f_{N_j} \in A^0_\varphi(X)$ functions $N_0 = n+1 < N_1 < N_2 \ldots$ (this is for $j = 0$).

According to 11.6 the image of the morphism

$$F_j = (f_1, \ldots, f_{N_j}) : X \to \mathbb{C}^{N_j}$$

is contained in an irreducible algebraic variety $M_j \subset \mathbb{C}^{N_j}$ dimension n. Either the $\tilde{M}_j$ standardization M_j. There is a commutative diagram

$$\begin{array}{ccccc} \tilde{M}_j & \hookrightarrow & \mathbb{C}^{\tilde{N}_j} & \ni & (z_1, \ldots, z_{\tilde{N}_j}) \\ \downarrow & & \downarrow & & \downarrow \\ M_j & \hookrightarrow & \mathbb{C}^{N_j} & \ni & (z_1, \ldots, z_{N_j}) \,. \end{array}$$

As X variety is smooth (and therefore normal), the morphism $F_j : X \to M_j$ lifts to a morphism
$$\tilde{F}_j = (f_1, \ldots, f_{\tilde{N}_j}) : X \to \tilde{M}_j.$$

By construction, the coordinated functions $z_{N_j+1}, \ldots, z_{\tilde{N}_j}$ (Resp. $f_{N_j+1}, \ldots, f_{\tilde{N}_j}$) are algebraic integers $\mathbb{C}[z_1, \ldots, z_{N_j}]/I(M_j)$ ring (resp. on $\mathbb{C}[f_1, \ldots, f_{N_j}]$), so $f_{N_j+1}, \ldots, f_{\tilde{N}_j} \in A^0_\varphi(X)$ according 11.2 (d). In addition, the restriction

$$\tilde{F}_j : X \smallsetminus f_{n+1}^{-1}(0) \to M_j$$

is slack because $df_1 \wedge \ldots \wedge df_n \neq 0$ on $X \smallsetminus f_{n+1}^{-1}(0)$ according (a). As $\tilde{M}_j$ is locally irreducible image $\tilde{F}_j(X \smallsetminus f_{n+1}^{-1}(0))$ is necessarily contained in all smooth points of $\tilde{M}_j$. If $\tilde{F}_j$ is injective on $X \smallsetminus f_{n+1}^{-1}(0)$, the construction is completed with $F = \tilde{F}_j$, $M = \tilde{M}_j$, $N = \tilde{N}_j$.

If not, are two points in $z_1 \neq z_2$ $X \smallsetminus f_{n+1}^{-1}(0)$ such as $\tilde{F}_j(z_1) = \tilde{F}_j(z_2)$. The proposal 11.5 (c) shows that there $g \in A^b_\varphi(X)$ a function such as $g(z_1) \neq g(z_2)$. We set $N_{j+1} = \tilde{N}_j + 1$, $f_{\tilde{N}_j+1} = g$. According to 11.6, g algebraic est on $\mathbb{C}[f_1, \ldots, f_{\tilde{N}_j}]$, g therefore vérifie an irreducible equation of the form

$$\sum_{k=0}^{d} a_k(\tilde{F}_j)\, g^k = 0, \qquad a_k \in \mathbb{C}[f_1, \ldots, f_{\tilde{N}_j}], \quad a_d(\tilde{F}_j) \not\equiv 0. \tag{11.8}$$

As $\tilde{F}_j$ is spread near and z_1 z_2, there are Points of z_1 w_1 neighbor, neighbor w_2 of z_2 such as $\tilde{F}_j(w_1) = \tilde{F}_j(w_2) \notin a_d^{-1}(0)$ and $g(w_1) \neq g(w_2)$. This results in that equation (11.8) is of degree $d \geqslant 2$. As $K_\varphi(X)$ is a finite degree of extension $\mathbb{C}(f_1, \ldots, f_n)$ the method stops necessarily after a finite number of steps. □

12. Quasi-surjectivity of embedding.

We take the ratings of the proposal 11.7. The objective of this section is to show that the image of the morphism $F : X \smallsetminus f_{n+1}^{-1}(0) \to M$ is an open Zariski M. $Q \in \mathbb{C}[z_1, \ldots, z_N]$ is a nonzero polynomial over M, divisible by z_{n+1} as the hypersurface contains $Q^{-1}(0)$ place singular M_{sing}. we set

$$\check{M} = M \smallsetminus Q^{-1}(0), \qquad \check{X} = X \smallsetminus Q(F)^{-1}(0) \subset X \smallsetminus f_{n+1}^{-1}(0),$$

so $\check{M}$ is smooth and the restriction morphism

$$\check{F} : \check{X} \to \check{M}$$

is an isomorphism of the open $\check{X}$ $\Omega = \check{F}(\check{X})$. The $\check{M}$ variety can be (and will be) identified with a submanifold affine algebraic $\mathbb{C}^{N+1}$ app $\check{M} \to \mathbb{C}^{N+1}$ defined by

$$(z_1, \ldots, z_N) \mapsto \big(z_1, \ldots, z_N, z_{N+1} = Q(z_1, \ldots, z_N)^{-1}\big)\ ;$$

morphism then $\check{F} : \check{X} \to \check{M} \subset \mathbb{C}^{N+1}$ given by $\check{F} = (F, Q(F)^{-1})$. One of the crucial points of the reasoning is to show that the positive current is closed $\check{F}_* dd^c\varphi$ extends from the open to the variety $\Omega = \check{F}(\check{X})$ M in full. We need for that precise estimates mass, which are provided by the following lemma.

Lemma 12.1. – *Either* $G = (g_1, \ldots, g_m) \in \big[A^0_\varphi(X)\big]^m$ *and* $\gamma = dd^c \log(1 + |G|^2)$. *So for whole* $k \geqslant 0$ *was :*

(a) $\displaystyle\int_X (dd^c\varphi)^{n-k} \wedge \gamma^k < +\infty, \qquad 0 \leqslant k \leqslant n,$

(b) $\displaystyle\int_{B(r)} d\varphi \wedge d^c\varphi \wedge (dd^c\varphi)^{n-k-1} \wedge \gamma^k \leqslant Cr, \qquad 0 \leqslant k \leqslant n-1,$

where is a constant $C \geqslant 0$.

Demonstration. Apply the theorem 2.2 (c) with $\rho = \varphi - r$, $\Omega = \{\rho < 0\} = B(r)$ and $V_1 = \ldots = V_k = \log(1 + |G|^2) \geqslant 0$. He comes

$$\int_{B(r)} \beta_k \wedge \gamma^k \leqslant C_3 \int_{B(r)} \big(\log(1 + |G|^2)\big)^k \beta_0$$

where, for $a > 0$ and $k \geqslant 0$, it was asked :

$$\beta_k = (r - \varphi)^{k+a} (dd^c\varphi)^{n-k} + (k + a)(r - \varphi)^{k-1+a} d\varphi \wedge d^c\varphi \wedge (dd^c\varphi)^{n-k-1}.$$

Was :

$$\beta_0 = 2(r - \varphi)^a (dd^c\varphi)^n + \frac{1}{2(1 + a)} dd^c\beta_1.$$

The Stokes theorem therefore impose any $r > 0$:

$$\int_{B(r)} \beta_0 = 2 \int_{B(r)} (r - \varphi)^a (dd^c\varphi)^n \leqslant 2\, r^a \ \mathrm{Vol}(X).$$

On the other hand, the function is concave $t \mapsto (\log(e^k + t))^k$ on $[0, +\infty[$. therefore obtained for all $p > 0$ inequality convexity :

$$\frac{\int_{B(r)} \big(\log(e^k + |G|^p)\big)^k \beta_0}{\int_{B(r)} \beta_0} \leqslant \left\{ \log \left[e^k + \frac{\int_{B(r)} |G|^p \beta_0}{\int_{B(r)} \beta_0} \right] \right\}^k.$$

As $g_j \in A^0_\varphi(X)$ [cf. definition 11.1], there $p > 0$ small enough and big enough as $C_4, C_5 \geqslant 0$

$$\int_{B(r)} |G|^p \beta_0 \leqslant \exp(C_4 r + C_5).$$

Previous inequalities then lead

$$\int_{B(r)} \beta_k \wedge \gamma^k \leqslant C_6 \int_{B(r)} \big(\log(e^k + |G|^p)\big)^k \beta_0 \leqslant C_7\, r^{k+a}.$$

Given the definition of β_k, this implies Lemma 12.1 after substitution $2r$ à r. □

Endow $\mathbb{C}^{N+1}$ and $\check{M} \subset \mathbb{C}^{N+1}$ the metric Fubini-Study $\omega = dd^c \log(1+|z|^2)$. We then have Theorem Next extension, the demonstration was inspired by H. Skoda [SK 5] and H. El Mir [EM] ; see also Article synthesis N. Sibony [Sib].

Proposition 12.2. – *Either T simple extension $\check{M}$ current $\check{F}_* dd^c\varphi$ defined by*

$$T = \check{F}_* dd^c\varphi \quad sur\ \check{F}(\check{X}), \qquad T = 0 \quad sur\ \check{M} \smallsetminus \check{F}(\check{X}).$$

So T is closed on a positive current total mass of $\check{M}$ $\int_{\check{M}} T \wedge \omega^{n-1}$ over.

Demonstration. First calculate the mass T :

$$\int_{\check{M}} T \wedge \omega^{n-1} = \int_{\check{F}(\check{X})} \check{F}_* dd^c\varphi \wedge \omega^{n-1} = \int_{\check{X}} dd^c\varphi \wedge (\check{F}^*\omega)^{n-1}.$$

The $(1,1)$ Platform $\check{F}^*\omega$ is given here by

$$\begin{aligned}\check{F}^*\omega &= dd^c \log(1+|\check{F}|^2) = dd^c \log(1+|F|^2+|Q(F)|^{-2})\\ &= dd^c \log(1+|Q(F)|^2+|F|^2|Q(F)|^2).\end{aligned}$$

Finiteness of the mass then follows from Lemma 12.1 (a). For all 1 actual Platform v $\mathcal{C}^\infty$ class of compact support in M and for all multiindices $J, K \subset \{1,\dots,N+1\}$ such as $|J| = |K| = n-2$, it now shows the invalidity of the integral

$$I = \int_{\check{M}} dv \wedge T \wedge dz_J \wedge d\overline{z}_K,$$

which will prove that $dT = 0$. χ be a function of class $\mathcal{C}^\infty$ on $\mathbb{R}$ as $0 \leqslant \chi \leqslant 1$, $\chi(t) = 1$ if $t < 0$, if $\chi(t) = 0$ $t > 1$ and $0 \leqslant \chi' \leqslant 2$. By definition T he comes

$$\begin{aligned}I &= \int_{\check{X}} \check{F}^*(dv) \wedge dd^c\varphi \wedge d\check{F}_J \wedge \overline{d\check{F}_K}\\ &= \lim_{r\to+\infty} \int_{\check{X}} \chi\Big(\frac{\varphi}{r}\Big)\, d(\check{F}^*v) \wedge dd^c\varphi \wedge d\check{F}_J \wedge \overline{d\check{F}_K}\ .\end{aligned}$$

The $\chi(\frac{\varphi}{r})\, d(\check{F}^*v)$ form has support in $\check{F}^{-1}(\operatorname{Supp} v) \cap \overline{B(r)} \Subset \check{X}$. So integration by parts gives

$$I = \lim_{r\to+\infty} \pm \int_{\check{X}} \check{F}^*v \wedge \chi'\Big(\frac{\varphi}{r}\Big)\, \frac{d\varphi}{r} \wedge dd^c\varphi \wedge d\check{F}_J \wedge \overline{d\check{F}_K}\ .$$

Thanks to the Cauchy-Schwarz inequality, it is full increased by $\frac{2}{r}\sqrt{I_1 I_2(r)}$ with

$$\begin{aligned}I_1 &= \int_{\check{X}} dd^c\varphi \wedge \check{F}^*\big(v \wedge \overline{v}^c \wedge dz_J \wedge d\overline{z}_J\big)\ ,\\ I_2(r) &= \int_{\check{F}^{-1}(\operatorname{Supp} v)\cap B(r)} d\varphi \wedge d^c\varphi \wedge dd^c\varphi \wedge d\check{F}_K \wedge \overline{d\check{F}_K}\ .\end{aligned}$$

As v has compact support in $\check{M}$, there are constants $C_1, C_2 \geqslant 0$ such as

$$v \wedge \overline{v}^c \wedge dz_J \wedge d\overline{z}_J \leqslant C_1(\check{F}^*\omega)^{n-1} \ ,$$
$$d\check{F}_K \wedge \overline{d\check{F}_K} = \check{F}^*(dz_K \wedge d\overline{z}_K) \leqslant C_2(\check{F}^*\omega)^{n-2} \quad \text{sur } \check{F}^{-1}(\operatorname{Supp} v).$$

Lemma 12.1 (a) and (b) then drives

$$I_1 \leqslant C_1 \int_X dd^c\varphi \wedge (\check{F}^*\omega)^{n-1} < +\infty \ ,$$
$$I_2(r) \leqslant C_2 \int_{B(r)} d\varphi \wedge d^c\varphi \wedge dd^c\varphi \wedge (\check{F}^*\omega)^{n-2} \leqslant C\, C_2\, r \ ,$$

from where

$$|I| \leqslant \lim_{r\to+\infty} \frac{2}{r}\sqrt{I_1 I_2(r)} = 0. \qquad \square$$

By Theorem 15.3 of the appendix, there is a psh function V and $(1,0)$ Platform class u $\mathcal{C}^\infty$ on having $\check{M}$ following properties, suitable for $C_1, C_2, C_3 \geqslant 0$ constants.

Properties 12.3. – *was the (in) equalities*

(a) $dd^cV \geqslant T$;

(b) $V(z) \leqslant C_1 \log(1+|z|^2)$;

(c) $dd^cV - T = \overline{\partial}u$;

(d) $|u|_\omega \leqslant C_2(1+|z|^2)^{C_3}$.

Consider then the function defined on $\tau = V - \check{F}_*\varphi$ open $\Omega = \check{F}(\check{X}) \subset \check{M}$. According 12.3 (a) τ is psh on Ω and more $\tau \leqslant V$. As $\check{F}_*\varphi$ approaches $+\infty$ near $\partial\Omega$, τ approaches $-\infty$ at any point $\partial\Omega$. Therefore, τ extends to a psh function on $\check{M}$, still denoted τ, as $\tau = -\infty$ on $\check{M} \smallsetminus \Omega$.

Corollary 12.4. – *is $\check{M} \smallsetminus \Omega$ closed part of pluripolar $\check{M}$.* $\square$

the next step is to show that in fact $\check{M} \smallsetminus \Omega$ an algebraic hypersurface $\check{M}$. According 12.3 (c) and definition T was :

$$2i\,\partial\overline{\partial}(V - \check{F}_*\varphi) = \overline{\partial}u \quad \text{sur } \Omega,$$

therefore the $(1,0)$ -form h defined by

$$h = \partial(V - \check{F}_*\varphi) + \frac{u}{2i} = \partial\tau + \frac{u}{2i} \tag{12.5}$$

is holomorphic on Ω ; as is u class $\mathcal{C}^\infty$ on $\check{M}$ This demonstrates the way that class is τ $\mathcal{C}^\infty$ on Ω. We now prove that if h 1 Platform extends to a rational meromorphic on $\check{M}$. This will essentially result estimates 12.3 (b, d) and Theorem algebraicity of 8.5. For

construction and F $\check{X}$, shapes $(df_1, \ldots, df_n)$ define a global reference of$T^*\check{X}$. Forms are therefore also $(dz_1, \ldots, dz_n)$ a cue $T^*\check{M}$ above the open $\Omega = \check{F}(\check{X})$, and we can write

$$h = \sum_{j=1}^{n} h_j dz_j$$

with $h_j \in \mathcal{O}(\Omega)$ functions. The reasoning of the principle is to verify that the functions are $h_j \circ \check{F}$ φ -polynomiale growth, from the increase in $\tau = V - \check{F}_*\varphi$ provided by 12.3 (b). The fact that we did not have a lower bound of τ introduced additional difficulty that we will short-circuit in seeking only an estimate of functions $\exp(\frac{1}{2}\tau \circ \check{F})\ |h_j(\check{F})|$.

Lemma 12.6. – *We consider the function on X of exhaustion*

$$\check{\varphi} = \log(1 + e^{\varphi}) + \log(1 + |\check{F}|^2)$$

and the metric associated

$$\check{\alpha} = dd^c\check{\varphi} = \log(1 + e^{\varphi}) + \check{F}^*\omega.$$

The following properties are checked for constant $p > 0$ quite small and quite large C_4, C_5.

(a) $\displaystyle\int_{\check{X}} \check{\alpha}^n < +\infty$;

(b) $\displaystyle\int_{\check{X}} e^{\tau\circ\check{F} - C_4\check{\varphi}}\ \check{F}^*(ih \wedge \overline{h}) \wedge \check{\alpha}^{n-1} < +\infty$;

(c) $\displaystyle\int_{\check{X}} \left[\exp\left(\frac{1}{2}\tau \circ \check{F}\right)\ |Q(F)|^{C_4+1}\ |h_j(\check{F})|\right]^p\ e^{-C_5\varphi}\ \beta^n < +\infty$.

Demonstration.

(A) is an immediate consequence of Lemma 12.1 if one observes that

$$dd^c \log(1 + e^{\varphi}) = \frac{e^{\varphi}\ dd^c\varphi}{1 + e^{\varphi}} + \frac{e^{\varphi}\ d\varphi \wedge d^c\varphi}{(1 + e^{\varphi})^2} \leqslant dd^c\varphi + e^{-\varphi}\ d\varphi \wedge d^c\varphi.$$

(B) The estimate 12.3 (b) implies

$$\tau = V - \check{F}^*\varphi \leqslant V \leqslant C_1 \log(1 + |z|^2), \tag{12.7}$$

therefore the function satisfies the $\theta = \log(1 + e^{\tau\circ\check{F}})$ increase

$$\theta \leqslant \log\left(1 + (1 + |\check{F}|^2)^{C_1}\right) \leqslant C_1\check{\varphi} + \log 2.$$

Corollary 7.3 applied to $(\check{X}, \check{\varphi})$ then drives

$$\int_{\check{X}} i\partial\overline{\partial}\theta \wedge \check{\alpha}^{n-1} < +\infty.$$

Immediate calculation also gives

$$i\partial\overline{\partial}\theta = \check{F}^*\Big(\frac{i\partial\overline{\partial}\tau}{1+e^{-\tau}} + \frac{ie^{\tau}\ \partial\tau\wedge\overline{\partial}\tau}{(1+e^{\tau})^2}\Big) \geqslant \frac{1}{2}\ \check{F}^*\big(ie^{\tau}\ \partial\tau\wedge\overline{\partial}\tau\big)\ e^{-2C_1\check{\varphi}},$$

and it follows

$$\int_{\check{X}} e^{\tau\circ\check{F}-2C_1\check{\varphi}}\ \check{F}^*(\partial\tau\wedge\overline{\partial}\tau)\wedge\check{\alpha}^{n-1} < +\infty.$$

By definition $\check{\alpha}$ was second $\check{\alpha} \geqslant \check{F}^*\omega$. The estimate 12.3 (d) therefore leads

$$|\check{F}^*u|_{\check{\alpha}} \leqslant \big(|u|_\omega\big)\circ\check{F} \leqslant C_2(1+|\check{F}|^2)^{C_3} \leqslant C_2\ e^{C_3\check{\varphi}},$$

$$\int_{\check{X}} |\check{F}^*u|^2_{\check{\alpha}}\ e^{\tau\circ\check{F}-(C_1+2C_3)\check{\varphi}}\ \check{\alpha}^n \leqslant C_2^2\int_{\check{X}}\check{\alpha}^n < +\infty.$$

The property (b) now follows from the definition of $h = \partial\tau + \frac{u}{2i}$ and equality

$$|\check{F}^*u|^2_{\check{\alpha}}\ \check{\alpha}^n = n\ \check{F}^*(iu\wedge\overline{u})\wedge\check{\alpha}^{n-1}.$$

(C) The trivial inequality

$$dd^c\log(1+e^{\varphi}) \geqslant \frac{1}{2}\,dd^c\varphi + \frac{1}{4}\,e^{-\varphi}\ d\varphi\wedge d^c\varphi$$

successively drives

$$\begin{aligned}\check{\alpha}^{n-1} &\geqslant \frac{1}{2^{n-1}}\,(dd^c\varphi)^{n-1} + \frac{(n-1)\,e^{-\varphi}}{2^n}\,(dd^c\varphi)^{n-2}\wedge d\varphi\wedge d^c\varphi\\ &\geqslant 2^{-n}e^{-n\varphi}\ (dd^ce^{\varphi})^{n-1} = 2^{-n}e^{-n\varphi}\ \beta^{n-1},\end{aligned}$$

$$\check{F}^*(ih\wedge\overline{h})\wedge\check{\alpha}^{n-1} \geqslant 2^{-n}e^{-n\varphi}\ |\check{F}^*h|^2_\beta\ \beta^n.$$

By definition $\check{\varphi}$ was second

$$\begin{aligned}\check{\varphi} &= \log(1+e^{\varphi}) - 2\log|Q(F)| + \log\big(1+|Q(F)|^2+|Q(F)|^2|F|^2\big)\\ &\leqslant \varphi - 2\log|Q(F)| + C_6\log(1+|F|^2) + C_7,\end{aligned}$$

where $C_6 = 1+\deg(Q)$. Inequality (b) gives us so

$$\int_{\check{X}} e^{\tau\circ\check{F}-C_4\varphi}\ |Q(F)|^{2C_4}\ (1+|F|^2)^{-C_4C_6}e^{-n\varphi}\ |\check{F}^*h|^2_\beta\ \beta^n < +\infty$$

and as $|F| \in L^0_\varphi(X)$, it follows

$$\exp\Big(\frac{1}{2}\tau\circ\check{F}\Big)\ |Q(F)|^{C_4}\ |\check{F}^*h|_\beta \in L^0_\varphi(X).$$

Moreover, h_j function can be written as

$$h_j = (-1)^{j-1}\ \frac{h\wedge dz_1\wedge\ldots\wedge\widehat{dz_j}\wedge\ldots\wedge dz_n}{dz_1\wedge\ldots\wedge dz_n}.$$

Was therefore

$$|h_j(\check{F})| \leqslant |\check{F}^* h|_\beta \; |df_1|_\beta \dots \widehat{|df_j|_\beta} \dots |df_n|_\beta \; |df_1 \wedge \dots \wedge df_n|_\beta^{-1},$$

and like

$$|df_n|_\beta \,, \; \dots \,, \; |df_n|_\beta \in L^0_\varphi(X) \qquad \text{[lemme 11.3 (b)], et}$$

$$|f_{n+1}| \; |df_1 \wedge \dots \wedge df_n|_\beta^{-1} \in L^0_\varphi(X) \qquad \text{[inégalité 11.7 (a)],}$$

he comes

$$\exp\Big(\frac{1}{2}\tau \circ \check{F}\Big) \; |Q(F)|^{C_4} \; |f_{n+1}| \; |h_j \circ \check{F}| \in L^0_\varphi(X).$$

Hypothetically Q is divisible by z_{n+1}, i.e. $Q = z_{n+1}R$. The property (c) is then obtained by multiplying the above function $|R(F)| \in L^0_\varphi(X)$. □

In order to work on X rather than $\check{X}$ we will need elemental lemma extension below.

Lemma 12.8. – *Either $S = g^{-1}(0)$ a hypersurface ofX and θ a psh function on $X \smallsetminus S$ as $e^\theta \in L^1_{\mathrm{loc}}(X)$. $\theta + \log|g|^2$ then extends to a psh function on X.*

Demonstration. It suffices to show that is $\theta + \log|g|^2$ plus the vicinity of any regular point S. So we can X assume that is an open $\mathbb{C}^n$ containing polydisk unit $\overline{\Delta}^n$ closed, and $S = \{z_1 = 0\}$. Inequality average applied to polydisk

$$(z_1 + |z_1|\,\Delta) \times \Delta^{n-1} \subset X \smallsetminus S$$

for every point $z \in \Delta^n$, $0 < |z_1| < \frac{1}{2}$ implies

$$e^{\theta(z)} \leqslant \frac{1}{\pi^n |z_1|^2} \int_{\Delta^n} e^\theta \; d\lambda.$$

The $\theta + \log|z_1|^2$ function is increased due to neighborhood of$S$. □

Proposition 12.9. – *The 1 $h = \sum_{1 \leqslant j \leqslant n} h_j dz_j$ Platform extends to a rational meromorphic 1 Platform on M.*

Demonstration. As $\check{X} = X \smallsetminus Q(F)^{-1}(0)$, lemmas 12.6 (c) and 12.8 show that

$$p \; \log\Big[\exp\Big(\frac{1}{2}\tau \circ \check{F}\Big) \; |Q(F)|^{C_4+1} \; |h_j \circ \check{F}| \;\Big] + \log|Q(F)|^2$$

extends a psh function on X. There is therefore a stationary $s > 0$ big enough and small enough $\varepsilon > 0$ as if g means the holomorphic function defined on X

$$g = Q(F)^s h_j(F),$$

then $\frac{1}{2}\tau \circ F + \log|g|$ psh function on X and

$$\int_X \exp\Big[\varepsilon\Big(\frac{1}{2}\tau \circ \check{F} + \log|g|\Big) - C_8\varphi\Big] \; \beta^n < +\infty.$$

Proceeding as in Lemma 11.3 (a) thus obtained

$$\limsup_{r\to+\infty} \frac{1}{r}\, \mu_r\Big[\Big(\frac{1}{2}\tau\circ F + \log|g|\Big)_+\Big] \leqslant \frac{C_8 - n}{\varepsilon}\,\mathrm{Vol}(X) < +\infty.$$

$P \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ is a polynomial as $\deg_{X_\ell} P \leqslant k_\ell$ and is θ defined function

$$\theta = \log|P(g, f_1, \ldots, f_n)| + k_0\Big(\frac{1}{2}\tau\circ F + C_1 \log|Q(F)|\Big).$$

According to the estimate (12.7) and the above results, is θ psh X and checks on an estimate

$$\theta_+ \leqslant \sum_{j=1}^{m} k_j \log_+|f_j| + k_0\Big(\frac{1}{2}\tau\circ F + C_9 \log|Q(F)|\Big) + C_{10}.$$

With the corollary 7.3, we obtain the increase

$$\int_X dd^c\theta \wedge \alpha^{n-1} \leqslant C_{11}k_0 + \sum_{j=1}^{n} k_j \delta_\varphi(f_j)$$

with constant $C_{11} \geqslant 0$. If $a \in X$, resulting in the inequality

$$\mathrm{ord}_a\, P(g, f_1, \ldots, f_n) \leqslant C_{12}(k_0 + k_1 + \cdots + k_n), \qquad C_{12} \geqslant 0.$$

The theorem of reasoning 8.5 then shows that g is algebraic $\mathbb{C}(f_1, \ldots, f_n)$ on, and it is the same for the function $h_j(\check F) = Q(F)^{-s} g$. Following is algebraic over h_j $\mathbb{C}(z_1, \ldots, z_n)$, i.e. h_j satisfies an equation

$$\sum_{\ell=0}^{d} a_\ell(z_1, \ldots, z_n)\, h_j^\ell = 0, \qquad a_\ell \in \mathbb{C}[z_1, \ldots, z_n], \quad a_d \not\equiv 0.$$

The $a_d h_j$ element is algebraic integer $\mathbb{C}[z_1, \ldots, z_n]$; we deduce an increase

$$|a_d(z) h_j(z)| \leqslant C_{13}(1 + |z|)^{C_{14}}.$$

As h_j is holomorphic on the open and the $\Omega \subset \check M$ Additional $\check M \smallsetminus \Omega$ is multipolar, is $a_d h_j$ extended to a polynomial on $\check M$. Therefore $h = \sum h_j dz_j$ is extends to a rational meromorphic 1 Platform on $\check M$. □

Proposal 12.10. – *Either Ω_1 most large open Zariski $\check M$ h which is holomorphic. So $\Omega = \Omega_1$.*

Demonstration. We obviously $\Omega \subset \Omega_1$. For mutual inclusion, first show that τ is $\mathcal{C}^\infty$ class on Ω_1. It is known that the equation (12.5)

$$\partial\tau = h - \frac{u}{2i}$$

held on Ω, and $v := h - \frac{u}{2i} \in \mathcal{C}^\infty_{1,0}(\Omega_1)$ since $u \in \mathcal{C}^\infty_{1,0}(\check{M})$. It comes on $v+\overline{v} = d\tau\ \Omega$, where $d(v+\overline{v}) = 0$ on Ω_1 by continuity. Is $(\Omega_{1,j})_{j\in J}$ a covering Ω_1 by open simply connected. There are $\tau_j \in \mathcal{C}^\infty(\Omega_{1,j})$ functions such as $d\tau_j = v + \overline{v}$ on $\Omega_{1,j}$. The function is then locally $\tau - \tau_j$ constant on $\Omega_{1,j} \cap \Omega$, therefore constant because $\Omega_{1,j} \cap \Omega = \Omega_{1,j} \smallsetminus (\check{M} \smallsetminus \Omega)$ is connected. Following $\tau \in \mathcal{C}^\infty(\Omega_1)$, and as $\tau = -\infty$ on $\check{M} \smallsetminus \Omega$, it follows that $\Omega_1 \smallsetminus \Omega = \emptyset$. □

Corollary 12.11. – *is $F(X \smallsetminus f_{n+1}^{-1}(0))$ an open Zariski M.*

Demonstration. If x is any point of the open $X \smallsetminus f_{n+1}^{-1}(0)$, then $F(x) \notin M_{\text{sing}} \cup \{z_{n+1} = 0\}$, so there is a Q_x polynomial divisible by z_{n+1} and vanishing on M_{sing} such that $Q_x(F(x)) \neq 0$. According to the corollary 12.10, $\Omega_x := F(X \smallsetminus Q_x(F)^{-1}(0))$ is an open Zariski M, so also the meeting

$$\bigcup_{x\in X\smallsetminus f_{n+1}^{-1}(0)} \Omega_x = F(X \smallsetminus f_{n+1}^{-1}(0)). \qquad \square$$

As $X \smallsetminus f_{n+1}^{-1}(0)$ of Stein and his biholomorphic picture by F, one sees that the additional $M \smallsetminus F(X \smallsetminus f_{n+1}^{-1}(0))$ is necessarily an algebraic hypersurface M.

13. Démonstration criterion of algebraicity (smooth case).

According to the proposal 11.7 (a) at any point can $x_0 \in X$ associate a morphism

$$F^{(0)} = (f_1^{(0)}, \dots, f_{N_0}^{(0)}) \in \left[A_\varphi^0(X)\right]^{N_0}$$

and a $g_0 = f_{n+1}^{(0)}$ function such as open or $X \smallsetminus g_0^{-1}(0) \ni x_0$ biholomorphic by $F^{(0)}$ a Zariski open a algebraic variety in $\mathbb{C}^{N_0}$. There is therefore a countable recovery X by such open $X \smallsetminus g_k^{-1}(0)$ associated with morphisms $F^{(k)} : X \to \mathbb{C}^{N_k}$. Consider the morphism product

$$F_k = F^{(0)} \times F^{(1)} \times \cdots \times F^{(k)} : X \to \mathbb{C}^{N_0+\cdots+N_k}.$$

According to Proposition 8.5, the $F_k(X)$ image is contained in a variety ir algebraic re- Dukeble of $M_k \subset \mathbb{C}^{N_0\,+\,\cdots\,+\,N_k}$ n dimension, and the corollary shows that 12.11 $F_k(X \smallsetminus g_j^{-1}(0))$ is open Zariski of M_k if $j \leqslant k$. Let

$$Y_k = \bigcap_{j\leqslant k} g_j^{-1}(0), \qquad X_k = X \smallsetminus Y_k = \bigcup_{j\leqslant k} \left(X \smallsetminus g_j^{-1}(0)\right).$$

By construction $F_k : X_k \to F_k(X_k) \subset M_k$ is an isomorphism, and $F_k(X_k)$ is an open Zariski M_k. We can state :

Propostion 13.1. – *If X checks assumptions*9.1 '(a', b ') *then X is meeting an increasing sequence quasi-affine varieties X_k where each X_k identifies with an open Zariski of X_{k+1} with the induced algebraic structure.* □

In other words, X has a ringed space structure which is "locally" that of an algebraic variety, but "Zariski topology" may not be substantially compact.

Note that there is indeed such varieties. Just take X to the surface (smooth) $\sin x = yz$ of equation $\mathbb{C}^3$, and Y_k meeting countable straight

$$(\{j\pi\} \times \{0\} \times \mathbb{C}) \ \cup \ (\{j\pi\} \times \mathbb{C} \times \{0\}),$$

$j \in \mathbb{Z}$, $|j| > k$. The $X_k = X \smallsetminus Y_k$ open then identifies the algebraic variety

$$V_k = \Big\{(x,y,z) \in \mathbb{C}^3\,;\ x\Big(1 - \frac{x^2}{\pi^2}\Big) \cdots \Big(1 - \frac{x^2}{k^2\pi^2}\Big) = yz\Big\}$$

via $V_k \hookrightarrow X$ defined application

$$(x,y,z) \mapsto (x,y,z') \quad \text{où} \quad z' = z \prod_{|j|>k} \Big(1 - \frac{x^2}{j^2\pi^2}\Big),$$

with $V_k \hookrightarrow V_{k+1}$ inclusions data morphisms algebraic

$$(x,y,z) \mapsto (x,y,z') \quad \text{où} \quad z' = z\Big(1 - \frac{x^2}{(k+1)^2\pi^2}\Big). \qquad \square$$

We now show that the result is necessarily (X_k) stationary if the cohomology spaces are $H^{2q}(X\,;\mathbb{R})$ finite dimensional [hypothesis 9.1 '(c')].

Lemma 13.2. – *Either X analytical variety dimensional complex n, Y an analytic set of dimension $\leqslant p$ in X and $d = n - p = \operatorname{codim}_{\mathbb{C}} Y$. Then the space of cohomology relative $H^q(X, X \smallsetminus Y\,;\mathbb{R})$ is zero if and $q < 2d$*

$$H^{2d}(X, X \smallsetminus Y\,;\mathbb{R}) \simeq \mathbb{R}^J,$$

$(Y_j)_{j\in J}$ where is the family of irreducible components of p dimension Y.

Demonstration. We refer, for example E. Spanier [Sp] to basic arguments of algebraic topology that will be used. We proceed by induction on p, the result being trivial for $p = 0$. If $p \geqslant 1$ or Z meeting singular place Y_{sing} and irreducible components of $Y < p$ dimension, so that $\dim Z \leqslant p - 1$. The exact suite triplet is written

$$H^q(X, X \smallsetminus Z) \to H^q(X, X \smallsetminus Y) \to H^q(X, X \smallsetminus Y) \to H^{q+1}(X, X \smallsetminus Z).$$

By induction hypothesis for $H^q(X, X \smallsetminus Z) = H^{q+1}(X, X \smallsetminus Z) = 0$ $q \leqslant 2d$ so $H^q(X, X \smallsetminus Y) \simeq H^q(X \smallsetminus Z, X \smallsetminus Y)$. Quits (X,Y) replaced by $(X \smallsetminus Z, Y \smallsetminus Z)$, smooth we can assume Y dimension p.

Y then has a tubular neighborhood homeomorphic U normal NY bundle. With the excision theorem, we obtain

$$H^q(X, X \smallsetminus Y) \simeq H^q(U, U \smallsetminus Y) \simeq H^q(NY, N^\bullet Y)$$

where $N^\bullet Y$ is the complement of the zero section of NY. As NY the bundle of real rank $2d$, the isomorphism theorem Thom-Gysin involves

$$H^q(NY, N^\bullet Y) \simeq H^{q-2d}(Y),$$

and $q = 2d$, $H^0(Y) \simeq \mathbb{R}^J$. □

Now back to the situation of the proposal 13.1, where $X_k = X \smallsetminus Y_k$, and put $\dim Y_k = p_k$, $d_k = n - p_k$. The exact suite of the pair gives $(X, X \smallsetminus Y_k)$

$$H^{2d_k-1}(X \smallsetminus Y_k) \to H^{2d_k}(X, X \smallsetminus Y_k) \to H^{2d_k}(X).$$

Since $X \smallsetminus Y_k$ is isomorphic to an algebraic variety, $H^{2d_k-1}(X \smallsetminus Y_k)$ is finite dimensional, and it is the same for $H^{2d_k}(X)$ hypothetically. Lemma 13.2 shows that Y_k therefore has a finite number of components irreducible p_k maximum dimension. As Y_k is a suite decreasing empty intersection, we see that there $\ell > k$ as $\dim Y_\ell < p_k$. After a finite number of steps we shall have $Y_\ell = \emptyset$ or $X = X_\ell$. Let

$$F = F_\ell = F^{(0)} \times \cdots \times F^{(\ell)}, \qquad M = M_\ell, \qquad N = N_0 + \cdots + N_\ell.$$

The morphism $F : X \to M \subset \mathbb{C}^N$ is then an isomorphism analytical X on a Zariski open $\Omega \subset M$. Even M replace its normalization as in the proof 11.7 (b) we can assume normal M. Since Ω of Stein, the Additional $H = M \smallsetminus \Omega$ is necessarily a hypersurface of$M$. $K(\Omega) \simeq K(M)$ denote the body functions on rational Ω, and the ring of functions $R(\Omega)$ regular algebraic over Ω. write

$$F = (f_1, \ldots, f_N) \in \left[A^0_\varphi(X)\right]^N.$$

The co-morphism F^* sends $K(\Omega)$ in the body $\mathbb{C}(f_1, \ldots, f_N) \subset K_\varphi(X)$. The proposal below shows the algebraic structures and $(X, K_\varphi(X) \cap \mathcal{O}(X))$ $(\Omega, R(\Omega))$ are isomorphic.

Proposition 13.3. – *It has the following properties* :

(a) $F^* : K(\Omega) \to K_\varphi(X)$ *is an isomorphism.*

(b) $F^* R(\Omega) = K_\varphi(X) \cap \mathcal{O}(X)$.

Demonstration.

(A) It suffices to show surjectivity of F^*. However, if $g \in K_\varphi(X)$, the function is algebraic over g $\mathbb{C}(f_1, \ldots, f_N)$ from 11.6. Following is meromorphic $g \circ F^{-1}$ on Ω and algebraic over $K(\Omega)$. It follows that $g \circ F^{-1} \in K(\Omega) = K(M)$, reasoning such as at the end of the demonstration 12.9.

(B) is deducted immediately from (a), provided to verify the equality $R(\Omega) = K(\Omega) \cap \mathcal{O}_{\mathrm{anal}}(\Omega)$. inclusion $R(\Omega) \subset \ldots$ is clear. Conversely, given $g \in K(\Omega)$ and $x \in \Omega$ or $g = u/v$ where $u, v \in \mathcal{O}_{x,\mathrm{alg}}$, irreducible to write g x item (which is smooth by hypothesis). This writing is also irreducible in $\mathcal{O}_{x,\mathrm{anal}}$. As $g \in \mathcal{O}_{x,\mathrm{anal}}$, so we $v(x) \neq 0$ a result and $g \in \mathcal{O}_{x,\mathrm{alg}}$ $g \in R(\Omega)$. □

To complete the proof of Theorem 9.1 ', it remains to show that Ω is algebraically isomorphic to an algebraic variety refines, i.e. it must prove the existence of an algebraic dip $\Omega = M \smallsetminus H \to \mathbb{C}^{N'}$ own. It's easy if M is smooth, but M is singular when it may be that the algebraic hypersurface H is not locally complete intersection, and in this situation Goodman [Go] gave examples for which algebra $R(M \smallsetminus H)$ is not finitely. Thank N. Mok of telling me about this difficulty, which made void my initial demonstration. The reasoning [Mok2] is to observe what is Ω *rationally convex* in the following sense: for any compact $K \subset \Omega$ envelope

$$\hat{K} = \Big\{ x \in \Omega \,;\; |g(x)| \leqslant \sup_K |g| \text{ pour tout } g \in R(\Omega) \Big\}$$

is compact. This follows from 11.5 (d) and that $\Omega \simeq X$ of Stein. It then applies the part (B) of Theorem below.

Theorem 13.4. – *Either a $M \subset \mathbb{C}^N$ affine algebraic variety (possibly Singudimensional die) pure n and H an algebraic hypersurface M. So $M \smallsetminus H$ is isomorphic to an affine algebraic variety under any one of the following two assumptions* :

(a) *H is (as Subschema reduced) locally complete intersection in M.*

(b) *$M \smallsetminus H$ is rationally convex*([Mok2]).

Demonstration assuming (a). For everything $x \in H$, there assumed a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ and a neighborhood Zariski $V(x) \subset M$ as $H \cap V(x) = P^{-1}(0) \cap V(x)$. Either H' the meeting of irreducible components of $P^{-1}(0)$ not contained in H. As $x \notin H'$, there Q a polynomial vanishing on H' as $Q(x) = 1$. Hilbert's theorem of zeros causes the existence of a whole $s \in \mathbb{N}$ as $Q^s/P \in R(M \smallsetminus H)$. As the Zariski topology is quasi-compact, one can extract a finite covering $V(x_1), \ldots, V(x_m)$ of H and a finite family of polynomials P_j, $Q_j^{s_j}$ associated with points x_j. Our building shows while the morphism

$$\big(z_1, \ldots, z_N, Q_1^{s_1}/P_1, \ldots, Q_m^{s_m}/P_m\big) : M \smallsetminus H \to \mathbb{C}^{N+m}$$

is a proper embedding.

Demonstration under the assumption (b). first constructed by descending induction on k a series of algebraic subvarieties $M_k \subset M$ closed, pure dimension k such as $M_k \cap H$ is a hypersurface of M_k. We set $M_n = M$; if M_k was built, we choose a polynomial vanishing on $P_k \in R(M)$ H but not vanish identically on any irreducible component ofM_k ; we then noted the meeting M_{k-1} components irreducible $M_k \cap P_k^{-1}(0)$ not contained in H.

Now we prove by induction on increasing k the existence of rational $g_1, \ldots, g_{m_k} \in R(M \smallsetminus H)$ such that the morphism

$$\Phi_k : (z_1, \ldots, z_N) \mapsto (z_1, \ldots, z_N, g_1(z), \ldots, g_{m_k}(z)) : M \smallsetminus H \to \mathbb{C}^{N+m_k}$$

is a proper embedding restriction to $M_k \smallsetminus H$ (for $k = n$ Theorem will be demonstrated as well). If $k = \dim M_k = 1$, this property is clear, because the assumption (b) and the principle of maximum result the existence of rational $g_1, \ldots, g_{m_1} \in R(M \smallsetminus H)$ including

restrictions M_1 have poles at different points $x_1, \ldots, x_{m_1}$ of$M_1 \cap H$. Suppose Φ_k built. Either $\pi_k : \mathbb{C}^{N+m_k} \to \mathbb{C}^N$ projection, $\overline{M}$ adhesion and $\Phi_k(M \smallsetminus H)$ in $\mathbb{C}^{N+m_k}$ $\overline{H} = \overline{M} \cap \pi_k^{-1}(H)$, so that

$$\Phi_k : M \smallsetminus H \to \overline{M} \smallsetminus \overline{H}$$

is an isomorphism (reverse $\Phi_k^{-1} = \pi_{k|\overline{M} \smallsetminus \overline{H}}$). By induction hypothesis $\overline{M}_k = \Phi_k(M_k \smallsetminus H)$ is an algebraic submanifold closed $\overline{M}_{k+1} = \overline{\Phi_k(M_{k+1} \smallsetminus H)}$. As the meeting is $\overline{M}_{k+1} \cap (P_k \circ \pi_k)^{-1}(0)$ disjoint $(\overline{M}_{k+1} \cap \overline{H}) \cup \overline{M}_k$, it is seen that $\overline{M}_{k+1} \cap \overline{H}$ is locally complete intersection in $\overline{M}_{k+1}$ (and it is locally defined by $P_k \circ \pi_k$). For $x \in \overline{M}_{k+1} \cap \overline{H}$ it therefore a polynomial $Q \in R(\overline{M})$ as $Q(x) = 1$, which vanishes on all irreducible components of $(P_k \circ \pi_k)^{-1}(0)$ encountering not $\overline{M}_{k+1} \cap \overline{H}$. We can complete Φ_k in a Φ_{k+1} morphism as in the case (a), by adding to Φ_k of $g_j = (Q_j \circ \Phi_k)^{s_j}/P_k$ functions $m_k < j < m_{k+1}$; then $\Phi_{k+1} : M_{k+1} \smallsetminus H \to \mathbb{C}^{N+m_{k+1}}$ is proper, because $g_j \circ \pi_k = Q_j^{s_j}/P_k \circ \pi_k$ the functions define a proper map on $\overline{M}_{k+1} \smallsetminus \overline{H}$. □

The property 13.3 (a) results that is generated by $K_\varphi(X)$ $f_1, \ldots, f_N$, so that $K_\varphi(X)$ is also the body of $A_\varphi^0(X)$ quotients, which was not obvious a priori. We can actually get slightly more accurate result.

Proposition 13.5. – *Assuming* 9.1'(b') [*resp.* 9.1 (b)], $K_\varphi(X)$ *is generated by* $A_\varphi^b(X)$ *where* $b = \frac{2c}{1+c}$ [*resp. by* $A_\varphi^2(X)$].

Demonstration. To see it, you only through reasoning Previous to construct an injective embedding

$$G = (g_1, \ldots, g_s) : X \to \mathbb{C}^s, \qquad g_j \in A_\varphi^b(X).$$

The proposal 11.5 (c) allows building for any points $x_0 \in X$ and $(x_1, x_2) \in X \times X \smallsetminus \Delta$ (where diagonal $\Delta =$) functions such as $g_1, \ldots, g_n, g \in A_\varphi^b(X)$ $dg_1 \wedge \ldots \wedge dg_n(x_0) \neq 0$ and $g_1(x) \neq g(x_2)$. Such as open

$$\big\{x\,;\ dg_1 \wedge \ldots \wedge dg_n(x) \neq 0\big\} \quad \text{et} \quad \big\{(x,y)\,;\ g(x) \neq g(y)\big\} \subset X \times X \smallsetminus \Delta$$

are open Zariski, there are finished recoveries X and $X \times X \smallsetminus \Delta$ respectively, such open. the collection g, g_j of functions thus obtained gives the morphism G cherché. □

Note 13.6. – was in full generality inclusions

$$A_\varphi^\infty(X) \subset A_\varphi^b(X) \subset A_\varphi^0(X) \subset A_\varphi(X), \qquad 0 < b \leqslant 2,$$

but we do not know for the last two if still tied or not. The surprising fact is that the algebra can have $A_\varphi^\infty(X)$ a degree of transcendence $< n$. Choose such $X = \mathbb{C}$, with strictly psh function

$$\varphi(z) = \sum_{j \in \mathbb{N}} 2^{-j} \log(\varepsilon_j + |z - j|^2), \qquad 0 < \varepsilon_j \leqslant 1, \quad \varepsilon_0 = 1.$$

$r \geqslant 3$ be given. By cutting the sum for the indices $j \leqslant \log_2 r$ one hand, $j > \log_2 r$ other hand, one obtains easily to $|z| = r$ estimates

$$\varphi(z) = 2\log(1 + |z|^2) + O\Big(\frac{\log r}{r}\Big) \qquad \text{si } \forall j,\ |z - j| > \tfrac{1}{2}, \tag{13.7}$$

$$\varphi(z) = 2\log(1 + |z|^2) + 2^{-j}\log(\varepsilon_j + |z - j|^2) + O\Big(\frac{\log r}{r}\Big) \qquad \text{si } |z - j| \leqslant \tfrac{1}{2}. \tag{13.8}$$

So choose ε_j

$$(13.9) \qquad 2\log(1+j^2) + 2^{-j}\log\varepsilon_j = \log(1+\log j), \quad j \geqslant 1, \quad \text{i.e. } \varepsilon_j = \left[\frac{1+\log j}{(1+j^2)^2}\right]^{2^j}.$$

was then $\varphi(j) \sim \log\log j$ when $j \to +\infty$, so φ that is comprehensive. The functions $f \in A^\infty_\varphi(\mathbb{C})$ are polynomial growth and must check more $|f(j)| \leqslant (\log j)^{\mathrm{Cte}}$ when $j \to +\infty$. therefore $A^\infty_\varphi(\mathbb{C})$ is reduced to constants. Conditions 9.1 (a) and (b) are still verified. An immediate calculation gives indeed

$$dd^c\varphi = 2i\,dz \wedge d\overline{z} \sum_{j\in\mathbb{N}} \frac{2^{-j}\varepsilon_j}{(\varepsilon_j + |z-j|^2)^2},$$

so that $\int_{\mathbb{C}} dd^c\varphi = 8\pi$. The increase 9.1 (b) takes place with the function

$$\psi(z) = -\log\left(\sum_{j\in\mathbb{N}} \frac{2^{-j}\varepsilon_j}{(\varepsilon_j + |z-j|^2)^2}\right).$$

Considering the single term $j = 0$, we obtain the increase $\psi(z) \leqslant 2\log(1+|z|^2)$. For $\varepsilon_j^{1/3} \leqslant |z-j| \leqslant \frac{1}{2}$ was second thanks to (13.8) and (13.9) $\times\, \frac{2}{3}$:

$$\varphi(z) \geqslant 2\log(1+|z|^2) + 2^{-j}\frac{2}{3}\log\varepsilon_j + O(1) \geqslant \frac{2}{3}\log(1+|z|^2) + O(1),$$

while it comes to $|z-j| \leqslant \varepsilon_j^{1/3}$

$$\frac{2^{-j}\varepsilon_j}{(\varepsilon_j + |z-j|^2)^2} \geqslant \frac{2^{-j}\varepsilon_j}{4\,\varepsilon_j^{4/3}} = 2^{-j-2}\varepsilon_j^{-1/3},$$

so if $\psi(z) \leqslant 0$ j is large enough. We see it is a constant B as $\psi \leqslant 3\varphi + B$. □

14. Algébricité singular complex spaces.

X is an analytic space of dimension n. If a set is X $\mathbb{C}^N$ in algebraic, calculations show that the conditions§10 Geometric 9.1 (a, b, c) are satisfied.

Conversely, to demonstrate the adequacy of geometric conditions, it faces two main challenges. On the one hand the L^2 estimates Hörmander priori are valid only on a Stein opened the $X \smallsetminus H$ smooth shape, which is a hypersurface H X of containing the singular place X_{sing}. To apply lemma extension, one must assume that X is normal.

Lemma 14.1. – *Either f holomorphic on $X \smallsetminus H$ as $f \in L^2_{\mathrm{loc}}(X_{\mathrm{reg}})$. So if X is normal, f extends to a holomorphic function on X.*

Proof. $f \in L^2_{\mathrm{loc}}(X_{\mathrm{reg}})$ Under the hypothesis, it is conventional f that extends from $X_{\mathrm{reg}} \smallsetminus H$ to X_{reg}, and any function holomorphic on X_{reg} extends X if X is normal (See [Nar] proposal VI.4).

Another difficulty is that the $e^{-\psi}$ weight may not be locally summable in some singular points, compared with a metric smooth room. For example, consider the case X of the conical variety of $z_0^p + \cdots + z_n^p = 0$ equation in $\mathbb{C}^{n+1}$, $p \in \mathbb{N}^*$. The Ricci curvature X is then given through proposal 10.1 (a) by the formula

$$\mathrm{Ricci}(\beta_{|X}) = -\frac{1}{2} dd^c \psi$$

with $\psi(z) = \log(|z_0|^{2p-2} + \cdots + |z_n|^{2p-2})$. We thus see $e^{-\psi}$ that the function is locally summable in 0 if $p \leqslant n$. In the case of a space for which X $e^{-\psi}$ is not integrable in the neighborhood of any point of a curve, the proposal 11.5 (c) no longer applies. It is therefore necessary to assume that the singularities X are isolated.

Proof of Theorem 9.1 '(sufficiency of the conditions in the case of isolated singularities). Hypothesis (c ') implies that the components irreducible X is finite. Is

$$\pi : \tilde{X} \to X$$

normalizing X. The $\varphi \circ \pi$ function is not generally strictly psh near $\pi^{-1}(X_{\mathrm{sing}})$, but even change $\varphi \circ \pi$ and $\psi \circ \pi$ near the finite $\pi^{-1}(X_{\mathrm{sing}})$, we see that the assumptions are met by $\tilde{X}$. Ultimately, it can be assumed normal and irreducible X.

The demonstration is now quite similar to that that was given during the§11, 12, 13, so we'll just indicate the broad lines and the changes needed. lemmas 11.2 and 11.3 are true without any changes, as well as properties 11.5 (a, b, d). The statement 11.5 (c) remains valid if $\{x_1, \ldots, x_m\} \subset X_{\mathrm{reg}}$, and if some of the points are x_j singular, we have the following partial result (which corresponds to If $\rho = 0$).

Lemma 14.2. – *Given a finite set $\{x_1, \ldots, x_m\} \subset X$. Then there is a function $f \in A^b_\varphi(X)$, $b = \frac{2c}{1+c}$ having a jet s order given to each point $x_1, x_2, \ldots, x_m$.*

Demonstration. We use the same arguments as in 11.5 (c) substituting $z^{(j)}$ local coordinate system by a system $(z_1^{(j)}, \ldots, z_{N_j}^{(j)})$ generator of the maximal ideal $\mathfrak{m}_{X,x_j}$ of $\mathcal{O}_{X,x_j}$ and by ρ_1

$$\rho_1 = s(n+2) \left[\sum_{j=1}^{m} \chi_j \log |z^{(j)}|^2 + C_1 \varphi \right].$$

Near x_j, construction then gives $f = P_j(z^{(j)}) - g$ g with holomorphic as

$$|g|^2 \; |z^{(j)}|^{-2s(n+2)} \; e^{-\psi} \in L^2_{\mathrm{loc}}(X).$$

According to estimates L^2 H. Skoda [Sk4] This implies that $g \in \mathfrak{m}^s_{X,x_j}$. □

11.7 The proposal therefore remains applicable if $x_0 \in X_{\mathrm{reg}}$, and Embodying the arguments of

S12, 13, constructing a variety normal algebraic M and a morphism $F = (f_1, \ldots, f_N) : X \to M$ whose restriction to X_{reg} is an isomorphism of X_{reg} on open Zariski of M. Thanks to Lemma 14.2, we can (completing F by a finite number of functions f_j) assume

that defines F a dipping X near each singular point. The F morphism is then an isomorphism of X the Zariski open $F(X) \subset M$. The end of the proof is identical to that given the §13. □

The reasoning just outlined gives the other the interesting result below.

Theorem 14.3. – *Either X an analytic space normal size n, checking assumptions*9.1 '(a', b ', c'). *So X_{reg} is analytically isomorphic to an algebraic variety Almost affine isomorphism being given by a morphism φ -polynomial F of X in an affine algebraic variety $M \subset \mathbb{C}^N$ normal size n.* □

15. Appendice: current and plurisubharmonic functions to minimal growth on an affine algebraic variety.

M is an algebraic subvariety of smooth affine dimension n. We equip M of Kähler metrics

$$\beta = dd^c|z|^2, \qquad \omega = dd^c \log(1 + |z|^2)$$

respectively induced by the metric flat $\mathbb{C}^N$ and the metric Fubini-Study of the projective space $\mathbb{P}^N$.

Definition 15.1. – *psh functions and currents Minimum* : *growth*

(a) A *V psh function on M is said to minimal growth if there are constants such that* $C_0, C_1 \geqslant 0$

$$V(z) \leqslant C_1 \log_+ |z| + C_0.$$

l positive current *T of bidegree* $(1,1)$ *on M is said to minimal growth if*

$$\int_M T \wedge \omega^{n-1} < +\infty.$$

Corollary 7.3 follows immediately the

Proposition 15.2. – *If V is of psh minimal growth on M, then $T = dd^c V$ is minimal growth.* □

Conversely, given a closed $T \geqslant 0$ current growth minimum, we can not find a solution to the equation $dd^c V = T$ if the cohomology class T is zero. The objective of this section is to prove the following general result, which is a partial reciprocal of 15.2 proposal.

Theorem 15.3. – *Either T a* $(1,1)$ *-current closed on positive M as*

$$\int_M T \wedge \omega^{n-1} < +\infty.$$

Then there is a psh function V and $(1,0)$ *Platform of u $\mathcal{C}^\infty$ class on M having the properties below, where $C_1, C_2, C_3 \geqslant 0$ are constants.*

(a) $dd^c V \geqslant T$;

(b) $V(z) \leqslant C_1 \log_+ |z|$;

(c) $dd^c V - T = \overline{\partial} u$;

(d) $|u|_\omega \leqslant C_2(1+|z|)^{C_3}$.

The demonstration will be done in stages. First observe that condition 15.1 (b) is equivalent to the following:

$$\sigma(r) = \int_{|\zeta|<r} T(\zeta) \wedge \beta^{n-1} \leqslant C\, r^{2n-2}. \tag{15.4}$$

It is not restrictive to assume the other $n \geqslant 2$. In the case otherwise, we can apply Theorem 15.3 to the variety $M' = M \times \mathbb{C}$ and current pullback $T' = \pi_M^* T$. Function $V(z) = V'(z, 0)$ and $u = u'_{|M\times\{0\}}$ trains meet then the question.

Given a current $T \geqslant 0$ bidegree $(1,1)$ on M checking (15.4), one can associate a potential V_T by the same formulas as used by P. Lelong [Le3] in $\mathbb{C}^n$:

$$V_T(z) = \int_M T(\zeta) \wedge \beta^{n-1}\, L_n(z,\zeta) \tag{15.5}$$

with

$$L_n(z,\zeta) = \frac{1}{(n-1)(4\pi)^n} \left[\frac{1}{(1+|\zeta|^2)^{n-1}} - \frac{1}{|z-\zeta|^{2n-2}}\right].$$

Lemma 15.6. – *The formula defines a (15.5) $V_T \in L^1_{\mathrm{loc}}(M)$ upper semi-continuous function, and there are constants such as C_0, C_1*

$$V_T(z) \leqslant C_1 \log_+ |z| + C_0.$$

Demonstration. The core L_n clearly verifies estimates following:

$$|L_n(z,\zeta)| \leqslant C_2 |z|\, |\zeta|^{1-2n} \qquad \text{si } |\zeta| \geqslant 2|z| \geqslant 1,$$

$$L_n(z,\zeta) \leqslant C_3 |\zeta|^{2-2n} \qquad \text{si } 1 \leqslant |\zeta| \leqslant 2|z|.$$

For $|z| = r \geqslant 1$, we deduce

$$V_T(z) \leqslant C_4 \left[1 + \int_1^{2r} \frac{1}{t^{2n-2}}\, d\sigma(t) + \int_{2r}^{+\infty} \frac{r}{t^{2n-1}}\, d\sigma(t)\right]$$

After integration by parts, it comes, in view of (15.4):

$$\begin{aligned} V_T(z) &\leqslant C_4 \left[1 + \frac{\sigma(2r)}{(2r)^{2n-2}} + (2n-2)\int_1^{2r} \frac{\sigma(t)\, dt}{t^{2n-1}} + (2n-1) r \int_{2r}^{+\infty} \frac{\sigma(t)\, dt}{t^{2n}}\right] \\ &\leqslant C_5 (1 + \log r). \end{aligned}$$

Previous estimates further show that the integral (15.5) converges absolutely on all $\{\zeta \in M\,;\ |\zeta| > 2|z|\}$, uniformly when z describes a compact M. The property $V_T \in L^1_{\mathrm{loc}}(M)$ then follows by Fubini theorem that $|L_n(z,\zeta)|$ is locally $M \times M$, integrable z uniformly with respect to ζ. □

Lemma 15.7. – *$z \in M$ For every point, there $B'_z \subset T_zM$ of balls, $B''_z \subset (T_zM)^\perp$ center 0 and $r(z) = C_6(1+z|)^{-C_7}$ radius where $C_6, C_7 > 0$, and application holomorphic $g_z : B'_z \to B''_z$ as either $M \cap (z + B'_z + B''_z)$ graph g, i.e. if $\zeta - z = \zeta' + \zeta''$ is writing a Point $\zeta \in M$ after $\mathbb{C}^N = (T_zM) \oplus (T_zM)^\perp$ decomposition, then*

$$M \cap (z + B'_z + B''_z) = \big\{\zeta \in \mathbb{C}^N\,;\ \zeta'' = g_z(\zeta'),\ \zeta' \in B'_z\big\}.$$

Demonstration. $(P_1, \ldots, P_m)$ is a polynomial system generators for the ideal of M variety in $\mathbb{C}[X_1, \ldots, X_N]$. Since M is smooth, partial Jacobians order $J_{K,L}$ $N - n$ (See S10) do not cancel all simultaneously M. According to the nullstellensatz Hilbert polynomials generate the $J_{K,L}$ Ideally unit M ; so there is constant $C_8, C_9 > 0$ as

$$\max_{K,L} |J_{K,L}(z)| \geqslant C_8(1+|z|)^{-C_9}, \qquad z \in M.$$

The lemma then follows from the implicit function theorem (in its Version quanti-tative). □

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Now we observe that the formula (15.5) can be rewritten as

$$V_T(z) = \int_M T(\zeta) \wedge \big[K_n(z,\zeta) - H_n(\zeta)\big] \tag{15.8}$$

with

$$K_n(z,\zeta) = -\frac{1}{(n-1)(4\pi)^n} \left[\frac{dd^c|z-\zeta|^2}{|z-\zeta|^2}\right]^{n-1},$$

$$H_n(\zeta) = -\frac{1}{(n-1)(4\pi)^n}\ \frac{\beta(\zeta)^{n-1}}{(1+|\zeta|^2)^{n-1}}.$$

K_n core properties will allow us to easily calculate dd^cV_T according to T.

Lemma 15.9. – *$dd^cK_n = [\Delta] + R_n$ where $[\Delta]$ is the integration of current on the diagonal $M \times M$ and where R_n is a (n,n) -ac $\geqslant 0$ to locally integrable coefficients on $M \times M$, checking the estimate*

$$\|R_n(z,\zeta)\|_{\beta\oplus\beta} \leqslant C_{10} \min\left[\frac{1}{|z-\zeta|^{2n}}\ ,\ \frac{(1+|z|)^{C_{11}}}{|z-\zeta|^{2n-1}}\right].$$

Demonstration. Outside the diagonal Δ, a calculation classic (the check is left to the reader) gives

$$dd^cK_n = \frac{(dd^c|z-\zeta|^2)^n - n\,|z-\zeta|^{-2}\,d|z-\zeta|^2 \wedge d^c|z-\zeta|^2 \wedge (dd^c|z-\zeta|^2)^{n-1}}{(4\pi)^n\,|z-\zeta|^{2n}}$$

$$= \Big(\frac{1}{4\pi}\,dd^c \log|z-\zeta|^2\Big)^n, \tag{15.10}$$

as we shall see later that $\mathbb{1}_\Delta\, dd^c K_n = [\Delta]$. In particular, it has

$$R_n = \mathbb{1}_{M\times M\smallsetminus\Delta}\, dd^c K_n \geqslant 0 \quad \text{et} \quad \|R_n(z,\zeta)\| \leqslant C|z-\zeta|^{-2n}.$$

For the second part of the increase, we place a point $z \in M$ and use Lemma 15.7. In restriction M was the Point z

$$dz = dz' = \text{composante de } dz \text{ sur } T_z M,$$

while in a neighboring point $\zeta \in z + (B'_z + B''_z)$ was :

$$d\zeta = d\zeta' + d(g_z(\zeta')).$$

By (15.10) so it comes:

$$R_n(z,\zeta) = \Bigg[\frac{dd^c\big(|z'-\zeta'|^2 + |g_z(\zeta')|^2\big)}{(4\pi)\,(|\zeta'|^2 + |g_z(\zeta')|^2)} - \frac{d\big(|z'-\zeta'|^2 + |g_z(\zeta')|^2\big)\wedge d^c\big(|z'-\zeta'|^2 + |g_z(\zeta')|^2\big)}{(4\pi)\,(|\zeta'|^2 + |g_z(\zeta')|^2)^2}\Bigg]^n,$$

where the differentiation of $g_z(\zeta')$ only covers ζ'. Lemma gives 15.7 by constricktion $g_z(0) = D_0 g_z = 0$; lemma Schwarz then implies inequality

$$|g_z(\zeta')| \leqslant |\zeta'|, \qquad \zeta' \in B'_z\ ;$$

$$\|D_{\zeta'} g_z(\zeta')\| \leqslant C(\lambda)\frac{|\zeta'|}{r(z)}, \qquad \zeta' \in \lambda B'_z, \quad 0<\lambda<1.$$

Now we observe that if $R_n(z,\zeta) \equiv 0$ $g_z \equiv 0$. It follows $\zeta' \in \frac{1}{2}B'_z$ for inequality

$$\|R_n(z,\zeta)\| \leqslant \frac{C_{12}\,|\zeta'|\,r(z)^{-1}}{\big(|\zeta'|^2 + |g_z(\zeta')|^2\big)^n} \leqslant \frac{C_{12}\,r(z)^{-1}}{|z-\zeta|^{2n-1}},$$

who complete the estimate of Lemma 15.9. The classic formula Bochner-Martinelli in $\mathbb{C}^n$ gives the other

$$dd^c K_n(z',\zeta') = [\Delta].$$

By a calculation similar to that above, the inequality is obtained

$$\|K_n(z,\zeta) - K_n(z',\zeta')\| \leqslant \frac{C_{13}\,r(z)^{-1}}{|z-\zeta|^{2n-3}},$$

and each differentiation K_n exhibitor $|z-\zeta|$ increases by one unit. So we see that $dd^c K_n - [\Delta]$ is coefficients L^1_{loc} on $M\times M$, and consequently it does not bear Δ mass. The preuve is completed. □

Proposal 15.11. – *If T is closed, then*

$$dd^c V_T = T + \Theta_T \qquad \textit{où} \quad \Theta_T(z) = \int_M R_n(z,\zeta)\wedge T(\zeta) \geqslant 0.$$

In particular, V_T is psh.

Demonstration. $\chi : \mathbb{R} \to [0,1]$ be a function of class $\mathcal{C}^\infty$ as $\chi(t) = 1$ for $t < 1$, $\chi(t) = 0$ for $t > 2$, and either a w $(n-1, n-1)$ form $\mathcal{C}^\infty$ compact support on M. Writing (15.8) gives us

$$\int_M V_T\, dd^c w = \lim_{r \to +\infty} I(r),$$

$$I(r) = \int_{M\times M} \chi\Big(\frac{|\zeta|}{r}\Big)\, T(\zeta) \wedge \big(K_n(z,\zeta) - H_n(\zeta)\big) \wedge dd^c w(z).$$

Stokes' theorem and Lemma 15.9 imply

$$\begin{aligned} I(r) &= \int_{M\times M} dd^c \Big[\chi\Big(\frac{|\zeta|}{r}\Big)\, T(\zeta) \wedge K_n(z,\zeta)\Big] \wedge w(z) \\ &= \int_{M\times M} \chi\Big(\frac{|\zeta|}{r}\Big)\, T(\zeta) \wedge \big([\Delta] + R_n(z,\zeta)\big) \wedge w(z) \\ &\quad + \int_{M\times M} d\Big[\chi\Big(\frac{|\zeta|}{r}\Big)\Big] \wedge T(\zeta) \wedge d^c K_n(z,\zeta) \wedge w(z) \\ &\quad - \int_{M\times M} d^c\Big[\chi\Big(\frac{|\zeta|}{r}\Big)\Big] \wedge T(\zeta) \wedge dK_n(z,\zeta) \wedge w(z) \\ &\quad + \int_{M\times M} dd^c\Big[\chi\Big(\frac{|\zeta|}{r}\Big)\Big] \wedge T(\zeta) \wedge K_n(z,\zeta) \wedge w(z) \end{aligned}$$

because $dT = d^cT = 0$. To justify this calculation, first we can assume that T is class $\mathcal{C}^\infty$, even then regularize the T $\chi\big(\frac{|\zeta|}{r}\big) \subset \{|\zeta| \leqslant 2r\}$ vicinity of the bracket. Now using (15.4) and the obvious increases

$$\Big\| d\chi\Big(\frac{|\zeta|}{r}\Big)\Big\| = O\Big(\frac{1}{r}\Big), \qquad \Big\| dd^c\chi\Big(\frac{|\zeta|}{r}\Big)\Big\| = O\Big(\frac{1}{r^2}\Big),$$
$$\|d^c K_n(z,\zeta)\| = O\Big(\frac{1}{|z-\zeta|^{2n-1}}\Big), \qquad \|dd^c K_n(z,\zeta)\| = O\Big(\frac{1}{|z-\zeta|^{2n-2}}\Big), \quad n \neq 1$$

to see that the last two integrals in calculating admit $I(r)$ an increase in the $O(r^{-2})$ form. We therefore have the expected formula

$$\lim_{r\to+\infty} I(r) = \int_M T(\zeta) \wedge w(\zeta) + \int_{M\times M} T(\zeta) \wedge R_n(z,\zeta) \wedge w(z). \qquad \square$$

Demonstration of Theorem 15.3. According to Proposition 15.2 and Lemma 15.6, the current is positive Θ_T closed to growth minimum. We can construct by induction on functions k psh V_k and positive currents T_k closed minimal growth such as

$$T_0 = T, \qquad V_k = V_{T_{k-1}}, \qquad T_k = \Theta_{T_{k-1}},$$
$$dd^c V_k = T_{k-1} + T_k.$$

Perform the alternating sum of these identities. For odd indices it comes :

$$dd^c\big(V_1 - V_2 + \cdots - V_{2k} + V_{2k+1}\big) = T + T_{2k+1} \geqslant 0,$$

Lemma 15.15 and below implies that the function psh $V = V_1 - V_2 + \cdots + V_{2k+1}$ is minimal growth. According to the proposal 15.11 was the recurrence relation

$$T_{k+1}(z) = \int_M R_n(z,\zeta) \wedge T_k(\zeta).$$

now exploits the fact that R_n is a regularizing core Type convolution.

Lemma 15.12. – *It has the following properties.*

(a) To *whole* k, $1 \leqslant k < 2n$, *there are* $A_k, B_k \geqslant 0$ *constant such that for all we have* $\varepsilon \in]0,1[$

$$\|T_k(z)\| \leqslant A_k(1+|z|)^{B_k}\left[\varepsilon^{-2} + \int_{|\zeta-z|<\varepsilon\, r(z)} \frac{T(\zeta)\wedge\beta(\zeta)^{n-1}}{|\zeta-z|^{2n-k}}\right].$$

where $r(z) = C_6(1+|z|)^{-C_7}$ [*cf. lemma*15.7].

(b) To $k \geqslant 3$ T_k *the current is continuous and coefficients*

$$\|T_k(z)\| = O\big((1+|z|)^{B_k}\big).$$

Demonstration.

(A) We proceed by induction on k. Let

$$\sigma_k(z,r) = \int_{|\zeta-z|<r} T_k(\zeta)\wedge\beta(\zeta)^{n-1}.$$

It is known that the function is increasing $r \mapsto r^{2-2n}\sigma_k(z,r)$ and she admits to limit when $\int_M T_k \wedge \omega^{n-1} < +\infty$ $k \to +\infty$. Write where $T_{k+1}(z) = I_1(z) + I_2(z)$

$$I_1(z) = \int_{|\zeta-z|\geqslant\varepsilon\, r(z)} R_n(z,\zeta)\wedge T_k(\zeta),$$
$$I_2(z) = \int_{|\zeta-z|<\varepsilon\, r(z)} R_n(z,\zeta)\wedge T_k(\zeta).$$

Now we use the lemma 15.9 to estimate $I_1(z)$ and $I_2(z)$. $\|I_1(z)\|$ the standard is increased to a constant by

$$\int_{\varepsilon\, r(z)}^{+\infty} \frac{d\sigma_k(z,r)}{r^{2n}} \leqslant 2n\int_{\varepsilon\, r(z)}^{+\infty} \frac{\sigma_k(z,r)}{r^{2n+1}}\,dr$$
$$\leqslant \frac{n}{\varepsilon^2 r(z)^2}\int_M T_k\wedge\omega^{n-1} = O\big(\varepsilon^{-2}(1+|z|)^{2C_7}\big),$$

while

$$\|I_2(z)\| \leqslant C_{10}(1+|z|)^{C_{11}}\int_{|\zeta-z|<\varepsilon\, r(z)} \frac{\|T_k(\zeta)\|\ \beta(\zeta)^n}{|\zeta-z|^{2n-1}}. \tag{15.13}$$

When $k = 0$, this shows the estimate (a) $\|T_1(z)\|$. In the general case, the estimate in order k combined (15.13) results

$$\|I_2(z)\| \leqslant C_{14}(1+|z|)^{B_k+C_{11}}\big(\varepsilon^{-2}I_3(z)+I_4(z)\big)$$

with

$$I_3(z) = \int_{|\zeta-z|<\varepsilon\, r(z)} \frac{\beta(\zeta)^n}{|\zeta-z|^{2n-1}},$$

$$I_4(z) = \int_{|\zeta-z|<\varepsilon\, r(z)} \frac{\beta(\zeta)^n}{|\zeta-z|^{2n-1}} \int_{|w-\zeta|<\varepsilon\, r(\zeta)} \frac{T(w)\wedge\beta(w)^{n-1}}{|w-\zeta|^{2n-k}}.$$

For ε quite small, the inequalities $|\zeta - z| < \varepsilon\, r(z)$ and $|w-\zeta| < \varepsilon\, r(\zeta)$ involve $|w-z| < 3\varepsilon\, r(z)$. With the notation of Lemma 15.7, integral and $I_3(z)$ $I_4(z)$ So admit surcharges

$$I_3(z) \leqslant C_{15}\int_{|\zeta'|<\varepsilon\, r(z)} \frac{\beta(\zeta')^n}{|\zeta'|^{2n-1}} \leqslant C_{16}\,\varepsilon\, r(z),$$

$$I_4(z) \leqslant C_{17}\int_{|w-z|<3\varepsilon\, r(z)} T(w)\wedge\beta(w)^{n-1}\int_{\zeta'\in\mathbb{C}^n} \frac{\beta(\zeta')^n}{|\zeta'|^{2n-1}|w'-\zeta'|^{2n-k}}.$$

By homogeneity is obtained

$$\int_{\zeta'\in\mathbb{C}^n} \frac{\beta(\zeta')^n}{|\zeta'|^{2n-1}|w'-\zeta'|^{2n-k}} = \frac{C_{18}}{|w'|^{2n-k-1}} \leqslant \frac{C_{19}}{|w-z|^{2n-k-1}},$$

and estimating (a) to deduce the order $k+1$.

(B) Let us use the inequality (a) $k \geqslant 3$. He comes

$$\int_{|\zeta-z|<\varepsilon\, r(z)} \frac{\beta(\zeta)^n}{|\zeta-z|^{2n-1}} \int_{|w-\zeta|<\varepsilon\, r(\zeta)} \frac{T(w)\wedge\beta(w)^{n-1}}{|w-\zeta|^{2n-k}}.$$

The estimate (b) result. more than the full observed Previous converges uniformly to 0 when $\varepsilon \to 0$. This integral corresponds in estimating (a) to the kernel iteration R_n on $|\zeta - z| < \varepsilon\, r(z)$ balls. All the others Under contributing in T_k involve at least integrating the complementary $\{|\zeta - z| \geqslant \varepsilon\, r(z)\}$, and are due in continuous z. So T is continuous that $k \geqslant 3$. □

Demonstration of Theorem 15.3 (continued). At this point, it was therefore V built a psh function of minimal growth and current Θ positive closed continuous coefficients such that

$$dd^cV = T + \Theta, \qquad \|\Theta(z)\| = O\big((1+|z|)^{C_{20}}\big).$$

We will start by showing it can be assumed from Θ $\mathcal{C}^\infty$ class. $(\Omega_j, g_j)_{j\in\mathbb{N}}$ either an atlas locally finished M where $\Omega_j \Subset M$ where $g_j : \Omega_j \to \mathbb{C}^n$ is a biholomorphic application Ω_j on the unit ball of $\mathbb{C}^n$ and either $(\psi_j)_{j\in\mathbb{N}}$ $\mathcal{C}^\infty$ a partition of unity subject to M. There are τ_j psh functions on Ω_j such as $dd^c\tau_j = \Theta$. $\tau_j^\varepsilon = \tau_j * \rho_\varepsilon$ denote a $\mathcal{C}^\infty$ regularized family τ_j respect to map g_j, and put

$$W = \sum_{j\in\mathbb{N}} \psi_j(\tau_j - \tau_j^{\varepsilon_j}), \qquad \varepsilon_j > 0,$$

On the open Ω_k it comes

$$dd^cW - \Theta = dd^c(W - \tau_k) = dd^c\left[\sum_{j\in\mathbb{N}} \psi_j(\tau_j - \tau_k - \tau_j^{\varepsilon_j})\right],$$

and since $\tau_j - \tau_k \in \mathcal{C}^\infty(\Omega_j \cap \Omega_k)$, one sees that $dd^cW - \Theta \in \mathcal{C}^\infty_{1,1}(M)$. As the current is Θ Continuous coefficients τ_j and 1 Platforms $d\tau_j$, $d^c\tau_j$ are continuous. When ε_j are chosen small enough, therefore obtained $|W| \leqslant 1$ and $-\omega \leqslant dd^cW \leqslant \omega$ with $\omega = dd^c\log(1+|z|^2)$. The $V' = V - W + \log(1 + |z|^2)$ function is to psh minimal growth and verifies where $dd^cV' = T + \Theta'$

$$\Theta' = \Theta - dd^cW + \omega$$

is a closed positive current $\mathcal{C}^\infty$ class as $\|\Theta'(z)\| = O\big((1 + |z|)^{C_{20}}\big)$.

We suppose now that Θ is $\mathcal{C}^\infty$ class. We then applies the L^2 estimates Hörmander-Nakano-Skoda [Nak], [Sk4] Θ aware, considered a Platform $(n, 1)$ closed values the holomorphic bundle $E = T^*M \otimes \bigwedge^n TM$. the bundle cotangent T^*M is semi-positive in the sense of the metrics for Griffiths β (it is a quotient of the flat bundle $T^*\mathbb{C}^N_{|M}$), so according [DS] $T^*M \otimes \bigwedge^n T^*M$ the fiber is semi-positive in the sense Nakano. The proposal 10.1 (b) shows that the bundle itself is E semi-positive in the sense of Nakano metric $\beta\, e^{-2\psi}$ where $\psi = \log\big(\sum_{K,L} |J_{K,L}|^2\big)$. An estimated of [Sk4] applied to the adjoint bundle

$$\Big(E, \beta\, \exp\big(-2\psi - C_{21}\log(1 + |z|^2)\big)\Big),$$

one obtains the existence of a shape such that $u \in \mathcal{C}^\infty_{1,0}(M)$ and$\overline{\partial}u = \Theta$

$$\int_M |u|^2_\beta\ (1 + |z|^2)^{-C_{22}}\ \beta^n < +\infty.$$

To complete the proof of Theorem 15.3 and in particular 15.3 (d) just convert this L^2 estimate estimate L^∞

$$|u|^2_\beta \leqslant C_{23}(1 + |z|)^{-C_{24}}.$$

Given that $\overline{\partial}u = \Theta$ admits an increase in L^∞ standard, just use the INEGAity below, by placing the balls in the $|\zeta - z| < \frac{1}{2}\, r(z)$ Lemma 15.7. □

Lemma 15.14. – *Either v a function of class $\mathcal{C}^1$ in $B(r) \subset \mathbb{C}^n$ ball. So*

$$|v(0)| \leqslant \left[\frac{n!}{\pi^n r^{2n}} \int_{B(r)} |v(z)|^2\, d\lambda(z)\right]^{1/2} + \frac{4n}{2n+1}\ r \cdot \sup_{B(r)} |\overline{\partial}v|.$$

Demonstration. Apply the Cauchy formula with the rest $t \mapsto v(tz)$ function $z \in B(r)$, $t \in \mathbb{C}$, $|t| < 1$. He comes

$$v(0) = \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\theta}z)\, d\theta - \frac{1}{\pi}\int_{|t|<1} \frac{\overline{\partial}v(tz)\cdot z}{t}\, d\lambda(t),$$

$$|v(0)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |v(e^{i\theta}z)|\, d\theta + 2|z|\ \sup_{B(r)} |\overline{\partial}v|.$$

After calculating the average value (VM) for $z \in B(r)$ is obtained

$$|v(0)| \leqslant \mathrm{VM}\left[\,|v|\;;\;B(r)\right] + \frac{4n}{2n+1}\; r \cdot \sup_{B(r)} |\overline{\partial} v|,$$

and

$$\mathrm{VM}\left[\,|v|\;;\;B(r)\right] \leqslant \mathrm{VM}\left[\,|v|^2\;;\;B(r)\right]^{1/2}$$

thanks to the Cauchy-Schwarz inequality. □

It remains for us to verify the following elementary result, which was used during the demonstration.

Lemma 15.15. – *Let V_1, V_2 two psh functions minimal growth on M. It is assumed that $V = V_1 - V_2$ is psh. So V is minimal growth.*

Demonstration. According to Noether's normalization theorem, there are polynomial functions such as $f_1, \ldots, f_n$ on M $R(M) = \mathbb{C}[z_1, \ldots, z_N]/I(M)$ an entire algebra $\mathbb{C}[f_1, \ldots, f_n]$. The morphism $F = (f_1, \ldots, f_n) : M \to \mathbb{C}^n$ is proper and finished, and was coaching

$$C_{25}(1 + |z|)^{C_{26}} \leqslant |F(z)| \leqslant C_{27}(1 + |z|)^{C_{28}}$$

with $C_{25}, \ldots, C_{28} > 0$. With the obvious inequality

$$V \leqslant V_+ \leqslant F^*(F_* V_+)$$

it suffices to show that $F_* V_+$ is minimal growth in $\mathbb{C}^n$. As

$$V_+ \leqslant (V_1)_+ + (V_2)_+ - V_2,$$

is deduced for the mean value of $F_* V_+$ on the ball $B(r) \subset \mathbb{C}^n$ of center 0 the upper bound

$$\mathrm{VM}\left[F_* V_+\,;\,B(r)\right] \leqslant \mathrm{VM}\left[F_*(V_1)_+ + F_*(V_2)_+\,;\,B(r)\right] - \mathrm{VM}\left[F_* V_2\,;\,B(r)\right].$$

The $F_*(V_1)_+$ functions are $F_*(V_2)_+$ psh to minimal growth, while the $r \mapsto \mathrm{VM}[F_* V_2\,;\,;B(r)]$ function is increasing. therefore obtained a mark

$$\mathrm{VM}\left[F_* V_+\,;\,B(r)\right] \leqslant C_{29}\, \log_+ r + C_{30},$$

and the lemma is derived from the average of inequalities

$$F_* V_+(z) \leqslant VM\left[F_* V_+\,;\,B(z,r)\right] \leqslant 2^{2n}\ \mathrm{VM}\left[F_* V_+\,;\,B(0,2r)\right] \leqslant C_{31} \log_+ r + C_{32}$$

with $r := |z|$. □

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Jean-Pierre Demailly
Institut Fourier
Laboratoire de Mathématiques associé au C.N.R.S.
BP 74
38402 St Martin d'Hères Cedex

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