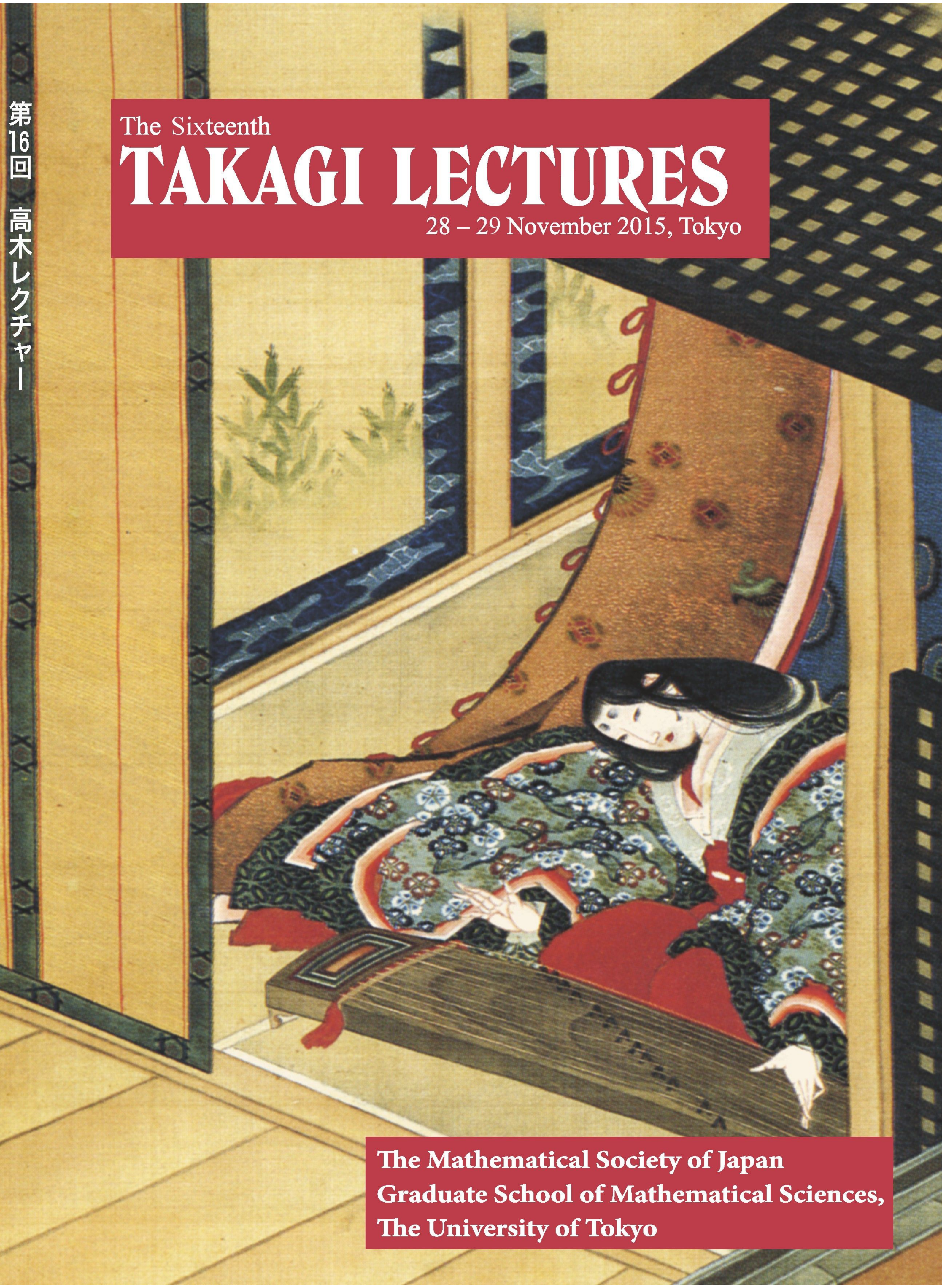
第16回 高木レクチャー

The Sixteenth

TAKAGI LECTURES

28 – 29 November 2015, Tokyo



The Mathematical Society of Japan
Graduate School of Mathematical Sciences,
The University of Tokyo

Preface to the 16th Takagi Lectures

The Takagi Lectures are expository lectures by the finest contemporary mathematicians.

The Mathematical Society of Japan (MSJ) inaugurated the Takagi Lectures as prestigious research survey lectures. The Takagi Lectures are the first series of the MSJ official lectures in mathematics to be honored with this respected Japanese mathematician's name [2]. The lectures are intended for a wide range of mathematicians, and are as a rule held twice a year. The first Takagi Lectures took place in November 2006 at Research Institute for Mathematical Sciences (RIMS), Kyoto. Since then Takagi Lectures have been delivered by the following distinguished mathematicians: P.F. Baum, Y. Benoist, S. Bloch, J.-P. Bourguignon, S. Brendle, A. Connes, É. Ghys, A. Guionnet, S. Gukov, M. Harris, M. Hopkins, U. Jannsen, V.F.R. Jones, C. Kenig, C. Khare, M. Khovanov, M. Kontsevich, L. Lafforgue, P.-L. Lions, A. Lubotzky, J. Makino, P. Malliavin, C. Manolescu, D. McDuff, J. McKernan, A. Naor, K.-H. Neeb, N.A. Nekrasov, H. Oh, H. Ooguri, S. Popa, P. Scholze, R. Seiringer, S. Smale, G. Tian, A. Venkatesh, A.M. Vershik, C. Villani, O. Viro, D.-V. Voiculescu, C. Voisin, and M. Yor.

The Takagi Lectures bear the name of the principal creator of Class Field Theory, Professor Teiji Takagi (1875–1960). In Japan, he is also known as the founder of the Japanese School of modern mathematics [1,3]. Internationally, he served as one of the first Fields Medal Committee Members in 1936 together with G.D. Birkhoff, É. Cartan, C. Carathéodory, and F. Severi.

The 16th Takagi Lectures are to be held November 28–29, 2015, at The University of Tokyo. The distinguished lecturers are F. Catanese, J.-P. Demailly, M. Kashiwara, and S.-T. Yau. The 16th Takagi Lectures will commemorate the centennial of the birth of Kunihiko Kodaira, one of the greatest mathematicians of the twentieth century.

The lecture notes of the Takagi Lectures are to be published by the *Japanese Journal of Mathematics* (JJM). It is the oldest continuously published mathematical journal in Japan (founded in 1924) and its third series was relaunched in 2006 as a mathematical journal of research survey articles of the highest scientific level in cooperation with Springer. The editors of JJM, Y. Kawahigashi, H. Nakajima, K. Ono, T. Saito, and I, also serve as the organizers of the Takagi Lectures. Videos of the lectures will be available on the Internet.

This scheme of the Takagi Lectures is intended to support the mission of continuing the advancement of mathematics, not only in Japan but throughout the world.

The Takagi Lectures are financially supported by the surplus from the International Congress of Mathematicians, which was held in Kyoto in 1990, with funding provided by the MSJ.

I would like to take this opportunity to thank the distinguished lecturers and all those who have supported our endeavors. I hope that the Takagi Lectures will gain the respect of a worldwide audience and will continue to promote future progress in mathematics.

Toshiyuki Kobayashi
The University of Tokyo
Graduate School of Mathematical Sciences
and
Kavli IPMU

Kunihiko Kodaira (1915–1997)

Biography of Kunihiko Kodaira

1915 March 16	Born in Tokyo, Japan
1935	Entered Department of Mathematics, Tokyo Imperial University
1938	Graduated from Department of Mathematics, and entered in Department of Physics
1941	Lecturer of Tokyo Imperial University (Department of Physics)
1949	Doctor of Science at the University of Tokyo (superviser: Shokichi Iyanaga)
1949	Member of the Institute for Advanced Study
1954	Awarded Fields Medal at ICM at Amsterdam
1955	Professor of Princeton University and the Institute for Advanced Study
1957	Decorated with Order of Culture, Japan
1962	Professor of Johns Hopkins University
1965	Professor of Stanford University
1965	Member of Japan Academy
1968	Professor of The University of Tokyo
1975	Professor of Gakushuin University
1978	Corresponding Member of National Academy of Arts and Sciences, USA
1985	Awarded Wolf Prize, Israel
1987	Decorated with First Order of Merit with the Sacred Treasure
1986–90	President of the International Congress of Mathematicians, Kyoto
1997 July 26	Died in Kofu City, Japan

Bibliography

[K] Special Issue of Journal of Mathematical Sciences, The University of Tokyo. In commemoration of Professor Kunihiko Kodaira's centennial birthday March 16, 2015. J. Math. Sci. Univ. Tokyo, **22** (2015), J.-P. Demailly, G. van der Geer, C. Hacon, Y. Kawamata, T. Kobayashi, Y. Miyaoka and W. Schmid (eds.).

Teiji Takagi (1875–1960)

Biography of Teiji Takagi

1875 April 21	Born in Gifu, Japan
1894	Entered the Department of Mathematics, Imperial University
1897	Entered the Graduate School of Tokyo Imperial University
1898–1901	Studied in Berlin and Göttingen
1903	Received the degree of Doctor of Science, Tokyo Imperial University
1904	Appointed Professor at Tokyo Imperial University
1920	Published his main paper on the class field theory
1925	Elected Member of the Imperial Academy of Japan
1936	Served on the 1st Fields Medal Committee
1938	Published the book *A Course on Analysis* (in Japanese)
1940	Received Culture Medal
1960 February 28	Died in Tokyo, Japan

Decorated posthumously with the Order of the Rising Sun of the First Grade

Bibliography

[1] S. Iyanaga, Chronological synopsis of the life of Teiji Takagi. In: Teiji Takagi Collected Papers, Second Enlarged Edition, Springer-Verlag Tokyo, 1990.
[2] T. Kobayashi, On the establishment of the Takagi Lectures. Japan. J. Math., **2** (2007), 145–148.
[3] K. Miyake, Teiji Takagi, Founder of the Japanese School of Modern Mathematics. Japan. J. Math., **2** (2007), 151–164.

The 16th Takagi Lectures

November 28 (Sat)--29 (Sun), 2015
Graduate School of Mathematical Sciences
The University of Tokyo, Tokyo, Japan

Program

November 28 (Sat), 2015

11:30--12:30	Registration
12:30--12:40	Opening
12:40--13:40	**Masaki Kashiwara** (RIMS, Kyoto University) *Riemann--Hilbert Correspondece for Holonomic D-modules* (I)
14:00--15:00	**Fabrizio Catanese** (Universität Bayreuth) *Kodaira Fibrations and Beyond: Methods for Moduli Theory* (I)
15:00--15:45	Coffee/Tea Break
15:45--16:45	**Jean-Pierre Demailly** (Université de Grenoble I) *Recent Progress towards the Kobayashi and Green--Griffiths--Lang Conjectures* (I)
17:05--18:05	**Shing-Tung Yau** (Harvard University) *From Riemann and Kodaira to Modern Developments on Complex Manifolds* (I)

November 29 (Sun), 2015

10:00--11:00	**Masaki Kashiwara** (RIMS, Kyoto University) *Riemann--Hilbert Correspondece for Holonomic D-modules* (II)
11:20--12:20	**Fabrizio Catanese** (Universität Bayreuth) *Kodaira Fibrations and Beyond: Methods for Moduli Theory* (II)
12:20--14:00	Lunch Break
14:00--15:00	**Jean-Pierre Demailly** (Université de Grenoble I) *Recent Progress towards the Kobayashi and Green--Griffiths--Lang Conjectures* (II)
15:20--16:20	**Shing-Tung Yau** (Harvard University) *From Riemann and Kodaira to Modern Developments on Complex Manifolds* (II)
16:30--17:30	Workshop closing with drinks

Organizing Committee
Y. Kawahigashi, T. Kobayashi, H. Nakajima, K. Ono, T. Saito

The Mathematical Society of Japan
Graduate School of Mathematical Sciences, The University of Tokyo

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Riemann–Hilbert correspondence for irregular holonomic $\mathscr{D}$-modules*

Masaki Kashiwara**

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Communicated by: Toshiyuki Kobayashi

Abstract. This is a survey paper on the Riemann–Hilbert correspondence on (irregular) holonomic $\mathscr{D}$-modules, based on the 16th Takagi Lectures (2015/11/28). In this paper, we use subanalytic sheaves, an analogous notion to the one of indsheaves.

Keywords and phrases: irregular Riemann–Hilbert problem, irregular holonomic $\mathscr{D}$-modules, ind-sheaves, subanalytic sheaves, Stokes phenomenon

Mathematics Subject Classification (2010): 32C38, 35A27, 32S60

Contents

* This article is based on the 16th Takagi Lectures that the author delivered at the University of Tokyo on November 28 and 29, 2015.

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M. Kashiwara
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
(e-mail: `masaki@kurims.kyoto-u.ac.jp`)

Introduction

The classical Riemann–Hilbert problem asks for the existence of a linear ordinary differential equation with regular singularities and a given monodromy on a curve.

Pierre Deligne ([De70]) formulated it as a correspondence between integrable connections with regular singularities on a complex manifold X with a pole on a hypersurface Y and local systems on $X \setminus Y$.

Later the author constructed an equivalence of triangulated categories between $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$, the derived category of $\mathscr{D}_X$-modules with regular holonomic cohomologies, and $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$, the derived category of sheaves on X with $\mathbb{C}$-constructible cohomologies ([Ka80, Ka84]). The equivalence is given by the solution functor

$$\mathscr{S}ol_X \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)^{\mathrm{op}}.$$

Here $\mathscr{S}ol_X(\mathscr{M}) = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. Note that $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ is self-dual by the duality functor.

However, it was a long-standing problem to generalize it to the (not necessarily regular) holonomic $\mathscr{D}$-module case. One of the difficulties was that we could not find an appropriate substitute of the target category $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. Recently, the author solved it jointly with Andrea D'Agnolo ([DK13]) by using an enhanced version of indsheaves.

There are two ingredients for the solution.

One is the notion of indsheaves. This notion was introduced with Pierre Schapira in [KS01] to treat "sheaves" of functions with tempered growth, such as $\mathscr{D}b^{\mathrm{t}}$ of tempered distributions or $\mathscr{O}^{\mathrm{t}}$ of tempered holomorphic functions.

The other ingredient is adding an extra variable. We consider indsheaves on $M \times \mathbb{R}$, not on the base manifold M. This method was originally introduced by Dmitry Tamarkin ([Ta08]) in order to treat non-homogeneous Lagrangian submanifolds of the cotangent bundle in the framework of sheaf theory. In our context, this method affords an appropriate language to capture various growth of solutions at singular points.

Among the results used in the course of the proof is the description of the structure of flat connections due to Takuro Mochizuki ([Mo09, Mo11]) and Kiran S. Kedlaya ([Ke10, Ke11]).

In this survey paper, we explain an outline of the irregular Riemann–Hilbert problem. We use here, instead of the notion of indsheaves, the analogous notion of "subanalytic sheaves".

For a complex manifold X, we construct a triangulated category $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$, called the triangulated category of enhanced subanalytic sheaves, a fully faithful functor $e\colon \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$ and its left quasi-inverse $\mathscr{H}om^{\mathsf{E}}(\mathbb{C}^{\mathsf{T}}_X, \bullet)\colon \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$. Next we construct $\mathscr{O}^{\mathsf{T}}_X \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$, the enhanced subanalytic sheaf of tempered holomorphic functions such that $\mathscr{H}om^{\mathsf{E}}(\mathbb{C}^{\mathsf{T}}_X, \mathscr{O}^{\mathsf{T}}_X) \simeq \mathscr{O}_X$. By using $\mathscr{O}^{\mathsf{T}}_X$ instead of $\mathscr{O}_X$, we define the enhanced solution functor from the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ of $\mathscr{D}_X$-modules to the category $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$ of enhanced subanalytic sheaves by

$$\mathscr{S}ol^{\mathsf{T}}_X(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}^{\mathsf{T}}_X) \quad \text{for } \mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X).$$

Restricting it to $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$, the subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ consisting of complexes with holonomic cohomologies, we obtain a fully faithful functor

$$\mathscr{S}ol^{\mathsf{T}}_X\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) \rightarrowtail \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})^{\mathrm{op}}.$$

Furthermore, we have an isomorphism

$$\mathscr{H}om^{\mathsf{E}}(\mathscr{S}ol^{\mathsf{T}}_X(\mathscr{M}), \mathscr{O}^{\mathsf{T}}_X) \simeq \mathscr{M} \quad \text{for any } \mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X).$$

Thus we obtain a quasi-commutative diagram:

$$\begin{tikzcd}[column sep=huge]
\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X) \arrow[r, "\sim", "\mathscr{S}ol_X"'] \arrow[d, tail] \arrow[rr, bend left, "\mathrm{id}"] & \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)^{\mathrm{op}} \arrow[r, "\sim", "\mathrm{R}\mathscr{H}om(\ast\,,\,\mathscr{O}^{\mathrm{t}}_X)"'] \arrow[d, tail, "e"] & \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X) \arrow[d, tail] \\
\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) \arrow[r, "\mathscr{S}ol^{\mathsf{T}}_X"'] \arrow[rr, bend right, "\text{canonical}"'] & \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})^{\mathrm{op}} \arrow[r, "\mathscr{H}om^{\mathsf{E}}(\ast\,,\,\mathscr{O}^{\mathsf{T}}_X)"'] & \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X).
\end{tikzcd}$$

This paper is organized as follows. In the first section, we review the local theory of linear ordinary differential equations. In the next sections, we shall review sheaves, $\mathscr{D}$-modules and subanalytic sheaves. After introducing the subanalytic sheaves of tempered distributions and that of tempered holomorphic functions in §4, we define the enhanced version of the de Rham functor and solution functor. Then, in §6, we state our main theorems by using these functors. In the next section §7, we give a very brief outline of the proof of the main theorems by using the results of T. Mochizuki and K.S. Kedlaya.

In the last section §8, we explain how Proposition 1.1 on the Stokes phenomena in the one-dimensional case can be interpreted in terms of the enhanced solution functors.

We refer the reader to [DK 13, KS 14, KS 15, DK 15] for a more detailed theory. Remark that the description of the Riemann–Hilbert correspondence in this paper is different from that of loc. cit. in the following points.

(a) We use in loc. cit. indsheaves instead of subanalytic sheaves. Since the category of subanalytic sheaves can be embedded into that of indsheaves, these two descriptions are almost equivalent.
(b) In loc. cit., the category $\mathsf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M)$ of enhanced indsheaves is defined as a quotient category of the category $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty})$ of indsheaves on $M \times \mathbb{R}_\infty$. However, $\mathsf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M)$ can be also embedded into $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty})$ by the right adjoint $\mathrm{R}^{\mathsf{E}}\colon \mathsf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty})$ of the quotient functor. In our paper, we use the subanalytic sheaf version of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M\times\mathbb{R}_\infty})$ instead of $\mathsf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_M)$ by using the embedding R^{E}.

1. Linear ordinary differential equations

1.1. One dimensional case

Let us recall the local theory of linear ordinary differential equations. Let $X \subset \mathbb{C}$ be an open subset with $0 \in X$ and let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module such that $\mathrm{SingSupp}(\mathscr{M}) \subset \{0\}$ and $\mathscr{M} \simeq \mathscr{M}(\ast\{0\}) := \mathscr{O}_X(\ast\{0\}) \otimes_{\mathscr{O}_X} \mathscr{M}$. Here

$\mathscr{O}_X(*\{0\})$ is the sheaf of meromorphic functions with possible poles at 0. It is equivalent to saying that $\mathscr{M}$ is a $\mathscr{D}_X$-module which is locally isomorphic to $\mathscr{O}_X(*\{0\})^r$ for some $r \in \mathbb{Z}_{\geqslant 0}$ as an $\mathscr{O}_X$-module. Let us take a system of generators $\{u_1, \dots, u_r\}$ of $\mathscr{M}$ as a free $\mathscr{O}_X(*\{0\})$-module on a neighborhood of 0. Then, writing $\vec{\boldsymbol{u}}$ for the column vector with these generators as components, we have

$$\frac{d}{dz}\vec{\boldsymbol{u}} = A(z)\vec{\boldsymbol{u}} \tag{1.1}$$

for some $A(z) \in \mathrm{Mat}_r(\mathscr{O}_X(*\{0\}))$, i.e., for an $(r \times r)$-matrix $A(z)$ whose components are in $\mathscr{O}_X(*\{0\})$. Then for any $\mathscr{D}_X$-module $\mathscr{L}$ such that $\mathscr{L} \simeq \mathscr{L}(*\{0\})$, we have

$$\begin{aligned}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{L}) = \{\vec{u} \in \mathscr{L}^r \;;\; &\vec{u} \text{ satisfies the same differential}\\ &\text{equation as (1.1)}\},\end{aligned}$$

where we associate to $\vec{u}$ the morphism from $\mathscr{M}$ to $\mathscr{L}$ defined by $\vec{\boldsymbol{u}} \mapsto \vec{u}$.

1.2. *Regular singularities*

If we can choose a system of generators $\{u_1, \dots, u_r\}$ of $\mathscr{M}$ such that $zA(z)$ has no pole at 0, then we say that 0 is a regular singularity of $\mathscr{M}$, or $\mathscr{M}$ is regular. In such a case, there are r linearly independent solutions of the form

$$\vec{u}_j = z^{\lambda_j} \sum_{s=0}^{r-1} \vec{a}_{j,s}(z)(\log z)^s \quad (j = 1, \dots, r),$$

where $\vec{a}_{j,s}(z)$ is a vector of holomorphic functions defined on a neighborhood of 0. Hence, after a change of generators $\vec{\boldsymbol{v}} = D(z)\vec{\boldsymbol{u}}$ with some invertible matrix $D(z) \in \mathrm{GL}_r(\mathscr{O}_X(*\{0\}))$, the new variable $\vec{\boldsymbol{v}}$ satisfies the equation

$$z\frac{d}{dz}\vec{\boldsymbol{v}} = C\,\vec{\boldsymbol{v}}$$

for some constant matrix $C \in \mathrm{Mat}_r(\mathbb{C})$. Then, by reducing C to a Jordan form, we see that $\mathscr{M}$ is isomorphic to a direct sum of $\mathscr{D}_X$-modules $\mathscr{D}_X(*\{0\})/\mathscr{D}_X(*\{0\})\Big(z\dfrac{d}{dz} - \lambda\Big)^{m+1}$ with $\lambda \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geqslant 0}$. Note that

$$\mathscr{D}_X(*\{0\})/\mathscr{D}_X(*\{0\})\Big(z\frac{d}{dz} - \lambda\Big)^{m+1} \simeq \mathscr{D}_X/\mathscr{D}_X\Big(z\frac{d}{dz} - \lambda - k\Big)^{m+1}$$

for any $k \in \mathbb{Z}$ such that $\lambda + k \notin \mathbb{Z}_{\geqslant 0}$.

Recall that the solution sheaf of $\mathscr{M}$ is defined by

$$\mathscr{S}ol_X(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X).$$

Then the local system on $X \setminus \{0\}$

$$L := \mathscr{S}ol_X(\mathscr{M})|_{X\setminus\{0\}} = \left\{\vec{u} \in (\mathscr{O}_{X\setminus\{0\}})^r \,;\, \frac{d}{dz}\vec{u} = A(z)\vec{u}\right\} \tag{1.2}$$

has the monodromy $\exp(2\pi\sqrt{-1}\,C)$. Hence L completely determines $\mathscr{M}$.

1.3. Irregular singularities

In the irregular case, we have the following results on the solutions of the ordinary linear differential equation (1.1):

(i) there exist linearly independent r formal solutions $\widehat{\boldsymbol{u}}_j$ ($j = 1, \dots, r$) of (1.1) with the form

$$\widehat{\boldsymbol{u}}_j = \mathrm{e}^{\varphi_j(z)} z^{\lambda_j} \sum_{s=0}^{r-1} \vec{a}_{j,s}(z)(\log z)^s,$$

where $\varphi_j(z) \in z^{-1/m}\mathbb{C}[z^{-1/m}]$ for some $m \in \mathbb{Z}_{>0}$, $\lambda_j \in \mathbb{C}$, and

$$\vec{a}_{j,s}(z) = \sum_{n\in m^{-1}\mathbb{Z}_{\geqslant 0}} \vec{a}_{j,s,n} z^n \in \mathbb{C}[[z^{1/m}]]^r \text{ with } \vec{a}_{j,s,n} \in \mathbb{C}^r,$$

(ii) for any $\theta_0 \in \mathbb{R}$ and each $j = 1, \dots, r$, there exist an angular neighborhood

$$D_{\theta_0} = \{z = r\mathrm{e}^{i\theta} \,;\, |\theta - \theta_0| < \varepsilon \text{ and } 0 < r < \delta\} \tag{1.3}$$

for sufficiently small $\varepsilon, \delta > 0$ and a holomorphic (column) solution $\boldsymbol{u}_j \in \mathscr{O}_X(D_{\theta_0})^r$ of (1.1) defined on D_{θ_0} such that

$$\boldsymbol{u}_j \sim \widehat{\boldsymbol{u}}_j,$$

in the following sense: for any $N > 0$, there exists $C > 0$ such that

$$|\boldsymbol{u}_j(z) - \widehat{\boldsymbol{u}}_j^N(z)| \leqslant C|\mathrm{e}^{\varphi_j(z)} z^{\lambda_j+N}| = C\mathrm{e}^{\mathrm{Re}(\varphi_j(z))}|z^{\lambda_j+N}|, \tag{1.4}$$

where $\widehat{\boldsymbol{u}}_j^N(z)$ is the finite partial sum

$$\widehat{\boldsymbol{u}}_j^N(z) = \mathrm{e}^{\varphi_j(z)} z^{\lambda_j} \sum_{s=0}^{r-1} \sum_{\substack{n\in m^{-1}\mathbb{Z}_{\geqslant 0},\\ n\leqslant N}} \vec{a}_{j,s,n} z^n (\log z)^s.$$

Here we choose branches of $z^{1/m}$ and $\log z$ on D_{θ_0}.

Note that a holomorphic solution $\boldsymbol{u}_j$ is not uniquely determined by the formal solution $\widehat{\boldsymbol{u}}_j$. Indeed, $\boldsymbol{u}_j + \sum_{k\neq j} c_k \boldsymbol{u}_k$ also satisfies the same estimate (1.4) whenever

$$\operatorname{Re}(\varphi_k(z)) < \operatorname{Re}(\varphi_j(z)) \text{ on } D_{\theta_0} \text{ if } c_k \neq 0.$$

1.4. Stokes phenomena

We choose another sufficiently small angular domain D_{θ_1} such that $D_{\theta_0} \cap D_{\theta_1} \neq \varnothing$, and, for each j, we take a holomorphic solution $\boldsymbol{u}'_j$ defined on D_{θ_1} and with the asymptotic behavior (1.4) on D_{θ_1}. Then we can write

$$\boldsymbol{u}'_j = \sum_k a_{j,k} \boldsymbol{u}_k \quad \text{on } D_{\theta_0} \cap D_{\theta_1}$$

with $a_{j,k} \in \mathbb{C}$. Note that

$$\operatorname{Re}(\varphi_k(z)) \leqslant \operatorname{Re}(\varphi_j(z)) \text{ on } D_{\theta_0} \cap D_{\theta_1} \text{ if } a_{j,k} \neq 0. \tag{1.5}$$

The matrix $(a_{j,k})_{1\leqslant j,k\leqslant r}$ is called the Stokes matrix. If we cover a neighborhood of $\{0\}$ by such angular domains, then a pair of adjacent angular domains gives a Stokes matrix, and thus we obtain a family of matrices satisfying (1.5).

Conversely, we can find a holonomic $\mathscr{D}$-module $\mathscr{M}$ whose Stokes matrices are a given family of matrices satisfying (1.5).

1.5. Stokes filtrations

Deligne [DMR07] interpreted these results as follows (see also Malgrange [DMR07] and Sabbah [Sa00, Sa13]).

Let $\varpi\colon \widetilde{X} \to X$ be the real blow up of X along $\{0\}$ defined in §7.1 below. Namely,

$$\widetilde{X} := \{(r,\zeta) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}\,;\, |\zeta| = 1, r\zeta \in X\} \quad \text{and} \quad \varpi(r,\zeta) = r\zeta.$$

Recall that $L = (\mathscr{S}ol_X \mathscr{M})|_{X\setminus\{0\}}$. Let $S := \varpi^{-1}(0)$ and $j\colon X\setminus\{0\} \to \widetilde{X}$ and set

$$\widetilde{L} = (j_* L)|_S. \tag{1.6}$$

Then $\widetilde{L}$ is a local system on S of rank r.

For the sake of simplicity, we assume that m in §1.3 (i) is equal to 1.

Set $\Phi = (\mathscr{O}_X(*\{0\})/\mathscr{O}_X)_0$. For $\mathrm{e}^{i\theta_0} \in S$ and $\varphi, \psi \in \Phi$, we write $\varphi \underset{\mathrm{e}^{i\theta_0}}{\preceq} \psi$ if there exists $c \in \mathbb{R}$ such that $\operatorname{Re}(\tilde{\varphi}(r\mathrm{e}^{i\theta})) \leqslant \operatorname{Re}(\tilde{\psi}(r\mathrm{e}^{i\theta})) + c$ for $0 < r \ll 1$

and $|\theta - \theta_0| \ll 1$ and representatives $\tilde{\varphi}, \tilde{\psi} \in \mathscr{O}_X(*\{0\})_0$ of φ and ψ. Then $\underset{\mathrm{e}^{i\theta_0}}{\preceq}$ is an order on Φ.

For $\varphi \in \Phi$ and $\mathrm{e}^{i\theta} \in S$, we set

$$(F_\varphi)_{\mathrm{e}^{i\theta}} = \left\{ u(z) \in (\widetilde{L})_{\mathrm{e}^{i\theta}} \ ; \ \begin{array}{l} |u(z)| \leqslant C|z^{-M}\mathrm{e}^{\varphi(z)}| \text{on a neighborhood of} \\ \mathrm{e}^{i\theta} \text{for some } C > 0 \text{ and } M \in \mathbb{Z}_{>0} \end{array} \right\}.$$

Then $\{F_\varphi\}_{\varphi\in\Phi}$ satisfies the following conditions by the properties of the solutions explained in §1.3:

(i) $\{F_\varphi\}_{\varphi\in\Phi}$ is a filtration of $\widetilde{L}$, namely,
  (a) F_φ is a subsheaf of $\widetilde{L}$ for any $\varphi \in \Phi$,
  (b) $\widetilde{L} = \sum_{\varphi\in\Phi} F_\varphi$,
  (c) $(F_\varphi)_{\mathrm{e}^{i\theta}} \subset (F_\psi)_{\mathrm{e}^{i\theta}}$ if $\varphi \underset{\mathrm{e}^{i\theta}}{\preceq} \psi$,

(ii) for any $\mathrm{e}^{i\theta_0} \in S$, there exist an open neighborhood U of $\mathrm{e}^{i\theta_0}$, a finite subset I of Φ and a constant subsheaf H_φ $(\varphi \in I)$ of $\widetilde{L}|_U$ such that
  (a) $\widetilde{L}|_U = \bigoplus_{\varphi\in I} H_\varphi$,
  (b) for any $\mathrm{e}^{i\theta} \in U$ and $\varphi \in \Phi$, we have

$$(F_\varphi)_{\mathrm{e}^{i\theta}} = \bigoplus_{\psi\in I,\ \psi \underset{\mathrm{e}^{i\theta}}{\preceq} \varphi} (H_\psi)_{\mathrm{e}^{i\theta}}.$$

If the above conditions are satisfied we say that $\{F_\varphi\}_{\varphi\in\Phi}$ is a Stokes filtration of the local system $\widetilde{L}$. Also in case $m > 1$, we can define the notion of Stokes filtration with a suitable modification.

Proposition 1.1. *The category of holonomic $\mathscr{D}_X$-module $\mathscr{M}$ such that*

$$\mathrm{SingSupp}(\mathscr{M}) \subset \{0\} \textit{ and } \mathscr{M} \simeq \mathscr{M}(*\{0\})$$

is equivalent to the category of pairs $(L, \{F_\varphi\})$ of a local system L on $X \setminus \{0\}$ and a Stokes filtration $\{F_\varphi\}$ on $\widetilde{L} := (j_*L)|_S$.

In order to generalize this result to holonomic $\mathscr{D}$-modules in the several dimension case, we use enhanced subanalytic sheaves. In the next sections, we shall review sheaves, $\mathscr{D}$-modules and subanalytic sheaves.

2. A brief review on sheaves and $\mathscr{D}$-modules

2.1. Sheaves

We refer to [KS90] for all notions of sheaf theory used here. For simplicity, we take the complex number field $\mathbb{C}$ as the base field, although most of the results would remain true when $\mathbb{C}$ is replaced with a commutative ring of finite global dimension.

A topological space is *good* if it is Hausdorff, locally compact, countable at infinity and has finite flabby dimension.

One denotes by $\mathrm{Mod}(\mathbb{C}_M)$ the abelian category of sheaves of $\mathbb{C}$-vector spaces on a good topological space M and by $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_M)$ its bounded derived category. Note that $\mathrm{Mod}(\mathbb{C}_M)$ has a finite homological dimension.

For a locally closed subset A of M, one denotes by $\mathbb{C}_A$ the constant sheaf on A with stalk $\mathbb{C}$ extended by 0 on $X \setminus A$.

One denotes by $\mathrm{Supp}(F)$ the support of F.

There are many formulas concerning the six operations. For example, we have the formulas below in which $F, F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_M)$, $G, G_1, G_2 \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_N)$:

$$
\begin{aligned}
&\mathrm{R}\mathscr{H}om(F_1 \otimes F_2, F) \simeq \mathrm{R}\mathscr{H}om(F_1, \mathrm{R}\mathscr{H}om(F_2, F)),\\
&\mathrm{R}f_*\mathrm{R}\mathscr{H}om(f^{-1}G, F) \simeq \mathrm{R}\mathscr{H}om(G, \mathrm{R}f_*F),\\
&\mathrm{R}f_!\,(F \otimes f^{-1}G) \simeq (\mathrm{R}f_!\,F) \otimes G \quad \text{(projection formula)},\\
&f^{\,!}\,\mathrm{R}\mathscr{H}om(G_1, G_2) \simeq \mathrm{R}\mathscr{H}om(f^{-1}G_1, f^{\,!}\,G_2),
\end{aligned}
\tag{2.1}
$$

and for a Cartesian square of good topological spaces,

$$
\begin{tikzcd}
M' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & N' \arrow[d, "g"] \\
M \arrow[r, "f"] & N
\end{tikzcd}
$$

we have the *base change formulas*

$$
g^{-1}\mathrm{R}f_! \simeq \mathrm{R}f'_!\,g'^{-1} \quad \text{and} \quad g^{\,!}\,\mathrm{R}f_* \simeq \mathrm{R}f'_*\,g'^{\,!}. \tag{2.2}
$$

2.2. *$\mathscr{D}$-modules*

References for $\mathscr{D}$-module theory are made to [Ka03]. See also [Ka70, Ka75, Ka78, KK81, Bj93, HTT08]. Here, we shall briefly recall some basic constructions in the theory of $\mathscr{D}$-modules.

Let $(X, \mathscr{O}_X)$ be a *complex* manifold. We denote by

- d_X the complex dimension of X,
- Ω_X the invertible $\mathscr{O}_X$-module of differential forms of top degree,
- $\Omega_{X/Y}$ the invertible $\mathscr{O}_X$-module $\Omega_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}(\Omega_Y^{\otimes -1})$ for a morphism $f\colon X \to Y$ of complex manifolds,
- Θ_X the sheaf of holomorphic vector fields,
- $\mathscr{D}_X$ the sheaf of algebras of finite-order differential operators.

Denote by $\mathrm{Mod}(\mathscr{D}_X)$ the abelian category of left $\mathscr{D}_X$-modules and by $\mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}})$ that of right $\mathscr{D}_X$-modules. There is an equivalence

$$\text{(2.3)} \qquad \mathrm{r}\colon \mathrm{Mod}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}), \quad \mathscr{M} \longmapsto \mathscr{M}^{\mathrm{r}} := \Omega_X \otimes_{\mathscr{O}_X} \mathscr{M}.$$

By this equivalence, it is enough to study left $\mathscr{D}_X$-modules.

The ring $\mathscr{D}_X$ is coherent and one denotes by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ the thick abelian subcategory of $\mathrm{Mod}(\mathscr{D}_X)$ consisting of coherent modules.

To a coherent $\mathscr{D}_X$-module $\mathscr{M}$ one associates its characteristic variety $\mathrm{char}(\mathscr{M})$, a closed $\mathbb{C}^\times$-conic *co-isotropic* (one also says *involutive*) $\mathbb{C}$-analytic subset of the cotangent bundle T^*X. The involutivity property is a central theorem of the theory and is due to [SKK73]. A purely algebraic proof was obtained later in [Gabb81].

If $\mathrm{char}(\mathscr{M})$ is Lagrangian, $\mathscr{M}$ is called *holonomic*. It is immediately checked that the full subcategory $\mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X)$ of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of holonomic $\mathscr{D}$-modules is a thick abelian subcategory.

A $\mathscr{D}_X$-module $\mathscr{M}$ is *quasi-good* if, for any relatively compact open subset $U \subset X$, there is a filtrant family $\{\mathscr{F}_i\}_i$ of coherent $(\mathscr{O}_X|_U)$-submodules of $\mathscr{M}|_U$ such that $\mathscr{M}|_U = \sum_i \mathscr{F}_i$. Here, a family $\{\mathscr{F}_i\}_i$ is filtrant if, for any i, i', there exists i'' such that $\mathscr{F}_i + \mathscr{F}_{i'} \subset \mathscr{F}_{i''}$.

A $\mathscr{D}_X$-module $\mathscr{M}$ is *good* if it is quasi-good and coherent. The subcategories of $\mathrm{Mod}(\mathscr{D}_X)$ consisting of quasi-good (resp. good) $\mathscr{D}_X$-modules are abelian and thick. Therefore, one has the triangulated categories

- $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X) = \{\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\,;\, H^j(\mathscr{M}) \text{ is coherent for all } j \in \mathbb{Z}\}$,
- $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) = \{\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\,;\, H^j(\mathscr{M}) \text{ is holonomic for all } j \in \mathbb{Z}\}$,
- $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X) = \{\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\,;\, H^j(\mathscr{M}) \text{ is regular holonomic for all } j \in \mathbb{Z}\}$,
- $\mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X) = \{\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\,;\, H^j(\mathscr{M}) \text{ is quasi-good for all } j \in \mathbb{Z}\}$,
- $\mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X) = \{\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\,;\, H^j(\mathscr{M}) \text{ is good for all } j \in \mathbb{Z}\}$.

One may also consider the unbounded derived categories $\mathsf{D}(\mathscr{D}_X)$, $\mathsf{D}^+(\mathscr{D}_X)$ and $\mathsf{D}^-(\mathscr{D}_X)$ and their full triangulated subcategories consisting of objects with coherent, holonomic, regular holonomic, quasi-good and good cohomologies.

We have the functors

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\,\bullet\,,\,\bullet\,) &: \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathsf{D}^+(\mathbb{C}_X), \\ \bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \bullet &: \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}) \times \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X). \end{aligned}$$

We also have the functor

$$\bullet \overset{\mathrm{D}}{\otimes} \bullet : \mathsf{D}^-(\mathscr{D}_X) \times \mathsf{D}^-(\mathscr{D}_X) \to \mathsf{D}^-(\mathscr{D}_X)$$

constructed as follows. For $\mathscr{D}_X$-modules $\mathscr{M}$ and $\mathscr{N}$, the tensor product $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ is endowed with a structure of $\mathscr{D}_X$-module by

$$v(s \otimes t) = (vs) \otimes t + s \otimes (vt) \quad \text{for } v \in \Theta_X,\, s \in \mathscr{M} \text{ and } t \in \mathscr{N}.$$

The functor $\bullet \overset{\mathrm{D}}{\otimes} \bullet$ is its left derived functor. One defines the duality functor for $\mathscr{D}$-modules by setting

$$\mathbb{D}_X \mathscr{M} = \mathrm{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})[d_X] \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \qquad \text{for } \mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X).$$

Now, let $f : X \to Y$ be a morphism of complex manifolds. The *transfer bimodule* $\mathscr{D}_{X\to Y}$ is a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule defined as follows. As an $(\mathscr{O}_X, f^{-1}\mathscr{D}_Y)$-bimodule, $\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$. The left $\mathscr{D}_X$-module structure of $\mathscr{D}_{X\to Y}$ is given by

$$v(a \otimes P) = v(a) \otimes P + \sum_i a a_i \otimes w_i P,$$

where $v \in \Theta_X$ and $\sum_i a_i \otimes w_i$ is its image in $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Theta_Y$.

One also uses the opposite transfer bimodule $\mathscr{D}_{Y\leftarrow X} = f^{-1}\mathscr{D}_Y \otimes_{f^{-1}\mathscr{O}_Y} \Omega_{X/Y}$, an $(f^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodule.

Note that for another morphism of complex manifolds $g : Y \to Z$, one has the natural isomorphisms

$$\mathscr{D}_{X\longrightarrow Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{D}_{Y\longrightarrow Z} \simeq \mathscr{D}_{X\longrightarrow Z},$$
$$f^{-1}\mathscr{D}_{Z\longleftarrow Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} \mathscr{D}_{Y\longleftarrow X} \simeq \mathscr{D}_{Z\longleftarrow X}.$$

One can now define the external operations on $\mathscr{D}$-modules by setting:

$$\mathrm{D}f^*\mathscr{N} := \mathscr{D}_{X\longrightarrow Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{N} \quad \text{for } \mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y),$$
$$\mathrm{D}f_!\mathscr{M} := \mathrm{R}f_!\,(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\longrightarrow Y}) \quad \text{for } \mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}),$$

and one defines $\mathrm{D}f_*\mathscr{M}$ by replacing $\mathrm{R}f_!$ with $\mathrm{R}f_*$ in the above formula. By using the opposite transfer bimodule $\mathscr{D}_{Y\leftarrow X}$ one defines similarly the inverse image of a right $\mathscr{D}_Y$-module or the direct images of a left $\mathscr{D}_X$-module.

One calls respectively $\mathrm{D}f^*$, $\mathrm{D}f_*$ and $\mathrm{D}f_!$ the inverse image, direct image and proper direct image functors in the category of $\mathscr{D}$-modules.

Note that

$$\mathrm{D}f^*\mathscr{O}_Y \simeq \mathscr{O}_X, \quad \mathrm{D}f^*\Omega_Y \simeq \Omega_X.$$

Also note that the property of being quasi-good is stable by inverse image and tensor product, as well as by direct image by maps proper on the support of the module. The property of being good is stable by duality.

Let $f\colon X \to Y$ be a morphism of complex manifolds. One associates the maps

$$\begin{tikzcd} T^*X \arrow[dr, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d] & T^*Y \arrow[d, "\pi_Y"] \\ & X \arrow[r, "f"] & Y. \end{tikzcd}$$

One says that f is non-characteristic for $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ if the map f_d is proper (hence, finite) on $f_\pi^{-1}(\mathrm{char}(\mathscr{N}))$.

The classical de Rham and solution functors are defined by

$$\begin{aligned} \mathscr{DR}_X\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) &\longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), & \mathscr{M} &\longmapsto \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}, \\ \mathscr{S}ol_X\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) &\longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)^{\mathrm{op}}, & \mathscr{M} &\longmapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X). \end{aligned}$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, one has

$$\mathscr{S}ol_X(\mathscr{M}) \simeq \mathscr{DR}_X(\mathbb{D}_X\mathscr{M})[-d_X]. \tag{2.4}$$

Let us list up the relations of the de Rham functors with the inverse and direct image functors.

Theorem 2.1 (Projection formulas [Ka03, Theorems 4.2.8, 4.40]). *Let $f\colon X \to Y$ be a morphism of complex manifolds. For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\mathrm{op}})$, there are natural isomorphisms*:

$$\begin{aligned} \mathrm{D}f_!(\mathrm{D}f^*\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{M}) &\simeq \mathscr{L} \overset{\mathrm{D}}{\otimes} \mathrm{D}f_!\mathscr{M}, \\ \mathrm{R}f_!\,(\mathrm{D}f^*\mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}) &\simeq \mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathrm{D}f_!\mathscr{M}. \end{aligned}$$

In particular, there is an isomorphism (*commutation of the de Rham functor and direct images*)

$$\mathrm{R}f_!\,(\mathscr{DR}_X(\mathscr{M})) \simeq \mathscr{DR}_Y(\mathrm{D}f_!\mathscr{M}).$$

Theorem 2.2 (Commutation with duality [Ka03, Sc86]). *Let $f\colon X \to Y$ be a morphism of complex manifolds.*

(i) *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ and assume that* $\mathrm{Supp}(\mathscr{M})$ *is proper over Y. Then $\mathrm{D}f_!\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_Y)$, and $\mathbb{D}_Y(\mathrm{D}f_!\mathscr{M}) \simeq \mathrm{D}f_!\mathbb{D}_X\mathscr{M}$.*

(ii) *If f is non-characteristic for $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$, then $\mathrm{D}f^*\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and $\mathbb{D}_X(\mathrm{D}f^*\mathscr{N}) \simeq \mathrm{D}f^*\mathbb{D}_Y\mathscr{N}$.*

Corollary 2.3. *Let $f\colon X \to Y$ be a morphism of complex manifolds.*

(i) *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ and assume that* $\mathrm{Supp}(\mathscr{M})$ *is proper over Y. Then we have the isomorphism for $\mathscr{N} \in \mathsf{D}(\mathscr{D}_Y)$:*

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{D}f^*\mathscr{N})\,[d_X] \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathrm{D}f_*\mathscr{M}, \mathscr{N})\,[d_Y].$$

In particular, with the same hypotheses, we have the isomorphism (*commutation of the Sol functor and direct images*)

$$\mathrm{R}f_*\mathscr{S}ol_X(\mathscr{M})\,[d_X] \simeq \mathscr{S}ol_Y(\mathrm{D}f_*\mathscr{M})\,[d_Y].$$

(ii) *Let $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ and assume that f is non-characteristic for $\mathscr{N}$. Then we have the isomorphism for $\mathscr{M} \in \mathsf{D}(\mathscr{D}_X)$:*

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathrm{D}f^*\mathscr{N}, \mathscr{M})[d_X] \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N}, \mathrm{D}f_*\mathscr{M})[d_Y].$$

3. Subanalytic sheaves

3.1. Subanalytic spaces

Let M be a real analytic manifold. On M there is the family of subanalytic subsets due to Hironaka ([Hi73]) and Gabrielov ([Gabr68]) (see [BM88, VD98] for an exposition). This family is the smallest family of subsets of M which satisfies the following properties:

(a) for any real analytic manifold N and any proper morphism $f\colon N \to M$, the image of N is subanalytic,
(b) the intersection of two subanalytic subsets is subanalytic,
(c) the complement of a subanalytic subset is subanalytic,
(d) the union of a locally finite family of subanalytic subsets is subanalytic.

This family is a nice family. For example, it is closed by taking the closure and interior; any relatively compact subanalytic subset has finitely many connected components, and each connected component is subanalytic; any closed subanalytic subset is the proper image of a real analytic manifold as in (a).

For real analytic manifolds M, N and a closed subanalytic subset S of M, we say that a map $f\colon S \to N$ is subanalytic if its graph is subanalytic in $M \times N$. One denotes by $\mathscr{A}_S^{\mathbb{R}}$ the sheaf of $\mathbb{R}$-valued subanalytic continuous maps on S. A *subanalytic space* $(M, \mathscr{A}_M^{\mathbb{R}})$, or simply M for short, is an $\mathbb{R}$-ringed space locally isomorphic to $(S, \mathscr{A}_S^{\mathbb{R}})$ for a closed subanalytic subset S of a real analytic manifold. In this paper, we assume that a subanalytic space is good, i.e., it is Hausdorff, locally compact, countable at infinity with finite flabby dimension.

A morphism of subanalytic spaces is a morphism of $\mathbb{R}$-ringed spaces. Then we obtain the category of subanalytic spaces.

We can define the notion of subanalytic subsets of a subanalytic space.

A sheaf F on a subanalytic space M is $\mathbb{R}$-*constructible* if there exists a locally finite family of locally closed subanalytic subsets M_j $(j \in J)$ such that $M = \bigcup_{j \in J} M_j$ and the sheaf $F|_{M_j}$ is locally constant of finite rank for each $j \in J$. We denote by $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ the full subcategory of $\mathrm{Mod}(\mathbb{C}_M)$ consisting of $\mathbb{R}$-constructible sheaves. It is a subcategory stable by taking kernels, cokernels and extensions.

One defines the category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ as the full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_M)$ consisting of objects F such that $H^i(F)$ is $\mathbb{R}$-constructible for all $i \in \mathbb{Z}$. It is a triangulated subcategory and equivalent to $\mathsf{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M))$.

3.2. Subanalytic sheaves

Subanalytic sheaves are sheaves on a certain Grothendieck topology associated with subanalytic spaces. Here we shall introduce it directly without using the language of Grothendieck topology.

Let M be a subanalytic space. Let Op_M be the category of open subsets. The morphisms are inclusions, that is, $\mathrm{Hom}_{\mathrm{Op}_M}(U, V) = \mathrm{pt}$ or $\varnothing$ according to $U \subset V$ or not. Let $\mathrm{Op}_M^{\mathrm{sub,c}}$ be the full subcategory of Op_M consisting of relatively compact subanalytic open subsets.

Recall that a sheaf is a contravariant functor from Op_M to $\mathrm{Mod}(\mathbb{C})$ satisfying a certain "patching condition". By replacing Op_M with $\mathrm{Op}_M^{\mathrm{sub,c}}$ and modifying the "patching condition", we obtain the notion of subanalytic sheaves introduced in [KS01] (see also [Pr08] for its more detailed study).

Definition 3.1. *A subanalytic presheaf F is a contravariant functor from* $\mathrm{Op}_M^{\mathrm{sub,c}}$ *to* $\mathrm{Mod}(\mathbb{C})$. *We say that a subanalytic presheaf F is a subanalytic sheaf if it satisfies:*

(i) $F(\varnothing) = 0$,
(ii) *For* $U, V \in \mathrm{Op}_M^{\mathrm{sub,c}}$, *the sequence*

$$0 \longrightarrow F(U \cup V) \xrightarrow{r_1} F(U) \oplus F(V) \xrightarrow{r_2} F(U \cap V)$$

is exact. Here r_1 is given by the restriction maps and r_2 is given by the difference of the restriction maps $F(U) \to F(U \cap V)$ and $F(V) \to F(U \cap V)$.

Denote by $\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$ the category of subanalytic sheaves. Recall that $\mathrm{Mod}(\mathbb{C}_M)$ denotes the category of sheaves on M. Since a sheaf is a contravariant functor from Op_M, the inclusion functor $\mathrm{Op}_M^{\mathrm{sub,c}} \to \mathrm{Op}_M$ induces a fully faithful functor

$$\iota_M : \mathrm{Mod}(\mathbb{C}_M) \longrightarrow \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}).$$

For example,

$$\mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})}(\iota_M \mathbb{C}_U, F) \simeq F(U) \quad \text{for any } U \in \mathrm{Op}_M^{\mathrm{sub,c}}.$$

The functor ι_M does not commute with inductive limits (as seen in Example 3.11). We denote by "$\varinjlim$" the inductive limit in $\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$ in order to avoid confusion.

Note that

$$(\text{“}\varinjlim_i\text{”}\, F_i)(U) \simeq \varinjlim_i (F_i(U))$$

for any $U \in \mathrm{Op}_M^{\mathrm{sub,c}}$ and a filtrant inductive system $\{F_i\}_i$ of subanalytic sheaves.

The functor ι_M admits a left adjoint, denoted by α_M. For $F \in \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$, the sheaf $\alpha_M(F)$ is the sheaf given by

$$\mathrm{Op}_M \ni U \longmapsto \varprojlim_{V \in \mathrm{Op}_M^{\mathrm{sub,c}},\, V \subset\subset U} F(V).$$

The functor α_M has a left adjoint β_M. For $F \in \mathrm{Mod}(\mathbb{C}_M)$, $\beta_M F$ is the subanalytic sheaf associated with the subanalytic presheaf $\mathrm{Op}_M^{\mathrm{sub,c}} \ni U \to F(\overline{U})$. Hence we have two pairs of adjoint functors (α_M, ι_M) and (β_M, α_M):

$$\begin{tikzcd} \mathrm{Mod}(\mathbb{C}_M) \arrow[r, shift left=2, "\iota_M"] \arrow[r, shift right=2, "\beta_M"'] & \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}). \arrow[l, "\alpha_M" description] \end{tikzcd}$$

Both $\mathrm{Mod}(\mathbb{C}_M)$ and $\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$ are abelian categories, and α_M and β_M are exact. The functor ι_M is left exact but *not* right exact. However, we have the following result.

Proposition 3.2. *The restriction of* ι_M:

$$\iota_M^{\mathbb{R}\text{-}c} : \mathrm{Mod}_{\mathbb{R}\text{-}c}(\mathbb{C}_M) \longrightarrow \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}) \tag{3.1}$$

is exact.

In fact, we have a more precise relation of these two categories (see [KS01]).

Proposition 3.3. *Let* $\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-}c}(\mathbb{C}_M)$ *be the category of* $\mathbb{R}$*-constructible sheaves on* M *with compact supports. Then,* $\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$ *is equivalent to* $\mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-}c}(\mathbb{C}_M))$*, the category of ind-objects in* $\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-}c}(\mathbb{C}_M)$*.*

For ind-objects we refer to [SGA4] or [KS06]. In particular, we have

$$\mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})}(\iota_M G, \text{“}\varinjlim_{i\in I}\text{”}\, F_i) \simeq \varinjlim_{i\in I} \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})}(\iota_M G, F_i)$$

for any $G \in \mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ and a filtrant inductive system $\{F_i\}_{i\in I}$ of subanalytic sheaves.

By the functor $\iota_M^{\mathbb{R}\text{-c}}$, we regard $\mathbb{R}$-constructible sheaves as subanalytic sheaves. We can define the restriction functor

$$\mathrm{Mod}(\mathbb{C}_U^{\mathrm{sub}}) \longrightarrow \mathrm{Mod}(\mathbb{C}_V^{\mathrm{sub}}) \quad \text{for open subsets } U \text{ and } V \subset U.$$

For $F \in \mathrm{Mod}(\mathbb{C}_U^{\mathrm{sub}})$, we denote by $F|_V \in \mathrm{Mod}(\mathbb{C}_V^{\mathrm{sub}})$ the image of F by the restriction functor.

Hence, $\mathrm{Op}_M \ni U \mapsto \mathrm{Mod}(\mathbb{C}_U^{\mathrm{sub}})$ is a prestack on the topological space M.

Proposition 3.4. *The prestack* $\mathrm{Op}_M \ni U \mapsto \mathrm{Mod}(\mathbb{C}_U^{\mathrm{sub}})$ *is a stack.*

We denote by $\mathscr{H}om$ the hom functor as a stack, i.e., for subanalytic sheaves F_1, F_2 on M, we define

$$\Gamma(U; \mathscr{H}om(F_1, F_2)) = \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_U^{\mathrm{sub}})}(F_1|_U, F_2|_U)$$

for any open subset U of M. It is a sheaf on M.

3.3. *Bordered spaces*

A *bordered space* $\mathsf{M} = (M, \check{M})$ is a pair of a good topological space $\check{M}$ and an open subset M of $\check{M}$.

Notation 3.5. *Let* $\mathsf{M} = (M, \check{M})$ *and* $\mathsf{N} = (N, \check{N})$ *be bordered spaces. For a continuous map* $f\colon M \to N$, *denote by* $\Gamma_f \subset M \times N$ *its graph, and by* $\overline{\Gamma}_f$ *the closure of* Γ_f *in* $\check{M} \times \check{N}$. *Consider the projections*

$$\check{M} \xleftarrow{q_1} \check{M} \times \check{N} \xrightarrow{q_2} \check{N}.$$

Bordered spaces form a category as follows: a morphism $f\colon \mathsf{M} \to \mathsf{N}$ is a continuous map $f\colon M \to N$ such that $q_1|_{\overline{\Gamma}_f}\colon \overline{\Gamma}_f \to \check{M}$ is proper; the composition of two morphisms is the composition of the underlying continuous maps.

Remark 3.6. (i) Let $f\colon M \to N$ be a continuous map.

(a) If f can be extended to a continuous map $\check{f}\colon \check{M} \to \check{N}$, then f is a morphism of bordered space from M to N.

(b) If $\check{N}$ is compact, then f is a morphism of bordered space from M to N.

(ii) The forgetful functor from the category of bordered spaces to that of good topological spaces is given by

$$\mathsf{M} = (M, \check{M}) \longmapsto \mathring{\mathsf{M}} := M.$$

It has a fully faithful left adjoint $M \mapsto (M, M)$. By this functor, we regard good topological spaces as particular bordered spaces, and denote (M, M) simply by M.

Be aware that $\mathsf{M} = (M, \check{M}) \mapsto \check{M}$ is not a functor.

(iii) Note that $\mathsf{M} \simeq (M, \overline{M})$, where $\overline{M}$ is the closure of M in $\check{M}$. More generally, for a morphism of bordered spaces $f\colon \mathsf{M} \to \mathsf{N}$, M is isomorphic to the bordered space $(\Gamma_f, \overline{\Gamma}_f)$.

(iv) The category of bordered spaces has an initial object, the empty set. It has also a final object, pt, the topological space consisting of one point. It also admits products:

$$(M, \check{M}) \times (N, \check{N}) \simeq (M \times N, \check{M} \times \check{N}).$$

Let $\mathsf{M} = (M, \check{M})$ be a bordered space. The morphisms of bordered spaces

$$M \longrightarrow \mathsf{M} \xrightarrow{j_{\mathsf{M}}} \check{M} \tag{3.2}$$

are defined by the continuous maps $M \xrightarrow{\mathrm{id}} M \hookrightarrow \check{M}$.

Definition 3.7. *We say that a morphism $f\colon \mathsf{M} \to \mathsf{N}$ is* semi-proper *if $q_2|_{\overline{\Gamma}_f}\colon \overline{\Gamma}_f \to \check{N}$ is proper. We say that f is* proper *if moreover $\mathring{f}\colon \mathring{\mathsf{M}} \to \mathring{\mathsf{N}}$ is proper.*

For example, j_{M} is semi-proper.

The class of semi-proper (resp. proper) morphisms is closed under composition.

Definition 3.8. *A subset S of a bordered space $\mathsf{M} = (M, \check{M})$ is a subset of M. We say that S is open (resp. closed, locally closed) if it is so in M. We say that S is relatively compact if it is contained in a compact subset of $\check{M}$.*

As seen by the following obvious lemma, the notion of relatively compact subsets only depends on M (and not on $\check{M}$).

Lemma 3.9. *Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of bordered spaces.*

(i) *If S is a relatively compact subset of M, then its image $\mathring{f}(S) \subset \mathring{\mathsf{N}}$ is a relatively compact subset of N.*

(ii) *Assume furthermore that f is semi-proper. If S is a relatively compact subset of N, then its inverse image $\mathring{f}^{-1}(S) \subset \mathring{\mathsf{M}}$ is a relatively compact subset of M.*

3.4. Subanalytic sheaves on bordered subanalytic spaces

A bordered subanalytic space is a bordered space $\mathsf{M} = (M, \check{M})$ such that $\check{M}$ is a subanalytic space and M is a subanalytic open subset of $\check{M}$. Then we can consider the category of bordered subanalytic spaces. A morphism $\mathsf{M} = (M, \check{M}) \to \mathsf{N} = (N, \check{N})$ of bordered subanalytic spaces is a morphism f of bordered spaces such that the graph Γ_f is a subanalytic subset of $\check{M} \times \check{N}$.

Let $\mathsf{M} = (M, \check{M})$ be a bordered subanalytic space. We denote by $\mathrm{Op}_{\mathsf{M}}^{\mathrm{sub,c}}$ the full subcategory of Op_M consisting of open subsets of M which are subanalytic and relatively compact in $\check{M}$. A subanalytic sheaf on M is defined as follows.

Definition 3.10. *A subanalytic presheaf F on a bordered subanalytic space M is a contravariant functor from $\mathrm{Op}_{\mathsf{M}}^{\mathrm{sub,c}}$ to $\mathrm{Mod}(\mathbb{C})$. We say that a subanalytic presheaf F is a subanalytic sheaf if it satisfies:*

(i) $F(\varnothing) = 0$,

(ii) *For $U, V \in \mathrm{Op}_{\mathsf{M}}^{\mathrm{sub,c}}$, the sequence*

$$0 \longrightarrow F(U \cup V) \xrightarrow{r_1} F(U) \oplus F(V) \xrightarrow{r_2} F(U \cap V)$$

is exact.

We denote by $\mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$ the category of subanalytic sheaves on M. We have a canonical fully faithful functor

$$\iota_{\mathsf{M}} \colon \mathrm{Mod}(\mathbb{C}_{\mathring{\mathsf{M}}}) \longrightarrow \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}). \tag{3.3}$$

Here $\mathrm{Mod}(\mathbb{C}_{\mathring{\mathsf{M}}})$ denotes the category of sheaves on the topological space $\mathring{\mathsf{M}}$. The functor ι_{M} is left exact but not exact.

We say that a sheaf on $\mathring{\mathsf{M}}$ is an $\mathbb{R}$-constructible sheaf on M if it can be extended to an $\mathbb{R}$-constructible sheaf on $\check{M}$. Let us denote by $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\mathsf{M}})$ the category of $\mathbb{R}$-constructible sheaves on M. Then the restriction of ι_{M}

$$\iota_{\mathsf{M}}^{\mathbb{R}\text{-c}} \colon \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\mathsf{M}}) \longrightarrow \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$$

is exact. By this functor, we regard $\mathbb{R}$-constructible sheaves on M as subanalytic sheaves on M.

3.5. Functorial properties of subanalytic sheaves

3.5.1. Tensor product and inner hom Let $\mathsf{M} = (M, \check{M})$ be a bordered subanalytic space. The category $\mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$ has tensor product and inner hom:

$$\bullet \otimes \bullet \colon \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \times \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \longrightarrow \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \quad \text{and}$$
$$\mathcal{H}om(\bullet, \bullet) \colon \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})^{\mathrm{op}} \times \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \longrightarrow \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}).$$

For $F_1, F_2 \in \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$, their tensor product $F_1 \otimes F_2$ is the subanalytic sheaf associated with the subanalytic presheaf $\mathrm{Op}_{\mathsf{M}}^{\mathrm{sub,c}} \ni U \mapsto F_1(U) \otimes F_2(U)$. The inner hom $\mathscr{H}\!om(F_1, F_2)$ is given by

$$\mathrm{Op}_{\mathsf{M}}^{\mathrm{sub,c}} \ni U \longmapsto \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{(U,\check{M})})}(F_1|_{(U,\check{M})}, F_2|_{(U,\check{M})}).$$

We have

$$\mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})}(F_1 \otimes F_2, F_3) \simeq \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})}(F_1, \mathscr{H}\!om(F_2, F_3))$$

for $F_1, F_2, F_3 \in \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$.

The bifunctor $\bullet \otimes \bullet$ is exact, and $\mathscr{H}\!om(\bullet, \bullet)$ is left exact.

3.5.2. Direct images and inverse images Let $\mathsf{M} = (M, \check{M})$ and $\mathsf{N} = (N, \check{N})$ be bordered subanalytic spaces and let $f : \mathsf{M} \to \mathsf{N}$ be a morphism of bordered subanalytic spaces.

For $F \in \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$, its direct image $f_* F \in \mathrm{Mod}(\mathbb{C}_{\mathsf{N}}^{\mathrm{sub}})$ is defined by

$$(f_* F)(V) = \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})}(\mathbb{C}_{f^{-1}V}, F) \quad \text{for any } V \in \mathrm{Op}_{\mathsf{N}}^{\mathrm{sub,c}}. \tag{3.4}$$

The functor $f_* \colon \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \to \mathrm{Mod}(\mathbb{C}_{\mathsf{N}}^{\mathrm{sub}})$ has a left adjoint

$$f^{-1} \colon \mathrm{Mod}(\mathbb{C}_{\mathsf{N}}^{\mathrm{sub}}) \longrightarrow \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}).$$

The functor f^{-1} is called the inverse image functor. For a subanalytic sheaf G on N, its inverse image $f^{-1}G$ is the subanalytic sheaf associated with the subanalytic presheaf

$$\mathrm{Op}_{\mathsf{M}}^{\mathrm{sub,c}} \ni U \longmapsto \varinjlim_{V \in \mathrm{Op}_{\mathsf{N}}^{\mathrm{sub,c}},\, U \subset f^{-1}V} G(V).$$

The functor f^{-1} is exact.

For $F \in \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$, the direct image with proper support $f_{!!} F$ is defined by

$$\Gamma(V; f_{!!} F) = \varinjlim_{U} \mathrm{Hom}\,(\mathbb{C}_{f^{-1}V}; F \otimes \mathbb{C}_U) \quad \text{for } V \in \mathrm{Op}_{\mathsf{N}}^{\mathrm{sub,c}}.$$

Here U ranges over the open subsets in $\mathrm{Op}_{\mathsf{M}}^{\mathrm{sub,c}}$ such that $f^{-1}V \cap \overline{U} \to V$ is proper, where $\overline{U}$ denotes the closure of U in M. In general, the diagram

$$\begin{tikzcd}
\mathrm{Mod}(\mathbb{C}_{\mathsf{M}}) \arrow[r, "\iota_{\mathsf{M}}"] \arrow[d, "f_!"] \arrow[dr, phantom, "\mathrm{NC}"] & \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \arrow[d, "f_{!!}"] \\
\mathrm{Mod}(\mathbb{C}_{\mathsf{N}}) \arrow[r, "\iota_{\mathsf{N}}"'] & \mathrm{Mod}(\mathbb{C}_{\mathsf{N}}^{\mathrm{sub}})
\end{tikzcd}$$

is not commutative, that is why we use the different notation $f_{!!}$. Note that the above diagram commutes if f is semi-proper.

Example 3.11. Let $M = \mathbb{R}_{>0}$, $N = \mathbb{R}$ and let $f\colon M \to N$ be the canonical inclusion. Then we have

$$\begin{aligned} f_{!!}\,\mathbb{C}_M &\simeq \text{“}\varinjlim_{c\to 0+}\text{”}\, \mathbb{C}_{\{t>c\}} \quad \text{and} \\ f_{!}\,\mathbb{C}_M &\simeq \mathbb{C}_{\{t>0\}}. \end{aligned}$$

They are not isomorphic. Indeed, we have for $U = \{t\,;\, 0 < t < 1\} \in \mathrm{Op}_N^{\mathrm{sub,c}}$

$$\Gamma(U;\, \text{“}\varinjlim_{c\to 0+}\text{”}\, \mathbb{C}_{\{t>c\}}) \simeq \varinjlim_{c\to 0+} \Gamma(U;\mathbb{C}_{\{t>c\}}) \simeq 0 \quad \text{and} \quad \Gamma(U;\mathbb{C}_{\{t>0\}}) \simeq \mathbb{C}.$$

Note that the inductive limit of $\mathbb{C}_{\{t>c\}}$ in $\mathrm{Mod}(\mathbb{C}_N)$ is isomorphic to $\mathbb{C}_{\{t>0\}}$.

Recall the morphism $j_{\mathsf{M}}\colon \mathsf{M} \to \check{M}$ of bordered subanalytic spaces. We have

$$\begin{aligned} j_{\mathsf{M}!!}\, j_{\mathsf{M}}^{-1} F &\simeq \mathbb{C}_{\mathring{\mathsf{M}}} \otimes F, \\ j_{\mathsf{M}*}\, j_{\mathsf{M}}^{-1} F &\simeq \mathscr{I}hom(\mathbb{C}_{\mathring{\mathsf{M}}}, F) \end{aligned} \qquad \text{for } F \in \mathrm{Mod}(\mathbb{C}_{\check{M}}^{\mathrm{sub}}). \tag{3.5}$$

Moreover, the functor $j_{\mathsf{M}}^{-1}\colon \mathrm{Mod}(\mathbb{C}_{\check{M}}^{\mathrm{sub}}) \to \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$ induces an equivalence of abelian categories:

$$\mathrm{Mod}(\mathbb{C}_{\check{M}}^{\mathrm{sub}}) \,/\, \mathrm{Mod}(\mathbb{C}_{\check{M}\setminus M}^{\mathrm{sub}}) \simeq \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}).$$

Here $\mathrm{Mod}(\mathbb{C}_{\check{M}\setminus M}^{\mathrm{sub}})$ is regarded as a full subcategory of $\mathrm{Mod}(\mathbb{C}_{\check{M}}^{\mathrm{sub}})$ by the fully faithful exact functor $i_* \simeq i_{!!}\colon \mathrm{Mod}(\mathbb{C}_{\check{M}\setminus M}^{\mathrm{sub}}) \to \mathrm{Mod}(\mathbb{C}_{\check{M}}^{\mathrm{sub}})$, where $i\colon \check{M} \setminus M \hookrightarrow \check{M}$ is the closed inclusion.

3.6. Derived functors

The fully faithful exact functor

$$\iota_{\mathsf{M}}^{\mathbb{R}\text{-c}}\colon \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\mathsf{M}}) \longrightarrow \mathrm{Mod}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$$

induces a fully faithful functor $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\mathsf{M}}) \rightarrowtail \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$ by which we regard $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\mathsf{M}})$ as a full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$.

The functors introduced in the previous subsection have derived functors:

$$\begin{aligned} \bullet \otimes \bullet &: \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \times \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}), \\ \mathrm{R}\mathscr{I}hom(\bullet,\bullet) &: \mathsf{D}^{-}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})^{\mathrm{op}} \times \mathsf{D}^{+}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \longrightarrow \mathsf{D}^{+}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}), \\ f^{-1} &: \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{N}}^{\mathrm{sub}}) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}), \\ \mathrm{R}f_* &: \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{N}}^{\mathrm{sub}}), \\ \mathrm{R}f_{!!} &: \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{N}}^{\mathrm{sub}}). \end{aligned}$$

The functor $\mathrm{R}f_{!!}$ has a right adjoint:

$$f^{\,!}\;:\;\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{N}}^{\mathrm{sub}})\longrightarrow\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}}).$$

If $\mathring{f}\colon\mathring{\mathsf{M}}\to\mathring{\mathsf{N}}$ is topologically submersive, i.e., $\mathring{f}$ is isomorphic to $\mathring{\mathsf{N}}\times\mathbb{R}^n\to\mathring{\mathsf{N}}$ locally on $\mathring{\mathsf{M}}$, then

$$f^{\,!}F\simeq\omega_{\mathring{\mathsf{M}}/\mathring{\mathsf{N}}}\otimes f^{-1}F.$$

Here $\omega_{\mathring{\mathsf{M}}/\mathring{\mathsf{N}}}:=\mathring{f}^{\,!}\,\mathbb{C}_{\mathring{\mathsf{N}}}\in\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_{\mathsf{M}})\subset\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}^{\mathrm{sub}})$ is the relative dualizing complex.

These six operations satisfy the properties similar to (2.1) and (2.2) for the Grothendieck's six operations for sheaves.

3.7. Ring actions

Let M be a subanalytic space, and let $\mathscr{A}$ be a sheaf of $\mathbb{C}$-algebras. Let F be a subanalytic sheaf on M. We say that F has an action of $\mathscr{A}$, or F is a subanalytic $\mathscr{A}$-module if a homomorphism of sheaves of $\mathbb{C}$-algebras

$$\mathscr{A}\longrightarrow\mathscr{H}om(F,F)\tag{3.6}$$

is given. Since $\mathscr{H}om(F,F)\simeq\alpha_M\,\mathscr{I}hom(F,F)$, the data (3.6) is equivalent to

$$\beta_M\mathscr{A}\longrightarrow\mathscr{I}hom(F,F),$$

or $\beta_M\mathscr{A}\otimes F\to F$ with the associativity property. We denote by $\mathrm{Mod}(\mathscr{A}^{\,\mathrm{sub}})$ the category of subanalytic $\mathscr{A}$-modules, and by $\mathsf{D}^{\mathrm{b}}(\mathscr{A}^{\,\mathrm{sub}})$ its bounded derived category.

We have the tensor functor and the hom functor:

$$\begin{aligned}\bullet\otimes_{\mathscr{A}}\bullet\;:\;\mathsf{D}^{\mathrm{b}}(\mathscr{A}^{\,\mathrm{op}})\times\mathsf{D}^{\mathrm{b}}(\mathscr{A}^{\,\mathrm{sub}})&\longrightarrow\mathsf{D}^{-}(\mathbb{C}_M^{\mathrm{sub}}),\\ \mathrm{R}\mathscr{H}om_{\mathscr{A}}(\bullet,\bullet)\;:\;\mathsf{D}^{\mathrm{b}}(\mathscr{A})^{\,\mathrm{op}}\times\mathsf{D}^{\mathrm{b}}(\mathscr{A}^{\,\mathrm{sub}})&\longrightarrow\mathsf{D}^{+}(\mathbb{C}_M^{\mathrm{sub}}).\end{aligned}$$

4. Subanalytic sheaves of tempered functions

4.1. Tempered distributions

Hereafter, M denotes a real analytic manifold.

An important property of subanalytic subsets is given by the lemma below. (See Lojasiewicz [Lo59] and also [Ma66] for a detailed study of its consequences.)

Lemma 4.1. *Let U and V be two relatively compact open subanalytic subsets of $\mathbb{R}^n$. There exist a positive integer N and $C > 0$ such that*

$$\operatorname{dist}(x, \mathbb{R}^n \setminus (U \cup V))^N \leqslant C(\operatorname{dist}(x, \mathbb{R}^n \setminus U) + \operatorname{dist}(x, \mathbb{R}^n \setminus V)).$$

We denote by $\mathscr{D}\mathit{b}_M$ the sheaf of Schwartz's distributions on M. Denote by $\mathscr{D}\mathit{b}_M^{\mathrm{t}}(U)$ the image of the restriction map $\Gamma(M; \mathscr{D}\mathit{b}_M) \to \Gamma(U; \mathscr{D}\mathit{b}_M)$, and call it the space of *tempered distributions* on U.

Using Lemma 4.1, one proves:

Lemma 4.2. *The subanalytic presheaf $U \mapsto \mathscr{D}\mathit{b}_M^{\mathrm{t}}(U)$ is a subanalytic sheaf on M.*

One denotes by $\mathscr{D}\mathit{b}_M^{\mathrm{t}}$ this subanalytic sheaf. By the definition, there is a monomorphism

$$\mathscr{D}\mathit{b}_M^{\mathrm{t}} \rightarrowtail \iota_M \mathscr{D}\mathit{b}_M,$$

and an isomorphism

$$\alpha_M \mathscr{D}\mathit{b}_M^{\mathrm{t}} \simeq \mathscr{D}\mathit{b}_M.$$

Let us denote by $\mathscr{D}_M$ the sheaf of rings of differential operators with real analytic coefficients. Then, $\mathscr{D}\mathit{b}_M^{\mathrm{t}}$ is a subanalytic $\mathscr{D}_M$-module in the sense of §3.7.

4.2. Tempered holomorphic functions

Let X be a complex manifold, and let us denote by $X_{\mathbb{R}}$ the underlying real analytic manifold. We have defined the subanalytic sheaf of tempered distributions $\mathscr{D}\mathit{b}_{X_{\mathbb{R}}}^{\mathrm{t}}$. It is a subanalytic $\mathscr{D}_{X_{\mathbb{R}}}$-module. Let us consider the Dolbeault complex with coefficients in $\mathscr{D}\mathit{b}_{X_{\mathbb{R}}}^{\mathrm{t}}$:

$$\mathscr{D}\mathit{b}_{X_{\mathbb{R}}}^{\mathrm{t}} \xrightarrow{\bar{\partial}} \Omega^1_{X^{\mathrm{c}}} \otimes_{\mathscr{O}_{X^{\mathrm{c}}}} \mathscr{D}\mathit{b}_{X_{\mathbb{R}}}^{\mathrm{t}} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{d_X}_{X^{\mathrm{c}}} \otimes_{\mathscr{O}_{X^{\mathrm{c}}}} \mathscr{D}\mathit{b}_{X_{\mathbb{R}}}^{\mathrm{t}}.$$

Here X^{c} is the complex conjugate manifold of X. It is a complex in the category $\mathrm{Mod}(\mathscr{D}_X^{\mathrm{sub}})$ of subanalytic $\mathscr{D}_X$-modules. Hence we can consider this complex as an object of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{sub}})$, the bounded derived category of $\mathrm{Mod}(\mathscr{D}_X^{\mathrm{sub}})$. We denote it by $\mathscr{O}_X^{\mathrm{t}}$ and call it the subanalytic sheaf of tempered holomorphic functions. Note that its cohomology groups are not concentrated at degree 0 in general.

4.3. Tempered de Rham and solution functors

Setting $\Omega_X^{\mathrm{t}} := \Omega_X \otimes_{\mathscr{O}_X} \mathscr{O}_X^{\mathrm{t}} \in \mathsf{D}^{\mathrm{b}}((\mathscr{D}_X^{\mathrm{op}})^{\mathrm{sub}})$, we define the tempered de Rham and solution functors by

$$\begin{aligned} \mathscr{DR}_X^{\mathrm{t}} \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) &\longrightarrow \mathsf{D}^{-}(\mathbb{C}_X^{\mathrm{sub}}), & \mathscr{M} &\longmapsto \Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}, \\ \mathscr{S}ol_X^{\mathrm{t}} \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) &\longrightarrow \mathsf{D}^{+}(\mathbb{C}_X^{\mathrm{sub}})^{\mathrm{op}}, & \mathscr{M} &\longmapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X^{\mathrm{t}}). \end{aligned}$$

One has

$$\mathscr{DR}_X \simeq \alpha_X \circ \mathscr{DR}_X^{\mathrm{t}} \quad \text{and} \quad \mathscr{S}ol_X \simeq \alpha_X \circ \mathscr{S}ol_X^{\mathrm{t}}.$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, one has

$$\mathscr{S}ol_X^{\mathrm{t}}(\mathscr{M}) \simeq \mathscr{DR}_X^{\mathrm{t}}(\mathbb{D}_X\mathscr{M})[-d_X]. \tag{4.1}$$

The next result is a reformulation of a theorem of [Ka84] (see also [KS01, Th. 7.4.1])

Theorem 4.3. *Let $f\colon X \to Y$ be a morphism of complex manifolds. There is an isomorphism in $\mathsf{D}^{\mathrm{b}}((f^{-1}\mathscr{D}_Y^{\mathrm{op}})^{\mathrm{sub}})$:*

$$\Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\,[d_X] \xrightarrow{\sim} f^{\,!}\,\Omega_Y^{\mathrm{t}}\,[d_Y]. \tag{4.2}$$

Note that this isomorphism (4.2) is equivalent to the isomorphism

$$\mathscr{D}_{Y\longleftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{O}_X^{\mathrm{t}}\,[d_X] \xrightarrow{\sim} f^{\,!}\,\mathscr{O}_Y^{\mathrm{t}}\,[d_Y] \quad \text{in } \mathsf{D}^{\mathrm{b}}((f^{-1}\mathscr{D}_Y)^{\mathrm{sub}}).$$

Corollary 4.4. *Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$. Then* (4.2) *induces the isomorphism*

$$\mathscr{DR}_X^{\mathrm{t}}(\mathrm{D}f^*\mathscr{N})\,[d_X] \simeq f^{\,!}\,\mathscr{DR}_Y^{\mathrm{t}}(\mathscr{N})\,[d_Y] \quad \text{in } \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\mathrm{sub}}).$$

Corollary 4.5. *For any complex manifold X, we have*

$$\mathscr{DR}_X^{\mathrm{t}}(\mathscr{O}_X) \simeq \mathbb{C}_X[d_X].$$

The next results are a kind of Grauert direct image theorem for tempered holomorphic functions, and its $\mathscr{D}$-module version.

Theorem 4.6 (Tempered Grauert theorem [KS96, Th. 7.3]). *Let $f\colon X \to Y$ be a morphism of complex manifolds, let $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ and assume that f is proper on* $\mathrm{Supp}(\mathscr{F})$. *Then there is a natural isomorphism*

$$\mathrm{R}f_{!!}\,(\mathscr{O}_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{F}) \simeq \mathscr{O}_Y^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_Y} \mathrm{R}f_{!}\,\mathscr{F}.$$

Proposition 4.7 ([KS01, Th. 7.4.6]). *Let $f\colon X \to Y$ be a morphism of complex manifolds. Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$ and assume that f is proper on* $\operatorname{Supp}(\mathscr{M})$. *Then there is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_Y^{\mathrm{sub}})$*

$$\mathscr{DR}_Y^{\mathrm{t}}(\mathrm{D}f_*\mathscr{M}) \xrightarrow{\sim} \mathrm{R}f_*\,\mathscr{DR}_X^{\mathrm{t}}(\mathscr{M}).$$

For a closed hypersurface $S \subset X$, denote by $\mathscr{O}_X(*S)$ the sheaf of meromorphic functions with poles at S. It is a holonomic $\mathscr{D}_X$-module and flat as an $\mathscr{O}_X$-module. For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ or $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{sub}})$, set

$$\mathscr{M}(*S) = \mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{O}_X(*S).$$

Proposition 4.8. *Let S be a closed complex hypersurface in X. There are isomorphisms*

$$\mathscr{O}_X^{\mathrm{t}}(*S) \simeq \mathrm{R}\mathscr{I}hom(\mathbb{C}_{X\setminus S}, \mathscr{O}_X^{\mathrm{t}}) \quad \textit{in } \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{sub}}),$$
$$\mathscr{O}_X(*S) \simeq \mathrm{R}\mathscr{H}om(\mathbb{C}_{X\setminus S}, \mathscr{O}_X^{\mathrm{t}}) \quad \textit{in } \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X).$$

Corollary 4.9. *Let S be a closed complex hypersurface in X. There are isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\mathrm{sub}})$*

$$\begin{aligned}\mathscr{DR}_X^{\mathrm{t}}(\mathscr{O}_X(*S)) &\simeq \mathscr{DR}_X(\mathscr{O}_X(*S))\\ &\simeq \mathrm{R}\mathscr{H}om(\mathbb{C}_{X\setminus S}, \mathbb{C}_X)\,[d_X].\end{aligned}$$

5. Enhanced subanalytic sheaves

5.1. Enhanced tensor product and inner hom

Consider the 2-point compactification of the real line $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{+\infty, -\infty\}$. Denote by $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \sqcup \{\infty\}$ the real projective line. Then $\overline{\mathbb{R}}$ has a structure of subanalytic space such that the natural map $\overline{\mathbb{R}} \to \mathbb{P}^1(\mathbb{R})$ is a morphism of subanalytic spaces.

Notation 5.1. *We will consider the bordered subanalytic space*

$$\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}}).$$

Note that $\mathbb{R}_\infty$ is isomorphic to $(\mathbb{R}, \mathbb{P}^1(\mathbb{R}))$ as a bordered subanalytic space.

Consider the morphisms of bordered subanalytic spaces

$$\begin{aligned} a\colon \mathbb{R}_\infty &\longrightarrow \mathbb{R}_\infty, \\ \mu, q_1, q_2\colon \mathbb{R}_\infty \times \mathbb{R}_\infty &\longrightarrow \mathbb{R}_\infty, \end{aligned} \tag{5.1}$$

where $a(t) = -t$, $\mu(t_1, t_2) = t_1 + t_2$ and q_1, q_2 are the natural projections.

For a subanalytic space M, we will use the same notations for the associated morphisms

$$a\colon M\times\mathbb{R}_\infty \longrightarrow M\times\mathbb{R}_\infty,$$
$$\mu, q_1, q_2\colon M\times\mathbb{R}_\infty\times\mathbb{R}_\infty \longrightarrow M\times\mathbb{R}_\infty.$$

We also use the natural morphisms

(5.2)
$$\begin{tikzcd} M\times\mathbb{R}_\infty \arrow[rr] \arrow[dr, "\pi_M"'] & & M\times\overline{\mathbb{R}} \arrow[dl, "\overline{\pi}_M"] \\ & M. & \end{tikzcd}$$

Definition 5.2. *The functors*

$$\overset{+}{\otimes}\colon \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})\times\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty}) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty}),$$
$$\mathscr{I}hom^{+}\colon \mathsf{D}^{-}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})^{\mathrm{op}}\times\mathsf{D}^{+}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty}) \longrightarrow \mathsf{D}^{+}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})$$

are defined by

$$K_1\overset{+}{\otimes}K_2 = \mathrm{R}\mu_{!!}\,(q_1^{-1}K_1\otimes q_2^{-1}K_2),$$
$$\mathscr{I}hom^{+}(K_1,K_2) = \mathrm{R}q_{1*}\mathrm{R}\mathscr{I}hom(q_2^{-1}K_1,\mu^{!}K_2).$$

One sets

(5.3)
$$\mathbb{C}_{\{t\geqslant 0\}} = \mathbb{C}_{\{(x,t)\in M\times\mathbb{R}\,;\,t\geqslant 0\}}.$$

We use similar notation for $\mathbb{C}_{\{t=0\}}$, $\mathbb{C}_{\{t>0\}}$, $\mathbb{C}_{\{t\leqslant 0\}}$, $\mathbb{C}_{\{t=a\}}$, etc. These are $\mathbb{R}$-constructible sheaves on $M\times\mathbb{R}_\infty$. We also regard them as objects of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})$.

Lemma 5.3. *For $K\in\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})$, there are isomorphisms*

$$\mathbb{C}_{\{t=0\}}\overset{+}{\otimes}K \simeq K \simeq \mathscr{I}hom^{+}(\mathbb{C}_{\{t=0\}},K).$$

More generally, for $a\in\mathbb{R}$, we have

$$\mathbb{C}_{\{t=a\}}\overset{+}{\otimes}K \simeq \mathrm{R}\mu_{a*}K \simeq \mathscr{I}hom^{+}(\mathbb{C}_{\{t=-a\}},K),$$

where $\mu_a\colon M\times\mathbb{R}_\infty\to M\times\mathbb{R}_\infty$ is the morphism induced by the translation $t\mapsto t+a$.

Corollary 5.4. *The category $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})$ has a structure of commutative tensor category with $\overset{+}{\otimes}$ as tensor product and $\mathbb{C}_{\{t=0\}}$ as unit object.*

As seen in the following lemma, the functor $\mathcal{I}hom^{+}$ is the inner hom of the tensor category $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})$.

Lemma 5.5. *For $K_1, K_2, K_3 \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})$ one has*

$$\begin{aligned}&\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})}(K_1 \overset{+}{\otimes} K_2,\ K_3)\\ \simeq&\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})}(K_1,\ \mathcal{I}hom^{+}(K_2, K_3)),\end{aligned}$$

$$\begin{aligned}&\mathcal{I}hom^{+}(K_1 \overset{+}{\otimes} K_2,\ K_3) \simeq \mathcal{I}hom^{+}(K_1,\ \mathcal{I}hom^{+}(K_2, K_3)),\\ &\mathrm{R}\pi_{M*}\mathrm{R}\mathcal{I}hom(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathrm{R}\pi_{M*}\mathrm{R}\mathcal{I}hom(K_1, \mathcal{I}hom^{+}(K_2, K_3)).\end{aligned}$$

We define the outer hom functors on $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})$ as follows.

Definition 5.6. *One defines the hom functor*

$$\begin{aligned}&\mathcal{I}hom^{\mathsf{E}} \colon \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty}) \longrightarrow \mathsf{D}^{+}(\mathbb{C}^{\mathrm{sub}}_{M})\\ &\mathcal{I}hom^{\mathsf{E}}(K_1, K_2) = \mathrm{R}\pi_{M*}\mathrm{R}\mathcal{I}hom(K_1, K_2),\end{aligned}$$

and one sets

$$\mathcal{H}om^{\mathsf{E}} = \alpha_M \circ \mathcal{I}hom^{\mathsf{E}} :\ \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty}) \longrightarrow \mathsf{D}^{+}(\mathbb{C}_M).$$

Note that

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M\times\mathbb{R}_\infty})}(K_1, K_2) \simeq H^0(M; \mathcal{H}om^{\mathsf{E}}(K_1, K_2)).$$

5.2. *Enhanced sheaf of tempered distributions*

Let M be a real analytic manifold. Let $j_M \colon M \times \mathbb{R}_\infty \to M \times \mathbb{P}^1(\mathbb{R})$ be the canonical morphism.

Let t be the affine coordinate of $\mathbb{P}^1(\mathbb{R})$. Then, $\partial_t := \partial/\partial t$ is a vector field on $M \times \mathbb{P}^1(\mathbb{R})$, and hence it acts on $\mathcal{D}b^{\mathrm{t}}_{M\times\mathbb{P}^1(\mathbb{R})}$.

Lemma 5.7. *The morphism of subanalytic sheaves*

$$\partial_t - 1 \colon\ \mathcal{D}b^{\mathrm{t}}_{M\times\mathbb{P}^1(\mathbb{R})} \to \mathcal{D}b^{\mathrm{t}}_{M\times\mathbb{P}^1(\mathbb{R})}$$

is an epimorphism.

We define the subanalytic sheaf on $M \times \mathbb{R}_\infty$ by

$$\mathscr{D}b_M^{\mathsf{T}} = \mathrm{Ker}(\partial_t - 1 : j_M^{-1} \mathscr{D}b_{M \times \mathbb{P}^1(\mathbb{R})}^{\mathsf{t}} \to j_M^{-1} \mathscr{D}b_{M \times \mathbb{P}^1(\mathbb{R})}^{\mathsf{t}}).$$

Since any solution of $(\partial_t - 1)u(t,x) = 0$ can be written as $u(t,x) = \mathrm{e}^t \varphi(x)$, we have a monomorphism in $\mathrm{Mod}(\mathbb{C}_{M \times \mathbb{R}_\infty}^{\mathrm{sub}})$

$$\mathscr{D}b_M^{\mathsf{T}} \rightarrowtail \pi_M^{-1} \iota_M \mathscr{D}b_M \quad \text{by } u(t,x) \mapsto \varphi(x).$$

Note that $\mathscr{D}b_M^{\mathsf{T}}$ is a subanalytic $\pi_M^{-1} \mathscr{D}_M$-module. We call it the enhanced subanalytic sheaf of tempered distributions.

Proposition 5.8.

$$\begin{aligned}\mathscr{D}b_M^{\mathsf{T}} &\simeq \mathscr{I}hom^+(\mathbb{C}_{\{t \geqslant a\}}, \mathscr{D}b_M^{\mathsf{T}}) \quad \textit{for any } a \in \mathbb{R} \\ &\simeq \mathscr{I}hom^+(\mathbb{C}_M^{\mathsf{T}}[1], \mathscr{D}b_M^{\mathsf{T}}).\end{aligned}$$

Here we set

$$\mathbb{C}_M^{\mathsf{T}} := \text{“}\varinjlim_{c \to +\infty}\text{”}\ \mathbb{C}_{\{t<c\}}.$$

The enhanced subanalytic sheaf $\mathbb{C}_M^{\mathsf{T}}$ satisfies

$$\mathbb{C}_M^{\mathsf{T}}[1] \overset{+}{\otimes} \mathbb{C}_M^{\mathsf{T}}[1] \simeq \mathbb{C}_M^{\mathsf{T}}[1].$$

We can recover $\mathscr{D}b_M^{\mathsf{t}}$ and $\mathscr{D}b_M$ from $\mathscr{D}b_M^{\mathsf{T}}$ as follows:

$$\begin{aligned}\mathscr{I}hom^{\mathsf{E}}(\mathbb{C}_M^{\mathsf{T}}, \mathscr{D}b_M^{\mathsf{T}}) &\simeq \mathscr{D}b_M^{\mathsf{t}}, \\ \mathscr{H}om^{\mathsf{E}}(\mathbb{C}_M^{\mathsf{T}}, \mathscr{D}b_M^{\mathsf{T}}) &\simeq \mathscr{D}b_M.\end{aligned} \tag{5.4}$$

Remark 5.9. The definition of $\mathscr{D}b^{\mathsf{T}}$ is slightly different from the one in [DK13, KS15, DK15]. The notation $\mathscr{D}b^{\mathsf{T}}$ in loc. cit. is equal to $\mathscr{D}b^{\mathsf{T}}[1]$ in our notation.

5.3. Enhanced sheaf of tempered holomorphic functions

Let X be a complex manifold, and let us denote by $X_\mathbb{R}$ the underlying real analytic manifold. We have defined the enhanced subanalytic sheaf of tempered distributions $\mathscr{D}b_{X_\mathbb{R}}^{\mathsf{T}}$. It is a subanalytic $\pi_X^{-1} \mathscr{D}_{X_\mathbb{R}}$-module. Let us consider the Dolbeault complex with coefficients in $\mathscr{D}b_{X_\mathbb{R}}^{\mathsf{T}}$:

$$\mathscr{D}b_{X_\mathbb{R}}^{\mathsf{T}} \xrightarrow{\bar\partial} \Omega_{X^c}^1 \otimes_{\mathscr{O}_{X^c}} \mathscr{D}b_{X_\mathbb{R}}^{\mathsf{T}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega_{X^c}^{d_X} \otimes_{\mathscr{O}_{X^c}} \mathscr{D}b_{X_\mathbb{R}}^{\mathsf{T}}.$$

Here X^{c} is the complex conjugate manifold of X. It is a complex in the category $\mathrm{Mod}((\pi_X^{-1}\mathscr{D}_X)^{\mathrm{sub}})$ of subanalytic $\pi_X^{-1}\mathscr{D}_X$-modules, where $\mathscr{D}b^{\mathsf{T}}_{X_{\mathbb{R}}}$ is situated at degree 0 and $\Omega^{d_X}_{X^{\mathrm{c}}} \otimes_{\mathscr{O}_{X^{\mathrm{c}}}} \mathscr{D}b^{\mathsf{T}}_{X_{\mathbb{R}}}$ at degree d_X. Hence we can consider this complex as an object of $\mathsf{D}^{\mathrm{b}}((\pi_X^{-1}\mathscr{D}_X)^{\mathrm{sub}})$, the bounded derived category of $\mathrm{Mod}((\pi_X^{-1}\mathscr{D}_X)^{\mathrm{sub}})$. We denote it by $\mathscr{O}^{\mathsf{T}}_X$ and call it the enhanced sheaf of tempered holomorphic functions. Note that its cohomology groups are not concentrated at degree 0.

Remark 5.10. If $X = \mathrm{pt}$, then

$$\mathscr{O}^{\mathsf{T}}_X \simeq \mathscr{D}b^{\mathsf{T}}_{X_{\mathbb{R}}} \simeq \mathbb{C}^{\mathsf{T}}_X := \text{``}\varinjlim_{c\to+\infty}\text{''}\ \mathbb{C}_{\{t<c\}}$$

as objects of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{R}_\infty})$. Indeed, for $-\infty \leqslant a < b \leqslant +\infty$, e^t is a tempered distribution on the open interval (a,b) if and only if $(a,b) \subset \{t < c\}$ for some $c \in \mathbb{R}$.

By (5.4), we have

$$\begin{aligned} &\mathscr{I}hom^{\mathsf{E}}(\mathbb{C}^{\mathsf{T}}_X, \mathscr{O}^{\mathsf{T}}_X) \simeq \mathscr{O}^{\mathrm{t}}_X \quad \text{and} \\ &\mathscr{H}om^{\mathsf{E}}(\mathbb{C}^{\mathsf{T}}_X, \mathscr{O}^{\mathsf{T}}_X) \simeq \mathscr{O}_X. \end{aligned} \tag{5.5}$$

5.4. Enhanced de Rham and solution functors

We set

$$\Omega^{\mathsf{T}}_X := \pi_X^{-1}\Omega_X \otimes_{\pi_X^{-1}\mathscr{O}_X} \mathscr{O}^{\mathsf{T}}_X \in \mathsf{D}^{\mathrm{b}}((\pi_X^{-1}\mathscr{D}_X^{\mathrm{op}})^{\mathrm{sub}}).$$

We define the enhanced de Rham and solution functors

$$\begin{aligned} &\mathscr{DR}^{\mathsf{T}}_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}), \\ &\mathscr{S}ol^{\mathsf{T}}_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})^{\mathrm{op}} \end{aligned}$$

by

$$\begin{aligned} &\mathscr{DR}^{\mathsf{T}}_X(\mathscr{M}) := \Omega^{\mathsf{T}}_X \overset{\mathsf{L}}{\otimes}_{\pi_X^{-1}\mathscr{D}_X} \pi_X^{-1}\mathscr{M}, \\ &\mathscr{S}ol^{\mathsf{T}}_X(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\pi_X^{-1}\mathscr{D}_X}(\pi_X^{-1}\mathscr{M}, \mathscr{O}^{\mathsf{T}}_X). \end{aligned}$$

Note that

$$\mathscr{S}ol^{\mathsf{T}}_X(\mathscr{M}) \simeq \mathscr{DR}^{\mathsf{T}}_X(\mathbb{D}_X\mathscr{M})[-d_X] \quad \text{for } \mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X).$$

By (5.5), we have for any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$

$$(5.6)\qquad \begin{aligned} \mathscr{DR}^{\mathsf{t}}_X \mathscr{M} &\simeq \mathscr{I}hom^{\mathsf{E}}(\mathbb{C}^{\mathsf{T}}_X, \mathscr{DR}^{\mathsf{T}}_X \mathscr{M}), \\ \mathscr{DR}_X \mathscr{M} &\simeq \mathscr{H}om^{\mathsf{E}}(\mathbb{C}^{\mathsf{T}}_X, \mathscr{DR}^{\mathsf{T}}_X \mathscr{M}). \end{aligned}$$

For a particular case of holonomic $\mathscr{D}$-modules, we can calculate explicitly the enhanced de Rham. Let $Y \subset X$ be a complex analytic hypersurface of a complex manifold X, and set $U = X \setminus Y$. For $\varphi \in \mathscr{O}_X(*Y)$, one sets

$$\begin{aligned} &\mathscr{D}_X \mathrm{e}^{\varphi} = \mathscr{D}_X / \{P \in \mathscr{D}_X \,;\, P\mathrm{e}^{\varphi} = 0 \text{ on } U\}, \\ &\mathscr{E}^{\varphi}_{U|X} = \mathscr{D}_X \mathrm{e}^{\varphi}(*Y). \end{aligned}$$

Hence $\mathscr{D}_X \mathrm{e}^{\varphi}$ is a $\mathscr{D}_X$-submodule of $\mathscr{E}^{\varphi}_{U|X}$, and $\mathscr{D}_X \mathrm{e}^{\varphi}$ as well as $\mathscr{E}^{\varphi}_{U|X}$ is a holonomic $\mathscr{D}_X$-module. Note that $\mathscr{E}^{\varphi}_{U|X}$ is isomorphic to $\mathscr{O}_X(*Y)$ as an $\mathscr{O}_X$-module, and the connection $\mathscr{O}_X(*Y) \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{O}_X(*Y)$ is given by $u \mapsto du + u d\varphi$. We call $\mathscr{E}^{\varphi}_{U|X}$ the exponential module with exponent φ.

For $c \in \mathbb{R}$, write for short

$$\{t < \operatorname{Re}\varphi + c\} := \{(x,t) \in U \times \mathbb{R} \,;\, t < \operatorname{Re}\varphi(x) + c\} \subset X \times \mathbb{R}.$$

Similarly to Remark 5.10, one can calculate explicitly $\mathscr{DR}^{\mathsf{T}}_X(\mathscr{M})$ when $\mathscr{M}$ is an exponential $\mathscr{D}$-module.

Proposition 5.11. *Let $Y \subset X$ be a closed complex analytic hypersurface, and set $U = X \setminus Y$. For $\varphi \in \mathscr{O}_X(*Y)$, there are isomorphisms*

$$\mathscr{DR}^{\mathsf{T}}_X(\mathscr{E}^{\varphi}_{U|X}) \simeq \mathrm{R}\mathscr{I}hom(\pi_X^{-1}\mathbb{C}_U, \text{“}\varinjlim_{c\to+\infty}\text{”}\ \mathbb{C}_{\{t<\operatorname{Re}\varphi+c\}})[d_X].$$

The next results are easy consequences of Theorem 4.3, Corollary 4.4, Corollary 4.7.

Theorem 5.12. *Let $f\colon X \to Y$ be a morphism of complex manifolds. Let $f_{\mathbb{R}}\colon X \times \mathbb{R}_\infty \to Y \times \mathbb{R}_\infty$ be the morphism induced by f.*

(i) *There is an isomorphism in $\mathsf{D}^{\mathrm{b}}((\pi_X^{-1} f^{-1}\mathscr{D}_Y)^{\mathrm{sub}})$*

$$(f_{\mathbb{R}})^{!}\mathscr{O}^{\mathsf{T}}_Y[d_Y] \simeq \pi_X^{-1}\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\pi_X^{-1}\mathscr{D}_X} \mathscr{O}^{\mathsf{T}}_X[d_X].$$

(ii) *For any $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$ there is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_X)$*

$$\mathscr{DR}^{\mathsf{T}}_X(\mathrm{D}f^*\mathscr{N})[d_X] \simeq (f_{\mathbb{R}})^{!}\,\mathscr{DR}^{\mathsf{T}}_Y(\mathscr{N})[d_Y].$$

(iii) *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$, and assume that* $\operatorname{Supp}(\mathscr{M})$ *is proper over Y. Then, there are isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_Y)$*

$$\mathscr{DR}^{\mathsf{T}}_Y(\mathrm{D}f_*\mathscr{M}) \simeq \mathrm{R}f_{\mathbb{R}*}\,\mathscr{DR}^{\mathsf{T}}_X(\mathscr{M}),$$

6. Main theorems

The Riemann–Hilbert correspondence for holonomic $\mathscr{D}$-modules can be stated as follows.

Theorem 6.1. *There exists a canonical isomorphism functorial with respect to $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$:*

$$\mathscr{M} \overset{\mathsf{D}}{\otimes} \mathscr{O}_X^{\mathsf{t}} \xrightarrow{\sim} \mathcal{I}\mathit{hom}^{\mathsf{E}}(\mathcal{S}\mathit{ol}_X^{\mathsf{T}}(\mathscr{M}), \mathscr{O}_X^{\mathsf{T}}) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{sub}}). \tag{6.1}$$

Applying the functor α_X to (6.1), we obtain

Theorem 6.2 (Enhanced Riemann–Hilbert correspondence). *There exists a canonical isomorphism functorial with respect to $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$:*

$$\mathscr{M} \xrightarrow{\sim} \mathcal{H}\mathit{om}^{\mathsf{E}}(\mathcal{S}\mathit{ol}_X^{\mathsf{T}}(\mathscr{M}), \mathscr{O}_X^{\mathsf{T}}) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X). \tag{6.2}$$

Thus we obtain the quasi-commutative diagram

$$\begin{tikzcd}[column sep=huge]
\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X) \arrow[r, "\sim", "\mathcal{S}\mathit{ol}_X"'] \arrow[d, tail] \arrow[rr, bend left, "\mathrm{id}"] & \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)^{\mathrm{op}} \arrow[r, "\sim", "\mathrm{R}\mathcal{H}\mathit{om}(\ast, \mathscr{O}_X^{\mathsf{t}})"'] \arrow[d, tail, "e"] & \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X) \arrow[d, tail] \\
\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) \arrow[r, "\mathcal{S}\mathit{ol}_X^{\mathsf{T}}"'] \arrow[rr, bend right, "\text{canonical}"'] & \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})^{\mathrm{op}} \arrow[r, "\mathcal{H}\mathit{om}^{\mathsf{E}}(\ast, \mathscr{O}_X^{\mathsf{T}})"'] & \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X).
\end{tikzcd}$$

Here the fully faithful functor $e\colon \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$ is defined by

$$e(F) := \mathbb{C}_X^{\mathsf{T}} \otimes \pi_X^{-1} F.$$

Theorem 6.2 shows that $\mathcal{S}\mathit{ol}_X^{\mathsf{T}}$ as well as $\mathcal{DR}_X^{\mathsf{T}}$ is faithful. In fact, we can also show the following full faithfulness of the enhanced de Rham functor.

Theorem 6.3. *For $\mathscr{M}, \mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$, one has an isomorphism*

$$\mathrm{R}\mathcal{H}\mathit{om}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \xrightarrow{\sim} \mathcal{H}\mathit{om}^{\mathsf{E}}(\mathcal{DR}_X^{\mathsf{T}}\mathscr{M}, \mathcal{DR}_X^{\mathsf{T}}\mathscr{N}).$$

In particular, the functor

$$\mathcal{DR}_X^{\mathsf{T}} : \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$$

is fully faithful.

Remark 6.4. Theorems 6.2 and 6.3 due to [DK13, Th. 9.6.1, Th. 9.7.1] are a natural formulation of the Riemann–Hilbert correspondence for irregular $\mathscr{D}$-modules. Theorem 6.1 is due to [KS14, Th. 4.5], which is a generalization of a theorem of J.-E. Björk ([Bj93]).

7. A brief outline of the proof of the main theorems

We reduce the main theorems to the exponential $\mathscr{D}$-module case, using the results of Mochizuki and Kedlaya.

7.1. *Real blow up*

A classical tool in the study of differential equations is the real blow up.

Recall that $\mathbb{C}^\times$ denotes $\mathbb{C} \setminus \{0\}$ and $\mathbb{R}_{>0}$ the multiplicative group of positive real numbers. Consider the action of $\mathbb{R}_{>0}$ on $\mathbb{C}^\times \times \mathbb{R}$:

$$\mathbb{R}_{>0} \times (\mathbb{C}^\times \times \mathbb{R}) \longrightarrow \mathbb{C}^\times \times \mathbb{R}, \quad (a, (z, t)) \longmapsto (az, a^{-1}t)$$

and set

$$\widetilde{\mathbb{C}}^{\mathrm{tot}} = (\mathbb{C}^\times \times \mathbb{R})/\mathbb{R}_{>0}, \ \widetilde{\mathbb{C}} = (\mathbb{C}^\times \times \mathbb{R}_{\geqslant 0})/\mathbb{R}_{>0}, \widetilde{\mathbb{C}}^{>0} = (\mathbb{C}^\times \times \mathbb{R}_{>0})/\mathbb{R}_{>0}.$$

One denotes by ϖ^{tot} the map:

$$\varpi^{\mathrm{tot}} \colon \widetilde{\mathbb{C}}^{\mathrm{tot}} \longrightarrow \mathbb{C}, \quad (z, t) \longmapsto tz. \tag{7.1}$$

Then we have

$$\widetilde{\mathbb{C}}^{\mathrm{tot}} \supset \widetilde{\mathbb{C}} \supset \widetilde{\mathbb{C}}^{>0} \xrightarrow{\sim} \mathbb{C}^\times.$$

Let $X = \mathbb{C}^n \simeq \mathbb{C}^r \times \mathbb{C}^{n-r}$ and let D be the divisor $\{z_1 \cdots z_r = 0\}$, where $(z_1, \dots, z_n)$ is a coordinate system on X. Set

$$\widetilde{X}^{\mathrm{tot}} = (\widetilde{\mathbb{C}}^{\mathrm{tot}})^r \times \mathbb{C}^{n-r}, \ \widetilde{X}^{>0} = (\widetilde{\mathbb{C}}^{>0})^r \times \mathbb{C}^{n-r}, \ \widetilde{X} = (\widetilde{\mathbb{C}})^r \times \mathbb{C}^{n-r}.$$

Then $\widetilde{X}$ is the closure of $\widetilde{X}^{>0}$ in $\widetilde{X}^{\mathrm{tot}}$. The map ϖ^{tot} in (7.1) defines the map

$$\varpi \colon \widetilde{X} \longrightarrow X.$$

The map ϖ is proper and induces an isomorphism

$$\varpi|_{\widetilde{X}^{>0}} \colon \widetilde{X}^{>0} = \varpi^{-1}(X \setminus D) \xrightarrow{\sim} X \setminus D.$$

We call $\widetilde{X}$ the *real blow up* of X along D.

Remark 7.1. The real manifold $\widetilde{X}$ (with boundary) as well as the map $\varpi \colon \widetilde{X} \to X$ may be intrinsically defined for a complex manifold X and a normal crossing divisor D, but $\widetilde{X}^{\mathrm{tot}}$ is only intrinsically defined as a germ of a manifold in a neighborhood of $\widetilde{X}$.

Definition 7.2. *Let $\mathcal{A}_{\widetilde{X}}$ be the subsheaf of $j_*(\mathscr{O}_{X\setminus D})$ consisting of holomorphic functions tempered at any point of $\widetilde{X} \setminus \widetilde{X}^{>0} = \varpi^{-1}(D)$. Here, $j\colon X \setminus D \simeq \widetilde{X}^{>0} \hookrightarrow \widetilde{X}$ is the open embedding. We set*

$$\mathscr{D}^{\mathcal{A}}_{\widetilde{X}} := \mathcal{A}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathscr{O}_X} \varpi^{-1}\mathscr{D}_X.$$

Then $\mathcal{A}_{\widetilde{X}}$ and $\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}$ are sheaves of rings on $\widetilde{X}$. We have a commutative diagram

$$\begin{tikzcd}
\varpi^{-1}\mathscr{O}_X \arrow[r, tail] \arrow[d, tail] & \varpi^{-1}\mathscr{D}_X \arrow[d, tail] \\
\mathcal{A}_{\widetilde{X}} \arrow[r, tail] & \mathscr{D}^{\mathcal{A}}_{\widetilde{X}}.
\end{tikzcd}$$

We have

$$\mathrm{R}\varpi_*\mathcal{A}_{\widetilde{X}} \simeq \mathscr{O}_X(*D).$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ we set:

$$\mathscr{M}^{\mathcal{A}} := \mathscr{D}^{\mathcal{A}}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathscr{D}_X} \varpi^{-1}\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}). \tag{7.2}$$

Then we obtain

$$\mathrm{R}\varpi_*\mathscr{M}^{\mathcal{A}}_{\widetilde{X}} \simeq \mathscr{M}(*D). \tag{7.3}$$

7.2. Normal form

The result in §1.3 for ordinary linear differential equations is generalized to higher dimensions by T. Mochizuki ([Mo09, Mo11]) and K.S. Kedlaya ([Ke10, Ke11]). In this subsection, we collect some of their results that we shall need.

Let X be a complex manifold and $D \subset X$ a normal crossing divisor. We shall use the notations introduced in the previous subsection: in particular the real blow up $\varpi\colon \widetilde{X} \to X$ and the notation $\mathscr{M}^{\mathcal{A}}$ of (7.2).

Definition 7.3. *We say that a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ has* a normal form *along D if*

(i) $\mathscr{M} \simeq \mathscr{M}(*D)$,

(ii) $\mathrm{SingSupp}(\mathscr{M}) \subset D$,

(iii) for any $x \in \varpi^{-1}(D) \subset \widetilde{X}$, there exist an open neighborhood $U \subset X$ of $\varpi(x)$ and finitely many $\varphi_i \in \Gamma(U; \mathscr{O}_X(*D))$ such that

$$(\mathscr{M}^{\mathcal{A}})|_V \simeq \Big(\bigoplus_i (\mathscr{E}^{\varphi_i}_{U\setminus D|U})^{\mathcal{A}}\Big)\Big|_V$$

for some open neighborhood V of x with $V \subset \varpi^{-1}(U)$.

A ramification of X along D on a neighborhood U of $x \in D$ is a finite map

$$p\colon X' \longrightarrow U$$

of the form

$$p(z'_1, \dots, z'_n) = (z'^{m_1}_1, \dots, z'^{m_r}_r, z'_{r+1}, \dots, z'_n)$$

for some $(m_1, \dots, m_r) \in (\mathbb{Z}_{>0})^r$. Here $(z'_1, \dots, z'_n)$ is a local coordinate system of X', and $(z_1, \dots, z_n)$ is a local coordinate system of X such that $D = \{z_1 \cdots z_r = 0\}$.

Definition 7.4. *We say that a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ has* a quasi-normal form *along D if it satisfies (i) and (ii) in Definition 7.3, and if for any $x \in D$ there exists a ramification $p\colon X' \to U$ on a neighborhood U of x such that $\mathrm{D}p^*(\mathscr{M}|_U)$ has a normal form along $p^{-1}(D \cap U)$.*

Remark 7.5. In the above definition, $\mathrm{D}p^*(\mathscr{M}|_U)$ as well as $\mathrm{D}p_*\mathrm{D}p^*(\mathscr{M}|_U)$ is concentrated at degree zero. Moreover, $\mathscr{M}|_U$ is a direct summand of $\mathrm{D}p_*\mathrm{D}p^*(\mathscr{M}|_U)$.

7.3. Results of Mochizuki and Kedlaya

The next result is an essential tool in the study of holonomic $\mathscr{D}$-modules and is easily deduced from the fundamental work of Mochizuki [Mo09, Mo11] (see also Sabbah [Sa00] for preliminary results and see Kedlaya [Ke10, Ke11] for the analytic case).

Theorem 7.6. *Let X be a complex manifold, $\mathscr{M}$ a holonomic $\mathscr{D}_X$-module and $x \in X$. Then there exist an open neighborhood U of x, a closed analytic hypersurface $Y \subset U$, a complex manifold X' and a projective morphism $f\colon X' \to U$ such that*

(i) $\mathrm{SingSupp}(\mathscr{M}) \cap U \subset Y$,
(ii) *$D := f^{-1}(Y)$ is a normal crossing divisor of X',*
(iii) *f induces an isomorphism $X' \setminus D \to U \setminus Y$,*
(iv) *$(\mathrm{D}f^*\mathscr{M})(*D)$ has a quasi-normal form along D.*

Remark that, under assumption (iii), $(\mathrm{D}f^*\mathscr{M})(*D)$ is concentrated at degree zero.

Using Theorem 7.6, one easily deduces the next lemma.

Lemma 7.7. *Let $P_X(\mathscr{M})$ be a statement concerning a complex manifold X and a holonomic object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. Consider the following conditions.*

(a) *Let $X = \bigcup_{i \in I} U_i$ be an open covering. Then $P_X(\mathscr{M})$ is true if and only if $P_{U_i}(\mathscr{M}|_{U_i})$ is true for any $i \in I$.*

(b) *If* $P_X(\mathscr{M})$ *is true, then* $P_X(\mathscr{M}[n])$ *is true for any* $n \in \mathbb{Z}$*.*
(c) *Let* $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \xrightarrow{+1}$ *be a distinguished triangle in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$*. If* $P_X(\mathscr{M}')$ *and* $P_X(\mathscr{M}'')$ *are true, then* $P_X(\mathscr{M})$ *is true.*
(d) *Let* $\mathscr{M}$ *and* $\mathscr{M}'$ *be holonomic* $\mathscr{D}_X$*-modules. If* $P_X(\mathscr{M} \oplus \mathscr{M}')$ *is true, then* $P_X(\mathscr{M})$ *is true.*
(e) *Let* $f\colon X \to Y$ *be a projective morphism and* $\mathscr{M}$ *a good holonomic* $\mathscr{D}_X$*-module. If* $P_X(\mathscr{M})$ *is true, then* $P_Y(\mathrm{D}f_*\mathscr{M})$ *is true.*
(f) *If* $\mathscr{M}$ *is a holonomic* $\mathscr{D}_X$*-module with a normal form along a normal crossing divisor of* X*, then* $P_X(\mathscr{M})$ *is true.*

If conditions (a)–(f) *are satisfied, then* $P_X(\mathscr{M})$ *is true for any complex manifold* X *and any* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$*.*

Sketch of the proof of the main theorems in §6. By applying Lemma 7.7, we reduce the assertions to the case of holonomic $\mathscr{D}$-modules with a normal form, then to the case of the exponential $\mathscr{D}$-modules. □

8. Stokes filtrations and enhanced de Rham functor

In this last section, we explain the relation between the enhanced solution sheaf and the Stokes filtration discussed in §1.5. Let us keep the notations in §1.3. In particular, recall that $0 \in X \subset \mathbb{C}$, $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module, $\varpi\colon \widetilde{X} \to X$ is the projection and $j\colon X \setminus \{0\} \hookrightarrow \widetilde{X}$ is the open embedding. We set $X^* := X \setminus \{0\}$. Let $\varpi_{\mathbb{R}}\colon \widetilde{X} \times \mathbb{R}_\infty \to X \times \mathbb{R}_\infty$ be the morphism induced by ϖ and let $i\colon S := \varpi^{-1}(0) \hookrightarrow \widetilde{X}$ be the closed embedding.

Set

$$\mathscr{M}' = \mathbb{D}_X((\mathbb{D}_X\mathscr{M})(*\{0\})).$$

Then we have a morphism $\mathscr{M}' \to \mathscr{M}$ such that it induces an isomorphism $\mathscr{M}'(*\{0\}) \xrightarrow{\sim} \mathscr{M}$.

We set

$$\mathscr{S}^{\mathsf{T}} := \mathscr{S}\!ol^{\mathsf{T}}(\mathscr{M}') \simeq \mathrm{R}\mathscr{H}\!om(\mathbb{C}_{X^*\times\mathbb{R}}, \mathscr{S}\!ol^{\mathsf{T}}(\mathscr{M})) \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}).$$

Since $\mathrm{e}^t\boldsymbol{u}_j(z)|_{D_{\theta_0}\times\mathbb{R}}$ is tempered on

$$\{t + \operatorname{Re}\varphi_j < c\} := \{(z,t) \in X^* \times \mathbb{R}\,;\, t + \operatorname{Re}(\varphi_j(z)) < c\}$$

for any c (see Proposition 5.11), we have

$$\mathbb{C}_{D_{\theta_0}\times\mathbb{R}} \otimes \mathscr{S}^{\mathsf{T}} \simeq \bigoplus_{1\leqslant j\leqslant r} \mathbb{C}_{D_{\theta_0}\times\mathbb{R}} \otimes \mathscr{S}^{\mathsf{T}}_{\varphi_j},$$

where

$$(8.1)\qquad \mathscr{S}_\varphi^{\mathsf{T}} := \text{“}\varinjlim_{c\to+\infty}\text{”}\ \mathbb{C}_{\{t+\mathrm{Re}\widetilde{\varphi}<c\}} \in \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}) \quad \text{for } \varphi\in\Phi.$$

Here $\widetilde{\varphi}\in\mathscr{O}_X(*\{0\})_0$ is a representative of $\varphi\in\Phi:=\mathscr{O}_X(*\{0\}/\mathscr{O}_X)_0$. Note that the right-hand side of of (8.1) does not depend on the choice of a representaive $\widetilde{\varphi}$.

Set

$$\widetilde{\mathscr{S}}_\varphi^{\mathsf{T}} := (\varpi_{\mathbb{R}})^{-1}\mathscr{S}_\varphi^{\mathsf{T}} \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\widetilde{X}\times\mathbb{R}_\infty}) \quad \text{and}$$
$$\widetilde{\mathscr{S}}^{\mathsf{T}} := (\varpi_{\mathbb{R}})^{!}\mathscr{S}^{\mathsf{T}} \simeq \mathrm{R}\mathscr{H}\!om(\mathbb{C}_{X^*\times\mathbb{R}}, (\varpi_{\mathbb{R}})^{-1}\mathscr{S}^{\mathsf{T}}) \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\widetilde{X}\times\mathbb{R}_\infty}).$$

Then set

$$K_\varphi := \mathscr{H}\!om^{\mathsf{E}}(\widetilde{\mathscr{S}}_\varphi^{\mathsf{T}}, \widetilde{\mathscr{S}}^{\mathsf{T}}) \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{\widetilde{X}}).$$

Since $\widetilde{\mathscr{S}}_\varphi^{\mathsf{T}}|_{X^*\times\mathbb{R}_\infty} \simeq \mathbb{C}^{\mathsf{T}}_{X^*}$, we have

$$K_\varphi|_{X^*} \simeq L := \mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)|_{X^*}.$$

Then we obtain a morphism of sheaves on S

$$i^{-1}K_\varphi \to i^{-1}j_*(K_\varphi|_{X^*}) \simeq \widetilde{L} := i^{-1}j_*L.$$

Lemma 8.1. *The object $i^{-1}K_\varphi \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_S)$ is concentrated at degree 0. The above morphism $i^{-1}K_\varphi \to \widetilde{L}$ is a monomorphism and its image coincides with F_φ.*

Thus, $\mathscr{S}\!ol^{\mathsf{T}}(\mathscr{M})$ recovers the Stokes filtration $\{F_\varphi\}_{\varphi\in\Phi}$ on $\widetilde{L}$.

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Kodaira fibrations and beyond: methods for moduli theory*

Fabrizio Catanese**

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'In Celebration of the 100th Anniversary of Kodaira's Birth'

Abstract. Kodaira fibred surfaces are remarkable examples of projective classifying spaces, and there are still many intriguing open questions concerning them, especially the slope question. The topological characterization of Kodaira fibrations is emblematic of the use of topological methods in the study of moduli spaces of surfaces and higher dimensional complex algebraic varieties, and their compactifications. Our tour through algebraic surfaces and their moduli (with results valid also for higher dimensional varieties) deals with fibrations, questions on monodromy and factorizations in the mapping class group, old and new results on Variation of Hodge Structures, especially a recent answer given (in joint work with Dettweiler) to a long standing question posed by Fujita. In the landscape of our tour, Galois coverings, deformations and rigid manifolds (there are by the way rigid Kodaira fibrations), projective classifying spaces, the action of the absolute Galois group on moduli spaces, stand also in the forefront. These questions lead to interesting algebraic surfaces, for instance remarkable surfaces constructed from VHS, surfaces isogenous to a product with automorphisms acting trivially on cohomology, hypersurfaces in Bagnera-de Franchis varieties, Inoue-type surfaces.

Keywords and phrases: algebraic surfaces, Kähler manifolds, moduli, deformations, topological methods, fibrations, Kodaira fibrations, Chern slope, automorphisms, uniformization, projective

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F. CATANESE
Mathematisches Institut, Lehrstuhl Mathematik VIII, Universitätsstraße 30, D-95447 Bayreuth, Deutschland
(e-mail: `fabrizio.catanese@uni-bayreuth.de`)

classifying spaces, monodromy, fundamental groups, variation of Hodge structure, absolute Galois group, locally symmetric varieties

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Contents

1. Kodaira fibrations

It is well known that the topological Euler characteristic e is multiplicative for fibre bundles: this means that, if $f : X \to B$ is a fibre bundle with fibre F, then

$$e(X) = e(B)e(F).$$

In 1957 Chern, Hirzebruch and Serre ([CHS57]) showed that the same holds true for the signature, also called index $\sigma = b^+ - b^-$ (it is the index of the intersection form on the middle cohomology group) if the fundamental group of the base B acts trivially on the (rational) cohomology of the fibre F.

In 1967 Kodaira [Kod67] constructed examples of fibrations of a complex algebraic surface over a curve which are differentiable but not holomorphic fibre bundles for which[1] the multiplicativity of the signature does not hold true. In his honour such fibrations are nowadays called Kodaira fibrations. In fact, for a compact oriented two dimensional manifold the intersection form is antisymmetric, hence $\sigma = 0$, whereas Kodaira fibrations have necessarily $\sigma > 0$.

As I am now going to explain, there are many interesting properties and open questions concerning Kodaira fibrations.

1.1. Generalities on algebraic surfaces

The signature formula of Hirzebruch, Atiyah and Singer for a compact complex surface S is

$$\sigma(S) = \frac{1}{3}(K_S^2 - 2e(S)) = \frac{1}{3}(c_1(S)^2 - 2c_2(S)).$$

Here the Euler number $e(S)$ is the alternating sum of the Betti numbers

$$e(S) = 1 - b_1(S) + b_2(S) - b_3(S) + 1 = 2 - 2b_1(S) + b_2(S),$$

and it equals the second Chern class $c_2(S)$ of the complex tangent bundle of S. Whereas $K_S = -c_1(S)$ is, for an algebraic surface, the Cartier divisor of a rational section of the sheaf Ω^2_S of holomorphic differential 2-forms.

In the Kähler case it was well known that the signature $\sigma(S) = b^+(S) - b^-(S)$ is determined by the Hodge numbers, indeed $b^+(S) = 2p_g(S) + 1$, $b^+(S) + b^-(S) = b_2(S)$, where

- $p_g(S) := h^{2,0}(S) := h^0(\Omega^2_S)$ is the **geometric genus** of S,
- $q(S) := h^{0,1}(S) := h^1(\mathcal{O}_S)$ is the **irregularity** of S,
- $h^{1,0}(S) := h^0(\Omega^1_S)$ is the Albanese number (dimension of the Albanese variety).

[1] Indeed, this is true for all such fibrations.

Even if I will concentrate on algebraic or Kähler manifolds, I would like to point out how the beautiful series of papers by Kodaira 'On the structure of compact complex analytic surfaces' [Kod64-8] (extending the Enriques classification to non-algebraic complex surfaces) begins with the following miraculous consequence of the signature formula:

$$(b^+(S) - 2p_g(S)) + (2q(S) - b_1(S)) = 1,$$

because both terms are easily shown to be non-negative, by the fact that the intersection form is positive definite on $H^0(\Omega^2_S) \oplus \overline{H^0(\Omega^2_S)}$, respectively that one has the exact sequence

$$0 \longrightarrow H^0(d\mathcal{O}_S) \longrightarrow H^1(S, \mathbb{C}) \longrightarrow H^1(\mathcal{O}_S),$$

and the Dolbeault inclusion $\overline{H^0(d\mathcal{O}_S)} \subset H^1(\mathcal{O}_S)$.

The non-Kähler case is just the case where the first Betti number $b_1(S)$ is odd, $b_1(S) = 2q(S) - 1, b^+(S) = 2p_g(S)$; this conjecture of Kodaira was proven through a long series of papers, culminating in [Siu83].

The so called 'surface geography' problem was raised by van de Ven ([vdV66]), and concerns the points of the plane with coordinates $(\chi(S), K_S^2)$; here $\chi(S) := \chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the Euler–Poincaré characteristic of the sheaf of holomorphic functions.

An important invariant, in the case $\chi(S) \geq 1$ (recall Castelnuovo's theorem: $\chi(S) < 0$ implies that S is ruled) is the so called slope

$$\nu := \nu(S) := K_S^2/\chi(S),$$

whose growth is equivalent to the one of the Chern slope, which is the ratio $\nu_C(S) := c_1(S)^2/c_2(S)$ of the Chern numbers, because of the Noether formula

$$\begin{aligned} 12\chi(S) &= c_1(S)^2 + c_2(S) \\ \implies c_1(S)^2/c_2(S) &= K_S^2/(12\chi(S) - K_S^2) = \frac{\nu}{12 - \nu}. \end{aligned}$$

As conjectured by van de Ven, inspired by work of Bogomolov, and proven by Miyaoka and Yau, there is the slope inequality,

$$\nu(S) = K_S^2/\chi(S) \leq 9 \iff \nu_C(S) = c_1(S)^2/c_2(S) \leq 3$$

for surfaces with $\chi(S) \geq 1$.

It is commonly called the **Bogomolov–Miyaoka–Yau inequality**.

Moreover, by the theorems of Aubin and Yau ([Aub78], [Yau77], [Yau78]) and Miyaoka's proof [Miy77], [Miy83] of ampleness of K_S in the case where equality holds, follows:

Theorem 1. *Is S a non-ruled surface and $K_S^2/\chi(S) = 9$, then the universal cover $\tilde{S}$ of S is biholomorphic to the ball, i.e., the unit disk $\mathcal{D}_2 \subset \mathbb{C}^2$.*

Indeed, the argument of proof about the existence of the Kähler–Einstein metric seemed to suggest that if the ratio $K_S^2/\chi(S)$ would be rather close to 9, then the universal cover $\tilde{S}$ of S would be diffeomorphic to Euclidean space. More than that, probably the imagination of many was struck by an impressive result by Mostow and Siu [M-S80]:

Theorem 2. *There exists an infinite series of rigid surfaces S with positive signature whose slope is very close to* 9,[2] *and which admit a metric of non-positive sectional curvature. In particular, by the theorem of Cartan–Hadamard, their universal cover* $\tilde{S}$ *is diffeomorphic to Euclidean space* $\mathbb{R}^4$.

Recently, Roulleau and Urzua [RU15] disposed in the negative of a stronger form of this no name-conjecture:

Theorem 3. *The slopes of simply connected surfaces are dense in the interval* $[8, 9]$.

They used a method introduced by Comessatti and, later, by Hirzebruch ([Hirz84]), to consider abelian coverings of the projective plane $\mathbb{P}^2$ branched over special configurations of curves, especially configurations of lines; we shall discuss this method at a later moment.

Of course if the slope $\nu(S) = 9$ then the universal cover $\tilde{S}$ is the ball, and the fundamental group is countable; while the result of [RU15] does not exclude a priori the existence of a small region around the Miyaoka–Yau line $K_S^2 = 9\chi(S)$ where the fundamental group is infinite and the universal cover is homeomorphic to a ball, it morally settles the question in the negative. A similar but even bolder conjecture, namely the so-called 'watershed conjecture', that every surface of positive index should have infinite fundamental group had been disproved thanks to Miyaoka's observation that the Galois closure S of a general projection of a surface $X \to \mathbb{P}^2$ tends to have positive index (see [Miy83], [MT87], and [CZ87] for other examples).

1.2. Kodaira fibrations and Kodaira's construction

We consider in this subsection the following situation: $f : S \to B$ is a holomorphic map of a compact complex surface S onto a curve B of genus b, which is a differentiable fibre bundle, with fibres F of genus g. In this case, the Euler number of S equals

$$e(S) = 4(g-1)(b-1).$$

This fibration clearly induces a morphism $\Phi : B \to \mathfrak{M}_g$ into the moduli space of curves of genus g.

[2] The maximum achieved by Mostow and Siu is 8.8575.

There are two cases: either Φ is constant, or, as one says, we have a non-constant moduli fibration. The second case is exactly the case of Kodaira fibrations. Following a generalization of Kodaira's method developed in our joint paper with Rollenske, we are going to show how this situation can be effectively constructed, and in such a way that we can calculate the slope $\nu(S)$ explicitly.

On the other hand, the fact that Kodaira fibrations do in fact exist, can be shown non-effectively in the following way.

1.2.1. General Kodaira fibrations Let $\overline{\mathfrak{M}_g}^*$ be the Satake compactification of the moduli space of curves of genus $g \geq 3$: this is the closure of $\mathfrak{M}_g$, embedded via the Torelli map into $\mathscr{A}_g$, inside the Satake compactification $\overline{\mathscr{A}_g}^*$.

From the moduli point of view, given a moduli-stable curve C of genus g, we associate to C the product of the Jacobian varieties of the components in the normalization $\tilde{C}$ of C. It follows that the boundary $\partial\overline{\mathfrak{M}_g}^*$ has dimension $3g-5$, hence codimension 2 inside $\overline{\mathfrak{M}_g}^*$. Similarly the singular set of $\overline{\mathfrak{M}_g}^*$ corresponds to the locus Σ of curves with automorphisms for $g \geq 4$, and is contained in Σ for $g = 3$; Σ has codimension at least 2 for $g \geq 4$, and codimension equal to 1 for $g = 3$ (its divisorial part is the locus of hyperelliptic curves).

By the projectivity of $\overline{\mathfrak{M}_g}^*$ we can find, for $g \geq 4$, a smooth linear section B of $\overline{\mathfrak{M}_g}^*$ having dimension 1 and which avoids both $\partial\overline{\mathfrak{M}_g}^*$ and the locus Σ.

So $B \subset \mathfrak{M}_g$ and, since $B \cap \Sigma = \emptyset$, we have a family of curves over B with non-constant moduli, and the fibre curves are all smooth.

We shall see in the sequel that a Kodaira fibration has always positive index, and indeed the Chern slope $\nu_C(S)$ lies in the open interval $(2, 3)$.

One can show that, via Kodaira fibrations obtained as just described from a general complete intersection curve B (under the composition of the Torelli map with the Satake embedding of the moduli space $\mathfrak{M}_g$ of curves of genus $g \geq 4$), one obtains a slope which is rather small, at most around 2.18 (see [CatRol09]).

The above argument shows, via a non-explicit construction, the existence of Kodaira fibrations for any fibre genus $g \geq 4$. The argument can also be adapted for the case $g = 3$, while for $g = 2$ there are no Kodaira fibrations; in fact $\mathfrak{M}_2$ is affine, by a theorem of Igusa [Ig60]: hence if B is a complete curve, any morphism $B \to \mathfrak{M}_2$ must be constant.

For $g = 3$ we take a smooth curve D which does not intersect $\partial\overline{\mathfrak{M}_g}^*$ and intersects Σ transversely in smooth points $p_1, \ldots, p_k$ corresponding to general hyperelliptic curves $C_1, \ldots, C_k$ (hence such that $\mathrm{Aut}(C_i) = \mathbb{Z}/2$). In this case we do not have a family over D, but only over $D \setminus \Sigma$. Take now a double covering $f : B \to D$ branched over the points $p_1, \ldots, p_k$ and over other points. Let $p'_j := f^{-1}(p_j)$: then we observe that for each $j = 1, \ldots, k$ the Kuranishi family of C_j has a map ψ to $\mathfrak{M}_3$ which is a double covering ramified over the hyperelliptic locus. In local coordinates we may assume that the map is given by

$$\psi : (y_1, y_2, \ldots, y_5, z) \longmapsto (y_1, y_2, \ldots, y_5, z^2), \quad i.e., \ w = z^2$$

and the curve D is locally given by

$$y_2 = \cdots = y_5 = 0, \quad y_1 = w,$$

hence

$$B = \{y_2 = \cdots = y_5 = 0, y_1 = z^2\}$$

and the family over $B \setminus f^{-1}(\Sigma)$ extends to a family over B, with all the fibres smooth genus 3 curves.

As discussed with Terasoma after the Takagi Lectures, the Kodaira construction does not work for $g = 3$. There is however an explicit construction for $g = 3$ due to Zaal [Zaal95], which uses Prym varieties, and other explicit constructions of curves B as above, by González-Diez and Harvey [GD-H91].

We do not know the answer to the following question, which might turn out to be not too difficult:

Question 4. Given an integer $g \geq 3$, which is the number $b(g)$, the minimum value $b \in \mathbb{N}$ such that there is a Kodaira fibration with fibres of genus g and base curve of genus b?

We just observe here that $b \geq 2$, and indeed $b \geq 3$ for $g = 3, 4$. Because for Kodaira fibrations we have $e(S) = 4(g-1)(b-1)$, the positive index inequality $K_S^2 > 2e(S) = 8(g-1)(b-1)$ and the Kefeng Liu inequality [Liu96] $K_S^2 < 9\chi(S)$ hold; combining with the Noether formula $K_S^2 = 12\chi(S) - e(S)$ one gets then

$$3(g-1)(b-1) < 3\chi(S) < 4(g-1)(b-1).$$

Our previous assertion follows immediately since, if $b = 2$ and $g = 3, 4$, then $3\chi(S) \geq 3g = 4(g-1) + (4-g) \geq 4(g-1)$, absurd.

Observe moreover that once you have a Kodaira fibration $f : S \to B$ with given fibre genus g, and base genus b, for each integer $n \geq 1$ we can take the Kodaira fibration $f_n : S(n) \to B$, where the fibres of $S(n)$ are the unramified coverings of the fibres F of f corresponding to the surjection $\pi_1(F) \to H_1(F, \mathbb{Z}/n)$. Then the fibre genus of f_n equals $g_n := 1 + (g-1)n^{2g}$, while the genus of the base curve remains b. This shows that

$$\liminf_{g \to \infty} (b(g)) = \min(b(g)).$$

1.2.2. Kodaira's construction and its generalizations Kodaira constructed explicit examples of these Kodaira fibrations, and we are going now to describe how his method can be generalized.

The basic notion for the Kodaira type construction is the following:

Definition 5. *A* **logarithmic Kodaira fibration** *is a quadruple (X, D, f, B) consisting of*

(1) a smooth fibration $\psi : X \to B$ of a surface to a curve, with fibres F_t and
(2) a divisor $D \subset X$ such that
 (a) the projection $D \to B$ is étale and
 (b) the fibration of pointed curves $(F_t, F_t \cap D)$ is not isotrivial, i.e., the fibres are not all isomorphic (as pointed curves).

Now, even if at first sight this does not seem to help, it really does: because now the fibration ψ might have constant moduli, but the points $F_t \cap D$ may be moving.

The easiest case to consider is the case where X will be a product of curves $X := B_1 \times B_2$ and D shall be a divisor such that the first projection $D \to B_1$ is étale and the second projection $D \to B_2$ is finite.

We shall now see that in order to construct Kodaira fibrations it suffices to construct log-Kodaira fibrations.

Proposition 6. *Let $(X, D) \to B$ be a log-Kodaira fibration and let $f : \tilde{F} \to F$ be a Galois-covering of a fibre F, with Galois group G, and branched over $D \cap F$. Then we can extend f to a ramified covering of surfaces $\bar{f} : S \to \tilde{X}$ obtaining a diagram*

$$\begin{tikzcd}
\tilde{F} \arrow[rr, hook] \arrow[d, "f"] & & S \arrow[d, "\bar{f}"] \\
F \arrow[rr, hook] \arrow[d, equal] & & \tilde{X} := g^* X \arrow[dl] \arrow[d] \\
F \arrow[r, hook] & X \arrow[d] & \tilde{B} \arrow[dl, "g"] \\
& B &
\end{tikzcd}$$

where

- *$g : \tilde{B} \to B$ is an étale covering,*
- *$\tilde{X}$ is the pullback of X via g,*
- *$\bar{f}$ is a ramified covering with Galois group G branched over $\tilde{D} := g^* D$ and such that $\bar{f}|_{\tilde{F}} = f$.*

The idea of the proof (for which we refer to [CatRol09]) is that the covering $\bar{f}|_{\tilde{F}} = f$ is determined by a monodromy homomorphism

$$\mu : \pi_1(F \setminus F \cap D) \longrightarrow G,$$

which extends on neighbouring fibres by the differential local triviality of the logarithmic fibration: but then there will be an action of $\pi_1(B)$ transforming the monodromy into another one. Since however there are only finitely many such monodromy homomorphisms to the finite group G, we get a tautological finite étale covering $g : \tilde{B} \to B$ associated to the 'monodromy of the monodromy', and then the ramified covering extends to the pull back of X.

The problem is thus reduced to finding disjoint étale correspondences between the two curves B_1, B_2, which give the connected components of the curve D we are looking for.

In turn, the easiest case of an étale correspondence is given by the following situation: D' is a curve with an automorphism group H, and there are two subgroups $H_1, H_2 < H$ acting freely on D'. We then set $B_j := D'/H_j$. Then D' admits a morphism $\psi : D' \to B_1 \times B_2$, such that the composition of ψ with both projections is étale.

If the intersection of the two subgroups $H_0 := H_1 \cap H_2$ is non-trivial, the map ψ factors through the quotient D'/H_0; in any case, ψ is injective if and only if there are no points $x \neq y$ such that $H_1 x = H_1 y$ and $H_2 x = H_2 y$ (i.e., there do exist $h_1 \in H_1 \setminus \{1_H\}, h_2 \in H_2$ such that $y = h_1 x$, $x = h_2 h_1 x$). Equivalently,

$$D' \longrightarrow B_1 \times B_2 \text{ embeds } D' \iff H_2(H_1 \setminus \{1_H\}) \cap \mathscr{S} = \emptyset,$$

where $\mathscr{S} = \{h \in H \mid \exists x \textit{ such that } hx = x\}$ is the set of stabilizers.

1.3. Slopes of Kodaira fibrations

An interesting and open question, raised by Le Brun, asks for the possible values of the Chern slope of a Kodaira fibred surface $f : S \to B$.

The Chern slope $\nu_C(S) := c_1^2(S)/c_2(S) = K_S^2/e(S)$ of a Kodaira fibred surface lies in the interval $(2, 3)$, in view of the well known Arakelov inequality (that shall be discussed in a later section) and of the improvement by Kefeng Liu ([Liu96]) of the Bogomolov–Miyaoka–Yau inequality to $K_S^2/e(S) < 3$.

Le Brun raised the question whether the slopes of Kodaira fibred surfaces can be effectively bounded away from 3: is it true that there exists $\epsilon > 0$ such that for a Kodaira fibred surface S we have $\nu_C(S) \leq 3 - \epsilon$?

The examples by Atiyah, Hirzebruch and Kodaira have slope not greater than $2 + 1/3 = 2.33\ldots$ (see [BPHV], page 221) and, as observed already, if one considers Kodaira fibrations obtained from a general complete intersection curve under the composition of the Torelli map with the Satake embedding in the moduli space $\mathfrak{M}_g$ of curves of genus $g \geq 3$, one obtains a smaller slope (around 2.18).

Now, given a Kodaira fibration $f : S \to B$, and any holomorphic map of curves $\varphi : B' \to B$, one can take the pull-back of f, namely the fibred product $S' := B' \times_B S$. The slope remains the same if φ is étale, but in case where we have a ramified map, then the slope decreases.

In fact, if we denote by d the degree of φ and by r the degree of the ramification divisor on B', then, denoting by b' the genus of B', we have $b'-1 = d(b-1)+r/2$, hence $e(S') = 4d(g-1)(b-1+\frac{r}{2d})$, while $K^2_{S'} = dK^2_S + 4r(g-1)$.

Hence

$$[\nu_C(S) - \nu_C(S')]\Big[4(g-1)(b-1)\Big(b-1+\frac{r}{2d}\Big)\Big] = \frac{r}{2d}(K^2_S - 8(b-1)(g-1)),$$

which is strictly positive as soon as $r > 0$.

We observe moreover that the slope

$$\nu_C(S') = \frac{K^2_S + 4\frac{r}{d}(g-1)}{4(g-1)(b-1+\frac{r}{2d})}$$

tends to 2 as soon as $\frac{r}{d}$ tends to infinity.

Therefore, once one has found a given slope, it looks more like a question of book-keeping to show that one can realize smaller slopes. While the hard question seems to be the one of finding higher slopes: for this reason we concentrate our attention on the problem of finding Kodaira fibrations with high slope.

The best known result in the direction of high slope is the following result of [CatRol09]:

Theorem 7 (Catanese–Rollenske). *There are Kodaira fibrations with slope equal to* $2 + 2/3 = 2.66\ldots$.

Our method of construction has been a variant of the one used by Kodaira, namely to consider *double Kodaira fibred surfaces*.

The first main point is that the slope of a Kodaira fibred surface $f : S \to B'$ obtained as a Galois branched covering of a logarithmic Kodaira fibred surface $\psi : (X, D) \to B$ is determined by the logarithmic structure of (X, D), namely, given the components $D_1, \ldots, D_r$ of D, one associates to D_i the branching integer m_i of f along the divisor D_i, and for instance the canonical divisor K_S is the pull-back of the logarithmic divisor

$$K_X + \sum_i \Big(1 - \frac{1}{m_i}\Big) D_i.$$

As a consequence, if $d = |G|$, G being the Galois group, then, since the curves D_i are disjoint ($D \to B$ being étale):

$$K_S^2 = d\Big(K_X + \sum_i \Big(1 - \frac{1}{m_i}\Big) D_i\Big)^2$$
$$= d\Big(K_X^2 + 2\sum_i \Big(1 - \frac{1}{m_i}\Big) K_X D_i + \sum_i \Big(1 - \frac{1}{m_i}\Big)^2 D_i^2\Big).$$

Similarly, since in this case D is smooth,

$$e(S) = d\Big[e(X) - \sum_i \Big(1 - \frac{1}{m_i}\Big) e(D_i)\Big]$$
$$= d\Big[e(X) + \sum_i \Big(1 - \frac{1}{m_i}\Big)(K_X D_i + D_i^2)\Big].$$

Since $(1 - \frac{1}{m_i})^2 < (1 - \frac{1}{m_i})$, we see that the slope $\nu_C(S)$ has little chance to become larger unless $D_i^2 < 0$.

This is however often the case: write $K_X = p^* K_B + K_{X|B}$, and observe that $K_{D_i} = p^* K_B$. Then $D_i^2 = K_B D_i - K_X D_i = -K_{X|B} D_i$, which is negative if the relative canonical divisor $K_{X|B}$ is nef (this fact is, under suitable assumptions, a consequence of Arakelov's theorem, and is used for the proof of the Mordell conjecture over function fields).

1.4. Double Kodaira fibrations and group theory

In [CatRol09] a logarithmic Kodaira fibration was defined to be **very simple** if $X = B \times B$ and each D_i is the graph of an automorphism of B, where the genus b of B is at least two.

In this case $D_i^2 = -2(b-1)$, $K_X D_i = 4(b-1)$, $K_X^2 = 8(b-1)^2$, $e(X) = 4(b-1)^2$ and for the Chern slope we have

$$\nu_C(S) = 2 + \frac{2(b-1)\sum_i (1 - \frac{1}{m_i^2})}{4(b-1)^2 + \sum_i (1 - \frac{1}{m_i}) 2(b-1)}$$
$$= 2 + \frac{\sum_i (1 - \frac{1}{m_i^2})}{2(b-1) + \sum_i (1 - \frac{1}{m_i})}.$$

Now, as the indices $m_i \to \infty$, the slope tends to

$$\nu_C = 2 + \frac{r}{2(b-1) + r} = 2 + \frac{\alpha}{2+\alpha} = 3 - \frac{2}{2+\alpha}, \quad \alpha := \frac{r}{(b-1)}.$$

Hence one can maximize the slope if one can maximize the ratio $\alpha = \frac{r}{(b-1)}$. One can indeed be more clever (as in loc. cit.) and take $m_i = 3$, so that

$$\nu_C(S) = 2 + \frac{8r}{18(b-1)+6r} = 2 + \frac{4\alpha}{9+3\alpha}.$$

Then, for $\alpha = 3$, we obtain

$$\nu_C(S) = 2 + \frac{12}{18} = \frac{8}{3}.$$

However, there are limits to maximizing α: first of all, by Hurwitz' theorem, $\alpha < 84$, since $|Aut(B)| \leq 84(b-1)$ [3].

This inequality is however much milder than the one given by the BMY-inequality, which implies $\alpha \leq 9$ (indeed we can show that $\alpha < 8$).

Now, the account for the drop from 84 to a much lower constant is due indeed to a cogent restriction, namely, that we want all the curves D_i to be disjoint!

We already showed that we would like to find an $\alpha > 3$, in particular the action of the group $G := Aut(B)$ on B cannot be free (otherwise $|G| \leq 2(b-1)!$).

Therefore, we denote as above by $\mathscr{S}$ the subset of stabilizers in G,

$$\mathscr{S} := \{g \in G \mid \exists x \in B, \ \textit{such that} \ \ gx = x\}.$$

Let $D_i = \{(x, g_i x)\}$. Then

$$D_i \cap D_j \neq \emptyset \Longleftrightarrow \exists x \in B, \ \textit{such that} \ \ g_i x = g_j x \Longleftrightarrow g_i^{-1} g_j \in \mathscr{S}.$$

The group theoretical question that we have therefore in mind is a sort of sphere packing problem for groups G acting on a curve. We define the sphere with center g to be the set $g\mathscr{S}$.

Our problem reduces to find a maximal number r such that there are elements $g_1, \dots, g_r$ so that $\forall i$ the sphere $g_i\mathscr{S}$ contains only the element g_i, and no other $g_j \neq g_i$.

Moreover, by the proof of Hurwitz's theorem, and some easy arguments, one sees that the quotient $B/G = \mathbb{P}^1$, and then that G is a finite quotient of a polygonal group

$$T(n_1, n_2, \dots, n_k) = \langle \gamma_1, \dots, \gamma_k \mid \gamma_1 \dots \gamma_k = 1, \gamma_i^{n_i} = 1 \ , \ \forall i = 1, \dots, k \rangle$$

which is hyperbolic, i.e., $\sum_{i=1}^k (1 - \frac{1}{n_i}) > 2$.

We wonder whether Lubotzky's theory of expander graphs might yield a possible method to approach the question of finding a large α, and also of giving a better upper bound for α ([Lub94]).

[3] Observe that if equality holds there are several automorphisms with fixed points.

1.5. Moduli of Kodaira fibrations

The study of moduli of Kodaira fibrations was initiated by Jost and Yau [J-Y83] (after that Kas [Ks68] had proven that the small deformations of Kodaira fibred surfaces are unobstructed), showing that all deformations yield again a Kodaira fibred surface.

Kodaira surfaces are a typical issue of the case where topology determines the moduli space, and the best characterization was obtained by Kotschick [Kot99]:

Theorem 8. *Let S be a complex surface. A Kodaira fibration on S with fibres of genus g and base curve of genus b is equivalent to the datum of*

(1) an exact sequence

$$1 \longrightarrow \Pi_g \longrightarrow \pi_1(S) \longrightarrow \Pi_b \longrightarrow 1$$

(here Π_g denotes the fundamental group of a compact curve of genus g) such that:

(2)

$$e(S) = 4(b-1)(g-1),$$

(3) the monodromy homomorphism $m : \Pi_b \to Out(\Pi_g)$, induced by conjugation in the previous exact sequence, has infinite image.

Kotschick used the methods that we shall review in Sect. 3, especially the following facts:

1) a fibration of a Kähler manifold X over a curve of genus $b \geq 2$ without multiple fibres is determined by a surjection $\pi_1(X) \to \Pi_b$ with finitely generated kernel [Cat03b], [Cat08],
2) the Zeuthen–Segre formula says that, in the case where X is a surface S, $e(S) \geq 4(b-1)(g-1)$, equality holding, when $g \geq 2$, if and only if we have a differentiable bundle,
3) the fibration has constant moduli if and only if the image of the monodromy m is finite.

We see then that every surface with the same fundamental group and Euler number as S is again Kodaira fibred with the same fibre genus g and base genus b.

Of course, a major question which remains is:

Question 9. Let S be a Kodaira fibred surface: do then the surfaces with the same fundamental group and Euler number as S form a connected component of the moduli space, or the union of a connected component with its complex conjugate component?

One may moreover ask the following question.

Question 10. How many Kodaira fibrations can a given algebraic surface possess?

Do there exist surfaces S with three distinct Kodaira fibrations?

Kodaira's original example, whose generalization was explained in the previous subsection, shows that S can have two distinct Kodaira fibrations, and one can indeed see that this is again a topological condition.

Proposition 11. *Let S be a complex surface. A double Kodaira fibration on S is equivalent to the datum of two exact sequences*

$$1 \longrightarrow \Pi_{g_i} \longrightarrow \pi_1(S) \xrightarrow{\bar{\psi}_i} \Pi_{b_i} \longrightarrow 1, \quad i = 1, 2,$$

(here Π_g denotes as before the fundamental group of a compact curve of genus g) such that:

(1) the monodromy homomorphisms $m_i : \Pi_{b_i} \to Out(\Pi_{g_i})$ have infinite image;
(2) $b_i \geq 2, g_i \geq 3$,
(3) the composition homomorphism

$$\Pi_{g_1} \longrightarrow \pi_1(S) \xrightarrow{\bar{\psi}_2} \Pi_{b_2}$$

is neither zero nor injective, and
(4) the Euler characteristic of S satisfies

$$e(S) = 4(b_1 - 1)(g_1 - 1) = 4(b_2 - 1)(g_2 - 1).$$

The above result shows that surfaces admitting a double Kodaira fibration form a closed and open subset in the moduli spaces of surfaces of general type; since for these one has a realization as a branched covering $S \to B_1 \times B_2$, branched over a divisor $D \subset B_1 \times B_2$, it makes sense to distinguish the étale case where D is smooth and the two projections $D \to B_i$ are étale. It is not clear a priori that this property is also open and closed, but in [CatRol09] we were able to prove it.

Theorem 12. *Double étale Kodaira fibrations form a closed and open subset in the moduli space of surfaces of general type.*

Concerning the previous Question 9 there are only partial results, Jost and Yau studied the deformations of the original example of Kodaira, while [CatRol09] considered a more general question for an important special class of double étale Kodaira fibrations. To describe the latter results we recall a definition from [CatRol09].

Definition 13. *A double étale Kodaira fibration is said to be* **standard** *if the logarithmic double Kodaira fibred surface* $(B_1 \times B_2, D)$ *reduces, after étale base changes for* B_1 *and* B_2*, to the very simple case of a logarithmic double Kodaira fibred surface* $(B \times B, D)$ *for which* D *is a union of graphs of automorphisms of* B.

Theorem 14. *The subset of the moduli space corresponding to* standard *double étale Kodaira fibred surfaces* S *with a fixed fundamental group consists of at most two irreducible connected components, exchanged by complex conjugation, which are isomorphic to the moduli space of pairs* (B, G)*, where* B *is a curve of genus* b *at least two and* G *is a group of biholomorphisms of* B *of a given topological type.*

In the above theorem G is the group generated by the automorphisms whose graphs yield D. For more details on the topological type of the action of a group G on an algebraic curve, see for instance [Cat15] or [CLP15,CLP16].

An interesting by-product of the study of the moduli space of standard double étale Kodaira fibrations is the following result which contradicted a spread belief ([M-S80]), and shows that Kefeng Liu's theorem that a Kodaira fibration cannot exist on a (free) ball quotient cannot be shown invoking non-rigidity.

Theorem 15. *There are double Kodaira fibred surfaces* S *which are rigid.*

Using these explicit descriptions Rollenske went further in [Rol10] and showed that, in the case where the branched cover has a cyclic Galois group, then the closure of this irreducible component inside the Kollár–Shepherd-Barron–Alexeev [K-SB88] compactification is again a connected component. One should observe that there are extremely few examples where the connected components of the KSBA compact moduli space have been investigated, apart from the obvious case of rigid surfaces (see [LW12] for the case of surfaces isogenous to a product).

Finally, concerning the existence problem for Kodaira fibrations, we have the following

Question 16. Given an exact sequence

$$1 \longrightarrow \Pi_g \longrightarrow \pi \longrightarrow \Pi_b \to 1$$

such that the image of $m : \Pi_b \to \mathcal{M}ap_g := Out^+(\Pi_g)$ is infinite, when does there exist a Kodaira fibred surface S with $\pi_1(S) \cong \pi$?

An obvious necessary condition is that the Abelianization $\pi^{ab} = \pi/[\pi, \pi]$ has even rank, by Hodge theory.

To the monodromy m is associated a continuous map $f : B \to \mathcal{T}_g/Im(m)$, where $\mathcal{T}_g$ denotes Teichmüller space, equivalently, an m-equivariant map of the universal cover $\tilde{B}$, $\tilde{f} : \tilde{B} \to \mathcal{T}_g$. It is at present not clear to me if it is proven that f can be deformed to a harmonic map; the main difficulty seems however to be to show the holomorphicity of such a harmonic map, for which the condition on the Betti number being even is the first obstruction.

Recently Arapura[4] observed that there are other necessary conditions. Let m_H be the monodromy on $V := \Pi_g^{ab} \otimes \mathbb{Q}$, i.e., the monodromy on cohomology, and let G be the connected component of the identity in the Zariski closure of $Im(m_H)$. Then we can first of all replace the above condition on the parity of the first Betti number as the condition that the space V_G of G-coinvariants (the largest quotient on which G acts trivially) has even dimension.

Then there are necessary conditions in special cases: for instance, in the case where the space V_G of coinvariants is zero, G must be semi-simple of classical Hermitian type. In other words, if $V_G = 0$, then $G(\mathbb{R})^0/K$ must be a Hermitian Symmetric Domain of classical type (here K is a maximal compact subgroup).

2. Projective varieties which are classifying spaces

2.1. Generalities on projective varieties which are classifying spaces

Definition 17. *Define $\mathcal{PC}$ as the class of projective varieties which are classifying spaces for their fundamental group $\pi_1(Z)$: equivalently, $\mathcal{PC}$ is the class of projective varieties Z whose universal covering $\tilde{Z}$ is contractible.*

The class $\mathcal{PC}$ is stable for Cartesian products, and for étale coverings, hence also for the relation of isogeny.

Definition 18. *Two varieties X, Y are said to be isogenous if there exist a third variety Z, and étale finite morphisms $f_X : Z \to X$, $f_Y : Z \to Y$.*

The class $\mathcal{PC}$ is however not stable for taking hyperplane sections: because for a compact manifold which is a classifying space its real dimension is read off by the top non-zero cohomology group $H^m(\pi, \mathbb{Z}/2)$, where $\pi = \pi_1(Z)$: since if Z is a classifying space, or a $K(\pi, 1)$ as one also says in topology, then $H^m(\pi, R) \cong H^m(Z, R)$, for any ring of coefficients R.

Projective curves C of genus g are, by virtue of the uniformization theorem, in the class $\mathcal{PC}$ if and only if $g \geq 1$. For $g \geq 2$ they are the quotients $C = \mathcal{H}/\Gamma$, where $\mathcal{H} := \{z \in \mathbb{C} \mid Im(z) > 0\}$ and $\Gamma \subset \mathbb{P}SL(2, \mathbb{R})$ is a discrete subgroup isomorphic to Π_g (then necessarily the action is free and cocompact).

Primary examples of projective varieties which are $K(\pi, 1)$'s are curves and Abelian varieties, and the varieties which are isogenous to a product of these. Particularly interesting are the varieties isogenous to a product of curves of genera at least 2.

[4] Talk at the Conference for Bob Friedman's 60-th birthday, in May 2015, and [Ara15].

It is interesting to observe ([Cat15], Corollary 82) that Abelian varieties are exactly the projective $K(\pi, 1)$ varieties, for which π is an abelian group.

2.1.1. Locally symmetric manifolds of negative type A very interesting class of projective varieties which are $K(\pi, 1)$'s are the **locally symmetric manifolds** Z with ample canonical divisor K_Z. These are in some sense a generalization in higher dimension of curves of genus $g \geq 2$: because the upper half plane $\mathscr{H} := \{z \in \mathbb{C} \mid Im(z) > 0\}$ is biholomorphically equivalent to the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

Locally symmetric manifolds Z with ample canonical divisor K_Z (also called locally Hermitian symmetric manifolds of negative curvature) are the quotients of a **bounded symmetric domain** $\mathscr{D}$ by a cocompact discrete subgroup $\Gamma \subset Aut(\mathscr{D})$ acting freely.

Recall that a bounded symmetric domain $\mathscr{D}$ is a bounded domain $\mathscr{D} \Subset \mathbb{C}^n$ such that its group $Aut(\mathscr{D})$ of biholomorphisms contains, for each point $p \in \mathscr{D}$, a holomorphic automorphism σ_p such that $\sigma_p(p) = p$, and such that the derivative of σ_p at p is equal to $-Id$. This property implies that σ is an involution (i.e., it has order 2), and that $Aut(\mathscr{D})^0$ (the connected component of the identity) is transitive on $\mathscr{D}$; therefore one can write $\mathscr{D} = G/K$, where G is a connected Lie group, and K is a maximal compact subgroup.

The classification of these bounded symmetric domains, done by Élie Cartan in [Car35], is based on the fact that such a $\mathscr{D}$ splits uniquely as the product of irreducible bounded symmetric domains.

$\mathscr{D}$ is a complete Riemannian manifold of negative sectional curvature, hence it is contractible, by the Cartan–Hadamard theorem, and $Z = \mathscr{D}/\Gamma$ is a classifying space for the group $\Gamma \cong \pi_1(X)$.

There are four series of non-sporadic bounded irreducible domains, in their bounded realization, plus two exceptional types:

(i) $I_{n,p}$ is the domain $\mathscr{D} = \{Z \in Mat(n, p; \mathbb{C}) : \mathrm{I}_p - {}^t\mathrm{Z} \cdot \bar{Z} > 0\}$.

(ii) II_n is the intersection of the domain $I_{n,n}$ with the subspace of skew symmetric matrices.

(iii) III_n is instead the intersection of the domain $I_{n,n}$ with the subspace of symmetric matrices.

(iv) The Cartan–Harish-Chandra realization of a domain of type IV_n in $\mathbb{C}^n$ is the subset $\mathscr{D}$ defined by the inequalities (compare [Helga78], page 527)

$$|z_1^2 + z_2^2 + \cdots + z_n^2| < 1,$$
$$1 + |z_1^2 + z_2^2 + \cdots + z_n^2|^2 - 2\left(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2\right) > 0.$$

(v) $\mathscr{D}_{16}$ is the exceptional domain of dimension $d = 16$.

(vi) $\mathscr{D}_{27}$ is the exceptional domain of dimension $d = 27$.

Each of these domains is contained in the so-called **compact dual**, which is a Hermitian symmetric spaces of compact type, the easiest example being, for type I, the Grassmann manifold.

Among the bounded symmetric domains are the so called **bounded symmetric domains of tube type**, those which are biholomorphic to a **tube domain**, a generalized Siegel upper half-space

$$T_{\mathscr{C}} = \mathbb{V} \oplus \sqrt{-1}\mathscr{C}$$

where $\mathbb{V}$ is a real vector space and $\mathscr{C} \subset \mathbb{V}$ is a *symmetric cone*, i.e., a self dual homogeneous convex cone containing no full lines.

In the case of type III domains, the tube domain is Siegel's upper half space:

$$\mathscr{H}_g := \{\tau \in Mat(g, g; \mathbb{C}) \mid \tau = {}^t\tau, Im(\tau) > 0\},$$

a generalisation of the upper half-plane of Poincaré.

Borel proved in [Bore63] that for each bounded symmetric domain $\mathscr{D}$ there exists a compact free quotient $X = \mathscr{D}/\Gamma$, called a compact Clifford–Klein form of the symmetric domain $\mathscr{D}$.

A classical result of J. Hano (see [Hano57] Theorem IV, page 886, and Lemma 6.2, page 317 of [Mil76]) asserts that a bounded homogeneous domain that is the universal cover of a compact complex manifold is symmetric.

2.1.2. Kodaira fibrations The Kodaira fibrations $f : S \to B$ are a remarkable example of surfaces in the class $\mathscr{PC}$.

Because S is a smooth projective surface and it is known that all the fibres of f are smooth curves of genus $g \geq 3$, whereas the base curve has genus $b \geq 2$.

By the fundamental group exact sequence

$$1 \longrightarrow \Pi_g \longrightarrow \pi_1(S) \longrightarrow \Pi_b \longrightarrow 1$$

the universal cover $\tilde{S}$ is a differentiable fibre bundle over $\tilde{B}$ with fibre $\tilde{F}$, hence it is diffeomorphic to a ball of real dimension 4.

By simultaneous uniformization ([Bers60]) the universal covering $\tilde{S}$ of a Kodaira fibred surface S is biholomorphic to a bounded domain in $\mathbb{C}^2$ (fibred over the unit disk $\Delta := \{z \in \mathbb{C} \mid |z| < 1\}$ with fibres isomorphic to Δ), which is not homogeneous.

$\tilde{S}$ is not homogeneous by the Hirzebruch proportionality principle; indeed there are only two bounded homogeneous domains in dimension 2: the bidisk and the 2-ball. The bidisk is biholomorphic to $\mathscr{H} \times \mathscr{H}$, and its group of biholomorphisms is a semi-direct product of $Aut(\mathscr{H}) \times Aut(\mathscr{H})$ by $\mathbb{Z}/2$, in particular one has Chern forms for the tangent bundle, invariant by automorphisms, which can be written as

$$c_1 := \phi(z_1)\, dz_1 \wedge d\bar{z}_1 \oplus \phi(z_2)\, dz_2 \wedge d\bar{z}_2,$$
$$c_2 := \phi(z_1)\phi(z_2)\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = \frac{1}{2}c_1^2.$$

Hence the Chern index $\nu_C = 2$ if the universal cover of S is the bidisk, i.e., $c_1(S)^2 = 2c_2(S)$; whereas, if the universal cover is the ball, $c_1(S)^2 = 3c_2(S)$ by a similar argument ([Hirz58]).

2.1.3. Known projective classifying spaces in complex dimension two In complex dimension 2, we have the following list of projective classifying spaces:

(1) Abelian surfaces.
(2) Hyperelliptic surfaces: these are the quotients of a complex torus of dimension 2 by a finite group G acting freely, and in such a way that the quotient is not again a complex torus.

    They have $p_g = 0, q = 1$.

    These surfaces were classified by Bagnera and de Franchis ([BdF08], see also [ES09] and [BPHV]) and they are obtained as quotients $(E_1 \times E_2)/G$ where E_1, E_2 are two elliptic curves, and G is an abelian group acting on E_1 by translations, and on E_2 effectively and in such a way that $E_2/G \cong \mathbb{P}^1$. In other words, these are exactly the surfaces isogenous to a product of curves of genus 1 which are not Abelian surfaces.
(3) Surfaces isogenous to a product of curves $C_1 \times C_2$, where C_1 has genus 1, C_2 has genus $g_2 \geq 2$.

    These are quotients $(C_1 \times C_2)/G$, where the finite group G acts freely, and where we can assume that G acts by a faithful diagonal action

$$g(x, y) = (g(x), g(y))$$

    (we regard G as $G \subset Aut(C_1), G \subset Aut(C_2)$). These surfaces have Kodaira dimension 1.
(4) Ball quotients.
(5) Bidisk quotients, divided into the **reducible case**, the case of surfaces $(C_1 \times C_2)/G$, isogenous to a product of curves of genera $g_i \geq 2$, and the **irreducible case** (for these rigidity holds, as proven by Jost and Yau [J-Y85]).
(6) Kodaira fibred surfaces.
(7) Mostow–Siu surfaces of negative curvature ([M-S80]): these are branched coverings of ball quotients, admitting a metric of negative scalar curvature. Their Chern slopes $\nu_C(S)$ are very close to 3, but strictly smaller than 2.96. These surfaces are rigid, in particular they are not the Kodaira examples of Kodaira fibrations.

(8) More examples are gotten from coverings of the plane branched over configurations of lines, as we shall discuss in a later section (see [Zheng99], and [Pan09], [Pan11]).

Proposition 19. *Every surface in $\mathscr{PC}$ is necessarily minimal, and indeed it contains no rational curve.*

Proof. In fact, if we have a rational curve $\phi : \mathbb{P}^1 \to S$, then the map ϕ lifts to the universal cover, and is then null-homotopic because $\tilde{S}$ is contractible. Hence also ϕ is null-homotopic, and the image rational curve $C := \phi(\mathbb{P}^1)$ is homologous to zero; this is a contradiction, as soon as S is a Kähler surface, a fortiori if S is projective. $\square$

Corollary 20. *Surfaces in $\mathscr{PC}$ which are not of general type are exactly type (1), (2) and (3) above. Any surface in $\mathscr{PC}$ which is of general type has ample canonical divisor.*

Proof. The result follows by minimality and by Enriques' classification, if the Kodaira dimension is < 1.

If S is of general type, then it contains no rational curves, in particular no (-2)-curves, hence K_S is ample.

If the Kodaira dimension is one, then S is minimal, properly elliptic, which means that a multiple of the canonical divisor yields a morphism $f : S \to B$ with general fibre an elliptic curve. By Kodaira's classification of singular fibres of elliptic fibrations ([Kod60]) follows that every fibre is either smooth elliptic, or multiple of a smooth elliptic curve. We use now the orbifold fundamental group exact sequence (see [CKO03], [Cat03b] or [Cat08])

$$\pi_1(F) \longrightarrow \pi_1(S) \longrightarrow \pi_1^{orb}(f) \longrightarrow 1.$$

Here the orbifold fundamental group is defined as $\pi_1(B^*)/K$, B^* being the set of non-critical values of f, and K is normally generated by $\gamma_i^{m_i}$, for each geometric loop going around a point $p_i \in B \setminus B^*$ for which the fibre $f^{-1}(p_i)$ is multiple of multiplicity m_i. Hence there is an intermediate covering $\hat{S} \to \hat{B}$, possibly of infinite degree, such that all the fibres are smooth elliptic curves, and where $\hat{B}$ is simply connected.

There is also a finite ramified covering $B' \to B$ such that the pull back $f' : S' \to B'$ has all the fibres which are smooth. The j-invariant is constant on B' since B' is projective. Hence all the smooth fibres are isomorphic to a fixed elliptic curve E and therefore, since S is projective, we obtain another finite cover $B'' \to B'$ such that $S'' = E \times B''$. Since the Kodaira dimension of S is one, we obtain that B'' has genus at least two, and there exists another étale covering $C \to B''$ such that $C \to B$ is Galois hence $S = (E \times C)/G$. $\square$

2.2. *Galois conjugate of projective classifying spaces*

Let $X \subset \mathbb{P}^n$ be a complex projective variety and let $Aut(\mathbb{C})$ be the group of field automorphisms of $\mathbb{C}$.

Then, for each $\sigma \in Aut(\mathbb{C})$, the conjugate variety X^σ is, as a set, simply $\sigma(X)$: $\sigma(X)$ is the projective variety defined by the ideal I_X^σ, obtained from the ideal I_X of X applying the homomorphism σ to the coefficients of the polynomials in I_X.

If σ is complex conjugation, we get $\bar{X}$, which is diffeomorphic to X.

Observe that, by the theorem of Steiniz, one has a surjection $Aut(\mathbb{C}) \to Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and that we have an action of the absolute Galois group $GAL := Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on the set of varieties X defined over $\bar{\mathbb{Q}}$.

X and the conjugate variety X^σ have the same Hodge numbers and Chern numbers. In particular, for curves, the genus is preserved.

It is also immediate that Galois conjugation by $\sigma \in Aut(\mathbb{C})$ preserves products and the equivalence relation given by isogeny, indeed Galois conjugation does not change the algebraic fundamental group, as shown by Grothendieck [SGA1].

Theorem 21. *Conjugate varieties X, X^σ have isomorphic algebraic fundamental groups*

$$\pi_1(X)^{alg} \cong \pi_1(X^\sigma)^{alg},$$

($\pi_1(X)^{alg}$ is the profinite completion of the topological fundamental group $G := \pi_1(X)$, i.e., $\pi_1(X)^{alg}$ is the inverse limit of the factor groups G/K, K being a normal subgroup of finite index in G).

It is easy to see ([Cat15], Theorem 223) that

i) If X is an Abelian variety, the same holds for any Galois conjugate X^σ.
ii) If S is a Kodaira fibred surface, then any Galois conjugate S^σ is also Kodaira fibred.

The following attempt of conjecture is based mainly on the fact that it holds for all known examples.

Conjecture 22. Assume that X is a projective $K(\pi, 1)$, and assume $\sigma \in Aut(\mathbb{C})$.

Is then the conjugate variety X^σ still a classifying space $K(\pi', 1)$?

We know since long, thanks to the result obtained by J.-P. Serre [Ser64] in the 60's , that it is not true in general that $\pi_1(X^\sigma) \cong \pi_1(X)$. Serre showed that there exists a field automorphism σ in the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and a variety X defined over a number field, such that X and the Galois conjugate variety X^σ have non-isomorphic fundamental groups, in particular they are not homeomorphic.

This is also false for surfaces in the class $\mathscr{P}\mathscr{C}$, for instance in a joint paper with I. Bauer and F. Grunewald [BCG14] we obtained the following theorem.

Theorem 23. *If $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ is not in the conjugacy class of complex conjugation, then there exists a surface isogenous to a product S such that S and the Galois conjugate surface S^σ have non-isomorphic fundamental groups.*

I. Bauer and F. Grunewald and the author ([BCG06], [BCG14]) discovered also many explicit examples of algebraic surfaces isogenous to a product for which the same phenomenon holds (observe that the proof of the general theorem is, as it may be surmised, non-constructive).

Remark 24. González-Diez and Jaikin-Zapirain [GD-JZ16] later extended theorem 23 to all automorphisms σ different from complex conjugation.

2.3. Some characterizations of locally symmetric varieties and Kazhdan's theorem in refined form

Proceeding with other projective $K(\pi, 1)$'s, the question becomes more subtle and we have to appeal to a famous theorem by Kazhdan on arithmetic varieties (see [Kazh70], [Kazh83], [Milne01], [CaDS12], [C-DS14], [ViZu07]).

Here the result is in the end much stronger: not only the conjugate variety is again locally symmetric, but the universal cover is indeed the same bounded symmetric domain!

Theorem 25. *Assume that X is a projective manifold with K_X ample, and that the universal covering $\tilde{X}$ is a bounded symmetric domain.*
Let $\tau \in Aut(\mathbb{C})$ be an automorphism of $\mathbb{C}$.
Then the conjugate variety X^τ has universal covering $\tilde{X^\tau} \cong \tilde{X}$.

The above result rests in an essential way on the Aubin–Yau theorem ([Yau78], [Aub78]) about the existence of a Kähler–Einstein metric for a projective manifold with ample canonical divisor K_X, and on the results of Berger [Ber53].

These results allow precise algebro-geometric characterizations of such locally symmetric varieties. These results were pioneered by Yau [Yau88] [Yau93]; his treatment of the non-tube case is however incorrect (he claims that (1) of Theorem 26, which characterizes the tube case, holds also in the non-tube case).

Simpler proofs follow from recent results obtained together with Antonio Di Scala. These results yield a simple and precise characterization of varieties possessing a bounded symmetric domain as universal cover, without having to resort to the existence of a finite étale covering where the holonomy splits.

For the tube case, we have the following theorem (see [CaDS12]), whose simple underlying idea is best illustrated in the case where the universal covering is a polydisk.

In this case one observes that, due to the nature of the automorphism group of $\mathscr{H}^n$ as semi-direct product of $Aut(\mathscr{H})^n$ with the symmetric group in n letters, the following tensor

$$\Psi := \frac{dz_1 \cdots dz_n}{dz_1 \wedge \cdots \wedge dz_n}$$

is a semi-invariant for the group of automorphisms, it is multiplied by ± 1 according to the sign of the corresponding coordinates permutation.

It therefore descends to a section $0 \neq \phi \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$, where η is the 2-torsion invertible sheaf associated to the sign character of the fundamental group $\pi_1(X) \cong \Gamma < Aut(\mathscr{H}^n)$ (observe that, depending on the choice of Γ, η may be trivial).

The existence of such a tensor is unfortunately, in complex dimension $n \geq 4$, not a characterizing property of polydisk quotients.

Indeed, in dimension 4, we have the bounded domain $\Omega \subset \mathbb{C}^4$,

$$\Omega = \{Z \in Mat(2,2;\mathbb{C}) \ : \ \mathrm{I}_2 - {}^tZ \cdot \bar{Z} > 0\},$$

the bounded (Harish-Chandra) realization of the Hermitian symmetric space $SU(2,2)/S(U(2) \times U(2))$.

Here the holonomy action of $(A, D) \in S(U(2) \times U(2))$ is given by $Z \mapsto AZD^{-1}$. Hence, the square of the determinant of Z yields a section Ψ which descends to a section $0 \neq \phi \in H^0(S^n(\Omega^1_X)(-K_X))$.

The main difference with respect to the polydisk quotient case is that the corresponding hypersurface in the projectivized tangent bundle of X is non-reduced, since we start from an invariant hypersurface of degree 4 which is twice a smooth quadric. With these examples in mind it should be easier to get the flavour of the following theorem ([CaDS12]).

Theorem 26. *Let X be a compact complex manifold of dimension n with K_X ample.*

Then the following two conditions (1) *and* (1′)*, resp.* (2) *and* (2′) *are equivalent:*

(1) *X admits a slope zero tensor $0 \neq \psi \in H^0(S^{mn}(\Omega^1_X)(-mK_X))$, (for some positive integer m);*
(1′) *$X \cong \Omega/\Gamma$, where Ω is a bounded symmetric domain of tube type and Γ is a cocompact discrete subgroup of $Aut(\Omega)$ acting freely.*
(2) *X admits a semi-special tensor $0 \neq \phi \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$, where η is a 2-torsion invertible sheaf, such that there is a point $p \in X$ for which the corresponding hypersurface $F_p := \{\phi_p = 0\} \subset \mathbb{P}(TX_p)$ is reduced.*
(2′) *The universal cover of X is a polydisk.*

Moreover, in case (1), the degrees and the multiplicities of the irreducible factors of the polynomial ψ_p determine uniquely the universal covering $\tilde{X} = \Omega$ (observe that these numbers are independent of the choice of the point p).

The crucial underlying fact is the following discovery of Korányi–Vági.

Let $D \subset \mathbb{C}^n$ be a homogeneous bounded symmetric domain in its circle realization around the origin $0 \in \mathbb{C}^n$.

Let K be the isotropy group of D at the origin $0 \in \mathbb{C}^n$, so that we have $D = G/K$.

A polynomial $f \in \mathbb{C}[X_1, \dots, X_n]$ is said to be K-semi-invariant if there is a character $\chi \colon K \to \mathbb{C}$ such that, for all $g \in K$, $f(gX) = \chi(g) f(X)$.

Observe that, since K is compact, we have: $|\chi(g)| = 1$.

Let $D = D_1 \times D_2$ be the decomposition of D as a product of two domains where D_1 is of tube type and D_2 has no irreducible factor of tube type.

Theorem 27 ([KoVa79, Korányi–Vági]). *Let $D = D_1 \times D_2$ be the above decomposition and let moreover*

$$D_1 = D_{1,1} \times D_{1,2} \times \cdots \times D_{1,p}$$

be the decomposition of D_1 as a product of irreducible tube type domains $D_{1,j}$ $(j = 1, \cdots, p)$.

Then there exist, for each $j = 1, \dots p$, a unique K_j-invariant polynomial $N_j(z_{1,j})$, where K_j is the isotropy subgroup of $D_{1,j}$, such that: for all K-invariant polynomial f there exist a constant $c \in \mathbb{C}$ and exponents k_j with

(1)

$$f = c \prod_{j=1}^{p} N_j^{k_j},$$

hence in particular

(2)

$$f(z_1, z_2) = f(z_1),$$

where z_1 denotes a vector in the domain D_1 and $z_2 \in D_2$.

The above theorem follows almost directly from [KoVa79] by taking into account that a K-invariant polynomial is, up to a multiple, an *inner function*, i.e., a function such that $|f(z)| = 1$ on the Shilov boundary of D (on which K acts transitively).

Moreover the polynomials N_j have degree equal to the $rank(D_j)$ of the irreducible domain D_j, ($rank(D_j)$ denotes the dimension r of the maximal totally geodesic embedded polydisc $\mathscr{H}^r \subset D_j$, or, equivalently, if $D = G/K$, with $G = Aut(D)^0$, $rank(D) = rank(G^{\mathbb{C}}) =$ the dimension of a maximal algebraic torus contained in the complexification $G^{\mathbb{C}}$).

The characterization is essentially a consequence of the unicity of these inner functions, and of the inequality $rank(D_j) \leq dim(D_j)$, where equality holds if and only if $D_j = \mathscr{H}$.

In the case where there are non-tube domains in the irreducible decomposition, one has to use some ideas of Kobayashi and Ochiai [KobOchi81], developed by Mok [Mok89] who introduced and studied certain characteristic varieties which generalize the hypersurfaces defined by the Korányi–Vági-polynomials.

2.3.1. Algebraic curvature-type tensors and their first Mok characteristic varieties Consider the situation where we are given a direct sum

$$T = T_1 \oplus \cdots \oplus T_k$$

of irreducible representations T_i of a group H_i (T shall be the tangent space to a projective manifold at one point, and $H = H_1 \times \cdots \times H_k$ shall be the restricted holonomy group).

Definition 28. *1) An algebraic curvature-type tensor is a non-zero element*

$$\sigma \in End(T \otimes T^\vee).$$

2) Its first Mok characteristic cone $\mathscr{C}\mathscr{S} \subset T$ is defined as the projection on the first factor of the intersection of $ker(\sigma)$ with the set of rank-1 tensors, plus the origin:

$$\mathscr{C}\mathscr{S} := \{t \in T \mid \exists t^\vee \in T^\vee \setminus \{0\}, (t \otimes t^\vee) \in ker(\sigma)\}.$$

3) Its **first Mok characteristic variety** *is the subset $\mathscr{S} := \mathbb{P}(\mathscr{C}\mathscr{S}) \subset \mathbb{P}(T)$.*
4) More generally, for each integer h, consider

$$\{A \in T \otimes T^\vee \mid A \in ker(\sigma), rank(A) \leq h\},$$

and consider the algebraic cone which is its projection on the first factor

$$\mathscr{C}\mathscr{S}^h := \{t \in T \mid \exists A \in ker(\sigma), rank(A) \leq h, \exists t' \in T : t = At'\},$$

and define then $\mathscr{S}^h := \mathbb{P}(\mathscr{C}\mathscr{S}^h) \subset \mathbb{P}(T)$ to be the ***h*-th Mok characteristic variety**.
5) We define then the **full characteristic sequence** *as the sequence*

$$\mathscr{S} = \mathscr{S}^1 \subset \mathscr{S}^2 \subset \cdots \subset \mathscr{S}^{k-1} \subset \mathscr{S}^k = \mathbb{P}(T).$$

Remark 29. In the case where σ is the curvature tensor of an irreducible symmetric bounded domain $\mathscr{D}$, Mok ([Mok02]) proved that the difference sets $\mathscr{S}^h \setminus \mathscr{S}^{h-1}$ are exactly all the orbits of the parabolic subgroup P associated to the compact dual $\mathscr{D}^\vee = G/P$.

The concept of an algebraic curvature type tensor σ can be then used to prove the following theorem.

Theorem 30. *Let X be a compact complex manifold of dimension n with K_X ample.*

Then the universal covering $\tilde{X}$ is a bounded symmetric domain without factors isomorphic to higher dimensional balls if and only if there is a holomorphic tensor $\sigma \in H^0(End(T_X \otimes T_X^\vee))$ enjoying the following properties:

1) there is a point $p \in X$, and a splitting of the tangent space $T = T_{X,p}$

$$T = T'_1 \oplus \cdots \oplus T'_m$$

such that the first Mok characteristic cone $\mathscr{CS}$ is $\neq T$ and moreover $\mathscr{CS}$ splits into m irreducible components $\mathscr{CS}'(j)$ with

2) $\mathscr{CS}'(j) = T'_1 \times \cdots \times \mathscr{CS}'_j \times \cdots \times T'_m$

3) $\mathscr{CS}'_j \subset T'_j$ is the cone over a smooth projective variety $\mathscr{S}'_j$ unless $\mathscr{CS}'_j = 0$ and $dim(T'_j) = 1$.

Moreover, we can recover the universal covering of $\tilde{X}$ from the sequence of pairs $(dim(\mathscr{CS}'_j), dim(T'_j))$.

The case where there are ball factors is the case where the Yau inequality is used (see also [ViZu07]), and one can also in this case give a characterization which is given in terms of X and not of some unspecified étale cover K' of X (Master Thesis of Daniel Mckenzie, 2013 [Mck13]).

A very interesting program, suggested by Yau in [Yau93] is to extend these characterizations to the case of quotients $Y := \mathscr{D}/\Gamma$, where $\mathscr{D}$ is a bounded symmetric domain, but the quotient need not be compact, and the action may be non-free.

This should be done using logarithmic sheaves $\Omega^1_X(\log D)$, where (X, D) is a normal crossing compactification and resolution of $(Y, Sing(Y))$. These sheaves can possibly only be defined in orbifold sense (similarly to what is done in the work of Campana et al. [CGP13]), otherwise it is not clear that the tensors which we considered above, and which descend to $Y \setminus Sing(Y)$, do indeed extend to X logarithmically.

2.4. Kodaira fibred surfaces and their conjugates

The bulk of the previous subsection was to show examples where the universal cover of a projective variety X and of its Galois conjugates X^σ are isomorphic.

Kodaira surfaces miracolously show us that we should not hope (by mere wishful thinking) that the same result should hold for all varieties in the class $\mathscr{PC}$.

In fact, it was proven by Shabat ([Shab77], [Shab83]):

Theorem 31. *Let $f : S \to B$ be a Kodaira fibration, and let $\tilde{S}$ be the universal covering of S, a bounded domain in $\mathbb{C}^2$. Then the fundamental group $\pi_1(S)$ has finite index inside $Aut(\tilde{S})$.*

We have then the following consequence:

Theorem 32. *There exist families $S_t, t \in T$ of Kodaira fibrations whose universal covers $\tilde{S}_t$ are not isomorphic. In particular, there exist Kodaira fibred surfaces such that S and some Galois conjugate S^σ have non-isomorphic universal covering.*

Proof. It suffices to take a family $S_t, t \in T$, $dim(T) \geq 1$, of Kodaira fibrations where the fibres of the map of T to the moduli space are finite. Then, by Shabat's theorem (Theorem 31), it follows that, for $t' \in T$, the number of surfaces S_t whose universal cover is isomorphic to $\tilde{S}'_t$ is countable (finite?), since these surfaces correspond to conjugacy classes of finite index subgroups of $G' := Aut(\tilde{S}'_t)$ which are isomorphic to $\pi_1(S)$. And G' is finitely presented, as well as $\pi_1(S)$.

For the second assertion, let $\mathfrak{M}(S_t)$ be the irreducible component of the moduli space of surfaces of general type (which is defined over $\mathbb{Q}$) containing the image of T, and let S'_t be a surface whose moduli point is not algebraic. Then the set of isomorphism classes of Kodaira fibred surfaces with universal covering isomorphic to $\tilde{S}'_t$ is countable, while the set of isomorphism classes of its conjugate surfaces S'^{σ}_t is uncountable (note that the corresponding moduli points do not need a priori to belong to $\mathfrak{M}(S_t)$, but this is irrelevant and can be indeed arranged taking some explicit family of double Kodaira fibrations). $\square$

The result of Shabat was brought to attention by González-Diez and Reyes-Carocca [GD-RC15], who also discuss the 'arithmeticity' condition that a Kodaira fibration is defined over a number field. They show, as a consequence of Arakelov's finiteness theorem, that S is arithmetic if and only if the base curve is so. And from this they deduce that if two such have the same universal cover, and one is arithmetic, then also the other is arithmetic.

Their work suggests the following possible extension of Theorem 32:

Question 33. Does there exist an arithmetic Kodaira surface S (i.e., S is defined over $\bar{\mathbb{Q}}$) and an automorphism $\sigma \in Aut(\mathbb{C})$ such that the universal coverings of S and S^σ are not isomorphic?

2.5. Surfaces fibred onto curves of positive genus which are classifying spaces

We consider now the following situation: $f : S \to B$ is a fibration of an algebraic surface S onto a curve B and we want to find a criterion for the universal covering $\tilde{S}$ to be contractible.

We already remarked in Proposition 2.1.3 that if $S \in \mathscr{PC}$ then S must be minimal, and indeed S contains no rational curves. More generally, the same argument shows that, for each curve $C \subset S$, the fundamental group of the normalization C' of C has infinite image in $\pi_1(S)$.

Use now the orbifold fundamental group exact sequence (see [CKO03], [Cat03b] or [Cat08])

$$\pi_1(F) \longrightarrow \pi_1(S) \longrightarrow \pi_1^{orb}(f) \longrightarrow 1,$$

hence we obtain a fibration $\tilde{f} : \tilde{S} \to \hat{B}$, where $\hat{B} \to B$ is the ramified covering of B corresponding to the surjection

$$\varphi : \pi_1(B \setminus \{p_1, \dots, p_r\}) \longrightarrow \pi_1^{orb}(f)$$

(here $p_1, \dots, p_r$ are the points whose fibre is multiple, of multiplicity $m_i \geq 2$, and the kernel of φ is normally generated by $\{\gamma_i^{m_i}\}$, for γ_i a loop around p_i).

Notice that $\hat{B}$ is compact if and only if $\pi_1^{orb}(f)$ is finite, and if and only if $\hat{B} \cong \mathbb{P}^1$, since $\hat{B}$ is simply connected. But even when $\pi_1^{orb}(f)$ is infinite there is a finite ramified covering B' of B such that the pull back of the fibration f has no multiple fibres (it suffices that the ramification index at each p_i is equal to the multiplicity m_i of the multiple fibre).

By passing therefore to a finite étale covering (of the surface S), we may assume that $f : S \to B$ has no multiple fibres, and we shall assume in the sequel that the genus b of B is ≥ 1.

Question 34. Are there examples of a fibration $f : S \to B$ of an algebraic surface S onto a curve B, where S is a projective classifying space, and $\pi_1^{orb}(f)$ is finite? Equivalently, with $B \cong \mathbb{P}^1$ and where f has no multiple fibres?

Assume now that f has no multiple fibres, so that $\hat{B}$ is the universal cover $\tilde{B}$ of B, and that the genus b of B is ≥ 1.

Next, by the necessary condition that for each curve C contained in a fibre $\pi_1(C') \to \pi_1(S)$ has infinite image, it follows that all the fibres of $\tilde{f} : \tilde{S} \to \tilde{B}$ are homotopy equivalent to CW complexes of real dimension 1.

Let $\mathscr{C} \subset B$ be the set of critical values of f, $\mathscr{C} = \{p_1, \dots, p_h\}$. Choose a set Σ of non-intersecting paths joining a fixed base point p_0 with the points of $\mathscr{C}$, and similarly a set Σ' inside $\tilde{B}$ of non-intersecting paths joining a fixed base point $y_0 \in \tilde{S}$ mapping to p_0, with the points of the inverse image of $\mathscr{C}$, which is a countable set $\{y_n \mid n \in \mathbb{N}, n \geq 1\}$.

Lemma 35. *Let $f : S \to B$ be a fibration without multiple fibres over a curve B of genus $b \geq 1$, and assume that, for each irreducible curve C contained in a fibre of f, C' denoting the normalization of C, the homomorphism $\pi_1(C') \to \pi_1(S)$ has infinite image.*

Since Σ' is deformation retract of $\tilde{B}$, $\tilde{S}$ is homotopically equivalent to a CW complex of dimension ≤ 2, in particular $\tilde{S}$ is contractible if and only if $H_2(\tilde{S}, \mathbb{Q}) = 0$.

Proof. $\tilde{S}$ retracts onto $K' := \tilde{f}^{-1}(\Sigma')$, and this shows the first assertion.

The second follows since, $\tilde{S}$ being simply connected and homotopically equivalent to a CW complex of dimension ≤ 2, it is contractible if and only if $H_2(\tilde{S}, \mathbb{Z}) = 0$. In fact, by Hurewicz's theorem, the first non-zero homotopy group $\pi_m(\tilde{S})$ is isomorphic to $H_m(\tilde{S}, \mathbb{Z})$, which is obviously zero for $m \geq 3$. Hence if $H_2(\tilde{S}, \mathbb{Z}) = 0$ all homotopy groups $\pi_i(\tilde{S}) = 0$ and $\tilde{S}$ is contractible (the converse is obvious).

Finally, by the universal coefficient formula, $H_2(\tilde{S}, \mathbb{Z})$ is torsion free, hence $H_2(\tilde{S}, \mathbb{Z}) = 0$ if and only if $H_2(\tilde{S}, \mathbb{Q}) = 0$. □

Let $\tilde{F}_0$ be the fibre over y_0, and let $\tilde{F}_n$ be the fibre over y_n.

We can write $K' = \bigcup_{n \in \mathbb{N}} K_n$, where K_n is the inverse image of the union Σ'_n of the segments joining y_0 with y_i, for $i \leq n$.

Clearly

$$H_2(\tilde{S}, \mathbb{Q}) = 0 \Longleftrightarrow H_2(K_n, \mathbb{Q}) = 0, \quad \forall n \in \mathbb{N}.$$

We can now use the theorem of Mayer–Vietoris, using the following notation: K_n^* shall be the inverse image of Σ_n^*, the union of the open segments, together with the point y_0, whereas ∂K_n shall be the inverse image of $\{y_1, \ldots, y_n\}$.

Now, K_n is the union of two open subsets, the first is K_n^* which is homotopically equivalent to $\tilde{F}_0$, and the second which is homotopically equivalent to ∂K_n, the disjoint union of the fibres $\tilde{F}_i$, $1 \leq i \leq n$. Moreover, the intersection of the two open sets is homotopically equivalent to n disjoint copies of $\tilde{F}_0$.

Define $H_i := H_1(\tilde{F}_i, \mathbb{Q})$. Then we have the Mayer–Vietoris exact sequence

$$0 \longrightarrow H_2(K_n, \mathbb{Q}) \longrightarrow H_0^n \longrightarrow H_0 \oplus H_1 \oplus \cdots \oplus H_n \longrightarrow H_1(K_n, \mathbb{Q}) \longrightarrow 0.$$

Here, we have a homomorphism $r_i : H_0 \to H_i$ which is obtained by the fact that a neighbourhood of the fibre $\tilde{F}_i$ retracts onto $\tilde{F}_i$: r_i is a surjection whose kernel is the group V_i of vanishing cycles. The homomorphism of the i-th summand H_0 inside $H_0 \oplus H_1 \oplus \cdots \oplus H_n$ has first component which is the identity, and all the other components equal to zero with the exception of one which is indeed $r_i : H_0 \to H_i$.

We obtain the following theorem.

Theorem 36. *Let $f : S \to B$ be a fibration without multiple fibres over a curve B of genus $b \geq 1$, and assume that, for each irreducible curve C contained in a fibre of f, C' denoting the normalization of C, the homomorphism $\pi_1(C') \to \pi_1(S)$ has infinite image.*

Then $\tilde{S}$ is contractible if and only if, for each $n \in \mathbb{N}$, the subgroups of vanishing cycles form a direct sum $V_1 \oplus \cdots \oplus V_n$ inside H_0.

Proof. The kernel of $H_0^n \to H_0 \oplus H_1 \oplus \cdots \oplus H_n$ are the elements $(x_1, \ldots, x_n)$ such that $(x_1 + \cdots + x_n, r_1(x_1), \ldots, r_n(x_n)) = (0, 0, \ldots, 0)$. Hence these are the elements of $V_1 \oplus \cdots \oplus V_n$ which map to 0 inside H_0. □

We observe that, even if the criterion is essentially a characterization of fibred algebraic surfaces which are projective classifying spaces, the condition on the vanishing cycles is not so easy to verify, and, up to now, good examples are still missing (*cf.* however later sections for candidates).

Observe also the following relation with the well-known Shafarevich conjecture. Keep the assumption that $f : S \to B$ is a fibration without multiple fibres over a curve B of genus $b \geq 1$, and assume that, for a general fibre F, the homomorphism $\pi_1(F) \to \pi_1(S)$ has infinite image. If there is an irreducible curve C contained in a fibre of f such that, C' denoting the normalization of C, the homomorphism $\pi_1(C') \to \pi_1(S)$ has finite image, while the homomorphism $\pi_1(C) \to \pi_1(S)$ has infinite image, then S would be a counterexample to the Shafarevich conjecture that $\tilde{S}$ is holomorphically convex (*cf.* [Bog-Katz-98]). Since then $\tilde{S}$ would contain a connected infinite chain of compact curves.

3. Surfaces fibred onto curves

3.1. The Zeuthen–Segre formula

The Zeuthen–Segre formula is a beautiful formula, valid for any smooth algebraic surface S:

Theorem 37 (Zeuthen–Segre, classical). *Let S be a smooth projective surface, and let C_λ, $\lambda \in \mathbb{P}^1$, be a linear pencil of curves of genus g which meet transversally in δ distinct points. If μ is the number of singular curves in the pencil (counted with multiplicity), then*

$$\mu - \delta - 2(2g - 2) = I + 4,$$

where the integer I is an invariant of the algebraic surface, called Zeuthen–Segre invariant.

Here, the integer δ equals the self-intersection number C^2 of the curve C, while in modern terms the integer $I + 4$ is not only an algebraic invariant, but is indeed a topological invariant: the topological Euler–Poincaré characteristic $e(S)$.

Observe that $I+4+C^2$ is then the topological Euler–Poincaré characteristic of the surface blown up in the $\delta = C^2$ base points of the pencil. In this way the Zeuthen–Segre formula generalizes for every fibration $f : S \to B$ of an algebraic surface onto a curve B, and the formula measures the deviation from the case of a topological bundle, for which one would have $e(S) = 4(b-1)(g-1)$, where g is the genus of a fibre.

The importance of the formula is that, as suggested by the interpretation of μ as a number of points, the difference $\mu := e(S) - 4(b-1)(g-1)$ is always non-negative.

The formula is well known using topology (see [BPHV]), but it is very convenient to have an algebraic formula, which is proven using sheaves and exact sequences (see the lecture notes [CB]).

Definition 38. *Let $f : S \to B$ be a fibration of a smooth algebraic surface S onto a curve of genus b, and consider a singular fibre $F_t = \sum n_i C_i$, where the C_i are distinct irreducible curves.*

Then the divisorial singular locus of the fibre is defined as the divisorial part of the critical scheme, $D_t := \sum (n_i - 1) C_i$, and the Segre number of the fibre is defined as

$$\mu_t := deg \mathscr{F} + D_t K_S - D_t^2,$$

where the sheaf $\mathscr{F}$ is concentrated in the singular points of the reduction of the fibre, and is the quotient of $\mathcal{O}_S$ by the ideal sheaf generated by the components of the vector $d\tau/s$, where $s = 0$ is the equation of D_t, and where τ is the pull-back of a local parameter at the point $t \in B$.

More concretely,

$$\tau = \prod_j f_j^{n_j}, \quad s = \tau / \Big(\prod_j f_j \Big),$$

and the logarithmic derivative yields

$$d\tau = s \Big[\sum_j n_j \Big(df_j \prod_{h \neq j} f_h \Big) \Big].$$

The following is the refined Zeuthen–Segre formula:

Theorem 39 (Zeuthen–Segre, modern). *Let $f : S \to B$ be a fibration of a smooth algebraic surface S onto a curve of genus b, and with fibres of genus g.*

Then

$$c_2(S) = 4(g-1)(b-1) + \mu,$$

where $\mu = \sum_{t \in B} \mu_t$, and $\mu_t \geq 0$ is defined as above. Moreover, μ_t is strictly positive, except if the fibre is smooth or a multiple of a smooth curve of genus $g = 1$.

Most of the times, the formula is used in its non-refined form: if $g > 1$, then either $\mu > 0$, or $\mu = 0$ and we have a differentiable fibre bundle. In this case there are two alternatives: either we have a Kodaira fibration, or all the smooth fibres are isomorphic, and we have a holomorphic fibre bundle ([F-G65]).

3.2. *The positivity results of Arakelov, Fujita and Kawamata*

3.2.1. *Arakelov's theorem*

Definition 40. *A fibration $f : S \to B$ of a smooth algebraic surface S onto a curve of genus b is said to be (relatively) minimal, if there is no (-1)-curve contained in a fibre. Moreover, it is said to be* **isotrivial**, *or with constant moduli, if all the smooth fibres are isomorphic.*

Isotriviality is equivalent to the condition that the moduli morphism $\psi : B \to \mathfrak{M}_g$ is constant, and it implies that there exists a finite Galois base change $B' \to B$ such that the pull back $S' \to B'$ is birational to a product.

The theorem of Arakelov ([Ara71]) gives a numerical criterion for isotriviality.

Theorem 41. *Let $f : S \to B$ be a minimal fibration of a smooth algebraic surface S onto a curve of genus b, where the genus g of the fibres is strictly positive. Define $K_{S|B}$, the relative canonical divisor, as $K_{S|B} := K_S - f^*(K_B)$. Then $K_{S|B}$ is nef, and, if $g \geq 2$, $K_{S|B}$ is big unless the fibration is isotrivial, in particular $K^2_{S|B} \geq 0$ and $K^2_{S|B} > 0$ if the fibration is not isotrivial.*

Proof (Idea).

(I) The minimality of the fibration and $g \geq 1$ ensure that $K_{S|B} \cdot C = K_S \cdot C \geq 0$ for each curve C contained in a fibre.

(II) For the case where C is not contained on a fibre, we use that the line bundle $\mathcal{O}_C(K_{S|B})$ is generically a quotient of the pull-back of $V := f_*\mathcal{O}_S(K_{S|B})$. Indeed, if $p : C \to B$ is induced by f, there is a non-zero morphism $p^*V \to \mathcal{O}_C(K_{S|B})$ and one applies the theorem of Fujita 43 (that we shall soon describe) stating that p^*V is nef, hence $\mathcal{O}_C(K_{S|B})$ has non-negative degree.

(III) Assume that the divisor $K_{S|B}$ is not big. $K_{S|B}$ is nef and, if $g \geq 2$, it is not numerically trivial since, for each fibre Φ, $K_{S|B} \cdot \Phi = 2g - 2 \geq 2$.

Hence the graded ring associated to it has Iitaka dimension 1, and yields a map to a curve C, $\varphi : S \to C$. We consider $F := f \times \varphi : S \to B \times C$, and we consider the Hurwitz formula: $K_S = F^*(K_B + K_C) + R$, which proves that $K_{S|B} = F^*(K_C) + R$. Since all the sections of multiples of $K_{S|B}$ pull back from C, it follows that R is horizontal, hence all the fibres are ramified covers of the same curve C, and branched in the same set of points. From this it follows that all the smooth fibres are isomorphic. □

The following corollary contains an observation by Beauville [Bea82] and the fact, already mentioned several times, that a Kodaira fibred surface has positive index.

Corollary 42. *Under the same assumptions as in Arakelov's Theorem 41, but assuming $g \geq 2$: then $\chi(S) \geq (g-1)(b-1)$, equality if and only if $f : S \to B$ is a holomorphic bundle.*

In particular, a Kodaira fibred surface S has a strictly positive index, i.e., S has $c_1^2(S) = K_S^2 > 2e(S) = 2c_2(S)$.

Proof. The Arakelov inequality $K_{S|B}^2 = K_S^2 - 8(b-1)(g-1) \geq 0$ and the Zeuthen–Segre inequality $e(S) \geq 4(b-1)(g-1)$ add up, in view of Noether's theorem (the first equality in the next formula) to yield the new inequality

$$12\chi(S) = K_S^2 + e(S) \geq 12(b-1)(g-1).$$

Moreover, equality holds if both Arakelov's and Zeuthen–Segre's inequality are equalities, implying that we have an isotrivial fibration and a differentiable bundle: hence, as we already observed, f is a holomorphic bundle.

If we have a Kodaira fibration, then $e(S) = 4(b-1)(g-1)$ and the fibration is not isotrivial, hence $K_S^2 > 8(b-1)(g-1) = 2e(S)$. □

3.2.2. Fujita's direct image theorems An important progress in classification theory was stimulated by a theorem of Fujita, who showed ([Fujita78a]) that the direct image of the relative dualizing sheaf

$$\omega_{X|B} = \mathcal{O}_X(K_{X|B}) := \mathcal{O}_X(K_X - f^* K_B)$$

is numerically positive, i.e., every quotient bundle has non-negative degree (a fact that was used in the previous subsection to give a different proof of Arakelov's theorem).

Theorem 43 (Fujita's first theorem). *If X is a compact Kähler manifold and $f : X \to B$ is a fibration onto a projective curve B (i.e., f has connected fibres), then the direct image sheaf*

$$V := f_* \omega_{X|B}$$

is a nef vector bundle on B, equivalently V is 'numerically semi-positive', meaning that each quotient bundle Q of V has degree $deg(Q) \geq 0$.

In particular, if X is an algebraic surface S, then $deg(V) \geq 0$, which is the inequality observed previously

$$deg(V) = \chi(S) - (g-1)(b-1) \geq 0$$

(except that the characterization of the case of equality does not follow right away from Theorem 43, one needs Theorem 44).

In the note [Fujita78b] Fujita announced the following quite stronger result:

Theorem 44 (Fujita's second theorem, [Fujita78b]). *Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B, and consider the direct image sheaf*

$$V := f_* \omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^* K_B)).$$

Then V splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle and Q is a unitary flat bundle (see Definition 48).

Fujita sketched the proof, but referred to a forthcoming article concerning the positivity of the so-called local exponents; the result was used in the meantime, for instance I used it in my joint work with Pignatelli, but since Fujita's detailed article was never written, there was some objection (see [Barja98]) to the use of this beautiful result: this was a motivation for Dettweiler and myself to write down a complete proof, which is going to appear in the article contributed to Kawamata's 60-th birthday volume ([CatDet13]).

3.2.3. Kawamata's positivity theorems In the meantime the idea of the proof had become more transparent, through Kawamata's use of Griffihts' results on Variation of Hodge Structures (the relation being that the fibre of $V := f_* \omega_{X|B}$ over a point $b \in B$, such that $X_b := f^{-1}(b)$ is smooth, is the vector space $V_b = H^0(X_b, \Omega^{n-1}_{X_b})$).

Kawamata ([Kaw81], [Kaw82]) improved on Fujita's result, solving a long standing problem and proving the subadditivity of Kodaira dimension for such fibrations,

$$Kod(X) \geq Kod(B) + Kod(F),$$

(here F is a general fibre) showing the semi-positivity also for the direct image of higher powers of the relative dualizing sheaf

$$W_m := f_*(\omega_{X|B}^{\otimes m}) = f_*(\mathcal{O}_X(m(K_X - f^* K_B))).$$

Kawamata also extended his result to the case where the dimension of the base variety B is > 1 in [Kaw81], giving later a simpler proof of semi-positivity in [Kaw02]. There has been a lot of literature on the subject ever since, (see the references we cited in [CatDet13], see [E-V90] for the ampleness of W_m when the fibration is not birationally isotrivial, and see [FF14] and [FFS14]).

Kawamata also introduced a simple lemma, concerning the degree of line bundles on a curve whose metric grows at most logarithmically around a finite number of singular points, which played a crucial role for the proof of Fujita's second theorem.

Indeed the missing details concerning the proof of the second theorem of Fujita, using Kawamata's lemma and some crucial estimates given by Zucker ([Zuc79]) for the growth of the norm of sections of the L^2-extension of Hodge bundles, were provided in [CatDet13], whose main contribution was a negative answer to a question posed by Fujita in 1982 (Problem 5, page 600 of [Katata83], Proceedings of the 1982 Taniguchi Conference).

3.3. Fujita's semi-ampleness question

To understand the question posed by Fujita it is not only important to have in mind Fujita's second theorem, but it is also very convenient to recall the following classical definition used by Fujita in [Fujita78a], [Fujita78b].

Let V be a holomorphic vector bundle over a projective curve B.

Definition 45. *Let $p : \mathbb{P} := Proj(V) = \mathbb{P}(V^\vee) \to B$ be the associated projective bundle, and let H be a hyperplane divisor (such that $p_*(\mathcal{O}_{\mathbb{P}}(H)) = V$).*

Then V is said to be:

- *(NP) numerically semi-positive if and only if every quotient bundle Q of V has degree $deg(Q) \geq 0$,*
- *(NEF) nef if and only if H is nef on $\mathbb{P}$,*
- *(A) ample if and only if H is ample on $\mathbb{P}$,*
- *(SA) semi-ample if and only if H is semi-ample on $\mathbb{P}$ (there is a positive multiple mH such that the linear system $|mH|$ is base-point free).*

Remark 46. Recall that (A) $\Rightarrow$ (SA) $\Rightarrow$ (NEF) $\Leftrightarrow$(NP), the last follows from the following result due to Hartshorne.

Proposition 47. *A vector bundle V on a curve is nef if and only if it is numerically semi-positive, i.e., if and only if every quotient bundle Q of V has degree $deg(Q) \geq 0$, and V is ample if and only if every quotient bundle Q of V has degree $deg(Q) > 0$.*

Recall the following standard definition, used in the statement of Fujita's second theorem.

Definition 48. *A flat holomorphic vector bundle on a complex manifold M is a holomorphic vector bundle $\mathcal{H} := \mathcal{O}_M \otimes_{\mathbb{C}} \mathbb{H}$, where $\mathbb{H}$ is a local system of complex vector spaces associated to a representation $\rho : \pi_1(M) \to GL(r, \mathbb{C})$,*

$$\mathbb{H} := (\tilde{M} \times \mathbb{C}^r)/\pi_1(M),$$

$\tilde{M}$ being the universal cover of M (so that $M = \tilde{M}/\pi_1(M)$).

We say that $\mathcal{H}$ is unitary flat if it is associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$.

We come now to Fujita's question:

Question 49 (Fujita). Is the direct image $V := f_*\omega_{X|B}$ semi-ample?

In [CatDet13] we established a technical result which clarifies how Fujita's question is very closely related to Fujita's second theorem:

Theorem 50. *Let $\mathcal{H}$ be a unitary flat vector bundle on a projective manifold M, associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. Then $\mathcal{H}$ is nef; moreover $\mathcal{H}$ is semi-ample if and only if $Im(\rho)$ is finite.*

The idea of the proof is to reduce to the case where we have such a bundle over a curve (obtained by taking successive hyperplane sections of M), decompose the representation as a direct sum of irreducible unitary representations, and then use the theorem of Narasimhan and Seshadri [NS65] implying that a unitary flat holomorphic bundle over a curve is holomorphically trivial if and only if the representation is trivial.

Hence in our particular case, where $V = A \oplus Q$ with A ample and Q unitary flat, the semi-ampleness of V simply means that the flat bundle has finite monodromy (this is another way to wording the fact that the representation of the fundamental group $\rho : \pi_1(B) \to U(r, \mathbb{C})$ associated to the flat unitary rank-r bundle Q has finite image).

The main new result in our joint work [CatDet13] was, as already said, to provide a negative answer to Fujita's question in general:

Theorem 51. *There exist surfaces X of general type endowed with a fibration $f : X \to B$ onto a curve B of genus ≥ 3, and with fibres of genus 6, such that $V := f_*\omega_{X|B}$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where A is an ample rank-2 vector bundle, and the flat unitary rank-2 summands Q_1, Q_2 have infinite monodromy group (i.e., the image of ρ_j is infinite). In particular, V is not semi-ample.*

Recently ([CatDet15]) we have found an infinite series of counterexamples, which are based on the same ideas, but are quite simpler; we shall report on them in a later subsection (they shall be called here BCDH-surfaces).

Notice that Fujita's second theorem follows right away from the first in the case where the base curve is $\mathbb{P}^1$, since then every vector bundle splits as a direct sum of line bundles, and we can separate the summands with strictly positive degree from the trivial summands.

Also, one can say something more precise in the case where the base curve B has genus 1, or under other assumptions, which imply that indeed V is semi-ample.

Corollary 52. *Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B. Then $V := f_*\omega_{X|B}$ is a direct sum $V = A \oplus (\bigoplus_{i=1}^h Q_i)$, with A ample and each Q_i unitary flat without any non-trivial degree zero quotient. Moreover,* *(I) if Q_i has rank equal to 1, then it is a torsion bundle ($\exists\, m$ such that $Q_i^{\otimes m}$ is trivial) (Deligne),*

(II) if the curve B has genus 1, then $rank(Q_i) = 1,\ \forall i$.

(III) In particular, if B has genus at most 1, then V is semi-ample.

Idea of proof. (I) was proven by Deligne [Del71] (and by Simpson [Simp93] using the theorem of Gelfond–Schneider), while

(II) follows since $\pi_1(B)$ is abelian, if B has genus 1: hence every representation splits as a direct sum of 1-dimensional ones. □

3.3.1. How do the flat bundles appear In order to get a fibration $f : S \to B$ where we have a splitting $V = f_*\omega_{S|B} \oplus Q$ our idea is to use symmetry, for instance the fibres are cyclic coverings C of $\mathbb{P}^1$ with group $\mathbb{Z}/n$ and branched in 4 points.

Then we get a curve $C = C_x$ birational to the curve described by an equation of the form:

$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - x y_0)^{m_3},\ \ x \in \mathbb{C} \setminus \{0, 1\}. \tag{53}$$

Here, we shall make the restrictive assumption that

$$0 < m_j \le n - 3, \text{ and } m_0 + m_1 + m_2 + m_3 = n.$$

Then C admits a Galois cover $\phi : C \to \mathbb{P}^1$ with cyclic Galois group equal to the group of n-th roots of unity in $\mathbb{C}$,

$$G = \{\zeta \in \mathbb{C}^* \mid \zeta^n = 1\},$$

acting by scalar multiplication on z_1. The choice of a generator in G yields an isomorphism $G \cong \mathbb{Z}/n$.

Now, the vector space $V_x := H^0(\Omega^1_{C_x})$ splits according to the characters $\chi : G \to \mathbb{C}^*$,

$$V_x = \bigoplus_{\chi} V^{\chi}.$$

Here each character is of the form $\zeta \mapsto \zeta^i$ for a suitable $i \in \mathbb{Z}/n$. We denote by χ_i the character such that $\chi_i(\zeta) = \zeta^i$.

There is an easy formula (see [D-M86]) for $dim(V^{\chi_i})$:

$$dim(V^{\chi_i}) = \frac{1}{n}([im_0] + [im_1] + [im_2] + [im_3] - n), \quad i \neq 0,$$

where $0 \le [m] \le n - 1$ denotes the remainder of division by n (a standard representative of the residue class).

For the dual character then $dim(V^{\bar{\chi_i}}) = dim(V^{\chi_{-i}}) = \frac{1}{n}(n - [im_0] + n - [im_1] + n - [im_2] + n - [im_3] - n)$ hence

$$dim(V^{\chi_i}) + dim(V^{\bar{\chi_i}}) = 2.$$

Hence, by Hodge Theory, if $H^{\chi_i} = V^{\chi_i} \oplus V^{\bar{\chi_i}}$ is the corresponding eigenspace for the action of G on $H^1(C_x, \mathbb{C})$, its dimension equals 2.

By our choice for the integers m_i, for $i = 1$ we get a character χ_1 such that $V^{\chi_1} = 0$, hence

$$H^{\chi_1} = V^{\chi_{n-1}},$$

hence we get a flat summand for V.

The next question is: how can we assert that the monodromy group for the flat summand is infinite?

The idea can be explained in simple terms like this (see [CatDet15] for details) : if it were finite, then there would be a monodromy invariant positive definite scalar product. This would also hold when we conjugate the characters via the action of the Galois group of the field extension E of $\mathbb{Q}$, generated by the n-th roots of 1, on the rational monodromy representation (for n prime, all nontrivial characters are conjugate). If the monodromy is irreducible, there is only one monodromy invariant scalar product, hence for all conjugate characters we would have a positive definite scalar product, which amounts to the condition that one never has $dim(V^{\chi}) = dim(V^{\bar{\chi}}) = 1$.

On the other hand, if we assume that $m_0 + m_3$ is invertible in $\mathbb{Z}/n\mathbb{Z}$, there is a j such that $j(m_0 + m_3) \equiv -1 \pmod{n}$. For instance $m_0 + m_3$ is invertible in $\mathbb{Z}/n\mathbb{Z}$ if we take $m_0 = 1, m_1 = 1, m_2 = 1, m_3 = n - 3$ and we assume that n is an odd number.

Define now $m'_i := [m_i j]$, where $[a]$ denotes the remainder of division by n. Hence $m'_0 + m'_3 = n - 1$.

We have the obvious inequalities $2 \leq m'_1 + m'_2 \leq 2n - 2$. Hence

$$n + 1 \leq m'_0 + \cdots + m'_3 \leq 3n - 3$$

and therefore

$$m'_0 + \cdots + m'_3 = 2n.$$

Hence the underlying unitary form is indefinite for the character χ_j, whereas it is definite for the conjugate character χ_{n-j}.

Moreover, (see [CatDet15]) once the rank 1 summand V^{χ_j} is shown to be ample, the irreducibility of the flat bundle associated to H^{χ_j} (hence of the monodromy) follows for the following chain of arguments:

- If the flat vector bundle were reducible there would be an exact sequence of flat vector bundles

  $$0 \longrightarrow \mathcal{H}' \longrightarrow H^{\chi_j} \longrightarrow \mathcal{H}'' \longrightarrow 0$$

  where both $\mathcal{H}'$, $\mathcal{H}''$ have rank-1.

- V^{χ_j} is a holomorphic subbundle of H^{χ_j}, hence it has a non-trivial homomorphism to it
- every homomorphism of V^{χ_j} to a flat rank 1 bundle $\mathcal{H}'$, $\mathcal{H}''$ must be zero
- contradiction!

The final conclusion is that the monodromy of the flat factor $V^{\chi_{n-1}}$ is infinite, and that V is not semi-ample; the concrete examples of this situation shall be described in Sect. 5.

3.4. Castelnuovo–de Franchis and morphisms onto curves

In order to address the study of moduli spaces of Kodaira fibred surfaces, or of surfaces which are isotrivially fibred, it is convenient to review some simple results, pioneered by Siu, and which concern topological conditions implying the existence of holomorphic maps of a compact Kähler manifold X onto an algebraic curve.

Siu used harmonic theory in order to construct holomorphic maps from Kähler manifolds to projective curves. The first result in this direction was the theorem of [Siu87], also obtained by Jost and Yau (see [J-Y93] and also [J-Y83] for other results).

Theorem 54 (Siu). *Assume that X is a compact Kähler manifold such that there is a surjection $\phi : \pi_1(X) \to \pi_g$, where $g \geq 2$ and, as usual, π_g is the fundamental group of a projective curve of genus g. Then there is a projective curve C of genus $g' \geq g$ and a fibration $f : X \to C$ (i.e., the fibres of f are connected) such that ϕ factors through $\pi_1(f)$.*

In this case the homomorphism leads to a harmonic map to a curve, and one has to show that the Stein factorization yields a map to some Riemann surface which is holomorphic for some complex structure on the target.

It can be seen more directly how the Kähler assumption is used, because this assumption guarantees that holomorphic forms are closed, i.e., $\eta \in H^0(X, \Omega^p_X)$ $\Rightarrow d\eta = 0$.

At the turn of last century this fact was used by Castelnuovo and de Franchis ([Cast05], [dF05]):

Theorem 55 (Castelnuovo–de Franchis). *Assume that X is a compact Kähler manifold, $\eta_1, \eta_2 \in H^0(X, \Omega^1_X)$ are $\mathbb{C}$-linearly independent, and the wedge product $\eta_1 \wedge \eta_2$ is d-exact. Then $\eta_1 \wedge \eta_2 \equiv 0$ and there exists a fibration $f : X \to C$ such that $\eta_1, \eta_2 \in f^* H^0(C, \Omega^1_C)$. In particular, C has genus $g \geq 2$.*

Even if the proof is well-known, let us point out that the first assertion follows from the Hodge–Kähler decomposition, while $\eta_1 \wedge \eta_2 \equiv 0$ implies the existence of a non-constant rational function φ such that $\eta_2 = \varphi \eta_1$. This shows that the foliation defined by the two holomorphic forms has Zariski closed leaves, and the rest follows then rather directly taking the Stein factorization of the rational map $\varphi : X \to \mathbb{P}^1$.

Now, the above result, which is holomorphic in nature, combined with the Hodge decomposition, produces results which are topological in nature (they actually only depend on the cohomology algebra structure of $H^*(X, \mathbb{C})$).

To explain this in the most elementary case, we start from the following simple observation. If two linear independent vectors in the first cohomology group $H^1(X, \mathbb{C})$ of a Kähler manifold have wedge product which is trivial in cohomology, and we represent them as $\eta_1 + \overline{\omega_1}, \eta_2 + \overline{\omega_2}$, for $\eta_1, \eta_2, \omega_1, \omega_2 \in H^0(X, \Omega^1_X)$, then by the Hodge decomposition and the first assertion of the theorem of Castelnuovo–de Franchis

$$(\eta_1 + \overline{\omega_1}) \wedge (\eta_2 + \overline{\omega_2}) = 0 \in H^2(X, \mathbb{C})$$

implies

$$\eta_1 \wedge \eta_2 \equiv 0, \quad \omega_1 \wedge \omega_2 \equiv 0.$$

We can apply Castelnuovo–de Franchis unless η_1, η_2 are $\mathbb{C}$-linearly dependent, and similarly ω_1, ω_2. Without loss of generality we may assume then $\eta_2 \equiv 0$ and $\omega_1 \equiv 0$. But then $\eta_1 \wedge \overline{\omega_2} = 0$ implies that the Hodge norm

$$\int_X (\eta_1 \wedge \overline{\omega_2}) \wedge \overline{(\eta_1 \wedge \overline{\omega_2})} \wedge \xi^{n-2} = 0,$$

where ξ is here the Kähler form. A simple trick is to observe that

$$0 = \int_X (\eta_1 \wedge \overline{\omega_2}) \wedge \overline{(\eta_1 \wedge \overline{\omega_2})} \wedge \xi^{n-2} = -\int_X (\eta_1 \wedge \omega_2) \wedge \overline{(\eta_1 \wedge \omega_2)} \wedge \xi^{n-2},$$

therefore the same integral yields that the Hodge norm of $\eta_1 \wedge \omega_2$ is zero, hence $\eta_1 \wedge \omega_2 \equiv 0$; the final conclusion is that we can in any case apply Castelnuovo–de Franchis and find a map to a projective curve C of genus $g \geq 2$.

More precisely, one gets the following theorem ([Cat91]):

Theorem 56. (Isotropic subspace theorem) *On a compact Kähler manifold X there is a bijection between isomorphism classes of fibrations $f : X \to C$ to a projective curve of genus $g \geq 2$, and real subspaces $V \subset H^1(X, \mathbb{C})$ ('real' means that V is self conjugate, $\bar{V} = V$) which have dimension $2g$ and are of the form $V = U \oplus \bar{U}$, where U is a maximal isotropic subspace for the wedge product*

$$H^1(X, \mathbb{C}) \times H^1(X, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C}).$$

The above result, as simple as it may be, implies the few relations theorem of Gromov ([Grom89]), which in turn implies Theorem 54 of Siu (see e.g. [Cat15] for an ampler discussion).

There is another result ([Cat08]) which again, like the isotropic subspace theorem, determines explicitly the genus of the target curve (a result which is clearly useful for classification and moduli problems).

Theorem 57. *Let X be a compact Kähler manifold, and let $f : X \to C$ be a fibration onto a projective curve C, of genus g, and assume that there are exactly r fibres which are multiple with multiplicities $m_1, \dots, m_r \geq 2$. Then f induces an orbifold fundamental group exact sequence*

$$\pi_1(F) \longrightarrow \pi_1(X) \longrightarrow \pi_1(g; m_1, \dots, m_r) \longrightarrow 0,$$

where F is a smooth fibre of f, and where the orbifold fundamental group

$$\pi_1(g; m_1, \dots, m_r)$$

is defined as

$$\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_r \mid \Pi_1^g [\alpha_j, \beta_j] \Pi_1^r \gamma_i = \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = 1 \rangle.$$

Conversely, let X be a compact Kähler manifold and let $(g, m_1, \dots, m_r)$ be a hyperbolic type, i.e., assume that $2g - 2 + \Sigma_i (1 - \frac{1}{m_i}) > 0$.

Then each epimorphism $\phi : \pi_1(X) \to \pi_1(g; m_1, \dots, m_r)$ with finitely generated kernel is obtained from a fibration $f : X \to C$ of type $(g; m_1, \dots, m_r)$.

With these results and the Zeuthen–Segre formula, it is easy to explain on the one hand the characterization of surfaces isogenous to a product, and on the other hand Kotschick's result on moduli of Kodaira surfaces.

Theorem 58. *a) A projective smooth surface S is isogenous to a product of two curves of respective genera $g_1, g_2 \geq 2$, if and only if the following two conditions are satisfied:*

1) there is an exact sequence

$$1 \longrightarrow \pi_{g_1} \times \pi_{g_2} \longrightarrow \pi = \pi_1(S) \longrightarrow G \longrightarrow 1,$$

where G is a finite group and where π_{g_i} denotes the fundamental group of a projective curve of genus $g_i \geq 2$;

2) $e(S)(= c_2(S)) = \frac{4}{|G|}(g_1 - 1)(g_2 - 1)$.

b) Write $S = (C_1 \times C_2)/G$. Any surface X with the same topological Euler number and the same fundamental group as S is diffeomorphic to S and is also isogenous to a product.

c) The corresponding subset of the moduli space of surfaces of general type $\mathfrak{M}_S^{top} = \mathfrak{M}_S^{diff}$, corresponding to surfaces orientedly homeomorphic, resp. orientedly diffeomorphic to S, is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.

Idea of the proof of b). $\Gamma := \pi_1(S)$ admits a subgroup Γ' of index d such that $\Gamma' \cong (\pi_{g_1} \times \pi_{g_2})$. Let S' be the associated unramified covering of S. Then application of the isotropic subspace theorem or of Theorem 57 yields a pair of holomorphic maps $f_j : S' \to C_j$, hence a holomorphic map

$$F := f_1 \times f_2 : S' \longrightarrow C_1' \times C_2'.$$

Then the fibres of f_1 have genus $h_2 \geq g_2$, hence by the Zeuthen–Segre formula $e(S') \geq 4(h_2 - 1)(g_1 - 1)$, equality holding if and only if all the fibres are smooth.

But $e(S') = 4(g_1 - 1)(g_2 - 1) \leq 4(h_2 - 1)(g_1 - 1)$, so $h_2 = g_2$, all the fibres are smooth hence isomorphic to C_2'; therefore F is an isomorphism. □

The previous theorem generalizes also to varieties isogenous to a product of curves of dimension $n \geq 3$ ([Cat00], [Cat15]).

We try now to compare the cited theorem of Kotschick (Theorem 8) with the previous one.

Theorem 59. *Assume that S is a compact Kähler surface, and that*

(i) its fundamental group sits into an exact sequence, where $g, b \geq 2$:

$$1 \longrightarrow \pi_g \longrightarrow \pi_1(S) \longrightarrow \pi_b \longrightarrow 1$$

(ii) $e(S) = 4(b - 1)(g - 1)$.

Then S has a smooth holomorphic fibration $f : S \to B$, where B is a projective curve of genus b, and where all the fibres are smooth projective curves of genus g. f is a Kodaira fibration if and only if the associated homomorphism $\rho : \pi_b \to \mathcal{M}ap_g$ has image of infinite order, else it is a surface isogenous to a product of unmixed type and where the action on the first curve is free.

Proof. By Theorem 57 the above exact sequence yields a fibration $f : S \to B$ such that there is a surjection $\pi_1(F) \to \pi_g$, where F is a smooth fibre. Hence, denoting by h the genus of F, we conclude that $h \geq g$, and again we can use the Zeuthen–Segre formula to conclude that $h = g$ and that all fibres are smooth. So F is a smooth fibration. Let $C' \to C$ be the unramified covering associated to $\ker(\rho)$: then the pull back family $S' \to C'$ has a topological trivialization, hence is a pull back of the universal family $\mathcal{C}_g \to \mathcal{T}_g$ for an appropriate holomorphic map $\varphi : C' \to \mathcal{T}_g$.

If $\ker(\rho)$ has finite index, then C' is compact and, since Teichmüller space is a bounded domain in $\mathbb{C}^{3g-3}$, the holomorphic map is constant. Therefore S' is a product $C' \times C_2$ and, denoting by $G := \mathrm{Im}(\rho)$, $S = (C' \times C_2)$, and we get exactly the surfaces isogenous to a product such that the action of G on the curve C' is free.

If instead $G := Im(\rho)$ is infinite, then the map of C' into Teichmüller space is not constant, since the isotropy group of a point corresponding to a curve F is, as we saw, equal to the group of automorphisms of F (which is finite). Therefore, in this case, we have a Kodaira fibration. □

3.5. *Singular fibres and mapping class group monodromy*

Let us consider again the situation where we have a fibration $f : S \to B$ of an algebraic surface S onto a curve of genus b, and where the fibres have genus $g \geq 2$. We assume throughout that the fibration is relatively minimal and not a product.

We have devoted most of our attention to the consideration of the case where all the fibres F of f are smooth curves. This can occur whenever the genus $b \geq 1$: just take a surface isogenous to a product of unmixed type, i.e., $S = (C_1 \times C_2)/G$, where G acts diagonally and freely on C_1, with quotient $C_1/G = B$.

Then $f : S = (C_1 \times C_2)/G \to C_1/G = B$ is the desired holomorphic bundle.

For Kodaira fibrations the situation is slightly more complicated, it is possible that the genus $b = 2$, as we saw, but the fibre genus g should be at least 3.

At any rate, if we have a differentiable fibre bundle over a curve B of genus 1, then, $\tilde{B}$ being the universal covering of B, $\tilde{B} \cong \mathbb{C}$, the Torelli map $\tilde{B} \to \mathscr{H}_g$ into Siegel's half space must be constant, therefore we have a holomorphic bundle, which is then isogenous to a product, as described above.

The conclusion is that there must be at least one singular fibre if $b = 1$ and we have non-constant moduli.

On the other hand, when $b = 0$, and the fibration is not a product, the number of singular fibres must be at least three.

Indeed, if $B^* = B \setminus \mathscr{C}$ is the set of non-critical values of f, we want that there is a non-constant map $B^* \to \mathscr{H}_g$; if $B = \mathbb{P}^1$, therefore the number of singular fibres, i.e., $|\mathscr{C}|$, must at least three.

If we assume however that all the fibres are moduli semi-stable curves, then the number of singular fibres must be at least 4 for $g \geq 1$ and at least 5 for $g \geq 2$, as proven by Beauville and Tan [Bea81] and [Tan95] ([Za04] gave an improvement for $g \geq 3$, that the number must be at least 6).

Tan's inequality is

$$\frac{g}{2}(2b - 2 + s) > deg(f_*(\omega_{S|B})) = \chi(S) - (b-1)(g-1),$$

so that for $b = 1$ the number s of singular fibres is $> \frac{2}{g}$, which gives $s \geq 2$ only if $g = 2$.

Can one obtain better estimates on the number of singular fibres also when the genus of the base curve is $b = 1$?

A very interesting result was obtained by Ishida [Ishi06] who found, in the moduli space of surfaces with $q = p_g = 1, K^2 = 3$, described in joint work with Ciliberto ([CaCi93]) some surfaces whose Albanese fibration (onto a curve of genus $b = 1$) has a unique singular fibre, and fibre genus $g = 3$; but the singular fibre is not reduced.

As far as I know, there are no known examples of a non-isotrivial fibration over a curve B of genus $b = 1$, and possessing only one singular fibre, irreducible and with only nodes as singularities.

If such a fibration were to exist, the local monodromy around the singular fibres, a product of commuting Dehn twists, would be a commutator in the mapping class group. This raises a general question:

Question 60. Which products of commuting Dehn twists are a commutator in the mapping class group Map_g, for $g \geq 2$?

The question in genus $g = 1$ is clear, since then the mapping class group is $SL(2, \mathbb{Z})$, and there the transvection (Picard–Lefschetz transformation)

$$T_1 : e_1 \longmapsto e_1, e_2 \longmapsto e_1 + e_2$$

is not a commutator, and indeed no parabolic transformation in

$$\mathbb{P}SL(2, \mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3 = \langle A, B \mid A^2 = B^3 = 1 \rangle$$

is a commutator. The easiest way to see this is to express T_1 as a product of the two generators, $T_1^{-1} = AB$, so that the image of T_1 in the Abelianization $\mathbb{Z}/2 \times \mathbb{Z}/3$ is the element $(1, -1)$. In fact, no power of T_1 is a commutator in $SL(2, \mathbb{Z})$.

In higher genus there is a surjection $Map_g \to Sp(2g, \mathbb{Z})$ and there the obstructions seem to cease to exist, as shown by Corvaja and Zannier (personal communication). Writing T_2 for the transvection such that

$$T_2 : e_1 \longmapsto e_1, e_2 \longmapsto e_1 + e_2, e_3 \longmapsto e_3, e_4 \longmapsto e_3 + e_4,$$

they show that T_2 is a commutator in $Sp(4, \mathbb{Z})$, and that T_1 and the similarly defined T_3 are commutators in $Sp(6, \mathbb{Z})$.

In the mapping class group the question becomes more subtle: first of all, there are Dehn twists on non-separating and on separating curves; in the latter case the homology class of the curve is trivial, and the image of the Dehn twist in $Sp(2g, \mathbb{Z})$ is trivial.

Endo and Kotschick [E-K01] showed that in the separating case the Dehn twist cannot be a commutator: the idea is that otherwise one would have a symplectic fibration over a torus with only one singular fibre with the Dehn twist as local monodromy.

Cartwright and Steger (see [CKY14]) recently constructed via computer calculations a surface of general type with $q = p_g = 1$ and $K^2 = 9$ (hence a ball quotient). In this case the singular fibres of the Albanese map [CK14] are either three irreducible singular fibres with one node, or just one irreducible singular fibre having either three nodes, or a tacnode.

The existence of this surface shows that the product of three Dehn twists can be a commutator in Map_g, and shows that not only the proof, but also the statement of Theorem 7 of [Kot04] is wrong. Recently Stipsicz and Yun [S-Y17] announced the following result: a product of at most two Dehn twists cannot be a commutator in Map_g for $g \geq 1$.

4. Covers branched over line configurations

4.1. Generalities on Abelian coverings of the plane

Assume that in the projective plane $\mathbb{P} := \mathbb{P}^2$ are given distinct irreducible curves C_j of respective degrees m_j; we shall denote by C the union of the curves C_j.

Then an irreducible Abelian cover branched over the union C of these curves and with group G is determined by:

- elements $g_j \in G$ such that they generate G and such that
- the sum of the $m_j g_j$ is equal to zero in G:

$$\sum_j m_j g_j = 0 \in G.$$

In fact, one has an exact sequence

$$H^2(\mathbb{P}, \mathbb{Z}) \longrightarrow H^2(C, \mathbb{Z}) \longrightarrow H^3(\mathbb{P}, C, \mathbb{Z}) \longrightarrow 0$$

and the last group is isomorphic to $H_1(\mathbb{P} \setminus C, \mathbb{Z})$ via Lefschetz duality, while $H^2(C, \mathbb{Z}) \cong \bigoplus_j \mathbb{Z}C_j$. This means that $H_1(\mathbb{P} \setminus C, \mathbb{Z})$ is generated by loops around the curves C_j, but these satisfy one relation.

The relation has the above form because, denoting $r : H^2(\mathbb{P}, \mathbb{Z}) \to \bigoplus_j \mathbb{Z}C_j$, the image of a line L in $coker(r) = H_1(\mathbb{P} \setminus C, \mathbb{Z})$ is zero.

If we blow up some points P_i which are singular for C, obtaining a birational morphism $\pi : Y \to \mathbb{P}$, then the homology of the complement of the preimage of C, $H_1(Y \setminus \pi^{-1}C, \mathbb{Z})$, is generated by the loops around the curves D_j which are strict transform of the curves C_j, and by the loops around the exceptional curves E_i.

The monodromy μ of the Abelian cover takes these generators to respective elements g_j and ϵ_i of the group.

Recall the relation $\sum_j m_j g_j = 0$.

If we write
$$D_j = m_j L - \sum_i a_{j,i} E_i,$$
we can repeat the same argument with Lefschetz duality, and the relation saying that the image of E_i in $coker(r_Y)$ is zero (here $r_Y : H^2(Y, \mathbb{Z}) \to \bigoplus_j \mathbb{Z} D_j \oplus \bigoplus_i \mathbb{Z} E_i$) yields
$$\sum_j a_{j,i} g_j = \epsilon_i .$$

This third formula determines the image ϵ_i of a loop around E_i under the monodromy homomorphism μ.

Definition 61 (Definition of a maximal cover). *Let d_j be the order of the element g_j and consider the following group G'' (relating to the terminology used e.g. in the lecture notes [Cat08], G'' is the abelianization of the orbifold fundamental group of the cover), defined as:*

G'' is the quotient of the direct sum of $\mathbb{Z}/d_j\mathbb{Z}$ by the relation
$$\sum_j m_j g''_j = 0,$$
where g''_j is the standard generator of the summand $G''_j = \mathbb{Z}/d_j\mathbb{Z}$.

Clearly the monodromy μ factors through μ'', which sends the loop around C_j (resp. D_j) to g''_j, and the obvious surjection of $G'' \to G$.

We get corresponding (irreducible) normal coverings Z'', Z of the plane $\mathbb{P}$ such that
$$\mathbb{P} = Z''/G'' = Z/G, \quad Z = Z''/H,$$
where H is the kernel of the surjection $G'' \to G$.

A cover is said to be **maximal** *if $G'' = G$, i.e., $Z = Z''$.*

Remark 62. The quotient $Z = Z''/H$ is only ramified in a finite set.

The importance of the concept of maximal covering is the following: let Y be the surface obtained by blowing up the points P_i where C is not a normal crossing divisor, and assume that the divisor D in Y, union of the E_i's and the D_j's, is a normal crossing divisor (this happens if and only if C has only ordinary singularities).

In this case the local monodromies are the elements g''_j, respectively
$$\epsilon''_i = \sum_j a_{j,i} g''_j .$$

Now, we can write our covering $X \to Y$ (the normalized fibre product $Z \times_{\mathbb{P}^2} Y$) as a GLOBAL QUOTIENT $X = X''/H$, and since the covering $X'' \to X$ is only ramified in a finite set, we get that, if X'' is smooth, then X has only cyclic quotient singularities (this approach has the advantage of making the description of the singularities shorter).

It is therefore important to see general conditions which ensure the smoothness of X'': this is however technical, so we skip this analysis here.

As done by Hirzebruch ([Hirz83], [BHH87]) formulae simplify drastically if we require all the curves C_j to be lines L_j, and we let all the orders d_j of the elements g_j to be equal to the same integer n.

In this case the maximal cover is called the Kummer cover of exponent n of the plane branched on the r lines $L_1, \dots, L_r$ and its Galois group is the group

$$(\mathbb{Z}/n)^{r-1} = (\mathbb{Z}/n)^r / \mathbb{Z}e,$$

where $e := \sum_i e_i$.

Definition 63. *Let $\mathcal{C}$ be a configuration of distinct lines*

$$L_1 = \{l_1(x_1, x_2, x_3) = 0\}, \dots, L_r = \{l_r(x_1, x_2, x_3) = 0\} \subset \mathbb{P}^2,$$

where we assume that

$$\bigcap_1^r L_i = \emptyset.$$

Hence, without loss of generality, we can assume, after changing the numbering of the lines, and after a projective change of coordinates, that $L_i = \{x_i = 0\}$ for $i = 1, 2, 3$.

The linear forms $(l_1, \dots, l_r)$ yield an embedding $l : \mathbb{P}^2 \to \mathbb{P}^{r-1}$, and let $(y_1, \dots, y_r)$ be coordinates in $\mathbb{P}^{r-1}$. Consider next the Galois covering

$$\psi_n : \mathbb{P}^{r-1} \longrightarrow \mathbb{P}^{r-1}, \ \ \psi_n((z_1, \dots, z_r)) = (z_1^n, \dots, z_r^n),$$

with Galois group $(\mathbb{Z}/n)^{r-1}$.

Let Y be the fibre product of l and ψ_n:

$$Y = \{(x, z) \in \mathbb{P}^2 \times \mathbb{P}^{r-1} \mid l(x) = \psi_n(z)\}.$$

Under our assumption on the linear forms, Y indeed embeds in $\mathbb{P}^{r-1}$ as the complete intersection of $r-3$ hypersurfaces:

$$Y = \{(z) \mid z_j^n = l_j(z_1^n, z_2^n, z_3^n), \ j = 4, \dots, r\}.$$

The minimal resolution of singularities X of Y is called the **Hirzebruch–Kummer covering** *of $\mathbb{P}^2$ of exponent n branched on the configuration $\mathcal{C}$ of lines, and denoted $HK_{\mathcal{C}}(n)$.*

Quite similarly one defines the **Hirzebruch–Kummer covering** *of $\mathbb{P}^m$ of exponent n branched on a configuration $\mathcal{C}$ of hyperplanes.*

Remark 64. a) When $r = m + 2$ one gets a hypersurface, the Fermat hypersurface.

b) In the case of $\mathbb{P}^1$ one gets curves which are also called **generalized Fermat curves**.

c) If the configuration is a normal crossing configuration, Y is a smooth complete intersection, so it has a lot of deformations.

4.2. Invariants

Assume that, as in the previous subsection, we have an Abelian cover $S \to Y$ branched on a normal crossing divisor $D' := \sum_j D_j + \sum_i E_i$. To each divisor is associated the cyclic subgroup (inertia subgroup) generated by the local monodromy: $\langle g_j \rangle$ in the case of D_j, and $\langle \epsilon_i \rangle$ in the case of E_i. Let d_j be the order of g_j and d'_i the order of $\langle \epsilon_i \rangle$.

The covering S is smooth if and only if, at each intersection point of two branch curves, the corresponding two inertia subgroups yield a direct sum. In this case also the maximal cover S'' is smooth, at least provided that the order of ϵ''_i in G'' equals d'_i; and S'' is an étale covering of S with group H, such that $G''/H \cong G$.

The Chern numbers of S can be easily calculated, since K_S is the pull back of a divisor with rational coefficients

$$K_S = p^*\Big(K_Y + \sum_j \Big(1 - \frac{1}{d_j}\Big) D_j + \sum_i \Big(1 - \frac{1}{d'_i}\Big) E_i\Big)$$

$$= p^*\Big(- 3L + \sum_j \Big(1 - \frac{1}{d_j}\Big)\Big(m_j L - \sum_i a_{j,i} E_i\Big) + \sum_i \Big(2 - \frac{1}{d'_i}\Big) E_i\Big).$$

Whereas, for the Euler number one uses the fact that it is additive for a stratification with strata which are orientable (non-compact) manifolds (of different dimensions).

The simplest case is the case of a Kummer covering[5] of exponent n branched on r lines, where $m_j = 1$, $d_j = n$, and where we assume that also $d'_i = n$, for all $i = 1, \dots, k$. Then, observing that $a_{j,i} \in \{0, 1\}$ and writing $v_i := \sum_j a_{j,i}$ for the valency of the point p_i, we get:

$$K_S = p^*\Big(- 3L + \sum_j \Big(1 - \frac{1}{n}\Big)\Big(L - \sum_i a_{j,i} E_i\Big) + \sum_i \Big(2 - \frac{1}{n}\Big) E_i\Big)$$

$$= p^*\Big(\Big(- 3 + r\Big(1 - \frac{1}{n}\Big)\Big) L + \sum_i \Big(1 + \Big(1 - \frac{1}{n}\Big)(1 - v_i)\Big) E_i\Big).$$

[5] One can define in the utmost generality the Kummer covering of exponent n of a normal variety Y branched on B as the normal finite covering associated to the epimorphism $\pi_1(Y \setminus B) \to H_1(Y \setminus B, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/n$.

Whence

$$K_S^2 = n^{r-1}\Big[\Big(-3 + r\Big(1-\frac{1}{n}\Big)\Big)^2 - \sum_i \Big(1 + \Big(1-\frac{1}{n}\Big)(1-v_i)\Big)^2\Big].$$

Observe that we only needed to blow up the points of valency $v_i \geq 3$.

Whereas, to calculate the Euler number we write

$$k + 3 = e(Y) = e(Y - D') + \sum_i e(E_i^*) + \sum_j e(D_j^*) + N,$$

where * denotes the intersection of a component of D' with the smooth locus of D', and N is the number of singular points of D'.

Let δ be the number of double points of $C = \sum_j L_j$,

$$\delta = \frac{1}{2}r(r-1) - \frac{1}{2}\sum_i v_i(v_i - 1).$$

Then

$$k+3 = e(Y) = e(Y-D') + \Big(2k - \sum_i v_i + 2r - \sum_i v_i - 2\delta\Big) + \Big(\sum_i v_i + \delta\Big).$$

Hence, writing $v := \sum_i v_i$

$$k + 3 = e(Y) = e(Y - D') + (2k - 2v + 2r - 2\delta) + (v + \delta).$$

The Euler number of S is then just equal to

$$e(S) = n^{r-1}\Big[k + 3 - \Big(1-\frac{1}{n}\Big)(2k - 2v + 2r - 2\delta) - \Big(1-\frac{1}{n^2}\Big)(v+\delta)\Big].$$

In order to calculate the irregularity of S, and in general also $q = h^1(\mathcal{O}_S)$, $p_g = h^2(\mathcal{O}_S)$, the best method is (see [BC08]) to calculate explicitly the decomposition of $p_*\mathcal{O}_S$ into eigensheaves,

$$p_*\mathcal{O}_S = \mathcal{O}_Y \oplus \Big(\bigoplus_\chi \mathcal{O}_Y(-L_\chi)\Big).$$

4.3. Covers branched on lines in general position

Assume that we have an Abelian covering with $d_j = n, \forall j = 1, \dots, r$, branched on the r lines $L_1, \dots, L_r$. We assume that the r lines are in general position, this means that C has only double points, hence we get exactly $\delta = \frac{1}{2}r(r-1)$ intersection points. In this case the fundamental group of $\mathbb{P}^2 \setminus C$ is Abelian, free of rank-$(r-1)$.

Then any such covering is a quotient of the Hirzebruch–Kummer covering of exponent n, and we can assume that $r \geq 4$, otherwise our surface is either singular or equal to $\mathbb{P}^2$.

We saw in Remark 64 that for $r = 4$ the Hirzebruch–Kummer covering yields the Fermat surface of degree n in $\mathbb{P}^3$.

The simplest case is $n = 2$ and $r = 4$: the Hirzebruch–Kummer cover is the smooth quadric

$$\{u^2 = x^2 + y^2 + z^2\} \subset \mathbb{P}^3,$$

hence the del Pezzo surface $\mathbb{P}^1 \times \mathbb{P}^1$.

The next case of 5 lines in general position yields as Hirzebruch–Kummer cover a complete intersection Y of type $(2, 2)$ in $\mathbb{P}^4$, hence again a del Pezzo surface, of degree 4. There is only one intermediate covering branched on the 5 lines: it is a singular del Pezzo surface of degree 2.

The next case of 6 lines in general position is interesting, the Kummer covering is a smooth K3 surface, with a group $(\mathbb{Z}/2)^5$ of covering automorphisms, whereas the double cover branched on the 6 lines is a K3 surface with 15 nodes coming from the $\delta = 15$ double points; in the case where the 6 lines are tangent to a conic Q, it is a 16-nodal Kummer surface blown up in one node (the blown up node corresponds to one of the two components in the inverse image of Q).

In general, in order to obtain a surface of general type, we must have at least 4 lines, and moreover we need, in the case where $d_j = n, \ \forall j$, that $r(n-1) > 3n \Leftrightarrow (r-3)n > r$.

Hence the smallest such case is for $r = 4$, $n = 5$. If the 4 lines are in general position, as we already mentioned, the Hirzebruch–Kummer covering is the Fermat quintic

$$\{u^5 = x^5 + y^5 + z^5\} \subset \mathbb{P}^3,$$

and any intermediate free $\mathbb{Z}/5$ quotient yields a classical Godeaux surface.

4.3.1. Pardini's surfaces Even if the surfaces we have been playing with right now may a priori look rather uninteresting (they are however fun!) a remarkable example in the class of abelian coverings branched on lines in general position was found by Pardini ([Par91b]); it belongs to the next case, with $r = 5$ lines, $n = 5$, and group $G = (\mathbb{Z}/5)^2$.

The covering surfaces have, by the above formulae, $K_S^2 = 25, e(S) = 35$ hence, by Noether's formula $\chi(S) = \frac{1}{12}(25 + 35) = 5$.

Since the fundamental group of the surface is $(\mathbb{Z}/5)^2$ (the Kummer cover being simply connected, see Definition 63), the conclusion is that $q(S) = 0, p_g(S)$
$= 4$, and the canonical map maps to $\mathbb{P}^3$.

To make a long story short, recall that, when the canonical map of a surface of general type has degree ≥ 2, then only two alternatives are possible (see [Bea79]):

(1) the canonical image of S, $\Sigma := \Phi_{K_S}(S)$ has $p_g(S) = 0$, or
(2) the canonical image of S, $\Sigma := \Phi_{K_S}(S)$ is canonically embedded.

Babbage claimed that only the first case should occur, then three essentially equivalent examples of case 2) were found by Beauville, van der Geer–Zagier, and the present author, ([Cat81], [vdGZ77], [Bea79]), where the degree of the canonical map Φ_{K_S} equals 2.

It is still an open quesition which is the highest possible value for $deg(\Phi_{K_S})$ in case 2), but the world record is 5, obtained by Pardini (and later, independently, by Tan [Tan92]).

To give a smooth covering with group $G = (\mathbb{Z}/5)^2$, branched over 5 lines in general position, it is equivalent to give the 5 monodromy vectors $g_1, \dots, g_5$ $\in (\mathbb{Z}/5)^2$, with the property that:

i) $\sum_j g_j = 0$, and
ii) two distinct vectors g_i and g_j are linearly independent (this is the condition for the smoothness of S).

Pardini's choice can best be explained in the following abstract way: we take the five vectors to be the five points of an affine line in $(\mathbb{Z}/5)^2$ not passing through the origin!

Now, conditions i) and ii) are satisfied trivially, and there remains to see the advantage of this clever choice.

This rests on the fact that there is a homomorphism $\psi : G \to \mathbb{Z}/5$ mapping all the five elements g_j to $1 \in \mathbb{Z}/5$. To the homomorphism ψ there corresponds an intermediate $\mathbb{Z}/5$-covering Z of $\mathbb{P}^2$, i.e., we have a factorization $S \to Z \to \mathbb{P}^2$ of the original covering.

If $l_j(x_0, x_1, x_2) = 0$ is the equation of the line L_j, then we see right away that

$$Z = \left\{ x_3^5 = \prod_{j=1}^{5} l_j(x_0, x_1, x_2) \right\} \subset \mathbb{P}^3.$$

Now, Z is a quintic in $\mathbb{P}^3$, whose only singularities are the 10 points $x_3 = l_i(x_0, x_1, x_2) = l_j(x_0, x_1, x_2) = 0$. These singularities are rational double points of type A_4, hence Z is a canonical model with $p_g(Z) = 4$.

Since $p_g(S) = p_g(Z) = 4$, the conclusion is:

Theorem 65 (Pardini). *There exist surfaces S with $K_S^2 = 25$, K_S ample, $p_g(S) = 4, q(S) = 0$, such that the canonical map of S maps with degree 5 onto a canonically embedded surface Z.*

One can ask what happens if the five monodromy vectors $g_1, \dots, g_5$ are affinely independent, yet satisfy i) and ii). This is interesting, since it shows the usefulness of calculating the character sheaves.

We write a character $\chi : G \to \mathbb{Z}/5$ as a pair $(a, b), 0 \le a, b < 5$, so that, for $(x, y) \in (\mathbb{Z}/5)^2$, $\chi(x, y) = ax + by \in \mathbb{Z}/5$.

The character sheaves are of the form $\mathcal{O}_{\mathbb{P}^2}(-L_\chi)$, where the L_χ are calculated applying χ to the monodromy vectors, taking the remainder for division by 5 (we denote by $[d]$ the remainder for division of d by 5, hence $0 \le [d] < 5$), and then summing all these remainders.

Just in order to have a concrete example, take the five vectors

$$(1, 0), (0, 1), (1, 1), (2, -1), (1, -1):$$

then

$$5L_\chi = a + b + [a + b] + [2a - b] + [a - b].$$

This is important, because one can write

$$H^0(S, \mathcal{O}_S(K_S)) = \bigoplus_\chi z_\chi H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + L_\chi)),$$

where $z_j = 0$ is the equation of the ramification divisor R_j corresponding to the j-th line L_j, and $z_\chi := \prod_{\chi(j)=0} z_j^{5-1-[\chi(g_j)]}$.

In our example, $p_g = 4$ and there are exactly 4 characters such that $L_\chi = 3$, namely, $a = b = 4, a = 4, b = 0, a = 3, b = 4, a = 1, b = 3$.

One sees therefore that the base locus of the canonical system K_S equals the subscheme intersection of the four divisors

$$R_3 + 4R_5, R_1 + 2R_3 + 2R_4, 4R_2 + R_4, 3R_1 + R_2 + R_5,$$

that is, we get the points $R_2 \cap R_3$ and $R_4 \cap R_5$ with multiplicity 1.

Hence

Proposition 66. *A $(\mathbb{Z}/5)^2$-covering of the plane branched on 5 lines in general position, and with monodromy vectors*

$$(1, 0), (0, 1), (1, 1), (2, -1), (1, -1)$$

is smooth with $K_S^2 = 25$, $p_g(S) = 4, q(S) = 0$, has a canonical system with 2 simple base points, and its canonical map is birational onto a surface Σ of degree 23 in $\mathbb{P}^3$.

We remark that there are other Abelian covers branched on 5 lines in general position, which are **uniform** (i.e., with $d_j = n, \ \forall j = 1, \dots, r$) and with exponent $n = 5$, but we do not pursue their classification here.

4.4. Hirzebruch's and other ball quotients

One obtains an infinite fundamental group whenever we take a **uniform** (i.e., with $d_j = n,\ \forall j = 1, \dots, r$) Abelian cover branched on a union of lines, such that there exists a point p_i of valency $v_i =: w \geq 3$.

This is because projection with centre the point p_i leads to a fibration $S'' \to B$, where B is an Abelian covering of $\mathbb{P}^1$ branched on w points, and with group $(\mathbb{Z}/n)^{w-1}$.

The genus of B satisfies $2(b-1) = n^{w-1}(-2 + w(1 - \frac{1}{n}))$, hence $b > 1$ as soon as $(w-2)(n-1) > 2$, which we shall now assume, and which holds for $n \geq 4$.

Since the maximal covering S'' maps onto a curve with genus $b \geq 2$, the fundamental group of S'' is infinite, as well as the one of S, of which S'' is a finite étale covering.

Yet, it may still happen that the first Betti number of S is zero, as we shall see.

Hirzebruch [Hirz83] found explicit examples of ball quotients by taking Kummer coverings branched on a union of lines.

The one on which we shall concentrate mostly is the one obtained for $n = 5$ taking 6 lines which are the sides of a complete quadrangle (which can be also visualized as the sides of a triangle plus its three medians, which meet in the barycentre).

Other important examples were the Hirzebruch–Kummer coverings (*cf.* Definition 63) $HK_{\mathscr{H}}(5)$ and $HK_{\mathscr{DH}}(3)$ associated to the Hesse configuration of lines $\mathscr{H}$, a configuration of type $(9_4, 12_3)$ formed by the 9 flexes of a plane cubic curve, and the 12 lines joining pairs of flexpoints, respectively to the dual Hesse configuration $\mathscr{DH}$ of type $(12_3, 9_4)$ of the 9 lines dual to the flexpoints.

Recall that every smooth plane cubic curve is isomorphic to one in the Hesse pencil:

$$C_\lambda := \{x^3 + y^3 + z^3 + 6\lambda xyz = 0\}.$$

C_λ is smooth, except for $\lambda = \infty$, or $8\lambda^3 = -1$; its flexes are the intersection of C_λ with its Hessian cubic curve, which is precisely

$$H_\lambda := \{\lambda^2(x^3 + y^3 + z^3) - (1 + 2\lambda^3)xyz = 0\} = C_\mu, \quad \mu = -\frac{1 + 2\lambda^3}{6\lambda^2}.$$

Hence the nine points are the base points of the pencil,

$$\{xyz = x^3 + y^3 + z^3 = 0\},$$

while $C_\lambda = H_\lambda$ exactly when $8\lambda^3 = -1$: hence the 4 singular curves of the pencil are 4 triangles (every point is a flexpoint!), and these 4 triangles produce the 12 lines.

More examples of ball quotients can be found in [Hirz85] and [BHH87] (they are also related to hypergeometric integrals, see [D-M86]).

4.5. Symmetries of the del Pezzo surface of degree 5

The blow-up Y of 4 points $P_1, \dots, P_4 \in \mathbb{P}^2$ in general position is the del Pezzo surface of degree 5.

This surface is the moduli space of ordered quintuples of points in $\mathbb{P}^1$, as we shall now see. The 6 lines can be labelled $L_{i,j}$, with $i \neq j \in \{1,2,3,4\}$ ($L_{i,j}$ is the line $\overline{P_i P_j}$).

The following is well-known:

Theorem 67. *The automorphism group of the del Pezzo surface Y of degree 5 is isomorphic to $\mathfrak{S}_5$.*

Proof. There is an obvious action of the symmetric group $\mathfrak{S}_4$ permuting the 4 points, but indeed there is more (hidden) symmetry, by the symmetric group $\mathfrak{S}_5$. This can be seen denoting by $E_{i,5}$ the exceptional curve lying over the point p_i, and denoting, for $i \neq j \in \{1,2,3,4\}$, by $E_{i,j}$ the line $L_{h,k}$, if $\{1,2,3,4\} = \{i,j,h,k\}$.

For each choice of 3 of the four points, $\{1,2,3,4\} \setminus \{h\}$, consider the standard Cremona transformation σ_h based on these three points. To σ_h we associate the transposition $(h,5) \in \mathfrak{S}_5$, and the upshot is that σ_h transforms the 10 (-1) curves $E_{i,j}$ via the action of $(h,5)$ on pairs of elements in $\{1,2,3,4,5\}$.

Indeed there are five geometric objects permuted by $\mathfrak{S}_5$: namely, 5 fibrations $\varphi_i : Y \to \mathbb{P}^1$, induced, for $1 \leq i \leq 4$, by the projection with centre P_i, and, for $i = 5$, by the pencil of conics through the 4 points. Each fibration is a conic bundle, with exactly three singular fibres, corresponding to the possible partitions of type $(2,2)$ of the set $\{1,2,3,4,5\} \setminus \{i\}$.

To conclude that $\mathfrak{S}_5 = \mathrm{Aut}(Y)$, we observe that Y contains exactly 10 lines, i.e., irreducible curves E with $E^2 = EK_Y = -1$. We have the following easy result:

Lemma 68. *The curves $E_{i,j}$, which generate the Picard group, have an intersection behaviour which is dictated by the simple rule (recall that $E_{i,j}^2 = -1,\ \forall i \neq j$)*

$$E_{i,j} \cdot E_{h,k} = 1 \iff \{i,j\} \cap \{h,k\} = \emptyset,$$
$$E_{i,j} \cdot E_{h,k} = 0 \iff \{i,j\} \cap \{h,k\} \neq \emptyset.$$

In this picture the three singular fibres of φ_1 are

$$E_{3,4} + E_{2,5},\ E_{2,4} + E_{3,5},\ E_{2,3} + E_{4,5}.$$

The relations among the $E_{i,j}$'s in the Picard group come from the linear equivalences $E_{3,4} + E_{2,5} \equiv E_{2,4} + E_{3,5} \equiv E_{2,3} + E_{4,5}$ and their $\mathfrak{S}_5$-orbits.

Therefore each automorphism ψ of Y permutes the 10 lines, preserving the incidence relation. Up to multiplying ψ with an element of the subgroup $\mathfrak{S}_5$, we may assume that ψ fixes $E_{1,2}$, hence that ψ permutes the three curves $E_{3,4}, E_{3,5}, E_{4,5}$. By the same trick we may assume that ψ fixes $E_{1,2}, E_{3,4}, E_{3,5}, E_{4,5}$. Hence $\psi(E_{1,j}), 3 \leq j \leq 5$ is either $E_{1,j}$ or $E_{2,j}$. Multiplying ψ by the transposition $(1,2)$ we may assume that also $\psi(E_{1,3}) = E_{1,3}$. The incidence relation now says that ψ fixes all the 10 lines, and there remains to show that ψ is the identity. But blowing down the four curve $E_{i,5}$ we see that ψ indudes an automorphism of $\mathbb{P}^2$ fixing each of the points P_1, P_2, P_3, P_4. Hence ψ is the identity and we are done. $\square$

Remark 69. We have the following correspondences:

- The lines $E_{i,j}$ correspond to the transpositions in $\mathfrak{S}_5$.
- The 15 intersection points $E_{i,j} \cdot E_{h,k}$, for $|\{i.j.h.k\}| = 4$ correspond to isomorphisms of $(\mathbb{Z}/2)^2$ with a subgroup of $\mathfrak{S}_5$.
- The five 2-Sylow subgroups correspond to the five conic bundles φ_i, $i = 1, \dots, 5$, and the triples of singular fibres correspond to the three different embeddings $(\mathbb{Z}/2)^2 \to \mathfrak{S}_5$ with the same image.
- The six 5-Sylow subgroups correspond combinatorially to pairs of opposite **pentagons**. Here a pentagon is the equivalence class of a bijection $\mathscr{P} : \mathbb{Z}/5 \to \{1,2,3,4,5\}$ for the action of the dihedral group D_5 on the source $(n \in \mathbb{Z}/5 \mapsto \pm n + b, b \in \mathbb{Z}/5)$. Whereas a pair of opposite pentagons is the equivalence class for the action of the affine group $A(1, \mathbb{Z}/5)$ on the source.
- To a combinatorial pentagon corresponds a **geometric pentagon**, i.e., a union of lines $E_{\mathscr{P}(i),\mathscr{P}(i+1)}, i \in \mathbb{Z}/5$ each meeting the following line $E_{\mathscr{P}(i+2),\mathscr{P}(i+3)}$. The divisor of a geometric pentagon is an anti-canonical divisor $D_{\mathscr{P}}$, and the sum of two opposite geometric pentagons $D_{\mathscr{P}} + D_{\mathscr{P}^o}$ is just the sum of the 12 lines $E_{i,j}$.
- If $D_{\mathscr{P}} = div(s_{\mathscr{P}}), s_{\mathscr{P}} \in H^0(\mathscr{O}_Y(-K_Y))$, we obtain five independent quadratic equations for the anti-canonical embedding
$$Y \longrightarrow \mathbb{P}(H^0(\mathscr{O}_Y(-K_Y)))^\vee,$$
from the six equations $s_{\mathscr{P}} s_{\mathscr{P}^o} = \delta$, where $div(\delta) = \sum E_{i,j}$.
- The above symmetry is the projective icosahedral symmetry, i.e., the symmetry of the image of the icosahedron in $\mathbb{P}^2(\mathbb{R})$: the 10 lines correspond to pairs of opposite faces, the 15 points to pairs of opposite edges, the 6 pairs of opposite pentagons correspond to pairs of opposite vertices.

A basis for $H^0(\mathscr{O}_Y(-K_Y))$ is given by the six sections corresponding to the pentagons where $4, 5$ are never neighbours.

Written as sections of $H^0(\mathscr{O}_{\mathbb{P}^2}(3))$ vanishing at the points $P_1, \dots, P_4$ which we assume to be the coordinate points and the point $(1,1,1)$, they are:

$$s_{i,j} = x_i x_j (x_j - x_k), \quad \{i,j,k\} = \{1,2,3\}.$$

All this leads to beautiful Pfaffian equations for $Y \subset \mathbb{P}^5$, which are $\mathfrak{S}_5$ equivariant. Indeed $H^0(\mathcal{O}_Y(-K_Y))$ is the unique irreducible representation χ_6 of $\mathfrak{S}_5$ of dimension 6, whereas the representation ρ on the set of pairs of opposite pentagons splits as the direct sum of the trivial representation with an irreducible representation χ_5 of dimension 5, the representation $\tilde{\rho}$ on the pentagons is the direct sum of ρ with the tensor product of ρ with the signature character χ_1. The permutation representation of $\mathfrak{S}_5$ on $\{1, 2, 3, 4, 5\}$ splits as the trivial representation direct sum with an irreducible representation χ_4 of dimension 4. χ_6 is the only irreducible representation such that $\chi_6 \cong \chi_6 \otimes \chi_1$.

In this way one obtains all the irreducible representations of $\mathfrak{S}_5$, see [J-L], page 201. We shall not go further here with the equations of Y, for the anti-canonical embedding $Y \subset \mathbb{P}^5$, and for the embedding $Y \subset (\mathbb{P}^1)^5$ via $\varphi_1 \times \cdots \times \varphi_5$.

But we shall now prove the classical:

Theorem 70. *The del Pezzo surface Y of degree 5 is the moduli space for ordered 5-tuples of points in $\mathbb{P}^1$, i.e., the GIT quotient of $(\mathbb{P}^1)^5$ by $\mathbb{P}GL(2, \mathbb{C})$.*

Proof. Another model for Y is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in the three diagonal points $(\infty, \infty), (0, 0), (1, 1)$, and the 10 lines come from the three blown up points, plus the strict transforms of the diagonal and of the vertical and horizontal lines $x = 0, x = 1, x = \infty, y = 0, y = 1, y = \infty$.

Removing these 7 lines in $\mathbb{P}^1 \times \mathbb{P}^1$ we obtain a point (u, v) such that the five points $\infty, 0, 1, u, v$ are all distinct.

If we approach a smooth point in the diagonal line, say (u, u) we obtain the 5-tuple $\infty, 0, 1, u, u$ where the fourth and the fifth points are equal $P_4 = P_5$, and the other three are different (so that set theoretically we have four distinct points). By $\mathfrak{S}_5$-symmetry, the same occurs whenever we get a smooth point of the divisor $\sum E_{i,j} \subset Y$: $P_i = P_j$ and the other three are different from P_i and pairwise different.

If we tend to the point $(0, 1)$, we get the 5-tuple $\infty, 0, 1, 0, 1$, where $P_2 = P_4$, $P_3 = P_5$; again by symmetry, to the point $E_{i,j} \cap E_{h,k}$ corresponds a quintuple with $P_i = P_j$, $P_h = P_k$, and where set theoretically we have three distinct points.

Now, as shown in Mumford's book [Mum65], especially Proposition 3.4, page 73, in this case the semi-stable 5-tuples are stable, and a quintuple is unstable if and only if three points are equal. One can easily conclude that we have an isomorphism of Y with the GIT-quotient $(\mathbb{P}^1)^5 \,//\, \mathbb{P}GL(2, \mathbb{C})$. $\square$

4.5.1. Hirzebruch–Kummer covers of a del Pezzo surface of degree 5 Let us now take as branch locus $D := \sum_{i,j} E_{i,j}$, and let us notice that D is linearly equivalent to twice the anti-canonical divisor $-K_Y$.

Therefore, for any n-uniform covering S of Y branched on the 10 lines, and which is smooth,

$$K_S = p^*\Big(K_Y - 2\Big(1 - \frac{1}{n}\Big)K_Y\Big) = p^*\Big(-\Big(\frac{n-2}{n}\Big)K_Y\Big).$$

In particular, for the Kummer covering, one has

$$K^2_{S''} = 5(n-2)^2 n^3.$$

Whereas the Euler number can be calculated (in [CatDet15] it is calculated in an elegant way) as:

$$e(S'') = n^3(3 + 2(n-2)(n-3)) = n^3(2n^2 - 10n + 15).$$

The Chern slope equals then

$$\nu_C(S'') = \frac{5(n-2)^2}{3 + 2(n-2)(n-3)} = \frac{5}{2}\frac{n-2}{n-3+\frac{3}{2(n-2)}} > \frac{5}{2}.$$

The same formula shows that the slope is a decreasing function of n for $n \geq 5$, and for $n = 5$ we obtain $\nu_C(S'') = \frac{45}{15} = 3$. Hence the theorem of Hirzebruch [Hirz83]:

Theorem 71. *A smooth n-uniform Abelian cover S with $n \geq 5$ branched on the 10 lines of a del Pezzo surface of degree 5 is a surface with ample canonical divisor, with positive index, and indeed a ball quotient if and only if $n = 5$.*

We observe, as a side remark, that the Kummer coverings above embed into C_n^5, where C_n is the Fermat curve of degree n,

$$C_n = \{x^n + y^n + z^n = 0\} \subset \mathbb{P}^2,$$

via the Cartesion product of the Stein factorizations of the maps induced by the φ_i, $i = 1, \dots, 5$.

We shall next discuss some of the surfaces S mentioned in the previous theorem, addressing the question of their irregularity $q(S)$.

4.5.2. A problem posed by Enriques, and its partial solution Enriques posed in his book [Enr49] the following problems:

Question 72. Given a surface S with $p_g(S) = 4$ and with birational canonical map onto its image $\Sigma \subset \mathbb{P}^3$,

I) what is the maximum value for its canonical degree K_S^2?
II) what is the maximum value for $deg(\Sigma)$?

He indeed suspected that there should be an upper bound equal to 24, for which counterexamples were given in [Cat99].

Now, without loss of generality, we may assume that S is minimal, because otherwise K_S^2 decreases, and we observe that, since K_S is nef, we have

$$45 \geq K_S^2 \geq deg(\Phi_{K_S}) \cdot deg(\Sigma),$$

where the first inequality is a consequence of the Miyaoka–Yau inequality $K_S^2 \leq 9(1 - q(S) + p_g(S)) = 9(5 - q(S))$.

Hence, in order to achieve the equality $K_S^2 = 45$, when $p_g(S) = 4$, we must have a ball quotient which has $q(S) = 0$.

With I. Bauer [BC08] we showed that there exists such a surface.

Theorem 73. *There exists an Abelian cover of the del Pezzo surface Y of degree 5, with group $(\mathbb{Z}/5)^2$, and branched on the 10 lines, which is regular, i.e., $q(S) = 0$, has $p_g(S) = 4$, $K_S^2 = 45$, and has a birational canonical map onto a surface Σ of degree 19.*

Indeed in the course of the search we classified all such coverings of the del Pezzo surface Y of degree 5, with group $(\mathbb{Z}/5)^2$, and branched on the 10 lines.

We considered the group $\mathscr{G}$, generated by $\mathfrak{S}_5$ and $GL(2, \mathbb{Z}/5\mathbb{Z})$, acting on the set of admissible monodromy vectors.

A MAGMA computation showed that $\mathscr{G}$ has four orbits, and representatives for these orbits could be taken as:

$$\mathfrak{U}_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right);$$

$$\mathfrak{U}_2 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right);$$

$$\mathfrak{U}_3 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right);$$

$$\mathfrak{U}_4 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right).$$

In particular we saw that $\mathscr{G} \cong GL(2, \mathbb{Z}/5\mathbb{Z}) \times \mathfrak{S}_5$, and concluded the classification with the following result:

Theorem 74. *Let S_i be the minimal smooth surface of general type with $K^2 = 45$ and $\chi = 5$ obtained from the covering induced by the admissible six-tuple $\mathfrak{U}_i$, where $i \in \{1, 2, 3, 4\}$. Then we have that S_3 is regular (i.e., $q(S_3) = 0$), whereas $q(S_i) = 2$ for $i \neq 3$.*

The story concerning Enriques' question is not yet completely finished, because it is not clear whether one can achieve $deg(\Sigma) = 45$ (the current record in this direction is $deg(\Sigma) = 28$, [Cat99]).

4.6. Bogomolov–Miyaoka–Yau fails in positive characteristic

Even if the BMY inequality was proven by Miyaoka with purely algebraic methods, still the proof uses characteristic 0 arguments in an essential way.

Robert Easton [Easton08] gave easy examples showing that indeed the BMY inequality does not hold in positive characteristic. These examples are related to the Hirzebruch–Kummer coverings of the plane, and the main idea is that in characteristic $p > 0$ there are configurations of lines which cannot exist in characteristic zero.

These configurations are just the projective planes $\mathbb{P}^2_{\mathbb{Z}/p} \subset \mathbb{P}^2_K$, for each algebraically closed field K of characteristic $p > 0$.

The easiest case is the so called **Fano plane** $\mathbb{P}^2_{\mathbb{Z}/2} \subset \mathbb{P}^2_K =: \mathbb{P}^2$. We have a configuration $\mathscr{C}$ of type $7_3 7_3$, 7 lines passing through seven points, each triple for the configuration.

For each odd number $n \geq 5$ we consider the Hirzebruch–Kummer covering of exponent n, S_n, a finite Galois cover of the blow up Z of $\mathbb{P}^2$ in the seven points, with Galois group $(\mathbb{Z}/n)^6$.

The canonical divisor of Z is $K_Z = -3H + \sum_1^7 E_i$, where E_i is the inverse image of the point P_i. Denoting by L_i the proper transform of the line L_i, since $\sum_1^7 L_i = 7H - 3\sum_1^7 E_i$, the canonical divisor of S_n is the pull back of

$$K_Z + \left(1 - \frac{1}{n}\right)\left(\sum_1^7 L_i + \sum_1^7 E_i\right)$$
$$= \left(-3 + 7\left(1 - \frac{1}{n}\right)\right)H + \left(1 - 2\left(1 - \frac{1}{n}\right)\right)\sum_1^7 E_i.$$

Hence

$$K^2_{S_n} = n^6\left[\left(4 - \frac{7}{n}\right)^2 - 7\left(1 - \frac{2}{n}\right)^2\right] = n^6\left[9 - \frac{28}{n} + \frac{21}{n^2}\right].$$

The formulae we illustrated earlier yield

$$c_2(S_n) = n^6\left[10 + 14\left(1 - \frac{1}{n}\right) - 21\left(1 - \frac{1}{n^2}\right)\right] = n^6\left[3 - 14\frac{1}{n} + 21\frac{1}{n^2}\right].$$

Hence

$$K^2_{S_n} = 3c_2(S_n) + n^6\left(14\frac{1}{n} - 42\frac{1}{n^2}\right) = 3c_2(S_n) + n^4(14n - 42).$$

Hence a particular case of Easton's theorem:

Theorem 75. *In characteristic equal to* 2 *the Hirzebruch–Kummer coverings of the plane branched on the Fano of configuration of lines have Chern slope*

$$\nu(S_n) = \frac{K_{S_n}^2}{c_2(S_n)} = 3\Big(1 + \frac{14n - 42}{9n^2 - 42n + 63}\Big) > 3,$$

violating the Bogomolov–Miyaoka–Yau inequality. The maximum of the slope is attained for $n = 5$, $\nu(S_5) = 4 + \frac{3}{39}$.

5. Counterexamples to Fujita's semi-ampleness question, rigid manifolds

5.1. BCDH surfaces, counterexamples to Fujita's question

Recently, we found new counterexamples to Fujita's Question 49 (we say counterexamples since we heard through the grapevine that experts were expecting a positive answer; however this counterexample does not inficiate the abundance conjecture).

Theorem 76. *There exists an infinite series of surfaces with ample canonical bundle, whose Albanese map is a fibration $f : S \to B$ onto a curve B of genus $b = \frac{1}{2}(n-1)$, and with fibres of genus $g = 2b = n - 1$, where n is any integer relatively prime with* 6.

These Albanese fibrations yield negative answers to Fujita's question about the semi-ampleness of $V := f_*\omega_{S|B}$, since here $V := f_*\omega_{S|B}$ splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle, and Q is a unitary flat bundle with infinite monodromy group.

The fibration f is semi-stable: indeed all the fibres are smooth, with the exception of three fibres which are the union of two smooth curves of genus b which meet transversally in one point.

For $n = 5$ we get three surfaces which are rigid, and are quotient of the unit ball in $\mathbb{C}^2$ by a torsion free cocompact lattice Γ. We shall call them BCD-surfaces (cf. Theorem 74). The rank of A, respectively Q, is in this case equal to 2.

The easiest way to describe these surfaces, which are Abelian covers of the del Pezzo surface Y of degree 5 with group $(\mathbb{Z}/n)^2$, branched over the 10 lines of Y, is to look at a birational model which is an Abelian covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over the diagonal and of the vertical and horizontal lines $x = 0, x = 1, x = \infty, y = 0, y = 1, y = \infty$.

We consider again the equation

$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (y_1 - x y_0)^{m_3}, \quad x \in \mathbb{C} \setminus \{0, 1\}$$

but we homogenize it to obtain the equation

$$z_1^n = y_0^{m_0} y_1^{m_1} (y_1 - y_0)^{m_2} (x_0 y_1 - x_1 y_0)^{m_3} x_0^{n-m_3}.$$

The above equation describes a singular surface Σ' which is a cyclic covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $G := \mathbb{Z}/n$; Σ' is contained inside the line bundle $\mathbb{L}_1$ over $\mathbb{P}^1 \times \mathbb{P}^1$ whose sheaf of holomorphic sections $\mathcal{L}_1$ equals $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$.

The first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ induces a morphism $p : \Sigma' \to \mathbb{P}^1$ and we consider the curve B, normalization of the covering of $\mathbb{P}^1$ given by

$$w_1^n = x_0^{n_0} x_1^{n_1} (x_1 - x_0)^{n_2}.$$

We consider the normalization Σ of the fibre product $\Sigma' \times_{\mathbb{P}^1} B$.

Σ is an abelian covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $(\mathbb{Z}/n)^2$, and, if $GCD(n, 6) = 1$, for a convenient choice of the integers m_j, n_i, for instance for

$$m_0 = m_1 = m_2 = 1, \quad m_3 = n - 3, \quad n_0 = n_1 = 1, \quad n_2 = n - 2$$

we obtain a smooth $(\mathbb{Z}/n)^2$-Abelian covering of the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in the three diagonal points (∞, ∞), $(0, 0)$, $(1, 1)$, which is the del Pezzo surface Y of degree 5.

The corresponding singular fibres are only 3, and come from one of the 5 conic bundle structures on Y, here given by the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$: hence one sees right away that the singular fibres are reducible, and that they are the union of two smooth curves of genus b intersecting transversally in one point.

The surfaces in Theorem 76, which we shall call BCDH-surfaces, have an étale unramified covering given by the Hirzebruch–Kummer coverings of the del Pezzo surface Y of degree 5 with group $(\mathbb{Z}/n)^2$, branched over the 10 lines of Y, which we shall denote HK-surfaces.

In the case $n = 5$ we get ball quotients, and, for $5|n$, we get therefore that the universal covering is a branched covering of the ball. This motivates the following questions, which are clearly satisfied for $n = 5$:

Question 77. (1) Are the BCDH-surfaces rigid?
(2) Is their universal covering $\tilde{S}$, or more generally the universal covering of the HK-surfaces, a Stein manifold?
(3) Is the universal covering $\tilde{S}$ of the BCDH- and HK-surfaces contractible?
(4) Do the BCDH-surfaces admit a metric of negative curvature?

The last question is motivated by analogy with the examples considered by Mostow–Siu: namely, one has a Kähler–Einstein metric on Y deprived of the 10 lines, and on the covering one should interpolate with another metric localized on the ramification divisor.

Of course, an interesting question is whether the Kähler–Einstein metric ω_n on S_n is negatively curved. To analyse the question one can observe that ω_n is the pull-back of a metric on Y with given cone angles along the branch divisors.

Now, a positive answer to (4) would imply (3); however, for (3), the vanishing cycles criterion which was given earlier may be applied.

Question (2) should have a positive answer, while for the answer to question (1), it is positive, as proven in joint work with I. Bauer [B-C16]:

Theorem 78. *The HK-surfaces are rigid for $n \geq 4$, hence also the BCDH-surfaces are rigid.*

Observe that, since the BCDH-surfaces have the HK surfaces as étale covers, the non-rigidity of the former would imply non-rigidity of the latter (see Proposition 85). Moreover, for $n = 3$ rigidity does not hold, this surface was studied by Roulleau in [Rou11].

The proof for HK surfaces uses many tools, first of all the special geometry of the del Pezzo surface of degree 5, and rather complicated arguments using logarithmic sheaves in order to control the deformations of the Abelian coverings.

Corollary 79. *The Albanese fibrations of BCDH-surfaces yield rigid curves in the moduli spaces $\overline{\mathfrak{M}}_{n-1}$.*

We noticed that all the fibres of the Albanese map have compact Jacobian, hence the following:

Question 80. Do the Albanese fibrations of BCDH-surfaces yield rigid curves in the moduli spaces $\mathfrak{A}_{n-1}$?

Theorem 78 raises several questions: here, given a configuration of lines $\mathscr{C}$, we denote by $HK_{\mathscr{C}}(n)$ the Hirzebruch–Kummer covering of exponent n ramified on the lines of the configuration $\mathscr{C}$.

Given a configuration of lines $\mathscr{C}$, one constructs a subvariety of $((\mathbb{P}^2)^\vee)^r$, $\mathscr{I}(\mathscr{C})$, given by the r-tuples of lines with the same incidence correspondence as $\mathscr{C}$.

One observes that there is a natural action of $G := \mathbb{P}GL(3, \mathbb{C}) \times \mathfrak{S}_r$ on $\mathscr{I}(\mathscr{C})$, and defines $\mathscr{C}$ to be projectively unique (resp.: rigid) if $\mathscr{I}(\mathscr{C})$ is equal to the G-orbit of $\mathscr{C}$ (resp.: equal to this G-orbit locally at the point $\mathscr{C} \in \mathscr{I}(\mathscr{C})$).

The property of rigidity can be detected by the vanishing of a certain first cohomology group of a sheaf of logarithmic vector fields.

Natural questions are (see [B-C16]):

Question 81. I) For which rigid configuration $\mathscr{C}$ of lines in $\mathbb{P}^2$ is the associated Hirzebruch–Kummer covering $HK_{\mathscr{C}}(n)$ rigid for $n \gg 0$?

II) For which rigid configuration $\mathscr{C}$ of lines in $\mathbb{P}^2$ is the associated Hirzebruch–Kummer covering $HK_{\mathscr{C}}(n)$ a $K(\pi, 1)$ for $n \gg 0$?

III) For which rigid configuration $\mathscr{C}$ of lines in $\mathbb{P}^2$ does the associated Hirzebruch–Kummer covering $HK_{\mathscr{C}}(n)$ possess a Kähler metric of negative sectional curvature for $n \gg 0$?

Observe that if III) has a positive answer, then also II), by the Cartan–Hadamard theorem.

Abelian coverings branched over configurations were also used by Vakil in [Va06], who showed that for $n \gg 0$ the local deformations of $HK_{\mathscr{C}}(n)$ correspond to the product of the deformations of the configuration $\mathscr{C}$ with a smooth manifold. Vakil used a result of Mnëv [Mnev88] in order to show that, up to a product with a smooth manifold, one obtains all possible singularity types.

Let me end this subsection by commenting that not only BCD- and BCDH-surfaces are quite interesting from many algebro-geometric and complex analytic points of view, but that the features of their Albanese fibration have also found very interesting applications for the construction of remarkable symplectic manifolds, in the work of Akhmedov and coworkers (see [AkSa15] and literature cited therein).

5.2. Rigid compact complex manifolds

Recall the following notions of rigidity (see [B-C16] for more details).

Definition 82.

(1) Two compact complex manifolds X and X' are said to be *deformation equivalent* if and only if there is a proper smooth holomorphic map

$$f : \mathfrak{X} \longrightarrow \mathscr{B},$$

where $\mathscr{B}$ is a connected (possibly not reduced) complex space and there are points $b_0, b'_0 \in \mathscr{B}$ such that the fibres $X_{b_0} := f^{-1}(b_0), X_{b'_0} := f^{-1}(b'_0)$ are respectively isomorphic to X, X' ($X_{b_0} \cong X, X_{b'_0} \cong X'$).

(2) A compact complex manifold X is said to be *globally rigid* if for any compact complex manifold X', which is deformation equivalent to X, we have an isomorphism $X \cong X'$.

(3) A compact complex manifold X is instead said to be *(locally) rigid* (or just *rigid*) if for each deformation of X,

$$f : (\mathfrak{X}, X) \longrightarrow (\mathscr{B}, b_0)$$

there is an open neighbourhood $U \subset \mathscr{B}$ of b_0 such that $X_t := f^{-1}(t) \cong X$ for all $t \in U$.

(4) A compact complex manifold X is said to be *infinitesimally rigid* if

$$H^1(X, \Theta_X) = 0,$$

where Θ_X is the sheaf of holomorphic vector fields on X.

(5) X is said to be *strongly rigid* if the set of compact complex manifolds Y which are homotopically equivalent to X, $\{Y | Y \sim_{h.e.} X\}$ consists of a finite set of isomorphism classes of globally rigid varieties.

(6) X is said to be *étale rigid* if every étale (finite unramified) cover Y of X is rigid.

Remark 83. 1) If X is infinitesimally rigid, then X is also locally rigid. This follows by the Kodaira–Spencer–Kuranishi theory, since $H^1(X, \Theta_X)$ is the Zariski tangent space of the germ of analytic space which is the base $\mathrm{Def}(X)$ of the Kuranishi semi-universal deformation of X. If $H^1(X, \Theta_X) = 0$, $\mathrm{Def}(X)$ is a reduced point and all deformations are locally trivial.

2) Obviously strong rigidity implies global rigidity; both global rigidity and étale rigidity imply local rigidity.

3) The simplest example illustrating the difference between global and infinitesimal rigidity is the del Pezzo surface Z_6 of degree 6, blow up of the plane $\mathbb{P}^2$ in three non-collinear points. It is infinitesimally rigid, but it deforms to the weak del Pezzo surface of degree 6, the blow up Z'_6 of the plane $\mathbb{P}^2$ in three collinear points. Z'_6 is not isomorphic to Z_6 because for the second surface the anti-canonical divisor is not ample.

The following useful general result is established in [B-C16] using many earlier results (and the Riemann–Roch theorem for the second statement):

Theorem 84. *A compact complex manifold X is rigid if and only if the Kuranishi space* $\mathrm{Def}(X)$ *(base of the Kuranishi family of deformations) is* 0*-dimensional.*

In particular, if $X = S$ is a smooth compact complex surface and

$$10\chi(\mathcal{O}_S) - 2K_S^2 + h^0(X, \Theta_S) > 0,$$

then S is not rigid.

We have moreover (ibidem):

Proposition 85. *If $p : Z \to X$ is étale, i.e., a finite unramified holomorphic map between compact complex manifolds, then the infinitesimal rigidity of Z implies the infinitesimal rigidity of X. Moreover, if Z is rigid, then also X is rigid.*

Idea of proof. For the first assertion, one observes that $H^1(Z, \Theta_Z) = H^1(X, p_*(\Theta_Z)) = 0$, and that $p_*(\Theta_Z) = p_*(p^*\Theta_X) = \Theta_X \otimes (p_*\mathcal{O}_Z)$ has Θ_X as a direct summand.

For the second assertion one reduces to the Galois case $X = Z/G$, where, as shown in [Cat88]:

$$\mathrm{Def}(X) = \mathrm{Def}(Z)^G \subset \mathrm{Def}(Z).$$

Hence, if $\mathrm{Def}(Z)$ has dimension 0, a fortiori also $\mathrm{Def}(X)$. □

While the only rigid curve is $\mathbb{P}^1$, in the case of surfaces the list is only known for surfaces which are not of general type. Indeed, using surface classification, it is shown in [B-C16]:

Theorem 86. *Let S be a smooth compact complex surface, which is (locally) rigid. Then either*

(1) S is a minimal surface of general type, or
(2) S is a del Pezzo surface of degree $d \geq 5$ (i.e., $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, S_8, S_7, S_6, S_5, where S_{9-r} is the blow-up of $\mathbb{P}^2$ in r points which are in general linear position).
(3) S is an Inoue surface of type S_M or $S^{(-)}_{N,p,q,r}$ (cf. [In94]).
(4) Rigid surfaces in class (1) are also globally rigid, surfaces in class (3) are infinitesimally and globally rigid, surfaces in class (2) are infinitesimally rigid, but the only globally rigid surface in class (2) is the projective plane $\mathbb{P}^2$.

In particular, rigid surfaces have Kodaira dimension either 2 (general type), or $-\infty$. In higher dimension $n \geq 3$ it shown in [B-C16] that there are rigid compact complex manifolds for each possible Kodaira dimension, except possibly Kodaira dimension $= 1$. Probably this exception does not really occur, at least for large n.

Remark 87. For surfaces of general type it is expected to find examples which are rigid, but not infinitesimally rigid: such an example would be the one of a minimal surface S such that its canonical model $X(S)$ is infinitesimally rigid and singular (see [B-W74]).

The intriguing part of the story is that all the old examples of globally rigid surfaces, except $\mathbb{P}^2$, are projective classifying spaces. Indeed, before the examples which we denote here by $HK(n)$-surfaces, all known examples of rigid surfaces of general type were the following:

(1) The *ball quotients*, which are infinitesimally rigid, strongly rigid and étale rigid ([Siu80], [Most73]).

(2) Irreducible bi-disk quotients, i.e., those surfaces whose universal covering of S is $\mathbb{B}_1 \times \mathbb{B}_1 \cong \mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the upper half plane, and moreover if we write $S = \mathbb{H} \times \mathbb{H}/\Gamma$ the fundamental group Γ has dense image for any of the two projections $\Gamma \to PSL(2, \mathbb{R})$; they are infinitesimally rigid, strongly rigid and étale rigid ([J-Y85], [Mok88]).
(3) *Beauville surfaces*; they are infinitesimally rigid, strongly rigid but not étale rigid ([Cat00]).
(4) *Mostow–Siu surfaces*, [M-S80]; these are infinitesimally rigid, strongly rigid and étale rigid.
(5) The rigid *Kodaira fibrations* constructed by the author and Rollenske, [CatRol09], and mentioned in Theorem 15; these are rigid and strongly rigid, infinitesimal rigidity and étale rigidity are not proven in [CatRol09] but could be true.

Remark 88. Examples (1)–(3), and (5) are strongly rigid.

A natural question is therefore:

Question 89. Do there exist infinitesimally rigid surfaces of general type which are not projective classifying spaces?

We believe that the answer should be yes, not only because wishful thinking in the case of surfaces almost invariably turns out to be contradicted, but for the following reason, which relates to the later section on Inoue-type varieties.

Remark 90. (i) Assume that Z is a projective classifying space of dimension $n \geq 3$, and let Y be a smooth hyperplane section of Z: then Y is not a projective classifying space.
This is a consequence of the Lefschetz hyperplane theorem: $\pi_1(Y) \cong \pi_1(Z)$.
If Y were a classifying space for $\pi_1(Y) \cong \pi_1(Z)$, then

$$H^*(Y, \mathbb{Z}) \cong H^*(\pi_1(Y), \mathbb{Z}) \cong H^*(\pi_1(Z), \mathbb{Z}) \cong H^*(Z, \mathbb{Z}),$$

in particular $H^{2n}(Y, \mathbb{Z}) = \mathbb{Z}$, against the fact that the real dimension of Y is $2n - 2$, which implies that $H^{2n}(Y, \mathbb{Z}) = 0$.
(ii) The same occurs if Y is an iterated hyperplane section of Z, with $dim(Y) \geq 2$.
(iii) In particular, even if Z admits a metric of negative curvature, Y cannot have one such negative metric, by the Cartan–Hadamard theorem. In fact, concerning the metric inherited from Z, observe that only the Hermitian curvature decreases in subbundles.

Concerning Question 89, the case of BCDH-surfaces, and $HK(n)$-surfaces is not completely settled.

For the surfaces $HK(n)$ the answer is positive, in case that 5 divides n, due to the work of Fangyang Zheng [Zheng99] who extended the Mostow–Siu technique to the case of normal crossings; from this also strong and étale rigidity follow in this case.

For other values of $n \geq 4$ the work of Panov [Pan09] gives a positive answer for $n \gg 0$ (but unspecified): his method consists in finding polyhedral metrics of negative curvature.

6. Surfaces isogenous to a product and their use

Even if the topic of surfaces isogenous to a product does appear at first sight skew to the topic of surfaces which are ramified coverings branched on a union of lines, this idea is deceptive.

In fact, a simple way to construct a curve C_i with G-symmetry is to construct a Galois branched covering of $\mathbb{P}^1 \setminus \mathscr{B}_i$, with group G, and where we assume that the branch locus is exactly the finite set $\mathscr{B}_i$.

Hence, the surface isogenous to a product $S := (C_1 \times C_2)/G$ is a ramified covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on the horizontal and vertical lines

$$(\mathbb{P}^1 \times \mathscr{B}_2) \cup (\mathscr{B}_1 \times \mathbb{P}^1).$$

Note that we have $\Delta_G \subset G \times G$, the diagonal subgroup, and a Galois diagram

$$C_1 \times C_2 \longrightarrow S := (C_1 \times C_2)/\Delta_G \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 = (C_1 \times C_2)/(G \times G).$$

The covering $S \to \mathbb{P}^1 \times \mathbb{P}^1$ is Galois if and only if G is Abelian.

By the main theorem on surfaces which are isogenous to a product, such a surface S is (strongly) rigid if $|\mathscr{B}_1| = |\mathscr{B}_2| = 3$, and in this case in [Cat00] we called these surfaces ‘Beauville surfaces’.

The reason for this is that Beauville surfaces with Abelian group occur only when $G = (\mathbb{Z}/n)^2$ with $GCD(n, 6) = 1$, as shown in [Cat00], and the original example by Beauville in [Bea78] was exactly the case of $G = (\mathbb{Z}/5)^2$. For these surfaces $C_1 = C_2$ is the Fermat curve of degree n, and the only difficulty consists in finding actions such that Δ_G acts freely on the product.

In the article [BCG05] the existence problem for Beauville surfaces was translated into group theoretical terms: because each covering $C_1 \to \mathbb{P}^1$ is determined, in view of the Riemann existence theorem, by its branch locus (here fixed!) and its monodromy; and in this case the monodromy means the datum of three elements $a, b, c \in G$ which generate G and satisfy $abc = 1_G$.

Then one gets two triples $(a, b, c)(a', b', c')$ and the condition that the action on $C_1 \times C_2$ is free amounts to the disjointness of the stabilizers, $\mathscr{S}_1 \cap \mathscr{S}_2 = \{1\}$, here the stabilizer set $\mathscr{S}_1$ is the union of the conjugates of the powers of the respective elements a, b, c, and similarly for $\mathscr{S}_2$.

In particular the question: which (non-abelian) simple groups except $\mathfrak{A}_5$ occur as G for a Beauville surface? It was solved for many groups in loc. cit., has then attracted the attention of group theorists and was solved in the affirmative, even if there are still open questions concerning whether this can also be achieved via non-real structures, via real structures, via strongly real structures (see [GLL12], [GM12], [FMP13], [BBPV15], see also the book [BGV15] and references therein for a partial account).

Now, our contention here is that Beauville surfaces not only create links between algebraic geometry and group theory, but that they continue to yield very interesting algebro-geometric examples.

For instance, Beauville surfaces with Abelian group $G = (\mathbb{Z}/n)^2$ were used in [Cat14] to answer a question posed by Jonathan Wahl, hence giving rise to new examples of threefolds Z (obtained as cones over such a surface S) which fulfill the following properties:

1) Z is Cohen–Macaulay,
2) the dualizing sheaf ω_Z is torsion,
3) the index 1-cover Z' is not Cohen–Macaulay, in particular Z is not $\mathbb{Q}$-Gorenstein.

This is the technical result answering the question by Wahl, showing in particular the existence of regular surfaces with subcanonical ring which is not Cohen–Macaulay.

Theorem 91. *For each $r = n - 3$, where $n \geq 7$ is relatively prime to 30, and for each m, $1 \leq m \leq r - 1$, there are Beauville type surfaces S with $q(S) = 0$ ($q(S) := \dim H^1(S, \mathcal{O}_S)$) such that there exists a divisor L with $K_S = rL$, and $H^1(mL) \neq 0$.*

6.1. Automorphisms acting trivially on cohomology

Another application is in the direction of giving examples of surfaces admitting automorphisms which act trivially on integral cohomology (but are not isotopic to the identity: recall that a diffeomorphism of a manifold M is said to be isotopic to the identity if it lies in the connected component of the identity in $\mathcal{Diff}(M)$).

In this context, Cai, Liu and Zhang [C-L-Z13] have proven the following theorem:

Theorem 92. *Let S be a minimal smooth surface of general type with $q(S) \geq 2$. Then either S is rational cohomologically rigidified, i.e., every automorphism acts trivially on $H^*(S, \mathbb{Q})$, or the subgroup $Aut(S)_{\mathbb{Q}}$ of automorphisms acting trivially on the rational cohomology algebra is isomorphic to $\mathbb{Z}/2$, and S satisfies*

I) $K_S^2 = 8\chi, q(S) = 2$,
II) the Albanese map is surjective, and,
III) S *has a pencil of genus* $b = 1$.

To show that the estimate is effective, they classified the surfaces isogenous to a product such that $q(S) \geq 2$ and $Aut(S)_{\mathbb{Q}} \neq 0$.

In joint work with Gromadzki we have shown that indeed, for one of these examples (and probably for both), the action is even trivial on $H^*(S, \mathbb{Z})$; but it is not trivial on the fundamental group. To show triviality of the action on $H^*(S, \mathbb{Z})$ it suffices to show that the action is trivial on the torsion group $H_1(S, \mathbb{Z})$; because then the result follows from the universal coefficients theorem and by Poincaré duality.

The following is still an open question:

Question 93. Do there exist surfaces of general type with non-trivial automorphisms which are isotopic to the identity?

The question (see [Cat13]) is crucial in order to compare the Kuranishi and the Teichmüller space of surfaces (and higher dimensional varieties).

7. Topological methods for moduli

In this very short section, devoted to concrete moduli theory in the tradition of Kodaira and Horikawa, i.e., the fine classification of complex projective varieties (see e.g. [Hor75]), we shall try to show how in some lucky cases, with big fundamental group, topology helps to achieve the fine classification, allowing explicit descriptions of the structure of moduli spaces.

This was done quite effectively in several papers ([BC09b], [BC09a], [BC10], [BC10-b], [B-C12], [B-C13], [BCF15a]), and in the article [Cat15] we already amply reported on this direction of research.

For this reason the exposition in this section shall be rather brief, we refer to [Cat15] for several preparatory results, and for other related topics, such as orbifold fundamental groups, Teichmüller spaces, moduli spaces of curves with symmetry, and also for an account of the results on the regularity of classifying maps, such as harmonicity, addressed by Eells and Sampson, and their complex analyticity, addressed by Siu, which are key ingredients for the study of moduli through topological methods (see especially [ABCKT96] on this topic); and which lead to rigidity and quasi-rigidity properties of projective varieties which are classifying spaces (meaning that their moduli spaces are completely determined by their topology).

We shall focus here instead on a few concrete problems in moduli theory, in particular new constructions of surfaces with $p_g = q = 0$ or $p_g = q = 1$.

7.1. Burniat surfaces and Inoue type varieties

Among the algebraic surfaces obtained as Abelian coverings of the plane branched on interesting configurations of lines, the oldest examples were the so-called Burniat surfaces, some surfaces with $q = p_g = 0$ and $K_S^2 = 6, 5, 4, 3, 2$.

Again, I shall skip their description, especially since I already reported on them at the Kinosaki Conference in the Fall of 2011; even if there are still interesting open questions concerning the connected component of the moduli space containing Burniat surfaces with $K_S^2 = 3$.

Following a suggestion of Miles Reid, Masahisa Inoue [In94] gave another description of the Burniat surfaces, as quotients of a hypersurface in the product of three elliptic curves. Using this method, he went further with his construction, and obtained new (minimal) surfaces of general type with $q = p_g = 0$ and $K_S^2 = 7$, which are now called **(algebraic) Inoue surfaces**.

These were given as quotients of a complete intersection of two surfaces inside the product of four elliptic curves, but a closer inspection showed that indeed they are quotients of a surface inside the product of a curve of genus 5 with two elliptic curves. More precisely, an Inoue surface S admits an unramified $(\mathbb{Z}/2\mathbb{Z})^5$-Galois covering $\hat{S}$ which is an ample divisor in $E_1 \times E_2 \times D$, where E_1, E_2 are elliptic curves and D is a projective curve of genus 5.

Hence the fundamental group of an Inoue surface with $p_g = 0$ and $K_S^2 = 7$ sits in an extension (Π_g being as usual the fundamental group of a projective curve of genus g):

$$1 \longrightarrow \Pi_5 \times \mathbb{Z}^4 \longrightarrow \pi_1(S) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^5 \longrightarrow 1.$$

It turned out that the ideas needed to treat the moduli space of this special family of Inoue surfaces could be put in a rather general framework, valid in all dimensions, and together with I. Bauer we proposed the study, obtaining several results, of what we called Inoue-type varieties.

Definition 94 ([B-C12]). *Define a complex projective manifold X to be an* **Inoue-type manifold** *if*

(1) dim $(X) \geq 2$;
(2) there is a finite group G and an unramified G-covering $\hat{X} \to X$, (hence $X = \hat{X}/G$) such that
(3) $\hat{X}$ is an ample divisor inside a $K(\Gamma, 1)$-projective manifold Z, (hence by the theorem of Lefschetz $\pi_1(\hat{X}) \cong \pi_1(Z) \cong \Gamma$) and moreover
(4) the action of G on $\hat{X}$ yields a faithful action on $\pi_1(\hat{X}) \cong \Gamma$: in other words the exact sequence

$$1 \longrightarrow \Gamma \cong \pi_1(\hat{X}) \longrightarrow \pi_1(X) \longrightarrow G \longrightarrow 1$$

gives an injection $G \to \mathrm{Out}(\Gamma)$, defined by conjugation by lifts of elements of G;

(5) the action of G on $\hat{X}$ is induced by an action of G on Z.

We say that an Inoue-type manifold X is a **special Inoue type manifold** *if moreover*

$$Z = (A_1 \times \cdots \times A_r) \times (C_1 \times \cdots \times C_h) \times (M_1 \times \cdots \times M_s),$$

where each A_i is an Abelian variety, each C_j is a curve of genus $g_j \geq 2$, and M_i is a compact quotient of an irreducible bounded symmetric domain of dimension at least 2 by a torsion free subgroup; and a **classical Inoue type manifold** *if instead $Z = (A_1 \times \cdots \times A_r) \times (C_1 \times \cdots \times C_h)$ where as above each A_i is an Abelian variety, each C_j is a curve of genus $g_j \geq 2$.*

The main idea underlying the importance of this notion is the fact that, under suitable assumptions, one can say that if X is an Inoue type manifold, and Y is homotopically equivalent to X (indeed even some weaker cohomological conditions are sufficient), then also Y is an Inoue type manifold, and in special cases one can conclude that Y and X belong to the same irreducible connected component of the moduli space. We omit to state the general results, referring to [BC08] and [Cat15], and here we shall just treat a concrete case, in the next subsection.

7.2. Bagnera–de Franchis varieties and applications to moduli

In our article [BCF15b], appeared in the volume dedicated to Kodaira, we treated a special case of the theory of Inoue type varieties, the one where the action of G happens to be free also on Z, and Z is the simplest projective classifying space, an Abelian variety.

Define the **Generalized Hyperelliptic Varieties** (GHV) as the quotients A/G of an Abelian Variety A by a finite group G acting freely, and with the property that G is not a subgroup of the group of translations. Without loss of generality one can then assume that G contains no translations, since the subgroup G_T of translations in G would be a normal subgroup, and if we denote $G' = G/G_T$, then $A/G = A'/G'$, where A' is the Abelian variety $A' := A/G_T$.

A smaller class is the class of **Bagnera–de Franchis (BdF) Varieties**: these are the quotients $X = A/G$ were G contains no translations, and G is a cyclic group of order m, with generator g (observe that, when A has dimension $n = 2$, the two notions coincide, thanks to the classification result of Bagnera–de Franchis in [BdF08]).

Bagnera–de Franchis varieties have a simple description as quotients of Bagnera–de Franchis varieties of product type, according to the following definition:

Definition 95. *A Bagnera–de Franchis manifold (resp.: variety)* **of product type** *is a quotient $X = A/G$ where $A = A_1 \times A_2$, A_1, A_2 are complex tori (resp.: Abelian Varieties), and $G \cong \mathbb{Z}/m$ is a cyclic group operating freely on A, generated by an automorphism of the form*

$$g(a_1, a_2) = (a_1 + \beta_1, \alpha_2(a_2)),$$

where $\beta_1 \in A_1[m]$ is an element of order exactly m, and similarly $\alpha_2 : A_2 \to A_2$ is a linear automorphism of order exactly m without 1 as eigenvalue (these conditions guarantee that the action is free).

This is then the characterization of general Bagnera–de Franchis varieties.

Proposition 96. *Every Bagnera–de Franchis variety $X = A/G$, where $G \cong \mathbb{Z}/m$ contains no translations, is the quotient of a Bagnera–de Franchis variety of product type, $(A_1 \times A_2)/G$ by any finite subgroup T of $A_1 \times A_2$ which satisfies the following properties:*

1) T is the graph of an isomorphism between two respective subgroups $T_1 \subset A_1, T_2 \subset A_2$,
2) $(\alpha_2 - Id)T_2 = 0$,
3) if $g(a_1, a_2) = (a_1 + \beta_1, \alpha_2(a_2))$, then the subgroup of order m generated by β_1 intersects T_1 only in $\{0\}$.

In particular, we may write X as the quotient $X = (A_1 \times A_2)/(G \times T)$ by the abelian group $G \times T$.

This notion was then used in [BCF15a] to make a construction that we briefly describe.

Let A_1 be an elliptic curve, and let A_2 be an Abelian surface with a line bundle L_2 yielding a polarization of type $(1, 2)$. Take as L_1 the line bundle $\mathcal{O}_{A_1}(2O)$, and let L be the line bundle on $A' := A_1 \times A_2$ obtained as the exterior tensor product of L_1 and L_2, so that

$$H^0(A', L) = H^0(A_1, L_1) \otimes H^0(A_2, L_2).$$

Moreover, choose the origin in A_2 so that the space of sections $H^0(A_2, L_2)$ consists only of *even* sections.

We take then, using properties of the Stone–von Neumann representation of the Heisenberg group, a Bagnera–de Franchis threefold $X := A/G$, where $A = (A_1 \times A_2)/T$, and $G \cong T \cong \mathbb{Z}/2$, and a surface $S \subset X$ which is the quotient of a $(G \times T)$-invariant $D \in |L|$, so that $S^2 = \frac{1}{4}D^2 = 6$.

We could then prove the following.

Theorem 97. *Let S be a surface of general type with invariants $K_S^2 = 6$, $p_g = q = 1$ such that there exists an unramified double cover $\hat{S} \to S$ with $q(\hat{S}) = 3$,* *and such that the Albanese morphism* $\hat{\alpha} : \hat{S} \to A$ *is birational onto its image* Z*, a divisor in* A *with* $Z^3 = 12$.

Then the canonical model of $\hat{S}$ *is isomorphic to* Z*, and the canonical model of* S *is isomorphic to* $Y = Z/(\mathbb{Z}/2)$*, which a divisor in a Bagnera–de Franchis threefold* $X := A/G$*, where* $A = (A_1 \times A_2)/T$*,* $G \cong T \cong \mathbb{Z}/2$*, and where the action is given by*

(98)
$$G := \{Id, g\}, \quad g(a_1 + a_2) := a_1 + \tau/2 - a_2 + \lambda_2/2, \quad \forall a_1 \in A_1, a_2 \in A_2,$$

(99)
$$T := (\mathbb{Z}/2)(1/2 + \lambda_4/2) \subset A = (A_1 \times A_2).$$

These surfaces exist, have an irreducible four dimensional moduli space, and their Albanese map $\alpha : S \to A_1 = A_1/A_1[2]$ *has general fibre a non-hyperelliptic curve of genus* $g = 3$.

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Recent results on the Kobayashi and Green–Griffiths–Lang conjectures*

Jean-Pierre Demailly**

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Contribution to the 16^{th} *Takagi Lectures*
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Abstract. The study of entire holomorphic curves contained in projective algebraic varieties is intimately related to fascinating questions of geometry and number theory—especially through the concepts of curvature and positivity which are central themes in Kodaira's contributions to mathematics. The aim of these lectures is to present recent results concerning the geometric side of the problem. The Green–Griffiths–Lang conjecture stipulates that for every projective variety X of general type over $\mathbb{C}$, there exists a proper algebraic subvariety Y of X containing all entire curves $f : \mathbb{C} \to X$. Using the formalism of directed varieties and jet bundles, we show that this assertion holds true in case X satisfies a strong general type condition that is related to a certain jet-semi-stability property of the tangent bundle T_X. It is possible to exploit similar techniques to investigate a famous conjecture of Shoshichi Kobayashi (1970), according to which a generic algebraic hypersurface of dimension n and of sufficiently large degree $d \geqslant d_n$ in the complex projective space $\mathbb{P}^{n+1}$ is hyperbolic: in the early 2000's, Yum-Tong Siu proposed a strategy that led in 2015 to a proof based on a clever use of slanted vector fields on jet spaces, combined with Nevanlinna theory arguments. In 2016, the conjecture has been settled in a different way by Damian Brotbek, making a more direct use of Wronskian differential operators and associated multiplier ideals; shortly afterwards, Ya Deng showed how the proof could be modified to yield

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J.-P. DEMAILLY
Université de Grenoble-Alpes, Institut Fourier (Mathématiques)
UMR 5582 du C.N.R.S., 100 rue des Maths, 38610 Gières, France
(e-mail: jean-pierre.demailly@univ-grenoble-alpes.fr)

an explicit value of d_n. We give here a short proof based on a substantial simplification of their ideas, producing a bound very similar to Deng's original estimate, namely $d_n = \lfloor \frac{1}{3}(en)^{2n+2} \rfloor$.

Keywords and phrases: Kobayashi hyperbolic variety, directed manifold, genus of a curve, jet bundle, jet differential, jet metric, Chern connection and curvature, negativity of jet curvature, variety of general type, Kobayashi conjecture, Green–Griffiths conjecture, Lang conjecture

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Contents

0. Introduction

The goal of these lectures is to study the conjecture of Kobayashi [Kob70] on the hyperbolicity of generic hypersurfaces of high degree in projective space, and the related conjecture by Green–Griffiths [GrGr80] and Lang [Lang86] on the structure of entire curve loci.

Let us recall that a complex space X is said to be hyperbolic in the sense of Kobayashi if analytic disks $f : \mathbb{D} \to X$ through a given point form a normal family. By a well-known result of Brody [Bro78], a compact complex space is Kobayashi hyperbolic if and only if it does not contain any entire holomorphic curve $f : \mathbb{C} \to X$ ("Brody hyperbolicity").

In this paper entire holomorphic curves are assumed to be non-constant and simply called entire curves. If X is not hyperbolic, a basic question is thus to analyze the geometry of entire holomorphic curves $f : \mathbb{C} \to X$, and especially to understand the *entire curve locus of* X, defined as the Zariski closure

$$\mathrm{ECL}(X) = \overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}. \tag{0.1}$$

The Green–Griffiths–Lang conjecture, in its strong form, can be stated as follows.

0.2. GGL conjecture. Let X be a projective variety of general type. Then $Y = \mathrm{ECL}(X)$ is a proper algebraic subvariety $Y \subsetneq X$.

Equivalently, there exists $Y \subsetneq X$ such that every entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$. A weaker form of the GGL conjecture states that entire curves are algebraically degenerate, i.e., that $f(\mathbb{C}) \subset Y_f \subsetneq X$, where Y_f may depend on f.

If $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined over a number field $\mathbb{K}_0$ (i.e., by polynomial equations with coefficients in $\mathbb{K}_0$), one defines the Mordell locus, denoted $\mathrm{Mordell}(X)$, to be the smallest complex subvariety Y in X such that the set of $\mathbb{K}$-points $X(\mathbb{K}) \smallsetminus Y$ is finite for every number field $\mathbb{K} \supset \mathbb{K}_0$. Lang [Lang86] conjectured that one should always have $\mathrm{Mordell}(X) = \mathrm{ECL}(X)$ in this situation. This conjectural arithmetical statement would be a vast generalization of the Mordell–Faltings theorem, and is one of the strong motivations to study the geometric GGL conjecture as a first step. S. Kobayashi [Kob70] had earlier made the following tantalizing conjecture.

0.3. Conjecture (Kobayashi).

(a) A (very) generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ large enough is hyperbolic, especially it does not possess any entire holomorphic curve $f : \mathbb{C} \to X$.

(b) The complement $\mathbb{P}^n \smallsetminus H$ of a (very) generic hypersurface $H \subset \mathbb{P}^n$ of degree $d \geqslant d'_n$ large enough is hyperbolic.

It should be noticed that the existence of a smooth hyperbolic hypersurface $X \subset \mathbb{P}^{n+1}$ in 0.3 (a), or a hyperbolic complement $\mathbb{P}^n \smallsetminus H$ with H smooth irreducible in 0.3 (b), is already a hard problem; many efforts were initially concentrated on this problem. As Zaidenberg observed, a smooth deformation of a union of $(2n+1)$ hyperplanes in $\mathbb{P}^n$ is not necessarily Kobayashi hyperbolic, and the issue is non-trivial at all. The existence problem was initially solved for sufficiently high degree hypersurfaces through a number of examples:

- case (a) for $n = 2$ and degree $d \geqslant 50$ by Brody and Green [BrGr77];
- case (b) for $n = 2$ by [AzSu80] (as a consequence of [BrGr77]);
- cases (a) and (b) for $n \geqslant 3$ by Masuda and Noguchi [MaNo96].

Improvements in the degree estimates were later obtained in [Shi98], [Fuj01], [ShZa02], in addition to many other papers dealing with low dimensional varieties ($n = 2, 3$).

We now describe a number of known results concerning the question of generic hyperbolicity, according to the Kobayashi conjectures 0.3 (a), (b). M. Zaidenberg observed in [Zai87] that the complement of a general hypersurface of degree $2n$ in $\mathbb{P}^n$ is not hyperbolic; as a consequence, one must take

$d'_n \geqslant 2n + 1$ in 0.3 (b). This observation, along with Fujimoto's classical result that the complement of $(2n + 1)$ hyperplanes of $\mathbb{P}^n$ in general position is hyperbolic and hyperbolically embedded in $\mathbb{P}^n$ ([Fuj72]) led Zaidenberg to propose the bound $d'_n = 2n + 1$ for $n \geqslant 1$. Another famous result due to Clemens [Cle86], Ein [Ein88], [Ein91] and Voisin [Voi96] states that every subvariety Y of a generic algebraic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2n + 1$ is of general type for $n \geqslant 2$ (for surfaces $X \subset \mathbb{P}^3$, Geng Xu [Xu94] also obtained some refined information for the genera of algebraic curves drawn in X). The bound was subsequently improved to $d \geqslant d_n = 2n$ for $n \geqslant 5$ by Pacienza [Pac04]. That the same bound d_n holds for Kobayashi hyperbolicity would then be a consequence of the Green–Griffiths–Lang conjecture. By these results, one can hope in the compact case that the optimal bound d_n is $d_1 = 4$, $d_n = (2n + 1)$ for $n = 2, 3, 4$ and $d_n = 2n$ for $n \geqslant 5$. The case of complements $\mathbb{P}^n \setminus H$ (the so-called "logarithmic case") is a priori somewhat easier to deal with: in fact, on can then exploit the fact that the hyperbolicity of the hypersurface $X = \{w^d = P(z)\} \subset \mathbb{P}^{n+1}$ implies the hyperbolicity of the complement $\mathbb{P}^n \setminus H$, when $H = \{P(z) = 0\}$. Pacienza and Rousseau [PaRo07] proved that for H very general of degree $d \geqslant 2n+2-k$, any k-dimensional log-subvariety (Y, D) of $(\mathbb{P}^n, H)$ is of log-general type, i.e., any log-resolution $\mu : \widetilde{Y} \to Y$ of (Y, D) has a big log-canonical bundle $K_{\widetilde{Y}}(\mu^* D)$.

One of the early important result in the direction of Conjecture 0.2 is the proof of the Bloch conjecture, as proposed by Bloch [Blo26a] and Ochiai [Och77]: this is the special case of the conjecture when the irregularity of X satisfies $q = h^0(X, \Omega^1_X) > \dim X$. Various solutions have then been obtained in fundamental papers of Noguchi [Nog77a], [Nog81a], [Nog81b], Kawamata [Kaw80], Green–Griffiths [GrGr80], McQuillan [McQ96], and the book of Noguchi–Winkelmann [NoWi13], by means of different techniques. Especially, assuming X to be of (log-) general type, it is now known by [NWY07], [NWY13] and [LuWi12] that if the (log-) irregularity is $q \geqslant \dim X$, then no entire curve $f : \mathbb{C} \to X$ has a Zariski dense image, and the GGL conjecture holds in the compact (i.e., non-logarithmic) case. In the case of complex surfaces, major progress was achieved by Lu, Miyaoka and Yau [LuYa90], [LuMi95], [LuMi96], [Lu96]; McQuillan [McQ96] extended these results to the case of all surfaces satisfying $c_1^2 > c_2$, in a situation where there are many symmetric differentials, e.g. sections of $H^0(X, S^m T^*_X \otimes \mathcal{O}(-1))$, $m \gg 1$ (*cf.* also [McQ99], [DeEG00] for applications to hyperbolicity). A more recent result is the deep statement due to Diverio, Merker and Rousseau [DMR10], confirming Conjecture 0.2 when $X \subset \mathbb{P}^{n+1}$ is a generic non-singular hypersurface of sufficiently large degree $d \geqslant 2^{n^5}$ (*cf.* Sect. 10); in the case $n = 2$ of surfaces in $\mathbb{P}^3$, we are here in the more difficult situation where symmetric differentials do not exist (we have $c_1^2 < c_2$ in this case). Conjecture 0.2 was also considered by S. Lang

[Lang86], [Lang87] in view of arithmetic counterparts of the above geometric statements.

Although these optimal conjectures are still unsolved at present, substantial progress was achieved in the meantime, for a large part via the technique of producing jet differentials. This is done either by direct calculations or by various indirect methods: Riemann–Roch calculations, vanishing theorems... Vojta [Voj87] and McQuillan [McQ98] introduced the "diophantine approximation" method, which was soon recognized to be an important tool in the study of holomorphic foliations, in parallel with Nevanlinna theory and the construction of Ahlfors currents. Around 2000, Siu [Siu02], [Siu04] showed that generic hyperbolicity results in the direction of the Kobayashi conjecture could be investigated by combining the algebraic techniques of Clemens, Ein and Voisin with the existence of certain "vertical" meromorphic vector fields on the jet space of the universal hypersurface of high degree; these vector fields are actually used to differentiate the global sections of the jet bundles involved, so as to produce new sections with a better control on the base locus. Also, during the years 2007–2010, it was realized [Dem07a], [Dem07b], [Dem11] that holomorphic Morse inequalities could be used to prove the existence of jet differentials; in 2010, Diverio, Merker and Rousseau [DMR10] were able in that way to prove the Green–Griffiths conjecture for generic hypersurfaces of high degree in projective space, e.g. for $d \geqslant 2^{n^5}$—their proof makes an essential use of Siu's differentiation technique via meromorphic vector fields, as improved by Păun [Pau08] and Merker [Mer09] in 2008. The present study will be focused on the holomorphic Morse inequality technique; as an application, a partial answer to the Kobayashi and Green–Griffiths–Lang conjecture can be obtained in a very wide context: the basic general result achieved in [Dem11] consists of showing that for every projective variety of general type X, there exists a global algebraic differential operator P on X (in fact many such operators P_j) such that every entire curve $f : \mathbb{C} \to X$ must satisfy the differential equations $P_j(f; f', \ldots, f^{(k)}) = 0$. One also recovers from there the result of Diverio–Merker–Rousseau on the generic Green–Griffiths conjecture (with an even better bound asymptotically as the dimension tends to infinity), as well as a result of Diverio–Trapani [DT10] on the hyperbolicity of generic 3-dimensional hypersurfaces in $\mathbb{P}^4$. Siu [Siu04], [Siu15] has introduced a more explicit but more computationally involved approach based on the use of "slanted vector fields" on jet spaces, extending ideas of Clemens [Cle86] and Voisin [Voi96] (*cf.* Sect. 10 for details); [Siu15] explains how this strategy can be used to assert the Kobayashi conjecture for $d \geqslant d_n$, with a very large bound and non-effective bound d_n instead of $(2n + 1)$.

As we will see here, it is useful to work in a more general context and to consider the category of directed varieties. When the problems under consideration are birationally invariant, as is the case of the Green–Griffiths–Lang conjecture, varieties can be replaced by non-singular models; for this reason, we will mostly restrict ourselves to the case of non-singular varieties in the rest of the introduction. A *directed projective manifold* is a pair (X, V), where X is a projective manifold equipped with an analytic linear subspace $V \subset T_X$, i.e., a closed irreducible complex analytic subset V of the total space of T_X, such that each fiber $V_x = V \cap T_{X,x}$ is a complex vector space. If X is not connected, V should rather be assumed to be irreducible merely over each connected component of X, but we will hereafter assume that our manifolds are connected. A morphism $\Phi : (X, V) \to (Y, W)$ in the category of directed manifolds is an analytic map $\Phi : X \to Y$ such that $\Phi_* V \subset W$. We refer to the case $V = T_X$ as being the *absolute case*, and to the case $V = T_{X/S} = \operatorname{Ker} d\pi$ for a fibration $\pi : X \to S$, as being the *relative case*; V may also be taken to be the tangent space to the leaves of a singular analytic foliation on X, or maybe even a non-integrable linear subspace of T_X. We are especially interested in *entire curves* that are tangent to V, namely non-constant holomorphic morphisms $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ of directed manifolds. In the absolute case, these are just arbitrary entire curves $f : \mathbb{C} \to X$.

0.4. Generalized GGL conjecture. Let (X, V) be a projective directed manifold. Define the entire curve locus of (X, V) to be the Zariski closure of the locus of entire curves tangent to V, i.e.,

$$\mathrm{ECL}(X, V) = \overline{\bigcup_{f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)} f(\mathbb{C})}^{\mathrm{Zar}}.$$

Then, if (X, V) is of general type in the sense that the canonical sheaf sequence $K_V^\bullet$ is big (*cf.* Proposition 2.11 below), $Y = \mathrm{ECL}(X, V)$ is a proper algebraic subvariety $Y \subsetneq X$.

[We will say that (X, V) is *Brody hyperbolic* if $\mathrm{ECL}(X, V) = \emptyset$; by Brody's reparametrization technique, this is equivalent to Kobayashi hyperbolicity whenever X is compact.]

In case V has no singularities, the *canonical sheaf* K_V is defined to be $(\det(\mathscr{O}(V)))^*$, where $\mathscr{O}(V)$ is the sheaf of holomorphic sections of V, but in general this naive definition would not work. Take for instance a generic pencil of elliptic curves $\lambda P(z) + \mu Q(z) = 0$ of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$, and the linear space V consisting of the tangents to the fibers of the rational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ defined by $z \mapsto Q(z)/P(z)$. Then V is given by

$$0 \longrightarrow \mathscr{O}(V) \longrightarrow \mathscr{O}(T_{\mathbb{P}^2_{\mathbb{C}}}) \xrightarrow{P\,dQ - Q\,dP} \mathscr{O}_{\mathbb{P}^2_{\mathbb{C}}}(6) \otimes \mathscr{J}_S \longrightarrow 0,$$

where $S = \mathrm{Sing}(V)$ consists of the 9 points $\{P(z) = 0\} \cap \{Q(z) = 0\}$, and $\mathscr{J}_S$ is the corresponding ideal sheaf of S. Since $\det(\mathscr{O}(T_{\mathbb{P}^2})) = \mathscr{O}(3)$, we see that

$(\det(\mathcal{O}(V)))^* = \mathcal{O}(3)$ is ample, thus generalized GGL conjecture 0.4 would not have a positive answer (all leaves are elliptic or singular rational curves and thus covered by entire curves). An even more "degenerate" example is obtained with a generic pencil of conics, in which case $(\det(\mathcal{O}(V)))^* = \mathcal{O}(1)$ and $\#S = 4$.

If we want to get a positive answer to Problem 0.4, it is therefore indispensable to give a definition of K_V that incorporates in a suitable way the singularities of V; this will be done in Definition 2.12 (see also Proposition 2.11). The goal is then to give a positive answer to Problem 0.4 under some possibly more restrictive conditions for the pair (X, V). These conditions will be expressed in terms of the tower of Semple jet bundles

(0.5)

$$(X_k, V_k) \longrightarrow (X_{k-1}, V_{k-1}) \longrightarrow \cdots \longrightarrow (X_1, V_1) \longrightarrow (X_0, V_0) := (X, V)$$

which we define more precisely in Sect. 1, following [Dem95]. It is constructed inductively by setting $X_k = P(V_{k-1})$ (projective bundle of *lines* of V_{k-1}), and all V_k have the same rank $r = \operatorname{rank} V$, so that $\dim X_k = n + k(r-1)$, where $n = \dim X$. Entire curve loci have their counterparts for all stages of the Semple tower, namely, one can define

$$\text{(0.6)} \qquad \mathrm{ECL}_k(X, V) = \overline{\bigcup_{f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)} f_{[k]}(\mathbb{C})}^{\mathrm{Zar}},$$

where $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the k-jet of f. These are by definition algebraic subvarieties of X_k, and if we denote by $\pi_{k,\ell} : X_k \to X_\ell$ the natural projection from X_k to X_ℓ, $0 \leqslant \ell \leqslant k$, we get immediately

$$\text{(0.7)} \qquad \pi_{k,\ell}(\mathrm{ECL}_k(X, V)) = \mathrm{ECL}_\ell(X, V), \quad \mathrm{ECL}_0(X, V) = \mathrm{ECL}(X, V).$$

Let $\mathcal{O}_{X_k}(1)$ be the tautological line bundle over X_k associated with the projective structure. We define the k-stage Green–Griffiths locus of (X, V) to be

$$\text{(0.8)} \quad \mathrm{GG}_k(X, V) = \overline{(X_k \smallsetminus \Delta_k) \cap \bigcap_{m\in\mathbb{N}} (\text{base locus of } \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* A^{-1})},$$

where A is any ample line bundle on X and $\Delta_k = \bigcup_{2\leqslant\ell\leqslant k} \pi_{k,\ell}^{-1}(D_\ell)$ is the union of "vertical divisors" (see (6.9) and (7.17); the vertical divisors play no role and have to be removed in this context; for this, one uses the fact that $f_{[k]}(\mathbb{C})$ is not contained in any component of Δ_k, *cf.* [Dem95]). Clearly, $\mathrm{GG}_k(X, V)$ does not depend on the choice of A.

0.9. Basic vanishing theorem for entire curves. *Let (X, V) be an arbitrary directed variety with X non-singular, and let A be an ample line bundle on X. Then any entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfies the differential equations*

$P(f; f', \ldots, f^{(k)}) = 0$ *arising from sections* $\sigma \in H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* A^{-1})$. *As a consequence, one has*

$$\mathrm{ECL}_k(X, V) \subset \mathrm{GG}_k(X, V).$$

The main argument goes back to [GrGr80]. We will give here a complete proof of Theorem 0.9, based only on the arguments [Dem95], namely on the Ahlfors–Schwarz lemma (the alternative proof given in [SiYe96b] uses Nevanlinna theory and is analytically more involved). By (0.7) and Theorem 0.9 we infer that

$$\mathrm{ECL}(X, V) \subset \mathrm{GG}(X, V), \tag{0.10}$$

where $\mathrm{GG}(X, V)$ is the global Green–Griffiths locus of (X, V) defined by

$$\mathrm{GG}(X, V) = \bigcap_{k \in \mathbb{N}} \pi_{k,0}\left(\mathrm{GG}_k(X, V)\right). \tag{0.11}$$

The main result of [Dem11] (Theorem 2.37 and Corollary 3.4) implies the following useful information:

0.12. Theorem. *Assume that* (X, V) *is of "general type", i.e., that the pluricanonical sheaf sequence* $K_V^\bullet$ *is big on* X*. Then there exists an integer* k_0 *such that* $\mathrm{GG}_k(X, V)$ *is a proper algebraic subset of* X_k *for* $k \geqslant k_0$ *[though* $\pi_{k,0}(\mathrm{GG}_k(X, V))$ *might still be equal to* X *for all* k*].*

In fact, if F is an invertible sheaf on X such that $K_V^\bullet \otimes F$ is big (*cf.* Proposition 2.11), the probabilistic estimates of [Dem11, Corollary 2.38 and Corollary 3.4] produce global sections of

$$\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big) F\Big) \tag{0.13}$$

for $m \gg k \gg 1$. The (long and elaborate) proof uses a curvature computation and singular holomorphic Morse inequalities to show that the line bundles involved in (0.11) are big on X_k for $k \gg 1$. One applies this to $F = A^{-1}$ with A ample on X to produce sections and conclude that $\mathrm{GG}_k(X, V) \subsetneq X_k$.

Thanks to (0.10), the GGL conjecture is satisfied whenever $\mathrm{GG}(X, V) \subsetneq X$. By [DMR10], this happens for instance in the absolute case when X is a generic hypersurface of degree $d \geqslant 2^{n^5}$ in $\mathbb{P}^{n+1}$ (see also [Pau08] for better bounds in low dimensions, and [Siu02], [Siu04]). However, as already mentioned in [Lang86], very simple examples show that one can have $\mathrm{GG}(X, V) = X$ even when (X, V) is of general type, and this already occurs in the absolute case as soon as $\dim X \geqslant 2$. A typical example is a product of directed manifolds

$$(X, V) = (X', V') \times (X'', V''), \qquad V = \mathrm{pr}'^* V' \oplus \mathrm{pr}''^* V''. \tag{0.14}$$

The absolute case $V = T_X$, $V' = T_{X'}$, $V'' = T_{X''}$ on a product of curves is the simplest instance. It is then easy to check that $\mathrm{GG}(X, V) = X$, *cf.* Definition 3.2. Diverio and Rousseau [DR15] have given many more such examples, including the case of indecomposable varieties (X, T_X), e.g. Hilbert modular surfaces, or more generally compact quotients of bounded symmetric domains of rank $\geqslant 2$.

The problem here is the failure of some sort of stability condition that is introduced in Remark 11.10. This leads us to make the assumption that the directed pair (X, V) is *strongly of general type*: by this, we mean that the induced directed structure (Z, W) on each non-vertical subvariety $Z \subset X_k$ that projects onto X either has rank $W = 0$ or is of general type modulo $X_\bullet \to X$, in the sense that $K^\bullet_{W_\ell} \otimes \mathscr{O}_{Z_\ell}(p)_{\restriction Z_\ell}$ is big for some stage of the Semple tower of (Z, W) and some $p \geqslant 0$ (see Sect. 11 for details—one may have to replace Z_ℓ by a suitable modification). Our main result can be stated as follows:

0.15. Theorem (partial solution to the generalized GGL conjecture). *Let (X, V) be a directed pair that is strongly of general type. Then the Green–Griffiths–Lang conjecture holds true for (X, V), namely $\mathrm{ECL}(X, V)$ is a proper algebraic subvariety of X.*

The proof proceeds through a complicated induction on $n = \dim X$ and $k = \operatorname{rank} V$, which is the main reason why we have to introduce directed varieties, even in the absolute case. An interesting feature of this result is that the conclusion on $\mathrm{ECL}(X, V)$ is reached without having to know anything about the Green–Griffiths locus $\mathrm{GG}(X, V)$, even a posteriori. Nevertheless, this is not yet enough to confirm the GGL conjecture. Our hope is that pairs (X, V) that are of general type without being strongly of general type—and thus exhibit some sort of “jet-instability”—can be investigated by different methods, e.g. by the diophantine approximation techniques of McQuillan [McQ98]. However, Theorem 0.15 provides a sufficient criterion for Kobayashi hyperbolicity [Kob70], thanks to the following concept of algebraic jet-hyperbolicity.

0.16. Definition. *A directed variety (X, V) will be said to be algebraically jet-hyperbolic if the induced directed variety structure (Z, W) on every non-vertical irreducible algebraic variety Z of X_k with $\operatorname{rank} W \geqslant 1$ is such that $K^\bullet_{W_\ell} \otimes \mathscr{O}_{Z_\ell}(p)_{\restriction Z_\ell}$ is big for some stage of the Semple tower of (Z, W) and some $p \geqslant 0$ [possibly after taking a suitable modification of Z_ℓ; see Sects. 11 and 12 for the definition of induced directed structures and further details]. We also say that a projective manifold X is algebraically jet-hyperbolic if (X, T_X) is.*

In this context, Theorem 0.15 yields the following connection between algebraic jet-hyperbolicity and the analytic concept of Kobayashi hyperbolicity.

0.17. Theorem. *Let (X, V) be a directed variety structure on a projective manifold X. Assume that (X, V) is algebraically jet-hyperbolic. Then (X, V) is Kobayashi hyperbolic.*

The following conjecture would then make a bridge between these theorems and the GGL and Kobayashi conjectures.

0.18. Conjecture. Let $X \subset \mathbb{P}^{n+c}$ be a complete intersection of hypersurfaces of respective degrees $d_1, \ldots, d_c$, codim $X = c$.

(a) If X is non-singular and of general type, i.e., if $\sum d_j \geqslant n + c + 2$, then X is in fact strongly of general type.
(b) If X is (very) generic and $\sum d_j \geqslant 2n + c$, then X is algebraically jet-hyperbolic.

Since Conjecture 0.18 only deals with algebraic statements, our hope is that a proof can be obtained through a suitable deepening of the techniques introduced by Clemens, Ein, Voisin and Siu. Under the slightly stronger condition $\sum d_j \geqslant 2n+c+1$, Voisin showed indeed that every subvariety $Y \subset X$ is of general type, if X is generic. To prove the Kobayashi conjecture in its optimal incarnation, we would need to show that such Y's are strongly of general type.

In the direction of getting examples of low degrees, Dinh Tuan Huynh [DTH16a] showed that there are families of hyperbolic hypersurfaces of degree $(2n + 2)$ in $\mathbb{P}^{n+1}$ for $2 \leqslant n \leqslant 5$, and in [DTH16b] he showed that certain small deformations (in Euclidean topology) of a union of $\lceil (n + 3)^2/4 \rceil$ hyperplanes in general position in $\mathbb{P}^{n+1}$ are hyperbolic. In [Ber18], G. Bérczi stated a positivity conjecture for Thom polynomials of Morin singularities (see also [BeSz12]), and announced that it would imply a polynomial bound $d_n = 2n^9 + 1$ for the generic hyperbolicity of hypersurfaces. By using the "technology" of Semple towers and following new ideas introduced by D. Brotbek [Brot17] and Ya Deng [Deng16], we prove here the following effective (although non-optimal) version of the Kobayashi conjecture on generic hyperbolicity.

0.19. Theorem. *Let Z be a projective $(n + 1)$-dimensional manifold and A a very ample line bundle on Z. Then, for a general section $\sigma \in H^0(Z, A^d)$ and $d \geqslant d_n$, the hypersurface $X_\sigma = \sigma^{-1}(0)$ is Kobayashi hyperbolic and, in fact, satisfies the stronger property of being algebraically jet hyperbolic. The bound d_n for the degree can be taken to be $d_n := \lfloor \frac{1}{3}(en)^{2n+2} \rfloor$.*

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1. Basic hyperbolicity concepts

1.A. Kobayashi hyperbolicity

We first recall a few basic facts concerning the concept of hyperbolicity, according to S. Kobayashi [Kob70], [Kob76]. Let X be a complex space. Given two points $p, q \in X$, let us consider a *chain of analytic disks* from p to q, that is a sequence of holomorphic maps $f_0, f_1, \ldots, f_k : \Delta \to X$ from the unit disk $\Delta = D(0,1) \subset \mathbb{C}$ to X, together with pairs of points $a_0, b_0, \ldots, a_k, b_k$ of Δ such that

$$p = f_0(a_0), \quad q = f_k(b_k), \quad f_i(b_i) = f_{i+1}(a_{i+1}), \quad i = 0, \ldots, k-1.$$

Denoting this chain by α, we define its length $\ell(\alpha)$ to be

$$\ell(\alpha) = d_P(a_1, b_1) + \cdots + d_P(a_k, b_k), \tag{1.1$'$}$$

where d_P is the Poincaré distance on Δ, and the *Kobayashi pseudodistance* d_X^K on X to be

$$d_X^K(p,q) = \inf_\alpha \ell(\alpha). \tag{1.1$''$}$$

A *Finsler metric* (resp. *pseudometric*) on a vector bundle E is a homogeneous positive (resp. non-negative) function N on the total space E, that is,

$$N(\lambda\xi) = |\lambda|\, N(\xi) \quad \text{for all } \lambda \in \mathbb{C} \text{ and } \xi \in E,$$

but in general N is not assumed to be subadditive (i.e., convex) on the fibers of E. A Finsler (pseudo-)metric on E is thus nothing but a Hermitian (semi-)norm on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $Y = P(E)$. The *Kobayashi–Royden infinitesimal pseudometric* on X is the Finsler pseudometric on the tangent bundle T_X defined by

$$\begin{aligned} &\mathbf{k}_X(\xi) = \inf\{\lambda > 0\,;\, \exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi\}, \\ &x \in X,\ \xi \in T_{X,x}. \end{aligned} \tag{1.2}$$

Here, if X is not smooth at x, we take $T_{X,x} = (\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)^*$ to be the Zariski tangent space, i.e., the tangent space of a minimal smooth ambient vector space containing the germ (X, x); all tangent vectors may not be reached by analytic disks and in those cases we put $\mathbf{k}_X(\xi) = +\infty$. When X is a smooth manifold, it follows from the work of H.L. Royden ([Roy71], [Roy74]) that $\mathbf{k}_X$ is upper-continuous on T_X and that d_X^K is the integrated pseudodistance associated with the pseudometric, i.e.,

$$d_X^K(p,q) = \inf_\gamma \int_\gamma \mathbf{k}_X(\gamma'(t))\, dt,$$

where the infimum is taken over all piecewise smooth curves joining p to q ; in the case of complex spaces, a similar formula holds, involving jets of analytic curves of arbitrary order, *cf.* S. Venturini [Ven96]. When X is a non-singular projective variety, it has been shown in [DeLS94] that the Kobayashi pseudodistance and the Kobayashi–Royden infinitesimal pseudometric can be computed by looking only at analytic disks that are contained in algebraic curves.

1.3. Definition. *A complex space X is said to be* hyperbolic *(in the sense of Kobayashi) if d_X^K is actually a distance, namely if $d_X^K(p,q) > 0$ for all pairs of distinct points (p,q) in X.*

When X is hyperbolic, it is interesting to investigate when the Kobayashi metric is complete: one then says that X is a *complete hyperbolic* space. However, we will be mostly concerned with compact spaces here, so completeness is irrelevant in that case.

Another important property is the *monotonicity* of the Kobayashi pseudometric with respect to holomorphic mappings. In fact, if $\Phi : X \to Y$ is a holomorphic map, it is easy to see from the definition that

$$d_Y^K(\Phi(p), \Phi(q)) \leqslant d_X^K(p,q), \quad \text{for all } p, q \in X. \tag{1.4}$$

The proof merely consists of taking the composition $\Phi \circ f_i$ for all chains of analytic disks connecting p and q in X. Clearly the Kobayashi pseudodistance $d_{\mathbb{C}}^K$ on $X = \mathbb{C}$ is identically zero, as one can see by looking at arbitrarily large analytic disks $\Delta \to \mathbb{C}$, $t \mapsto \lambda t$. Therefore, if there is an *entire curve* $\Phi : \mathbb{C} \to X$, namely a non-constant holomorphic map defined on the whole complex plane $\mathbb{C}$, then by monotonicity d_X^K is identically zero on the image $\Phi(\mathbb{C})$ of the curve, and therefore X cannot be hyperbolic. When X is hyperbolic, it follows that X cannot contain rational curves $C \simeq \mathbb{P}^1$, or elliptic curves $\mathbb{C}/\Lambda$, or more generally any non-trivial image $\Phi : W = \mathbb{C}^p/\Lambda \to X$ of a p-dimensional complex torus (quotient of $\mathbb{C}^p$ by a lattice). The only case where hyperbolicity is easy to assess is the case of curves ($\dim_{\mathbb{C}} X = 1$).

1.5. Case of complex curves. *Up to bihomorphism, any smooth complex curve X belongs to one (and only one) of the following three types:*

(a) *(rational curve) $X \simeq \mathbb{P}^1$;*
(b) *(parabolic type) $\widehat{X} \simeq \mathbb{C}$, $X \simeq \mathbb{C}$, $\mathbb{C}^*$ or $X \simeq \mathbb{C}/\Lambda$ (elliptic curve);*
(c) *(hyperbolic type) $\widehat{X} \simeq \Delta$. All compact curves X of genus $g \geqslant 2$ enter in this category, as well as $X = \mathbb{P}^1 \smallsetminus \{a,b,c\} \simeq \mathbb{C} \smallsetminus \{0,1\}$, or $X = \mathbb{C}/\Lambda \smallsetminus \{a\}$ (elliptic curve minus one point).*

In fact, as the disk is simply connected, every holomorphic map $f : \Delta \to X$ lifts to the universal cover $\widehat{f} : \Delta \to \widehat{X}$, so that $f = \rho \circ \widehat{f}$, where $\rho : \widehat{X} \to X$ is the projection map, and the conclusions (a), (b), (c) follow easily from the

Poincaré–Koebe uniformization theorem: every simply connected Riemann surface is biholomorphic to $\mathbb{C}$, the unit disk Δ or the complex projective line $\mathbb{P}^1$.

In some rare cases, the one-dimensional case can be used to study the case of higher dimensions. For instance, it is easy to see by looking at projections that the Kobayashi pseudodistance on a product $X \times Y$ of complex spaces is given by

$$d^K_{X\times Y}((x,y),(x',y')) = \max(d^K_X(x,x'), d^K_Y(y,y')), \tag{1.6}$$

$$\mathbf{k}_{X\times Y}(\xi,\xi') = \max(\mathbf{k}_X(\xi), \mathbf{k}_Y(\xi')), \tag{1.6$'$}$$

and from there it follows that a product of hyperbolic spaces is hyperbolic. As a consequence $(\mathbb{C} \smallsetminus \{0,1\})^2$, which is also a complement of five lines in $\mathbb{P}^2$, is hyperbolic.

1.B. Brody criterion for hyperbolicity

Throughout this subsection, we assume that X is a complex manifold. In this context, we have the following well-known result of Brody [Bro78]. Its main interest is to relate hyperbolicity to the non-existence of entire curves.

1.7. Brody reparametrization lemma. *Let ω be a Hermitian metric on X and let $f : \Delta \to X$ be a holomorphic map. For every $\varepsilon > 0$, there exists a radius $R \geqslant (1-\varepsilon)\|f'(0)\|_\omega$ and a homographic transformation ψ of the disk $D(0,R)$ onto $(1-\varepsilon)\Delta$ such that*

$$\|(f\circ\psi)'(0)\|_\omega = 1, \quad \|(f\circ\psi)'(t)\|_\omega \leqslant \frac{1}{1-|t|^2/R^2} \quad \textit{for every } t \in D(0,R).$$

Proof. Select $t_0 \in \Delta$ such that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ reaches its maximum for $t = t_0$. The reason for this choice is that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ is the norm of the differential $f'((1-\varepsilon)t) : T_\Delta \to T_X$ with respect to the Poincaré metric $|dt|^2/(1-|t|^2)^2$ on T_Δ, which is conformally invariant under $\mathrm{Aut}(\Delta)$. One then adjusts R and ψ so that $\psi(0) = (1-\varepsilon)t_0$ and $|\psi'(0)|\,\|f'(\psi(0))\|_\omega = 1$. As $|\psi'(0)| = \frac{1-\varepsilon}{R}(1-|t_0|^2)$, the only possible choice for R is

$$R = (1-\varepsilon)(1-|t_0|^2)\|f'(\psi(0))\|_\omega \geqslant (1-\varepsilon)\|f'(0)\|_\omega.$$

The inequality for $(f\circ\psi)'$ follows from the fact that the Poincaré norm is maximum at the origin, where it is equal to 1 by the choice of R. □

Using the Ascoli–Arzelà theorem we obtain immediately:

1.8. Corollary (Brody). *Let (X, ω) be a compact complex Hermitian manifold. Given a sequence of holomorphic mappings $f_\nu : \Delta \to X$ such that* $\lim \|f_\nu'(0)\|_\omega = +\infty$, *one can find a sequence of homographic transformations $\psi_\nu : D(0, R_\nu) \to (1 - 1/\nu)\Delta$ with* $\lim R_\nu = +\infty$, *such that, after passing possibly to a subsequence, $(f_\nu \circ \psi_\nu)$ converges uniformly on every compact subset of $\mathbb{C}$ towards a non-constant holomorphic map $g : \mathbb{C} \to X$ with $\|g'(0)\|_\omega = 1$ and* $\sup_{t\in\mathbb{C}} \|g'(t)\|_\omega \leqslant 1$.

An entire curve $g : \mathbb{C} \to X$ such that $\sup_{\mathbb{C}} \|g'\|_\omega = M < +\infty$ is called a *Brody curve*; this concept does not depend on the choice of ω when X is compact, and one can always assume $M = 1$ by rescaling the parameter t.

1.9. Brody criterion. *Let X be a compact complex manifold. The following properties are equivalent:*

(a) *X is hyperbolic;*
(b) *X does not possess any entire curve $f : \mathbb{C} \to X$;*
(c) *X does not possess any Brody curve $g : \mathbb{C} \to X$;*
(d) *The Kobayashi infinitesimal metric $\mathbf{k}_X$ is uniformly bounded below, namely*

$$\mathbf{k}_X(\xi) \geqslant c\|\xi\|_\omega, \quad c > 0,$$

for any Hermitian metric ω on X.

Proof. (a) $\Rightarrow$ (b). If X possesses an entire curve $f : \mathbb{C} \to X$, then by looking at arbitrary large disks $D(0, R) \subset \mathbb{C}$, it is easy to see that the Kobayashi distance of any two points in $f(\mathbb{C})$ is zero, so X is not hyperbolic.

(b) $\Rightarrow$ (c). This is trivial.

(c) $\Rightarrow$ (d). If (d) does not hold, there exists a sequence of tangent vectors $\xi_\nu \in T_{X,x_\nu}$ with $\|\xi_\nu\|_\omega = 1$ and $\mathbf{k}_X(\xi_\nu) \to 0$. By definition, this means that there exists an analytic curve $f_\nu : \Delta \to X$ with $f(0) = x_\nu$ and $\|f_\nu'(0)\|_\omega \geqslant (1 - \frac{1}{\nu})/\mathbf{k}_X(\xi_\nu) \to +\infty$. One can then produce a Brody curve $g = \mathbb{C} \to X$ by Corollary 1.8, contradicting (c).

(d) $\Rightarrow$ (a). In fact (d) implies after integrating that $d_X^K(p, q) \geqslant c\, d_\omega(p, q)$, where d_ω is the geodesic distance associated with ω, so d_X^K must be non-degenerate. □

Notice also that if $f : \mathbb{C} \to X$ is an entire curve such that $\|f'\|_\omega$ is unbounded, one can apply the Corollary 1.8 to $f_\nu(t) := f(t + a_\nu)$, where the sequence (a_ν) is chosen such that $\|f_\nu'(0)\|_\omega = \|f(a_\nu)\|_\omega \to +\infty$. Brody's result then produces reparametrizations $\psi_\nu : D(0, R_\nu) \to D(a_\nu, 1 - 1/\nu)$ and a Brody curve $g = \lim f \circ \psi_\nu : \mathbb{C} \to X$ such that $\sup \|g'\|_\omega = 1$ and $g(\mathbb{C}) \subset \overline{f(\mathbb{C})}$. It may happen that the image $g(\mathbb{C})$ of such a limiting curve is disjoint from $f(\mathbb{C})$. In fact Winkelmann [Win07] has given a striking example, actually a projective 3-fold X obtained by blowing-up a 3-dimensional abelian variety Y, such that every Brody curve $g : \mathbb{C} \to X$ lies in the exceptional divisor $E \subset X$; however, entire curves $f : \mathbb{C} \to X$ can be dense, as one can see by taking f to be the lifting of a generic complex line embedded in the abelian variety Y. For further precise information on the localization of Brody curves, we refer the reader to the remarkable results of [Duv08].

The absence of entire holomorphic curves in a given complex manifold is often referred to as *Brody hyperbolicity*. Thus, in the compact case, Brody hyperbolicity and Kobayashi hyperbolicity coincide (but Brody hyperbolicity is in general a strictly weaker property when X is non-compact).

1.C. Geometric applications

We give here two immediate consequences of the Brody criterion: the openness property of hyperbolicity and a hyperbolicity criterion for subvarieties of complex tori. By definition, a *holomorphic family* of compact complex manifolds is a holomorphic proper submersion $\mathscr{X} \to S$ between two complex manifolds.

1.10. Proposition. *Let $\pi : \mathscr{X} \to S$ be a holomorphic family of compact complex manifolds. Then the set of $s \in S$ such that the fiber $X_s = \pi^{-1}(s)$ is hyperbolic is open in the Euclidean topology.*

Proof. Let ω be an arbitrary Hermitian metric on $\mathscr{X}$, $(X_{s_\nu})_{s_\nu \in S}$ a sequence of non-hyperbolic fibers, and $s = \lim s_\nu$. By the Brody criterion, one obtains a sequence of entire maps $f_\nu : \mathbb{C} \to X_{s_\nu}$ such that $\|f_\nu'(0)\|_\omega = 1$ and $\|f_\nu'\|_\omega \leqslant 1$. Ascoli's theorem shows that there is a subsequence of f_ν converging uniformly to a limit $f : \mathbb{C} \to X_s$, with $\|f'(0)\|_\omega = 1$. Hence X_s is not hyperbolic and the collection of non-hyperbolic fibers is closed in S. □

Consider now an n-dimensional complex torus W, i.e., an additive quotient $W = \mathbb{C}^n/\Lambda$, where $\Lambda \subset \mathbb{C}^n$ is a (cocompact) lattice. By taking a composition of entire curves $\mathbb{C} \to \mathbb{C}^n$ with the projection $\mathbb{C}^n \to W$ we obtain an infinite dimensional space of entire curves in W.

1.11. Theorem. *Let $X \subset W$ be a compact complex submanifold of a complex torus. Then X is hyperbolic if and only if it does not contain any translate of a subtorus.*

Proof. If X contains some translate of a subtorus, then it contains lots of entire curves and so X is not hyperbolic.

Conversely, suppose that X is not hyperbolic. Then by the Brody criterion there exists an entire curve $f : \mathbb{C} \to X$ such that $\|f'\|_\omega \leqslant \|f'(0)\|_\omega = 1$, where ω is the flat metric on W inherited from $\mathbb{C}^n$. This means that any lifting

$\widetilde{f} = (\widetilde{f}, \dots, \widetilde{f_\nu}) : \mathbb{C} \to \mathbb{C}^n$ is such that

$$\sum_{j=1}^{n} |f_j'|^2 \leqslant 1.$$

Then, by Liouville's theorem, $\widetilde{f}'$ is constant and therefore $\widetilde{f}$ is affine linear. But then the closure of the image of f is a translate $a + H$ of a connected (possibly real) subgroup H of W. We conclude that X contains the analytic Zariski closure of $a + H$, namely $a + H^{\mathbb{C}}$, where $H^{\mathbb{C}} \subset W$ is the smallest closed complex subgroup of W containing H. □

2. Directed manifolds

2.A. Basic definitions concerning directed manifolds

Let us consider a pair (X, V) consisting of an n-dimensional complex manifold X equipped with a *linear subspace* $V \subset T_X$: if we assume X to be connected, this is by definition an irreducible closed analytic subspace of the total space of T_X such that each fiber $V_x = V \cap T_{X,x}$ is a vector subspace of $T_{X,x}$. If $\mathscr{W} \subset \Omega^1_X$ is the sheaf of 1-forms vanishing on V, then $\mathscr{W}$ is coherent (this follows from the direct image theorem by looking at the proper morphism $P(V) \subset P(T_X) \to X$), and V is locally defined by

$$V_x = \{\xi \in T_{X,x}\,;\ \alpha_j(x)\cdot\xi = 0,\ 1 \leqslant j \leqslant N\}, \quad \alpha_j \in H^0(U, \Omega^1_X), \quad x \in U,$$

where $(\alpha_1, \dots, \alpha_N)$, is a local family of generators of $\mathscr{W}$ on a small open set U. We can also associate to V a coherent sheaf $\mathscr{V} := \mathscr{W}^\perp = \mathrm{Hom}(\Omega^1_X/\mathscr{W}, \mathscr{O}_X) \subset \mathscr{O}(T_X)$, which is a *saturated subsheaf* of $\mathscr{O}(T_X)$, i.e., such that $\mathscr{O}(T_X)/\mathscr{V}$ has no torsion; then $\mathscr{V}$ is also reflexive, i.e., $\mathscr{V}^{**} = \mathscr{V}$. We will refer to such a pair as being a (complex) *directed manifold*, and we will in general think of V as a linear space (rather than considering the associated saturated subsheaf $\mathscr{V} \subset \mathscr{O}(T_X)$). A morphism $\Phi : (X, V) \to (Y, W)$ in the category of complex directed manifolds is a holomorphic map such that $\Phi_*(V) \subset W$.

Here, the rank $x \mapsto \dim_{\mathbb{C}} V_x$ is Zariski lower semi-continuous, and it may a priori jump. The rank $r := \mathrm{rank}(V) \in \{0, 1, \dots, n\}$ of V is by definition the dimension of V_x at a generic point. The dimension may be larger at non-generic points; this happens e.g. on $X = \mathbb{C}^n$ for the rank 1 linear space V generated by the Euler vector field: $V_z = \mathbb{C}\sum_{1\leqslant j\leqslant n} z_j \frac{\partial}{\partial z_j}$ for $z \neq 0$, and $V_0 = \mathbb{C}^n$. Our philosophy is that directed manifolds are also useful to study the "absolute case", i.e., the case $V = T_X$, because there are certain functorial constructions which are quite natural in the category of directed manifolds (see e.g. Sects. 5, 6, 7). We think of directed manifolds as a kind of "relative situation", covering e.g.

the case when V is the relative tangent space to a holomorphic map $X \to S$. It is important to notice that the local sections of $\mathscr{V}$ need not generate the fibers of V at singular points, as one sees already in the case of the Euler vector field when $n \geqslant 2$. We also want to stress that no assumption need be made on the Lie bracket tensor $[\bullet, \bullet] : \mathscr{V} \times \mathscr{V} \to \mathscr{O}(T_X)/\mathscr{V}$, i.e., we do not assume any kind of integrability for $\mathscr{V}$ or $\mathscr{W}$.

The singular set $\mathrm{Sing}(V)$ is by definition the set of points where $\mathscr{V}$ is not locally free, it can also be defined as the indeterminacy set of the (meromorphic) classifying map $\alpha : X \dashrightarrow G_r(T_X)$, $z \mapsto V_z$ to the Grassmannian of r dimensional subspaces of T_X. We thus have $V_{\restriction X \smallsetminus \mathrm{Sing}(V)} = \alpha^* S$, where $S \to G_r(T_X)$ is the tautological subbundle of $G_r(T_X)$. The singular set $\mathrm{Sing}(V)$ is an analytic subset of X of codim $\geqslant 2$, and hence V is always a holomorphic subbundle outside of codimension 2. Thanks to this remark, one can most often treat linear spaces as vector bundles (possibly modulo passing to the Zariski closure along $\mathrm{Sing}(V)$).

2.B. Hyperbolicity properties of directed manifolds

Most of what we have done in Sect. 1 can be extended to the category of directed manifolds.

2.1. Definition. *Let (X, V) be a complex directed manifold.*

(i) *The Kobayashi–Royden infinitesimal metric of (X, V) is the Finsler metric on V defined for any $x \in X$ and $\xi \in V_x$ by*

$$\begin{aligned}&\mathbf{k}_{(X,V)}(\xi)\\&= \inf\{\lambda > 0\,;\ \exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi,\ f'(\Delta) \subset V\}.\end{aligned}$$

Here $\Delta \subset \mathbb{C}$ is the unit disk and the map f is an arbitrary holomorphic map which is tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all $t \in \Delta$. We say that (X, V) is infinitesimally hyperbolic if $\mathbf{k}_{(X,V)}$ is positive definite on every fiber V_x and satisfies a uniform lower bound $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon \|\xi\|_\omega$ in terms of any smooth Hermitian metric ω on X, when x describes a compact subset of X.

(ii) *More generally, the Kobayashi–Eisenman infinitesimal pseudometric of (X, V) is the pseudometric defined on all decomposable p-vectors $\xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V_x$, $1 \leqslant p \leqslant r = \mathrm{rank}(V)$, by*

$$\begin{aligned}&\mathbf{e}^p_{(X,V)}(\xi)\\&= \inf\{\lambda > 0\,;\ \exists f : \mathbb{B}_p \to X,\ f(0) = x,\ \lambda f_*(\tau_0) = \xi,\ f_*(T_{\mathbb{B}_p}) \subset V\},\end{aligned}$$

where $\mathbb{B}_p$ is the unit ball in $\mathbb{C}^p$ and $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p$ is the unit p-vector of $\mathbb{C}^p$ at the origin. We say that (X, V) is infinitesimally p-measure *hyperbolic if $\mathbf{e}^p_{(X,V)}$ is positive definite on every fiber $\Lambda^p V_x$ and satisfies a locally uniform lower bound in terms of any smooth metric.*

If $\Phi : (X, V) \to (Y, W)$ is a morphism of directed manifolds, it is immediate to check that we have the monotonicity property

$$\mathbf{k}_{(Y,W)}(\Phi_*\xi) \leqslant \mathbf{k}_{(X,V)}(\xi), \quad \forall \xi \in V, \tag{2.2}$$

$$\mathbf{e}^p_{(Y,W)}(\Phi_*\xi) \leqslant \mathbf{e}^p_{(X,V)}(\xi), \quad \forall \xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V. \tag{2.2^p}$$

The following proposition shows that virtually all reasonable definitions of the hyperbolicity property are equivalent if X is compact (in particular, the additional assumption that there is locally uniform lower bound for $\mathbf{k}_{(X,V)}$ is not needed). We merely say in that case that (X, V) is *hyperbolic*.

2.3. Proposition. *For an arbitrary directed manifold (X, V), the Kobayashi–Royden infinitesimal metric $\mathbf{k}_{(X,V)}$ is upper semi-continuous on the total space of V. If X is compact, (X, V) is infinitesimally hyperbolic if and only if there are no entire curves $g : \mathbb{C} \to X$ tangent to V. In that case, $\mathbf{k}_{(X,V)}$ is a continuous (and positive definite) Finsler metric on V.*

Proof. The proof is almost identical to the standard proof for $\mathbf{k}_X$, for which we refer to Royden [Roy71], [Roy74]. One of the main ingredients is that one can find a Stein neighborhood of the graph of any analytic disk (thanks to a result of [Siu76], *cf.* also [Dem90a] for more general results). This allows to obtain "free" small deformations of any given analytic disk, as there are many holomorphic vector fields on a Stein manifold. □

Another easy observation is that the concept of p-measure hyperbolicity gets weaker and weaker as p increases (we leave it as an exercise to the reader, this is mostly just linear algebra).

2.4. Proposition. *If (X, V) is p-measure hyperbolic, then it is $(p+1)$-measure hyperbolic for all $p \in \{1, \ldots, \mathrm{rank}(V) - 1\}$.*

Again, an argument extremely similar to the proof of Proposition 1.10 shows that relative hyperbolicity is an open property.

2.5. Proposition. *Let $(\mathscr{X}, \mathscr{V}) \to S$ be a holomorphic family of compact directed manifolds (by this, we mean a proper holomorphic map $\mathscr{X} \to S$ together with an analytic linear subspace $\mathscr{V} \subset T_{\mathscr{X}/S} \subset T_{\mathscr{X}}$ of the relative tangent bundle, defining a deformation $(X_s, V_s)_{s \in S}$ of the fibers). Then the set of $s \in S$ such that the fiber (X_s, V_s) is hyperbolic is open in S with respect to the Euclidean topology.*

Let us mention here an impressive result proved by Marco Brunella [Bru03], [Bru05], [Bru06] concerning the behavior of the Kobayashi metric on foliated varieties.

2.6. Theorem (Brunella). *Let X be a compact Kähler manifold equipped with a (possibly singular) rank 1 holomorphic foliation which is not a foliation by rational curves. Then the canonical bundle $K_{\mathscr{F}} = \mathscr{F}^*$ of the foliation is pseudoeffective (i.e., the curvature of $K_{\mathscr{F}}$ is $\geqslant 0$ in the sense of currents).*

The proof is obtained by putting on $K_{\mathscr{F}}$ precisely the metric induced by the Kobayashi metric on the leaves whenever they are generically hyperbolic (i.e., covered by the unit disk). The case of parabolic leaves (covered by $\mathbb{C}$) has to be treated separately.

2.C. Pluricanonical sheaves of a directed variety

Let (X, V) be a directed projective manifold, where V is possibly singular, and let $r = \mathrm{rank}\, V$. If $\mu : \widehat{X} \to X$ is a proper modification (a composition of blow-ups with smooth centers, say), we get a directed manifold $(\widehat{X}, \widehat{V})$ by taking $\widehat{V}$ to be the closure of $\mu_*^{-1}(V')$, where $V' = V_{\restriction X'}$ is the restriction of V over a Zariski open set $X' \subset X \smallsetminus \mathrm{Sing}(V)$ such that $\mu : \mu^{-1}(X') \to X'$ is a biholomorphism. We say that $(\widehat{X}, \widehat{V})$ is a *modification* of (X, V) and write $\widehat{V} = \mu^* V$.

We will be interested in taking modifications realized by iterated blow-ups of certain non-singular subvarieties of the singular set $\mathrm{Sing}(V)$, so as to eventually "improve" the singularities of V; outside of $\mathrm{Sing}(V)$ the effect of blowing-up will be irrelevant. The canonical sheaf K_V, resp. the pluricanonical sheaf sequence $K_V^{[m]}$, will be defined here in several steps, using the concept of bounded pluricanonical forms that was already introduced in [Dem11].

2.7. Definition. *For a directed pair (X, V) with X non-singular, we define ${}^bK_V$ (resp. ${}^bK_V^{[m]}$) for any integer $m \geqslant 0$, to be the rank 1 analytic sheaves such that*

$$
\begin{aligned}
{}^bK_V(U) &= \textit{sheaf of locally bounded sections of } \mathscr{O}_X(\Lambda^r V'^*)(U \cap X'), \\
{}^bK_V^{[m]}(U) &= \textit{sheaf of locally bounded sections of } \mathscr{O}_X((\Lambda^r V'^*)^{\otimes m})(U \cap X'),
\end{aligned}
$$

where $r = \mathrm{rank}(V)$, $X' = X \smallsetminus \mathrm{Sing}(V)$, $V' = V_{\restriction X'}$, and "locally bounded" means bounded with respect to a smooth Hermitian metric h on T_X, on every set $W \cap X'$ such that W is relatively compact in U.

In the trivial case $r = 0$, we simply set ${}^bK_V^{[m]} = \mathscr{O}_X$ for all m; clearly $\mathrm{ECL}(X, V) = \emptyset$ in that case, so there is not much to say. The above definition of ${}^bK_V^{[m]}$ may look like an analytic one, but it can easily be turned into an equivalent algebraic definition:

2.8. Proposition. *Consider the natural morphism $\mathscr{O}(\Lambda^r T_X^*) \to \mathscr{O}(\Lambda^r V^*)$, where $r = \operatorname{rank} V$ and $\mathscr{O}(\Lambda^r V^*)$ is defined as the quotient of $\mathscr{O}(\Lambda^r T_X^*)$ by r-forms that have zero restrictions to $\mathscr{O}(\Lambda^r V^*)$ on $X \smallsetminus \mathrm{Sing}(V)$. The bidual $\mathscr{L}_V = \mathscr{O}_X(\Lambda^r V^*)^{**}$ is an invertible sheaf, and our natural morphism can be written*

$$\mathscr{O}(\Lambda^r T_X^*) \longrightarrow \mathscr{O}(\Lambda^r V^*) = \mathscr{L}_V \otimes \mathscr{J}_V \subset \mathscr{L}_V, \tag{2.8$_1$}$$

where $\mathscr{J}_V$ is a certain ideal sheaf of $\mathscr{O}_X$ whose zero set is contained in $\mathrm{Sing}(V)$, and the arrow on the left is surjective by definition. Then

$${}^bK_V^{[m]} = \mathscr{L}_V^{\otimes m} \otimes \overline{\mathscr{J}_V^m}, \tag{2.8$_2$}$$

where $\overline{\mathscr{J}_V^m}$ is the integral closure of $\mathscr{J}_V^m$ in $\mathscr{O}_X$. In particular, ${}^bK_V^{[m]}$ is always a coherent sheaf.

Proof. Let (u_k) be a set of generators of $\mathscr{O}(\Lambda^r V^*)$ obtained (say) as the images of a basis $(dz_I)_{|I|=r}$ of $\Lambda^r T_X^*$ in some local coordinates near a point $x \in X$. Write $u_k = g_k \ell$, where ℓ is a local generator of $\mathscr{L}_V$ at x. Then $\mathscr{J}_V = (g_k)$ by definition. The boundedness condition expressed in Definition 2.7 means that we take sections of the form $f \ell^{\otimes m}$, where f is a holomorphic function on $U \cap X'$ (and U a neighborhood of x), such that

$$|f| \leqslant C\Big(\sum |g_k|\Big)^m \tag{2.8$_3$}$$

for some constant $C > 0$. But then f extends holomorphically to U into a function that lies in the integral closure $\overline{\mathscr{J}_V^m}$ (it is well-known that the latter is characterized analytically by condition (2.8$_3$)). This proves Proposition 2.8. □

2.9. Lemma. *Let (X, V) be a directed variety.*

(a) *For any modification $\mu : (\widehat{X}, \widehat{V}) \to (X, V)$, there are always well-defined injective natural morphisms of rank 1 sheaves*

$${}^bK_V^{[m]} \hookrightarrow \mu_*({}^bK_{\widehat{V}}^{[m]}) \hookrightarrow \mathscr{L}_V^{\otimes m}.$$

(b) *The direct image $\mu_*({}^bK_{\widehat{V}}^{[m]})$ may only increase when we replace μ by a "higher" modification $\widetilde{\mu} = \mu' \circ \mu : \widetilde{X} \to \widehat{X} \to X$ and $\widehat{V} = \mu^* V$ by $\widetilde{V} = \widetilde{\mu}^* V$, i.e., there are injections*

$$\mu_*({}^bK_{\widehat{V}}^{[m]}) \hookrightarrow \widetilde{\mu}_*({}^bK_{\widetilde{V}}^{[m]}) \hookrightarrow \mathscr{L}_V^{\otimes m}.$$

We refer to this property as the monotonicity principle.

Proof. (a) The existence of the first arrow is seen as follows: the differential $\mu_* = d\mu : \widehat{V} \to \mu^* V$ is smooth, so it is bounded with respect to ambient Hermitian metrics on X and $\widehat{X}$. Going to the duals reverses the arrows while preserving boundedness with respect to the metrics. We thus get an arrow

$$\mu^*({}^b V^\star) \hookrightarrow {}^b \widehat{V}^\star.$$

By taking the top exterior power, followed by the m-th tensor product and the integral closure of the ideals involved, we get an injective arrow $\mu^*({}^b K_V^{[m]}) \hookrightarrow {}^b K_{\widehat{V}}^{[m]}$. Finally we apply the direct image functor μ_* and the canonical morphism $\mathscr{F} \to \mu_* \mu^* \mathscr{F}$ to get the first inclusion morphism. The second arrow comes from the fact that $\mu^*({}^b K_V^{[m]})$ coincides with $\mathscr{L}_V^{\otimes m}$ (and with $\det(V^*)^{\otimes m}$) on the complement of the codimension 2 set $S = \mathrm{Sing}(V) \cup \mu(\mathrm{Exc}(\mu))$, and the fact that for every open set $U \subset X$, sections of $\mathscr{L}_V$ defined on $U \smallsetminus S$ automatically extend to U by Riemann's extension theorem (or Hartog's extension theorem...), even without any boundedness assumption.
(b) Given $\mu' : \widetilde{X} \to \widehat{X}$, we argue as in (a) that there is a bounded morphism $d\mu' : \widetilde{V} \to \widehat{V}$. □

By the monotonicity principle and the strong Noetherian property of coherent ideals, we infer that there exists a maximal direct image when $\mu : \widehat{X} \to X$ runs over all non-singular modifications of X. The following definition is thus legitimate.

2.10. Definition. *We define the pluricanonical sheaves K_V^m of (X, V) to be the inductive limits*

$$K_V^{[m]} := \varinjlim_{\mu} \mu_*({}^b K_{\widehat{V}}^{[m]}) = \max_{\mu} \mu_*({}^b K_{\widehat{V}}^{[m]})$$

taken over the family of all modifications $\mu : (\widehat{X}, \widehat{V}) \to (X, V)$, with the trivial (filtering) partial order. The canonical sheaf K_V itself is defined to be the same as $K_V^{[1]}$. By construction, we have for every $m \geqslant 0$ inclusions

$${}^b K_V^{[m]} \hookrightarrow K_V^{[m]} \hookrightarrow \mathscr{L}_V^{\otimes m},$$

and $K_V^{[m]} = \mathscr{J}_V^{[m]} \cdot \mathscr{L}_V^{\otimes m}$ for a certain sequence of integrally closed ideals $\mathscr{J}_V^{[m]} \subset \mathscr{O}_X$.

It is clear from this construction that $K_V^{[m]}$ is birationally invariant, i.e., that $K_V^{[m]} = \mu_*(K_{V'}^{[m]})$ for every modification $\mu : (X', V') \to (X, V)$. Moreover the sequence is submultiplicative, i.e., there are injections

$$K_V^{[m_1]} \otimes K_V^{[m_2]} \hookrightarrow K_V^{[m_1+m_2]}$$

for all non-negative integers m_1, m_2; the corresponding sequence of ideals $\mathscr{J}_V^{[m]}$ is thus also submultiplicative. By blowing up $\mathscr{J}_V^{[m]}$ and taking a desingularization $\widehat{X}$ of the blow-up, one can always find a *log-resolution* of $\mathscr{J}_V^{[m]}$, i.e., a modification $\mu_m : \widehat{X}_m \to X$ such that $\mu_m^* \mathscr{J}_V^{[m]} \subset \mathscr{O}_{\widehat{X}_m}$ is an invertible ideal sheaf; it follows that

$$\mu_m^* K_V^{[m]} = \mu_m^* \mathscr{J}_V^{[m]} \cdot (\mu_m^* \mathscr{L}_V)^{\otimes m}$$

is an invertible sheaf on $\widehat{X}_m$. We do not know whether μ_m can be taken independent of m, nor whether the inductive limit introduced in Definition 2.10 is reached for a μ that is independent of m. If such a "uniform" μ exists, it could be thought of as some sort of replacement for the resolution of singularities of directed structures (which do not exist in the naive sense that V could be made non-singular). By means of a standard Serre–Siegel argument, one can easily show

2.11. Proposition. *Let (X, V) be a directed variety (X, V) and F be an invertible sheaf on X. The following properties are equivalent:*

(a) *there exists a constant $c > 0$ and $m_0 > 0$ such that $h^0(X, K_V^{[m]} \otimes F^{\otimes m}) \geqslant cm^n$ for $m \geqslant m_0$, where $n = \dim X$;*
(b) *the space of sections $H^0(X, K_V^{[m]} \otimes F^{\otimes m})$ provides a generic embedding of X in projective space for sufficiently large m;*
(c) *there exists $m > 0$ and a log-resolution $\mu_m : \widehat{X}_m \to X$ of $K_V^{[m]}$ such that $\mu_m^*(K_V^{[m]} \otimes F^{\otimes m})$ is a big invertible sheaf on $\widehat{X}_m$;*
(d) *there exists $m > 0$, a modification $\widetilde{\mu}_m : (\widetilde{X}_m, \widetilde{V}_m) \to (X, V)$ and a log-resolution $\mu'_m : \widehat{X}_m \to \widetilde{X}$ of ${}^bK_{\widetilde{V}_m}^{[m]}$ such that $\mu'^*_m({}^bK_{\widetilde{V}_m}^{[m]} \otimes \widetilde{\mu}_m^* F^{\otimes m})$ is a big invertible sheaf on $\widehat{X}_m$.*

We will express any of these equivalent properties by saying that the twisted pluricanonical sheaf sequence $K_V^\bullet \otimes F^\bullet$ is big.

In the special case $F = \mathscr{O}_X$, we introduce

2.12. Definition. *We say that (X, V) is of general type if $K_V^\bullet$ is big.*

2.13. Remarks.
(a) At this point, it is important to stress the difference between "our" canonical sheaf K_V, and the sheaf $\mathscr{L}_V$, which is considered by some experts as "the canonical sheaf of the foliation" defined by V, in the integrable case. Notice that $\mathscr{L}_V$ can also be obtained as the direct image $\mathscr{L}_V = i_* \mathscr{O}(\det(V^*))$ associated with the injection $i : X \smallsetminus \mathrm{Sing}(V) \hookrightarrow X$. The discrepancy already occurs with the rank 1 linear space $V \subset T_{\mathbb{P}^n_{\mathbb{C}}}$ consisting at each point $z \neq 0$ of the

tangent to the line $(0z)$ (so that necessarily $V_0 = T_{\mathbb{P}^n_{\mathbb{C}},0}$). As a sheaf (and not as a linear space), $i_*\mathscr{O}(V)$ is the invertible sheaf generated by the vector field $\xi = \sum z_j\,\partial/\partial z_j$ on the affine open set $\mathbb{C}^n \subset \mathbb{P}^n_{\mathbb{C}}$, and therefore $\mathscr{L}_V := i_*\mathscr{O}(V^*)$ is generated over $\mathbb{C}^n$ by the unique 1-form u such that $u(\xi) = 1$. Since ξ vanishes at 0, the generator u is *unbounded* with respect to a smooth metric h_0 on $T_{\mathbb{P}^n_{\mathbb{C}}}$, and it is easily seen that K_V is the non-invertible sheaf $K_V = \mathscr{L}_V \otimes \mathfrak{m}_{\mathbb{P}^n_{\mathbb{C}},0}$. We can make it invertible by considering the blow-up $\mu : \widetilde{X} \to X$ of $X = \mathbb{P}^n_{\mathbb{C}}$ at 0, so that $\mu^* K_V$ is isomorphic to $\mu^*\mathscr{L}_V \otimes \mathscr{O}_{\widetilde{X}}(-E)$, where E is the exceptional divisor. The integral curves C of V are of course lines through 0, and when a standard parametrization is used, their derivatives do not vanish at 0, while the sections of $i_*\mathscr{O}(V)$ do—a first sign that $i_*\mathscr{O}(V)$ and $i_*\mathscr{O}(V^*)$ are the *wrong objects* to consider.

(b) When V is of rank 1, we get a foliation by curves on X. If (X, V) is of general type (i.e., $K_V^\bullet$ is big), we will see in Proposition 4.9 that almost all leaves of V are hyperbolic, i.e., covered by the unit disk. This would not be true if $K_V^\bullet$ was replaced by $\mathscr{L}_V$. In fact, the examples of pencils of conics or cubic curves in $\mathbb{P}^2$ already produce this phenomenon, as we have seen in the introduction, right after generalized GGL conjecture 0.4. For this second reason, we believe that $K_V^\bullet$ is a more appropriate concept of "canonical sheaf" than $\mathscr{L}_V$ is.

(c) When $\dim X = 2$, a singularity of a (rank 1) foliation V is said to be *simple* if the linear part of the local vector field generating $\mathscr{O}(V)$ has two distinct eigenvalues $\lambda \neq 0$, $\mu \neq 0$ such that the quotient λ/μ is not a positive rational number. Seidenberg's theorem [Sei68] says there always exists a composition of blow-ups $\mu : \widehat{X} \to X$ such that $\widehat{V} = \mu^* V$ only has simple singularities. It is easy to check that the inductive limit canonical sheaf $K_V^{[m]} = \mu_*({}^b K_{\widehat{V}}^{[m]})$ is reached whenever $\widehat{V} = \mu^* V$ has simple singularities.

3. Algebraic hyperbolicity

In the case of projective algebraic varieties, hyperbolicity is expected to be related to other properties of a more algebraic nature. Theorem 3.1 below is a first step in this direction.

3.1. Theorem. *Let (X, V) be a compact complex directed manifold and let $\sum \omega_{jk}\, dz_j \otimes d\overline{z}_k$ be a Hermitian metric on T_X, with associated positive $(1,1)$-form $\omega = \frac{i}{2}\sum \omega_{jk}\, dz_j \wedge d\overline{z}_k$. Consider the following three properties, which may or not be satisfied by (X, V):*

(i) *(X, V) is hyperbolic.*
(ii) *There exists $\varepsilon > 0$ such that every compact irreducible curve $C \subset X$ tangent to V satisfies*

$$-\chi(\overline{C}) = 2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C),$$

where $\deg_\omega(C) = \int_C \omega$, *and where* $g(\overline{C})$ *is the genus of the normalization* $\overline{C}$ *of* C *and* $\chi(\overline{C})$ *its Euler characteristic (the degree coincides with the usual concept of degree if* X *is projective, embedded in* $\mathbb{P}^N$ *via a very ample line bundle* A*, and* $\omega = \Theta_{A,h_A} > 0$; *such an estimate is of course independent of the choice of* ω*, provided that* ε *is changed accordingly).*

(iii) *There does not exist any non-constant holomorphic map* $\Phi : Z \to X$ *from an abelian variety* Z *to* X *such that* $\Phi_*(T_Z) \subset V$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii). If (X, V) is hyperbolic, there is a constant $\varepsilon_0 > 0$ such that $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon_0 \|\xi\|_\omega$ for all $\xi \in V$. Now, let $C \subset X$ be a compact irreducible curve tangent to V and let $\nu : \overline{C} \to C$ be its normalization. As (X, V) is hyperbolic, $\overline{C}$ cannot be a rational or elliptic curve. Hence $\overline{C}$ admits the disk as its universal covering $\rho : \Delta \to \overline{C}$.

The Kobayashi–Royden metric $\mathbf{k}_\Delta$ is the Finsler metric $|dz|/(1-|z|^2)$ associated with the Poincaré metric $|dz|^2/(1-|z|^2)^2$ on Δ, and $\mathbf{k}_{\overline{C}}$ is such that $\rho^*\mathbf{k}_{\overline{C}} = \mathbf{k}_\Delta$. In other words, the metric $\mathbf{k}_{\overline{C}}$ is induced by the unique Hermitian metric on $\overline{C}$ of constant Gaussian curvature -4. If $\sigma_\Delta = \frac{i}{2}\,dz \wedge d\overline{z}/(1-|z|^2)^2$ and $\sigma_{\overline{C}}$ are the corresponding area measures, the Gauss–Bonnet formula (integral of the curvature $= 2\pi\,\chi(\overline{C})$) yields

$$\int_{\overline{C}} d\sigma_{\overline{C}} = -\frac{1}{4}\int_{\overline{C}} \mathrm{curv}(\mathbf{k}_{\overline{C}}) = -\frac{\pi}{2}\chi(\overline{C}).$$

On the other hand, if $j : C \to X$ is the inclusion, the monotonicity property (2.2) applied to the holomorphic map $j \circ \nu : \overline{C} \to X$ shows that

$$\mathbf{k}_{\overline{C}}(t) \geqslant \mathbf{k}_{(X,V)}((j\circ\nu)_* t) \geqslant \varepsilon_0 \|(j\circ\nu)_* t\|_\omega, \quad \forall t \in T_{\overline{C}}.$$

From this, we infer $d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 (j\circ\nu)^*\omega$, thus

$$-\frac{\pi}{2}\chi(\overline{C}) = \int_{\overline{C}} d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 \int_{\overline{C}} (j\circ\nu)^*\omega = \varepsilon_0^2 \int_C \omega.$$

Property (ii) follows with $\varepsilon = 2\varepsilon_0^2/\pi$.

(ii) $\Rightarrow$ (iii). First observe that (ii) excludes the existence of elliptic and rational curves tangent to V. Assume that there is a non-constant holomorphic map $\Phi : Z \to X$ from an abelian variety Z to X such that $\Phi_*(T_Z) \subset V$. We must have $\dim \Phi(Z) \geqslant 2$, otherwise $\Phi(Z)$ would be a curve covered by images of holomorphic maps $\mathbb{C} \to \Phi(Z)$, and so $\Phi(Z)$ would be elliptic or rational, contradiction. Select a sufficiently general curve Γ in Z (e.g. a curve obtained as an intersection of very generic divisors in a given very ample linear system $|L|$ in Z). Then all isogenies $u_m : Z \to Z$, $s \mapsto ms$ map Γ in a $1:1$ way to curves $u_m(\Gamma) \subset Z$, except maybe for finitely many double points of $u_m(\Gamma)$

when $\dim Z = 2$: we leave this as an exercise to the reader, using Bertini type arguments). It follows that the normalization of $u_m(\Gamma)$ is isomorphic to Γ. If Γ is general enough and $\tau_a : Z \to Z$, $w \mapsto w + a$ denote translations of Z, similar arguments show that for general $a \in Z$ the images

$$C_{m,a} := \Phi(\tau_a(u_m(\Gamma))) \subset X$$

are also generically $1 : 1$ images of Γ, thus $\overline{C}_{m,a} \simeq \Gamma$ and $g(\overline{C}_{m,a}) = g(\Gamma)$. We claim that on average $C_{m,a}$ has degree $\geqslant \mathrm{Const}\, m^2$. In fact, if μ is the translation invariant probability measure on Z

$$\int_{C_{m,a}} \omega = \int_\Gamma u_m^*(\tau_a^* \Phi^* \omega), \quad \text{and hence} \quad \int_{a\in Z} \Big(\int_{C_{m,a}} \omega \Big)\, d\mu(a) = \int_\Gamma u_m^* \beta,$$

where $\beta = \int_{a\in Z} (\tau_a^* \Phi^* \omega)\, d\mu(a)$ is a translation invariant $(1,1)$-form on Z. Therefore β is a constant coefficient $(1,1)$-form, so $u_m^* \beta = m^2 \beta$ and the right hand side is cm^2 with $c = \int_\Gamma \beta > 0$. For a suitable choice of $a_m \in Z$, we have $\deg_\omega C_{m,a_m} \geqslant cm^2$ and $(2g(\overline{C}_{m,a_m}) - 2)/\deg_\omega(C_{m,a_m}) \to 0$, contradiction. $\square$

3.2. Definition. *We say that a projective directed manifold (X, V) is "algebraically hyperbolic" if it satisfies property 3.1* (ii)*, namely, if there exists $\varepsilon > 0$ such that every algebraic curve $C \subset X$ tangent to V satisfies*

$$2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C).$$

A nice feature of algebraic hyperbolicity is that it satisfies an algebraic analogue of the openness property.

3.3. Proposition. *Let $(\mathscr{X}, \mathscr{V}) \to S$ be an algebraic family of projective algebraic directed manifolds (given by a projective morphism $\mathscr{X} \to S$). Then the set of $t \in S$ such that the fiber (X_t, V_t) is algebraically hyperbolic is open with respect to the "countable Zariski topology" of S (by definition, this is the topology for which closed sets are countable unions of algebraic sets).*

Proof. After replacing S by a Zariski open subset, we may assume that the total space $\mathscr{X}$ itself is quasi-projective. Let ω be the Kähler metric on $\mathscr{X}$ obtained by pulling back the Fubini–Study metric via an embedding in a projective space. If the integers $d > 0$, $g \geqslant 0$ are fixed, the set $A_{d,g}$ of $t \in S$ such that X_t contains an algebraic 1-cycle $C = \sum m_j C_j$ tangent to V_t with $\deg_\omega(C) = d$ and $g(\overline{C}) = \sum m_j\, g(\overline{C}_j) \leqslant g$ is a closed algebraic subset of S (this follows from the existence of a relative cycle space of curves of given degree, and from the fact that the geometric genus is Zariski lower semi-continuous). Now, the set of non-algebraically hyperbolic fibers is by definition

$$\bigcap_{k>0} \bigcup_{2g-2<d/k} A_{d,g}.$$

This concludes the proof (of course, one has to know that the countable Zariski topology is actually a topology, namely that the class of countable unions of algebraic sets is stable under arbitrary intersections; this can be easily checked by an induction on dimension). □

3.4. Remark. More explicit versions of the openness property have been dealt with in the literature. H. Clemens ([Cle86] and [CKM88]) has shown that on a very generic surface of degree $d \geqslant 5$ in $\mathbb{P}^3$, the curves of type (d,k) are of genus $g > kd(d-5)/2$ (recall that a very generic surface $X \subset \mathbb{P}^3$ of degree $\geqslant 4$ has Picard group generated by $\mathcal{O}_X(1)$ thanks to the Noether–Lefschetz theorem; thus any curve on the surface is a complete intersection with another hypersurface of degree k; such a curve is said to be of type (d,k); genericity is taken here in the sense of the countable Zariski topology). Improving on this result of Clemens, Geng Xu [Xu94] has shown that every curve contained in a very generic surface of degree $d \geqslant 5$ satisfies the sharp bound $g \geqslant d(d-3)/2-2$. This actually shows that a very generic surface of degree $d \geqslant 6$ is algebraically hyperbolic. Although a very generic quintic surface has no rational or elliptic curves, it seems to be unknown whether a (very) generic quintic surface is algebraically hyperbolic in the sense of Definition 3.2.

Improving on this result of Clemens, Geng Xu [Xu94] proved that every curve contained in a very generic surface of degree $d \geqslant 5$ satisfies the sharp bound $g \geqslant d(d-3)/2-2$. In April 2018, I. Coskun and E. Riedl improved the above bounds and got the more precise bound $g \geqslant 1+(dk(d-5)+k)/2$; this result actually shows that a very generic surface of degree $d \geqslant 5$ is algebraically hyperbolic in the sense of Definition 3.2. In higher dimension, L. Ein ([Ein88], [Ein91]) proved that every subvariety of a very generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2n+1$ $(n \geqslant 2)$, is of general type. This was reproved by a simple efficient technique by C. Voisin in [Voi96], along with other improvements.

3.5. Remark. In view of Proposition 1.10, it would be interesting to know whether algebraic hyperbolicity is open with respect to the Euclidean topology; still more interesting would be to know whether Kobayashi hyperbolicity is open for the countable Zariski topology (of course, both properties would follow immediately if one knew that algebraic hyperbolicity and Kobayashi hyperbolicity coincide, but they seem otherwise highly non-trivial to establish). The latter openness property has raised an important amount of work around the following more particular question: is a (very) generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d large enough (say $d \geqslant 2n+1$) Kobayashi hyperbolic? Again, "very generic" is to be taken here in the sense of the countable Zariski topology. Brody–Green [BrGr77] and Nadel [Nad89] produced examples of hyperbolic surfaces in $\mathbb{P}^3$ for all degrees $d \geqslant 50$, and Masuda–Noguchi [MaNo96] gave examples of such hypersurfaces in $\mathbb{P}^n$ for arbitrary $n \geqslant 2$, of degree $d \geqslant d_0(n)$ large enough. The hyperbolicity of complements $\mathbb{P}^n \smallsetminus H$ of generic divisors may be inferred from the compact case; in fact if $H = \{P(z_0, \dots, z_n) = 0\}$ is a smooth generic divisor of degree d, one may look at the hypersurface

$$X = \{z_{n+1}^d = P(z_0, \dots, z_n)\} \subset \mathbb{P}^{n+1}$$

which is a cyclic $d:1$ covering of $\mathbb{P}^n$. Since any holomorphic map $f : \mathbb{C} \to \mathbb{P}^n \smallsetminus H$ can be lifted to X, it is clear that the hyperbolicity of X would imply the hyperbolicity of $\mathbb{P}^n \smallsetminus H$. The hyperbolicity of complements of divisors in $\mathbb{P}^n$ has been investigated by many authors. In the case $n = 2$, Huynh, Vu and Xie [HVX17, Theorem 1.2] have announced that $\mathbb{P}^2 \smallsetminus C$ is hyperbolic for a very general curve C of degree $d \geqslant 11$ (and that a very general surface $X \subset \mathbb{P}^3$ of degree $d \geqslant 15$ is hyperbolic, [HVX17, Theorem 1.5]). The reader can also consult [CFZ17, Section 4] for more details and references in these directions.

In the "absolute case" $V = T_X$, it seems reasonable to expect that Properties 3.1 (i), (ii) are equivalent, i.e., that Kobayashi and algebraic hyperbolicity coincide. However, it was observed by Serge Cantat [Can00] that Property 3.1 (iii) is not sufficient to imply the hyperbolicity of X, at least when X is a general complex surface: a general (non-algebraic) K3 surface is known to have no elliptic curves and does not admit either any surjective map from an abelian variety; however such a surface is not Kobayashi hyperbolic. We are uncertain about the sufficiency of 3.1 (iii) when X is assumed to be projective.

4. The Ahlfors–Schwarz lemma for metrics of negative curvature

One of the most basic ideas is that hyperbolicity should somehow be related with suitable negativity properties of the curvature. For instance, it is a standard fact already observed in Kobayashi [Kob70] that the negativity of T_X (or the ampleness of T_X^*) implies the hyperbolicity of X. There are many ways of improving or generalizing this result. We present here a few simple examples of such generalizations.

4.A. Exploiting curvature via potential theory

If (V, h) is a holomorphic vector bundle equipped with a smooth Hermitian metric, we denote by $\nabla_h = \nabla_h' + \nabla_h''$ the associated Chern connection and by $\Theta_{V,h} = \frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor.

4.1. Proposition. *Let (X, V) be a compact directed manifold. Assume that V is non-singular and that V^* is ample. Then (X, V) is hyperbolic.*

Proof (from an original idea of [Kob75]). Recall that a vector bundle E is said to be ample if $S^m E$ has enough global sections $\sigma_1, \dots, \sigma_N$ so as to generate 1-jets of sections at any point, when m is large. One obtains a Finsler metric N on E^* by putting

$$N(\xi) = \Big(\sum_{1 \leqslant j \leqslant N} |\sigma_j(x) \cdot \xi^m|^2 \Big)^{1/2m}, \quad \xi \in E_x^*,$$

and N is then a strictly plurisubharmonic function on the total space of E^* minus the zero section (in other words, the line bundle $\mathscr{O}_{P(E^*)}(1)$ has a metric of positive curvature). By the ampleness assumption on V^*, we thus have a Finsler metric N on V which is strictly plurisubharmonic outside the zero section. By the Brody lemma, if (X, V) is not hyperbolic, there is an entire curve $g : \mathbb{C} \to X$ tangent to V such that $\sup_{\mathbb{C}} \|g'\|_\omega \leqslant 1$ for some given Hermitian metric ω on X. Then $N(g')$ is a bounded subharmonic function on $\mathbb{C}$ which is strictly subharmonic on $\{g' \neq 0\}$. This is a contradiction, for any bounded subharmonic function on $\mathbb{C}$ must be constant. □

4.B. Ahlfors–Schwarz lemma

Proposition 4.1 can be generalized a little bit further by means of the Ahlfors–Schwarz lemma (see e.g. [Lang87]; we refer to [Dem95] for the generalized version presented here; the proof is merely an application of the maximum principle plus a regularization argument).

4.2. Ahlfors–Schwarz lemma. *Let $\gamma(t) = \gamma_0(t)\, i\, dt \wedge d\bar{t}$ be a Hermitian metric on Δ_R, where $\log \gamma_0$ is a subharmonic function such that $i\,\partial\bar\partial \log \gamma_0(t) \geqslant A\,\gamma(t)$ in the sense of currents, for some positive constant A. Then γ can be compared with the Poincaré metric of Δ_R as follows:*

$$\gamma(t) \leqslant \frac{2}{A} \frac{R^{-2}|dt|^2}{(1 - |t|^2/R^2)^2}.$$

More generally, let $\gamma = i \sum \gamma_{jk}\, dt_j \wedge d\bar{t}_k$ be an almost everywhere positive Hermitian form on the ball $B(0, R) \subset \mathbb{C}^p$, such that $-\mathrm{Ricci}(\gamma) := i\,\partial\bar\partial \log \det(\gamma) \geqslant A\gamma$ in the sense of currents, for some constant $A > 0$ (this means in particular that $\det(\gamma) = \det(\gamma_{jk})$ is such that $\log \det(\gamma)$ is plurisubharmonic). Then the γ-volume form is controlled by the Poincaré volume form:

$$\det(\gamma) \leqslant \Big(\frac{p+1}{AR^2}\Big)^p \frac{1}{(1 - |t|^2/R^2)^{p+1}}.$$

4.C. Applications of the Ahlfors–Schwarz lemma to hyperbolicity

Let (X, V) be a *projective* directed variety. We assume throughout this subsection that X is *non-singular*.

4.3. Proposition. *Assume that V itself is non-singular and that the dual bundle V^* is "very big" in the following sense: there exists an ample line bundle L and a sufficiently large integer m such that the global sections in $H^0(X, S^m V^* \otimes L^{-1})$ generate all fibers over $X \smallsetminus Y$, for some analytic subset $Y \subsetneq X$. Then all entire curves $f : \mathbb{C} \to X$ tangent to V satisfy $f(\mathbb{C}) \subset Y$.*

Proof. Let $\sigma_1, \ldots, \sigma_N \in H^0(X, S^m V^* \otimes L^{-1})$ be a basis of sections generating $S^m V^* \otimes L^{-1}$ over $X \smallsetminus Y$. If $f : \mathbb{C} \to X$ is tangent to V, we define a semi-positive Hermitian form $\gamma(t) = \gamma_0(t)\,|dt|^2$ on $\mathbb{C}$ by putting

$$\gamma_0(t) = \sum \|\sigma_j(f(t)) \cdot f'(t)^m\|_{L^{-1}}^{2/m},$$

where $\|\ \|_L$ denotes a Hermitian metric with positive curvature on L. If $f(\mathbb{C}) \not\subset Y$, the form γ is not identically 0 and we then find

$$i\,\partial\bar\partial \log \gamma_0 \geqslant \frac{2\pi}{m} f^*\Theta_L,$$

where Θ_L is the curvature form. The positivity assumption combined with an obvious homogeneity argument yield

$$\frac{2\pi}{m} f^*\Theta_L \geqslant \varepsilon \|f'(t)\|_\omega^2\,|dt|^2 \geqslant \varepsilon' \gamma(t)$$

for any given Hermitian metric ω on X. Now, for any t_0 with $\gamma_0(t_0) > 0$, the Ahlfors–Schwarz lemma shows that f can only exist on a disk $D(t_0, R)$ such that $\gamma_0(t_0) \leqslant \frac{2}{\varepsilon'} R^{-2}$, contradiction. □

There are similar results for p-measure hyperbolicity, see e.g. [Carl72] and [Nog77b]:

4.4. Proposition. *Assume that V is non-singular and that $\Lambda^p V^*$ is ample. Then (X, V) is infinitesimally p-measure hyperbolic. More generally, assume that $\Lambda^p V^*$ is very big with base locus contained in $Y \subsetneq X$ (see Proposition 3.3). Then $\mathbf{e}^p$ is non-degenerate over $X \smallsetminus Y$.*

Proof. By the ampleness assumption, there is a smooth Finsler metric N on $\Lambda^p V$ which is strictly plurisubharmonic outside the zero section. We select also a Hermitian metric ω on X. For any holomorphic map $f : \mathbb{B}_p \to X$ we define a semi-positive Hermitian metric $\widetilde{\gamma}$ on $\mathbb{B}_p$ by putting $\widetilde{\gamma} = f^*\omega$. Since ω need not have any good curvature estimate, we introduce the function $\delta(t) = N_{f(t)}(\Lambda^p f'(t) \cdot \tau_0)$, where $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p$, and select a metric $\gamma = \lambda\widetilde{\gamma}$ conformal to $\widetilde{\gamma}$ such that $\det(\gamma) = \delta$. Then λ^p is equal to the ratio $N/\Lambda^p\omega$ on the element $\Lambda^p f'(t)\cdot\tau_0 \in \Lambda^p V_{f(t)}$. Since X is compact, it is clear that the conformal factor λ is bounded by an absolute constant independent of f. From the curvature assumption we then get

$$i\,\partial\overline{\partial}\log\det(\gamma) = i\,\partial\overline{\partial}\log\delta \geqslant (f,\Lambda^p f')^*(i\,\partial\overline{\partial}\log N) \geqslant \varepsilon f^*\omega \geqslant \varepsilon'\,\gamma.$$

By the Ahlfors–Schwarz lemma we infer that $\det(\gamma(0)) \leqslant C$ for some constant C, i.e., $N_{f(0)}(\Lambda^p f'(0)\cdot\tau_0) \leqslant C'$. This means that the Kobayashi–Eisenman pseudometric $\mathbf{e}^p_{(X,V)}$ is positive definite everywhere and uniformly bounded from below. In the case $\Lambda^p V^*$ is very big with base locus Y, we use essentially the same arguments, but we then only have N being positive definite on $X \smallsetminus Y$. □

4.5. Corollary ([Gri71], [KobO71]). *If X is a projective variety of general type, the Kobayashi–Eisenmann volume form $\mathbf{e}^n$, $n = \dim X$, can degenerate only along a proper algebraic set $Y \subsetneq X$.*

The converse of Corollary 4.5 is expected to be true, namely, the generic non-degeneracy of $\mathbf{e}^n$ should imply that X is of general type; this is only known for surfaces (see [GrGr80] and [MoMu82]):

4.6. General Type Conjecture (Green–Griffiths [GrGr80]). A projective algebraic variety X is measure hyperbolic (i.e., $\mathbf{e}^n$ degenerates only along a proper algebraic subvariety) if and only if X is of general type.

An essential step in the proof of the necessity of having general type subvarieties would be to show that manifolds of Kodaira dimension 0 (say, Calabi–Yau manifolds and holomorphic symplectic manifolds, all of which have $c_1(X) = 0$) are not measure hyperbolic, e.g. by exhibiting enough families of curves $C_{s,\ell}$ covering X such that $(2g(\overline{C}_{s,\ell}) - 2)/\deg(C_{s,\ell}) \to 0$.

4.7. Conjectural corollary (Lang). *A projective algebraic variety X is hyperbolic if and only if all its algebraic subvarieties (including X itself) are of general type.*

4.8. Remark. The GGL conjecture implies the "if" part of 4.7, and the General Type Conjecture 4.6 implies the "only if" part of 4.7. In fact if the GGL conjecture holds and every subvariety Y of X is of general type, then it is easy to infer that every entire curve $f : \mathbb{C} \to X$ has to be constant by induction on $\dim X$, because in fact f maps $\mathbb{C}$ to a certain subvariety $Y \subsetneq X$. Therefore X is hyperbolic. Conversely, if Conjecture 4.6 holds and X has a certain subvariety Y which is not of general type, then Y is not measure hyperbolic. However Proposition 2.4 shows that hyperbolicity implies measure hyperbolicity. Therefore Y is not hyperbolic and so X itself is not hyperbolic either.

We end this section by another easy application of the Ahlfors–Schwarz lemma for the case of rank 1 (possibly singular) foliations.

4.9. Proposition. *Let (X,V) be a projective directed manifold. Assume that V is of rank 1 and that $K_V^\bullet$ is big. Then S be the union of the singular set $\mathrm{Sing}(V)$ and of the base locus of $K_V^\bullet$ (namely the intersection of the images $\mu_m(B_m)$ of the base loci B_m of the invertible sheaves $\mu_m^* K_V^{[m]}$, $m>0$, obtained by taking log-resolutions). Then $\mathrm{ECL}(X,V)\subset S$, in other words, all non-hyperbolic leaves of V are contained in S.*

Proof. By Proposition 2.11 (d), we can take a blow-up $\widetilde{\mu}_m : \widetilde{X}_m \to X$ and a log-resolution $\mu'_m : \widehat{X}_m \to \widetilde{X}_m$ such that $F_m = \mu'^*_m({}^bK^{[m]}_{\widetilde{V}_m})$ is a big invertible sheaf. This means that (after possibly increasing m) we can find sections $\sigma_1,\ldots,\sigma_N \in H^0(\widehat{X}_m, F_m)$ that define a (singular) Hermitian metric with strictly positive curvature on F_m, *cf.* Definition 8.1 below. Now, for every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X,V)$ not contained in S, we can choose m and a lifting $\widetilde{f} : (\mathbb{C}, T_{\mathbb{C}}) \to (\widetilde{X},\widetilde{V})$ such that $\widetilde{f}(\mathbb{C})$ is not contained in the base locus of our sections. Again, we can define a semi-positive Hermitian form $\gamma(t)=\gamma_0(t)\,|dt|^2$ on $\mathbb{C}$ by putting

$$\gamma_0(t) = \sum \|\sigma_j(f(t))\cdot f'(t)^m\|^{2/m}_{L^{-1}}.$$

Then γ is not identically zero and we have $i\partial\overline{\partial}\log\gamma_0 \geqslant \varepsilon\gamma$ by the strict positivity of the curvature. One should also notice that γ_0 is locally bounded from above by the assumption that the σ_j's come from *locally bounded* sections on $\widetilde{X}_m$. This contradicts the Ahlfors–Schwarz lemma, and thus it cannot happen that $f(\mathbb{C}) \not\subset S$. □

5. Projectivization of a directed manifold

5.A. The 1-jet functor

The basic idea is to introduce a functorial process which produces a new complex directed manifold $(\widetilde{X},\widetilde{V})$ from a given one (X,V). The new structure $(\widetilde{X},\widetilde{V})$ plays the role of a space of 1-jets over X. Fisrt assume that V is *non-singular*. We let

$$\widetilde{X} = P(V), \quad \widetilde{V}\subset T_{\widetilde{X}}$$

be the projectivized bundle of lines of V, together with a subbundle $\widetilde{V}$ of $T_{\widetilde{X}}$ defined as follows: for every point $(x,[v])\in\widetilde{X}$ associated with a vector $v\in V_x\smallsetminus\{0\}$,

$$\widetilde{V}_{(x,[v])} = \{\xi \in T_{\widetilde{X},(x,[v])}\,;\,\pi_*\xi\in\mathbb{C}v\},\quad \mathbb{C}v\subset V_x\subset T_{X,x}, \tag{5.1}$$

where $\pi : \widetilde{X} = P(V) \to X$ is the natural projection and $\pi_* : T_{\widetilde{X}} \to \pi^* T_X$ is its differential. On $\widetilde{X} = P(V)$ we have a tautological line bundle $\mathcal{O}_{\widetilde{X}}(-1) \subset \pi^* V$ such that $\mathcal{O}_{\widetilde{X}}(-1)_{(x,[v])} = \mathbb{C}v$. The bundle $\widetilde{V}$ is characterized by the two exact sequences

$$0 \longrightarrow T_{\widetilde{X}/X} \longrightarrow \widetilde{V} \xrightarrow{\pi_*} \mathcal{O}_{\widetilde{X}}(-1) \longrightarrow 0, \tag{5.2}$$

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \pi^* V \otimes \mathcal{O}_{\widetilde{X}}(1) \longrightarrow T_{\widetilde{X}/X} \longrightarrow 0, \tag{5.2$'$}$$

where $T_{\widetilde{X}/X}$ denotes the relative tangent bundle of the fibration $\pi : \widetilde{X} \to X$. The first sequence is a direct consequence of the definition of $\widetilde{V}$, whereas the second is a relative version of the Euler exact sequence describing the tangent bundle of the fibers $P(V_x)$. From these exact sequences we infer

$$\dim \widetilde{X} = n + r - 1, \quad \operatorname{rank} \widetilde{V} = \operatorname{rank} V = r, \tag{5.3}$$

and by taking determinants we find $\det(T_{\widetilde{X}/X}) = \pi^* \det(V) \otimes \mathcal{O}_{\widetilde{X}}(r)$. Thus

$$\det(\widetilde{V}) = \pi^* \det(V) \otimes \mathcal{O}_{\widetilde{X}}(r-1). \tag{5.4}$$

By definition, $\pi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a morphism of complex directed manifolds. Clearly, our construction is functorial, i.e., for every morphism of directed manifolds $\Phi : (X, V) \to (Y, W)$, there is a commutative diagram

$$\begin{tikzcd} (\widetilde{X}, \widetilde{V}) \arrow[r, "\pi"] \arrow[d, dashed, "\widetilde{\Phi}"'] & (X, V) \arrow[d, "\Phi"] \\ (\widetilde{Y}, \widetilde{W}) \arrow[r, "\pi"] & (Y, W), \end{tikzcd} \tag{5.5}$$

where the left vertical arrow is the meromorphic map $P(V) \dashrightarrow P(W)$ induced by the differential $\Phi_* : V \to \Phi^* W$ ($\widetilde{\Phi}$ is actually holomorphic if $\Phi_* : V \to \Phi^* W$ is injective).

5.B. Lifting of curves to the 1*-jet bundle*

Suppose that we are given a holomorphic curve $f : \Delta_R \to X$ parametrized by the disk Δ_R of centre 0 and radius R in the complex plane, and that f is a tangent curve of the directed manifold, i.e., $f'(t) \in V_{f(t)}$ for every $t \in \Delta_R$. If f is non-constant, there is a well-defined and unique tangent line $[f'(t)] \in P(V_{f(t)})$ for every t, even at stationary points, and the map

$$\widetilde{f} : \Delta_R \longrightarrow \widetilde{X}, \quad t \longmapsto \widetilde{f}(t) := (f(t), [f'(t)]) \tag{5.6}$$

is holomorphic; in fact, at a stationary point t_0, we can write

$$f'(t) = (t - t_0)^s u(t)$$

with $s \in \mathbb{N}^*$ and $u(t_0) \neq 0$, and we define the tangent line at t_0 to be $[u(t_0)]$, so that $\widetilde{f}(t) = (f(t), [u(t)])$ near t_0; even for $t = t_0$, we still denote $[f'(t_0)] = [u(t_0)]$ for simplicity of notation. By definition $f'(t) \in \mathcal{O}_{\widetilde{X}}(-1)_{\widetilde{f}(t)} = \mathbb{C}u(t)$, so the derivative f' defines a section

$$f' : T_{\Delta_R} \longrightarrow \widetilde{f}^* \mathcal{O}_{\widetilde{X}}(-1). \tag{5.7}$$

Moreover $\pi \circ \widetilde{f} = f$, and thus

$$\pi_* \widetilde{f}'(t) = f'(t) \in \mathbb{C}u(t) \Longrightarrow \widetilde{f}'(t) \in \widetilde{V}_{(f(t),u(t))} = \widetilde{V}_{\widetilde{f}(t)}$$

and we see that $\widetilde{f}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$. We say that $\widetilde{f}$ is the *canonical lifting* of f to $\widetilde{X}$. Conversely, if $g : \Delta_R \to \widetilde{X}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$, then by definition of $\widetilde{V}$ we see that $f = \pi \circ g$ is a tangent trajectory of (X, V) and that $g = \widetilde{f}$ (unless g is contained in a vertical fiber $P(V_x)$, in which case f is constant).

For any point $x_0 \in X$, there are local coordinates $(z_1, \ldots, z_n)$ on a neighborhood Ω of x_0 such that the fibers $(V_z)_{z \in \Omega}$ can be defined by linear equations

$$V_z = \Big\{\xi = \sum_{1 \leqslant j \leqslant n} \xi_j \frac{\partial}{\partial z_j}\,;\ \xi_j = \sum_{1 \leqslant k \leqslant r} a_{jk}(z)\xi_k \text{ for } j = r+1, \ldots, n\Big\}, \tag{5.8}$$

where (a_{jk}) is a holomorphic $(n-r) \times r$ matrix. It follows that a vector $\xi \in V_z$ is completely determined by its first r components $(\xi_1, \ldots, \xi_r)$, and the affine chart $\xi_j \neq 0$ of $P(V)_{\restriction \Omega}$ can be described by the coordinate system

$$\Big(z_1, \ldots, z_n; \frac{\xi_1}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, \frac{\xi_{j+1}}{\xi_j}, \ldots, \frac{\xi_r}{\xi_j}\Big). \tag{5.9}$$

Let $f \simeq (f_1, \ldots, f_n)$ be the components of f in the coordinates $(z_1, \ldots, z_n)$ (we suppose here R so small that $f(\Delta_R) \subset \Omega$). It should be observed that f is uniquely determined by its initial value x and by the first r components $(f_1, \ldots, f_r)$. Indeed, as $f'(t) \in V_{f(t)}$, we can recover the other components by integrating the system of ordinary differential equations

$$f_j'(t) = \sum_{1 \leqslant k \leqslant r} a_{jk}(f(t)) f_k'(t), \quad j > r, \tag{5.10}$$

on a neighborhood of 0, with initial data $f(0) = x$. We denote by $m = m(f, t_0)$ the *multiplicity* of f at any point $t_0 \in \Delta_R$, that is, $m(f, t_0)$ is the smallest integer $m \in \mathbb{N}^*$ such that $f_j^{(m)}(t_0) \neq 0$ for some j. By (5.10), we can always suppose $j \in \{1, \ldots, r\}$, for example $f_r^{(m)}(t_0) \neq 0$. Then $f'(t) = (t - t_0)^{m-1} u(t)$ with $u_r(t_0) \neq 0$, and the lifting $\widetilde{f}$ is described in the coordinates of the affine chart $\xi_r \neq 0$ of $P(V)_{\restriction \Omega}$ by

$$\widetilde{f} \simeq \Big(f_1, \ldots, f_n; \frac{f_1'}{f_r'}, \ldots, \frac{f_{r-1}'}{f_r'}\Big). \tag{5.11}$$

5.C. Curvature properties of the 1*-jet bundle*

We end this section with a few curvature computations. Assume that V is non-singular and equipped with a smooth Hermitian metric h. Denote by $\nabla_h = \nabla'_h + \nabla''_h$ the associated Chern connection and by $\Theta_{V,h} = \frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor. For every point $x_0 \in X$, there exists a "normalized" holomorphic frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ on a neighborhood of x_0, such that

$$\langle e_\lambda, e_\mu\rangle_h = \delta_{\lambda\mu} - \sum_{1\leqslant j,k\leqslant n} c_{jk\lambda\mu} z_j \overline{z}_k + O(|z|^3), \tag{5.12}$$

with respect to any holomorphic coordinate system $(z_1, \ldots, z_n)$ centered at x_0. A computation of $d'\langle e_\lambda, e_\mu\rangle_h = \langle\nabla'_h e_\lambda, e_\mu\rangle_h$ and $\nabla_h^2 e_\lambda = d''\nabla'_h e_\lambda$ then gives

$$\nabla'_h e_\lambda = -\sum_{j,k,\mu} c_{jk\lambda\mu}\overline{z}_k\, dz_j \otimes e_\mu + O(|z|^2),$$

$$\Theta_{V,h}(x_0) = \frac{i}{2\pi}\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, dz_j \wedge d\overline{z}_k \otimes e_\lambda^* \otimes e_\mu. \tag{5.13}$$

The above curvature tensor can also be viewed as a Hermitian form on $T_X \otimes V$. In fact, one associates with $\Theta_{V,h}$ the Hermitian form $\langle\Theta_{V,h}\rangle$ on $T_X \otimes V$ defined for all $(\zeta, v) \in T_X \times_X V$ by

$$\langle\Theta_{V,h}\rangle(\zeta \otimes v) = \sum_{1\leqslant j,k\leqslant n,\, 1\leqslant\lambda,\mu\leqslant r} c_{jk\lambda\mu}\zeta_j\overline{\zeta}_k v_\lambda \overline{v}_\mu. \tag{5.14}$$

Let h_1 be the Hermitian metric on the tautological line bundle $\mathscr{O}_{P(V)}(-1) \subset \pi^* V$ induced by the metric h of V. We compute the curvature $(1,1)$-form $\Theta_{h_1}(\mathscr{O}_{P(V)}(-1))$ at an arbitrary point $(x_0, [v_0]) \in P(V)$, in terms of $\Theta_{V,h}$. For simplicity, we suppose that the frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ has been chosen in such a way that $[e_r(x_0)] = [v_0] \in P(V)$ and $|v_0|_h = 1$. We get holomorphic local coordinates $(z_1, \ldots, z_n\,;\, \xi_1, \ldots, \xi_{r-1})$ on a neighborhood of $(x_0, [v_0])$ in $P(V)$ by assigning

$$(z_1, \ldots, z_n\,;\, \xi_1, \ldots, \xi_{r-1}) \longmapsto (z, [\xi_1 e_1(z) + \cdots + \xi_{r-1}e_{r-1}(z) + e_r(z)]) \in P(V).$$

Then the function

$$\eta(z, \xi) = \xi_1 e_1(z) + \cdots + \xi_{r-1}e_{r-1}(z) + e_r(z)$$

defines a holomorphic section of $\mathscr{O}_{P(V)}(-1)$ in a neighborhood of $(x_0, [v_0])$. By using the expansion (5.12) for h, we find

$$|\eta|^2_{h_1} = |\eta|^2_h = 1 + |\xi|^2 - \sum_{1\leqslant j,k\leqslant n} c_{jkrr} z_j \overline{z}_k + O((|z| + |\xi|)^3),$$

$$\begin{aligned} &\Theta_{h_1}(\mathscr{O}_{P(V)}(-1))_{(x_0,[v_0])} \\ &= -\frac{i}{2\pi}\partial\overline{\partial}\log|\eta|^2_{h_1} \\ &= \frac{i}{2\pi}\Big(\sum_{1\leqslant j,k\leqslant n} c_{jkrr}\, dz_j \wedge d\overline{z}_k - \sum_{1\leqslant\lambda\leqslant r-1} d\xi_\lambda \wedge d\overline{\xi}_\lambda\Big). \end{aligned} \tag{5.15}$$

6. Jets of curves and Semple jet bundles

6.A. Semple tower of non-singular directed varieties

Let X be a complex n-dimensional manifold. Following ideas of Green–Griffiths [GrGr80], we let $J_kX \to X$ be the bundle of k-jets of germs of parametrized curves in X, that is, the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leqslant j \leqslant k$, when computed in some local coordinate system of X near x. The projection map $J_kX \to X$ is simply $f \mapsto f(0)$. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, the elements f of any fiber J_kX_x, $x \in \Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$f = (f_1, \ldots, f_n) : (\mathbb{C}, 0) \to \Omega \subset \mathbb{C}^n,$$

and they are completely determined by their Taylor expansion of order k at $t = 0$

$$f(t) = x + t f'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1}).$$

In these coordinates, the fiber J_kX_x can thus be identified with the set of k-tuples of vectors $(\xi_1, \ldots, \xi_k) = (f'(0), \ldots, f^{(k)}(0)) \in (\mathbb{C}^n)^k$. It follows that J_kX is a holomorphic fiber bundle with typical fiber $(\mathbb{C}^n)^k$ over X (however, J_kX is not a vector bundle for $k \geqslant 2$, because of the non-linearity of coordinate changes; see formula (7.2) in Sect. 7).

According to the philosophy developed throughout this paper, we describe the concept of jet bundle in the general situation of complex directed manifolds. If X is equipped with a holomorphic subbundle $V \subset T_X$, we associate to V a k-jet bundle J_kV as follows, assuming V *non-singular* throughout Subsect. 6.A.

6.1. Definition. *Let (X, V) be a complex directed manifold. We define $J_kV \to X$ to be the bundle of k-jets of curves $f : (\mathbb{C}, 0) \to X$ which are tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of* 0*, together with the projection map $f \mapsto f(0)$ onto X.*

It is easy to check that $J_k V$ is actually a subbundle of $J_k X$. In fact, by using (5.8) and (5.10), we see that the fibers $J_k V_x$ are parametrized by

$$((f_1'(0), \ldots, f_r'(0)); (f_1''(0), \ldots, f_r''(0)); \ldots; (f_1^{(k)}(0), \ldots, f_r^{(k)}(0))) \in (\mathbb{C}^r)^k$$

for all $x \in \Omega$, and hence $J_k V$ is a locally trivial $(\mathbb{C}^r)^k$-subbundle of $J_k X$. Alternatively, we can pick a local holomorphic connection ∇ on V such that for any germs $w = \sum_{1 \leqslant j \leqslant n} w_j \frac{\partial}{\partial z_j} \in \mathscr{O}(T_{X,x})$ and $v = \sum_{1 \leqslant \lambda \leqslant r} v_\lambda e_\lambda \in \mathscr{O}(V)_x$ in a local trivializing frame $(e_1, \ldots, e_r)$ of $V_{\restriction \Omega}$ we have

$$\nabla_w v(x) = \sum_{1 \leqslant j \leqslant n,\, 1 \leqslant \lambda \leqslant r} w_j \frac{\partial v_\lambda}{\partial z_j} e_\lambda(x) + \sum_{1 \leqslant j \leqslant n,\, 1 \leqslant \lambda, \mu \leqslant r} \Gamma_{j\lambda}^\mu(x) w_j v_\lambda\, e_\mu(x).$$

We can of course take the frame obtained from (5.8) by lifting the vector fields $\partial/\partial z_1, \ldots, \partial/\partial z_r$, and the "trivial connection" given by the zero Christoffel symbols $\Gamma = 0$. One then obtains a trivialization $J^k V_{\restriction \Omega} \simeq V_{\restriction \Omega}^{\oplus k}$ by considering

$$J_k V_x \ni f \longmapsto (\xi_1, \xi_2, \ldots, \xi_k) = (\nabla f(0), \nabla^2 f(0), \ldots, \nabla^k f(0)) \in V_x^{\oplus k}$$

and computing inductively the successive derivatives $\nabla f(t) = f'(t)$ and $\nabla^s f(t)$ via

$$\begin{aligned} \nabla^s f &= (f^*\nabla)_{d/dt}(\nabla^{s-1} f) \\ &= \sum_{1 \leqslant \lambda \leqslant r} \frac{d}{dt}(\nabla^{s-1} f)_\lambda e_\lambda(f) + \sum_{1 \leqslant j \leqslant n,\, 1 \leqslant \lambda, \mu \leqslant r} \Gamma_{j\lambda}^\mu(f) f_j'(\nabla^{s-1} f)_\lambda e_\mu(f). \end{aligned}$$

This identification depends of course on the choice of ∇ and cannot be defined globally in general (unless we are in the rare situation where V has a global holomorphic connection). □

We now describe a convenient process for constructing "projectivized jet bundles", which will later appear as natural quotients of our jet bundles $J_k V$ (or rather, as suitable desingularized compactifications of the quotients). Such spaces have already been considered since a long time, at least in the special case $X = \mathbb{P}^2$, $V = T_{\mathbb{P}^2}$ (see Gherardelli [Ghe41], Semple [Sem54]), and they have been mostly used as a tool for establishing enumerative formulas dealing with the order of contact of plane curves (see [Coll88], [CoKe94]); the article [ASS97] is also concerned with such generalizations of jet bundles, as well as [LaTh96] by Laksov and Thorup.

We define inductively the *projectivized k-jet bundle* X_k (or *Semple k-jet bundle*) and the associated subbundle $V_k \subset T_{X_k}$ by

$$(6.2) \qquad (X_0, V_0) = (X, V), \quad (X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1}).$$

In other words, (X_k, V_k) is obtained from (X, V) by iterating k-times the lifting construction $(X, V) \mapsto (\widetilde{X}, \widetilde{V})$ described in Sect. 5. By (5.2)–(5.7), we find

$$\dim X_k = n + k(r-1), \quad \operatorname{rank} V_k = r, \tag{6.3}$$

together with exact sequences

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0, \tag{6.4}$$

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0, \tag{6.4'}$$

where π_k is the natural projection $\pi_k : X_k \to X_{k-1}$ and $(\pi_k)_*$ its differential. Formula (5.4) yields

$$\det(V_k) = \pi_k^* \det(V_{k-1}) \otimes \mathcal{O}_{X_k}(r-1). \tag{6.5}$$

Every non-constant tangent trajectory $f : \Delta_R \to X$ of (X, V) lifts to a well-defined and unique tangent trajectory $f_{[k]} : \Delta_R \to X_k$ of (X_k, V_k). Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]} : T_{\Delta_R} \longrightarrow f^*_{[k]} \mathcal{O}_{X_k}(-1). \tag{6.6}$$

In coordinates, one can compute $f_{[k]}$ in terms of its components in the various affine charts (5.9) occurring at each step: we get inductively

$$f_{[k]} = (F_1, \ldots, F_N), \quad f_{[k+1]} = \left(F_1, \ldots, F_N, \frac{F'_{s_1}}{F'_{s_r}}, \ldots, \frac{F'_{s_{r-1}}}{F'_{s_r}}\right), \tag{6.7}$$

where $N = n + k(r-1)$ and $\{s_1, \ldots, s_r\} \subset \{1, \ldots, N\}$. If $k \geqslant 1$, $\{s_1, \ldots, s_r\}$ contains the last $(r-1)$ indices of $\{1, \ldots, N\}$ corresponding to the "vertical" components of the projection $X_k \to X_{k-1}$, and in general, s_r is an index such that $m(F_{s_r}, 0) = m(f_{[k]}, 0)$, that is, F_{s_r} has the smallest vanishing order among all components F_s (s_r may be vertical or not, and the choice of $\{s_1, \ldots, s_r\}$ need not be unique).

By definition, there is a canonical injection $\mathcal{O}_{X_k}(-1) \hookrightarrow \pi_k^* V_{k-1}$, and a composition with the projection $(\pi_{k-1})_*$ (analogue for order $(k-1)$ of the arrow $(\pi_k)_*$ in sequence (6.4)) yields for any $k \geqslant 2$ a natural line bundle morphism

$$\mathcal{O}_{X_k}(-1) \hookrightarrow \pi_k^* V_{k-1} \xrightarrow{(\pi_k)^*(\pi_{k-1})_*} \pi_k^* \mathcal{O}_{X_{k-1}}(-1), \tag{6.8}$$

which admits precisely $D_k = P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k$ as its zero divisor (clearly, D_k is a hyperplane subbundle of X_k). Hence we find

$$\mathcal{O}_{X_k}(1) = \pi_k^* \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k). \tag{6.9}$$

Now, we consider the composition of projections

$$\pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : X_k \longrightarrow X_j. \tag{6.10}$$

Then $\pi_{0,k} : X_k \to X_0 = X$ is a locally trivial holomorphic fiber bundle over X, and the fibers $X_{k,x} = \pi_{0,k}^{-1}(x)$ are k-stage towers of $\mathbb{P}^{r-1}$-bundles. Since we have (in both directions) morphisms $(\mathbb{C}^r, T_{\mathbb{C}^r}) \leftrightarrow (X, V)$ of directed manifolds which are bijective on the level of bundle morphisms, the fibers are all isomorphic to a "universal" non-singular projective algebraic variety of dimension $k(r-1)$ which we will denote by $\mathscr{R}_{r,k}$; it is not hard to see that $\mathscr{R}_{r,k}$ is rational (as will indeed follow from the proof of Theorem 7.11 below).

6.B. Semple tower of singular directed varieties

Let (X, V) be a directed variety. We assume X non-singular, but here V is allowed to have singularities. We are going to give a natural definition of the Semple tower (X_k, V_k) in that case.

Let us take $X' = X \smallsetminus \mathrm{Sing}(V)$ and $V' = V_{\restriction X'}$. By Subsect. 6.A, we have a well-defined Semple tower (X'_k, V'_k) over the Zariski open set X'. We also have an "absolute" Semple tower (X^a_k, V^a_k) obtained from $(X^a_0, V^a_0) = (X, T_X)$, which is non-singular. The injection $V' \subset T_X$ induces by functoriality (*cf.* (5.5)) an injection

$$(X'_k, V'_k) \subset (X^a_k, V^a_k). \tag{6.11}$$

6.12. Definition. *Let (X, V) be a directed variety, with X non-singular. When $\mathrm{Sing}(V) \neq \emptyset$, we define X_k and V_k to be the respective closures of X'_k, V'_k associated with $X' = X \smallsetminus \mathrm{Sing}(V)$ and $V' = V_{\restriction X'}$, where the closure is taken in the non-singular absolute Semple tower (X^a_k, V^a_k) obtained from $(X^a_0, V^a_0) = (X, T_X)$.*

We leave the reader check that the following functoriality property still holds.

6.13. Fonctoriality. *If $\Phi : (X, V) \to (Y, W)$ is a morphism of directed varieties such that $\Phi_* : T_X \to \Phi^* T_Y$ is injective (i.e., Φ is an immersion), then there is a corresponding natural morphism $\Phi_{[k]} : (X_k, V_k) \to (Y_k, W_k)$ at the level of Semple bundles. If one merely assumes that the differential $\Phi_* : V \to \Phi^* W$ is non-zero, there is still a natural meromorphic map $\Phi_{[k]} : (X_k, V_k) \dashrightarrow (Y_k, W_k)$ for all $k \geqslant 0$.*

In case V is singular, the k-th stage X_k of the Semple tower will also be singular, but we can replace (X_k, V_k) by a suitable modification $(\widehat{X}_k, \widehat{V}_k)$ if we want to work with a non-singular model $\widehat{X}_k$ of X_k. The exceptional set of $\widehat{X}_k$

over X_k can be chosen to lie above $\mathrm{Sing}(V) \subset X$, and proceeding inductively with respect to k, we can also arrange the modifications in such a way that we get a tower structure $(\widehat{X}_{k+1}, \widehat{V}_{k+1}) \to (\widehat{X}_k, \widehat{V}_k)$; however, in general, it will not be possible to achieve that $\widehat{V}_k$ is a subbundle of $T_{\widehat{X}_k}$.

7. Jet differentials

7.A. Green–Griffiths jet differentials

We first introduce the concept of jet differentials in the sense of Green–Griffiths [GrGr80]. The goal is to provide an intrinsic geometric description of holomorphic differential equations that a germ of curve $f : (\mathbb{C}, 0) \to X$ may satisfy. In the sequel, we fix a directed manifold (X, V) and suppose implicitly that all germs of curves f are tangent to V.

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \longmapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \quad a_1 \in \mathbb{C}^*, \quad a_j \in \mathbb{C}, \quad j \geqslant 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ is a k-dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on $J_k V$. The action consists of reparametrizing k-jets of maps $f : (\mathbb{C}, 0) \to X$ by a biholomorphic change of parameter $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, that is, $(f, \varphi) \mapsto f \circ \varphi$. There is an exact sequence of groups

$$1 \longrightarrow \mathbb{G}'_k \longrightarrow \mathbb{G}_k \longrightarrow \mathbb{C}^* \longrightarrow 1,$$

where $\mathbb{G}_k \to \mathbb{C}^*$ is the obvious morphism $\varphi \mapsto \varphi'(0)$, and $\mathbb{G}'_k = [\mathbb{G}_k, \mathbb{G}_k]$ is the group of k-jets of biholomorphisms tangent to the identity. Moreover, the subgroup $\mathbb{H} \simeq \mathbb{C}^*$ of homotheties $\varphi(t) = \lambda t$ is a (non-normal) subgroup of $\mathbb{G}_k$, and we have a semi-direct decomposition $\mathbb{G}_k = \mathbb{G}'_k \ltimes \mathbb{H}$. The corresponding action on k-jets is described in coordinates by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

Following [GrGr80], we introduce the vector bundle $E^{\mathrm{GG}}_{k,m} V^* \to X$ whose fibers are complex valued polynomials $Q(f', f'', \ldots, f^{(k)})$ on the fibers of $J_k V$, of weighted degree m with respect to the $\mathbb{C}^*$ action defined by $\mathbb{H}$, that is, such that

$$Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)}) \tag{7.1}$$

for all $\lambda \in \mathbb{C}^*$ and $(f', f'', \ldots, f^{(k)}) \in J_k V$. Here we view $(f', f'', \ldots, f^{(k)})$ as indeterminates with components

$$((f'_1, \ldots, f'_r); (f''_1, \ldots, f''_r); \cdots ; (f^{(k)}_1, \ldots, f^{(k)}_r)) \in (\mathbb{C}^r)^k.$$

Notice that the concept of polynomial on the fibers of $J_k V$ makes sense, for all coordinate changes $z \mapsto w = \Psi(z)$ on X induce polynomial transition automorphisms on the fibers of $J_k V$, given by a formula

(7.2)

$$(\Psi\circ f)^{(j)} = \Psi'(f)\cdot f^{(j)} + \sum_{s=2}^{s=j} \sum_{j_1+j_2+\cdots+j_s=j} c_{j_1\cdots j_s}\Psi^{(s)}(f)\cdot(f^{(j_1)},\ldots,f^{(j_s)})$$

with suitable integer constants $c_{j_1\cdots j_s}$ (this is easily checked by induction on s). In the "absolute case" $V = T_X$, we simply write $E^{\rm GG}_{k,m}T^*_X = E^{\rm GG}_{k,m}$. If $V \subset V' \subset V^a := T_X$ are holomorphic subbundles, there are natural inclusions

$$J_kV \subset J_kV' \subset J_kV^a, \quad X_k \subset X'_k \subset X^a_k.$$

The restriction morphisms induce surjective arrows

$$E^{\rm GG}_{k,m}T^*_X \longrightarrow E^{\rm GG}_{k,m}V'^* \longrightarrow E^{\rm GG}_{k,m}V^*,$$

and in particular $E^{\rm GG}_{k,m}V^*$ can be seen as a quotient of $E^{\rm GG}_{k,m}T^*_X$. (The notation V^* is used here to make the contravariance property implicit from the notation). Another useful consequence of these inclusions is that one can extend the definition of J_kV and X_k to the case where V is an arbitrary linear space, simply by taking the closure of $J_kV_{X\smallsetminus {\rm Sing}(V)}$ and $X_{k|X\smallsetminus{\rm Sing}(V)}$ in the smooth bundles J_kX and X^a_k, respectively.

If $Q \in E^{\rm GG}_{k,m}V^*$ is decomposed into multihomogeneous components of multidegree $(\ell_1,\ell_2,\ldots,\ell_k)$ in $f', f'',\ldots,f^{(k)}$ (the decomposition is of course coordinate dependent), these multidegrees must satisfy the relation

$$\ell_1 + 2\ell_2 + \cdots + k\ell_k = m.$$

The bundle $E^{\rm GG}_{k,m}V^*$ will be called the *bundle of jet differentials of order k and weighted degree m. It is clear* from (7.2) that a coordinate change $f\mapsto \Psi\circ f$ transforms every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1}(f'')^{\ell_2}\cdots(f^{(k)})^{\ell_k}$ of partial weighted degree $|\ell|_s := \ell_1+2\ell_2+\cdots+s\ell_s$, $1\leqslant s\leqslant k$, into a polynomial $((\Psi\circ f)^{(\bullet)})^\ell$ in $(f',f'',\ldots,f^{(k)})$ whose non-zero monomials have the same partial weighted degree of order s if $\ell_{s+1}=\cdots=\ell_k=0$, and a larger or equal partial degree of order s otherwise. Hence, for each $s=1,\ldots,k$, we get a well-defined (i.e., coordinate invariant) decreasing filtration $F^\bullet_s$ on $E^{\rm GG}_{k,m}V^*$ as follows:

(7.3)

$$F^p_s(E^{\rm GG}_{k,m}V^*) = \left\{\begin{matrix} Q(f',f'',\ldots,f^{(k)}) \in E^{\rm GG}_{k,m}V^* \text{ involving} \\ \text{only monomials } (f^{(\bullet)})^\ell \text{ with } |\ell|_s \geqslant p\end{matrix}\right\}, \quad \forall p\in\mathbb{N}.$$

The graded terms $\mathrm{Gr}^p_{k-1}(E^{\mathrm{GG}}_{k,m}V^*)$ associated with the filtration $F^p_{k-1}(E^{\mathrm{GG}}_{k,m}V^*)$ are precisely the homogeneous polynomials $Q(f',\ldots,f^{(k)})$ whose monomials $(f^\bullet)^\ell$ all have partial weighted degree $|\ell|_{k-1}=p$ (hence their degree ℓ_k in $f^{(k)}$ is such that $m-p=k\ell_k$, and $\mathrm{Gr}^p_{k-1}(E^{\mathrm{GG}}_{k,m}V^*)=0$ unless $k|m-p$). The transition automorphisms of the graded bundle are induced by coordinate changes $f\mapsto\Psi\circ f$, and they are described by substituting the arguments of $Q(f',\ldots,f^{(k)})$ according to formula (7.2), namely $f^{(j)}\mapsto(\Psi\circ f)^{(j)}$ for $j<k$, and $f^{(k)}\mapsto\Psi'(f)\circ f^{(k)}$ for $j=k$ (when $j=k$, the other terms fall in the next stage F^{p+1}_{k-1} of the filtration). Therefore $f^{(k)}$ behaves as an element of $V\subset T_X$ under coordinate changes. We thus find

(7.4) $$G^{m-k\ell_k}_{k-1}(E^{\mathrm{GG}}_{k,m}V^*)=E^{\mathrm{GG}}_{k-1,m-k\ell_k}V^*\otimes S^{\ell_k}V^*.$$

Combining all filtrations $F^\bullet_s$ together, we find inductively a filtration $F^\bullet$ on $E^{\mathrm{GG}}_{k,m}V^*$ such that the graded terms are

(7.5) $$\mathrm{Gr}^\ell(E^{\mathrm{GG}}_{k,m}V^*)=S^{\ell_1}V^*\otimes S^{\ell_2}V^*\otimes\cdots\otimes S^{\ell_k}V^*,\quad \ell\in\mathbb{N}^k,\quad |\ell|_k=m.$$

The bundles $E^{\mathrm{GG}}_{k,m}V^*$ have other interesting properties. In fact,

$$E^{\mathrm{GG}}_{k,\bullet}V^*:=\bigoplus_{m\geqslant 0}E^{\mathrm{GG}}_{k,m}V^*$$

is in a natural way a bundle of graded algebras (the product is obtained simply by taking the product of polynomials). There are natural inclusions $E^{\mathrm{GG}}_{k,\bullet}V^*\subset E^{\mathrm{GG}}_{k+1,\bullet}V^*$ of algebras, and hence $E^{\mathrm{GG}}_{\infty,\bullet}V^*=\bigcup_{k\geqslant 0}E^{\mathrm{GG}}_{k,\bullet}V^*$ is also an algebra. Moreover, the sheaf of holomorphic sections $\mathcal{O}(E^{\mathrm{GG}}_{\infty,\bullet}V^*)$ admits a canonical derivation D^{GG} given by a collection of $\mathbb{C}$-linear maps

$$D^{\mathrm{GG}}:\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*)\longrightarrow\mathcal{O}(E^{\mathrm{GG}}_{k+1,m+1}V^*),$$

constructed in the following way. A holomorphic section of $E^{\mathrm{GG}}_{k,m}V^*$ on a coordinate open set $\Omega\subset X$ can be seen as a differential operator on the space of germs $f:(\mathbb{C},0)\to\Omega$ of the form

(7.6) $$Q(f)=\sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m}a_{\alpha_1\cdots\alpha_k}(f)\,(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$$

in which the coefficients $a_{\alpha_1\cdots\alpha_k}$ are holomorphic functions on Ω. Then $D^{\mathrm{GG}}Q$ is given by the formal derivative $(D^{\mathrm{GG}}Q)(f)(t)=d(Q(f))/dt$ with respect to the 1-dimensional parameter t in $f(t)$. For example, in dimension 2, if $Q\in H^0(\Omega,\mathcal{O}(E^{\mathrm{GG}}_{2,4}))$ is the section of weighted degree 4

$$Q(f)=a(f_1,f_2)\,f_1'^3f_2'+b(f_1,f_2)\,f_1''^2,$$

we find that $D^{\mathrm{GG}}Q \in H^0(\Omega, \mathscr{O}(E^{\mathrm{GG}}_{3,5}))$ is given by

$$\begin{aligned}(D^{\mathrm{GG}}Q)(f) = {}& \frac{\partial a}{\partial z_1}(f_1, f_2)\, f_1'^4 f_2' + \frac{\partial a}{\partial z_2}(f_1, f_2)\, f_1'^3 f_2'^2 \\ &+ \frac{\partial b}{\partial z_1}(f_1, f_2)\, f_1' f_1''^2 + \frac{\partial b}{\partial z_2}(f_1, f_2)\, f_2' f_1''^2 \\ &+ a(f_1, f_2)\,(3f_1'^2 f_1'' f_2' + f_1'^3 f_2'') + b(f_1, f_2)\, 2f_1'' f_1'''.\end{aligned}$$

Associated with the graded algebra bundle $E^{\mathrm{GG}}_{k,\bullet}V^*$, we have an analytic fiber bundle

(7.7) $$X_k^{\mathrm{GG}} := \mathrm{Proj}(E^{\mathrm{GG}}_{k,\bullet}V^*) = (J_k V \smallsetminus \{0\})/\mathbb{C}^*$$

over X, which has weighted projective spaces $\mathbb{P}(1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ as fibers (these weighted projective spaces are singular for $k > 1$, but they only have quotient singularities, see [Dol81]; here $J_k V \smallsetminus \{0\}$ is the set of non-constant jets of order k; we refer e.g. to Hartshorne's book [Har77] for a definition of the Proj functor). As such, it possesses a canonical sheaf $\mathscr{O}_{X_k^{\mathrm{GG}}}(1)$ such that $\mathscr{O}_{X_k^{\mathrm{GG}}}(m)$ is invertible when m is a multiple of $\mathrm{lcm}(1, 2, \ldots, k)$. Under the natural projection $\pi_k : X_k^{\mathrm{GG}} \to X$, the direct image $(\pi_k)_*\mathscr{O}_{X_k^{\mathrm{GG}}}(m)$ coincides with polynomials

(7.8) $$P(z\,;\xi_1, \ldots, \xi_k) = \sum_{\alpha_\ell \in \mathbb{N}^r,\, 1 \leqslant \ell \leqslant k} a_{\alpha_1 \cdots \alpha_k}(z)\, \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}$$

of weighted degree $|\alpha_1| + 2|\alpha_2| + \cdots + k|\alpha_k| = m$ on $J^k V$ with holomorphic coefficients; in other words, we obtain precisely the sheaf of sections of the bundle $E^{\mathrm{GG}}_{k,m}V^*$ of jet differentials of order k and degree m.

7.9. Proposition. *By construction, if $\pi_k : X_k^{\mathrm{GG}} \to X$ is the natural projection, we have the direct image formula*

$$(\pi_k)_*\mathscr{O}_{X_k^{\mathrm{GG}}}(m) = \mathscr{O}(E^{\mathrm{GG}}_{k,m}V^*)$$

for all k and m.

7.B. Invariant jet differentials

In the geometric context, we are not really interested in the bundles $(J_k V \smallsetminus \{0\})/\mathbb{C}^*$ themselves, but rather in their quotients $(J_k V \smallsetminus \{0\})/\mathbb{G}_k$ (would such nice complex space quotients exist!). We will see that the Semple bundle X_k constructed in Sect. 6 plays the role of such a quotient. First we introduce a canonical subalgebra of the bundle algebra $E^{\mathrm{GG}}_{k,\bullet}V^*$.

7.10. Definition. *We introduce a subbundle $E_{k,m}V^* \subset E^{\mathrm{GG}}_{k,m}V^*$, called the bundle of invariant jet differentials of order k and degree m, defined as follows: $E_{k,m}V^*$ is the set of polynomial differential operators $Q(f', f'', \ldots, f^{(k)})$ which are invariant under arbitrary changes of parametrization, i.e., for every $\varphi \in \mathbb{G}_k$*

$$Q((f \circ \varphi)', (f \circ \varphi)'', \ldots, (f \circ \varphi)^{(k)}) = \varphi'(0)^m Q(f', f'', \ldots, f^{(k)}).$$

Alternatively, $E_{k,m}V^* = (E^{\mathrm{GG}}_{k,m}V^*)^{\mathbb{G}'_k}$ is the set of invariants of $E^{\mathrm{GG}}_{k,m}V^*$ under the action of $\mathbb{G}'_k$. Clearly, $E_{\infty,\bullet}V^* = \bigcup_{k\geqslant 0}\bigoplus_{m\geqslant 0} E_{k,m}V^*$ is a subalgebra of $E^{\mathrm{GG}}_{k,m}V^*$ (observe however that this algebra is not invariant under the derivation D^{GG}, since e.g. $f''_j = D^{\mathrm{GG}} f_j$ is not an invariant polynomial). In addition to this, there are natural induced filtrations $F^p_s(E_{k,m}V^*) = E_{k,m}V^* \cap F^p_s(E^{\mathrm{GG}}_{k,m}V^*)$ (all locally trivial over X). These induced filtrations will play an important role later on.

7.11. Theorem. *Suppose that V has rank $r \geqslant 2$. Let $\pi_{0,k} : X_k \to X$ be the Semple jet bundles constructed in Sect. 6, and let $J_k V^{\mathrm{reg}}$ be the bundle of regular k-jets of maps $f : (\mathbb{C}, 0) \to X$, that is, jets f such that $f'(0) \neq 0$.*

(i) *The quotient $J_k V^{\mathrm{reg}}/\mathbb{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding $J_k V^{\mathrm{reg}}/\mathbb{G}_k \hookrightarrow X_k$ over X, which identifies $J_k V^{\mathrm{reg}}/\mathbb{G}_k$ with X^{reg}_k (thus X_k is a relative compactification of $J_k V^{\mathrm{reg}}/\mathbb{G}_k$ over X).*
(ii) *The direct image sheaf*

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(m) \simeq \mathcal{O}(E_{k,m}V^*)$$

can be identified with the sheaf of holomorphic sections of $E_{k,m}V^*$.
(iii) *For every $m > 0$, the relative base locus of the linear system $|\mathcal{O}_{X_k}(m)|$ is equal to the set X^{sing}_k of singular k-jets. Moreover, $\mathcal{O}_{X_k}(1)$ is relatively big over X.*

Proof. (i) For $f \in J_k V^{\mathrm{reg}}$, the lifting $\widetilde{f}$ is obtained by taking the derivative $(f, [f'])$ without any cancellation of zeroes in f', and hence we get a uniquely defined $(k-1)$-jet $\widetilde{f} : (\mathbb{C}, 0) \to \widetilde{X}$. Inductively, we get a well-defined $(k-j)$-jet $f_{[j]}$ in X_j, and the value $f_{[k]}(0)$ is independent of the choice of the representative f for the k-jet. As the lifting process commutes with reparametrization, i.e., $(f \circ \varphi)^{\sim} = \widetilde{f} \circ \varphi$ and more generally $(f \circ \varphi)_{[k]} = f_{[k]} \circ \varphi$, we conclude that there is a well-defined set-theoretic map

$$J_k V^{\mathrm{reg}}/\mathbb{G}_k \longrightarrow X^{\mathrm{reg}}_k, \quad f \mod \mathbb{G}_k \longmapsto f_{[k]}(0).$$

This map is better understood in coordinates as follows. Fix coordinates $(z_1, \ldots, z_n)$ near a point $x_0 \in X$, such that $V_{x_0} = \mathrm{Vect}(\partial/\partial z_1, \ldots, \partial/\partial z_r)$. Let $f = (f_1, \ldots, f_n)$

be a regular k-jet tangent to V. Then there exists $i \in \{1,2,\ldots,r\}$ such that $f_i'(0) \neq 0$, and there is a unique reparametrization $t = \varphi(\tau)$ such that $f\circ\varphi = g = (g_1, g_2, \ldots, g_n)$ with $g_i(\tau) = \tau$ (we just express the curve as a graph over the z_i-axis, by means of a change of parameter $\tau = f_i(t)$, i.e., $t = \varphi(\tau) = f_i^{-1}(\tau)$). Suppose $i = r$ for the simplicity of notation. The space X_k is a k-stage tower of $\mathbb{P}^{r-1}$-bundles. In the corresponding inhomogeneous coordinates on these $\mathbb{P}^{r-1}$'s, the point $f_{[k]}(0)$ is given by the collection of derivatives

$$((g_1'(0),\ldots,g_{r-1}'(0)); (g_1''(0),\ldots,g_{r-1}''(0)); \ldots; (g_1^{(k)}(0),\ldots,g_{r-1}^{(k)}(0))).$$

[Recall that the other components $(g_{r+1},\ldots,g_n)$ can be recovered from $(g_1,\ldots,g_r)$ by integrating the differential system (5.10).] Thus the map $J_kV^{\mathrm{reg}}/\mathbb{G}_k \to X_k$ is a bijection onto X_k^{reg}, and the fibers of these isomorphic bundles can be seen as unions of r affine charts $\simeq (\mathbb{C}^{r-1})^k$, associated with each choice of the axis z_i used to describe the curve as a graph. The change of parameter formula $\frac{d}{d\tau} = \frac{1}{f_r'(t)}\frac{d}{dt}$ expresses all derivatives $g_i^{(j)}(\tau) = d^jg_i/d\tau^j$ in terms of the derivatives $f_i^{(j)}(t) = d^jf_i/dt^j$

$$\begin{aligned}
(g_1',\ldots,g_{r-1}') &= \Big(\frac{f_1'}{f_r'},\ldots,\frac{f_{r-1}'}{f_r'}\Big);\\
(g_1'',\ldots,g_{r-1}'') &= \Big(\frac{f_1''f_r' - f_r''f_1'}{f_r'^3},\ldots,\frac{f_{r-1}''f_r' - f_r''f_{r-1}'}{f_r'^3}\Big);\ \ldots;\\
(g_1^{(k)},\ldots,g_{r-1}^{(k)}) &= \Big(\frac{f_1^{(k)}f_r' - f_r^{(k)}f_1'}{f_r'^{k+1}},\ldots,\frac{f_{r-1}^{(k)}f_r' - f_r^{(k)}f_{r-1}'}{f_r'^{k+1}}\Big)\\
&\qquad + (\text{order} < k).
\end{aligned}\tag{7.12}$$

Also, it is easy to check that $f_r'^{2k-1}g_i^{(k)}$ is an invariant polynomial in f', $f'',\ldots,f^{(k)}$ of total degree $(2k-1)$, i.e., a section of $E_{k,2k-1}$.

(ii) Since the bundles X_k and $E_{k,m}V^*$ are both locally trivial over X, it is sufficient to identify sections σ of $\mathcal{O}_{X_k}(m)$ over a fiber $X_{k,x} = \pi_{0,k}^{-1}(x)$ with the fiber $E_{k,m}V_x^*$, at any point $x\in X$. Let $f\in J_kV_x^{\mathrm{reg}}$ be a regular k-jet at x. By (6.6), the derivative $f_{[k-1]}'(0)$ defines an element of the fiber of $\mathcal{O}_{X_k}(-1)$ at $f_{[k]}(0)\in X_k$. Hence we get a well-defined complex valued operator

$$Q(f',f'',\ldots,f^{(k)}) = \sigma(f_{[k]}(0))\cdot(f_{[k-1]}'(0))^m.\tag{7.13}$$

Clearly, Q is holomorphic on $J_kV_x^{\mathrm{reg}}$ (by the holomorphicity of σ), and the $\mathbb{G}_k$-invariance condition of Definition 7.10 is satisfied since $f_{[k]}(0)$ does not depend on reparametrization and

$$(f\circ\varphi)_{[k-1]}'(0) = f_{[k-1]}'(0)\varphi'(0).$$

Now, $J_k V_x^{\mathrm{reg}}$ is the complement of a linear subspace of codimension r in $J_k V_x$, and hence Q extends holomorphically to all of $J_k V_x \simeq (\mathbb{C}^r)^k$ by Riemann's extension theorem (here we use the hypothesis $r \geqslant 2$; if $r = 1$, the situation is anyway not interesting since $X_k = X$ for all k). Thus Q admits an everywhere convergent power series

$$Q(f', f'', \dots, f^{(k)}) = \sum_{\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{N}^r} a_{\alpha_1 \cdots \alpha_k} (f')^{\alpha_1} (f'')^{\alpha_2} \cdots (f^{(k)})^{\alpha_k}.$$

The $\mathbb{G}_k$-invariance asserted in Definition 7.10 implies in particular that Q must be multihomogeneous in the sense of (7.1), and thus Q must be a polynomial. We conclude that $Q \in E_{k,m} V_x^*$, as desired.

Conversely, for all w in a neighborhood of any given point $w_0 \in X_{k,x}$, we can find a holomorphic family of germs $f_w : (\mathbb{C}, 0) \to X$ such that $(f_w)_{[k]}(0) = w$ and $(f_w)'_{[k-1]}(0) \neq 0$ (just take the projections to X of integral curves of (X_k, V_k) integrating a non-vanishing local holomorphic section of V_k near w_0). Then every $Q \in E_{k,m} V_x^*$ yields a holomorphic section σ of $\mathcal{O}_{X_k}(m)$ over the fiber $X_{k,x}$ by putting

$$\sigma(w) = Q(f_w', f_w'', \dots, f_w^{(k)})(0) \, ((f_w)'_{[k-1]}(0))^{-m}. \tag{7.14}$$

(iii) By what we saw in (i)–(ii), every section σ of $\mathcal{O}_{X_k}(m)$ over the fiber $X_{k,x}$ is given by a polynomial $Q \in E_{k,m} V_x^*$, and this polynomial can be expressed on the Zariski open chart $f_r' \neq 0$ of $X_{k,x}^{\mathrm{reg}}$ as

$$Q(f', f'', \dots, f^{(k)}) = f_r'^m \widehat{Q}(g', g'', \dots, g^{(k)}), \tag{7.15}$$

where $\widehat{Q}$ is a polynomial and g is the reparametrization of f such that $g_r(\tau) = \tau$. In fact $\widehat{Q}$ is obtained from Q by substituting $f_r' = 1$ and $f_r^{(j)} = 0$ for $j \geqslant 2$, and conversely Q can be recovered easily from $\widehat{Q}$ by using the substitutions (7.12).

In this context, the jet differentials $f \mapsto f_1', \dots, f \mapsto f_r'$ can be viewed as sections of $\mathcal{O}_{X_k}(1)$ on a neighborhood of the fiber $X_{k,x}$. Since these sections vanish exactly on X_k^{sing}, the relative base locus of $\mathcal{O}_{X_k}(m)$ is contained in X_k^{sing} for every $m > 0$. We see that $\mathcal{O}_{X_k}(1)$ is big by considering the sections of $\mathcal{O}_{X_k}(2k-1)$ associated with the polynomials $Q(f', \dots, f^{(k)}) = f_r'^{2k-1} g_i^{(j)}$, $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant k$; indeed, these sections separate all points in the open chart $f_r' \neq 0$ of $X_{k,x}^{\mathrm{reg}}$.

Now, we check that every section σ of $\mathcal{O}_{X_k}(m)$ over $X_{k,x}$ must vanish on $X_{k,x}^{\mathrm{sing}}$. Pick an arbitrary element $w \in X_k^{\mathrm{sing}}$ and a germ of curve $f : (\mathbb{C}, 0) \to X$ such that $f_{[k]}(0) = w$, $f'_{[k-1]}(0) \neq 0$ and $s = m(f, 0) \gg 0$ (such an f exists by [Dem95, Corollary 5.14]). There are local coordinates $(z_1, \dots, z_n)$ on X such that $f(t) = (f_1(t), \dots, f_n(t))$, where $f_r(t) = t^s$. Let Q, $\widehat{Q}$ be the polynomials associated with σ in these coordinates and let $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$ be a monomial occurring in Q, with $\alpha_j \in \mathbb{N}^r$, $|\alpha_j| = \ell_j$, $\ell_1 + 2\ell_2 + \cdots + k\ell_k = m$. Putting $\tau = t^s$, the curve $t \mapsto f(t)$ becomes a Puiseux expansion $\tau \mapsto g(\tau) = (g_1(\tau), \ldots, g_{r-1}(\tau), \tau)$ in which g_i is a power series in $\tau^{1/s}$, starting with exponents of τ at least equal to 1. The derivative $g^{(j)}(\tau)$ may involve negative powers of τ, but the exponent is always $\geqslant 1 + \frac{1}{s} - j$. Hence the Puiseux expansion of $\widehat{Q}(g', g'', \ldots, g^{(k)})$ can only involve powers of τ of exponent $\geqslant -\max_\ell((1-\frac{1}{s})\ell_2 + \cdots + (k-1-\frac{1}{s})\ell_k)$. Finally $f_r'(t) = st^{s-1} = s\tau^{1-1/s}$, and so the lowest exponent of τ in $Q(f', \ldots, f^{(k)})$ is at least equal to

$$\begin{aligned}
&\Big(1-\frac{1}{s}\Big)m - \max_\ell\Big(\Big(1-\frac{1}{s}\Big)\ell_2 + \cdots + \Big(k-1-\frac{1}{s}\Big)\ell_k\Big)\\
&\geqslant \min_\ell \Big(1-\frac{1}{s}\Big)\ell_1 + \Big(1-\frac{1}{s}\Big)\ell_2 + \cdots + \Big(1-\frac{k-1}{s}\Big)\ell_k,
\end{aligned}$$

where the minimum is taken over all monomials $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$, $|\alpha_j| = \ell_j$, occurring in Q. Choosing $s \geqslant k$, we already find that the minimal exponent is positive, and hence $Q(f', \ldots, f^{(k)})(0) = 0$, so that $\sigma(w) = 0$ by (7.14). $\square$

Theorem 7.11 (iii) shows that $\mathcal{O}_{X_k}(1)$ is never relatively ample over X for $k \geqslant 2$. In order to overcome this difficulty, we define for every $a_\bullet = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ a line bundle $\mathcal{O}_{X_k}(a_\bullet)$ on X_k such that

(7.16) $$\mathcal{O}_{X_k}(a_\bullet) = \pi_{1,k}^*\mathcal{O}_{X_1}(a_1) \otimes \pi_{2,k}^*\mathcal{O}_{X_2}(a_2) \otimes \cdots \otimes \mathcal{O}_{X_k}(a_k).$$

By (6.9), we have $\pi_{j,k}^*\mathcal{O}_{X_j}(1) = \mathcal{O}_{X_k}(1) \otimes \mathcal{O}_{X_k}(-\pi_{j+1,k}^*D_{j+1} - \cdots - D_k)$. Therefore by putting $D_j^* = \pi_{j+1,k}^*D_{j+1}$ for $1 \leqslant j \leqslant k-1$ and $D_k^* = 0$, we find an identity

(7.17) $$\begin{aligned}
&\mathcal{O}_{X_k}(a_\bullet) = \mathcal{O}_{X_k}(b_k) \otimes \mathcal{O}_{X_k}(-b_\bullet \cdot D^*),\\
&\text{where}\quad b_\bullet = (b_1, \ldots, b_k) \in \mathbb{Z}^k, \quad b_j = a_1 + \cdots + a_j,\\
&b_\bullet \cdot D^* = \sum_{1 \leqslant j \leqslant k-1} b_j\, \pi_{j+1,k}^* D_{j+1}.
\end{aligned}$$

In particular, if $b_\bullet \in \mathbb{N}^k$, i.e., $a_1 + \cdots + a_j \geqslant 0$, we get a morphism

(7.18) $$\mathcal{O}_{X_k}(a_\bullet) = \mathcal{O}_{X_k}(b_k) \otimes \mathcal{O}_{X_k}(-b_\bullet \cdot D^*) \longrightarrow \mathcal{O}_{X_k}(b_k).$$

The following result gives a sufficient condition for the relative nefness or ampleness of weighted jet bundles. Let us recall that a line bundle $L \to X$ on a projective variety X is said to be nef if $L \cdot C \geqslant 0$ for all irreducible algebraic curves $C \subset X$, and that a vector bundle $E \to X$ is said to be nef if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E) := P(E^*)$; any vector bundle generated by global sections is nef (*cf.* [DePS94] for more details).

7.19. Proposition. *Take a very ample line bundle A on X, and consider on X_k the line bundle*

$$L_k = \mathcal{O}_{X_k}(3^{k-1}, 3^{k-2}, \ldots, 3, 1) \otimes \pi_{k,0}^* A^{\otimes 3^k}$$

defined inductively by $L_0 = A$ and $L_k = \mathcal{O}_{X_k}(1) \otimes \pi_{k,k-1}^* L_{k-1}^{\otimes 3}$. Then $V_k^* \otimes L_k^{\otimes 2}$ is a nef vector bundle on X_k, which is in fact generated by its global sections, for all $k \geqslant 0$. Equivalently, for all $k \geqslant 1$,

$$\begin{aligned} L'_k &= \mathcal{O}_{X_k}(1) \otimes \pi_{k,k-1}^* L_{k-1}^{\otimes 2} \\ &= \mathcal{O}_{X_k}(2 \cdot 3^{k-2}, 2 \cdot 3^{k-3}, \ldots, 6, 2, 1) \otimes \pi_{k,0}^* A^{\otimes 2 \cdot 3^{k-1}} \end{aligned}$$

is nef over X_k and generated by sections.

The statement concerning L'_k is obtained by projectivizing the vector bundle $E = V_{k-1}^* \otimes L_{k-1}^{\otimes 2}$ on X_{k-1}, whose associated tautological line bundle is $\mathcal{O}_{\mathbb{P}(E)}(1) = L'_k$ on $\mathbb{P}(E) = P(V_{k-1}) = X_k$. Also one gets inductively that

(7.20) $\qquad L_k = \mathcal{O}_{\mathbb{P}(V_{k-1} \otimes L_{k-1}^{\otimes 2})}(1) \otimes \pi_{k,k-1}^* L_{k-1}$ *is very ample on X_k.*

Proof. Let $X \subset \mathbb{P}^N$ be the embedding provided by A, so that $A = \mathcal{O}_{\mathbb{P}^N}(1)_{\restriction X}$. As is well-known, if Q is the tautological quotient vector bundle on $\mathbb{P}^N$, the twisted cotangent bundle

$$T_{\mathbb{P}^N}^* \otimes \mathcal{O}_{\mathbb{P}^N}(2) = \Lambda^{N-1} Q$$

is nef; hence its quotients $T_X^* \otimes A^{\otimes 2}$ and $V_0^* \otimes L_0^{\otimes 2} = V^* \otimes A^{\otimes 2}$ are nef (any tensor power of nef vector bundles is nef, and so is any quotient). We now proceed by induction, assuming $V_{k-1}^* \otimes L_{k-1}^{\otimes 2}$ to be nef, $k \geqslant 1$. By taking the second wedge power of the central term in (6.4′), we get an injection

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow \Lambda^2(\pi_k^\star V_{k-1} \otimes \mathcal{O}_{X_k}(1)).$$

By dualizing and twisting with $\mathcal{O}_{X_{k-1}}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2}$, we find a surjection

$$\pi_k^\star \Lambda^2(V_{k-1}^\star \otimes L_{k-1}) \longrightarrow T_{X_k/X_{k-1}}^\star \otimes \mathcal{O}_{X_k}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2} \longrightarrow 0.$$

By the induction hypothesis, we see that $T_{X_k/X_{k-1}}^\star \otimes \mathcal{O}_{X_k}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2}$ is nef. Next, the dual of (6.4) yields an exact sequence

$$0 \longrightarrow \mathcal{O}_{X_k}(1) \longrightarrow V_k^\star \longrightarrow T_{X_k/X_{k-1}}^\star \longrightarrow 0.$$

As an extension of nef vector bundles is nef, the nefness of $V_k^* \otimes L_k^{\otimes 2}$ will follow if we check that $\mathcal{O}_{X_k}(1) \otimes L_k^{\otimes 2}$ and $T_{X_k/X_{k-1}}^\star \otimes L_k^{\otimes 2}$ are both nef.

However, this follows again from the induction hypothesis if we observe that the latter implies

$$L_k \geqslant \pi_{k,k-1}^* L_{k-1} \quad \text{and} \quad L_k \geqslant \mathscr{O}_{X_k}(1) \otimes \pi_{k,k-1}^* L_{k-1},$$

in the sense that $L'' \geqslant L'$ if the "difference" $L'' \otimes (L')^{-1}$ is nef. All statements remain valid if we replace "nef" with "generated by sections" in the above arguments. □

7.21. Corollary. *A $\mathbb{Q}$-line bundle $\mathscr{O}_{X_k}(a_\bullet) \otimes \pi_{k,0}^* A^{\otimes p}$ with $a_\bullet \in \mathbb{Q}^k$, $p \in \mathbb{Q}$, is nef, resp. ample, on X_k as soon as*

$$a_j \geqslant 3a_{j+1} \textit{ for } j = 1, 2, \ldots, k-2 \textit{ and } a_{k-1} \geqslant 2a_k \geqslant 0,\ p \geqslant 2\sum a_j,$$

resp.

$$a_j \geqslant 3a_{j+1} \textit{ for } j = 1, 2, \ldots, k-2 \textit{ and } a_{k-1} > 2a_k > 0,\ p > 2\sum a_j.$$

Proof. This follows easily by taking convex combinations of the L_j and L_j' and applying Proposition 7.19 and our observation (7.20). □

7.22. Remark. As in Green–Griffiths [GrGr80], Riemann's extension theorem shows that for every meromorphic map $\Phi : X \dashrightarrow Y$ there are well-defined pull-back morphisms

$$\Phi^* : H^0(Y, E_{k,m}^{\mathrm{GG}} T_Y^*) \longrightarrow H^0(X, E_{k,m}^{\mathrm{GG}} T_X^*),$$
$$\Phi^* : H^0(Y, E_{k,m} T_Y^*) \longrightarrow H^0(X, E_{k,m} T_X^*).$$

In particular the dimensions $h^0(X, E_{k,m}^{\mathrm{GG}} T_X^*)$ and $h^0(X, E_{k,m} T_X^*)$ are bimeromorphic invariants of X.

7.23. Remark. As $\mathbb{G}_k$ is a non-reductive group, it is not a priori clear that the graded ring $\mathscr{A}_{n,k,r} = \bigoplus_{m \in \mathbb{Z}} E_{k,m} V^\star$ (even pointwise over X) is finitely generated. This can be checked by hand ([Dem07a], [Dem07b]) for $n = 2$ and $k \leqslant 4$. Rousseau [Rou06] also checked the case $n = 3$, $k = 3$, and then Merker [Mer08], [Mer10] proved the finiteness for $n = 2, 3, 4$, $k \leqslant 4$ and $n = 2$, $k = 5$. Recently, Bérczi and Kirwan [BeKi12] made an attempt to prove the finiteness in full generality, but it appears that the general case is still unsettled.

7.C. Semple tower of a directed variety of general type

Even if (X, V) is of general type, it is not true that (X_k, V_k) is of general type: the fibers of $X_k \to X$ are towers of $\mathbb{P}^{r-1}$ bundles, and the canonical bundles of projective spaces are always negative! However, a twisted version holds true.

7.24. Lemma. *If (X, V) is of general type, there is a modification $(\widehat{X}, \widehat{V})$ such that all pairs $(\widehat{X}_k, \widehat{V}_k)$ of the associated Semple tower have a twisted canonical bundle $K_{\widehat{V}_k} \otimes \mathcal{O}_{\widehat{X}_k}(p)$ that is big when one multiplies $K_{\widehat{V}_k}$ by a suitable $\mathbb{Q}$-line bundle $\mathcal{O}_{\widehat{X}_k}(p)$, $p \in \mathbb{Q}_+$.*

Proof. First assume that V has no singularities. The exact sequences (6.4) and (6.4′) provide

$$K_{V_k} := \det V_k^* = \det(T^*_{X_k/X_{k-1}}) \otimes \mathcal{O}_{X_k}(1) = \pi^*_{k,k-1} K_{V_{k-1}} \otimes \mathcal{O}_{X_k}(-(r-1)),$$

where $r = \operatorname{rank}(V)$. Inductively we get

$$K_{V_k} = \pi^*_{k,0} K_V \otimes \mathcal{O}_{X_k}(-(r-1)1_\bullet), \quad 1_\bullet = (1, \ldots, 1) \in \mathbb{N}^k. \tag{7.25}$$

We know by [Dem95] that $\mathcal{O}_{X_k}(c_\bullet)$ is relatively ample over X when we take the special weight $c_\bullet = (2\,3^{k-2}, \ldots, 2\,3^{k-j-1}, \ldots, 6, 2, 1)$; hence

$$K_{V_k} \otimes \mathcal{O}_{X_k}((r-1)1_\bullet + \varepsilon c_\bullet) = \pi^*_{k,0} K_V \otimes \mathcal{O}_{X_k}(\varepsilon c_\bullet)$$

is big over X_k for any sufficiently small positive rational number $\varepsilon \in \mathbb{Q}^*_+$. Thanks to Formula (1.9), we can in fact replace the weight $(r-1)1_\bullet + \varepsilon c_\bullet$ by its total degree $p = (r-1)k + \varepsilon |c_\bullet| \in \mathbb{Q}_+$. The general case of a singular linear space follows by considering suitable "sufficiently high" modifications $\widehat{X}$ of X, the related directed structure $\widehat{V}$ on $\widehat{X}$, and embedding $(\widehat{X}_k, \widehat{V}_k)$ in the absolute Semple tower $(\widehat{X}^a_k, \widehat{V}^a_k)$ of $\widehat{X}$. We still have a well-defined morphism of rank 1 sheaves

$$\pi^*_{k,0} K_{\widehat{V}} \otimes \mathcal{O}_{\widehat{X}_k}(-(r-1)1_\bullet) \longrightarrow K_{\widehat{V}_k} \tag{7.26}$$

because the multiplier ideal sheaves involved at each stage behave according to the monotonicity principle applied to the projections $\pi^a_{k,k-1} : \widehat{X}^a_k \to \widehat{X}^a_{k-1}$ and their differentials $(\pi^a_{k,k-1})_*$, which yield well-defined transposed morphisms from the $(k-1)$-st stage to the k-th stage at the level of exterior differential forms. Our contention follows. □

7.D. Induced directed structure on a subvariety of a jet bundle

We discuss here the concept of induced directed structure for subvarieties of the Semple tower of a directed variety (X, V). This will be very important to proceed inductively with the base loci of jet differentials. Let Z be an irreducible algebraic subset of some k-jet bundle X_k over X, $k \geqslant 0$. We define the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure

$$W := \overline{T_{Z'} \cap V_k} \tag{7.27}$$

taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection $T_{Z'} \cap V_k$ has a constant rank and is a subbundle of $T_{Z'}$. Alternatively, we could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage (X_k^a, V_k^a) of the absolute Semple tower, which has the advantage of being non-singular. We say that (Z, W) is the *induced* directed variety structure; this concept of induced structure already applies of course in the case $k = 0$. If $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfies $f_{[k]}(\mathbb{C}) \subset Z$, then

$$\text{either} \quad f_{[k]}(\mathbb{C}) \subset Z_\alpha \quad \text{or} \quad f'_{[k]}(\mathbb{C}) \subset W, \tag{7.28}$$

where Z_α is one of the connected components of $Z \smallsetminus Z'$ and Z' is chosen as in (7.27); especially, if $W = 0$, we conclude that $f_{[k]}(\mathbb{C})$ must be contained in one of the Z_α's. In the sequel, we always consider such a subvariety Z of X_k as a directed pair (Z, W) by taking the induced structure described above. By (7.28), if we proceed by induction on $\dim Z$, the study of curves tangent to V that have an order k lifting $f_{[k]}(\mathbb{C}) \subset Z$ is reduced to the study of curves tangent to (Z, W). Let us first quote the following easy observation.

7.29. Observation. *For $k \geqslant 1$, let $Z \subsetneq X_k$ be an irreducible algebraic subset that projects onto X_{k-1}, i.e., $\pi_{k,k-1}(Z) = X_{k-1}$. Then the induced directed variety $(Z, W) \subset (X_k, V_k)$, satisfies*

$$1 \leqslant \operatorname{rank} W < r := \operatorname{rank}(V_k).$$

Proof. Take a Zariski open subset $Z' \subset Z_{\mathrm{reg}}$ such that $W' = T_{Z'} \cap V_k$ is a vector bundle over Z'. Since $X_k \to X_{k-1}$ is a $\mathbb{P}^{r-1}$-bundle, Z has codimension at most $(r-1)$ in X_k. Therefore $\operatorname{rank} W \geqslant \operatorname{rank} V_k - (r-1) \geqslant 1$. On the other hand, if we had $\operatorname{rank} W = \operatorname{rank} V_k$ generically, then $T_{Z'}$ would contain $V_{k|Z'}$, in particular it would contain all vertical directions $T_{X_k/X_{k-1}} \subset V_k$ that are tangent to the fibers of $X_k \to X_{k-1}$. By taking the flow along vertical vector fields, we would conclude that Z' is a union of fibers of $X_k \to X_{k-1}$ up to an algebraic set of smaller dimension, but this is excluded since Z projects onto X_{k-1} and $Z \subsetneq X_k$. □

We introduce the following definition that slightly differs from the one given in [Dem14]—it is actually more flexible and more general.

7.30. Definition. *For $k \geqslant 1$, let $Z \subset X_k$ be an irreducible algebraic subset of X_k and (Z, W) the induced directed structure. We assume moreover that $Z \not\subset D_k = P(T_{X_{k-1}/X_{k-2}})$ (and put $D_1 = \emptyset$ in what follows to avoid to have to single out the case $k = 1$). In this situation we say that (Z, W) is of general type modulo the Semple tower $X_\bullet \to X$ if either $W = 0$, or* rank $W \geqslant 1$ *and there exists $\ell \geqslant 0$ and $p \in \mathbb{Q}_{\geqslant 0}$ such that*

$$K^\bullet_{\widehat{W}_\ell} \otimes \mathcal{O}_{\widehat{Z}_\ell}(p) = K^\bullet_{\widehat{W}_\ell} \otimes \mathcal{O}_{\widehat{X}_{k+\ell}}(p)_{\restriction \widehat{Z}_\ell} \quad \textit{is big over } \widehat{Z}_\ell, \tag{7.31}$$

possibly after replacing (Z_ℓ, W_ℓ) by a suitable (non-singular) modification $(\widehat{Z}_\ell, \widehat{W}_\ell)$ obtained via an embedded resolution of singularities

$$\mu_\ell : (\widehat{Z}_\ell \subset \widehat{X}_{k+\ell}) \longrightarrow (Z_\ell \subset X_{k+\ell}).$$

Notice that by (7.26), Condition (7.31) is satisfied if we assume the existence of $p \geqslant 0$ such that

$$\pi^*_{k+\ell} K^\bullet_{\widehat{W}} \otimes \mathcal{O}_{\widehat{X}_{k+\ell}}(p)_{\restriction \widehat{Z}_\ell} \quad \text{is big over } \widehat{Z}_\ell \subset \widehat{X}_{k+\ell}. \tag{7.32}$$

In fact we infer (7.31) with $\mathcal{O}_{\widehat{Z}_\ell}(p)$ replaced by

$$\mathcal{O}_{\widehat{Z}_\ell}((0, \ldots, 0, p) + (r_W - 1)1_\bullet) \subset \mathcal{O}_{\widehat{Z}_\ell}(p + (r_W - 1)\ell).$$

As a consequence, (7.31) is satisfied if $K^\bullet_{\widehat{W}}$ is big (i.e., (Z, W) is of general type), or if $\mathcal{O}_{\widehat{Z}_\ell}(1)$ is big on some $\widehat{Z}_\ell$, $\ell \geqslant 1$, but (7.32) is weaker than these two bigness conditions, since we only require that some combination is big. Also, we have the following easy observation.

7.33. Proposition. *Let (X, V) be a projective directed variety. Assume that there exist $\ell \geqslant 1$ and a weight $a_\bullet \in \mathbb{Q}^\ell_{>0}$ such that $\mathcal{O}_{X_\ell}(a_\bullet)$ is ample over X_ℓ. Then every induced directed variety $(Z, W) \subset (X_k, V_k)$ is if general type modulo $X\bullet \to X$ for every $k \geqslant 1$.*

Proof. Corollary 7.21 shows that for $\ell' > \ell$ and a suitable weight $b_\bullet \in \mathbb{Q}^{\ell'}_{>0}$, the line bundle $\mathcal{O}_{X_{\ell'}}(b_\bullet)$ is relatively ample with respect to the projection $X_{\ell'} \to X_\ell$. From this, one deduces that the assumption also holds for arbitrary $\ell' > \ell$ and a suitable weight $a'_\bullet \in \mathbb{Q}^{\ell'}_{>0}$. Now, we use (7.32), in combination with Lemma 2.9 (b); in fact, $\mathcal{O}_{\widehat{X}_{k+\ell}}(1)_{\restriction \widehat{Z}_\ell}$ is big over $\widehat{Z}_\ell \subset \widehat{X}_{k+\ell}$ for $\ell \gg 1$, since we get many sections by pulling back the sections of $\mathcal{O}_{\widehat{X}_{\ell'}}(ma'_\bullet)$, $\ell' = k + \ell$, and by restricting them to $\widehat{Z}_\ell$. □

7.E. Relation between invariant and non-invariant jet differentials

We show here that the existence of $\mathbb{G}_k$-invariant global jet differentials is essentially equivalent to the existence of non-invariant ones. We have seen that the direct image sheaf

$$\pi_{k,0}\mathcal{O}_{X_k}(m) := E_{k,m}V^* \subset E^{\mathrm{GG}}_{k,m}V^*$$

has a stalk at point $x \in X$ that consists of algebraic differential operators $P(f_{[k]})$ acting on germs of k-jets $f : (\mathbb{C}, 0) \to (X, x)$ tangent to V, satisfying the invariance property

$$P((f \circ \varphi)_{[k]}) = (\varphi')^m P(f_{[k]}) \circ \varphi \tag{7.34}$$

whenever $\varphi \in \mathbb{G}_k$ is in the group of k-jets of biholomorphisms $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. The right action $J_k V \times \mathbb{G}_k \to J_k V$, $(f, \varphi) \mapsto f \circ \varphi$ induces a dual left action of $\mathbb{G}_k$ on $\bigoplus_{m'\leqslant m} E^{\mathrm{GG}}_{k,m'}V^*$ by

$$\begin{aligned}\mathbb{G}_k \times \bigoplus_{m'\leqslant m} E^{\mathrm{GG}}_{k,m'}V^*_x &\longrightarrow \bigoplus_{m'\leqslant m} E^{\mathrm{GG}}_{k,m'}V^*_x,\\ (\varphi, P) \qquad &\longmapsto \varphi^* P,\ (\varphi^* P)(f_{[k]}) = P((f \circ \varphi)_{[k]}),\end{aligned}\tag{7.35}$$

so that $\psi^*(\varphi^* P) = (\psi \circ \varphi)^* P$. Notice that for a global curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ and a global operator $P \in H^0(X, E^{\mathrm{GG}}_{k,m}V^* \otimes F)$ we have to modify a little bit the definition to consider germs of curves at points $t_0 \in \mathbb{C}$ other than 0. This leads to putting

$$\varphi^* P(f_{[k]})(t_0) = P((f \circ \varphi_{t_0})_{[k]})(0), \quad \text{where } \varphi_{t_0}(t) = t_0 + \varphi(t),\ t \in D(0, \varepsilon).$$

The $\mathbb{C}^*$-action on a homogeneous polynomial of degree m is simply $h^*_\lambda P = \lambda^m P$ for a dilation $h_\lambda(t) = \lambda t$, $\lambda \in \mathbb{C}^*$, but since $\varphi \circ h_\lambda \neq h_\lambda \circ \varphi$ in general, $\varphi^* P$ is no longer homogeneous when P is. However, by expanding the derivatives of $t \mapsto f(\varphi(t))$ at $t = 0$, we find an expression

$$(\varphi^* P)(f_{[k]}) = \sum_{\alpha\in\mathbb{N}^k,\, |\alpha|_w=m} \varphi^{(\alpha)}(0)\, P_\alpha(f_{[k]}), \tag{7.36}$$

where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, $\varphi^{(\alpha)} = (\varphi')^{\alpha_1}(\varphi'')^{\alpha_2} \cdots (\varphi^{(k)})^{\alpha_k}$, $|\alpha|_w = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k$ is the weighted degree and P_α is a homogeneous polynomial. Since any additional derivative taken on φ' means one less derivative left for f, it is easy to see that for P homogeneous of degree m we have

$$m_\alpha := \deg(P_\alpha) = m - (\alpha_2 + 2\alpha_3 + \cdots + (k-1)\alpha_k) = \alpha_1 + \alpha_2 + \cdots + \alpha_k,$$

in particular $\deg(P_\alpha) < m$ unless $\alpha = (m, 0, \ldots, 0)$, in which case $P_\alpha = P$. Let us fix a non-zero global section $P \in H^0(X, E^{\mathrm{GG}}_{k,m}V^* \otimes F)$ for some line bundle F over X, and pick a non-zero component P_{α_0} of minimum degree m_{α_0} in the decomposition of P (of course $m_{\alpha_0} = m$ if and only if P is already invariant). We have by construction

$$P_{\alpha_0} \in H^0(X, E^{\mathrm{GG}}_{k,m_{\alpha_0}} V^* \otimes F).$$

We claim that P_{α_0} is $\mathbb{G}_k$-invariant. Otherwise, there is for each α a decomposition

$$(\psi^* P_\alpha)(f_{[k]}) = \sum_{\beta \in \mathbb{N}^k,\, |\beta|_w = m_\alpha} \psi^{(\beta)}(0)\, P_{\alpha,\beta}(f_{[k]}), \tag{7.37}$$

and the non-invariance of P_{α_0} would yield some non-zero term P_{α_0,β_0} of degree

$$\deg(P_{\alpha_0,\beta_0}) < \deg(P_{\alpha_0}) \leqslant \deg(P) = m.$$

By replacing f with $f \circ \psi$ in (7.36) and plugging (7.37) into it, we would get an identity of the form

$$\begin{aligned}(\psi \circ \varphi)^* P(f_{[k]}) &= \sum_{\alpha \in \mathbb{N}^k} (\psi \circ \varphi)^{(\alpha)}(0)\, P_\alpha(f_{[k]}) \\ &= \sum_{\alpha,\beta \in \mathbb{N}^k} \varphi^{(\alpha)}(0)\psi^{(\beta)}(0)\, P_{\alpha,\beta}(f_{[k]}),\end{aligned}$$

but the term in the middle would have all components of degree $\geqslant m_{\alpha_0}$, while the term on the right possesses a component of degree $< m_{\alpha_0}$ for a sufficiently generic choice of φ and ψ, contradiction. Therefore, we have shown the existence of a non-zero invariant section

$$P_{\alpha_0} \in H^0(X, E_{k,m_{\alpha_0}} V^* \otimes F), \quad m_{\alpha_0} \leqslant m. \qquad \square$$

8. k-jet metrics with negative curvature

The goal of this section is to show that hyperbolicity is closely related to the existence of k-jet metrics with suitable negativity properties of the curvature. The connection between these properties is in fact a simple consequence of the Ahlfors–Schwarz lemma. Such ideas have been already developed long ago by Grauert–Reckziegel [GRe65], Kobayashi [Kob75] for 1-jet metrics (i.e., Finsler metrics on T_X) and by Cowen–Griffiths [CoGr76], Green–Griffiths [GrGr80] and Grauert [Gra89] for higher order jet metrics.

8.A. Definition of k-jet metrics

Even in the standard case $V = T_X$, the definition given below differs from that of [GrGr80], in which the k-jet metrics are not supposed to be $\mathbb{G}'_k$-invariant. We prefer to deal here with $\mathbb{G}'_k$-invariant objects, because they reflect better the intrinsic geometry. Grauert [Gra89] actually deals with $\mathbb{G}'_k$-invariant metrics, but he apparently does not take care of the way the quotient space $J_k^{\mathrm{reg}}V/\mathbb{G}_k$ can be compactified; also, his metrics are always induced by the Poincaré metric, and it is not at all clear whether these metrics have the expected curvature properties (see Problem 8.14 below). In the present situation, it is important to allow also Hermitian metrics possessing some singularities ("singular Hermitian metrics" in the sense of [Dem90b]).

8.1. Definition. *Let $L \to X$ be a holomorphic line bundle over a complex manifold X. We say that h is a singular metric on L if for any trivialization $L_{\restriction U} \simeq U \times \mathbb{C}$ of L, the metric is given by $|\xi|_h^2 = |\xi|^2 e^{-\varphi}$ for some real valued weight function $\varphi \in L^1_{\mathrm{loc}}(U)$. The curvature current of L is then defined to be the closed $(1,1)$-current $\Theta_{L,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi$, computed in the sense of distributions. We say that h admits a closed subset $\Sigma \subset X$ as its degeneration set if φ is locally bounded on $X \smallsetminus \Sigma$ and is unbounded on a neighborhood of any point of Σ.*

An especially useful situation is the case when the curvature of h is positive definite. By this, we mean that there exists a smooth positive definite Hermitian metric ω and a continuous positive function ε on X such that $\Theta_{L,h} \geqslant \varepsilon\omega$ in the sense of currents, and we write in this case $\Theta_{L,h} \gg 0$. We need the following basic fact (quite standard when X is projective algebraic); however we want to avoid any algebraicity assumption here, so as to cover potential applications to non-algebraic complex tori.

8.2. Proposition. *Let L be a holomorphic line bundle on a compact complex manifold X.*

(i) *L admits a singular Hermitian metric h with positive definite curvature current $\Theta_{L,h} \gg 0$ if and only if L is big. Now, define B_m to be the base locus of the linear system $|H^0(X, L^{\otimes m})|$ and let*

$$\Phi_m : X \smallsetminus B_m \longrightarrow \mathbb{P}^N$$

be the corresponding meromorphic map. Let Σ_m be the closed analytic set equal to the union of B_m and of the set of points $x \in X \smallsetminus B_m$ such that the fiber $\Phi_m^{-1}(\Phi_m(x))$ is positive dimensional.

(ii) *If $\Sigma_m \neq X$ and G is any line bundle, the base locus of $L^{\otimes k} \otimes G^{-1}$ is contained in Σ_m for k large. As a consequence, L admits a singular Hermitian metric h with degeneration set Σ_m and with $\Theta_{L,h}$ positive definite on X.*

(iii) *Conversely, if L admits a Hermitian metric h with degeneration set Σ and positive definite curvature current $\Theta_{L,h}$, there exists an integer $m > 0$ such that the base locus B_m is contained in Σ and $\Phi_m : X \smallsetminus \Sigma \to \mathbb{P}_m$ is an embedding.*

Proof. (i) is proved e.g. in [Dem90b], [Dem92], and (ii) and (iii) are well-known results in the basic theory of linear systems. □

We now come to the main definitions. By (6.6), every regular k-jet $f \in J_k V$ gives rise to an element $f'_{[k-1]}(0) \in \mathcal{O}_{X_k}(-1)$. Thus, measuring the "norm of k-jets" is the same as taking a Hermitian metric on $\mathcal{O}_{X_k}(-1)$.

8.3. Definition. *A smooth, (resp. continuous, resp. singular) k-jet metric on a complex directed manifold (X, V) is a Hermitian metric h_k on the line bundle $\mathcal{O}_{X_k}(-1)$ over X_k (i.e., a Finsler metric on the vector bundle V_{k-1} over X_{k-1}), such that the weight functions φ representing the metric are smooth (resp. continuous, L^1_{loc}). We let $\Sigma_{h_k} \subset X_k$ be the singularity set of the metric, i.e., the closed subset of points in a neighborhood of which the weight φ is not locally bounded.*

We will always assume here that the weight function φ is quasi-plurisubharmonic. Recall that a function φ is said to be quasi-plurisubharmonic if φ is locally the sum of a plurisubharmonic function and of a smooth function (so that in particular $\varphi \in L^1_{\mathrm{loc}}$). Then the curvature current

$$\Theta_{h_k^{-1}}(\mathcal{O}_{X_k}(1)) = \frac{i}{2\pi}\partial\overline{\partial}\varphi.$$

is well-defined as a current and is locally bounded from below by a negative $(1,1)$-form with constant coefficients.

8.4. Definition. *Let h_k be a k-jet metric on (X, V). We say that h_k has negative jet curvature (resp. negative total jet curvature) if $\Theta_{h_k}(\mathcal{O}_{X_k}(-1))$ is negative definite along the subbundle $V_k \subset T_{X_k}$ (resp. on all of T_{X_k}), i.e., if there is $\varepsilon > 0$ and a smooth Hermitian metric ω_k on T_{X_k} such that*

$$\langle \Theta_{h_k^{-1}}(\mathcal{O}_{X_k}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_k}, \quad \forall \xi \in V_k \subset T_{X_k} \quad (\textit{resp.}\ \forall \xi \in T_{X_k}).$$

(If the metric h_k is not smooth, we suppose that its weights φ are quasi-plurisubharmonic, and the curvature inequality is taken in the sense of distributions.)

It is important to observe that for $k \geqslant 2$ there cannot exist any smooth Hermitian metric h_k on $\mathcal{O}_{X_k}(1)$ with positive definite curvature along $T_{X_k/X}$, since $\mathcal{O}_{X_k}(1)$ is not relatively ample over X. However, it is relatively big, and Proposition 8.2 (i) shows that $\mathcal{O}_{X_k}(-1)$ admits a singular Hermitian metric with negative total jet curvature (whatever the singularities of the metric are) if and only if $\mathcal{O}_{X_k}(1)$ is big over X_k. It is therefore crucial to allow singularities in the metrics in Definition 8.4.

8.B. Special case of 1*-jet metrics*

A 1-jet metric h_1 on $\mathscr{O}_{X_1}(-1)$ is the same as a Finsler metric $N = \sqrt{h_1}$ on $V \subset T_X$. Assume until the end of this paragraph that h_1 is smooth. By the well-known Kodaira embedding theorem, the existence of a smooth metric h_1 such that $\Theta_{h_1^{-1}}(\mathscr{O}_{X_1}(1))$ is positive on all of T_{X_1} is equivalent to $\mathscr{O}_{X_1}(1)$ being ample, that is, V^* ample.

8.5. Remark. In the absolute case $V = T_X$, there are only few examples of varieties X such that T_X^* is ample, mainly quotients of the ball $\mathbb{B}_n \subset \mathbb{C}^n$ by a discrete cocompact group of automorphisms.

The 1-jet negativity condition considered in Definition 8.4 is much weaker. For example, if the Hermitian metric h_1 comes from a (smooth) Hermitian metric h on V, then formula (5.15) implies that h_1 has negative total jet curvature (i.e., $\Theta_{h_1^{-1}}(\mathscr{O}_{X_1}(1))$ is positive) if and only if $\langle\Theta_{V,h}\rangle(\zeta \otimes v) < 0$ for all $\zeta \in T_X \smallsetminus \{0\}$, $v \in V \smallsetminus \{0\}$, that is, if (V,h) is *negative in the sense of Griffiths*. On the other hand, $V_1 \subset T_{X_1}$ consists by definition of tangent vectors $\tau \in T_{X_1,(x,[v])}$ whose horizontal projection ${}^H\tau$ is proportional to v. Thus $\Theta_{h_1}(\mathscr{O}_{X_1}(-1))$ is negative definite on V_1 if and only if $\Theta_{V,h}$ satisfies the much weaker condition that the *holomorphic sectional curvature* $\langle\Theta_{V,h}\rangle(v \otimes v)$ is negative on every complex line. □

8.C. Vanishing theorem for invariant jet differentials

We now come back to the general situation of jets of arbitrary order k. Our first observation is the fact that the k-jet negativity property of the curvature becomes actually weaker and weaker as k increases.

8.6. Lemma. *Let (X,V) be a compact complex directed manifold. If (X,V) has a $(k-1)$-jet metric h_{k-1} with negative jet curvature, then there is a k-jet metric h_k with negative jet curvature such that $\Sigma_{h_k} \subset \pi_k^{-1}(\Sigma_{h_{k-1}}) \cup D_k$. (The same holds true for negative total jet curvature).*

Proof. Let ω_{k-1}, ω_k be given smooth Hermitian metrics on $T_{X_{k-1}}$ and T_{X_k}. The hypothesis implies

$$\langle\Theta_{h_{k-1}^{-1}}(\mathscr{O}_{X_{k-1}}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_{k-1}}, \quad \forall \xi \in V_{k-1}$$

for some constant $\varepsilon > 0$. On the other hand, as $\mathscr{O}_{X_k}(D_k)$ is relatively ample over X_{k-1} (D_k is a hyperplane section bundle), there exists a smooth metric $\widetilde{h}$ on $\mathscr{O}_{X_k}(D_k)$ such that

$$\langle\Theta_{\widetilde{h}}(\mathscr{O}_{X_k}(D_k))\rangle(\xi) \geqslant \delta|\xi|^2_{\omega_k} - C|(\pi_k)_*\xi|^2_{\omega_{k-1}}, \quad \forall \xi \in T_{X_k}$$

for some constants $\delta, C > 0$. Combining both inequalities (the second one being applied to $\xi \in V_k$ and the first one to $(\pi_k)_*\xi \in V_{k-1}$), we get

$$\begin{aligned}&\langle\Theta_{(\pi_k^* h_{k-1})^{-p}\widetilde{h}}(\pi_k^*\mathscr{O}_{X_{k-1}}(p)\otimes\mathscr{O}_{X_k}(D_k))\rangle(\xi)\\ &\geqslant \delta|\xi|^2_{\omega_k} + (p\varepsilon - C)|(\pi_k)_*\xi|^2_{\omega_{k-1}}, \qquad \forall\xi\in V_k.\end{aligned}$$

Hence, for p large enough, $(\pi_k^* h_{k-1})^{-p}\widetilde{h}$ has positive definite curvature along V_k. Now, by (6.9), there is a sheaf injection

$$\mathscr{O}_{X_k}(-p) = \pi_k^*\mathscr{O}_{X_{k-1}}(-p)\otimes\mathscr{O}_{X_k}(-pD_k) \hookrightarrow (\pi_k^*\mathscr{O}_{X_{k-1}}(p)\otimes\mathscr{O}_{X_k}(D_k))^{-1}$$

obtained by twisting with $\mathscr{O}_{X_k}((p-1)D_k)$. Therefore

$$h_k := ((\pi_k^* h_{k-1})^{-p}\widetilde{h})^{-1/p} = (\pi_k^* h_{k-1})\widetilde{h}^{-1/p}$$

induces a singular metric on $\mathscr{O}_{X_k}(1)$ in which an additional degeneration divisor $p^{-1}(p-1)D_k$ appears. Hence we get $\Sigma_{h_k} = \pi_k^{-1}\Sigma_{h_{k-1}} \cup D_k$ and

$$\Theta_{h_k^{-1}}(\mathscr{O}_{X_k}(1)) = \frac{1}{p}\Theta_{(\pi_k^* h_{k-1})^{-p}\widetilde{h}} + \frac{p-1}{p}[D_k]$$

is positive definite along V_k. The same proof works in the case of negative total jet curvature. □

One of the main motivations for the introduction of k-jets metrics is the following list of algebraic sufficient conditions.

8.7. Algebraic sufficient conditions. We suppose here that X is projective algebraic, and we make one of the additional assumptions (i), (ii) or (iii) below.
(i) Assume that there exist integers $k, m > 0$ and $b_\bullet \in \mathbb{N}^k$ such that the line bundle $L := \mathscr{O}_{X_k}(m) \otimes \mathscr{O}_{X_k}(-b_\bullet \cdot D^*)$ is ample over X_k. Then there is a smooth Hermitian metric h_L on L with positive definite curvature on X_k. By means of the morphism $\mu : \mathscr{O}_{X_k}(-m) \to L^{-1}$, we get an induced metric $h_k = (\mu^* h_L^{-1})^{1/m}$ on $\mathscr{O}_{X_k}(-1)$ which is degenerate on the support of the zero divisor $\operatorname{div}(\mu) = b_\bullet \cdot D^*$. Hence $\Sigma_{h_k} = \operatorname{Supp}(b_\bullet \cdot D^*) \subset X_k^{\mathrm{sing}}$ and

$$\Theta_{h_k^{-1}}(\mathscr{O}_{X_k}(1)) = \frac{1}{m}\Theta_{h_L}(L) + \frac{1}{m}[b_\bullet \cdot D^*] \geqslant \frac{1}{m}\Theta_{h_L}(L) > 0.$$

In particular h_k has negative total jet curvature.
(ii) Assume more generally that there exist integers $k, m > 0$ and an ample line bundle A on X such that $H^0(X_k, \mathscr{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1})$ has non-zero sections $\sigma_1, \dots, \sigma_N$. Let $Z \subset X_k$ be the base locus of these sections; necessarily

$Z \supset X_k^{\mathrm{sing}}$ by Theorem 7.11 (iii). By taking a smooth metric h_A with positive curvature on A, we get a singular metric h'_k on $\mathcal{O}_{X_k}(-1)$ such that

$$h'_k(\xi) = \Big(\sum_{1 \leqslant j \leqslant N} |\sigma_j(w) \cdot \xi^m|^2_{h_A^{-1}} \Big)^{1/m}, \quad w \in X_k, \quad \xi \in \mathcal{O}_{X_k}(-1)_w.$$

Then $\Sigma_{h'_k} = Z$, and by computing $\frac{i}{2\pi}\partial\overline{\partial} \log h'_k(\xi)$ we obtain

$$\Theta_{h'^{-1}_k}(\mathcal{O}_{X_k}(1)) \geqslant \frac{1}{m} \pi^*_{0,k} \Theta_A.$$

By (7.17) and an induction on k, there exists a weight $b_\bullet \in \mathbb{Q}^k_+$ such that $\mathcal{O}_{X_k}(1) \otimes \mathcal{O}_{X_k}(-b_\bullet \cdot D^*)$ is relatively ample over X. Hence

$$L = \mathcal{O}_{X_k}(1) \otimes \mathcal{O}_{X_k}(-b_\bullet \cdot D^*) \otimes \pi^*_{0,k} A^{\otimes p}$$

is ample on X for $p \gg 0$. The arguments used in (i) show that there is a k-jet metric h''_k on $\mathcal{O}_{X_k}(-1)$ with $\Sigma_{h''_k} = \mathrm{Supp}(b_\bullet \cdot D^*) = X_k^{\mathrm{sing}}$ and

$$\Theta_{h''^{-1}_k}(\mathcal{O}_{X_k}(1)) = \Theta_L + [b_\bullet \cdot D^*] - p\, \pi^*_{0,k} \Theta_A,$$

where Θ_L is positive definite on X_k. The metric $h_k = (h'^{mp}_k h''_k)^{1/(mp+1)}$ then satisfies $\Sigma_{h_k} = \Sigma_{h'_k} = Z$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{X_k}(1)) \geqslant \frac{1}{mp+1} \Theta_L > 0.$$

(iii) If $E_{k,m}V^*$ is ample, there is an ample line bundle A and a sufficiently high symmetric power such that $S^p(E_{k,m}V^*) \otimes A^{-1}$ is generated by sections. These sections can be viewed as sections of $\mathcal{O}_{X_k}(mp) \otimes \pi^*_{0,k} A^{-1}$ over X_k, and their base locus is exactly $Z = X_k^{\mathrm{sing}}$ by Theorem 7.11 (iii). Hence the k-jet metric h_k constructed in (ii) has negative total jet curvature and satisfies $\Sigma_{h_k} = X_k^{\mathrm{sing}}$.

An important fact, first observed by [GRe65] for 1-jet metrics and by [GrGr80] in the higher order case, is that k-jet negativity implies hyperbolicity. In particular, the existence of enough global jet differentials implies hyperbolicity.

8.8. Theorem. *Let (X, V) be a compact complex directed manifold. If (X, V) has a k-jet metric h_k with negative jet curvature, then every entire curve $f : \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$. In particular, if $\Sigma_{h_k} \subset X_k^{\mathrm{sing}}$, then (X, V) is hyperbolic.*

Proof. The main idea is to use the Ahlfors–Schwarz lemma, following the approach of [GrGr80]. However we will give here all necessary details because our setting is slightly different. Assume that there is a k-jet metric h_k as in the hypotheses of Theorem 8.8. Let ω_k be a smooth Hermitian metric on T_{X_k}. By hypothesis, there exists $\varepsilon > 0$ such that

$$\langle \Theta_{h_k^{-1}}(\mathcal{O}_{X_k}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_k}, \qquad \forall \xi \in V_k.$$

Moreover, by (6.4), $(\pi_k)_*$ maps V_k continuously to $\mathcal{O}_{X_k}(-1)$ and the weight e^φ of h_k is locally bounded from above. Hence there is a constant $C > 0$ such that

$$|(\pi_k)_*\xi|^2_{h_k} \leqslant C|\xi|^2_{\omega_k}, \qquad \forall \xi \in V_k.$$

Combining these inequalities, we find

$$\langle \Theta_{h_k^{-1}}(\mathcal{O}_{X_k}(1))\rangle(\xi) \geqslant \frac{\varepsilon}{C}|(\pi_k)_*\xi|^2_{h_k}, \qquad \forall \xi \in V_k.$$

Now, let $f : \Delta_R \to X$ be a non-constant holomorphic map tangent to V on the disk Δ_R. We use the line bundle morphism (6.6)

$$F = f'_{[k-1]} : T_{\Delta_R} \longrightarrow f^*_{[k]}\mathcal{O}_{X_k}(-1)$$

to obtain a pull-back metric

$$\gamma = \gamma_0(t)\, dt \otimes d\bar{t} = F^* h_k \quad \text{on } T_{\Delta_R}.$$

If $f_{[k]}(\Delta_R) \subset \Sigma_{h_k}$ then $\gamma \equiv 0$. Otherwise, $F(t)$ has isolated zeroes at all singular points of $f_{[k-1]}$ and so $\gamma(t)$ vanishes only at these points and at points of the degeneration set $(f_{[k]})^{-1}(\Sigma_{h_k})$ which is a polar set in Δ_R. At other points, the Gaussian curvature of γ satisfies

$$\begin{aligned}\frac{i\,\partial\bar\partial \log \gamma_0(t)}{\gamma(t)} &= \frac{-2\pi\,(f_{[k]})^*\Theta_{h_k}(\mathcal{O}_{X_k}(-1))}{F^*h_k}\\ &= \frac{\langle \Theta_{h_k^{-1}}(\mathcal{O}_{X_k}(1))\rangle(f'_{[k]}(t))}{|f'_{[k-1]}(t)|^2_{h_k}} \geqslant \frac{\varepsilon}{C},\end{aligned}$$

since $f'_{[k-1]}(t) = (\pi_k)_* f'_{[k]}(t)$. The Ahlfors–Schwarz lemma 4.2 implies that γ can be compared with the Poincaré metric as follows:

$$\gamma(t) \leqslant \frac{2C}{\varepsilon}\frac{R^{-2}|dt|^2}{(1-|t|^2/R^2)^2} \implies |f'_{[k-1]}(t)|^2_{h_k} \leqslant \frac{2C}{\varepsilon}\frac{R^{-2}}{(1-|t|^2/R^2)^2}.$$

If $f : \mathbb{C} \to X$ is an entire curve tangent to V such that $f_{[k]}(\mathbb{C}) \not\subset \Sigma_{h_k}$, the above estimate implies as $R \to +\infty$ that $f_{[k-1]}$ must be a constant, and hence also f. Now, if $\Sigma_{h_k} \subset X_k^{\mathrm{sing}}$, the inclusion $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$ implies $f'(t) = 0$ at every point. Therefore f is a constant and (X, V) is hyperbolic. □

Combining Theorem 8.8 with 8.7 (ii) and (iii), we get the following consequences.

8.9. Vanishing theorem. *Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that $H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* L^{-1}) \simeq H^0(X, E_{k,m}V^* \otimes L^{-1})$ has non-zero sections $\sigma_1, \ldots, \sigma_N$. Let $Z \subset X_k$ be the base locus of these sections. Then every entire curve $f : \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k$-invariant polynomial differential operator P with values in L^{-1}, every entire curve f must satisfy the algebraic differential equation $P(f_{[k]}) = 0$.*

8.10. Corollary. *Let (X, V) be a compact complex directed manifold. If $E_{k,m}V^*$ is ample for some positive integers k, m, then (X, V) is hyperbolic.*

8.11. Remark. Green and Griffiths [GrGr80] stated that Theorem 8.9 is even true for sections $\sigma_j \in H^0(X, E_{k,m}^{\mathrm{GG}}(V^*) \otimes L^{-1})$, in the special case $V = T_X$ they consider. This is proved below in Subsect. 8.D; the reader is also referred to Siu and Yeung [SiYe97] for a proof based on a use of Nevanlinna theory and the logarithmic derivative lemma (the original proof given in [GrGr80] does not seem to be complete, as it relies on an unsettled pointwise version of the Ahlfors–Schwarz lemma for general jet differentials); other proofs seem to have been circulating in the literature in the last years. Let us first give a very short proof in the case where f is supposed to have a bounded derivative (thanks to the Brody criterion, this is enough if one is merely interested in proving hyperbolicity; thus Corollary 8.10 will be valid with $E_{k,m}^{\mathrm{GG}}V^*$ in place of $E_{k,m}V^*$). In fact, if f' is bounded, one can apply the Cauchy inequalities to all components f_j of f with respect to a finite collection of coordinate patches covering X. As f' is bounded, we can do this on sufficiently small discs $D(t, \delta) \subset \mathbb{C}$ of constant radius $\delta > 0$. Therefore all derivatives f', f'', $\ldots$, $f^{(k)}$ are bounded. From this we conclude that $\sigma_j(f)$ is a bounded section of $f^* L^{-1}$. Its norm $|\sigma_j(f)|_{L^{-1}}$ (with respect to any positively curved metric $|\ |_L$ on L) is a bounded subharmonic function, which is moreover strictly subharmonic at all points where $f' \neq 0$ and $\sigma_j(f) \neq 0$. This is a contradiction unless f is constant or $\sigma_j(f) \equiv 0$.

The above results justify the following definition and problems.

8.12. Definition. *We say that X, resp. (X, V), has non-degenerate negative k-jet curvature if there exists a k-jet metric h_k on $\mathcal{O}_{X_k}(-1)$ with negative jet curvature such that $\Sigma_{h_k} \subset X_k^{\mathrm{sing}}$.*

8.13. Conjecture. Let (X, V) be a compact directed manifold. Then (X, V) is hyperbolic if and only if (X, V) has non-degenerate negative k-jet curvature for k large enough.

This is probably a hard problem. In fact, it is shown in [Dem97, Section 8], that the smallest admissible integer k must depend on the geometry of X and need not be uniformly bounded as soon as $\dim X \geqslant 2$ (even in the absolute case $V = T_X$). On the other hand, if (X, V) is hyperbolic, we get for each integer $k \geqslant 1$ a generalized Kobayashi–Royden metric $\mathbf{k}_{(X_{k-1}, V_{k-1})}$ on V_{k-1} (see Definition 2.1), which can be also viewed as a k-jet metric h_k on $\mathcal{O}_{X_k}(-1)$; we will call it the *Grauert k-jet metric* of (X, V), although it formally differs from the jet metric considered in [Gra89] (see also [DGr91]). By looking at the projection $\pi_k : (X_k, V_k) \to (X_{k-1}, V_{k-1})$, we see that the sequence h_k is monotonic, namely $\pi_k^* h_k \leqslant h_{k+1}$ for every k. If (X, V) is hyperbolic, then h_1 is non-degenerate and therefore by monotonicity $\Sigma_{h_k} \subset X_k^{\mathrm{sing}}$ for $k \geqslant 1$. Conversely, if the Grauert metric satisfies $\Sigma_{h_k} \subset X_k^{\mathrm{sing}}$, it is easy to see that (X, V) is hyperbolic. The following problem is thus especially meaningful.

8.14. Problem. Estimate the k-jet curvature $\Theta_{h_k^{-1}}(\mathcal{O}_{X_k}(1))$ of the Grauert metric h_k on (X_k, V_k) as k tends to $+\infty$.

8.D. Vanishing theorem for non-invariant jet differentials

As an application of the arguments developed in Subsect. 7.E, we indicate here a proof of the basic vanishing theorem for non-invariant jet differentials. This version has been first proved in full generality by Siu [Siu97] (*cf.* also [Dem97]), with a different and more involved technique based on Nevanlinna theory and the logarithmic derivative lemma.

8.15. Theorem. *Let (X, V) be a projective directed and A an ample divisor on X. Then one has $P(f\,;\, f', f'', \ldots, f^{(k)}) = 0$ for every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ and every global section $P \in H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-A))$.*

Sketch of proof. In general, we know by Theorem 8.9 that the result is true when P is invariant, i.e., for $P \in H^0(X, E_{k,m} V^* \otimes \mathcal{O}(-A))$. Now, we prove Theorem 8.15 by induction on k and m (simultaneously for all directed varieties). Let $Z \subset X_k$ be the base locus of all polynomials $Q \in H^0(X, E_{k,m'}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-A))$ with $m' < m$. A priori, this defines merely an algebraic set in the Green–Griffiths bundle $X_k^{\mathrm{GG}} = (J_k V \smallsetminus \{0\})/\mathbb{C}^*$, but since the global polynomials $\varphi^* Q$ also enter the game, we know that the base locus is $\mathbb{G}_k$-invariant, and thus descends to X_k. Let $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$. By the induction hypothesis, we know that $f_{[k]}(\mathbb{C}) \subset Z$. Therefore f can also be viewed as a entire curve drawn in the directed variety (Z, W) induced by (X_k, V_k). By (7.36), we have a decomposition

$$(\varphi^* P)(g_{[k]}) = \sum_{\alpha\in\mathbb{N}^k,\, |\alpha|_w=m} \varphi^{(\alpha)}(0)\ P_\alpha(g_{[k]}),$$
$$\text{with } \deg(P_\alpha) < \deg(P) \text{ for } \alpha \neq (m, 0, \ldots, 0),$$

and since $P_\alpha(g_{[k]}) = 0$ for all germs of curves g of (Z, W) when $\alpha \neq (m, 0, \ldots, 0)$, we conclude that P defines an invariant jet differential when it is restricted to (Z, W), in other words it still defines a section of

$$H^0(Z, (\mathscr{O}_{X_k}(m) \otimes \pi^*_{k,0}\mathscr{O}_X(-A))_{\restriction Z}).$$

We can then apply the Ahlfors–Schwarz lemma in the way we did it in Subsect. 8.C to conclude that $P(f_{[k]}) = 0$. □

9. Morse inequalities and the Green–Griffiths–Lang conjecture

The goal of this section is to study the existence and properties of entire curves $f : \mathbb{C} \to X$ drawn in a complex irreducible n-dimensional variety X, and more specifically to show that they must satisfy certain global algebraic or differential equations as soon as X is projective of general type. By means of holomorphic Morse inequalities and a probabilistic analysis of the cohomology of jet spaces, it is possible to prove a significant step of the generalized Green–Griffiths–Lang conjecture. The use of holomorphic Morse inequalities was first suggested in [Dem07a], and then carried out in an algebraic context by S. Diverio in his PhD work ([Div08], [Div09]). The general more analytic and more powerful results presented here first appeared in [Dem11], [Dem12].

9.A. Introduction

Let (X, V) be a directed variety. By definition, proving the algebraic degeneracy of an entire curve $f ; (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ means finding a non-zero polynomial P on X such that $P(f) = 0$. As already explained in Sect. 8, all known methods of proof are based on establishing first the existence of certain algebraic differential equations $P(f\,;\ f', f'', \ldots, f^{(k)}) = 0$ of some order k, and then trying to find enough such equations so that they cut out a proper algebraic locus $Y \subsetneq X$. We use for this global sections of $H^0(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathscr{O}(-A))$, where A is ample, and apply the fundamental vanishing theorem 8.15. It is expected that the global sections of $H^0(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathscr{O}(-A))$ are precisely those which ultimately define the algebraic locus $Y \subsetneq X$ where the curve f should lie. The problem is then reduced to (i) showing that there are many non-zero sections of $H^0(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathscr{O}(-A))$ and (ii) understanding what is their joint base locus. The first part of this program is the main result of this section.

9.1. Theorem. *Let (X, V) be a directed projective variety such that K_V is big and let A be an ample divisor. Then for $k \gg 1$ and $\delta \in \mathbb{Q}_+$ small enough, $\delta \leqslant c(\log k)/k$, the number of sections $h^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-m\delta A))$ has maximal growth, i.e., is larger that $c_k m^{n+kr-1}$ for some $m \geqslant m_k$, where $c, c_k > 0$, $n = \dim X$ and $r = \operatorname{rank} V$. In particular, entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfy (many) algebraic differential equations.*

The statement is very elementary to check when $r = \operatorname{rank} V = 1$, and therefore when $n = \dim X = 1$. In higher dimensions $n \geqslant 2$, only very partial results were known before Theorem 9.1 was obtained in [Dem11], [and they dealt merely with the absolute case $V = T_X$]. In dimension 2, Theorem 9.1 is a consequence of the Riemann–Roch calculation of Green–Griffiths [GrGr80], combined with a vanishing theorem due to Bogomolov [Bog79]—the latter actually only applies to the top cohomology group H^n, and things become much more delicate when extimates of intermediate cohomology groups are needed. In higher dimensions, Diverio [Div08], [Div09] proved the existence of sections of $H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-1))$ whenever X is a hypersurface of $\mathbb{P}_{\mathbb{C}}^{n+1}$ of high degree $d \geqslant d_n$, assuming $k \geqslant n$ and $m \geqslant m_n$. More recently, Merker [Mer15] was able to treat the case of arbitrary hypersurfaces of general type, i.e., $d \geqslant n + 3$, assuming this time k to be very large. The latter result is obtained through explicit algebraic calculations of the spaces of sections, and the proof is computationally very intensive. Bérczi [Ber15], [Ber18] also obtained related results with a different approach based on residue formulas, assuming e.g. $d \geqslant n^{9n}$.

All these approaches are algebraic in nature. Here, however, our techniques are based on more elaborate curvature estimates in the spirit of Cowen–Griffiths [CoGr76]. They require holomorphic Morse inequalities (see 9.10 below)—and we do not know how to translate our method in an algebraic setting. Notice that holomorphic Morse inequalities are essentially insensitive to singularities, as we can pass to non-singular models and blow-up X as much as we want: if $\mu : \widetilde{X} \to X$ is a modification then $\mu_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and for $q \geqslant 1$, $R^q \mu_* \mathcal{O}_{\widetilde{X}}$ is supported on a codimension 1 analytic subset (even a codimension 2 subset if X is smooth). It follows from the Leray spectral sequence that the cohomology estimates for L on X or for $\widetilde{L} = \mu^* L$ on $\widetilde{X}$ differ by negligible terms, i.e.,

$$h^q(\widetilde{X}, \widetilde{L}^{\otimes m}) - h^q(X, L^{\otimes m}) = O(m^{n-1}). \tag{9.2}$$

Finally, singular holomorphic Morse inequalities (in the form obtained by L. Bonavero [Bon93]) allow us to work with singular Hermitian metrics h; this is the reason why we will only require to have big line bundles rather than ample line bundles. In the case of linear subspaces $V \subset T_X$, we introduce singular Hermitian metrics as follows.

9.3. Definition. *A singular Hermitian metric on a linear subspace $V \subset T_X$ is a metric h on the fibers of V such that the function $\log h : \xi \mapsto \log |\xi|_h^2$ is locally integrable on the total space of V.*

Such a metric can also be viewed as a singular Hermitian metric on the tautological line bundle $\mathscr{O}_{P(V)}(-1)$ on the projectivized bundle $P(V) = V \smallsetminus \{0\}/\mathbb{C}^*$, and therefore its dual metric h^* defines a curvature current $\Theta_{\mathscr{O}_{P(V)}(1),h^*}$ of type $(1,1)$ on $P(V) \subset P(T_X)$, such that

$$p^*\Theta_{\mathscr{O}_{P(V)}(1),h^*} = \frac{i}{2\pi}\partial\overline{\partial}\log h, \quad \text{where } p : V \smallsetminus \{0\} \longrightarrow P(V). \tag{9.4}$$

If $\log h$ is quasi-plurisubharmonic (or quasi-psh, which means plurisubharmonic modulo addition of a smooth function) on V, then $\log h$ is indeed locally integrable, and we have moreover

$$\Theta_{\mathscr{O}_{P(V)}(1),h^*} \geqslant -C\omega \tag{9.5}$$

for some smooth positive $(1,1)$-form on $P(V)$ and some constant $C > 0$; conversely, if (9.5) holds, then $\log h$ is quasi-plurisubharmonic.

9.6. Definition. *We will say that a singular Hermitian metric h on V is* admissible *if h can be written as $h = e^{\varphi} h_{0|V}$, where h_0 is a smooth positive definite Hermitian on T_X and φ is a quasi-plurisubharmonic weight with analytic singularities on X, as in Definition 9.3. Then h can be seen as a singular Hermitian metric on $\mathscr{O}_{P(V)}(1)$, with the property that it induces a smooth positive definite metric on a Zariski open set $X' \subset X \smallsetminus \mathrm{Sing}(V)$; we will denote by $\mathrm{Sing}(h)$ the complement of the largest such Zariski open set X' (so that $\mathrm{Sing}(h) \supset \mathrm{Sing}(V)$).*

If h is an admissible metric on V, we define $\mathscr{O}_h(V^*)$ to be the sheaf of germs of holomorphic sections of $V^*_{\restriction X \smallsetminus \mathrm{Sing}(h)}$ which are h^*-bounded near $\mathrm{Sing}(h)$; by the assumption on the analytic singularities, this is a coherent sheaf (as the direct image of some coherent sheaf on $P(V)$), and actually, since $h^* = e^{-\varphi}h_0^*$, it is a subsheaf of the sheaf $\mathscr{O}(V^*) := \mathscr{O}_{h_0}(V^*)$ associated with a smooth positive definite metric h_0 on T_X. If r is the generic rank of V and m a positive integer, we define similarly

$$\begin{aligned} {}^bK_{V,h}^{[m]} &= \text{sheaf of germs of holomorphic sections of } (\det V^*_{\restriction X'})^{\otimes m} \\ &= (\Lambda^r V^*_{\restriction X'})^{\otimes m} \\ &\quad\ \text{which are } \det h^*\text{-bounded,} \end{aligned} \tag{9.7}$$

so that ${}^bK_V^{[m]} := {}^bK_{V,h_0}^{[m]}$ according to Definition 2.7. For a given admissible Hermitian structure (V,h), we define similarly the sheaf $E_{k,m}^{\mathrm{GG}}V_h^*$ to be the sheaf of polynomials defined over $X \smallsetminus \mathrm{Sing}(h)$ which are "h-bounded". This means that when they are viewed as polynomials $P(z\,;\,\xi_1,\dots,\xi_k)$ in terms of $\xi_j = (\nabla_{h_0}^{1,0})^j f(0)$, where $\nabla_{h_0}^{1,0}$ is the $(1,0)$-component of the induced Chern connection on (V, h_0), there is a uniform bound

$$|P(z\,;\,\xi_1,\dots,\xi_k)| \leqslant C\Big(\sum \|\xi_j\|_h^{1/j}\Big)^m \tag{9.8}$$

near points of $X \smallsetminus X'$ (see Sect. 2 for more details on this). Again, by a direct image argument, one sees that $E_{k,m}^{\mathrm{GG}} V_h^*$ is always a coherent sheaf. The sheaf $E_{k,m}^{\mathrm{GG}} V^*$ is defined to be $E_{k,m}^{\mathrm{GG}} V_h^*$ when $h = h_0$ (it is actually independent of the choice of h_0, as follows from arguments similar to those given in Sect. 2). Notice that this is exactly what is needed to extend the proof of the vanishing theorem 8.15 to the case of a singular linear space V; the value distribution theory argument can only work when the functions $P(f\,;\, f',\dots,f^{(k)})(t)$ do not exhibit poles, and this is guaranteed here by the boundedness assumption.

Our strategy can be described as follows. We consider the Green–Griffiths bundle of k-jets $X_k^{\mathrm{GG}} = J^k V \smallsetminus \{0\}/\mathbb{C}^*$, which by (7.7) consists of a fibration in *weighted projective spaces*, and its associated tautological sheaf

$$L = \mathcal{O}_{X_k^{\mathrm{GG}}}(1),$$

viewed rather as a virtual $\mathbb{Q}$-line bundle $\mathcal{O}_{X_k^{\mathrm{GG}}}(m_0)^{1/m_0}$ with $m_0 = \mathrm{lcm}(1, 2, \dots, k)$. Then, if $\pi_k : X_k^{\mathrm{GG}} \to X$ is the natural projection, we have

$$E_{k,m}^{\mathrm{GG}} = (\pi_k)_* \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \quad \text{and} \quad R^q(\pi_k)_* \mathcal{O}_{X_k^{\mathrm{GG}}}(m) = 0 \text{ for } q \geqslant 1.$$

Hence, by the Leray spectral sequence we get for every invertible sheaf F on X the isomorphism

$$H^q(X, E_{k,m}^{\mathrm{GG}} V^* \otimes F) \simeq H^q(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* F). \tag{9.9}$$

The latter group can be evaluated thanks to holomorphic Morse inequalities. Let us recall the main statement.

9.10. Holomorphic Morse inequalities ([Dem85]). *Let X be a compact complex manifolds, $E \to X$ a holomorphic vector bundle of rank r, and (L, h) a Hermitian line bundle. The dimensions $h^q(X, E \otimes L^m)$ of cohomology groups of the tensor powers $E \otimes L^m$ satisfy the following asymptotic estimates as $m \to +\infty$:*

(WM) *Weak Morse inequalities*:

$$h^q(X, E \otimes L^m) \leqslant r\frac{m^n}{n!}\int_{X(L,h,q)} (-1)^q \Theta_{L,h}^n + o(m^n),$$

where $X(L,h,q)$ *denotes the open set of points* $x \in X$ *at which the curvature form* $\Theta_{L,h}(x)$ *has signature* $(q, n-q)$;

(SM) *Strong Morse inequalities*:

$$\sum_{0\leqslant j\leqslant q} (-1)^{q-j} h^j(X, E\otimes L^m) \leqslant r\frac{m^n}{n!}\int_{X(L,h,\leqslant q)} (-1)^q \Theta^n_{L,h} + o(m^n),$$

where $X(L,h,\leqslant q) = \bigcup_{j\leqslant q} X(L,h,j)$;

(RR) *Asymptotic Riemann–Roch formula*:

$$\chi(X, E\otimes L^m) := \sum_{0\leqslant j\leqslant n} (-1)^j h^j(X, E\otimes L^m) = r\frac{m^n}{n!}\int_X \Theta^n_{L,h} + o(m^n).$$

Moreover (*cf.* Bonavero's PhD thesis [Bon93]), if $h = e^{-\varphi}$ is a singular Hermitian metric with analytic singularities of pole set $P = \varphi^{-1}(-\infty)$, the estimates still hold provided all cohomology groups are replaced by cohomology groups $H^q(X, E\otimes L^m \otimes \mathscr{I}(h^m))$ twisted with the corresponding L^2 multiplier ideal sheaves

$$\begin{aligned}\mathscr{I}(h^m) &= \mathscr{I}(k\varphi)\\ &= \Big\{f \in \mathscr{O}_{X,x},\ \ \exists V \ni x,\ \int_V |f(z)|^2 e^{-m\varphi(z)}\,d\lambda(z) < +\infty\Big\},\end{aligned}$$

and provided the Morse integrals are computed on the regular locus of h, namely restricted to $X(L,h,q)\smallsetminus\Sigma$:

$$\int_{X(L,h,q)\smallsetminus\Sigma} (-1)^q \Theta^n_{L,h}.$$

The special case of 9.10 (SM) when $q = 1$ yields a very useful criterion for the existence of sections of large multiples of L.

9.11. Corollary. *Let* $L \to X$ *be a holomorphic line bundle equipped with a singular Hermitian metric* $h = e^{-\varphi}$ *with analytic singularities of pole set* $\Sigma = \varphi^{-1}(-\infty)$. *Then we have the following lower bounds*

(a) *at the* h^0 *level*:

$$\begin{aligned}h^0(X, E\otimes L^m) &\geqslant h^0(X, E\otimes L^m\otimes\mathscr{I}(h^m))\\ &\geqslant h^0(X, E\otimes L^m\otimes\mathscr{I}(h^m))\\ &\quad - h^1(X, E\otimes L^m\otimes\mathscr{I}(h^m))\\ &\geqslant r\frac{k^n}{n!}\int_{X(L,h,\leqslant 1)\smallsetminus\Sigma} \Theta^n_{L,h} - o(k^n).\end{aligned}$$

Especially L *is big as soon as* $\int_{X(L,h,\leqslant 1)\smallsetminus\Sigma}\Theta^n_{L,h} > 0$ *for some singular Hermitian metric* h *on* L.

(b) *at the h^q level*:

$$h^q(X, E \otimes L^m \otimes \mathscr{I}(h^m)) \geqslant r\frac{k^n}{n!} \sum_{j=q-1,q,q+1} (-1)^q \int_{X(L,h,j)\smallsetminus\Sigma} \Theta^n_{L,h} - o(k^n).$$

Now, given a directed manifold (X, V), we can associate with any admissible metric h on V a metric (or rather a natural family) of metrics on $L = \mathscr{O}_{X_k^{\mathrm{GG}}}(1)$. The space X_k^{GG} always possesses quotient singularities if $k \geqslant 2$ (and even some more if V is singular), but we do not really care since Morse inequalities still work in this setting thanks to Bonavero's generalization. As we will see, it is then possible to get nice asymptotic formulas as $m \to +\infty$. They appear to be of a *probabilistic nature* if we take the components of the k-jet (i.e., the successive derivatives $\xi_j = f^{(j)}(0)$, $1 \leqslant j \leqslant k$) as random variables. This probabilistic behaviour was somehow already visible in the Riemann–Roch calculation of [GrGr80]. In this way, assuming K_V big, we produce a lot of sections $\sigma_j = H^0(X_k^{\mathrm{GG}}, \mathscr{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* F)$, corresponding to certain divisors $Z_j \subset X_k^{\mathrm{GG}}$. The hard problem which is left in order to complete a proof of the generalized Green–Griffiths–Lang conjecture is to compute the base locus $Z = \bigcap Z_j$ and to show that $Y = \pi_k(Z) \subset X$ must be a proper algebraic variety.

9.B. Hermitian geometry of weighted projective spaces

The goal of this section is to introduce natural Kähler metrics on weighted projective spaces, and to evaluate the corresponding volume forms. Here we put $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$ so that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. The normalization of the d^c operator is chosen such that we have precisely $(dd^c \log |z|^2)^n = \delta_0$ (the Dirac mass at 0) for the Monge–Ampère operator in $\mathbb{C}^n$. Given a k-tuple of "weights" $a = (a_1, \dots, a_k)$, i.e., of integers $a_s > 0$ with $\gcd(a_1, \dots, a_k) = 1$, we introduce the weighted projective space $P(a_1, \dots, a_k)$ to be the quotient of $\mathbb{C}^k \smallsetminus \{0\}$ by the corresponding weighted $\mathbb{C}^*$ action:

(9.12)
$$P(a_1, \dots, a_k) = \mathbb{C}^k \smallsetminus \{0\}/\mathbb{C}^*, \quad \lambda \cdot z = (\lambda^{a_1} z_1, \dots, \lambda^{a_k} z_k), \quad \lambda \in \mathbb{C}^*.$$

As is well-known, this defines a toric $(k-1)$-dimensional algebraic variety with quotient singularities. On this variety, we introduce the possibly singular (but almost everywhere smooth and non-degenerate) Kähler form $\omega_{a,p}$ defined by

(9.13)
$$\pi_a^* \omega_{a,p} = dd^c \varphi_{a,p}, \quad \varphi_{a,p}(z) = \frac{1}{p} \log \sum_{1 \leqslant s \leqslant k} |z_s|^{2p/a_s},$$

where $\pi_a : \mathbb{C}^k \smallsetminus \{0\} \to P(a_1, \dots, a_k)$ is the canonical projection and $p > 0$ is a positive constant. It is clear that $\varphi_{p,a}$ is real analytic on $\mathbb{C}^k \smallsetminus \{0\}$ if p is an integer and a common multiple of all weights a_s, and we will implicitly pick such a p later on to avoid any difficulty. Elementary calculations give the following well-known formula for the volume

$$\int_{P(a_1,\dots,a_k)} \omega_{a,p}^{k-1} = \frac{1}{a_1 \cdots a_k} \tag{9.14}$$

(notice that this is independent of p, as it is obvious by Stokes theorem, since the cohomology class of $\omega_{a,p}$ does not depend on p).

Our later calculations will require a slightly more general setting. Instead of looking at $\mathbb{C}^k$, we consider the weighted $\mathbb{C}^*$ action defined by

$$\mathbb{C}^{|r|} = \mathbb{C}^{r_1} \times \cdots \times \mathbb{C}^{r_k}, \quad \lambda \cdot z = (\lambda^{a_1} z_1, \dots, \lambda^{a_k} z_k), \quad \lambda \in \mathbb{C}^*. \tag{9.15}$$

Here $z_s \in \mathbb{C}^{r_s}$ for some k-tuple $r = (r_1, \dots, r_k)$ and $|r| = r_1 + \cdots + r_k$. This gives rise to a weighted projective space

$$\begin{aligned} &P(a_1^{[r_1]}, \dots, a_k^{[r_k]}) = P(a_1, \dots, a_1, \dots, a_k, \dots, a_k), \\ &\pi_{a,r} : \mathbb{C}^{r_1} \times \cdots \times \mathbb{C}^{r_k} \smallsetminus \{0\} \longrightarrow P(a_1^{[r_1]}, \dots, a_k^{[r_k]}), \end{aligned} \tag{9.16}$$

obtained by repeating r_s times each weight a_s. On this space, we introduce the degenerate Kähler metric $\omega_{a,r,p}$ such that

$$\pi_{a,r}^* \omega_{a,r,p} = dd^c \varphi_{a,r,p}, \quad \varphi_{a,r,p}(z) = \frac{1}{p} \log \sum_{1 \leqslant s \leqslant k} |z_s|^{2p/a_s}, \tag{9.17}$$

where $|z_s|$ stands now for the standard Hermitian norm $(\sum_{1 \leqslant j \leqslant r_s} |z_{s,j}|^2)^{1/2}$ on $\mathbb{C}^{r_s}$. This metric is cohomologous to the corresponding "polydisc-like" metric $\omega_{a,p}$ already defined, and therefore Stokes theorem implies

$$\int_{P(a_1^{[r_1]}, \dots, a_k^{[r_k]})} \omega_{a,r,p}^{|r|-1} = \frac{1}{a_1^{r_1} \cdots a_k^{r_k}}. \tag{9.18}$$

Using standard results of integration theory (Fubini, change of variable formula...), one obtains:

9.19. Proposition. *Let $f(z)$ be a bounded function on $P(a_1^{[r_1]}, \dots, a_k^{[r_k]})$ which is continuous outside of the hyperplane sections $z_s = 0$. We also view f as a $\mathbb{C}^*$-invariant continuous function on $\prod(\mathbb{C}^{r_s} \smallsetminus \{0\})$. Then*

$$\begin{aligned} &\int_{P(a_1^{[r_1]}, \dots, a_k^{[r_k]})} f(z)\, \omega_{a,r,p}^{|r|-1} \\ &= \frac{(|r|-1)!}{\prod_s a_s^{r_s}} \int_{(x,u) \in \Delta_{k-1} \times \prod S^{2r_s-1}} f(x_1^{a_1/2p} u_1, \dots, x_k^{a_k/2p} u_k) \prod_{1 \leqslant s \leqslant k} \frac{x_s^{r_s-1}}{(r_s-1)!}\, dx\, d\mu(u), \end{aligned}$$

where Δ_{k-1} is the $(k-1)$-simplex $\{x_s \geqslant 0, \sum x_s = 1\}$, $dx = dx_1 \wedge \cdots \wedge dx_{k-1}$ its standard measure, and where $d\mu(u) = d\mu_1(u_1)\cdots d\mu_k(u_k)$ is the rotation invariant probability measure on the product $\prod_s S^{2r_s-1}$ of unit spheres in $\mathbb{C}^{r_1} \times \cdots \times \mathbb{C}^{r_k}$. As a consequence

$$\lim_{p\to+\infty} \int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1} = \frac{1}{\prod_s a_s^{r_s}} \int_{\prod S^{2r_s-1}} f(u)\,d\mu(u).$$

Also, by elementary integrations by parts and induction on k, $r_1, \ldots, r_k$, it can be checked that

$$\int_{x\in\Delta_{k-1}} \prod_{1\leqslant s\leqslant k} x_s^{r_s-1}\,dx_1\cdots dx_{k-1} = \frac{1}{(|r|-1)!} \prod_{1\leqslant s\leqslant k} (r_s-1)! \tag{9.20}$$

This implies that $(|r|-1)!\prod_{1\leqslant s\leqslant k} \frac{x_s^{r_s-1}}{(r_s-1)!}\,dx$ is a probability measure on Δ_{k-1}.

9.C. *Probabilistic estimate of the curvature of k-jet bundles*

Let (X, V) be a compact complex directed non-singular variety. To avoid any technical difficulty at this point, we first assume that V is a holomorphic vector subbundle of T_X, equipped with a smooth Hermitian metric h.

According to the notation already specified in Sect. 7, we denote by J^kV the bundle of k-jets of holomorphic curves $f : (\mathbb{C}, 0) \to X$ tangent to V at each point. Let us set $n = \dim_{\mathbb{C}} X$ and $r = \mathrm{rank}_{\mathbb{C}} V$. Then $J^kV \to X$ is an algebraic fiber bundle with typical fiber $\mathbb{C}^{rk}$, and we get a projectivized k-jet bundle

$$X_k^{\mathrm{GG}} := (J^kV \smallsetminus \{0\})/\mathbb{C}^*, \quad \pi_k : X_k^{\mathrm{GG}} \longrightarrow X, \tag{9.21}$$

which is a $P(1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ weighted projective bundle over X, and we have the direct image formula $(\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m) = \mathcal{O}(E_{k,m}^{\mathrm{GG}}V^*)$ (*cf.* Proposition 7.9). In the sequel, we do not make a direct use of coordinates, because they need not be related in any way to the Hermitian metric h of V. Instead, we choose a local holomorphic coordinate frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ of V on a neighborhood U of x_0, such that

$$\langle e_\alpha(z), e_\beta(z)\rangle = \delta_{\alpha\beta} + \sum_{1\leqslant i,j\leqslant n,\, 1\leqslant\alpha,\beta\leqslant r} c_{ij\alpha\beta} z_i \bar z_j + O(|z|^3) \tag{9.22}$$

for suitable complex coefficients $(c_{ij\alpha\beta})$. It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor $\frac{i}{2\pi}D_{V,h}^2$ of (V, h) at x_0 is then given by

$$\Theta_{V,h}(x_0) = -\frac{i}{2\pi} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\,dz_i \wedge d\bar z_j \otimes e_\alpha^* \otimes e_\beta. \tag{9.23}$$

Consider a local holomorphic connection ∇ on $V_{\restriction U}$ (e.g. the one which turns (e_α) into a parallel frame), and take $\xi_k = \nabla^k f(0) \in V_x$ defined inductively by $\nabla^1 f = f'$ and $\nabla^s f = \nabla_{f'}(\nabla^{s-1} f)$. This gives a local identification

$$J_k V_{\restriction U} \longrightarrow V_{\restriction U}^{\oplus k}, \quad f \longmapsto (\xi_1, \ldots, \xi_k) = (\nabla f(0), \ldots, \nabla f^k(0)),$$

and the weighted $\mathbb{C}^*$ action on $J_k V$ is expressed in this setting by

$$\lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda \xi_1, \lambda^2 \xi_2, \ldots, \lambda^k \xi_k).$$

Now, we fix a finite open covering $(U_\alpha)_{\alpha \in I}$ of X by open coordinate charts such that $V_{\restriction U_\alpha}$ is trivial, along with holomorphic connections ∇_α on $V_{\restriction U_\alpha}$. Let θ_α be a partition of unity of X subordinate to the covering (U_α). Let us fix $p > 0$ and small parameters $1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k > 0$. Then we define a global weighted Finsler metric on $J^k V$ by putting for any k-jet $f \in J_x^k V$

$$\Psi_{h,p,\varepsilon}(f) := \Big(\sum_{\alpha \in I} \theta_\alpha(x) \sum_{1 \leqslant s \leqslant k} \varepsilon_s^{2p} \| \nabla_\alpha^s f(0) \|_{h(x)}^{2p/s} \Big)^{1/p}, \tag{9.24}$$

where $\| \ \|_{h(x)}$ is the Hermitian metric h of V evaluated on the fiber V_x, $x = f(0)$. The function $\Psi_{h,p,\varepsilon}$ satisfies the fundamental homogeneity property

$$\Psi_{h,p,\varepsilon}(\lambda \cdot f) = \Psi_{h,p,\varepsilon}(f)\, |\lambda|^2 \tag{9.25}$$

with respect to the $\mathbb{C}^*$ action on $J^k V$, in other words, it induces a Hermitian metric on the dual L^* of the tautological $\mathbb{Q}$-line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ over X_k^{GG}. The curvature of L_k is given by

$$\pi_k^* \Theta_{L_k, \Psi_{h,p,\varepsilon}^*} = dd^c \log \Psi_{h,p,\varepsilon}. \tag{9.26}$$

Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to $L \to X_k^{\mathrm{GG}}$ with the above metric. It might look a priori like an untractable problem, since the definition of $\Psi_{h,p,\varepsilon}$ is a rather unnatural one. However, the "miracle" is that the asymptotic behavior of $\Psi_{h,p,\varepsilon}$ as $\varepsilon_s / \varepsilon_{s-1} \to 0$ is in some sense uniquely defined and very natural. It will lead to a computable asymptotic formula, which is moreover simple enough to produce useful results.

9.27. Lemma. *On each coordinate chart U equipped with a holomorphic connection ∇ of $V_{\restriction U}$, let us define the components of a k-jet $f \in J^k V$ by $\xi_s = \nabla^s f(0)$, and consider the rescaling transformation*

$$\rho_{\nabla,\varepsilon}(\xi_1, \xi_2, \ldots, \xi_k) = (\varepsilon_1^1 \xi_1, \varepsilon_2^2 \xi_2, \ldots, \varepsilon_k^k \xi_k) \quad \textit{on } J_x^k V,\ x \in U,$$

(it commutes with the $\mathbb{C}^*$-action but is otherwise unrelated and not canonically defined over X as it depends on the choice of ∇). Then, if p is a multiple of $\mathrm{lcm}(1, 2, \ldots, k)$ *and $\varepsilon_s/\varepsilon_{s-1} \to 0$ for all $s = 2, \ldots, k$, the rescaled function $\Psi_{h,p,\varepsilon} \circ \rho_{\nabla,\varepsilon}^{-1}(\xi_1, \ldots, \xi_k)$ converges towards*

$$\Big(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2p/s}\Big)^{1/p}$$

on every compact subset of $J^k V_{\restriction U} \smallsetminus \{0\}$, uniformly in C^∞ topology.

Proof. Let $U \subset X$ be an open set on which $V_{\restriction U}$ is trivial and equipped with some holomorphic connection ∇. Let us pick another holomorphic connection $\widetilde{\nabla} = \nabla + \Gamma$, where $\Gamma \in H^0(U, \Omega_X^1 \otimes \mathrm{Hom}(V, V))$. Then $\widetilde{\nabla}^2 f = \nabla^2 f + \Gamma(f)(f') \cdot f'$, and inductively we get

$$\widetilde{\nabla}^s f = \nabla^s f + P_s(f\,;\nabla^1 f, \ldots, \nabla^{s-1} f),$$

where $P(x\,;\xi_1, \ldots, \xi_{s-1})$ is a polynomial of weighted degree s in $(\xi_1, \ldots, \xi_{s-1})$ with holomorphic coefficients in $x \in U$. In other words, the corresponding change in the parametrization of $J^k V_{\restriction U}$ is given by a $\mathbb{C}^*$-homogeneous transformation

$$\widetilde{\xi}_s = \xi_s + P_s(x\,;\xi_1, \ldots, \xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon}, \ldots, \xi_{k,\varepsilon}) = (\varepsilon_1^1 \xi_1, \ldots, \varepsilon_k^k \xi_k), \quad (\widetilde{\xi}_{1,\varepsilon}, \ldots, \widetilde{\xi}_{k,\varepsilon}) = (\varepsilon_1^1 \widetilde{\xi}_1, \ldots, \varepsilon_k^k \widetilde{\xi}_k).$$

Then

$$\begin{aligned}\widetilde{\xi}_{s,\varepsilon} &= \xi_{s,\varepsilon} + \varepsilon_s^s\, P_s(x\,;\varepsilon_1^{-1}\xi_{1,\varepsilon}, \ldots, \varepsilon_{s-1}^{-(s-1)}\xi_{s-1,\varepsilon})\\ &= \xi_{s,\varepsilon} + O(\varepsilon_s/\varepsilon_{s-1})^s\, O(\|\xi_{1,\varepsilon}\| + \cdots + \|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s\end{aligned}$$

and the error terms are thus polynomials of fixed degree with arbitrarily small coefficients as $\varepsilon_s/\varepsilon_{s-1} \to 0$. Now, the definition of $\Psi_{h,p,\varepsilon}$ consists of glueing the sums

$$\sum_{1\leqslant s\leqslant k} \varepsilon_s^{2p} \|\xi_k\|_h^{2p/s} = \sum_{1\leqslant s\leqslant k} \|\xi_{k,\varepsilon}\|_h^{2p/s}$$

corresponding to $\xi_k = \nabla_\alpha^s f(0)$ by means of the partition of unity $\sum \theta_\alpha(x) = 1$. We see that by using the rescaled variables $\xi_{s,\varepsilon}$ the changes occurring when replacing a connection ∇_α by an alternative one ∇_β are arbitrary small in C^∞ topology, with error terms uniformly controlled in terms of the ratios $\varepsilon_s/\varepsilon_{s-1}$ on all compact subsets of $V^k \smallsetminus \{0\}$. This shows that in C^∞ topology, $\Psi_{h,p,\varepsilon} \circ \rho_{\nabla,\varepsilon}^{-1}(\xi_1, \ldots, \xi_k)$ converges uniformly towards $(\sum_{1\leqslant s\leqslant k} \|\xi_k\|_h^{2p/s})^{1/p}$, whatever the trivializing open set U and the holomorphic connection ∇ used to evaluate the components and to perform the rescaling are. $\square$

Now, we fix a point $x_0 \in X$ and a local holomorphic frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ satisfying (9.22) on a neighborhood U of x_0. We introduce the rescaled components $\xi_s = \varepsilon_s^s \nabla^s f(0)$ on $J^k V_{\restriction U}$ and compute the curvature of

$$\Psi_{h,p,\varepsilon} \circ \rho_{\nabla,\varepsilon}^{-1}(z\,;\,\xi_1,\ldots,\xi_k) \simeq \Big(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2p/s}\Big)^{1/p},$$

(by Lemma 9.27, the errors can be taken arbitrary small in C^∞ topology). We write $\xi_s = \sum_{1\leqslant\alpha\leqslant r} \xi_{s\alpha} e_\alpha$. By (9.22) we have

$$\|\xi_s\|_h^2 = \sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i \overline{z}_j \xi_{s\alpha} \overline{\xi}_{s\beta} + O(|z|^3|\xi|^2).$$

The question is to evaluate the curvature of the weighted metric defined by

$$\begin{aligned}
&\Psi(z\,;\,\xi_1,\ldots,\xi_k)\\
&= \Big(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2p/s}\Big)^{1/p}\\
&= \Big(\sum_{1\leqslant s\leqslant k} \Big(\sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i \overline{z}_j \xi_{s\alpha}\overline{\xi}_{s\beta}\Big)^{p/s}\Big)^{1/p} + O(|z|^3).
\end{aligned}$$

We set $|\xi_s|^2 = \sum_\alpha |\xi_{s\alpha}|^2$. A straightforward calculation yields

$$\begin{aligned}
&\log \Psi(z\,;\,\xi_1,\ldots,\xi_k)\\
&= \frac{1}{p}\log \sum_{1\leqslant s\leqslant k} |\xi_s|^{2p/s}\\
&\quad + \sum_{1\leqslant s\leqslant k} \frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i\overline{z}_j \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} + O(|z|^3).
\end{aligned}$$

By (9.26), the curvature form of $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is given at the central point x_0 by the following formula.

9.28. Proposition. *With the above choice of coordinates and with respect to the rescaled components $\xi_s = \varepsilon_s^s \nabla^s f(0)$ at $x_0 \in X$, we have the approximate expression*

$$\begin{aligned}
&\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(x_0,[\xi]) \simeq \omega_{a,r,p}(\xi)\\
&\qquad + \frac{i}{2\pi} \sum_{1\leqslant s\leqslant k} \frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\, dz_i \wedge d\overline{z}_j,
\end{aligned}$$

where the error terms are $O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$ uniformly on the compact variety X_k^{GG}. Here $\omega_{a,r,p}$ is the (degenerate) Kähler metric associated with the weight $a = (1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ of the canonical $\mathbb{C}^*$ action on $J^k V$.

Thanks to the uniform approximation, we can (and will) neglect the error terms in the calculations below. Since $\omega_{a,r,p}$ is positive definite on the fibers of $X_k^{\mathrm{GG}} \to X$ (at least outside of the axes $\xi_s = 0$), the index of the $(1,1)$ curvature form $\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(z,[\xi])$ is equal to the index of the $(1,1)$-form

$$\gamma_k(z,\xi) := \frac{i}{2\pi} \sum_{1\leqslant s\leqslant k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z) \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\, dz_i \wedge d\overline{z}_j \tag{9.29}$$

depending only on the differentials $(dz_j)_{1\leqslant j\leqslant n}$ on X. The q-index integral of $(L_k, \Psi^*_{h,p,\varepsilon})$ on X_k^{GG} is therefore equal to

$$\begin{aligned}&\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}\\ &= \frac{(n+kr-1)!}{n!(kr-1)!} \int_{z\in X} \int_{\xi\in P(1^{[r]},\dots,k^{[r]})} \omega^{kr-1}_{a,r,p}(\xi) \mathbb{1}_{\gamma_k,q}(z,\xi) \gamma_k(z,\xi)^n,\end{aligned}$$

where $\mathbb{1}_{\gamma_k,q}(z,\xi)$ is the characteristic function of the open set of points where $\gamma_k(z,\xi)$ has signature $(n-q,q)$ in terms of the dz_j's. Notice that since $\gamma_k(z,\xi)^n$ is a determinant, the product $\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n$ gives rise to a continuous function on X_k^{GG}. Formula (9.20) with $r_1 = \cdots = r_k = r$ and $a_s = s$ yields the slightly more explicit integral

$$\begin{aligned}&\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}\\ &= \frac{(n+kr-1)!}{n!(k!)^r}\\ &\quad\times \int_{z\in X} \int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} \mathbb{1}_{g_k,q}(z,x,u) g_k(z,x,u)^n \frac{(x_1\cdots x_k)^{r-1}}{(r-1)!^k}\, dx\, d\mu(u),\end{aligned}$$

where $g_k(z,x,u) = \gamma_k(z, x_1^{1/2p}u_1,\dots,x_k^{k/2p}u_k)$ is given by

$$g_k(z,x,u) = \frac{i}{2\pi} \sum_{1\leqslant s\leqslant k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j \tag{9.30}$$

and $\mathbb{1}_{g_k,q}(z,x,u)$ is the characteristic function of its q-index set. Here

$$d\nu_{k,r}(x) = (kr-1)!\, \frac{(x_1\cdots x_k)^{r-1}}{(r-1)!^k}\, dx \tag{9.31}$$

is a probability measure on Δ_{k-1}, and we can rewrite

(9.32)
$$\begin{aligned}
&\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}\\
&= \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\\
&\quad\times \int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} \mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,d\nu_{k,r}(x)\,d\mu(u).
\end{aligned}$$

Now, formula (9.30) shows that $g_k(z,x,u)$ is a "Monte Carlo" evaluation of the curvature tensor, obtained by averaging the curvature at random points $u_s\in S^{2r-1}$ with certain positive weights x_s/s; we should then think of the k-jet f as some sort of random variable such that the derivatives $\nabla^k f(0)$ are uniformly distributed in all directions. Let us compute the expected value of $(x,u)\mapsto g_k(z,x,u)$ with respect to the probability measure $d\nu_{k,r}(x)\,d\mu(u)$. Since
$\int_{S^{2r-1}} u_{s\alpha}\overline{u}_{s\beta}\,d\mu(u_s)=\frac{1}{r}\delta_{\alpha\beta}$ and $\int_{\Delta_{k-1}} x_s\,d\nu_{k,r}(x)=\frac{1}{k}$, we find

$$\mathbf{E}(g_k(z,\bullet,\bullet))=\frac{1}{kr}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\cdot\frac{i}{2\pi}\sum_{i,j,\alpha}c_{ij\alpha\alpha}(z)\,dz_i\wedge d\overline{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.,

$$\mathbf{E}(g_k(z,\bullet,\bullet))=\frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\Theta_{\det(V^*),\det h^*},\tag{9.33}$$

where $\Theta_{\det(V^*),\det h^*}$ is the $(1,1)$-curvature form of $\det(V^*)$ with the metric induced by h. It is natural to guess that $g_k(z,x,u)$ behaves asymptotically as its expected value $\mathbf{E}(g_k(z,\bullet,\bullet))$ when k tends to infinity. If we replace brutally g_k by its expected value in (9.32), we get the integral

$$\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\,\frac{1}{(kr)^n}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)^n\int_X \mathbb{1}_{\eta,q}\eta^n,$$

where $\eta:=\Theta_{\det(V^*),\det h^*}$ and $\mathbb{1}_{\eta,q}$ is the characteristic function of its q-index set in X. The leading constant is equivalent to $(\log k)^n/n!(k!)^r$ modulo a multiplicative factor $(1+O(1/\log k))$. By working out a more precise analysis of the deviation, the following result has been proved in [Dem11] and [Dem12].

9.34. Probabilistic-estimate. *Fix smooth Hermitian metrics h on V and $\omega=\frac{i}{2\pi}\sum\omega_{ij}\,dz_i\wedge d\overline{z}_j$ on X. Denote by $\Theta_{V,h}=-\frac{i}{2\pi}\sum c_{ij\alpha\beta}\,dz_i\wedge d\overline{z}_j\otimes e^*_\alpha\otimes e_\beta$ the curvature tensor of V with respect to an h-orthonormal frame (e_α), and put*

$$\eta(z)=\Theta_{\det(V^*),\det h^*}=\frac{i}{2\pi}\sum_{1\leqslant i,j\leqslant n}\eta_{ij}\,dz_i\wedge d\overline{z}_j,\quad \eta_{ij}=\sum_{1\leqslant\alpha\leqslant r}c_{ij\alpha\alpha}.$$

Finally consider the k-jet line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \to X_k^{\mathrm{GG}}$ equipped with the induced metric $\Psi^*_{h,p,\varepsilon}$ (as defined above, with $1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k > 0$). When k tends to infinity, the integral of the top power of the curvature of L_k on its q-index set $X_k^{\mathrm{GG}}(L_k, q)$ is given by

$$\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_X \mathbb{1}_{\eta,q}\eta^n + O((\log k)^{-1})\Big)$$

for all $q = 0, 1, \dots, n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of Θ_V, η and ω. Moreover, the left hand side is identically zero for $q > n$.

The final statement follows from the observation that the curvature of L_k is positive along the fibers of $X_k^{\mathrm{GG}} \to X$, by the plurisubharmonicity of the weight (this is true even when the partition of unity terms are taken into account, since they depend only on the base); therefore the q-index sets are empty for $q > n$. It will be useful to extend the above estimates to the case of sections of

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)F\Big), \tag{9.35}$$

where $F \in \mathrm{Pic}_{\mathbb{Q}}(X)$ is an arbitrary $\mathbb{Q}$-line bundle on X and $\pi_k : X_k^{\mathrm{GG}} \to X$ is the natural projection. We assume here that F is also equipped with a smooth Hermitian metric h_F. In formulas (9.32), (9.33) and estimate 9.34, the renormalized curvature $\eta_k(z, x, u)$ of L_k takes the form

$$\eta_k(z, x, u) = \frac{1}{\frac{1}{kr}(1 + \frac{1}{2} + \cdots + \frac{1}{k})} g_k(z, x, u) + \Theta_{F,h_F}(z), \tag{9.36}$$

and by the same calculations its expected value is

$$\eta(z) := \mathbf{E}(\eta_k(z, \bullet, \bullet)) = \Theta_{\det V^*, \det h^*}(z) + \Theta_{F,h_F}(z). \tag{9.37}$$

Then the variance estimate for $\eta_k - \eta$ is unchanged, and the L^p bounds for η_k are still valid, since our forms are just shifted by adding the constant smooth term $\Theta_{F,h_F}(z)$. The probabilistic estimate 9.34 is therefore still true exactly in the same form, provided we use (9.35)–(9.37) instead of the previously defined L_k, η_k and η. An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} &h^q\Big(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)F\Big)\Big) \\ &= h^q\Big(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)F\Big)\Big), \end{aligned}$$

provided m is sufficiently divisible to give a multiple of F which is a $\mathbb{Z}$-line bundle.

9.38. Theorem. *Let (X, V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V, h) and (F, h_F) smooth Hermitian structures on V and F respectively. We define*

$$L_k = \mathscr{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^* \mathscr{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)F\Big),$$
$$\eta = \Theta_{\det V^*, \det h^*} + \Theta_{F, h_F},$$

and let $X(\eta, q)$ be the open set of points $x \in X$, where $\eta(x)$ has signature $(q, n-q)$. We also set $X(\eta, \leqslant q) = \bigcup_{j \leqslant q} X(\eta, j)$. Then for all $q \geqslant 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have

(a)
$$h^q(X_k^{\mathrm{GG}}, \mathscr{O}(L_k^{\otimes m})) \leqslant \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r} \Big(\int_{X(\eta,q)} (-1)^q \eta^n + O((\log k)^{-1})\Big),$$
(b)
$$h^0(X_k^{\mathrm{GG}}, \mathscr{O}(L_k^{\otimes m})) \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r} \Big(\int_{X(\eta,\leqslant 1)} \eta^n - O((\log k)^{-1})\Big),$$
(c)
$$\chi(X_k^{\mathrm{GG}}, \mathscr{O}(L_k^{\otimes m})) = \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r} (c_1(V^* \otimes F)^n + O((\log k)^{-1})).$$

Green and Griffiths [GrGr80] already checked the Riemann–Roch calculation (9.38c) in the special case $V = T_X^*$ and $F = \mathscr{O}_X$. Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies $\chi = h^0 - h^1 + h^2 \leqslant h^0 + h^2$, and hence it is enough to get the vanishing of the top cohomology group H^2 to infer $h^0 \geqslant \chi$; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\Big(X, E_{k,m}^{\mathrm{GG}} T_X^* \otimes \mathscr{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)F\Big)\Big) = 0$$

as soon as $K_X \otimes F$ is big and $m \gg 1$.

In fact, thanks to Bonavero's singular holomorphic Morse inequalities [Bon93], everything works almost unchanged in the case where $V \subset T_X$ has singularities and h is an admissible metric on V (see Definition 9.6). We only have to find a blow-up $\mu : \widetilde{X}_k \to X_k$ so that the resulting pull-backs $\mu^* L_k$ and $\mu^* V$ are locally free, and $\mu^* \det h^*$, $\mu^* \Psi_{h,p,\varepsilon}$ only have divisorial singularities. Then η is a $(1,1)$-current with logarithmic poles, and we have to deal with smooth metrics on $\mu^* L_k^{\otimes m} \otimes \mathscr{O}(-mE_k)$, where E_k is a certain effective divisor on

X_k (which, by our assumption in Definition 9.6, does not project onto X). The cohomology groups involved are then the twisted cohomology groups

$$H^q(X_k^{\mathrm{GG}}, \mathscr{O}(L_k^{\otimes m}) \otimes \mathscr{J}_{k,m}),$$

where $\mathscr{J}_{k,m} = \mu_*(\mathscr{O}(-mE_k))$ is the corresponding multiplier ideal sheaf, and the Morse integrals need only be evaluated in the complement of the poles, i.e., on $X(\eta, q) \smallsetminus S$ with $S = \mathrm{Sing}(V) \cup \mathrm{Sing}(h)$. Since

$$(\pi_k)_*(\mathscr{O}(L_k^{\otimes m}) \otimes \mathscr{J}_{k,m}) \subset E_{k,m}^{\mathrm{GG}} V^* \otimes \mathscr{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)F\Big)$$

we still get a lower bound for the h^0 of the latter sheaf (or for the h^0 of the un-twisted line bundle $\mathscr{O}(L_k^{\otimes m})$ on X_k^{GG}). If we assume that $K_V \otimes F$ is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of (X, V). The following corollary implies Theorem 9.1 as a consequence.

9.39. Corollary. *If F is an arbitrary $\mathbb{Q}$-line bundle over X, one has*

$$\begin{aligned}
&h^0\Big(X_k^{\mathrm{GG}}, \mathscr{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathscr{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)F\Big)\Big)\\
&\geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,(k!)^r}\big(\mathrm{Vol}(K_V \otimes F) - O((\log k)^{-1})\big) - o(m^{n+kr-1}),
\end{aligned}$$

when $m \gg k \gg 1$, in particular there are many sections of the k-jet differentials of degree m twisted by the appropriate power of F if $K_V \otimes F$ is big.

Proof. The volume is computed here as usual, i.e., after performing a suitable log-resolution $\mu : \widetilde{X} \to X$ which converts K_V into an invertible sheaf. There is of course nothing to prove if $K_V \otimes F$ is not big, so we can assume $\mathrm{Vol}(K_V \otimes F) > 0$. Let us fix smooth Hermitian metrics h_0 on T_X and h_F on F. They induce a metric $\mu^*(\det(h_0^{-1}) \otimes h_F)$ on $\mu^*(K_V \otimes F)$ which, by our definition of K_V, is a smooth metric (the divisor produced by the log-resolution gets simplified with the degeneration divisor of the pull-back of the quotient metric on $\det(V^*)$ induced by $\mathscr{O}(\Lambda^r T_X^*) \to \mathscr{O}(\Lambda^r V^*)$). By the result of Fujita [Fuji94] on approximate Zariski decomposition, for every $\delta > 0$, one can find a modification $\mu_\delta : \widetilde{X}_\delta \to X$ dominating μ such that

$$\mu_\delta^*(K_V \otimes F) = \mathscr{O}_{\widetilde{X}_\delta}(A + E),$$

where A and E are $\mathbb{Q}$-divisors, A ample and E effective, with

$$\mathrm{Vol}(A) = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta.$$

If we take a smooth metric h_A with positive definite curvature form Θ_{A,h_A}, then we get a singular Hermitian metric $h_A h_E$ on $\mu_\delta^*(K_V \otimes F)$ with poles along

E, i.e., the quotient $h_A h_E / \mu_\delta^*(\det(h_0^{-1}) \otimes h_F)$ is of the form $e^{-\varphi}$, where φ is quasi-plurisubharmonic with log poles $\log|\sigma_E|^2 \pmod{C^\infty(\widetilde{X}_\delta)}$ precisely given by the divisor E. We then only need to take the singular metric h on T_X defined by

$$h = h_0 e^{\frac{1}{r}(\mu_\delta)_* \varphi}$$

(the choice of the factor $\frac{1}{r}$ is there to correct adequately the metric on $\det V$). By construction h induces an admissible metric on V and the resulting curvature current $\eta = \Theta_{K_V, \det h^*} + \Theta_{F,h_F}$ is such that

$$\mu_\delta^* \eta = \Theta_{A,h_A} + [E], \quad [E] = \text{current of integration on } E.$$

Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0)\smallsetminus S} \eta^n = \int_{\widetilde{X}_\delta} \Theta_{A,h_A}^n = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta$$

and Corollary 9.39 follows from the fact that δ can be taken arbitrary small. □

The following corollary implies Theorem 0.12.

9.40. Corollary. *Let (X, V) be a projective directed manifold such that $K_V^\bullet$ is big, and A an ample $\mathbb{Q}$-divisor on X such that $K_V^\bullet \otimes \mathcal{O}(-A)$ is still big. Then, if we put $r = \operatorname{rank} V$ and $\delta_k = \frac{1}{kr}(1 + \frac{1}{2} + \cdots + \frac{1}{k})$, the space of global invariant jet differentials*

$$H^0(X, E_{k,m} V^* \otimes \mathcal{O}(-m\delta_k A))$$

has (many) non-zero sections for $m \gg k \gg 1$ and m sufficiently divisible.

Proof. Corollary 9.39 produces a non-zero section $P \in H^0(E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}_X(-m\delta_k A))$ for $m \gg k \gg 1$, and the arguments given in Subsect. 7.E (*cf.* (7.36)) yield a non-zero section

$$Q \in H^0(E_{k,m'} V^* \otimes \mathcal{O}_X(-m\delta_k A)), \quad m' \leqslant m.$$

By raising Q to some power p and using a section $\sigma \in H^0(X, \mathcal{O}_X(dA))$, we obtain a section

$$Q^p \sigma^{mq} \in H^0(X, E_{k,pm'} V^* \otimes \mathcal{O}(-m(p\delta_k - qd)A)).$$

One can adjust p and q so that $m(p\delta_k - qd) = pm'\delta_k$ and $pm'\delta_k A$ is an integral divisor. □

9.41. Example. In some simple cases, the above estimates can lead to very explicit results. Take for instance X to be a smooth complete intersection of multidegree $(d_1, d_2, \ldots, d_s)$ in $\mathbb{P}^{n+s}_{\mathbb{C}}$ and consider the absolute case $V = T_X$. Then $K_X = \mathcal{O}_X(d_1+\cdots+d_s-n-s-1)$ and one can check via explicit bounds of the error terms (*cf.* [Dem11], [Dem12]) that a sufficient condition for the existence of sections is

$$k \geqslant \exp\Big(7.38\, n^{n+\frac{1}{2}}\Big(\frac{\sum d_j + 1}{\sum d_j - n - s - a - 1}\Big)^n\Big).$$

This is good in view of the fact that we can cover arbitrary smooth complete intersections of general type. On the other hand, even when the degrees d_j tend to $+\infty$, we still get a large lower bound $k \sim \exp(7.38\, n^{n+\frac{1}{2}})$ on the order of jets, and this is far from being optimal: Diverio [Div08], [Div09] has shown e.g. that one can take $k = n$ for smooth hypersurfaces of high degree, using the algebraic Morse inequalities of Trapani [Tra95]. The next paragraph uses essentially the same idea, in our more analytic setting.

9.D. Non-probabilistic estimate of the Morse integrals

We assume here that the curvature tensor $(c_{ij\alpha\beta})$ satisfies a lower bound

$$\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\xi_i\overline{\xi}_j u_\alpha \overline{u}_\beta \geqslant -\sum \gamma_{ij}\xi_i\overline{\xi}_j\, |u|^2, \quad \forall \xi \in T_X,\ u \in V, \tag{9.42}$$

for some semi-positive $(1,1)$-form $\gamma = \frac{i}{2\pi}\sum \gamma_{ij}(z)\, dz_i \wedge d\overline{z}_j$ on X. This is the same as assuming that the curvature tensor of (V^*, h^*) satisfies the semi-positivity condition

$$\Theta_{V^*,h^*} + \gamma \otimes \mathrm{Id}_{V^*} \geqslant 0 \tag{9.42$'$}$$

in the sense of Griffiths, or equivalently $\Theta_{V,h} - \gamma \otimes \mathrm{Id}_V \leqslant 0$. Thanks to the compactness of X, such a form γ always exists if h is an admissible metric on V. Now, instead of replacing Θ_V with its trace free part $\widetilde{\Theta}_V$ and exploiting a Monte Carlo convergence process, we replace Θ_V with $\Theta_V^\gamma = \Theta_V - \gamma \otimes \mathrm{Id}_V \leqslant 0$, i.e., $c_{ij\alpha\beta}$ by $c^\gamma_{ij\alpha\beta} = c_{ij\alpha\beta} + \gamma_{ij}\delta_{\alpha\beta}$. Also, we take a line bundle $F = A^{-1}$ with $\Theta_{A,h_A} \geqslant 0$, i.e., F semi-negative. Then our earlier formulas Proposition 9.28, (9.35), (9.36) become instead

$$(9.43)\qquad g_k^\gamma(z,x,u) = \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}^\gamma(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j \geqslant 0,$$

$$(9.44)\qquad L_k = \mathscr{O}_{X_k^{\mathrm{GG}}}(1)\otimes\pi_k^*\mathscr{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)A\Big),$$

$$(9.45)\qquad \Theta_{L_k} = \eta_k(z,x,u)$$

$$= \frac{1}{\frac{1}{kr}(1+\frac{1}{2}+\cdots+\frac{1}{k})}g_k^\gamma(z,x,u) - (\Theta_{A,h_A}(z)+r\gamma(z)).$$

In fact, replacing Θ_V by $\Theta_V - \gamma\otimes\mathrm{Id}_V$ has the effect of replacing $\Theta_{\det V^*} = \mathrm{Tr}\,\Theta_{V^*}$ by $\Theta_{\det V^*}+r\gamma$. The major gain that we have is that $\eta_k = \Theta_{L_k}$ is now expressed as a difference of semi-positive $(1,1)$-forms, and we can exploit the following simple lemma, which is the key to derive algebraic Morse inequalities from their analytic form (*cf.* [Dem94], Theorem 12.3).

9.46. Lemma. *Let $\eta = \alpha - \beta$ be a difference of semi-positive $(1,1)$-forms on an n-dimensional complex manifold X, and let $\mathbb{1}_{\eta,\leqslant q}$ be the characteristic function of the open set, where η is non-degenerate with a number of negative eigenvalues at most equal to q. Then*

$$(-1)^q\mathbb{1}_{\eta,\leqslant q}\,\eta^n \leqslant \sum_{0\leqslant j\leqslant q}(-1)^{q-j}\alpha^{n-j}\beta^j,$$

in particular

$$\mathbb{1}_{\eta,\leqslant 1}\,\eta^n \geqslant \alpha^n - n\alpha^{n-1}\wedge\beta\quad \textit{for } q=1.$$

Proof. Without loss of generality, we can assume $\alpha>0$ positive definite, so that α can be taken as the base Hermitian metric on X. Let us denote by

$$\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n\geqslant 0$$

the eigenvalues of β with respect to α. The eigenvalues of $\eta=\alpha-\beta$ are then given by

$$1-\lambda_1\leqslant\cdots\leqslant 1-\lambda_q\leqslant 1-\lambda_{q+1}\leqslant\cdots\leqslant 1-\lambda_n;$$

hence the open set $\{\lambda_{q+1}<1\}$ coincides with the support of $\mathbb{1}_{\eta,\leqslant q}$, except that it may also contain a part of the degeneration set $\eta^n=0$. On the other hand we have

$$\binom{n}{j}\alpha^{n-j}\wedge\beta^j = \sigma_n^j(\lambda)\,\alpha^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in the λ_j's. Thus, to prove the lemma, we only have to check that

$$\sum_{0\leqslant j\leqslant q}(-1)^{q-j}\sigma_n^j(\lambda) - \mathbb{1}_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\leqslant j\leqslant n}(1-\lambda_j)\geqslant 0.$$

This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda) = \sigma_{n-1}^j(\lambda) + \sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). □

We apply here Lemma 9.46 with

$$\alpha = g_k^\gamma(z,x,u), \quad \beta = \beta_k = \frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)(\Theta_{A,h_A}+r\gamma),$$

which are both semi-positive by our assumption. The analogue of (9.32) leads to

$$\begin{aligned}
&\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)} \Theta_{L_k,\Psi_{h,p,\varepsilon}^*}^{n+kr-1} \\
&= \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!} \\
&\quad\times \int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} \mathbb{1}_{g_k^\gamma-\beta_k,\leqslant 1}\,(g_k^\gamma-\beta_k)^n\,d\nu_{k,r}(x)\,d\mu(u) \\
&\geqslant \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!} \\
&\quad\times \int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} ((g_k^\gamma)^n - n(g_k^\gamma)^{n-1}\wedge\beta_k)\,d\nu_{k,r}(x)\,d\mu(u).
\end{aligned}$$

The resulting integral now produces a "closed formula" which can be expressed solely in terms of Chern classes (at least if we assume that γ is the Chern form of some semi-positive line bundle). It is just a matter of routine to find a sufficient condition for the positivity of the integral. One can first observe that g_k^γ is bounded from above by taking the trace of $(c_{ij\alpha\beta})$, in this way we get

$$0 \leqslant g_k^\gamma \leqslant \Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)(\Theta_{\det V^*}+r\gamma),$$

where the right hand side no longer depends on $u\in(S^{2r-1})^k$. Also, g_k^γ can be written as a sum of semi-positive $(1,1)$-forms

$$g_k^\gamma = \sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\theta^\gamma(u_s), \quad \theta^\gamma(u) = \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}^\gamma u_\alpha\overline{u}_\beta\,dz_i\wedge d\overline{z}_j,$$

and hence for $k\geqslant n$ we have

$$(g_k^\gamma)^n \geqslant n! \sum_{1\leqslant s_1<\cdots<s_n\leqslant k}\frac{x_{s_1}\cdots x_{s_n}}{s_1\cdots s_n}\,\theta^\gamma(u_{s_1})\wedge\theta^\gamma(u_{s_2})\wedge\cdots\wedge\theta^\gamma(u_{s_n}).$$

Since $\int_{S^{2r-1}} \theta^\gamma(u)\,d\mu(u) = \frac{1}{r}\,\mathrm{Tr}(\Theta_{V^*}+\gamma) = \frac{1}{r}\Theta_{\det V^*}+\gamma$, we infer from this

$$\begin{aligned}&\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} (g_k^\gamma)^n\,d\nu_{k,r}(x)\,d\mu(u)\\ &\geqslant n!\sum_{1\leqslant s_1<\cdots<s_n\leqslant k}\frac{1}{s_1\cdots s_n}\Big(\int_{\Delta_{k-1}} x_1\cdots x_n\,d\nu_{k,r}(x)\Big)\Big(\frac{1}{r}\Theta_{\det V^*}+\gamma\Big)^n.\end{aligned}$$

By putting everything together, we conclude:

9.47. Theorem. *Assume that $\Theta_{V^*}\geqslant -\gamma\otimes \mathrm{Id}_{V^*}$ with a semi-positive $(1,1)$-form γ on X. Then the Morse integral of the line bundle*

$$L_k = \mathscr{O}_{X_k^{\mathrm{GG}}}(1)\otimes\pi_k^*\mathscr{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)A\Big),\quad A\geqslant 0$$

satisfies for $k\geqslant n$ the inequality

$(*)$

$$\begin{aligned}&\frac{1}{(n+kr-1)!}\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}\\ &\geqslant\frac{1}{n!(k!)^r(kr-1)!}\\ &\quad\times\int_X c_{n,r,k}(\Theta_{\det V^*}+r\gamma)^n - c'_{n,r,k}(\Theta_{\det V^*}+r\gamma)^{n-1}\wedge(\Theta_{A,h_A}+r\gamma),\end{aligned}$$

where

$$\begin{aligned}c_{n,r,k}&=\frac{n!}{r^n}\Big(\sum_{1\leqslant s_1<\cdots<s_n\leqslant k}\frac{1}{s_1\cdots s_n}\Big)\int_{\Delta_{k-1}}x_1\cdots x_n\,d\nu_{k,r}(x),\\ c'_{n,r,k}&=\frac{n}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^{n-1}d\nu_{k,r}(x).\end{aligned}$$

Especially we have a lot of sections in $H^0(X_k^{\mathrm{GG}}, mL_k)$, $m\gg 1$, as soon as the difference occurring in $(*)$ is positive.

The statement is also true for $k<n$, but then $c_{n,r,k}=0$ and the lower bound $(*)$ cannot be positive. By Corollary 9.11, it still provides a non-trivial lower bound for $h^0(X_k^{\mathrm{GG}}, mL_k) - h^1(X_k^{\mathrm{GG}}, mL_k)$, though. For $k\geqslant n$ we have $c_{n,r,k}>0$ and $(*)$ will be positive if $\Theta_{\det V^*}$ is large enough. By formula (9.20) we have

$$c_{n,r,k}=\frac{n!\,(kr-1)!}{(n+kr-1)!}\sum_{1\leqslant s_1<\cdots<s_n\leqslant k}\frac{1}{s_1\cdots s_n}\geqslant\frac{(kr-1)!}{(n+kr-1)!},\tag{9.48}$$

(with equality for $k = n$). On the other hand, for any multi-index $(\beta_1, \dots, \beta_k) \in \mathbb{N}^k$ with $\sum \beta_s = p$, the Hölder inequality implies

$$\begin{aligned}\int_{\Delta_{k-1}} x_1^{\beta_1} \cdots x_k^{\beta_k}\, d\nu_{k,r}(x) &\leqslant \prod_{s=1} \Big(\int_{\Delta_{k-1}} x_s^p\, d\nu_{k,r}(x)\Big)^{\beta_s/p}\\ &= \int_{\Delta_{k-1}} x_1^p\, d\nu_{k,r}(x).\end{aligned}$$

An expansion of $(\sum_{1\leqslant s\leqslant k} \frac{x_s}{s})^{n-1}$ by means of the multinomial formula then yields

$$\int_{\Delta_{k-1}} \Big(\sum_{1\leqslant s\leqslant k} \frac{x_s}{s}\Big)^{n-1} d\nu_{k,r}(x) \leqslant \int_{\Delta_{k-1}} \Big(\sum_{1\leqslant s\leqslant k} \frac{1}{s}\Big)^{n-1} x_1^{n-1}\, d\nu_{k,r}(x).$$

On the other hand, it is obvious that $\int_{\Delta_{k-1}} (\sum_{1\leqslant s\leqslant k} \frac{x_s}{s})^{n-1}\, d\nu_{k,r}(x) \geqslant \int_{\Delta_{k-1}} x_1^{n-1}\, d\nu_{k,r}(x)$, thus the error in the above upper bound is at most by a factor $(1 + \frac{1}{2} + \cdots + \frac{1}{k})^n \leqslant (1 + \log k)^n$. From this, we infer again by formula (9.20) that

$$\begin{aligned}c'_{n,r,k} &\leqslant \frac{n}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)^n \int_{\Delta_{k-1}} x_1^{n-1}\, d\nu_{k,r}(x),\\ &= \frac{n}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)^n \frac{(n+r-2)!}{(r-1)!}\, \frac{(kr-1)!}{(n+kr-2)!}.\end{aligned} \tag{9.49}$$

Since $\frac{n+kr-1}{k} = r + \frac{n-1}{k} \leqslant n + r - 1$, our bounds (9.48) and (9.49) imply

$$\frac{c'_{n,r,k}}{c_{n,r,k}} \leqslant \frac{n}{k}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)^n \frac{(n+r-2)!}{r!}\,(n+kr-1), \tag{9.50}$$

$$\frac{c'_{n,r,k}}{c_{n,r,k}} \leqslant n\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)^n \frac{(n+r-1)!}{r!}. \tag{9.51}$$

The right hand side of (9.51) increases with r. For $r \leqslant n$, the Stirling formula yields

$$\begin{aligned}\frac{c'_{n,r,k}}{c_{n,r,k}} &< (1 + \log k)^n \frac{(2n)!}{2\,n!}\\ &< (1 + \log k)^n \frac{\sqrt{2n}\,(\frac{2n}{e})^{2n}}{2\sqrt{n}\,(\frac{n}{e})^n} = \frac{1}{\sqrt{2}}(4e^{-1}\, n(1 + \log k))^n.\end{aligned} \tag{9.52$_n$}$$

Up to the constant $4e^{-1}$, this is essentially the same bound as the one obtained in [Dem12], which, however, included a numerical mistake, making unclear whether the constant $4e^{-1} > 1$ could be dropped there, as would follow from the claimed estimate. We will later need the following particular values (*cf.* Formula (9.20) and [Dem11, Lemma 2.20]):

$$c_{2,2,2} = \frac{1}{20}, \quad c'_{2,2,2} = \frac{9}{16}, \quad \frac{c'_{2,2,2}}{c_{2,2,2}} = \frac{45}{4}, \tag{9.52$_2$}$$

$$c_{3,3,3} = \frac{1}{990}, \quad c'_{3,3,3} = \frac{451}{4860}, \quad \frac{c'_{3,3,3}}{c_{3,3,3}} = \frac{4961}{54}. \tag{9.52$_3$}$$

10. Hyperbolicity properties of hypersurfaces of high degree

10.A. Global generation of the twisted tangent space of the universal family

In [Siu02], [Siu04], Y.-T. Siu developed a new strategy to produce jet differentials, involving meromorphic vector fields on the total space of jet bundles—these vector fields are used to differentiate the sections of $E^{\mathrm{GG}}_{k,m}$ so as to produce new ones with less zeroes. The approach works especially well on universal families of hypersurfaces in projective space, thanks to the good positivity properties of the relative tangent bundle, as shown by L. Ein [Ein88], [Ein91] and C. Voisin [Voi96]. This allows at least to prove the hyperbolicity of generic surfaces and generic 3-dimensional hypersurfaces of sufficiently high degree. We reproduce here the improved approach given by [Pau08] for the twisted global generation of the tangent space of the space of vertical two jets. The situation of k-jets in arbitrary dimension n is substantially more involved, details can be found in [Mer09].

Consider the universal hypersurface $\mathscr{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ of degree d given by the equation

$$\sum_{|\alpha|=d} A_\alpha\, Z^\alpha = 0,$$

where $[Z] \in \mathbb{P}^{n+1}$, $[A] \in \mathbb{P}^{N_d}$, $\alpha = (\alpha_0, \ldots, \alpha_{n+1}) \in \mathbb{N}^{n+2}$ and

$$N_d = \binom{n+d+1}{d} - 1.$$

Finally, we denote by $\mathscr{V} \subset \mathscr{X}$ the vertical tangent space, i.e., the kernel of the projection

$$\pi : \mathscr{X} \longrightarrow U \subset \mathbb{P}^{N_d},$$

where U is the Zariski open set parametrizing smooth hypersurfaces, and by $J_k\mathscr{V}$ the bundle of k-jets of curves tangent to $\mathscr{V}$, i.e., curves contained in the fibers $X_s = \pi^{-1}(s)$. The goal is to describe certain meromorphic vector fields on the total space of $J_k\mathscr{V}$. By an explicit calculation of vector fields in coordinates, according to Siu's strategy, Păun [Pau08] was able to prove:

10.1. Theorem. *The twisted tangent space $T_{J_2\mathcal{V}} \otimes \mathcal{O}_{\mathbb{P}^3}(7) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(1)$ is generated over by its global sections over the complement $J_2\mathcal{V} \smallsetminus \mathcal{W}$ of the Wronskian locus $\mathcal{W}$. Moreover, one can choose generating global sections that are invariant with respect to the action of $\mathbb{G}_2$ on $J_2\mathcal{V}$.*

By similar, but more computationally intensive arguments [Mer09], one can investigate the higher dimensional case. The following result strengthens the initial announcement of [Siu04].

10.2. Theorem. *Let $J_k^{\mathrm{vert}}(\mathcal{X})$ be the space of vertical k-jets of the universal hypersurface*

$$\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$$

parametrizing all projective hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d. Then for $k = n$, there exist constants c_n and c'_n such that the twisted tangent bundle

$$T_{J_k^{\mathrm{vert}}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(c_n) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(c'_n)$$

is generated by its global $\mathbb{G}_k$-invariant sections outside a certain exceptional algebraic subset $\Sigma \subset J_k^{\mathrm{vert}}(\mathcal{X})$. One can take either $c_n = \frac{1}{2}(n^2+5n)$, $c'_n = 1$ and Σ defined by the vanishing of certain Wronskians, or $c_n = n^2 + 2n$ and a smaller set $\widetilde{\Sigma} \subset \Sigma$ defined by the vanishing of the 1-jet part.

10.B. General strategy of proof

Let again $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ be the universal hypersurface of degree d in $\mathbb{P}^{n+1}$.

10.3. *Assume that we can prove the existence of a non-zero polynomial differential operator*

$$P \in H^0(\mathcal{X}, E_{k,m}^{\mathrm{GG}} T_{\mathcal{X}}^* \otimes \mathcal{O}(-A)),$$

where A is an ample divisor on $\mathcal{X}$, at least over some Zariski open set U in the base of the projection $\pi : \mathcal{X} \to U \subset \mathbb{P}^{N_d}$.

Observe that we now have a lot of techniques to do this; the existence of P over the family follows from lower semi-continuity in the Zariski topology, once we know that such a section P exists on a generic fiber $X_s = \pi^{-1}(s)$. Let $\mathcal{Y} \subset \mathcal{X}$ be the set of points $x \in \mathcal{X}$ where $P(x) = 0$, as an element in the fiber of the vector bundle $E_{k,m}^{\mathrm{GG}} T_{\mathcal{X}}^* \otimes \mathcal{O}(-A)$ at x. Then $\mathcal{Y}$ is a proper algebraic subset of $\mathcal{X}$, and after shrinking U we may assume that $Y_s = \mathcal{Y} \cap X_s$ is a proper algebraic subset of X_s for every $s \in U$.

10.4. *Assume also, according to Theorems 10.1 and 10.2, that we have enough global holomorphic $\mathbb{G}_k$-invariant vector fields θ_i on $J_k\mathcal{V}$ with values in the pull-back of some ample divisor B on $\mathcal{X}$, in such a way that they generate $T_{J_k\mathcal{V}} \otimes p_k^* B$ over the dense open set $(J_k\mathcal{V})^{\mathrm{reg}}$ of regular k-jets, i.e., k-jets with non-zero first derivative (here $p_k : J_k\mathcal{V} \to \mathcal{X}$ is the natural projection).*

Considering jet differentials P as functions on $J_k\mathscr{V}$, the idea is to produce new ones by taking differentiations

$$Q_j := \theta_{j_1}\cdots\theta_{j_\ell}P, \quad 0 \leqslant \ell \leqslant m,\ j = (j_1,\ldots,j_\ell).$$

Since the θ_j's are $\mathbb{G}_k$-invariant, they are in particular $\mathbb{C}^*$-invariant; thus

$$Q_j \in H^0(\mathscr{X}, E^{\mathrm{GG}}_{k,m}T^*_{\mathscr{X}} \otimes \mathscr{O}(-A+\ell B))$$

(and Q is in fact $\mathbb{G}'_k$ invariant as soon as P is). In order to be able to apply the vanishing theorems of Sect. 8, we need $(A - mB)$ to be ample, so A has to be large compared to B. If $f : \mathbb{C} \to X_s$ is an entire curve contained in some fiber $X_s \subset \mathscr{X}$, its lifting $j_k(f) : \mathbb{C} \to J_k\mathscr{V}$ has to lie in the zero divisors of all sections Q_j. However, every non-zero polynomial of degree m has at any point some non-zero derivative of order $\ell \leqslant m$. Therefore, at any point where the θ_i generate the tangent space to $J_k\mathscr{V}$, we can find some non-vanishing section Q_j. By the assumptions on the θ_i, the base locus of the Q_j's is contained in the union of $p_k^{-1}(\mathscr{Y}) \cup (J_k\mathscr{V})^{\mathrm{sing}}$; there is of course no way of getting a non-zero polynomial at points of $\mathscr{Y}$ where P vanishes. Finally, we observe that $j_k(f)(\mathbb{C}) \not\subset (J_k\mathscr{V})^{\mathrm{sing}}$ (otherwise f is constant). Therefore $j_k(f)(\mathbb{C}) \subset p_k^{-1}(\mathscr{Y})$ and thus $f(\mathbb{C}) \subset \mathscr{Y}$, i.e., $f(\mathbb{C}) \subset Y_s = \mathscr{Y} \cap X_s$.

10.5. Corollary. *Let $\mathscr{X} \subset \mathbb{P}^{n+1}\times\mathbb{P}^{N_d}$ be the universal hypersurface of degree d in $\mathbb{P}^{n+1}$. If $d \geqslant d_n$ is taken so large that conditions (10.3) and (10.4) are met with $(A-mB)$ ample, then the generic fiber X_s of the universal family $\mathscr{X} \to U$ satisfies the Green–Griffiths conjecture, namely all entire curves $f : \mathbb{C} \to X_s$ are contained in a proper algebraic subvariety $Y_s \subset X_s$, and the Y_s can be taken to form an algebraic subset $\mathscr{Y} \subset \mathscr{X}$.*

This is unfortunately not enough to get the hyperbolicity of X_s, because we would have to know that Y_s itself is hyperbolic. However, one can use the following simple observation due to Diverio and Trapani [DT10]. The starting point is the following general, straightforward remark. Let $\mathscr{E} \to \mathscr{X}$ be a holomorphic vector bundle let $\sigma \in H^0(\mathscr{X}, \mathscr{E}) \neq 0$; then, up to factorizing by an effective divisor D contained in the common zeroes of the components of σ, one can view σ as a section

$$\sigma \in H^0(\mathscr{X}, \mathscr{E} \otimes \mathscr{O}_{\mathscr{X}}(-D)),$$

and this section now has a zero locus without divisorial components. Here, when $n \geqslant 2$, a very generic fiber X_s has Picard number one by the Noether–Lefschetz theorem, and so, after shrinking U if necessary, we can assume that $\mathscr{O}_{\mathscr{X}}(-D)$ is the restriction of $\mathscr{O}_{\mathbb{P}^{n+1}}(-p)$, $p \geqslant 0$ by the effectivity of D. Hence D can be assumed to be nef. After performing this simplification, $(A-mB)$ is replaced by $(A - mB + D)$, which is still ample if $(A - mB)$ is ample. As a consequence, we may assume $\operatorname{codim} \mathscr{Y} \geqslant 2$, and after shrinking U again, that all Y_s have $\operatorname{codim} Y_s \geqslant 2$.

10.6. Additional statement. *In Corollary 10.5, under the same hypotheses (10.3) and (10.4), one can take all fibers Y_s to have $\operatorname{codim} Y_s \geqslant 2$.*

This is enough to conclude that X_s is hyperbolic if $n = \dim X_s \leqslant 3$. In fact, this is clear if $n = 2$ since the Y_s are then reduced to points. If $n = 3$, the Y_s are at most curves, but we know by Ein and Voisin that a very generic hypersurface $X_s \subset \mathbb{P}^4$ of degree $d \geqslant 7$ does not possess any rational or elliptic curve. Hence Y_s is hyperbolic and so is X_s, for s generic.

10.7. Corollary. *Assume that $n = 2$ or $n = 3$, and that $\mathscr{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ is the universal hypersurface of degree $d \geqslant d_n \geqslant 2n+1$ so large that conditions 10.3 and 10.4 are met with $(A - mB)$ ample. Then the very generic hypersurface $X_s \subset \mathbb{P}^{n+1}$ of degree d is hyperbolic.*

10.C. Proof of the Green–Griffiths conjecture for generic hypersurfaces in $\mathbb{P}^{n+1}$

One of the first significant steps towards the Green–Griffiths conjecture is the result of Diverio, Merker and Rousseau [DMR10], confirming the statement when $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a generic hypersurface of large degree d. Their proof yields a non-optimal lower bound $d \geqslant 2^{n^5}$ for the degree; it is based on an essential way on Siu's strategy as detailed in Subsect. 10.B, combined with the earlier techniques of [Dem95]. Using our improved bounds from Subsect. 9.D, we obtain here a better estimate (actually, an estimate $O(\exp(n^{1+\varepsilon}))$ of exponential order 1 rather than 5). For the algebraic degeneracy of entire curves in open complements $X = \mathbb{P}^n \smallsetminus H$, a better bound $d \geqslant 5n^2 n^n$ has been obtained by Darondeau [Dar14], [Dar16b].

10.8. Theorem. *A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ with*

$$d_2 = 286, \quad d_3 = 7316, \quad d_n = \left\lfloor \frac{n^4}{\sqrt{2}} (4e^{-1} n (1 + \log n))^n \right\rfloor \quad \textit{for } n \geqslant 4,$$

satisfies the Green–Griffiths conjecture.

Proof. Let us apply Theorem 9.47 with $V = T_X$, $r = n$ and $k = n$. The main starting point is the well-known fact that $T^*_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(2)$ is semi-positive (in fact, generated by its sections). Hence the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(-d) \longrightarrow T^*_{\mathbb{P}^{n+1}|X} \longrightarrow T^*_X \longrightarrow 0$$

implies that $T^*_X \otimes \mathscr{O}_X(2) \geqslant 0$. We can therefore take $\gamma = \Theta_{\mathscr{O}(2)} = 2\omega$, where ω is the Fubini–Study metric. Moreover $\det(V^*) = K_X = \mathscr{O}_X(d-n-2)$ has curvature $(d-n-2)\omega$, and thus $\Theta_{\det(V^*)} + r\gamma = (d+n-2)\omega$. The Morse integral to be computed when $A = \mathscr{O}_X(p)$ is

$$\int_X (c_{n,n,n}(d+n-2)^n - c'_{n,n,n}(d+n-2)^{n-1}(p+2n))\omega^n,$$

so the critical condition we need is

$$d+n-2 > \frac{c'_{n,n,n}}{c_{n,n,n}}(p+2n).$$

On the other hand, Siu's differentiation technique requires $\frac{m}{n^2}(1+\frac{1}{2}+\cdots+\frac{1}{n})A - mB$ to be ample, where $B = \mathscr{O}_X(n^2+2n)$ by Merker's result (Theorem 10.2). This ampleness condition yields

$$\frac{1}{n^2}\Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)p - (n^2+2n) > 0,$$

so one easily sees that it is enough to take $p = n^4 - 2n$ for $n \geqslant 3$. Our estimates (9.52_n) give the expected bound d_n. □

Thanks to 10.6, one also obtains the generic hyperbolicity of 2 and 3-dimensional hypersurfaces of large degree.

10.9. Theorem. *For $n = 2$ or $n = 3$, a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ is Kobayashi hyperbolic.*

By using more explicit calculations of Chern classes (and invariant jets rather than Green–Griffiths jets) Diverio–Trapani [DT10] obtained the better lower bound $d \geqslant d_3 = 593$ in dimension 3. In the case of surfaces, Păun [Pau08] obtained $d \geqslant d_2 = 18$, using deep results of McQuillan [McQ98].

One may wonder whether it is possible to use jets of order $k < n$ in the proofs of Theorems 10.8 and 10.9. Diverio [Div08] showed that the answer is negative (his proof is based on elementary facts of representation theory and a vanishing theorem of Brückmann–Rackwitz [BR90]):

10.10. Proposition ([Div08]). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. Then*

$$H^0(X, E^{\mathrm{GG}}_{k,m}T^*_X) = 0$$

for $m \geqslant 1$ and $1 \leqslant k < n$. More generally, if $X \subset \mathbb{P}^{n+s}$ is a smooth complete intersection of codimension s, there are no global jet differentials for $m \geqslant 1$ and $k < n/s$.

11. Strong general type condition and the GGL conjecture

11.A. A partial result towards the Green–Griffiths–Lang conjecture

The main result of this section is a proof of the partial solution to the Green–Griffiths–Lang conjecture asserted in Theorem 0.15. The following important "induction step" can be derived by Corollary 9.39. Here D_k denotes again the sequence of "vertical divsors" defined in (6.9).

11.1. Proposition. *Let (X,V) be a directed pair, where X is projective algebraic. Take an irreducible algebraic subset $Z \not\subset D_k$ of the associated k-jet Semple bundle X_k that projects onto X_{k-1}, $k \geqslant 1$, and assume that the induced directed space $(Z,W) \subset (X_k,V_k)$ is of general type modulo $X_\bullet \to X$,* $\operatorname{rank} W \geqslant 1$. *Then there exists a divisor $\Sigma \subset Z_\ell$ in a sufficiently high stage of the Semple tower (Z_ℓ, W_ℓ) associated with (Z,W), such that every nonconstant holomorphic map $f:\mathbb{C}\to X$ whose k-jet defines a morphism $f_{[k]}$: $(\mathbb{C},T_{\mathbb{C}}) \to (Z,W)$ also satisfies $f_{[k+\ell]}(\mathbb{C}) \subset \Sigma$.*

Proof. Our hypothesis is that we can find an embedded resolution of singularities

$$\mu_{\ell_0} : (\widehat{Z}_{\ell_0} \subset \widehat{X}_{k+\ell_0}) \longrightarrow (Z_{\ell_0} \subset X_{k+\ell_0}), \quad \ell_0 \geqslant 0$$

and $p \in \mathbb{Q}_{\geqslant 0}$ such that

$$K^\bullet_{\widehat{W}_{\ell_0}} \otimes \mathscr{O}_{\widehat{Z}_{\ell_0}}(p)_{\restriction \widehat{Z}_{\ell_0}} \quad \text{is big over } \widehat{Z}_{\ell_0}.$$

Since Corollary 9.39 and the related lower bound of h^0 are universal in the category of directed varieties, we can apply them by replacing (X,V) with $(\widehat{Z}_{\ell_0}, \widehat{W}_{\ell_0})$, r with $r_0 = \operatorname{rank} W$, and F by

$$F_{\ell_0} = \mathscr{O}_{\widehat{Z}_{\ell_0}}(p) \otimes \mu^*_{\ell_0}\pi^*_{k+\ell_0,0}\mathscr{O}_X(-\varepsilon A),$$

where A is an ample divisor on X and $\varepsilon \in \mathbb{Q}_{>0}$. The assumptions show that $K_{\widehat{W}_{\ell_0}} \otimes F_{\ell_0}$ is still big on $\widehat{Z}_{\ell_0}$ for ε small enough, therefore, by applying our theorem and taking $m \gg \ell \gg \ell_0$, we get a large number of (metric bounded) sections of

$$\begin{aligned}
&\mathscr{O}_{\widehat{Z}_\ell}(m) \otimes \widehat{\pi}^*_{k+\ell,k+\ell_0}\mathscr{O}\Big(\frac{m}{\ell r_0}\Big(1+\frac{1}{2}+\cdots+\frac{1}{\ell}\Big)F_{\ell_0}\Big)\\
&= \mathscr{O}_{\widehat{Z}_\ell}(ma_\bullet) \otimes \mu^*_\ell\pi^*_{k+\ell,0}\mathscr{O}\Big(-\frac{m\varepsilon}{\ell r_0}\Big(1+\frac{1}{2}+\cdots+\frac{1}{\ell}\Big)A\Big)_{\restriction \widehat{Z}_\ell}\\
&\subset \mathscr{O}_{\widehat{Z}_\ell}((1+\lambda)m) \otimes \mu^*_\ell\pi^*_{k+\ell,0}\mathscr{O}\Big(-\frac{m\varepsilon}{\ell r_0}\Big(1+\frac{1}{2}+\cdots+\frac{1}{\ell}\Big)A\Big)_{\restriction \widehat{Z}_\ell},
\end{aligned} \tag{11.2}$$

where $\mu_\ell : (\widehat{Z}_\ell \subset \widehat{X}_{k+\ell}) \to (Z_\ell \subset X_{k+\ell})$ is an embedded resolution dominating $\widehat{X}_{k+\ell_0}$, and $a_\bullet \in \mathbb{Q}_+^{\ell'}$ a positive weight of the form $(0, \ldots, \lambda, \ldots, 0, 1)$ with some non-zero component $\lambda \in \mathbb{Q}_+$ at index ℓ_0. Let $\widehat{\Sigma} \subset \widehat{Z}_\ell$ be the divisor of such a section. We apply the fundamental vanishing theorem 8.9 to lifted curves $\widehat{f}_{[k+\ell]} : \mathbb{C} \to \widehat{Z}_\ell$ and sections of (11.2), and conclude that $\widehat{f}_{[k+\ell]}(\mathbb{C}) \subset \widehat{\Sigma}$. Therefore $f_{[k+\ell]}(\mathbb{C}) \subset \Sigma := \mu_\ell(\widehat{\Sigma})$ and Proposition 11.1 is proved. □

We now introduce the ad hoc condition that will enable us to check the GGL conjecture.

11.3. Definition. *Let (X, V) be a directed pair, where X is projective algebraic. We say that (X, V) is "strongly of general type" if it is of general type and for every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, that projects onto X, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_\bullet \to X$.*

11.4. Example. The situation of a product $(X, V) = (X', V') \times (X'', V'')$ described in (0.14) shows that (X, V) can be of general type without being strongly of general type. In fact, if (X', V') and (X'', V'') are of general type, then $K_V = \mathrm{pr}'^* K_{V'} \otimes \mathrm{pr}''^* K_{V''}$ is big, so (X, V) is again of general type. However

$$Z = P(\mathrm{pr}'^* V') = X_1' \times X'' \subset X_1$$

has a directed structure $W = \mathrm{pr}'^* V_1'$ which does not possess a big canonical bundle over Z, since the restriction of K_W to any fiber $\{x'\} \times X''$ is trivial. The higher stages (Z_k, W_k) of the Semple tower of (Z, W) are given by $Z_k = X_{k+1}' \times X''$ and $W_k = \mathrm{pr}'^* V_{k+1}'$, so it is easy to see that $\mathrm{GG}_k(X, V)$ contains Z_{k-1}. Since Z_k projects onto X, we have here $\mathrm{GG}(X, V) = X$ (see [DR15] for more sophisticated indecomposable examples).

11.5. Hypersurface case. Assume that $Z \neq D_k$ is an irreducible hypersurface of X_k that projects onto X_{k-1}. To simplify things further, also assume that V is non-singular. Since the Semple jet-bundles X_k form a tower of $\mathbb{P}^{r-1}$-bundles, their Picard groups satisfy $\mathrm{Pic}(X_k) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}^k$ and we have $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(a_\bullet) \otimes \pi_{k,0}^* B$ for some $a_\bullet \in \mathbb{Z}^k$ and $B \in \mathrm{Pic}(X)$, where $a_k = d > 0$ is the relative degree of the hypersurface over X_{k-1}. Let $\sigma \in H^0(X_k, \mathcal{O}_{X_k}(Z))$ be the section defining Z in X_k. The induced directed variety (Z, W) has rank $W = r - 1 = \mathrm{rank}(V) - 1$ and formula (7.25) yields $K_{V_k} = \mathcal{O}_{X_k}(-(r-1)1_\bullet) \otimes \pi_{k,0}^*(K_V)$. We claim that

$$\begin{aligned} K_W &\supset (K_{V_k} \otimes \mathcal{O}_{X_k}(Z))_{\restriction Z} \otimes \mathcal{J}_S \\ &= (\mathcal{O}_{X_k}(a_\bullet - (r-1)1_\bullet) \otimes \pi_{k,0}^*(B \otimes K_V))_{\restriction Z} \otimes \mathcal{J}_S, \end{aligned} \tag{11.5.1}$$

where $S \subsetneq Z$ is the set (containing Z_{sing}) where σ and $d\sigma_{\restriction V_k}$ both vanish, and $\mathcal{J}_S$ is the ideal locally generated by the coefficients of $d\sigma_{\restriction V_k}$ along

$Z = \sigma^{-1}(0)$. In fact, the intersection $W = T_Z \cap V_k$ is transverse on $Z \smallsetminus S$; then (11.5.1) can be seen by looking at the morphism

$$V_{k|Z} \xrightarrow{d\sigma_{\restriction V_k}} \mathscr{O}_{X_k}(Z)_{\restriction Z},$$

and observing that the contraction by $K_{V_k} = \Lambda^r V_k^*$ provides a metric bounded section of the canonical sheaf K_W. In order to investigate the positivity properties of K_W, one has to show that B cannot be too negative, and in addition to control the singularity set S. The second point is a priori very challenging, but we get useful information for the first point by observing that σ provides a morphism $\pi_{k,0}^* \mathscr{O}_X(-B) \to \mathscr{O}_{X_k}(a_\bullet)$, whence a non-trivial morphism

$$\mathscr{O}_X(-B) \longrightarrow E_{a_\bullet} := (\pi_{k,0})_* \mathscr{O}_{X_k}(a_\bullet).$$

By [Dem95, Sect. 12], there exists a filtration on $E_{a_\bullet}$ such that the graded pieces are irreducible representations of $\mathrm{GL}(V)$ contained in $(V^*)^{\otimes \ell}$, $\ell \leqslant |a_\bullet|$. Therefore we get a non-trivial morphism

$$\mathscr{O}_X(-B) \to (V^*)^{\otimes \ell}, \quad \ell \leqslant |a_\bullet|. \tag{11.5.2}$$

If we know about certain (semi-)stability properties of V, this can be used to control the negativity of B. □

We further need the following useful concept that slightly generalizes entire curve loci.

11.6. Definition. *If Z is an algebraic set contained in some stage X_k of the Semple tower of (X, V), we define its "induced entire curve locus" $\mathrm{IEL}_{X,V}(Z) \subset Z$ to be the Zariski closure of the union $\bigcup f_{[k]}(\mathbb{C})$ of all jets of entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ such that $f_{[k]}(\mathbb{C}) \subset Z$.*

We have of course $\mathrm{IEL}_{X,V}(\mathrm{IEL}_{X,V}(Z)) = \mathrm{IEL}_{X,V}(Z)$ by definition. It is not hard to check that modulo certain "vertical divisors" of X_k, the $\mathrm{IEL}_{X,V}(Z)$ locus is essentially the same as the entire curve locus $\mathrm{ECL}(Z, W)$ of the induced directed variety, but we will not use this fact here. Notice that if $Z = \bigcup Z_\alpha$ is a decomposition of Z into irreducible components, then

$$\mathrm{IEL}_{X,V}(Z) = \bigcup_\alpha \mathrm{IEL}_{X,V}(Z_\alpha).$$

Since $\mathrm{IEL}_{X,V}(X_k) = \mathrm{ECL}_k(X, V)$, proving the Green–Griffiths–Lang property amounts to showing that $\mathrm{IEL}_{X,V}(X) \subsetneq X$ in the stage $k = 0$ of the tower. The basic step of our approach is expressed in the following statement.

11.7. Proposition. *Let (X, V) be a directed variety and $p_0 \leqslant n = \dim X$, $p_0 \geqslant 1$. Assume that there is an integer $k_0 \geqslant 0$ such that for every $k \geqslant k_0$ and every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, such that $\dim \pi_{k,k_0}(Z) \geqslant p_0$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_\bullet \to X$. Then $\dim \mathrm{ECL}_{k_0}(X, V) < p_0$.*

Proof. We argue here by contradiction, assuming that $\dim \mathrm{ECL}_{k_0}(X, V) \geqslant p_0$. If

$$p_0' := \dim \mathrm{ECL}_{k_0}(X, V) > p_0$$

and if we can prove the result for p_0', we will already get a contradiction. Hence we can assume without loss of generality that $\dim \mathrm{ECL}_{k_0}(X, V) = p_0$. The main argument consists of producing inductively an increasing sequence of integers

$$k_0 < k_1 < \cdots < k_j < \cdots$$

and directed varieties $(Z^j, W^j) \subset (X_{k_j}, V_{k_j})$ satisfying the following properties:

(11.7.1) Z^0 is one of the irreducible components of $\mathrm{ECL}_{k_0}(X, V)$ and $\dim Z^0 = p_0$;

(11.7.2) Z^j is one of the irreducible components of $\mathrm{ECL}_{k_j}(X, V)$ and $\pi_{k_j,k_0}(Z^j) = Z^0$;

(11.7.3) for all $j \geqslant 0$, $\mathrm{IEL}_{X,V}(Z^j) = Z^j$ and rank $W_j \geqslant 1$;

(11.7.4) for all $j \geqslant 0$, the directed variety (Z^{j+1}, W^{j+1}) is contained in some stage (of order $\ell_j = k_{j+1} - k_j$) of the Semple tower of (Z^j, W^j), namely

$$(Z^{j+1}, W^{j+1}) \subsetneq (Z^j_{\ell_j}, W^j_{\ell_j}) \subset (X_{k_{j+1}}, V_{k_{j+1}})$$

and

$$W^{j+1} = \overline{T_{Z^{j+1\prime}} \cap W^j_{\ell_j}} = \overline{T_{Z^{j+1\prime}} \cap V_{k_j}}$$

is the induced directed structure; moreover $\pi_{k_{j+1},k_j}(Z^{j+1}) = Z^j$,

(11.7.5) for all $j \geqslant 0$, we have $Z^{j+1} \subsetneq Z^j_{\ell_j}$ but $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j-1}$.

For $j = 0$, we simply take Z^0 to be one of the irreducible components S_α of $\mathrm{ECL}_{k_0}(X, V)$ such that $\dim S_\alpha = p_0$, which exists by our hypothesis that $\dim \mathrm{ECL}_{k_0}(X, V) = p_0$. Clearly, $\mathrm{ECL}_{k_0}(X, V)$ is the union of the $\mathrm{IEL}_{X,V}(S_\alpha)$ and we have $\mathrm{IEL}_{X,V}(S_\alpha) = S_\alpha$ for all those components. Thus $\mathrm{IEL}_{X,V}(Z^0) = Z^0$ and $\dim Z^0 = p_0$. Assume that (Z^j, W^j) has been constructed. The subvariety Z^j cannot be contained in the vertical divisor D_{k_j}. In fact no irreducible algebraic set Z such that $\mathrm{IEL}_{X,V}(Z) = Z$ can be contained in a vertical divisor D_k, because $\pi_{k,k-2}(D_k)$ corresponds to stationary jets in X_{k-2}; as every non-constant curve f has non-stationary points, its k-jet $f_{[k]}$ cannot be entirely contained in D_k; also the induced directed structure (Z, W) must satisfy $\operatorname{rank} W \geqslant 1$, otherwise $\mathrm{IEL}_{X,V}(Z) \subsetneq Z$. Condition (11.7.2) implies that $\dim \pi_{k_j,k_0}(Z^j) \geqslant p_0$. Therefore (Z^j, W^j) is of general type modulo $X_\bullet \to X$ by the assumptions of the proposition. Thanks to Proposition 2.5, we get an algebraic subset $\Sigma \subsetneq Z^j_\ell$ in some stage of the Semple tower (Z^j_ℓ) of Z^j such that every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfying $f_{[k_j]}(\mathbb{C}) \subset Z^j$ also satisfies $f_{[k_j+\ell]}(\mathbb{C}) \subset \Sigma$. By definition, this implies the first inclusion in the sequence

$$Z^j = \mathrm{IEL}_{X,V}(Z^j) \subset \pi_{k_j+\ell,k_j}(\mathrm{IEL}_{X,V}(\Sigma)) \subset \pi_{k_j+\ell,k_j}(\Sigma) \subset Z^j$$

(the other ones being obvious), so we have in fact an equality throughout. Let (S'_α) be the irreducible components of $\mathrm{IEL}_{X,V}(\Sigma)$. We have $\mathrm{IEL}_{X,V}(S'_\alpha) = S'_\alpha$ and one of the components S'_α must satisfy

$$\pi_{k_j+\ell,k_j}(S'_\alpha) = Z^j = Z^j_0.$$

We take $\ell_j \in [1, \ell]$ to be the smallest order such that $Z^{j+1} := \pi_{k_j+\ell,k_j+\ell_j}(S'_\alpha) \subsetneq Z^j_{\ell_j}$, and set $k_{j+1} = k_j + \ell_j > k_j$. By definition of ℓ_j, we have $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j-1}$, otherwise ℓ_j would not be minimal. We then get $\pi_{k_{j+1},k_j}(Z^{j+1}) = Z^j$ and thus $\pi_{k_{j+1},k_0}(Z^{j+1}) = Z^0$ by induction, and all properties (11.7.1)–(11.7.5) follow easily. Now, by Observation 7.29, we have

$$\operatorname{rank} W^j < \operatorname{rank} W^{j-1} < \cdots < \operatorname{rank} W^1 < \operatorname{rank} W^0 = \operatorname{rank} V.$$

This is a contradiction because we cannot have such an infinite sequence. Proposition 11.7 is proved. □

The special case $k_0 = 0$, $p_0 = n$ of Proposition 11.7 yields the following consequence.

11.8. Partial solution to the generalized GGL conjecture. Let (X, V) be a directed pair that is strongly of general type. Then the Green–Griffiths–Lang conjecture holds true for (X, V), namely $\mathrm{ECL}(X, V) \subsetneq X$; in other words there exists a proper algebraic variety $Y \subsetneq X$ such that every non-constant holomorphic curve $f : \mathbb{C} \to X$ tangent to V satisfies $f(\mathbb{C}) \subset Y$.

11.9. Remark. The proof is not very constructive, but it is however theoretically effective. By this we mean that if (X, V) is strongly of general type and is taken in a bounded family of directed varieties, i.e., X is embedded in some projective space $\mathbb{P}^N$ with some bound δ on the degree, and $P(V)$ also has bounded degree $\leqslant \delta'$ when viewed as a subvariety of $P(T_{\mathbb{P}^N})$, then one could theoretically derive bounds $d_Y(n, \delta, \delta')$ for the degree of the locus Y. Also, there would exist bounds $k_0(n, \delta, \delta')$ for the orders k and bounds $d_k(n, \delta, \delta')$ for the degrees of subvarieties $Z \subset X_k$ that have to be checked in the definition of a pair of strong general type. In fact, [Dem11] produces more or less explicit bounds for the order k such that Corollary 9.39 holds true. The degree of the divisor Σ is given by a section of a certain twisted line bundle $\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}_X(-A)$ that we know to be big by an application of holomorphic Morse inequalities—and the bounds for the degrees of (X_k, V_k) then provide bounds for m.

11.10. Remark. The condition that (X, V) is strongly of general type seems to be related to some sort of stability condition. We are unsure what is the most appropriate definition, but here is one that makes sense. Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geqslant 1$, and $Z = X = X_0$ for $k = 0$, we define the slope $\mu_A(Z, W)$ of the corresponding directed variety (Z, W) to be

$$\mu_A(Z, W) = \frac{\inf \lambda}{\operatorname{rank} W},$$

where λ runs over all rational numbers such that there exists $\ell \geqslant 0$, a modification $\widehat{Z}_\ell \to Z_\ell$ and $p \in \mathbb{Q}_+$ for which

$$K_{\widehat{W}_\ell} \otimes (\mathcal{O}_{\widehat{Z}_\ell}(p) \otimes \pi_{k+\ell,0}^* \mathcal{O}(\lambda A))_{\restriction \widehat{Z}_\ell} \quad \text{is big on } \widehat{Z}_\ell$$

(again, we assume here that $Z \not\subset D_k$ for $k \geqslant 2$). Notice that by definition (Z, W) is of general type modulo $X_\bullet \to X$ if and only if $\mu_A(Z, W) < 0$, and that $\mu_A(Z, W) = -\infty$ if $\mathcal{O}_{\widehat{Z}_\ell}(1)$ is big for some ℓ. Also, the proof of Lemma 7.24 shows that for any (Z, W) we have $\mu_A(Z_\ell, W_\ell) = \mu_A(Z, W)$ for all $\ell \geqslant 0$. We say that (X, V) is *A-jet-stable* (resp. *A-jet-semi-stable*) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leqslant \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above. It is then clear that if (X, V) is of general type and A-jet-semi-stable, then it is strongly of general type in the sense of Definition 11.3. It would be useful to have a better understanding of this condition of stability (or any other one that would have better properties).

11.B. Algebraic jet-hyperbolicity implies Kobayashi hyperbolicity

Let (X, V) be a directed variety, where X is an irreducible projective variety; the concept still makes sense when X is singular, by embedding (X, V) in a projective space $(\mathbb{P}^N, T_{\mathbb{P}^N})$ and taking the linear space V to be an irreducible algebraic subset of $T_{\mathbb{P}^n}$ that is contained in T_X at regular points of X.

11.11. Definition. *Let (X, V) be a directed variety. We say that (X, V) is algebraically jet-hyperbolic if for every $k \geqslant 0$ and every irreducible algebraic subvariety $Z \subset X_k$ that is not contained in the union Δ_k of vertical divisors, the induced directed structure (Z, W) either satisfies $W = 0$, or is of general type modulo $X_\bullet \to X$, i.e., there exists $\ell \geqslant 0$ and $p \in \mathbb{Q}_{\geqslant 0}$ such that $K^\bullet_{\widehat{W}_\ell} \otimes \mathcal{O}_{\widehat{Z}_\ell}(p)$ is big over $\widehat{Z}_\ell$, for some modification $(\widehat{Z}_\ell, \widehat{W}_\ell)$ of the ℓ-stage of the Semple tower of (Z, W).*

Proposition 7.33 can be restated:

11.12. Proposition. *If a projective directed variety (X, V) is such that $\mathcal{O}_{X_\ell}(a_\bullet)$ is ample for some $\ell \geqslant 1$ and some weight $a_\bullet \in \mathbb{Q}^\ell_{>0}$, then (X, V) is algebraically jet-hyperbolic.*

In a similar vein, one would prove that if $\mathcal{O}_{X_\ell}(a_\bullet)$ is big and the "augmented base locus"$B = \mathrm{Bs}(\mathcal{O}_{X_\ell}(a_\bullet) \otimes \pi^*_{l,0} A^{-1})$ projects onto a proper subvariety $B' = \pi_{\ell,0}(B) \subsetneq X$, then (X, V) is strongly of general type. In general, Proposition 11.7 gives the following:

11.13. Theorem. *Let (X, V) be an irreducible projective directed variety that is algebraically jet-hyperbolic in the sense of the above definition. Then (X, V) is Brody (or Kobayashi) hyperbolic, i.e.,* $\mathrm{ECL}(X, V) = \emptyset$.

Proof. Here we apply Proposition 11.7 with $k_0 = 0$ and $p_0 = 1$. It is enough to deal with subvarieties $Z \subset X_k$ such that $\dim \pi_{k,0}(Z) \geqslant 1$; otherwise $W = 0$ and we can reduce Z to a smaller subvariety by (2.2). Then we conclude that $\dim \mathrm{ECL}(X, V) < 1$. All entire curves tangent to V have to be constant, and we conclude in fact that $\mathrm{ECL}(X, V) = \emptyset$. □

12. Proof of the Kobayashi conjecture on generic hyperbolicity

We give here a simple proof of the Kobayashi conjecture, combining ideas of Green–Griffiths [GrGr80], Nadel [Nad89], Masuda–Noguchi [MaNo96], Demailly [Dem95], Siu–Yeung [SiYe96a], Shiffman–Zaidenberg [ShZa02], Brotbek [Brot17], Ya Deng [Deng16], in chronological order. Related ideas had been used earlier in [Xie15], and then in [BrDa17], to establish Debarre's conjecture on the ampleness of the cotangent bundle of generic complete intersections of codimension at least equal to dimension.

12.A. General Wronskian operators

This section follows closely the work of D. Brotbek [Brot17]. Let U be an open set of a complex manifold X, $\dim X = n$, and $s_0, \ldots, s_k \in \mathcal{O}_X(U)$ be holomorphic functions. To these functions, we can associate a Wronskian operator of order k defined by

(12.1)
$$W_k(s_0,\ldots,s_k)(f) = \begin{vmatrix} s_0(f) & s_1(f) & \cdots & s_k(f) \\ D(s_0(f)) & D(s_1(f)) & \cdots & D(s_k(f)) \\ \vdots & \vdots & & \vdots \\ D^k(s_0(f)) & D^k(s_1(f)) & \cdots & D^k(s_k(f)) \end{vmatrix},$$

where $f : (\mathbb{C},0) \ni t \mapsto f(t) \in U \subset X$ is a germ of holomorphic curve (or a k-jet of curve), and $D = \frac{d}{dt}$. For a biholomorphic change of variable $\varphi : (\mathbb{C},0) \to (\mathbb{C},0)$, we find by induction on ℓ a polynomial differential operator $p_{\ell,i}$ of order $\leqslant \ell$ acting on φ satisfying

$$D^\ell(s_j(f\circ\varphi)) = \varphi'^\ell D^\ell(s_j(f))\circ\varphi + \sum_{i<\ell} p_{\ell,i}(\varphi) D^i(s_j(f))\circ\varphi.$$

It follows easily from this that

$$W_k(s_0,\ldots,s_k)(f\circ\varphi) = (\varphi')^{1+2+\cdots+k} W_k(s_0,\ldots,s_k)(f)\circ\varphi,$$

and hence $W_k(s_0,\ldots,s_k)(f)$ is an invariant differential operator of degree $k' = \frac{1}{2}k(k+1)$. Especially, we get in this way a section that we denote somewhat sloppily

(12.2)
$$W_k(s_0,\ldots,s_k) = \begin{vmatrix} s_0 & s_1 & \cdots & s_k \\ D(s_0) & D(s_1) & \cdots & D(s_k) \\ \vdots & \vdots & & \vdots \\ D^k(s_0) & D^k(s_1) & \cdots & D^k(s_k) \end{vmatrix} \in H^0(U, E_{k,k'}T_X^*).$$

12.3. Proposition. *These Wronskian operators satisfy the following properties.*

(a) *$W_k(s_0,\ldots,s_k)$ is $\mathbb{C}$-multilinear and alternate in $(s_0,\ldots,s_k)$.*
(b) *For any $g \in \mathcal{O}_X(U)$, we have*

$$W_k(gs_0,\ldots,gs_k) = g^{k+1}W_k(s_0,\ldots,s_k).$$

Property 12.3 (b) is an easy consequence of the Leibniz formula

$$D^\ell(g(f)s_j(f)) = \sum_{k=0}^{\ell} \binom{\ell}{k} D^k(g(f))D^{\ell-k}(s_j(f)),$$

by performing linear combinations of rows in the determinants. This property implies in its turn that for any $(k+1)$-tuple of sections $s_0,\ldots,s_k \in H^0(U,L)$ of a holomorphic line bundle $L \to X$, one can define more generally an operator

(12.4) $$W_k(s_0,\ldots,s_k) \in H^0(U, E_{k,k'}T_X^* \otimes L^{k+1}).$$

In fact, when we compute the Wronskian in a local trivialization of $L_{\restriction U}$, Property 12.3 (b) shows that the determinant is independent of the trivialization. Moreover, if $g \in H^0(U, G)$ for some line bundle $G \to X$, we have

(12.5)
$$W_k(gs_0, \ldots, gs_k) = g^{k+1} W_k(s_0, \ldots, s_k) \in H^0(U, E_{k,k'} T_X^* \otimes L^{k+1} \otimes G^{k+1}).$$

We consider here a line bundle $L \to X$ possessing a linear system $\Sigma \subset H^0(X, L)$ of global sections such that $W_k(s_0, \ldots, s_k) \not\equiv 0$ for generic elements $s_0, \ldots, s_k \in \Sigma$. We can then view $W_k(s_0, \ldots, s_k)$ as a section of $H^0(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{k,0}^* L^{k+1})$ on the k-stage X_k of the Semple tower. Very roughly, the idea for the proof of the Kobayashi conjecture is to produce many such Wronskians, and to apply the fundamental vanishing theorem 8.15 to exclude the existence of entire curves. However, the vanishing theorem only holds for jet differentials in $H^0(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{k,0}^* A^{-1})$ with $A > 0$, while the existence of sufficiently many sections $s_j \in H^0(X, L)$ can be achieved only when L is ample, so the strategy seems a priori unapplicable. It turns out that one can sometimes arrange the Wronkian operator coefficients to be divisible by a section $\sigma_\Delta \in H^0(X, \mathscr{O}_X(\Delta))$ possessing a large zero divisor Δ, so that

$$\sigma_\Delta^{-1} W_k(s_0, \ldots, s_k) \in H^0(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{k,0}^*(L^{k+1} \otimes \mathscr{O}_X(-\Delta))),$$

and we can then hope that $L^{k+1} \otimes \mathscr{O}_X(-\Delta) < 0$. Our goal is thus to find a variety X and linear systems $\Sigma \subset H^0(X, L)$ for which the associated Wronskians $W_k(s_0, \ldots, s_k)$ have a very high divisibility. The study of the base locus of line bundles $\mathscr{O}_{X_k}(k') \otimes \pi_{k,0}^*(L^{k+1} \otimes \mathscr{O}_X(-\Delta))$ and their related positivity properties will be taken care of by using suitable blow-ups.

12.B. Using a blow-up of the Wronskian ideal sheaf

We consider again a linear system $\Sigma \subset H^0(X, L)$ producing some non-zero Wronskian sections $W_k(s_0, \ldots, s_k)$, so that $\dim \Sigma \geqslant k + 1$. As the Wronskian is alternate and multilinear in the arguments s_j, we get a meromorphic map $X_k \dashrightarrow P(\Lambda^{k+1}\Sigma^*)$ by sending a k-jet $\gamma = f_{[k]}(0) \in X_k$ to the point of projective coordinates $[W_k(u_{i_0}, \ldots, u_{i_k})(f)(0)]_{i_0,\ldots,i_k}$, where $(u_j)_{j\in J}$ is a basis of Σ and $i_0, \ldots, i_k \in J$ are in increasing order. This assignment factorizes through the Plücker embedding into a meromorphic map

$$\Phi : X_k \dashrightarrow \mathrm{Gr}_{k+1}(\Sigma)$$

into the Grassmannian of dimension $(k + 1)$ subspaces of Σ^* (or codimension $(k+1)$ subspaces of Σ, alternatively). In fact, if $L_{\restriction U} \simeq U \times \mathbb{C}$ is a trivialization of L in a neighborhood of a point $x_0 = f(0) \in X$, we can consider the map $\Psi_U : X_k \to \mathrm{Hom}(\Sigma, \mathbb{C}^{k+1})$ given by

$$\pi_{k,0}^{-1}(U) \ni f_{[k]} \longmapsto (s \longmapsto (D^\ell(s(f))_{0\leqslant \ell \leqslant k})),$$

and associate either the kernel $\Xi \subset \Sigma$ of $\Psi_U(f_{[k]})$, seen as a point $\Xi \in \mathrm{Gr}_{k+1}(\Sigma)$, or $\Lambda^{k+1}\Xi^\perp \subset \Lambda^{k+1}\Sigma^*$, seen as a point of $P(\Lambda^{k+1}\Sigma^*)$ (assuming that we are at a point where the rank is equal to $(k+1)$). Let $\mathcal{O}_{\mathrm{Gr}}(1)$ be the tautological very ample line bundle on $\mathrm{Gr}_{k+1}(\Sigma)$ (equal to the restriction of $\mathcal{O}_{P(\Lambda^{k+1}\Sigma^*)}(1)$). By construction, Φ is induced by the linear system of sections

$$W_k(u_{i_0}, \ldots, u_{i_k}) \in H^0(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{k,0}^* L^{k+1}),$$

and we thus get a natural isomorphism

$$\mathcal{O}_{X_k}(k') \otimes \pi_{k,0}^* L^{k+1} \simeq \Phi^* \mathcal{O}_{\mathrm{Gr}}(1) \quad \text{on } X_k \smallsetminus B_k, \tag{12.6}$$

where $B_k \subset X_k$ is the base locus of our linear system of Wronskians. The presence of the indeterminacy set B_k may create trouble in analyzing the positivity of our line bundles, so we are going to use an appropriate blow-up to resolve the indeterminacies. For this purpose, we introduce the ideal sheaf $\mathcal{J}_{k,\Sigma} \subset \mathcal{O}_{X_k}$ generated by the linear system Σ, and take a modification $\mu_{k,\Sigma} : \widehat{X}_{k,\Sigma} \to X_k$ in such a way that $\mu_{k,\Sigma}^* \mathcal{J}_{k,\Sigma} = \mathcal{O}_{\widehat{X}_{k,\Sigma}}(-F_{k,\Sigma})$ for some divisor $F_{k,\Sigma}$ in $\widehat{X}_{k,\Sigma}$. Then Φ is resolved into a morphism $\Phi \circ \mu_{k,\Sigma} : \widehat{X}_{k,\Sigma} \to \mathrm{Gr}_{k+1}(\Sigma)$, and on $\widehat{X}_{k,\Sigma}$, (12.6) becomes an everywhere defined isomorphism

$$\mu_{k,\Sigma}^*(\mathcal{O}_{X_k}(k') \otimes \pi_{k,0}^* L^{k+1}) \otimes \mathcal{O}_{\widehat{X}_{k,\Sigma}}(-F_{k,\Sigma}) \simeq (\Phi \circ \mu_{k,\Sigma})^* \mathcal{O}_{\mathrm{Gr}}(1). \tag{12.7}$$

In fact, we can simply take $\widehat{X}_k$ to be the normalized blow-up of $\mathcal{J}_{k,\Sigma}$, i.e., the normalization of the closure $\Gamma \subset X_k \times \mathrm{Gr}_{k+1}(\Sigma)$ of the graph of Φ and $\mu_{k,\Sigma} : \widehat{X}_k \to X_k$ to be the composition of the normalization map $\widehat{X}_k \to \Gamma$ with the first projection $\Gamma \to X_k$. [The Hironaka desingularization theorem would possibly allow us to replace $\widehat{X}_k$ by a non-singular modification, and $F_{k,\Sigma}$ by a simple normal crossing divisor on the desingularization; we will avoid doing so here, as we would otherwise need to show the existence of universal desingularizations when (X_t, Σ_t) is a family of linear systems of k-jets of sections associated with a family of algebraic varieties]. The following basic lemma was observed by Ya Deng [Deng16].

12.8. Lemma. *Locally over coordinate open sets $U \subset X$ on which $L_{\restriction U}$ is trivial, there is a maximal "Wronskian ideal sheaf" $\mathcal{J}_k^X \supset \mathcal{J}_{k,\Sigma}$ in $\mathcal{O}_{X_k}$ achieved by linear systems $\Sigma \subset H^0(U, L)$. It is attained globally on X whenever the linear system $\Sigma \subset H^0(X, L)$ generates k-jets of sections of L at every* *point. Finally, it is "universal" in the sense that is does not depend on L and behaves functorially under immersions: if* $\psi : X \to Y$ *is an immersion and* $\mathscr{J}_k^X$, $\mathscr{J}_k^Y$ *are the corresponding Wronskian ideal sheaves in* $\mathscr{O}_{X_k}$, $\mathscr{O}_{Y_k}$, *then* $\psi_k^* \mathscr{J}_k^Y = \mathscr{J}_k^X$ *with respect to the induced immersion* $\psi_k : X_k \to Y_k$.

Proof. The (local) existence of such a maximal ideal sheaf is merely a consequence of the strong Noetherian property of coherent ideals. As observed at the end of Subsect. 6.A, the bundle $X_k \to X$ is a locally trivial tower of $\mathbb{P}^{n-1}$-bundles, with a fiber $\mathscr{R}_{n,k}$ that is a rational $k(n-1)$-dimensional variety; over any coordinate open set $U \subset X$ equipped with local coordinates $(z_1, \ldots, z_n) \in B(0,r) \subset \mathbb{C}^n$, it is isomorphic to the product $U \times \mathscr{R}_{n,k}$, the fiber over a point $x_0 \in U$ being identified with the central fiber through a translation $(t \mapsto f(t)) \mapsto (t \mapsto x_0 + f(t))$ of germs of curves. In this setting, $\mathscr{J}_k^X$ is generated by the functions in $\mathscr{O}_{X_k}$ associated with Wronskians

$$X_{k\restriction U} \ni \xi = f_{[k]} \mapsto W_k(s_0, \ldots, s_k)(f) \in \mathscr{O}_{X_k}(k')_{\restriction \mathscr{R}_{n,k}}, \quad s_j \in H^0(U, \mathscr{O}_X),$$

by taking local trivializations $\mathscr{O}_{X_k}(k')_{\xi_0} \simeq \mathscr{O}_{X_k,\xi_0}$ at points $\xi_0 \in X_k$. In fact, it is enough to take Wronskians associated with *polynomials* $s_j \in \mathbb{C}[z_1, \ldots, z_n]$. To see this, one can e.g. invoke Krull's lemma for local rings, which implies $\mathscr{J}_{k,\xi_0}^X = \bigcap_{\ell \geqslant 0} (\mathscr{J}_{k,\xi_0}^X + \mathfrak{m}_{\xi_0}^{\ell+1})$, and to observe that ℓ-jets of Wronskians $W_k(s_0, \ldots, s_k) \pmod{\mathfrak{m}_{\xi_0}^{\ell+1}}$ depend only on the $(k+\ell)$-jets of the sections s_j in $\mathscr{O}_{X,x_0}/\mathfrak{m}_{x_0}^{k+\ell+1}$, where $x_0 = \pi_{k,0}(\xi_0)$. Therefore, polynomial sections s_j or arbitrary holomorphic functions s_j define the same ℓ-jets of Wronskians for any ℓ. Now, in the case of polynomials, it is clear that translations

$$(t \longmapsto f(t)) \longmapsto (t \longmapsto x_0 + f(t))$$

leave $\mathscr{J}_k^X$ invariant, hence $\mathscr{J}_k^X$ is the pull-back by the second projection $X_{k\restriction U} \simeq U \times \mathscr{R}_{n,k} \to \mathscr{R}_{n,k}$ of its restriction to any of the fibers $\pi_{k,0}^{-1}(x_0) \simeq \mathscr{R}_{n,k}$. As the k-jets of the s_j's at x_0 are sufficient to determine the restriction of our Wronskians to $\pi_{k,0}^{-1}(x_0)$, the first two claims of Lemma 12.8 follow. The universality property comes from the fact that $L_{\restriction U}$ is trivial (*cf.* Property 12.3 (b)) and that germs of sections of $\mathscr{O}_X$ extend to germs of sections of $\mathscr{O}_Y$ via the immersion ψ. (Notice that in this discussion, one may have to pick Taylor expansions of order $> k$ for f to reach all points of the fiber $\pi_{k,0}^{-1}(x_0)$, the order $(2k-1)$ being sufficient by [Dem95, Proposition 5.11], but this fact does not play any role here). A consequence of universality is that $\mathscr{J}_k^X$ does not depend on coordinates nor on the geometry of X. □

The above discussion combined with Lemma 12.8 leads to the following statement.

12.9. Proposition. *Assume that L generates all k-jets of sections (e.g. take $L = A^p$ with A very ample and $p \geqslant k$), and let $\Sigma \subset H^0(X, L)$ be a linear system that also generates k-jets of sections at any point of X. Then we have a universal isomorphism*

$$\mu_k^*(\mathscr{O}_{X_k}(k') \otimes \pi_{k,0}^* L^{k+1}) \otimes \mathscr{O}_{\widehat{X}_{k,\Sigma}}(-F_k) \simeq (\Phi \circ \mu_k)^* \mathscr{O}_{\mathrm{Gr}_{k+1}(\Sigma)}(1),$$

where $\mu_k : \widehat{X}_k \to X_k$ is the normalized blow-up of the (maximal) ideal sheaf $\mathscr{J}_k^X \subset \mathscr{O}_{X_k}$ associated with order k Wronskians, and F_k the universal divisor of $\widehat{X}_k$ resolving $\mathscr{J}_k^X$.

12.C. Specialization to suitable hypersurfaces

Let Z be a non-singular $(n+1)$-dimensional projective variety, and let A be a very ample divisor on Z; the fundamental example is of course $Z = \mathbb{P}^{n+1}$ and $A = \mathscr{O}_{\mathbb{P}^{n+1}}(1)$. Our goal is to show that a sufficiently general (n-dimensional) hypersurface $X = \{x \in Z\,;\ \sigma(x) = 0\}$ defined by a section $\sigma \in H^0(Z, A^d)$, $d \gg 1$, is Kobayashi hyperbolic. A basic idea, inspired by some of the main past contributions, such as Brody–Green [BrGr77], Nadel [Nad89], Masuda–Noguchi [MaNo96], Shiffman–Zaidenberg [ShZa02] and [Xie15], is to consider hypersurfaces defined by special equations, e.g. deformations of unions of hyperplane sections $\tau_1 \cdots \tau_d = 0$ or of Fermat–Waring hypersurfaces $\sum_{0\leqslant j\leqslant N} \tau_j^d = 0$, for suitable sections $\tau_j \in H^0(Z, A)$. Brotbek's main idea developed in [Brot17] is that a carefully selected hypersurface may have enough Wronskian sections to imply the ampleness of some tautological jet line bundle—a Zariski open property. Here, we take $\sigma \in H^0(X, A^d)$ equal to a sum of terms

$$\begin{aligned} &\sigma = \sum_{0\leqslant j\leqslant N} a_j m_j^\delta, \\ &a_j \in H^0(Z, A^\rho),\ \ m_j \in H^0(Z, A^b),\ \ n < N \leqslant k,\ \ d = \delta b + \rho, \end{aligned} \tag{12.10}$$

where $\delta \gg 1$ and the m_j are "monomials" of the same degree b, i.e., products of b "linear" sections $\tau_I \in H^0(Z, A)$, and the factors a_j are general enough. The integer ρ is taken in the range $[k, k+b-1]$, first to ensure that $H^0(Z, A^\rho)$ generates k-jets of sections, and second, to allow d to be an arbitrary large integer (once $\delta \geqslant \delta_0$ has been chosen large enough).

The monomials m_j will be chosen in such a way that for suitable $c \in \mathbb{N}$, $1 \leqslant c \leqslant N$, any subfamily of c terms m_j shares one common factor $\tau_I \in H^0(X, A)$. To this end, we consider all subsets $I \subset \{0, 1, \ldots, N\}$ with card $I = c$; there are $B := \binom{N+1}{c}$ subsets of this type. For all such I, we select sections $\tau_I \in H^0(Z, A)$ such that $\prod_I \tau_I = 0$ is a simple normal crossing divisor in Z (with all of its components of multiplicity 1). For $j = 0, 1, \ldots, N$ given, the number of subsets I containing j is $b := \binom{N}{c-1}$. We put

$$m_j = \prod_{I \ni j} \tau_I \in H^0(Z, A^b). \tag{12.11}$$

By construction, every family $m_{i_1}, \ldots, m_{i_c}$ of sections shares the common factor $\tau_I \in H^0(X, A)$, where $I = \{i_1, \ldots, i_c\}$. The first step consists in checking that we can achieve X to be smooth with these constraints.

12.12. Lemma. *Assume $N \geqslant c(n+1)$. Then, for a generic choice of the sections $a_j \in H^0(Z, A^\rho)$ and $\tau_I \in H^0(Z, A)$, the hypersurface $X = \sigma^{-1}(0) \subset Z$ defined by* (12.10), (12.11) *is non-singular. Moreover, under the same condition for N, the intersection of $\prod \tau_I = 0$ with X can be taken to be a simple normal crossing divisor in X.*

Proof. As the properties considered in the Lemma are Zariski open properties in terms of the $(N + B + 1)$-tuple (a_j, τ_I), it is sufficient to prove the result for a specific choice of the a_j's: we fix here $a_j = \tilde{\tau}_j \tau_{I(j)}^{\rho-1}$, where $\tilde{\tau}_j \in H^0(X, A)$, $0 \leqslant j \leqslant N$ are new sections such that $\prod \tilde{\tau}_j \prod \tau_I = 0$ is a simple normal crossing divisor, and $I(j)$ is any subset of cardinal c containing j. Let H be the hypersurface of degree d of $\mathbb{P}^{N+B}$ defined in homogeneous coordinates $(z_j, z_I) \in \mathbb{C}^{N+B+1}$ by $h(z) = 0$, where

$$h(z) = \sum_{0 \leqslant j \leqslant N} z_j z_{I(j)}^{\rho-1} \prod_{I \ni j} z_I^\delta,$$

and consider the morphism $\Phi : Z \to \mathbb{P}^{N+B}$ such that $\Phi(x) = (\tilde{\tau}_j(x), \tau_I(x))$. With our choice of the a_j's, we have $\sigma = h \circ \Phi$. Now, when the $\tilde{\tau}_j$ and τ_I are general enough, the map Φ defines an embedding of Z into $\mathbb{P}^{N+B}$ (for this, one needs $N + B \geqslant 2(\dim Z) + 1 = 2n + 3$, which is the case by our assumptions). Then, by definition, X is isomorphic to the intersection of H with $\Phi(Z)$. Changing generically the $\tilde{\tau}_j$ and τ_I's can be achieved by composing Φ with a generic automorphism $g \in \mathrm{Aut}(\mathbb{P}^{N+B}) = \mathrm{PGL}_{N+B+1}(\mathbb{C})$ (as $\mathrm{GL}_{N+B+1}(\mathbb{C})$ acts transitively on $(N + B + 1)$-tuples of linearly independent linear forms). As $\dim g \circ \Phi(Z) = \dim Z = n + 1$, Lemma 12.12 will follow from a standard Bertini argument if we can check that $\mathrm{Sing}(H)$ has codimension at least $(n+2)$ in $\mathbb{P}^{N+B}$. In fact, this condition implies $\mathrm{Sing}(H) \cap (g \circ \Phi(Z)) = \emptyset$ for g generic, while $g \circ \Phi(Z)$ can be chosen transverse to $\mathrm{Reg}(H)$. Now, a sufficient condition for smoothness is that one of the differentials dz_j, $0 \leqslant j \leqslant N$, appears with a non-zero factor in $dh(z)$ (just neglect the other differentials $*dz_I$ in this argument). We infer from this and the fact that $\delta \geqslant 2$ that $\mathrm{Sing}(H)$ consists of the locus defined by $\prod_{I \ni j} z_I = 0$ for all $j = 0, 1, \ldots, N$. It is the union of the linear subspaces $z_{I_0} = \cdots = z_{I_N} = 0$ for all possible choices of subsets I_j

such that $I_j \ni j$. Since $\operatorname{card} I_j = c$, the equality $\bigcup I_j = \{0, 1, \ldots, N\}$ implies that there are at least $\lceil (N+1)/c \rceil$ distinct subsets I_j involved in each of these linear subspaces, and the equality can be reached. Therefore $\operatorname{codim} \operatorname{Sing}(H) = \lceil (N+1)/c \rceil \geqslant n+2$ as soon as $N \geqslant c(n+1)$. By the same argument, we can assume that the intersection of Z with at least $(n+2)$ distinct hyperplanes $z_I = 0$ is empty. In order that $\prod \tau_I = 0$ defines a normal crossing divisor at a point $x \in X$, it is sufficient to ensure that for any family $\mathscr{G}$ of coordinate hyperplanes $z_I = 0$, $I \in \mathscr{G}$, with $\operatorname{card} \mathscr{G} \leqslant n+1$, we have a "free" index $j \notin \bigcup_{I \in \mathscr{G}} I$ such that $x_I \neq 0$ for all $I \ni j$, so that dh involves a non-zero term $* \, dz_j$ independent of the dz_I, $I \in \mathscr{G}$. If this fails, there must be at least $(n+2)$ hyperplanes $z_I = 0$ containing x, associated either with $I \in \mathscr{G}$, or with other I's covering $\complement(\bigcup_{I \in \mathscr{G}} I)$. The corresponding bad locus is of codimension at least $(n+2)$ in $\mathbb{P}^{N+B}$ and can be avoided by $g(\Phi(Z))$ for a generic choice of $g \in \operatorname{Aut}(\mathbb{P}^{N+B})$. Then $X \cap \bigcap_{I \in \mathscr{G}} \tau_I^{-1}(0)$ is smooth of codimension equal to $\operatorname{card} \mathscr{G}$. □

12.D. Construction of highly divisible Wronskians

To any families s, $\hat{\tau}$ of sections $s_1, \ldots, s_r \in H^0(Z, A^k)$, $\hat{\tau}_1, \ldots, \hat{\tau}_r \in H^0(Z, A)$, and to each subset $J \subset \{0, 1, \ldots, N\}$ with $\operatorname{card} J = c$, we associate a Wronskian operator of order k (i.e., a $(k+1) \times (k+1)$-determinant)

$$(12.13) \qquad \begin{aligned} &W_{k,s,\hat{\tau},a,J} = W_k(s_1 \hat{\tau}_1^{d-k}, \ldots, s_r \hat{\tau}_r^{d-k}, (a_j m_j^\delta)_{j \in \complement J}), \\ &r = k + c - N, \quad |\complement J| = N - c. \end{aligned}$$

We assume here again that the $\hat{\tau}_j$ are chosen so that $\prod \hat{\tau}_j \prod \tau_I = 0$ defines a simple normal crossing divisor in Z and X. Since $s_j \hat{\tau}_j^{d-k}$, $a_j m_j^\delta \in H^0(Z, A^d)$, formula (12.4) applied with $L = A^d$ implies that

$$(12.14) \qquad W_{k,s,\hat{\tau},a,J} \in H^0(Z, E_{k,k'} T_Z^* \otimes A^{(k+1)d}).$$

However, we are going to see that $W_{k,s,\hat{\tau},a,J}$ and its restriction $W_{k,s,\hat{\tau},a,J \restriction X}$ are divisible by monomials $\hat{\tau}^\alpha \tau^\beta$ of very large degree, where $\hat{\tau}$, resp. τ, denotes the collection of sections $\hat{\tau}_j$, resp. τ_I in $H^0(Z, A)$. In this way, we will see that we can even obtain a negative exponent of A after simplifying $\hat{\tau}^\alpha \tau^\beta$ in $W_{k,s,\hat{\tau},a,J \restriction X}$. This simplification process is a generalization of techniques already considered by [Siu87] and [Nad89] (and later [DeEG97]) in relation with the use of meromorphic connections of low pole order.

12.15. Lemma. *Assume that $\delta \geqslant k$. Then the Wronskian operator $W_{k,s,\hat{\tau},a,J}$, resp. $W_{k,s,\hat{\tau},a,J \restriction X}$, is divisible by a monomial $\hat{\tau}^\alpha \tau^\beta$, resp. $\hat{\tau}^\alpha \tau^\beta \tau_J^{\delta-k}$ (with a multiindex notation $\hat{\tau}^\alpha \tau^\beta = \prod \hat{\tau}_j^{\alpha_j} \prod \tau_I^{\beta_I}$), and*

$$\alpha, \beta \geqslant 0, \quad |\alpha| = r(d - 2k), \quad |\beta| = (N + 1 - c)(\delta - k)b.$$

Proof. $W_{k,s,\hat\tau,a,J}$ is obtained as a determinant whose r first columns are the derivatives $D^\ell(s_j\hat\tau_j^{d-k})$ and the last $(N+1-c)$ columns are the $D^\ell(a_jm_j^\delta)$, divisible respectively by $\hat\tau_j^{d-2k}$ and $m_j^{\delta-k}$. As m_j is of the form τ^γ, $|\gamma|=b$, this implies the divisibility of $W_{k,s,\hat\tau,a,J}$ by a monomial of the form $\hat\tau^\alpha\tau^\beta$, as asserted. Now, we explain why one can gain the additional factor $\tau_J^{\delta-k}$ dividing the restriction $W_{k,s,\hat\tau,a,J\restriction X}$. First notice that τ_J *does not* appear as a factor in $\hat\tau^\alpha\tau^\beta$, precisely because the Wronskian involves only terms $a_jm_j^\delta$ with $j\notin J$, and thus these m_j's do not contain τ_J. Let us pick $j_0=\min(\complement J)\in\{0,1,\dots,N\}$. Since X is defined by $\sum_{0\leqslant j\leqslant N}a_jm_j^\delta=0$, we have identically

$$a_{j_0}m_{j_0}^\delta=-\sum_{i\in J}a_im_i^\delta-\sum_{i\in\complement J\smallsetminus\{j_0\}}a_im_i^\delta$$

in restriction to X, whence (by the alternate property of $W_k(\bullet)$)

$$W_{k,s,\hat\tau,a,J\restriction X}=-\sum_{i\in J}W_k(s_1\hat\tau_1^{d-k},\dots,s_r\hat\tau_r^{d-k},a_im_i^\delta,(a_jm_j^\delta)_{j\in\complement J\smallsetminus\{j_0\}})_{\restriction X}.$$

However, all terms m_i, $i\in J$, contain by definition the factor τ_J, and the derivatives $D^\ell(\bullet)$ leave us a factor $m_i^{\delta-k}$ at least. Therefore, the above restricted Wronskian is also divisible by $\tau_J^{\delta-k}$, thanks to the fact that $\prod\hat\tau_j\prod\tau_I=0$ forms a simple normal crossing divisor in X. □

12.16. Corollary. *For $\delta\geqslant k$, there exists a monomial $\hat\tau^{\alpha_J}\tau^{\beta_J}$ dividing $W_{k,s,\hat\tau,a,J\restriction X}$ such that*

$$|\alpha_J|+|\beta_J|=(k+c-N)(d-2k)+(N+1-c)(\delta-k)b+(\delta-k),$$

and we have

$$\widetilde W_{k,s,\hat\tau,a,J\restriction X}:=(\hat\tau^{\alpha_J}\tau^{\beta_J})^{-1}W_{k,s,\hat\tau,a,J\restriction X}\in H^0(X,E_{k,k'}T_X^*\otimes A^{-p}),$$

where

(12.17)
$$p=|\alpha_J|+|\beta_J|-(k+1)d=(\delta-k)-(k+c-N)2k-(N+1+c)(kb+\rho).$$

In particular, we have $p>0$ for δ large enough (all other parameters being fixed or bounded), and under this assumption, the fundamental vanishing theorem 8.15 implies that all entire curves $f:\mathbb C\to X$ are annihilated by these Wronskian operators.

Proof. In fact,

$$(k+1)d=(k+c-N)d+(N+1-c)d=(k+c-N)d+(N+1-c)(\delta b+\rho),$$

and we get (12.17) by subtraction. □

12.E. Control of the base locus for sufficiently general coefficients a_j in σ

The next step is to control more precisely the base locus of these Wronskians and to find conditions on $N, k, c, d = b\delta + \rho$ ensuring that the base locus is empty for a generic choice of the sections a_j in $\sigma = \sum a_j m_j$. Although we will not formally use it, the next lemma is useful to realize that the base locus is related to a natural rank condition.

12.18. Lemma. *Set $u_j := a_j m_j^\delta$. The base locus in $X_k^{\rm reg}$ of the above Wronskians $W_{k,s,\hat\tau,a,J\restriction X}$, when s, $\hat\tau$ vary, consists of jets $f_{[k]}(0) \in X_k^{\rm reg}$ such that the matrix $(D^\ell(u_j \circ f)(0))_{0\leqslant \ell\leqslant k,\, j\in\complement J}$ is not of maximal rank (i.e., of rank $<$ card $\complement J = N+1-c$); if $\delta > k$, the base locus includes all jets $f_{[k]}(0)$ such that $f(0) \in \bigcup_{I\neq J} \tau_I^{-1}(0)$. When J also varies, the base locus of all $W_{k,s,\hat\tau,a,J\restriction X}$ in the Zariski open set $X'_k := X_k^{\rm reg} \smallsetminus \bigcup_{|I|=c} \tau_I^{-1}(0)$ consists of all k-jets such that* $\operatorname{rank}(D^\ell(u_j\circ f)(0))_{0\leqslant\ell\leqslant k,\, 0\leqslant j\leqslant N} \leqslant N-c$.

Proof. If $\delta > k$ and $m_j \circ f(0) = 0$ for some $j \in J$, we have in fact $D^\ell(u_j \circ f)(0) = 0$ for all derivatives $\ell \leqslant k$, because the exponents involved in all factors of the differentiated monomial $a_j m_j^\delta$ are at least equal to $\delta - k > 0$. Hence the rank of the matrix cannot be maximal. Now, assume that $m_j \circ f(0) \neq 0$ for all $j \in \complement J$, i.e.,

$$x_0 := f(0) \in X \smallsetminus \bigcup_{j\in\complement J} m_j^{-1}(0) = X \smallsetminus \bigcup_{I\neq J} \tau_I^{-1}(0). \tag{12.19}$$

We take sections $\hat\tau_j$ so that $\hat\tau_j(x_0) \neq 0$, and then adjust the k-jet of the sections $s_1, \ldots, s_r$ in order to generate any matrix of derivatives

$$(D^\ell(s_j(f)\hat\tau_j(f)^{d-k})(0))_{0\leqslant\ell\leqslant k,\, j\in\complement J}$$

(the fact that $f'(0) \neq 0$ is used for this!). Therefore, by expanding the determinant according to the last $(N+1-c)$ columns, we see that the base locus is defined by the equations

$$\det(D^\ell(u_j(f))(0))_{\ell\in L,\, j\in\complement J} = 0, \quad \forall L \subset \{0,1,\ldots,k\},\ |L| = N+1-c, \tag{12.20}$$

equivalent to the non-maximality of the rank. The last assertion follows by a simple linear algebra argument. □

For a finer control of the base locus, we adjust the family of coefficients

$$a = (a_j)_{0\leqslant j\leqslant N} \in S := H^0(Z, A^\rho)^{\oplus(N+1)} \tag{12.21}$$

in our section $\sigma = \sum a_j m_j^\delta \in H^0(Z, A^d)$, and denote by $X_a = \sigma^{-1}(0) \subset Z$ the corresponding hypersurface. By Lemma 12.12, we know that there is a

Zariski open set $U \subset S$ such that X_a is smooth and $\prod \tau_I = 0$ is a simple normal crossing divisor in X_a for all $a \in U$. We consider the Semple tower $X_{a,k} := (X_a)_k$ of X_a, the "universal blow-up" $\mu_{a,k} : \widehat{X}_{a,k} \to X_{a,k}$ of the Wronskian ideal sheaf $\mathscr{J}_{a,k}$ such that $\mu_{a,k}^* \mathscr{J}_{a,k} = \mathscr{O}_{\widehat{X}_{a,k}}(-F_{a,k})$ for some "Wronskian divisor" $F_{a,k}$ in $\widehat{X}_{a,k}$. By the universality of this construction (*cf.* Lemma 12.8), we can also embed $X_{a,k}$ in the Semple tower Z_k of Z, blow up the Wronskian ideal sheaf $\mathscr{J}_k^Z$ of Z_k to get a Wronskian divisor F_k^Z in $\widehat{Z}_k$, where $\mu_k : \widehat{Z}_k \to Z_k$ is the blow-up map. Then $F_{a,k}$ is the restriction of F_k^Z to $\widehat{X}_{a,k} \subset \widehat{Z}_k$. Our section $\widetilde{W}_{k,s,\hat{\tau},a,J \restriction X_a}$ is the restriction of a *meromorphic* section defined on Z, namely

(12.22)
$$(\hat{\tau}^{\alpha_J} \tau^{\beta_J})^{-1} W_{k,s,\hat{\tau},a,J} = (\hat{\tau}^{\alpha_J} \tau^{\beta_J})^{-1} W_k(s_1 \hat{\tau}_1^{d-k}, \ldots, s_r \hat{\tau}_r^{d-k}, (a_j m_j^{\delta})_{j \in \complement J}).$$

It induces over the Zariski open set $Z' = Z \smallsetminus \bigcup_I \tau_I^{-1}(0)$ a holomorphic section

$$\sigma_{k,s,\hat{\tau},a,J} \in H^0(\widehat{Z}'_k, \mu_k^*(\mathscr{O}_{Z_k}(k') \otimes \pi_{k,0}^* A^{-p}) \otimes \mathscr{O}_{\widehat{Z}_k}(-F_k^Z)) \tag{12.23}$$

(notice that the relevant factors $\hat{\tau}_j$ remain divisible on the whole variety Z). By construction, thanks to the divisibility property explained in Lemma 12.15, the restriction of this section to $\widehat{X}'_{a,k} = \widehat{X}_{a,k} \cap \widehat{Z}'_k$ extends holomorphically to $\widehat{X}_{a,k}$, i.e.,

(12.24)
$$\sigma_{k,s,\hat{\tau},a,J \restriction \widehat{X}_{a,k}} \in H^0(\widehat{X}_{a,k}, \mu_{a,k}^*(\mathscr{O}_{X_{a,k}}(k') \otimes \pi_{k,0}^* A^{-p}) \otimes \mathscr{O}_{\widehat{X}_{a,k}}(-F_{a,k})).$$

(Here the fact that we took $\widehat{X}_{k,a}$ to be normal implies that the divided section is indeed holomorphic on $\widehat{X}_{k,a}$, as $\widehat{X}_{k,a} \cap \mu_k^{-1}(\pi_{k,0}^{-1} \bigcap_{I \in \mathscr{G}} \tau_I^{-1}(0))$ has the expected codimension $= \operatorname{card} \mathscr{G}$ for any family $\mathscr{G}$).

12.25. Lemma. *Let V be a finite dimensional vector space over $\mathbb{C}$, $\Psi : V^p \to \mathbb{C}$ a non-zero alternating multilinear form, and let $m, c \in \mathbb{N}$, $c < m \leqslant p$, $r = p + c - m \geqslant 0$. Then the subset $T \subset V^m$ of vectors $(v_1, \ldots, v_m) \in V^m$ such that*

$$(**) \qquad \begin{array}{l} \Psi(h_1, \ldots, h_r, (v_j)_{j \in \complement J}) = 0 \\ \textit{for all } J \subset \{1, \ldots, m\},\ |J| = c, \textit{ and all } h_1, \ldots, h_r \in V, \end{array}$$

is a closed algebraic subset of codimension $\geqslant (c+1)(r+1)$.

Proof. A typical example is $\Psi = \det$ on a p-dimensional vector space V, then T consists of m-tuples of vectors of rank $< p - r$, and the assertion concerning the codimension is well-known (we will reprove it anyway). In general, the algebraicity of T is obvious. We argue by induction on p, the result being trivial for $p = 1$ (the kernel of a non-zero linear form is indeed of codimension $\geqslant 1$). If K is the kernel of Ψ, i.e., the subspace of vectors $v \in V$ such that $\Psi(h_1, \dots, h_{p-1}, v) = 0$ for all $h_j \in V$, then Ψ induces an alternating multilinear form $\overline{\Psi}$ on V/K, whose kernel is equal to $\{0\}$. The proof is thus reduced to the case when $\operatorname{Ker}\Psi = \{0\}$. Notice that we must have $\dim V \geqslant p$, otherwise Ψ would vanish. If $\operatorname{card} \complement J = m - c = 1$, condition ($**$) implies that $v_j \in \operatorname{Ker}\Psi = \{0\}$ for all j, and hence $\operatorname{codim} T = \dim V^m \geqslant mp = (c+1)(r+1)$, as desired. Now, assume $m - c \geqslant 2$, fix $v_m \in V \smallsetminus \{0\}$ and consider the non-zero alternating multilinear form on V^{p-1} such that

$$\Psi'_{v_m}(w_1, \dots, w_{p-1}) := \Psi(w_1, \dots, w_{p-1}, v_m).$$

If $(v_1, \dots, v_m) \in T$, then $(v_1, \dots, v_{m-1})$ belongs to the set T'_{v_m} associated with the new data $(\Psi'_{v_m}, p-1, m-1, c, r)$. The induction hypothesis implies that $\operatorname{codim} T'_{v_m} \geqslant (c+1)(r+1)$, and since the projection $T \to V$ to the first factor admits the T'_{v_m} as its fibers, we conclude that

$$\operatorname{codim} T \cap ((V \smallsetminus \{0\}) \times V^{m-1}) \geqslant (c+1)(r+1).$$

By permuting the arguments v_j, we also conclude that

$$\operatorname{codim} T \cap (V^{k-1} \times (V \smallsetminus \{0\}) \times V^{m-k}) \geqslant (c+1)(r+1)$$

for all $k = 1, \dots, m$. The union $\bigcup_k (V^{k-1} \times (V \smallsetminus \{0\}) \times V^{m-k}) \subset V^m$ leaves out only $\{0\} \subset V^m$ whose codimension is at least $mp \geqslant (c+1)(r+1)$, so Lemma 12.25 follows. □

12.26. Proposition. *Consider in $U \times \widehat{Z}'_k$ the set Γ of pairs (a, ξ) such that $\sigma_{k,s,\hat{\tau},a,J}(\xi) = 0$ for all choices of s, $\hat{\tau}$ and $J \subset \{0, 1, \dots, N\}$ with* card $J = c$. *Then Γ is an algebraic set of dimension*

$$\dim \Gamma \leqslant \dim S - (c+1)(k+c-N+1) + n + 1 + kn.$$

As a consequence, if $(c+1)(k+c-N+1) > n+1+kn$, there exists $a \in U \subset S$ such that the base locus of the family of sections $\sigma_{k,s,\hat{\tau},a,J}$ in $\widehat{X}_{a,k}$ lies over $\bigcup_I X_a \cap \tau_I^{-1}(0)$.

Proof. The idea is similar to [Brot17, Lemma 3.8], but somewhat simpler in the present context. Let us consider a point $\xi \in \widehat{Z}'_k$ and the k-jet $f_{[k]} = \mu_k(\xi) \in Z'_k$, so that $x = f(0) \in Z' = Z \smallsetminus \bigcup_I \tau_I^{-1}(0)$. Let us take the $\hat{\tau}_j$ such that $\hat{\tau}_j(x) \neq 0$. Then, we do not have to pay attention to the non-vanishing factors $\hat{\tau}^{\alpha_J} \tau^{\beta_J}$, and the k-jets of sections m_j and $\hat{\tau}_j^{d-k}$ are invertible near x. Let e_A be a local generator of A near x and $e_{\mathscr{L}}$ a local generator of the invertible sheaf

$$\mathscr{L} = \mu_k^* \mathscr{O}_{Z_k}(k') \otimes \mathscr{O}_{\widehat{Z}_k}(-F_k^Z)$$

near $\xi \in \widehat{Z}'_k$. Let $J^k\mathcal{O}_{Z,x} = \mathcal{O}_{Z,x}/\mathfrak{m}_{Z,x}^{k+1}$ be the vector space of k-jets of functions on Z at x. By definition of the Wronskian ideal and of the associated divisor F_k^Z, we have a *non-zero* alternating multilinear form

$$\Psi : (J^k\mathcal{O}_{Z,x})^{k+1} \longrightarrow \mathbb{C}, \quad (g_0,\dots,g_k) \longmapsto \mu_k^* W_k(g_0,\dots,g_k)(\xi)/e_{\mathcal{L}}(\xi).$$

The simultaneous vanishing of our sections at ξ is equivalent to the vanishing of

$$\Psi(s_1\hat{\tau}_1^{d-k}e_A^{-d},\dots,s_r\hat{\tau}_r^{d-k}e_A^{-d},(a_j m_j^\delta e_A^{-d})_{j\in\complement J}) \tag{12.27}$$

for all $(s_1,\dots,s_r)$. Since A is very ample and $\rho \geqslant k$, the power A^ρ generates k-jets at every point $x \in Z$, and thus the morphisms

$$H^0(Z,A^\rho) \longrightarrow J^k\mathcal{O}_{Z,x}, \quad a \longmapsto a m_j^\delta e_A^{-d} \quad \text{and}$$
$$H^0(Z,A^k) \longrightarrow J^k\mathcal{O}_{Z,x}, \quad s \longmapsto s\hat{\tau}_j^{d-k}e_A^{-d}$$

are surjective. Lemma 12.25 applied with $r = k + c - N$ and (p,m) replaced by $(k+1,N+1)$ implies that the codimension of families $a = (a_0,\dots,a_N) \in S = H^0(Z,A^\rho)^{\oplus(N+1)}$ for which $\sigma_{k,s,\hat{\tau},a,J}(\xi) = 0$ for all choices of s, $\hat{\tau}$ and J is at least $(c+1)(k+c-N+1)$, i.e., the dimension is at most $\dim S - (c+1)(k+c-N+1)$. When we let ξ vary over $\widehat{Z}'_k$ which has dimension $(n+1)+kn$ and take into account the fibration $(a,\xi) \mapsto \xi$, the dimension estimate of Proposition 12.26 follows. Under the assumption

$$(c+1)(k+c-N+1) > n+1+kn, \tag{12.28}$$

we have $\dim \Gamma < \dim S$, and so the image of the projection $\Gamma \to S$, $(a,\xi) \mapsto a$, is a constructible algebraic subset distinct from S. This concludes the proof. □

Our final goal is to completely eliminate the base locus. Proposition 12.26 indicates that we have to pay attention to the intersections $X_a \cap \tau_I^{-1}(0)$. For $x \in Z$, we let $\mathcal{G}$ be the family of hyperplane sections $\tau_I = 0$ that contain x. We introduce the set $P = \{0,1,\dots,N\} \smallsetminus \bigcup_{I\in\mathcal{G}} I$ and the smooth intersection

$$Z_{\mathcal{G}} = Z \cap \bigcap_{I\in\mathcal{G}} \tau_I^{-1}(0),$$

so that $N'+1 := \operatorname{card} P \geqslant N+1-c\operatorname{card}\mathcal{G}$ and $\dim Z_{\mathcal{G}} = n+1-\operatorname{card}\mathcal{G}$. If $a \in U$ is such that $x \in X_a$, we also look at the intersection

$$X_{\mathcal{G},a} = X_a \cap \bigcap_{I\in\mathcal{G}} \tau_I^{-1}(0),$$

which is a smooth hypersurface of $Z_{\mathcal{G}}$. In that situation, we consider Wronskians $W_{k,s,\hat{\tau},a,J}$ as defined above, but we now take $J \subset P$, $\operatorname{card} J = c$, $\complement J = P \smallsetminus J$, $r' = k + c - N'$.

12.29. Lemma. *In the above setting, if we assume $\delta > k$, the restriction $W_{k,s,\hat\tau,a,J\restriction X_{\mathscr{G},a}}$ is still divisible by a monomial $\hat\tau^{\alpha_J}\tau^{\beta_J}$ such that*

$$|\alpha_J| + |\beta_J| = (k+c-N')(d-2k) + (N'+1-c)(\delta-k)b + (\delta-k).$$

Therefore, if

$$p' = |\alpha_J|+|\beta_J|-(k+1)d = (\delta-k)-(k+c-N')2k-(N'+1+c)(kb+\rho)$$

as in (12.17)*, we obtain again holomorphic sections*

$$\widetilde{W}_{k,s,\hat\tau,a,J\restriction X_{\mathscr{G},a}} := (\hat\tau^{\alpha_J}\tau^{\beta_J})^{-1}W_{k,s,\hat\tau,a,J\restriction X_{\mathscr{G},a}} \in H^0(X_{\mathscr{G},a}, E_{k,k'}T^*_X\otimes A^{-p'}),$$

$$\sigma_{k,s,\hat\tau,a,J\restriction \pi_{k,0}^{-1}(X_{\mathscr{G},a})}$$
$$\in H^0(\pi_{k,0}^{-1}(X_{\mathscr{G},a}), \mu_{a,k}^*(\mathscr{O}_{X_{a,k}}(k')\otimes\pi_{k,0}^*A^{-p'})\otimes\mathscr{O}_{\widehat{X}_{a,k}}(-F_{a,k})).$$

Proof. The arguments are similar to those employed in the proof of Lemma 12.15. Let $f_{[k]}\in X_{a,k}$ be a k-jet such that $f(0)\in X_{\mathscr{G},a}$ (the k-jet need not be entirely contained in $X_{\mathscr{G},a}$). Putting $j_0 = \min(\complement J)$, we observe that we have on $X_{\mathscr{G},a}$ an identity

$$a_{j_0}m_{j_0}^\delta = -\sum_{i\in P\smallsetminus\{j_0\}} a_i m_i^\delta = -\sum_{i\in J} a_i m_i^\delta - \sum_{P\smallsetminus(J\cup\{j_0\})} a_i m_i^\delta,$$

because $m_i = \prod_{I\ni i}\tau_I = 0$ on $X_{\mathscr{G},a}$ for $i\in\complement P = \bigcup_{I\in\mathscr{G}} I$ (one of the factors τ_I is such that $I\in\mathscr{G}$, so $\tau_I = 0$). If we compose with a germ $t\mapsto f(t)$ such that $f(0)\in X_{\mathscr{G},a}$ (even though f does not necessarily lie entirely in $X_{\mathscr{G},a}$), we get

$$a_{j_0}m_{j_0}^\delta(f(t)) = -\sum_{i\in J} a_i m_i^\delta(f(t)) - \sum_{P\smallsetminus(J\cup\{j_0\})} a_i m_i^\delta(f(t)) + O(t^{k+1})$$

as soon as $\delta > k$. Hence we have an equality for all derivatives $D^\ell(\bullet)$, $\ell\leqslant k$ at $t=0$, and

$$W_{k,s,\hat\tau,a,J\restriction X_{\mathscr{G},a}}(f_{[k]})$$
$$= -\sum_{i\in J} W_k(s_1\hat\tau_1^{d-k},\ldots,s_{r'}\hat\tau_{r'}^{d-k}, a_i m_i^\delta, (a_j m_j^\delta)_{j\in P\smallsetminus(J\cup\{j_0\})})_{\restriction X_{\mathscr{G},a}}(f_{[k]}).$$

Then, again, $\tau_J^{\delta-k}$ is a new additional common factor of all terms in the sum, and we conclude as in Lemma 12.15 and Corollary 12.16. □

Now, we analyze the base locus of these new sections on

$$\bigcup_{a\in U} \mu_{a,k}^{-1}\pi_{k,0}^{-1}(X_{\mathscr{G},a}) \subset \mu_k^{-1}\pi_{k,0}^{-1}(Z_{\mathscr{G}}) \subset \widehat{Z}_k.$$

As x runs in $Z_{\mathscr{G}}$ and $N' < N$, Lemma 12.25 shows that (12.28) can be replaced by the less demanding condition

$$(c+1)(k+c-N'+1) > n+1-\operatorname{card}\mathscr{G}+kn = \dim \mu_k^{-1}\pi_{k,0}^{-1}(Z_{\mathscr{G}}). \tag{12.28$'$}$$

A proof entirely similar to that of Proposition 12.26 shows that for a generic choice of $a \in U$, the base locus of these sections on $\widehat{X}_{\mathscr{G},a,k}$ projects onto $\bigcup_{I\in\complement\mathscr{G}} X_{\mathscr{G},a} \cap \tau_I^{-1}(0)$. Arguing inductively on $\operatorname{card}\mathscr{G}$, the base locus can be shrinked step by step down to empty set (but it is in fact sufficient to stop when $X_{\mathscr{G},a} \cap \tau_I^{-1}(0)$ reaches dimension 0).

12.F. Nefness and ampleness of appropriate tautological line bundles

At this point, we have produced a smooth family $\mathscr{X}_S \to U \subset S$ of particular hypersurfaces in Z, namely $X_a = \{\sigma_a(z) = 0\}$, $a \in U$, for which a certain "tautological" line bundle has an empty base locus for sufficiently general coefficients:

12.30. Corollary. *Under condition* (12.28) *and the hypothesis $p > 0$ in* (12.17)*, the following properties hold.*

(a) *The line bundle*

$$\mathscr{L}_a := \mu_{a,k}^*(\mathscr{O}_{X_{a,k}}(k') \otimes \pi_{k,0}^* A^{-1}) \otimes \mathscr{O}_{\widehat{X}_{a,k}}(-F_{a,k})$$

is nef on $\widehat{X}_{a,k}$ for general $a \in U$, i.e., for $a \in U' \subset U$, where U' is a dense Zariski open subset.

(b) *Let $\Delta_a = \sum_{2\leqslant \ell\leqslant k} \lambda_\ell D_{a,\ell}$ be a positive rational combination of vertical divisors of the Semple tower and $q \in \mathbb{N}$, $q \gg 1$, an integer such that*

$$\mathscr{L}'_a := \mathscr{O}_{X_{a,k}}(1) \otimes \mathscr{O}_{a,k}(-\Delta_a) \otimes \pi_{k,0}^* A^q$$

is ample on $X_{a,k}$. Then the $\mathbb{Q}$-line bundle

$$\begin{aligned}\mathscr{L}'_{a,\varepsilon,\eta} := \mu_{a,k}^*(&\mathscr{O}_{X_{a,k}}(k') \otimes \mathscr{O}_{X_{a,k}}(-\varepsilon\Delta_a) \otimes \pi_{k,0}^* A^{-1+q\varepsilon}) \\ &\otimes \mathscr{O}_{\widehat{X}_{a,k}}(-(1+\varepsilon\eta)F_{a,k})\end{aligned}$$

is ample on $\widehat{X}_{a,k}$ for $a \in U'$, for some $q \in \mathbb{N}$ and $\varepsilon, \eta \in \mathbb{Q}_{>0}$ arbitrarily small.

Proof. (a) This would be obvious if we had global sections generating $\mathscr{L}_a$ on the whole of $\widehat{X}_{a,k}$, but our sections are only defined on a stratification of $\widehat{X}_{a,k}$. In any case, if $C \subset \widehat{X}_{a,k}$ is an irreducible curve, we take a maximal family $\mathscr{G}$ such that $C \subset X_{\mathscr{G},a,k}$. Then, by what we have seen, for $a \in U$ general enough, we can find global sections of $\mathscr{L}_a$ on $\widehat{X}_{\mathscr{G},a,k}$ such that C is not contained in their base locus. Hence $\mathscr{L}_a \cdot C \geqslant 0$ and $\mathscr{L}_a$ is nef for a in a dense Zariski open set $U' \subset U$.

(b) The existence of Δ_a and q follows from Proposition 7.19 and Corollary 7.21, which even provide universal values for λ_ℓ and q. After taking the blow up $\mu_{a,k} : \widehat{X}_{a,k} \to X_{a,k}$ (*cf.* (12.7)), we infer that

$$\begin{aligned}\mathscr{L}'_{a,\eta} &:= \mu^*_{a,k}\mathscr{L}' \otimes \mathscr{O}_{\widehat{X}_{a,k}}(-\eta F_{a,k})\\ &= \mu^*_{a,k}(\mathscr{O}_{X_{a,k}}(1) \otimes \mathscr{O}_{X_{a,k}}(-\Delta_a) \otimes \pi^*_{k,0}A^q) \otimes \mathscr{O}_{\widehat{X}_{a,k}}(-\eta F_{a,k})\end{aligned}$$

is ample for $\eta > 0$ small. The result now follows by taking a combination

$$\mathscr{L}_{a,\varepsilon,\eta} = \mathscr{L}_a^{1-\varepsilon/k'} \otimes (\mathscr{L}'_{a,\eta})^{\varepsilon}. \qquad \square$$

12.31. Corollary. *Let $\mathscr{X} \to \Omega$ be the universal family of hypersurfaces $X_\sigma = \{\sigma(z) = 0\}$, $\sigma \in \Omega$, where $\Omega \subset P(H^0(Z, A^d))$ is the dense Zariski open set over which the family is smooth. On the "Wronskian blow-up" $\widehat{X}_{\sigma,k}$ of $X_{\sigma,k}$, let us consider the line bundle*

$$\begin{aligned}\mathscr{L}_{\sigma,\varepsilon,\eta} := \mu^*_{\sigma,k}(&\mathscr{O}_{X_{\sigma,k}}(k') \otimes \mathscr{O}_{X_{\sigma,k}}(-\varepsilon\Delta_\sigma) \otimes \pi^*_{k,0}A^{-1+q\varepsilon})\\ &\otimes \mathscr{O}_{\widehat{X}_{\sigma,k}}(-(1+\varepsilon\eta)F_{\sigma,k})\end{aligned}$$

associated with the same choice of constants as in Corollary 12.30. Then $\mathscr{L}'_{\sigma,\varepsilon,\eta}$ is ample on $\widehat{X}_{\sigma,k}$ for σ in a dense Zariski open set $\Omega' \subset \Omega$.

Proof. By Corollary 12.30 (b), we can find $\sigma_0 \in H^0(Z, A^d)$ such that $X_{\sigma_0} = \sigma_0^{-1}(0)$ is smooth and $\mathscr{L}^m_{\sigma_0,\varepsilon,\eta}$ is an ample line bundle on $\widehat{X}_{\sigma_0,k}$ ($m \in \mathbb{N}^*$). As ampleness is a Zariski open condition, we conclude that $\mathscr{L}^m_{\sigma,\varepsilon,\eta}$ remains ample for a general section $\sigma \in H^0(Z, A^d)$, i.e., for $[\sigma]$ in some Zariski open set $\Omega' \subset \Omega$. Since $\mu_{\sigma,k}(F_{\sigma,k})$ is contained in the vertical divisor of $X_{\sigma,k}$, we conclude by Theorem 8.8 that X_σ is Kobayashi hyperbolic for $[\sigma] \in \Omega$. $\square$

12.G. Final conclusion and computation of degree bounds

At this point, we fix our integer parameters to meet all conditions that have been found. We must have $N \geqslant c(n+1)$ by Lemma 12.12, and for such a large value of N, condition (12.28) can hold only when $c \geqslant n$, so we take $c = n$ and

$N = n(n+1)$. Inequality (12.28) then requires k large enough, $k = n^3+n^2+1$ being the smallest possible value. We find

$$b = \binom{N}{c-1} = \binom{n^2+n}{n-1} = n\frac{(n^2+n)\cdots(n^2+2)}{n!}.$$

We have $n^2 + k = n^2(1 + k/n^2) < n^2 \exp(k/n^2)$ and by Stirling's formula, $n! > \sqrt{2\pi n}\,(n/e)^n$. Therefore

$$b < \frac{n^{2n-1}\exp((2+\cdots+n)/n^2)}{\sqrt{2\pi n}\,(n/e)^n} < \frac{e^{n+\frac{1}{2}+\frac{1}{2n}}}{\sqrt{2\pi}}\,n^{n-\frac{3}{2}}.$$

Finally, we divide $d-k$ by b, get in this way $d-k = b\delta + \lambda$, $0 \leqslant \lambda < b$, and put $\rho = \lambda + k \geqslant k$. Then $\delta + 1 \geqslant (d-k+1)/b$ and formula (12.17) yields

$$\begin{aligned} p &= (\delta - k) - (n^3+1)2k - (n^2+2n+1)(kb+\rho) \\ &\geqslant (d-k+1)/b - 1 - (2n^3+3)k - (n^2+2n+1)(kb+k+b-1). \end{aligned}$$

Therefore $p > 0$ is achieved as soon as

$$d \geqslant d_n = k + b(1 + (2n^3+3)k + (n^2+2n+1)(kb+k+b-1)),$$

where

$$k = n^3+n^2+1, \quad b = \binom{n^2+n}{n-1}.$$

The dominant term in d_n is $k(n^2+2n+1)b^2 \sim e^{2n+1}n^{2n+2}/2\pi$. By more precise numerical calculations and Stirling's asymptotic expansion, one can show in fact that $d_n \leqslant \lfloor (n+4)\,(en)^{2n+1}/2\pi \rfloor$ for $n \geqslant 4$ (which is also an equivalent and a close approximation as $n \to +\infty$), while $d_1 = 61$, $d_2 = 6685$, $d_3 = 2825761$. We can now state the main result of this section.

12.32. Theorem. *Let Z be a projective $(n+1)$-dimensional manifold and A a very ample line bundle on Z. Then, for a general section $\sigma \in H^0(Z, A^d)$ and $d \geqslant d_n$, the hypersurface $X_\sigma = \sigma^{-1}(0)$ is Kobayashi hyperbolic, and in fact, algebraically jet hyperbolic in the sense of Definition 11.11. The bound d_n for the degree can be taken to be*

$$d_n = \lfloor (n+4)\,(en)^{2n+1}/2\pi \rfloor \quad \textit{for } n \geqslant 4,$$

and for $n \leqslant 3$, one can take $d_1 = 4$, $d_2 = 6685$, $d_3 = 2825761$.

A simpler (and less refined) choice is $\tilde{d}_n = \lfloor \frac{1}{3}(en)^{2n+2} \rfloor$, which is valid for all n. These bounds are only slightly weaker than the ones found by Ya Deng in [Deng16], [Deng17], namely $\tilde{d}_n = O(n^{2n+6})$.

Proof. The bound $d_1 = 4$ (instead the insane value $d_1 = 61$) can be obtained in an elementary way by adjunction: sections of A can be used to embed any polarized surface (Z, A) in $\mathbb{P}^N$ (one can always take $N = 5$), and we have $K_{X_\sigma} = K_{Z\restriction X_\sigma} \otimes A^d$, along with a surjective morphism $\Omega^2_{\mathbb{P}^N} \to K_Z$. As $\Omega^2_{\mathbb{P}^N} \otimes \mathcal{O}(3) = \Lambda^{N-2}(T_{\mathbb{P}^N} \otimes \mathcal{O}(-1))$ is generated by sections, this implies that $K_Z \otimes A^3$ is also generated by sections, and hence K_{X_σ} is ample for $d \geqslant 4$. □

12.H. Further comments

12.33. Our bound d_n is rather large, but just as in Ya Deng's effective approach of Brotbek's theorem [Deng17], the bound holds for a property that looks substantially stronger than hyperbolicity, namely the ampleness of the pull-back of some (twisted) jet bundle $\mu_k^* \mathcal{O}_{\widehat{X}_k}(a_\bullet) \otimes \mathcal{O}_{\widehat{X}_k}(-F'_k)$. Subsect. 11.B provides much weaker conditions for hyperbolicity, but checking them is probably more involved.

12.34. After these notes were written, Riedl and Yang [RiYa18] proved the important and somewhat surprising result that the lower bound estimates $d_{\mathrm{GG}}(n)$ and $d_{\mathrm{Kob}}(n)$, respectively for the Green–Griffiths–Lang and Kobayashi conjectures for general hypersurfaces in $\mathbb{P}^{n+1}$, can be related by $d_{\mathrm{Kob}}(n) := d_{\mathrm{GG}}(2n-2)$. This should be understood in the sense that a solution of the generic $(2n-2)$-dimensional Green–Griffiths conjecture for $d \geqslant d_{\mathrm{GG}}(2n-2)$ implies a solution of the n-dimensional Kobayashi conjecture for the same lower bound. We refer to [RiYa18] for the precise statement, which requires an ad hoc assumption on the algebraic dependence of the Green–Griffiths locus with respect to a variation of coefficients in the defining polynomials. In combination with [DMR10], this gives a completely new proof of the Kobayashi conjecture, and the order 1 bound $d_{\mathrm{GG}}(n) = O(\exp(n^{1+\varepsilon}))$ of [Dem12] implies a similar bound $d_{\mathrm{Kob}}(n) = O(\exp(n^{1+\varepsilon}))$ for the Kobayashi conjecture—just a little bit weaker than what our direct proof gave (Theorem 12.32). In [MeTa19], Merker and Ta were able to improve the Green–Griffiths bound to $d_{\mathrm{GG}}(n) = o((\sqrt{n}\log n)^n)$, using a strengthening of Darondeau's estimates [Dar16a], [Dar16b], along with very delicate calculations. The Riedl–Yang result then implies $d_{\mathrm{Kob}}(n) = O((n\log n)^{n+1})$, which is the best bound known at this time.

12.35. In [Ber18], G. Bérczi stated a positivity conjecture for Thom polynomials of Morin singularities (see also [BeSz12]), and showed that it would imply a polynomial bound $d_n = 2n^9 + 1$ for the generic hyperbolicity of hypersurfaces.

12.36. In the unpublished preprint [Dem15], we introduced an alternative strategy for the proof of the Kobayashi conjecture which appears to be still incomplete at this point. We nevertheless hope that a refined version could one day lead to linear bounds such as $d_{\mathrm{Kob}}(n) = 2n + 1$. The rough idea was to establish a k-jet analogue of Claire Voisin's proof [Voi96] of the Clemens conjecture. Unfortunately, Lemma 5.1.18 as stated in [Dem15] is incorrect—the assertion concerning the Δ divisor introduced there simply does not hold. It is however conceivable that a weaker statement holds, in the form of a control of the degree of the divisor Δ, and in a way that would still be sufficient to imply similar consequences for the generic positivity of tautological jet bundles, as demanded e.g. in Subsect.11.B.

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From Riemann and Kodaira to modern developments on complex manifolds*

Shing-Tung Yau

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Abstract. We survey the theory of complex manifolds that are related to Riemann surface, Hodge theory, Chern class, Kodaira embedding and Hirzebruch–Riemann–Roch, and some modern development of uniformization theorems, Kähler–Einstein metric and the theory of Donaldson–Uhlenbeck–Yau on Hermitian Yang–Mills connections. We emphasize mathematical ideas related to physics. At the end, we identify possible future research directions and raise some important open questions.

Keywords and phrases: Kähler–Einstein metric, Donaldson–Uhlenbeck–Yau correspondence, mirror symmetry, Calabi–Yau manifold

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Contents

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S.-T. YAU
Department of Mathematics, Harvard University, One Oxford Street, Cambridge MA 02138, USA
(e-mail: yau@math.harvard.edu)

1. Introduction

This article originated in my two lectures given in honor of Professor Kodaira's 100th birthday, part of the 16th Takagi Lectures at the University of Tokyo, November 28–29, 2015.

About ten years ago, I read the biography of Professor Takagi and admired his great leadership in modernizing the development of mathematics in Japan.

The teaching of Chern on the subject of complex geometry had already deeply influenced me when, in the spring of 1970, I participated in a seminar organized by Kobayashi and Ochiai, and learnt the concept of Kobayashi hyperbolic manifolds. In the fall of 1971, I was a member of the Institute for Advanced Study in Princeton. There I met David Giesecker, Hitchin, Iitaka, and several young Japanese mathematicians. We had a good time together discussing mathematics and I started to learn much more about algebraic geometry—especially, from Iitaka, about the works of Kodaira. I was deeply impressed by the accomplishments of the Japanese algebraic geometers. Iitaka taught me the classification of algebraic varieties through the Kodaira dimension. He also introduced me to the work of Ueno, and of others.

In 1974, I met Hironaka and Mumford, and had the good fortune to listen to the talk given by Inoue on Inoue surfaces, at the big algebraic geometry conference organized by the AMS in Arcata, California. I met Miyaoka and Shioda in Paris in 1978, Mori at Harvard in 1979, and Mukai and Kawamata during the special year on algebraic geometry at Princeton in 1982. All these mathematicians had deep influence upon my work as related to this paper. I had also the good fortune to have taken Bando as my student in 1980, and Hosono and Yamaguchi as my postdocs in 1993 and 2003, respectively. All of these mathematicians have produced important works. Some of them are mentioned in this essay.

It is difficult to imagine how many absolutely first-class original mathematicians were produced by Japan in the 1940s. Among these great men, Professor Kodaira stood out as one of the most important leaders in modern algebraic and complex geometry, and his work succeeded that of the great mathematician Riemann.

2. The Work of Riemann

Riemann was one of the founders of complex analysis, along with Cauchy. Riemann pioneered several directions in the subject of holomorphic functions:

1. The idea of using differential equations and variational principle. The major work here is the Cauchy–Riemann equation, and the creation of Dirichlet principle to solve the boundary value problem for harmonic functions. (It took several great mathematicians, such as David Hilbert, to complete this work of Riemann.)
2. He gave the proof of the Riemann mapping theorem for simply connected domains. This theory of uniformization theorems has been extremely influential. There are methods based on various approaches, including methods of partial differential equations, hypergeometric functions and algebraic geometry. A natural generalization is to understand the moduli space of Riemann surfaces where Riemann made an important contribution by showing that it is a complex variety with dimension $3g - 3$.
3. The idea of using geometry to understand multivalued holomorphic functions, where he looked at the largest domain that a multivalued holomorphic function can define. He created the concept of Riemann surfaces, where he studied their topology and their moduli space. In fact, he introduced the concept of connectivity of space by cutting Riemann surface into pieces. The concept of Betti number was introduced by him for spaces in arbitrary dimension. The idea of understanding analytic problems through topology or geometry has far-reaching consequences. It influenced the later works of Poincaré, Picard, Lefstchetz, Hodge and others. Important examples of Riemann's research is to use monodromy groups to study analytic functions. Such study has deep influence on the development of discrete groups in the 20th century. The Riemann–Hilbert problem was inspired by this and up to now, is still an important subject in geometry and analysis. The study of ramified covering and the Riemann–Hurwitz formula gave an efficient technique in algebraic geometry and number theory.
4. The discovery of Riemann–Roch formula over algebraic curve. The generalizations by Kodaira, Hirzebruch, Grothendieck, Atiyah–Singer have led to tremendous progress in mathematics in the 20th century.

5. His study of period integrals related to Abel–Jacobi map and the hypergeometric equations:

$$z(1-z)y'' + [c - (a+b+c)z]y' - aby = 0.$$

6. The study of Riemann bilinear relations, the Riemann forms and the theta functions. During his study of the periods of Riemann surfaces, he found that the period matrix must satisfy period relations with a suitable invertible skew symmetric integral matrix which is called Riemann matrix later. Riemann realized that the period relations give necessary and sufficient condition for the existence of non-degenerate Abelian functions.
   (According to Siegel [77], his formulation was incomplete and he did not supply a proof. Later, Weierstrass also failed to establish a complete proof despite his many efforts in this direction. Complete proofs were finally attained by Appell for the case $g = 2$ and by Poincaré for arbitrary g.)

It should be noted that Riemann spent most of his last four years in Italy because he contracted tuberculosis and needed to avoid the severe winter in Germany. But as a result, he inspired a large group of differential geometers and projective algebraic geometers in Italy. Their works influenced the development of geometry and physics in the 20th century.

First of all, we should say that Riemann was the mathematician that brought us a new concept of space that was not perceived by any mathematician before him. I believe that was the reason that Gauss was so touched by his famous address on the foundations of geometry in 1854. I could not read German and was only able to read this address recently after it was translated into English. I was rather surprised that Riemann had rather liberal view about what geometry is supposed to be.

His guiding principle was nature itself (B. Riemann, *On the Hypotheses Which Lie at the Foundation of Geometry*, 1854.):

> The theorems of geometry cannot be deduced from the general notion of magnitude alone, but only from those properties which distinguished space from other conceivable entities, and these properties can only be found experimentally.... This takes us into the realm of another science–physics.

He thinks a deep understanding of geometry should be based on concepts of physics. And this is indeed the case as we experienced in the past century, especially in the past 50 years development of geometry. Although he was the one who introduced the concept of Riemann surface, which is the largest domain that a multivalued holomorphic function lives in, the precise modern concept was developed much later through the efforts of Klein, Poincaré and others.

While Felix Klein [43] already used atlas to describe Riemann surface, it has to wait until Hermann Weyl [89] who first gave the modern rigorous definition of Riemann surface, in terms of coordinate charts.

It was rather strange that a formal introduction of the concept of complex manifold was quite a bit later. Historically, generalization of one complex variable to several complex variables began by the study of functions on domains in $\mathbb{C}^n$. There were fundamental works of Levi, Oka, and Bergman.

The natural generalization of the concept of two dimensional surfaces to higher dimensional manifolds was done by O. Veblen and J.H.C. Whitehead in 1931–32. H. Whitney (1936) clarified the concept by proving that differentiable manifolds can be embedded into Euclidean space.

However, it was only in 1932 at the International Congress of Mathematicians in Zurich, did Carathéodory study "four dimensional Riemann surface" for its own sake. In 1944, Teichmüller mentioned "komplexe analytische Mannigfaltigkeit" in his work on "*Veränderliche Riemannsce Flächen*".

Chern was perhaps the first to use the English name "complex manifold" in his work [18].

The general abstract concept of almost complex structure was introduced by Ehresmann and Hopf in the 1940s. In 1948, Hopf [38] proved that the spheres S^4 and S^8 cannot admit almost complex structures.

The concept of Kähler geometry was introduced by Kähler [40] in 1933 where he demanded the Kähler form (which was first constructed by E. Cartan) to have a Kähler potential. Kähler had already observed special properties of such metric. He knew that the Ricci tensor associated to the metric tensor $g_{i\bar{j}}$ can be written rather simply as

$$R_{k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l}(\log \det g_{i\bar{j}}),$$

which gave a globally defined closed form on the manifold.

He knew that it defines a topological invariant for the geometry. It defines a cohomology class independent of the metric. It was found later that, after normalization, it represents the first Chern class of the manifold. The simplicity of the Ricci form allows Kähler to define the concept of Kähler–Einstein metric and he wrote down the equation locally in terms of the Kähler potential. He gave examples of the Kähler metric of the ball.

Slightly afterwards, Hodge developed Hodge theory, without knowing the work of Kähler, based on the induced metric from projective space to the algebraic manifolds. He studied the theory of harmonic forms with special attention to algebraic manifolds. The (p,q) decomposition of the differential forms have tremendous influence on the global understanding of Kähler manifolds. A very important observation is that the Hodge Laplacian commutes with the projection operator to the (p,q)-forms and hence the (p,q) decomposition descends to the de Rham cohomology. The theory was soon generalized to cohomology with twisted coefficients.

A very important cohomology with twisted coefficient is cohomology with coefficient in the tangent bundle or cotangent bundle, and their exterior powers. For the first cohomology with coefficient in tangent bundle, Kodaira and Spencer developed the fundamental theory of deformation of geometric structures, which gave far reaching generalization of the works of Riemann, Klein, Teichmüller and others on parametrization of complex structures over Riemann surfaces. They realize that the first cohomology with coefficient on tangent bundle, denoted by $H^1(T)$, parametrize the complex structure infinitesimally and that the second cohomology with coefficient on tangent bundle, denoted by $H^2(T)$, gives rise to obstruction to the deformation. The last statement was made very precisely by Kurinishi using Harmonic theory of Hodge–Kodaira. It describes the singular structure of the moduli space locally. Kodaira–Spencer studied how elements in $H^1(T)$ acts on other cohomology, which leads to study of variation of Hodge structures. The Hodge groups can be grouped in an appropriate way to form a natural filtration of the natural de Rham group. The Kodaira–Spencer map plays a very important role in understanding the deformation of such filtrations. Cohomology with coefficient of cotangent bundle or wedge product of cotangent bundle gives to Hodge (p, q)-forms. The duality of tangent bundle and cotangent bundle gives rise to something called mirror symmetry studied extensively in the last thirty years in relation to the theory of Calabi–Yau manifolds.

A very important tool in complex geometry was the introduction of Chern classes to complex bundles over a manifold and the representation of such classes by curvature of the bundle.

When Chern introduced the concept of Chern classes, he was influenced by the works of Pontryagin classes. In the course of defining Chern classes by de Rham forms given by symmetric polynomial of the curvature form, Chern defined the Chern connection for holomorphic bundles. He also proved that Chern classes of holomorphic bundles are represented by algebraic cycles on algebraic manifolds. This has been the major evidence of the Hodge conjecture: That every (p, p) class can be represented by algebraic cycles.

Chern proved that three different ways to define Chern classes are equivalent. In particular, he proved they are integral classes. Weil explained how they are related to Lie algebra invariant polynomials. Weil remarked that the integrality of Chern classes should play a role in quantum theory. Chern–Weil theory forms a bridge between topology, geometry, and mathematical physics.

The desire to generalize Riemann–Roch formula to higher dimensional algebraic manifolds has been relatively slow, until the very powerful method of sheaf theory was introduced by Leray, and important inputs were given by Weil, Borel and Serre. These basic techniques enabled Hirzebruch to arrive at the important Hirzebruch–Riemann–Roch formula in his 1954 paper [36], which can be stated in the following way:

$$\chi(V, E) = \int_V \mathrm{ch}(E)\mathrm{td}(V),$$

where E is a holomorphic vector bundle over a projective variety V.

The formulation of this formula by itself is remarkable. Hirzebruch developed the splitting principle and the theory of multiplicative sequences to find important power series of Chern classes. The Todd class is such a power series which is found by Hirzebruch to represent the arithmetic genus of the algebraic manifold, generalizing some old works of Todd in lower dimension. The Chern character $\mathrm{ch}(E)$ was invented by him to be a homomorphism from space of holomorphic vector bundles to even dimensional cohomology. The left-hand side of the formula is the Euler characteristic of cohomology with coefficient in E. This beautiful formula was observed by Serre when the algebraic manifold is two dimensional.

In the other direction, Kodaira was the first major mathematician who developed Hodge theory of harmonic forms right after its announcement by Hodge, and he generalized the theory of harmonic forms to manifolds with boundaries, where various boundary conditions have to be imposed.

Perhaps his most important work was his deep understanding that the Bochner argument in Riemannian geometry can be used to prove a vanishing theorem for cohomology classes under curvature condition of the manifold. He realized that the natural place for such vanishing theorem is to deal with cohomology with coefficient on bundle or sheaf. The vanishing theorem of Kodaira says that for positive line bundle L on a compact complex manifold M:

$$H^q(M, K_M \otimes L) = 0$$

for $q > 0$.

Coupled with the following theorem of Serre duality:

$$H^q(M, E) \cong H^{n-q}(M, K \otimes E^*),$$

Kodaira vanishing theorem implies that the Euler characteristic of cohomology with coefficients in a holomorphic vector bundle E with $E \otimes K^*$ positive, is simply the dimension of the group of holomorphic sections of E.

The above mentioned Hirzebruch–Riemann–Roch theorem then gives a formula to compute the dimension of the sections of the holomorphic bundle in terms of Chern numbers defined by Chern classes of the manifold and the bundle. This creates the most basic tool to understand algebraic manifolds.

Kodaira also showed that by blowing up points on the manifold, one can find enough holomorphic sections to separate points of the original manifolds and in fact gives an embedding of the manifold into complex projective space by using holomorphic sections of the bundle.

In particular, he proved that any Kähler manifold, whose Kähler class is defined by the Chern class of a holomorphic line bundle, can be holomorphically embedded into the complex projective space. The theorem of Chow then implies the manifold is in fact defined by an ideal of homogeneous polynomials, and hence an algebraic manifold.

What Kodaira has proved is one of the most spectacular theorems in geometry, and a glorious generalization of the work of Riemann on the condition of a complex torus to be abelian. More importantly the method of proving the Kodaira vanishing theorem has far reaching consequences in complex geometry. It was generalized to non-compact complex manifold, by various mathematicians including C. Morrey, Hörmander, Kohn, Vessintini, and others.

The Kodaira embedding theorem requires a high enough power of the ample line bundle to accomplish the embedding into projective space. An upper bound of this power of the line bundle is not clear from his argument.

Later on, Matsusaka [63,64] (improved by Kollár–Matsusaka [44]) proved the very-ampleness of mL for an ample line bundle L on an n-dimensional projective variety X, when m is no less than a bound, depending only on the intersection numbers L^n and $K_X \cdot L^{n-1}$ on X.

In 1980s, Kawamata proved his famous basepoint freeness theorem about the pluricanonical systems of minimal models in [41,42]. This is very important in the study of abundance conjecture. He proved that under the assumption that the numerical Kodaira dimension of a minimal variety X is equal to its Kodaira dimension, the pluricanonical system $|mK_X|$ is basepoint free for large m. This implies the basepoint freeness for minimal models of general type varieties. Later on, in a series paper of Miyaoka and Kawamata, they settled the proof of abundance conjecture for threefolds.

An important unsolved conjecture was proposed by Fujita in 1985, $mL+K_X$ is base-point free for $m \geq n+1$ and is very ample for $m \geq n+2$. Many mathematicians did important work on Fujita's conjecture, including Reider, Ein–Lazarsfeld, Kawamata, and many others. Demailly proved an effective formula for the bound on very ampleness [22]. Angehrn and Siu proved a quadratic bound for basepoint freeness [4].

There are many other contributions to algebraic geometry made by Japanese algebraic geometers. Mori first introduced the ingenious idea of "bend and break" argument in his proof of Hartshorne conjecture [71]. This leads to his proof of cone theorem in birational geometry and had deep influences in minimal model program. Mukai introduced the Fourier–Mukai transform in 1981 [73]. This became an important tool in the study of derived categories.

3. Calabi conjecture and Kähler–Einstein metrics

The theorems by Kodaira, Matsusaka, Kawamata provide abundance of holomorphic sections for the holomorphic line bundle to embed the manifold into complex projective with higher dimension. An interesting important problem is the zero codimension case where we want to embed X to complex projective space with the same dimension. Hirzebruch and Kodaira [37] conjectured that every algebraic manifold that is homeomorphic to $\mathbb{CP}^n$ is actually biholomorphic to it. They used Hirzebruch–Riemann–Roch formula, but they could only treat the case of odd dimensional manifolds due to the indeterminacy of the sign of the first Chern class. The even dimensional case was finally settled by me [95] in 1976. While the arguments of Kodaira are based on Hilbert space theory, which depends on linear analysis, the argument that I used was non-linear in nature. It has became an important new tool in complex geometry in the past forty years.

My argument depends on the existence of Kähler–Einstein metrics assuming the first Chern class is either positive, zero or negative. Although the Kähler–Einstein metric was already discussed by Kähler in his 1933 paper [40], where he wrote the equation explicitly, it was not until 1954 when Calabi [11] made a formal proposal to prove the existence of Kähler metric with prescribed volume form.

This could be used to prove the existence of Ricci-flat Kähler metric for any polarization if the first Chern class of the manifold is zero. Then Calabi asked the question when the first Chern class of the manifold is either negative or positive. The questions of Calabi were believed to be too good to be true in the old days, as nobody was able to construct an explicit Kähler–Einstein metric on any compact Kähler manifolds with no symmetries.

On 1976, I settled the cases when the first Chern class is either trivial or negative. (Aubin did the work independently for negative first Chern class.) I also considered the case when the manifold can have singularities, as was announced in my talk [98] at 1978 ICM in Helsinki.

3.1. Kähler–Einstein metrics on Fano manifolds

When the first Chern class is positive, it is called a Fano manifold. There are many interesting properties about Fano manifolds. Kollár, Mori and Miyaoka in [45] showed that smooth Fano varieties are rationally connected, in the sense that any two points are connected by a rational curve with (effectively) bounded degree. This implies an effective bound for the degree of the Fano n-fold, with respect to its anti-canonical bundle. Based on the work of Kollár and Matsusaka, it also implies that Fano n-folds form a bounded family.

In this case, there is an obstruction for the existence of Kähler–Einstein metric due to Matsushima [65]: the Lie algebra of the automorphism group of the manifold must be reductive. On [32], Futaki introduced his beautiful invariant defined on this Lie algebra. The Futaki invariant soon became a fundamental tool to study Kähler–Einstein metric on Fano manifolds. On the other hand, It took a long while to find a necessary and sufficient condition for the existence of Kähler–Einstein metric on Fano manifolds. Many people, including Calabi, was misled to believe that the non-existence of non-zero holomorphic vector fields is enough for the existence of Kähler–Einstein metric on Fano manifolds.

Right after I proved the Calabi conjecture on the existence of Kähler metric with prescribed volume form, I tried to work on the problem of the existence of Kähler–Einstein metric on Fano manifolds.

It is clear that based on the (non-trivial) higher order estimates that I had (independently due to Aubin for second order estimate) in the proof of the Calabi conjecture [97], the only missing point is some integral estimate of the Kähler potential. I found it is useful to set up the continuity argument

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) = \exp(h - tu) \det(g_{i\bar{j}}),$$

where $t = 0$ corresponds to a Kähler metric with positive Ricci curvature, as was given by the Calabi conjecture.

A simple calculation shows that the Ricci curvature of all members in the family have positive lower bound. This simplifies the analysis quite a bit as we have experiences with compact manifolds with Ricci curvature bounded from below by positive constant. In 1978, I returned to Stanford from my visit of Berkeley. At that time, I succeeded to convince Stanford mathematics department to hire Y.-T. Siu to come to Stanford from Yale.

We started to think about a proof of the existence of Kähler–Einstein metric by finding some integral estimate of the Kähler potential. Many estimates were found, but they are short of proving the existence of the metric. Some of those estimates can be sharpened if there are symmetries on the manifold, a procedure similar to the way that Moser sharpened the Trudinger inequality on the sphere when there is antipodal symmetry.

In the meanwhile, in 1977, I realized that Bogomolov [8] used the concept of stability of bundles to prove Chern number inequalities for algebraic surfaces which were sharpened by Miyaoka [68] and myself [95] independently. I started to believe there has to be links between the concept of stability with the existence of Hermitian Yang–Mills connections on bundles. When I proved the Calabi conjecture in 1976, I was at UCLA, and had a fruitful discussion with David Gieseker, who is a great expert on the stability of bundle theory. He re-proved the Chern number inequality of Bogomolov over characteristic p.

The fact that a holomorphic bundle admits a Hermitian Yang–Mills connection if the bundle is polystable was proved by Uhlenbeck–Yau on arbitrary compact Kähler manifolds, and by Simon Donaldson for algebraic surfaces. Simpson observed that the proof of Uhlenbeck–Yau can be used to settle the case when there is a Higgs field. (Up to now, the argument of Uhlenbeck–Yau has been the only argument to prove existence of the Yang–Mills–Higgs' equation.)

The Bogomolov inequality is optimal for general stable bundles. But it is not as sharp as the Miyaoka–Yau inequality when applied to the tangent bundle of the manifold. Hence I suspected that existence of Kähler–Einstein metric should be considered as a non-linear version of the existence of Hermitian Yang–Mills connection, and the stability of bundle should be replaced by manifold stability. Therefore, only in 1980, I realized that the right condition for existence of Kähler–Einstein metric is the stability of the algebraic manifold.

I made the conjecture that the existence of Kähler–Einstein metric is equivalent to stability of manifold. I told all my graduate students about this conjecture, especially to Gang Tian who showed interest in the problem of Kähler–Einstein metric. But it took a long time to convince him of the validity of my conjecture.

There are many ways to define stability of manifolds including the concepts of Chow stability or Hilbert stability. I was not sure which one is correct. But I started to explore it with my students in my seminars. First of all, one had to make sure that algebraic stability, which is defined by embeddings of algebraic manifolds into complex projective space, can be linked to existence of Kähler–Einstein metric.

In fact, in order to link stability condition to algebraic geometry, I [99, p. 139] proposed to prove any Hodge metric on an algebraic manifold can be approximated by normalized Fubini–Study metric induced on the manifold through embedding of the manifold into complex projective space by high powers of an ample line bundle.

I asked Tian to follow this line of argument to finish the first step of my conjecture on the equivalence of stability of Fano manifolds with the existence of Kähler–Einstein metrics.

I suggested Tian to use my method with Siu [79] on the uniformization of Kähler manifolds to produce peak functions to achieve such a goal. (The purpose of that paper with Siu was also embedding of Kähler manifolds.)

The proof was reasonably transparent using technology from my paper with Siu. This became Tian's thesis at Harvard.

The method can be said to be an understanding of the works of Kodaira in the analytic setting. The work was carried out as I expected and it was strengthened by Catlin [14], Zelditch [105] and by Lu [57].

So, we know that we can approximate any Hodge metric by the induced metric of the projective embedding of the manifold into some complex projective spaces. However, there is an ambiguity due to the action of complex projective group. This is of course what geometric invariant theory studies.

It turns out that when I studied first eigenvalue of the Laplacian with Bourguignon and Peter Li [10], we need to find a good position for the embedding upon action by the projective group, which we called the balanced condition. It can be written in the following form:

$$\int_{\sigma(M)} \frac{z_i \bar{z}_j}{|z_0|^2 + \cdots + |z_N|^2} \omega^n = \frac{\mathrm{vol}(M)}{N+1} \delta_{ij}$$

for some $\sigma \in SL(N+1, \mathbb{C})$.

With such a condition, we can use the embedding to give a good estimate of the first eigenvalue in terms of the total volume and the degree defined by the Chern form wedge with the Kähler classes.

I suggested this condition as a starting point to my former student Luo to understand the concept of stability required to prove my conjecture on the existence of Kähler–Einstein metric based on stability.

Luo [59] found it effective to change the measure in the above formula defined by the induced measure of the ambient projective space. And it turns out that for a polarized manifold (M, L) if there exists a metric on L such that the Bergman function of L^k is constant for some k, then it is Chow stable.

A theorem of Shouwu Zhang [106] says that the existence of a unique balanced embedding is equivalent to the manifold being Chow–Mumford stable.

My conjecture that the existence of Kähler–Einstein metric is equivalent to stability was announced several times in several conferences and was explicitly written in my article [101] for the proceedings of UCLA conference on differential geometry in 1990.

I also communicated to Tian in detail on how to understand the Futaki invariant in this setting. The final conjecture of mine was solved recently by Chen–Donaldson–Sun [15–17] based on earlier works of Donaldson including the right algebro-geometric definition of K-stability.

According to Donaldson [24], a Fano manifold is called K-stable if all its non-trivial test configurations (which describe certain degeneration of Kähler manifolds by flat families) have positive Futaki invariants. For a test configuration $\mathscr{X} \to \mathbb{C}$ with $\mathbb{C}^*$ action, the Futaki invariant $-F_1$ can be found from the total weight w_k of $\mathbb{C}^*$ acting on $H^0(X_0, L^k)$, using

$$\frac{w_k}{k d_k} = F_0 + F_1 k^{-1} + O(k^{-2}),$$

where d_k is the dimension of $H^0(X_0, L^k)$.

But the condition of K-stability is not easy to check, even in the case of surfaces. It would therefore be interesting to prove the existence of balanced condition for high power embeddings of a Fano manifold implies existence of

Kähler–Einstein metrics. It is highly desirable to clarify the condition of K-stability so that it can be checked effectively.

3.2. Balanced metric and Strominger system

Kähler–Einstein metrics are very useful in biregular geometry. We shall discuss it later. However, it cannot answer the important question whether an algebraic manifold is rational or not. The existence of Kähler metric is not a concept that is invariant under birational transformations, while the existence of balanced metric is. The concept of balanced metric was introduced by Michelsohn [67]. A Hermitian metric is called balanced if its Kähler form satisfied the following equation:

$$d(\omega^{n-1}) = 0$$

and it was proved by Alessandrini and Bassanelli [1] that its existence is invariant under birational transformations. However, there is much more freedom to deform a balanced metric than a Kähler metric. Just demanding that Ricci curvature equal to zero is not enough to determine a unique Balanced metric within the $(n-1, n-1)$-class.

On the other hand, balanced metric comes up naturally in the theory of Heterotic string theory in complex 3-dimension. And (this) balanced condition is related to the concept of supersymmetry. When there is a nowhere vanishing holomorphic $(3,0)$-form Ω on a Hermitian 3-fold X with Hermitian form ω, we look for a Hermitian metric which is balanced, and a stable holomorphic bundle E (stable with respect to the balanced metric) whose second Chern Class is equal to the second Chern Class given by the Hermitian metric. Altogether, the following equations of the Strominger system need to be satisfied:

(1) $d(\|\Omega\|_\omega \cdot \omega^2) = 0$
(2) $F_h^{2,0} = F_h^{0,2} = 0,\ F_h \wedge \omega^2 = 0$
(3) $\sqrt{-1}\,\partial\bar{\partial}\omega = \dfrac{\alpha'}{4}(\mathrm{tr}(R_\omega \wedge R_\omega) - \mathrm{tr}(F_h \wedge F_h))$

Here α' is a positive number, R_ω is the curvature tensor of the Hermitian metric ω, h is a Hermitian metric of E and F_h is its curvature form with respect to the Chern connection.

It provides a natural generalization of the Calabi–Yau geometry, which couples Hermitian metrics with Hermitian Yang–Mills theory. My belief is that the above system of equations can be solved when the obvious conditions hold. Jun Li and I [50] solved this system on any Calabi–Yau manifold by making a deformation from the original Calabi–Yau metric.

For some intrinsically non-Kähler manifold, Fu and I [29] solved the Strominger system based on some ansätze for a 3-dimensional complex manifolds obtained from the Calabi–Eckmann construction. (The construction of the non-Kähler manifolds based on Calabi–Eckmann construction was also observed by Goldstein and Prokushkin [35].) It is a non-singular complex torus bundle over the K3 surface. The proof of existence of non-singular solution to the Strominger system given by Fu–Yau [29] is based on non-trivial estimates related to complex Monge–Ampère equations. In order to understand the significance of Strominger system, Tseng and I [85, 86], and later with Tsai [84], developed a new theory of symplectic cohomology which we expect to be dual to this kind of geometry.

Note that the existence of Ricci-flat Kähler metric provides a reduction of holonomic group to a subgroup of $SU(n)$, and according to the work of Candelas–Horowitz–Strominger–Witten [13], provides a supersymmetric model for vacuum solutions for Type II string theory. They called such manifolds to be Calabi–Yau manifolds. The Strominger system was introduced by Strominger to study Heterotic string where the vacuum is a warped product instead of a direct product.

3.3. Questions of Kähler–Einstein metrics in algebraic geometry

There are several interesting consequences of the existence of Kähler–Einstein metric.

3.3.1. Understanding of Kähler–Einstein metrics near singularities A corollary of the above mentioned theorem of Chen–Donaldson–Sun is that the K-stability of such manifold implies that the tangent bundle is stable with respect to the polarization given by the anti-canonical line bundle. This is an interesting statement that is purely algebro-geometric, for which it would be nice to have a proof based only on algebraic geometry.

Also it implies that a K-stable Fano manifold is biregular to $\mathbb{CP}^n$ if the ratio of its two Chern numbers $c_2c_1^{n-2}$ and c_1^n is the same as $\mathbb{CP}^n$.

Another interesting question is the following: If a smooth algebraic manifold has Kodaira dimension either equal to the dimension of the manifold or $-\infty$, and if it is minimal in the sense in birational geometry and the ratio of two Chern numbers $c_2c_1^{n-2}$ and c_1^n is the same as $\mathbb{CP}^n$, then the manifold is either $\mathbb{CP}^n$ or complex ball quotient.

For the case of general type, this is likely to be true. But it will be good to allow singular minimal models and in the case of singular algebraic manifolds, we need to define the Chern numbers suitably. This is related to the question of what is the best Kähler–Einstein metric on an algebraic manifold with singularity.

Let us look at the simplest case when the singularity is isolated. If the Kähler metric is complete at the singularity, it is not hard to prove that the Kähler–

Einstein metric is unique. However, when it is not complete, it is not necessarily unique. It depends on the behavior of the volume form near the singularity.

What kind of volume forms are allowed? We need to know that the Ricci form of this volume form is positive definite and that the n-fold product of the Ricci form is asymptotic to this volume form near the singularity. (We may require that the metric defined by the Ricci form should have lower bound on its bisectional curvature.) It would be interesting to classify the asymptotic models of such volume forms. In principle, each of them will give rise to a canonical Kähler–Einstein metric with the given asymptotic behavior of the volume form.

It would be interesting to calculate the contribution of the singularity towards the Chern numbers. An important case is the canonical singularity appearing in the minimal model theory, which we recall below.

Suppose that Y is a normal variety and $f : X \to Y$ be a resolution of the singularities. Then

$$K_X = f^*(K_Y) + \sum_i a_i E_i,$$

where the sum is over the irreducible exceptional divisors and the rational numbers a_i are called the discrepancies.

Then the singularities of Y are called canonical if $a_i \geq 0$ for all i and called terminal if $a_i > 0$ for all i.

A 3-dimensional singularity is terminal of index 1 if and only if it is an isolated composite DuVal (cDV) point in $\mathbb{C}^4$. A 3-dimensional terminal singularity of index $r \geq 2$ is a quotient of an isolated cDV point in $\mathbb{C}^4$.

The important question is to find a good Kähler–Einstein metric in a neighborhood of the cDV singularity which is invariant under the group action. For orbifold singularities, one can use those metrics obtained by pushing down from the non-singular model before quotient by the group. On the other hand, there may be some other volume form that satisfies the above properties that is distinct from the orbifold construction. The complicated situation is the case that the Ricci form of the volume form may define a metric that is partially going to complete and partially degenerate at the singular point.

It will be important to construct nice model volume form in a neighborhood of the canonical singularities of the manifold whose Ricci form can give rise to a nice metric which is asymptotically Kähler–Einstein.

3.3.2. Kähler–Einstein metrics on quasi-projective varieties and Sasakian–Einstein metrics In my first paper on the Calabi conjecture, we know that given any Kähler class, we can find a Kähler metric which may degenerate along a divisor whose volume is given by the unique volume defined by the divisor of the pluricanonical sections. How to calculate the second Chern class related to this divisor would be important. The Chern numbers calculated by the degenerate

Ricci flat metrics should have residue from the divisor. It would be important to calculate this contribution.

The non-compact version of complete Ricci-flat metric is more complicated, partially because we lack of a good model space to build a good ansatz. At 1978 ICM in Helsinki, I [98] announced the way to build complete non-compact Ricci-flat manifolds.

I conjectured that the manifold can be written as the complement of a divisor D of a compact Kähler manifold M. (It was pointed out by Michael Anderson et al. [3] that we should assume the finiteness of the topology of the manifold, otherwise Gibbons–Hawking ansatz produces counterexamples.)

My program was to take D to be an anti-canonical divisor of M which cannot be contracted to a codimension two subvarity. There will be a holomorphic volume form on M which has poles along D. I expect that this is close enough to provide a necessary and sufficient condition for $M \setminus D$ to admit complete Kähler metric with zero Ricci curvature. When D is non-singular, I have worked out the program. The details were written up with Tian in two papers [81, 82].

However, when D has normal crossing singularities, the problem is unsolved, largely because we do not have a good model of complete Ricci-flat metric in a neighborhood of D when D has singularity. An important and interesting case is to allow the complete Kähler metric to have certain type of singularities. Besides quotient singularity, we can allow cone singularity.

In the last case, the interesting examples are metric cones over a Sasakian–Einstein manifold. Important progress was made by Gauntlett, Martelli, Sparks and myself starting with [33] on the existence of Sasakian–Einstein metrics. In [33, 61] we gave several obstructions to their existence by studying the Einstein–Hilbert functional restricted to the space of Sasakian–Einstein metrics where it becomes essentially the volume functional. It can further be shown to be a functional of the Reeb vector field associated to the Sasakian structure alone.

We obtain a useful obstruction from the Lichnerowicz bound on the Laplacian [54] which we could identify precisely as the physics criterion of a unitarity bound in the conformal field theory associated to the hypersurface singularity. We also show that the first variation of the volume functional is related to the Futaki invariant on the Kähler orbifold, hence volume minimization (and a-maximization in the physics language) implies vanishing of Futaki invariant. This includes the cases of regular and quasi-regular Sasakian structures as classified by Reeb vector orbits. In the irregular case, Collins and Székelyhidi [21] extended the notion of K-semistability to Sasakian structures, showing constant scalar curvature Sasakian metric implies K-semistability and also recovered our results based on the volume functional. The complete classification is still not known, even for complex hypersurfaces with isolated singularity which admits $\mathbb{C}^*$-action.

3.3.3. Compactification of Shimura varieties Another very important class of Kähler–Einstein metrics on quasi-projective varieties appears on the compactification of Shimura varieties of non-compact type. There is the work of Mumford on giving a toroidal compactification which is non-singular. In terms of the divisor at infinity, Yi Zhang and I [104] wrote down the behavior of the volume form of the Hermitian symmetric metric in a neighborhood of the divisor.

Here is a summary of my work with Yi Zhang: Let $\mathscr{A}_{g,\Gamma} := \mathfrak{H}_g/\Gamma$ be a quotient of the genus g Siegel space $\mathfrak{H}_g$ by a fixed arithmetic neat subgroup $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$. The Siegel variety $\mathscr{A}_{g,\Gamma}$ is then a quasi-projective algebraic manifold. The positive cone $C(\mathfrak{F}_0)$ of the standard minimal cusp $\mathfrak{F}_0$ of the Siegel space $\mathfrak{H}_g$ can be regarded as the set of all symmetric positive $g \times g$ real matrices. Let $\Sigma_{\mathfrak{F}_0}$ be any decomposition of $C(\mathfrak{F}_0)$ such that the corresponding Mumford toroidal compactification $\overline{\mathscr{A}}_{g,\Gamma}$ of $\mathscr{A}_{g,\Gamma}$ has normal crossing boundary divisor $D_\infty = \overline{\mathscr{A}}_{g,\Gamma} \setminus \mathscr{A}_{g,\Gamma}$. Let σ be an arbitrary top-dimensional polyhedral cone in $\Sigma_{\mathfrak{F}_0}$ and let $D_1, \cdots, D_N (N = \dim_{\mathbb{C}} \mathscr{A}_{g,\Gamma})$ be some different irreducible components of D_∞ corresponding to edges of σ.

Then the volume $\Phi_{g,\Gamma}$ on $\mathscr{A}_{g,\Gamma}$ can be represented by

$$\Phi_{g,\Gamma} = \frac{d\mathscr{V}_g}{(\prod_{j=1}^N ||s_i||_i^2) F_\sigma^{g+1}(\log ||s_1||_1, \cdots, \log ||s_N||_N)},$$

where

- $d\mathscr{V}_g$ is a continuous volume form on a partial compactification $\mathscr{U}_{\sigma_{\max}}$ of $\mathscr{A}_{g,\Gamma}$ with $\mathscr{A}_{g,\Gamma} \subset \mathscr{U}_{\sigma_{\max}} \subset \overline{\mathscr{A}}_{g,\Gamma}$,
- the $||\cdot||_i$ is a suitable Hermitian metric of the line bundle $[D_i]$ on $\overline{\mathscr{A}}_{g,n}$ for every integer $i \in [1, N]$,
- the s_i is global section of $\mathscr{O}_{\overline{\mathscr{A}}_{g,\Gamma}}(D_i)$ such that $D_i = \{s_i = 0\}$,
- the $F_\sigma \in \mathbb{Z}[x_1, \cdots, x_N]$ is a homogenous polynomial of degree g, and the coefficients of F_σ are integers dependent only on Γ and σ together with marking order of edges.

In fact, Yi Zhang and I computed the volume form of the Hermitian symmetric metric as is represented on the coordinate given by the toroidal compactification. It shows that $K + D$ is non-negative and positive on $M \setminus D$.

Whether $D_\infty := \overline{\mathscr{A}}_{g,\Gamma} \setminus \mathscr{A}_{g,\Gamma}$ is normal crossing or not, Yi Zhang and I showed that there is always a local model of partial compactification associated to each maximal regular cone σ in the cusp $\mathfrak{F}_0$.

The quotient manifold $\mathfrak{H}_g/(\Gamma \cap U^{\mathfrak{F}_0}(\mathbb{Q}))$ gives an étale map of the Siegel variety. For each maximal cone σ in the cusp $\mathfrak{F}_0$, we have associate exponential maps of the inclusion $\mathfrak{H}_g \subset U^{\mathfrak{F}_0}(\mathbb{C}) \cong \mathbb{C}^n$, so that these maps endow a local model of partial compactification

$$\mathfrak{H}_g/(\Gamma \cap U^{\mathfrak{F}_0}(\mathbb{Q})) \subset (\mathbb{C}^*)^n.$$

The exponential map $\mathbb{C}^n \to (\mathbb{C}^*)^n$ is given by

$$z \longmapsto w = (w_1, \cdots, w_n), \quad \text{where } w_i = \exp(2\pi\sqrt{-1}\, l_i(z)),\ \forall i,$$

where $\{l_i\}_{i=1}^n$ is the dual base of edges of the cone σ.

The $(w_1, \cdots, w_n)$ gives a local coordinate system of the partial compactification, but it can not be a local coordinate system of $\overline{\mathscr{A}_{g,\Gamma}}$ if the D_∞ is not normal crossing.

The quotient manifold $\mathfrak{H}_g/(\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q}))$ also has an induced Kähler–Einstein metric with volume form

$$\Phi_\sigma = \frac{(\frac{\sqrt{-1}}{2})^n 2^{\frac{g(g-1)}{2}} \mathrm{vol}_\Gamma(\sigma)^2 \bigwedge_{1\le i\le n} dw_i \wedge d\overline{w_i}}{(\prod_{1\le i\le n} |w_i|^2)(F_\sigma(\log|w_1|, \cdots, \log|w_n|))^{g+1}}.$$

The coefficients of the polynomial F_σ are integers dependent only on Γ and σ, and the function $H := -\log F_\sigma$ must satisfy the following elliptic Monge–Ampére equation

$$\det\Big(\frac{\partial^2 H}{\partial x_i \partial x_j}\Big)_{i,j} = 2^{\frac{g(g-1)}{2}} \mathrm{vol}_\Gamma(\sigma)^2 \exp((g+1)H)$$

on the domain $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \le -C < 0,\ \forall i\}$.

Note that it was known to me [96] that for a Divisor D in an algebraic manifold M, if $K + D$ is strictly positive, then there is a canonical complete Kähler–Einstein metric on $M \setminus D$ whose volume form behaves like:

$$\Phi \sim \frac{dV}{\prod_{j=1}^k ||s_j||^2(-\log||s_j||)^2}$$

for some integer $k > 0$, where s_i is a section of the line bundle $[D_i]$ if $D = \sum D_i$, dV is a global smooth volume form on M, and the norm is defined such that its zero set is D and the minus Ricci tensor of M plus the Ricci curvature of the metric on D is positive.

In the above work with Yi Zhang, the volume form is more complicated, because we only know $K + D$ is non-negative and positive outside D. How to study such quasi-projective manifold? It is important to find the right algebro-geometric and combinatorial conditions on the Chern forms of M and the Chern forms of the various divisors D so that the Ricci curvature of the volume form gives rise to a positive definite Ricci form whose n-fold power is asymptotic to the volume form itself.

An ansatz we propose to construct complete Kähler–Einstein metric on $M \setminus D$ is to construct a volume form described similar to the above Φ_σ, where H satisfies the above Monge–Ampère equation with $x_i = \log|s_i|$. We need to find the condition on the divisors D_i so that Φ_σ gives rise to a positive Ricci form whose n-fold power is asymptotic to Φ_σ. Hopefully we can find a good existence theorem for complete Kähler–Einstein metric with finite volume on $M \setminus D$, when $K + D$ is non-negative.

The existence of complete Kähler–Einstein metric on the Shimura varieties comes from the Hermitian symmetric domain that covers it. The tangent bundle and various homogeneous bundles over the symmetric domain are invariant under the discrete group acting on the domain. Hence they can be descended to the Shimura varieties. As was explained by Mumford, these bundles and their connections can be extended naturally to the Mumford toroidal compactification of the Shimura variety. (Mumford proved that the extension satisfies the property of being "good".)

And the Chern forms defined by the connection were identified by Mumford to represent the Chern class of the extended bundles in the sense of distribution. In the case of the cotangent bundle, the cotangent bundle is extended to be $\Omega^1(\log(D))$, where D is the divisor at infinity.

Since the bundles are homogenous, the Chern numbers of these extended bundles are determined by some numerical combination of its curvature tensor at one point times the volume of the Shimura variety. The existence of Kähler–Einstein metric on the Shimura variety shows that the manifold is stable in various senses and the homogeneous bundles are stable with respect to the polarization $K + D$ of the variety. Besides algebro-geometric characterization of Shimura variety, it would be good to characterize algebraic geometrically those holomorphic bundles that are homogenous.

Incidentally, from my observation in 1978 that the positivity of $K + D$ implies the existence of a unique canonical Kähler–Einstein metric on the complement of D. We can find a map from the space of divisors D such that $K + D$ is ample to the space of stable bundles defined by cotangent $\Omega^1(\log(D))$. It will be nice to find conditions on D so that we can weaken the conclusion $K + D > 0$ to $K + D \geq 0$.

For a compact algebraic manifolds M, it can be shown to be a Shimura variety if the canonical line bundle is ample, and if the bundle, formed by symmetric powers of the cotangent (or tangent) bundle tensored by some line bundle so that the determinant bundle is trivial, is irreducible and has non-trivial sections. This is a simple observation (due independently to Kobayashi–Ochiai and myself [102]) because the existence of Kähler–Einstein metric will make this non-trivial section to be a parallel section and hence the holonomic group will be reduced. Algebraic characterization of Shimura varieties allows one to give a simple proof of the theorem of David Kazhdan, that Shimura varieties are invariant under Galois conjugation. Unfortunately our knowledge for non-compact manifold is not good enough to give such a proof in such case.

3.3.4. Explicit construction of Kähler–Einstein metrics and uniformization
For Kähler–Einstein manifolds with negative or zero first Chern class, I proposed [99, p. 139] that the metric can be computed in the following manner: When the canonical line bundle K is ample, we can embed the manifold into the complex projective space by n-th power of K.

The embedding can be changed by projective transformation in general. But there was a concept of balanced position (inspired by my work with Bourguignon and Peter Li [10] on first eigenvalue of the Laplacian) that I suggested to my former student Luo [59].

The embedding is unique up to unitary transformation after putting into such balanced position. The induced metric from complex projective space defines a sequence of Kähler metrics on the manifold, which after division by n, will converge as $n \to \infty$ to a Kähler–Einstein metric of the manifold.

In the above construction, when the manifold has zero first Chern class (Calabi–Yau manifold), the canonical line bundle should be replaced by any positive line bundle.

The balanced position is achieved by some projective transformation. We expect that the projective transformation depends algebraically on the original embedding of the manifold. The whole procedure should give a reasonable "explicit" form of the Kähler–Einstein metric. Once we obtain an explicit form of the Kähler–Einstein metric, we can compute the uniformization of the manifold.

A simple case is the elliptic curve where we know how to calculate its unique holomorphic 1-form by residue. The absolute value of it gives the Ricci-flat metric on the elliptic curve. We can calculate the uniformization of the elliptic curve, using the period calculation.

Computation of the periods of the holomorphic 1-form is obtained by computing the Picard–Fuchs equation. Once one finds the period, one can obtain a map from the complex line, mod the lattice spanned by the periods, to the elliptic curve. The components of this map is the Weierstrass $\wp$ function and its derivatives. This procedure is classical and went back to Abel, Jacobi and Riemann.

The uniformization of a general algebraic curve: finding a covering holomorphic map from the upper half plane to the curve, is more difficult and is done only for special curves.

Suppose we can calculate the Poincaré metric on the curve, as was explained above, we can calculate this map by studying the periods through the Picard–Fuchs equation. It is of course much more challenging to calculate the uniformization map explicitly in higher dimensions.

Given an algebraic manifold, we know it can be uniformized as a quotient of some classical domain. It is a classical question on how to find such a uniformization.

As mentioned above, we know how to find an algebro-geometric criterion (by using Chern numbers) for an algebraic manifold to be a ball quotient. But we can generalize this criterion to more general manifolds covered by Hermitian symmetric domains.

Once we identify such an algebraic manifold, we need to find suitable multi-valued holomorphic map from the manifold into a Hermitian symmetric domain.

Chenglong Yu, Peng Gao and I proposed the following program for the ball quotient:

1. Find a reasonably explicit way to construct the Kähler–Einstein metric on the algebraic manifold. This involves having a good understanding of the right projective embedding for the algebraic manifold.
2. Based on Kähler–Einstein metric, we compute its connection A_h and construct a system of holomorphic linear differential equations

$$ds + \begin{pmatrix} -A_h^T + \frac{1}{n+1}\mathrm{tr}(A_h) - a\, d\hat{z}\ \partial a + A_h^T a - a(d\hat{z}\cdot a) \\ d\hat{z} \qquad\qquad d\hat{z}\, a - \frac{1}{n+1}\mathrm{tr}(A_h) \end{pmatrix} s = 0$$

   where

$$s = (f^1, f^2, \dots, f^{n+1})^T, \quad d\hat{z} = (dz^1, dz^2, \dots, dz^n),$$
$$a = (a_1 \cdots a_n)^T, \quad a_i = -\Gamma_i{}^j_j + \frac{1}{n+1}\sum_k \Gamma_k{}^k_i \delta^i_j.$$

   Here j does not depend on i. The value of a is such that it gives a gauge transformation making the connection matrix holomorphic.
3. We find a base for the solution of the this system, given by the span of $\{s_1, s_2, \dots, s_{n+1}\}$. This allows us to define locally a map to the projective space of dimension n. Up to projective transformations we find a map to the complex ball of dimension n as well.

These maps are multivalued functions. The inverse of this map should be given by automorphic forms and one should be able to find information of the discrete group that acts on the ball based on the information of the algebraic variety and the monodromy of the map. An example in the case of elliptic curve is given by the Weierstrass-$\wp$ function.

For higher genus curve, if the Kähler–Einstein metric is $e^{2u}\, dz \wedge d\bar{z}$, then the system above becomes

$$d\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} 0 & ((u_z)^2 - u_{zz})\, dz \\ dz & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0,$$

or

$$(f_2)'' + ((u_z)^2 - u_{zz}) f_2 = 0.$$

The coefficient in the differential equation gives the Schwarzian derivative for the uniformization map $S(u) = u_{zz} - u_z^2$.

A complete understanding of this program should create many interesting special functions and should be related to the GKZ system and the tautological system introduced by Bong Lian and myself [53] on computing the period of integrals of holomorphic forms.

3.3.5. Relation with birational geometry The moduli space of algebraic manifolds of general type was studied by Gieseker and Viehweg who proved that they are quasi-projective. The detailed structure of the moduli space is not well understood. However, the canonical Kähler–Einstein metric on the manifold can be useful for such study. It induces a canonical metric on the moduli space which is called Weil–Petersson metric: A Kähler metric on the manifold gives rise to a metric on $H^1(T)$ which is the tangent space of the Kuranishi space of the manifold.

The Weil–Petersson metric can be computed in some cases. But the properties were best understood only for moduli space of curves. There are several metrics defined on the moduli space of curves: Weil–Petersson metric defined by using the Poincaré metric (or the Bergman metric) on the curve, the Teichmüller metric (which was proved by Royden [74] to be equal to the Kobayashi metric), the Carathéodory metric, the Bergman metric and the Kähler–Einstein metric.

The last three metrics can be defined by general method, not just for moduli space of curves. Hence the computation of them is interesting. Although K. Liu, X. Sun and I [55,56] showed that the last four metrics on the Teichmüller space are all uniformly equivalent to each other, it is likely that the Carathéodory metric is different from the Kobayashi metric. But the precise statement is not known.

However, Liu–Sun–Yau did calculate the asymptotic behavior of the Kähler–Einstein metric on the Teichmüller space. In fact, the minus Ricci tensor of the Weil–Peterson metric defines a complete metric on the Teichmüller space. The Kähler–Einstein metric could be obtained by perturbing from this metric. The Teichmüller metric and the Weil–Petersson metric are computable based on local information of the Riemann surface.

The other metrics are defined by global means. Hence the remarkable theorem of Royden, proving that a metric defined by global means to be equal to locally defined metric, has provided powerful information. It will be very useful to compute all these metrics in the simplest Teichmüller space of genus two curves.

The second class of canonical metrics are those manifolds with zero Kodaira dimension. By taking the absolute value of the pluriholomorphic n-form, and taking roots, we obtain a canonical volume form which may degenerate along a divisor. We can solve the Monge–Ampère equation to obtain some Kähler metric which may be degenerate along the divisor. It will be interesting to know the birational class of these manifolds.

Let us now consider the possibility of using metrics to understand birational geometry. First of all, for two classes of manifolds, there are natural measures that are birational invariant:

For manifolds with Kodaira dimensional equal to zero, we have a canonical volume form by using the absolute value of the pluricanonical form. For manifolds of general type, I introduced 40 years ago [94] an intrinsic measure that is invariant under birational transformations. It was a generalization of the construction given by Kobayashi and Eisenman in 1970s.

In both cases, we can pick any Kähler class and solve the Calabi conjecture with this volume form as prescribed. In the first case, such metric was studied in the second part of my paper on Calabi conjecture. The metric can be proved to be unique and smooth outside the divisor defined by the pluricanonical form. In the second case, the metric also exists uniquely. But smoothness depends on the measure that I constructed.

In any case, given two birational manifolds of general type M and M', we can find M'' and smooth rational maps from M'' to M and M' respectively. If the pull-backs of polarizations on M and M' are the same on M'', then they are isometric to each other. It is easy to derive from this fact that the group of automorphisms of a manifold of general type is finite.

It is not hard to prove that any algebraic manifold of general type admits a Kähler–Einstein metric with singularity (as was demonstrated by Tsuji and myself thirty years ago). However, in order for such metrics to be useful, one needs to know the singular behavior of the metric. Kähler–Einstein metrics do not respect rational maps. However, the Bergman metric has better behavior under birational transformation.

Let us look at the line bundle K^m where K is the canonical line bundle. For any holomorphic section s of K^m, we can take $2/m$ power of its absolute value, which defines a pseudo-norm on the canonical line bundle. If we normalize its integral to be one and maximize the pseudo-norms among all such s, we obtain a canonical pseudo-norm on the canonical line bundle. It defines a birational invariant volume form. The curvature form of this volume form should define a pseudo-Kähler metric on the manifold. See [19] for a detailed discussion.

We can deform this pseudo-Kähler metric within its class to obtain a pseudo-Kähler metric which is Einstein when it is smooth. When m is large, the Kähler–Einstein metric should be less singular and if we know the singular behavior of the original pseudo-Kähler metric, we should have a way to control the singularity of this pseudo-Kähler–Einstein metric. It should be useful to study birational geometry.

For example, when the Kodaira dimension of the manifold is zero, and the non-zero form s is a section of K^m, then the volume form is the absolute value of s to the power of $2/m$. At the non-singular point of the divisor of s, the local model of the Ricci-flat metric should be the push-forward of the Kähler metric on an m-fold branch cover of the manifold branched along the divisor $s = 0$.

Based on this, one can compute the second Chern form of the Ricci-flat metric degenerate along the divisor $s = 0$. The second Chern form of this degenerate metric wedge with Kähler class to the top dimension is positive unless it is flat. This should give interesting information for manifolds with Kodaira dimension zero.

Many years ago, I conjectured that there are only a finite number of deformation types for compact Kähler manifolds with $c_1 = 0$ at each dimension (*cf.* [100, p.3]). The question is still unknown and is getting more and more important in string theory. The minimal model of algebraic manifold with Kodaira dimension zero should play an important role if we want to ask similar questions for such manifolds.

4. Hermitian Yang–Mills connections

Hermitian metric on a complex manifold has a natural generalization to holomorphic bundles over complex manifolds. Given a Hermitian metric on the bundle, there is a natural connection which preserves the metric and also the $(0, 1)$ part of the covariant derivative of which would be the same as the naturally defined $\bar{\partial}$ operator that depends only on the complex structure of the bundle and the complex manifold. The curvature is a $(1, 1)$-form with values in the endomorphism of the bundle.

4.1. Donaldson–Uhlenbeck–Yau correspondence

There is a natural generalization of the Kähler–Einstein condition to this setting by wedging the curvature 2-form with the Kähler form to the top dimension and require it to be a scalar multiple of identity tensor with the volume form. This equation is the natural generalization of anti-self-dual equations for bundles over a Kähler surface.

In fact, around 1977, C.N. Yang [93] was trying to solve the anti-self-dual Yang–Mills equation on $\mathbb{R}^4$, and he showed that it can be reduced to Cauchy–Riemann equations. And therefore he demonstrated that the above equation is part of Yang–Mills equations. It is therefore natural to call such connection to be Hermitian Yang–Mills connection.

The equation became rather well known in the math community after 1977, when people recognized the importance of applications of Kähler–Einstein metric to complex geometry. The proof of the existence of such connections would be clearly different as the Calabi–Yau theorem was based on the complex Monge–Ampère equation which depends only on a scalar. The Hermitian Yang–Mills connection is a vector-valued equation.

In December of 1977, when I was preparing the talk for the ICM in Helsinki, I thought about the possible conditions for existence of Hermitian Yang–Mills connections. I concluded that it had to be related to the slope stability of the holomorphic bundle, as was motivated by the work of Bogomolov and Miyaoka on Chern number inequalities. I was informed much later that this possibility was also believed to be true by Hitchin and Kobayashi. (Apparently they toy around the conjecture starting 1980, also motivated by my proof of the Calabi conjecture. But they had no idea how to approach the problem.)

However, the proof would have to be quite tough as there is no good way to handle such a non-linear system of elliptic equations. It turns out that Donaldson and Uhlenbeck–Yau were working on this problem independently. I learnt from Hitchin during a trip to England that Donaldson was able to prove the existence for Hermitian connections of any holomorphic vector bundle that can be deformed to the tangent bundle of a K3 surface. (Note that the Ricci-flat metric on a K3 surface provides a natural solution of the Hermitian Yang–Mills connection on the tangent bundle.) This is of course encouraging as it indicates the possibility of the conjecture.

It turns out that Donaldson [23] was concentrated on algebraic surfaces and Uhlenbeck–Yau [88] on arbitrary dimensional Kähler manifolds. While Donaldson used the Bott–Chern form and the Hermitian Yang–Mills flow, Uhlenbeck–Yau constructed a destabilizing sheaf assuming the non-existence of Hermitian Yang–Mills connection.

The proof of regularity of the destabilizing subsheaf took non-trivial effort and as a result, our paper appeared later than the work of Donaldson's proof for algebraic surfaces. After we published our work, Donaldson found that some of our formula can be used to re-prove the Uhlenbeck–Yau theorem for algebraic manifolds by restriction of the bundle to hyperplane sections of the algebraic manifold. (It was proved by Maruyama and Mehta–Ramanathan that a stable bundle is stable on a generic hyperplane section.)

This later argument of Donaldson depends intrinsically on the manifold being projective for higher dimensional manifolds. As was acknowledged by Donaldson, the argument of Uhlenbeck–Yau is most natural and in fact, all the later development for Hermitian Yang–Mills connections for higher dimensional manifolds are based on the procedure of Uhlenbeck–Yau.

Some later paper such as the one by Bando–Siu [5] used the Hermitian Yang–Mills flow to generalize our result, but the essential feature of Uhlenbeck–Yau procedure is still needed in an essential manner. It should also be pointed out that the continuity argument used by Uhlenbeck–Yau is just as convenient as the Hermitian Yang–Mills flow.

A few years later, Carlos Simpson [78] generalized the Uhlenbeck–Yau argument to establish similar theorem when the Higgs field was introduced. Hermitian Yang–Mills connections were proposed by me to Edward Witten in 1984 to study heterotic string, which had since became an important subject in mathematical physics. But from the very beginning, we knew the importance of Hermitian Yang–Mills connections, as it provides important Chern number inequalities, and also the conditions for the bundle to be projectively flat.

4.2. Chern number inequalities and characterization of flat bundles

The very first applications was the sharpening of the Chern number inequality of Bogomolov and a very important generalization of the theorem of Seshadri–Narasimhan (1965) that every stable bundle over an algebraic curve is flat if the degree of the bundle is zero. A very remarkable corollary of the existence of Hermitian Yang–Mills connection for stable holomorphic bundle is that such bundle must be projective flat, if the Bogomolov inequality $2r\, c_2(E) \geq (r-1)c_1(E)^2$ becomes equality.

This can be considered as a generalization of my theorem that the equality of certain Chern numbers can be used to characterize ball quotients. In fact, Carlos Simpson observed that by generalizing this theorem to the Hermitian Yang–Mills–Higgs connection, one can reproduce my previous theorem that an algebraic surface of general type is covered by the ball if the ratio of the two Chern numbers is the same as the projective plane. In fact, by choosing the Higgs field carefully, one can generalize the theorem to characterize quotient of general Hermitian symmetric space assuming suitable stability.

Characterization of flat bundles based on Hermitian Yang–Mills–Higgs connection also allows Simpson to construct variation of Hodge structures. This is remarkable and led me to believe that there is a good connection with the characterization of quotients of more general Hermitian symmetric domains based on the existence of Kähler–Einstein metrics.

It is really remarkable that the construction of stable bundles satisfying certain Chern number equality gives rise to non-trivial projective representation of the fundamental group of the manifold, which we know little about.

In particular, if some natural bundle constructed from the tangent bundle of the manifold is stable with respect to certain polarization and if the numbers defined by wedging the second Chern class of the natural bundle with the polarization $n-2$ times, and the square of the first Chern class of the natural bundle wedged with the polarization $n-2$ times are equal to zero, then the natural bundle admits a flat Hermitian connection, which means that the fundamental group of the manifold has a non-trivial unitary representation, unless the natural bundle is trivial. Note that we do not need to assume existence of Kähler–Einstein metric on the manifold in this setting. Natural bundles are bundles constructed from natural decomposition of tensor product of tangent and cotangent bundles.

It raises an interesting question in this regard: given an algebraic manifold M with a fixed Kähler class, we consider all holomorphic bundles with trivial first Chern class over M which are polystable with respect to this Kähler class. We consider two such bundles equivalent if they become isomorphic to each other after adding trivial bundles. They form a ring consisting of countable number of algebraic subvarieties which are moduli space of the bundles with a fixed Hilbert polynomial.

There is a subring formed by those stable bundles whose second Chern class wedged with the Kähler class $n-2$ times vanishes. Does the structure of this subring determine the algebraic fundamental group of the manifold? (It is quite likely that we need to consider bundles with Hilbert space fiber in order to obtain information for the full fundamental group.) What is the structure of this subring for Shimura varieties? Can they determine the Shimura variety?

4.3. Generalization to non-Kähler and non-compact manifolds

Since the theory of Uhlenbeck–Yau was generalized by Jun Li and myself to general complex manifolds, we are able to apply it to handle some interesting non-Kähler manifolds. The most notable one was the class VII surfaces of Kodaira. They were studied by Kodaira, Inoue and Bombieri. Kobayashi and Ochiai realized the importance of holomorphic connections for such manifolds. Bogomolov claimed that for such manifolds without curves, they are given by the examples constructed by Inoue. The proof by Bogomolov [6,9] is not clear.

Jun Li, Fangyang Zheng and I [51] gave a clear proof based on the existence of Hermitian Yang–Mills connections. It should be possible to generalize our argument to handle those class VII surfaces with finite number of curves also. Many years ago, I proposed to study those connections mod the curves long ago. If this proposal is successful, it should complete the Kodaira classification of complex non-Kähler surfaces.

The study of Hermitian Yang–Mills connections over quasi-projective curve was discussed by Simpson. The generalization to the case when the base pair is (M, D) with D non-singular, is not hard. The case when D is normal crossing divisor is more difficult, and was studied by Takuro Mochizuki [70].

4.4. Analytic criterions for various stability conditions

There is no simple criterion to check whether a bundle is stable or not. In many cases, the existence of Hermitian Yang–Mills connection helps to understand properties of stability of bundles. Slope stability is only one kind of stability that appeared in algebraic geometry. A natural class of stability was introduced by David Gieseker in early 1970's. He compared Hilbert polynomials of the subsheaves.

The analytic analog of Gieseker stability is not well understood, although Conan Leung studied this problem in his PhD thesis [49], under my guidance about 20 years ago. There are a sequence of differential equations which can be considered as a natural generalization of the Hermitian Yang–Mills equations. (Todd classes are part of the equations as Hilbert polynomial need to be expressed.) Assuming the curvature is uniformly bounded, Leung proved that the existence of the equations is equivalent to Gieseker stability of the bundle. This bound of the curvature has not been proved and whether this set of equations is the most natural set of equations is not clear.

As was proposed by me [101], the existence of a Kähler–Einstein metric or metrics with constant scalar curvature on an algebraic manifold is related to stability of the algebraic manifold. My former student Huazhang Luo, followed my suggestion of using the concept of balanced condition to study stability of manifolds. It would be good to relate manifold stability to bundle stability. Now Chen–Donaldson–Sun proved that K-stability of the manifold implies the existence of Kähler–Einstein metric. It implies, in particular, the stability of the tangent bundle of the manifold.

In order to relate two concepts of stability, I propose to define a bundle to be balanced if the sections of the bundle, after twisted by a very ample line bundle, can embed the manifold into a balanced submanifold of the Grassmannian.

Hermitian Yang–Mills bundles are mirror to special Lagrangian submanifolds in the theory of mirror symmetry under the program of Strominger–Yau–Zaslow. Gieseker stability is slightly weaker than slope stability. It may be interesting to know which class of Lagrangian cycles will be their mirror images. By studying stability question for Lagrangian cycles carefully and applying mirror symmetry, Mike Douglas found new concepts of stability of bundles. Based on his work and the works of F. Denef, Douglas–Reinbacher–Yau [25] proposed a conjecture on the existence of stable bundles based on Chern classes of the bundle which can be stated as follows:

Consider an ample class D on a simply connected Calabi–Yau threefold X and an integer $r > 1$ and three classes

$$c_i \in H^{2i}(X, \mathbb{Z}),\ i = 1, 2, 3$$

such that

$$\left(2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(D)\right) = 2r^2 D^2$$

and

$$(c_1^3 + 3r(r\, ch_3 - ch_2 c_1)) < 8\sqrt{2} \cdot r^3 D^3.$$

Then there exists a rank r reflexive sheaf V on X stable with respect to some ample class such that

$$c_i(V) = c_i, \ i = 1, 2, 3.$$

Prior to the SYZ program on mirror symmetry, Kontsevich introduced the concept of homological mirror symmetry, where he introduced the derived category over algebraic manifolds. It was realized later to correspond to branes in string theory. This has been developed into a rich theory. Bridgeland studied the concept of stability of derived category and it is now called Bridgeland stability. It would be interesting to find a suitable analytic counterpart of Bridgeland stability.

5. Mirror symmetry

Supersymmetry provides powerful tools to understand Calabi–Yau manifolds. The intuitions from physics have been powerful. The important concepts introduced by string theorists have deep influence on the geometry of such manifolds. The most important one was the idea of mirror symmetry. It called for the existence of another Calabi–Yau manifold (which we call the mirror manifold) whose Hodge diamond for cohomology is the transpose of the Hodge diamond of the original Calabi–Yau manifold.

5.1. Counting of curves

More importantly the conformal field theory based on one Calabi–Yau manifold is dual to that of its mirror manifold. The Type IIA conformal field theory of Calabi–Yau manifold is isomorphic to the Type IIB theory of the mirror manifold. This is a remarkable theory predicted by Vafa, Dixon and others. But it was Greene–Plesser and Candelas et al. who developed the details of such theory. The most remarkable consequence is that it solved an old problem in enumerative geometry.

The reason is that the Type IIB theory can be computed by deformation theory of Kodaira–Spencer while the type IIA has quantum corrections. The quantum corrections are provided by the rational curves on the Calabi–Yau manifold. Since Type IIA theory of one manifold is isomorphic to the type IIB of the mirror manifold, we can compute the number of rational curves on the Calabi–Yau manifolds by the variation of Hodge structure for its mirror family.

The initial theory was mostly based on physical intuition. But two groups of mathematicians proved such statements rigorously in 1996, by Lian–Liu–Yau [52] and Givental [34] independently. Despite that the proof is rigorous, the intuition from string theory played the most important role. The idea of supersymmetry has become one of the most fundamental philosophies underlying the current modern development of algebraic geometry.

The simplest and most elegant examples of geometric structures showing up in string theory studies are the Calabi–Yau manifolds, where rich structures related to deep string theory and Quantum Field Theory dualities are discovered. This includes the so called Gromov–Witten invariants related to the counting of rational curves mentioned above.

The counting of algebraic curves of higher genus is far more complicated. One approach was initiated by Bershadsky, Cecotti, Ooguri and Vafa (BCOV), who developed a theory called Kodaira–Spencer theory of gravity. The computation gives beautiful predictions based on some master equations. But it suffers from an ambiguity which is called holomorphic ambiguity. Up to now, we still have difficulty to overcome this ambiguity, although some important progress was made by Zinger, Jun Li and their coauthors for genus one or low genus curves.

Modularity of partition functions of the so-called topological strings contains non-trivial arithmetic information of the Gromov–Witten invariants and it is still a mystery to understand the appearance of modular forms completely. Yamaguchi and I were able to demonstrate some polynomial structure on such partition functions [92], which suggested that there is rich algebraic structure behind them. Due to especially its impressive power in enumerative geometry, there was a great desire to understand mirror symmetry mathematically.

5.2. Mathematical approaches to mirror symmetry

As mentioned above, two different approaches were proposed. One is the famous Kontsevich's homological mirror symmetry conjecture [46] which says that the derived category of coherent sheaves of a Calabi–Yau manifold is equivalent to the Fukaya category of its mirror manifold. Fukaya pioneered the research to study the extensive complicated structure of the Floer theory of Lagrangian cycles through A_∞-algebra [30, 31], which is an important ingredient in the conjecture of Konsevich. Another approach was proposed by Strominger–Yau–Zaslow [80] that the Calabi–Yau manifold is fibered by special Lagrangian torus and the mirror manifold is obtained by replacing the torus by its dual torus. Much progress was made by Auroux, Seidel et al. in this direction. It is important that singularities are allowed in the fibration for both topological and more subtle reasons. The SYZ conjecture has much evidence to be true. Gross and Siebert made a lot of progress in the last few years using tropical methods.

Mirror symmetry has inspired many important developments in Kähler geometry. The program of SYZ calls for close relationship between special Lagrangians with bundles. In the paper of Leung–Yau–Zaslow, we explained how, under the SYZ map, equation for special Lagrangian cycle which intersects the SYZ torus at one point can be transformed to an equation for a holomorphic line bundle. The equation turns out to be studied by M. Mariño, R. Minasian, G.

Moore and Strominger [60]. Recently Tristan Collins, Adam Jacob, and myself [20] studied this equation and we can prove the existence for many important cases assuming some form of stability for the $(1,1)$-class. The equation has the form:

$$\mathrm{Im}(J + \sqrt{-1}\,\omega)^n = \tan(\hat{\theta})\mathrm{Re}(J + \sqrt{-1}\,\omega)^n,$$

where $\hat{\theta}$ is a topological constant determined by J and $[\omega]$.

The equation admits supersymmetry and the pair consisting of the Kähler class J and the closed $(1,1)$-class ω can be looked as a natural complexification of the Kähler class. It defines an open set in the complexified Kähler cone. This may give a good candidate for the mirror of the moduli space of polarized complex structures of its mirror manifold. Note that we like to see the "Kähler moduli" to be isomorphic to the moduli space of the complex structure on its mirror manifold. It is also important to find a suitable discrete group acting on this open set in the Kähler cone. The mirror symmetric version of the special Lagrangian that intersects the SYZ torus for more than a point is supposed to be a higher rank bundle. The equation defined on it is being explored.

One should note that the homological mirror conjecture of Kontsevich has inspired a great deal of study of derived category in geometry. While we may not be used to abstract reasoning of category theory in geometry, we hope that its relationship to SYZ construction may eventually broaden the scope of geometry.

6. Future directions in mathematical physics and arithmetic geometry

In conclusion, we should say that the beautiful subject initiated by Riemann in the 19th century on Riemann surfaces had deep influence on the development of complex geometry in the 20th century. While Hodge provided the fundamental structure relating complex analysis with topology via Hodge groups, Kodaira provided fundamental methods to construct holomorphic sections of bundles. With the works of Chern classes and Hirzebruch–Riemann–Roch formula, the works of Hodge and Kodaira have been developed to be most powerful tools in understanding Kähler geometry. The modern development has been emphasizing the use of non-linear elliptic equations, relating the concept of Kähler–Einstein metrics and Hermitian Yang–Mills equations to various fundamental concepts of stability introduced to study moduli spaces.

The most recent development on Calabi–Yau space due to cooperations between mathematicians and string theorists has been spectacular. Ideas of many fields in mathematics were used. We hope to see some more ideas of number theory in this beautiful subject. For many Calabi–Yau manifolds, the partition functions related to conformal field theory are related to modular forms. For example, it was observed in 1996 by Zaslow and I [103] that the partition function counting rational curves of various degrees in K3 surfaces can be written in terms of η-functions. This was the first time that such modular function appears in counting curves in algebraic geometry. This motivated Göttsche to generalize the Yau–Zaslow formula to general surfaces and for curves of arbitrary genus. This was recently first proved by Y. Tzeng [87] and later by Kool–Shende–Thomas [47]. For Calabi–Yau manifolds with higher dimension, these formulas are much more complicated and their organization is still being explored. In recent work with Zhou and others [2], we were able to show the ring generated by quasi-modular forms associated to $PSL(2, \mathbb{Z})$ or a congruent subgroup therein is isomorphic to the ring of higher genus Gromov–Witten invariants for certain non-compact Calabi–Yau geometries based on the projective plane and also del Pezzo surfaces. This was later extended to the orbifold case [76] and also to include open Gromov–Witten theory in [48].

For more classical arithmetic geometry, we may point out that Serge Lang has noticed long time ago that the importance of Kobayashi hyperbolicity is relevant to the question of Diophanitine problem, such as the Mordell conjecture proved by Faltings for algebraic curves of higher genus. Kobayashi conjectured that for an algebraic manifold of general type, the Kobayashi metric should be non-degenerate in a Zariski open set. In particular, there is a subvariety of the manifold such that all rational curves and elliptic curves are subset of this algebraic subvariety. This is sometimes called the Lang conjecture.

Lang also conjectures that if the manifold is defined over integers, the rational points of the manifold should all be in this subvariety. There was little progress on the Kobayashi–Lang conjecture except in the case of surfaces where Bogomolov [7] and Miyaoka [69] made important contributions. Steven Lu and I studied the differential geometric aspect of it [58].

For algebraic surfaces with positive index, one can find a Finsler metric with strongly negative holomorphic sectional curvature, but the metric may degenerate in some subvarieties. This statement implies the Kobayashi–Lang conjecture. Therefore one would like to make the following conjecture: an algebraic manifold is of general type if and only if it admits a complex Finsler metric which may be degenerate along a subvariety which has strongly negative holomorphic sectional curvature. It is quite possible that Finsler metric may be replaced by Kähler metric. The converse was asked by me, there were some progress due to several people, but only recently Damin Wu and I [91] were able to prove that if an algebraic manifold admits a Kähler metric with strongly negative holomorphic sectional curvature, its canonical line bundle must be ample. (Our original argument assumes manifold to be algebraic, but it was pointed out by Valentino Tosatti and Xiaokui Yang [83] that our argument, which is based on solving Monge–Amperè equation, can work for Kähler manifold also.) It is not hard to generalize the theorem to complete non-compact Kähler manifolds whose holomorphic sectional curvature is bounded by two negative constants. A natural question is that a compact manifold admits a pseudo-Kähler metric

(Kähler metric that may degenerate along some subvarieties) with strongly negative holomorphic sectional curvature if and only if the manifold is of general type. Manin has conjectured that Kähler–Einstein metric will play important role in arithmetic geometry. I believe that is the case. There is still much to learn about the relation between complex geometry, algebra and number theory.

Appendix. some open questions

Question 1. Classify smooth Calabi–Yau manifold which is a non-trivial fiber space whose fiber is another Calabi–Yau manifold. The base algebraic variety should have Kodaira dimension less than or equal to zero. But there should be much stronger constraints. For dimension four, this is related to F-theory appeared in string theory. It would be useful to know the discriminant locus and its singularities. Their structure is important for phenomenology in string theory. We believe that there is only a finite number of topological types of Calabi–Yau manifolds in each dimension, hence the constrains on the base and the discriminate locus should be strong enough to show there are only finite number of such structures, up to deformations. For references, see [12, 26–28, 72].

Question 2. Classify manifolds with positive scalar curvature that can be written as conformal boundary of a complete Einstein manifold which is asymptotic to a hyperbolic space form. Witten and I [90] proved that it has to be connected. But higher connectivity is not known. Schoen and I [75] demonstrated the possibility of doing codimension-3 surgery on space of manifolds with positive scalar curvature in 1979. It will be nice to do the same within this category. (It is not so hard to do connected sum of such manifolds.) We can add some more conditions on the Einstein manifold in the bulk such as anti-self-dual Einstein metric. One can also demand the boundary to be Sasaki–Einstein manifolds and would they always bound an asymptotic hyperbolic Einstein metric? Since Sasaki manifold can bound asymptotic hyperbolic Einstein metric, it would be interested to construct such manifolds using Kähler geometry. The key question is to understand the metric in the bulk.

Question 3. What is the appropriate supersymmetric version of bundle theory over Sasaki–Einstein manifolds, and when can they be looked as boundary data for connections within the bulk? What are their moduli space? Will the moduli space itself have good structure? For supersymmetric bundles over Sasaki–Einstein manifolds, are there appropriate generalization of Chern classes and will they be dual to suitable special cycles?

Question 4. What is the appropriate mirror dual of a complete non-compact Calabi–Yau manifold asymptotic to a metric cone? And what are their special Lagrangian submanifold that is asymptotic to a metric cone?

Question 5. For a Calabi–Yau fourfold, there are parallel 4-forms coming from linear combination of real and imaginary parts of the standard holomorphic 4-form and wedge product of the Kähler form. It is very interesting to understand the cycles calibrated by non-trivial combination of such forms. We do not have interesting non-trivial global examples! For a Calabi–Yau fourfold, if there exists an anti-holomorphic involution which fixes the above cohomology class up to sign, but not the Kähler class, will the fixed point of such involution gives rise to such calibrated cycles? The special Lagrangian cycles are dual to algebraic cycles under mirror symmetry, what are the mirror dual of such cycles? They are presumably dual to itself. What kind of properties do we expect from them? In analog to the Hodge conjecture, do we expect homology in middle dimensional for Calabi–Yau manifolds be represented by cycles calibrated by parallel forms? What is the geometric meaning of $(2,2)$-cohomology of a Calabi–Yau fourfold? Are they related to the above cycles?

Question 6. Can one classify elliptic Calabi–Yau fourfolds (i.e., Calabi–Yau fourfolds that are elliptically fibered over some algebraic 3-folds) up to birational transformations? Some progress has been made in [39].

Question 7. Let M be a fiber space with base B and fiber F. Suppose there is a unitary representation of the fundamental group of M into $SU(n)$ and a stable bundle V over B with trivial determinant line bundle. Can one construct a stable bundle (with suitable polarization) over M with trivial determinant line bundle whose quotient bundle is given by the pullback of a bundle over B, and the unitary flat vector bundle over F is a subbundle for V restricted to F? Hopefully one can manage to find such a bundle whose second Chern class is zero. In this way, one can construct unitary representation of fundamental group of algebraic manifolds.

Question 8. Classify all algebraic manifolds whose anti-canonical line bundle admits an effective divisor. Classify those divisors that have disconnected components. If the anti-canonical line bundle is ample along this divisor, would the complement of the divisor admit a complete non-compact Calabi–Yau metric?

Question 9. It is well known that a stable bundle over an algebraic manifold is stable when it restricts to a generic hyperplane section of the manifold [62,66]. What is the mirror picture of this statement?

Question 10. For stable bundles over an algebraic manifold, there is the Bogomolov–Gieseker inequality. What is the mirror dual of such inequality for the pair: special Lagrangian cycle and complex flat bundle over the cycle.

Question 11. For a bounded complex of holomorphic vector bundles over an algebraic manifold, there is the concept of quasi-equivalence between them.

Within this equivalence, can we find a more canonical one such as choosing harmonic form among closed forms. Can we give a canonical Hermitian metric of each of the bundles appeared in the complex so that we have a good theory of stability, presumably recover the concept of Bridgeland stability.

Question 12. Classify stable algebraic 3-dimensional manifolds with constant scalar curvature. Describe the space of all polarizations that support such metrics. Can one construct such metrics by blowing up sufficiently large number of subvarieties for any given algebraic manifold. Can one find such metrics near (possibly singular) Kähler–Einstein metrics which we know are stable?

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第16回高木レクチャーのご案内

「高木レクチャー」は世界的に卓越した数学者を講演者として招聘し、気概に満ちた研究総説講演を若手研究者・大学院生を含む専門分野を超えた数学者が聴くことにより、創造のインスピレーションを引き起こし、新たな数学の発展に寄与することを目指した企画です。

下記の日程で「第１６回高木レクチャー」を開催いたしますので、ご案内申し上げます。

組織委員：小野 薫、河東泰之、小林俊行、斎藤 毅、中島 啓

記

日　時：平成２７年１１月２８日（土）－２９日（日）

場　所：東京大学大学院数理科学研究科 大講義室

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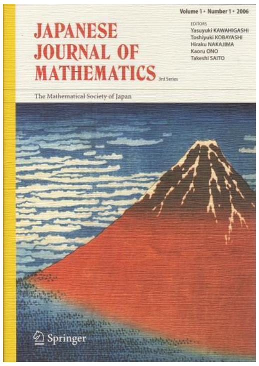
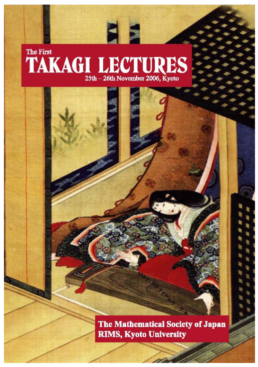


Japanese Journal of Mathematics

高木貞治
（1875－1960）



高木ブックレット

プログラム

平成２７年１１月２８日（土）

11:30--12:30 Registration

12:30--12:40 Opening

12:40--13:40 **Masaki Kashiwara** (RIMS, Kyoto University)
Riemann--Hilbert Correspondece for Holonomic D-modules (I)
（ホロノミック D 加群に対するリーマン=ヒルベルト対応）

14:00--15:00 **Fabrizio Catanese** (Universität Bayreuth)
Kodaira Fibrations and Beyond: Methods for Moduli Theory (I)
（小平のファイバー空間とその展開―モジュライ空間の方法）

15:00--15:45 Coffee/Tea Break

15:45--16:45 **Jean-Pierre Demailly** (Université de Grenoble I)
Recent Progress towards the Kobayashi and Green--Griffiths--Lang Conjectures (I) （小林予想および Green--Griffiths--Lang 予想の最近の進展）

17:05--18:05 **Shing-Tung Yau** (Harvard University)
From Riemann and Kodaira to Modern Developments on Complex Manifolds (I)
（リーマンと小平邦彦から複素多様体論の現代の発展へ）

平成２７年１１月２９日（日）

10:00--11:00 **Masaki Kashiwara** (RIMS, Kyoto University)
Riemann--Hilbert Correspondece for Holonomic D-modules (II)
（ホロノミック D 加群に対するリーマン=ヒルベルト対応）

11:20--12:20 **Fabrizio Catanese** (Universität Bayreuth)
Kodaira Fibrations and Beyond: Methods for Moduli Theory (II)
（小平のファイバー空間とその展開―モジュライ空間の方法）

12:20--14:00 Lunch Break

14:00--15:00 **Jean-Pierre Demailly** (Université de Grenoble I)
Recent Progress towards the Kobayashi and Green--Griffiths--Lang Conjectures (II) （小林予想および Green--Griffiths--Lang 予想の最近の進展）

15:20--16:20 **Shing-Tung Yau** (Harvard University)
From Riemann and Kodaira to Modern Developments on Complex Manifolds (II)
（リーマンと小平邦彦から複素多様体論の現代の発展へ）

16:30--17:30 Workshop closing with drinks

The Sixteenth

Takagi Lectures

Teiji Takagi
(1875-1960)

Date : November 28 (Sat) - 29 (Sun), 2015
Place : Graduate School of Mathematical Sciences, The University of Tokyo

Takagi Lecturers
In Celebration of the 100th Anniversary of Kodaira's Birth

F. Catanese
(Universität Bayreuth)

J.-P. Demailly
(Université de Grenoble I)

M. Kashiwara
(RIMS, Kyoto University)

S.-T. Yau
(Harvard University)

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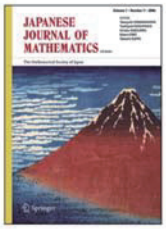

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