2'nd Report of " Recent results on the Kobayashi and Green-Griffiths-Lang Conjectures"

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The aim of the present article is

(1) to provide rather detailed complete surveys of the two conjectures, Kobayashi's and Green–Griffiths', and

(2) to simplify and improve D. Brotbek's solution of Kobayashi Conjecture.

From the first Report:

The present article surveys the development of the titled conjectures and related topics, especially the Kobayashi conjecture which claims the Kobayashi hyperbolicity of generic hypersurfaces of high degree in complex projective space. The main theme is the notion of "directed manifolds" due to the present author and its applications to the hyperbolicity problem centered around the Kobayashi Conjecture. In the last year the Kobayashi Conjecture was finally solved affirmatively by D. Brotbek [Brot17], and then some improving works have been taken on since then. The present author also gave some simplification of the proof in §12.

As for §§1–11, the contents are mainly based on [Dem97], [Dem11], [Dem12] and [Dem14], which were already published: it is good to find a nice summarization of them in one place here.

The content of §12 is new and devoted to a simplification of Brotbek's proof of the Kobayashi Conjecture [Brot17]. Here, unfortunately the presentation of this section is rather rough, and some more clarifications should be expected, as the referee thinks:

Report 2.

(1) As for §12, the new version describes much more details than the former one, dividing the sections into the subsections A–H (formerly, A and B) with two pages increase. The presentation of this part is improved considerably and now should be acceptable to many readers.

(2) To complete the goal of the present survey paper, there still remain the following two points that should be improved:

(a) **0.2. GGL Conjecture.** In the next page 3, there is a mention on the result in the case of $q = h^0(X, \Omega^1_X) > \dim X$. In this approach with assuming X being of (log-) general type, the best result in the general dimensional case, now available, is that *if the (log-) irregularity* $q \geqq \dim X$*, then no entire curve* $f : \mathbf{C} \to X$ *has a Zariski dense image, and the strong Green–Griffits Conjecture (=GGL's) holds in compact case at least* by

Noguchi–Winkelmann–Yamanoi, J. Math. Pure Appl. 88 (2007), Vietnam J. Math. 41 (2013), Lu–Winkelmann, Forum Math. 24 (2012).

In the projective case one can find strong Green–Griffiths Conjecture as stated in GGL Conjecture 0.2 in Noguchi–Winkelmann–Yamanoi, J. Math. Pure Appl. 88 (2007), and Noguchi–Winkelmann [NoWi14](GL350 (2014)), §6.6.4,

Needless to say, it is totally non-trivial to allow the case, $q = \dim X$.

(b) **0.3. Conjecture** (Kobayashi). (a) and (b).

The first issue at the beginning was the "*existence*" of such X or H stated in (a) or (b) of the Conjecture that are irreducible or smooth.

As Zaidenberg observed, a smooth deformation of $2n+1$ hyperplanes in $\mathbf{P}^n$ is not necessarily Kobayashi hyperbolic (a personal communication), and the issue was non-trivial at all.

The existence was proved by showing the examples, which developed as follows:

i. (a) for $n = 3$ by Brody–Green [BrGr77].
ii. (b) for $n = 2$ by Azukawa, K. and Suzuki, Masaaki, Some examples of algebraic degeneracy and hyperbolic manifolds, Rocky Mountain J. Math. **10** (1980), 655–659.
iii. (a) for $n \geq 4$ and (b) for $n \geq 3$ by Masuda–Noguchi [MaNo96].

Then later on, there were improvements in the degree estimates by a number of papers such as M. Shirosaki (1998), H. Fujimoto (2001), and Shiffman-Zaidenberg (2002), and much more for $n = 2, 3$.

(3) What follows are typos and stylistic matters:
p.5, (0.11): GG(X, V)**,**

(4) p.6: Neve**r**therless

(5) p.8–9: The "*Proof*" at the end of p.8 has no ending mark □.

(6) p.19: For **4.4. Proposition**, it is fare to cite [Carl72] and [Nog77b].

(7) p.23: Is the exponent of $|z|$ correct?

(8) p.37, p.47: Why the inequality $\geqslant$ or equality $=$ is doubled in the equation?

(9) p.38: What is "□" for? Here is the end of "Assumptions".....

(10) p.77, [MeTe19]: $(\sqrt{n}\log n)^n$

(11) p.78, at the end of [ShZa02]: (2002)**,** 2031–2035**.**

(12) p.79, [Zai93]: ...... Related topics 1992, ...

(13) There are some simple typos in the citation symbols such as [GG79] at p.5, [NogWi13] and some inconsistency in the symbols such that [GG79] should be [GrGr80], where "79" represents the year of the meeting being held, and the publication year was 1980. In the most other cases, that number represents the year of the publication of the referred paper, and it is better since the final content as a reference can be different to what had been really given at the meeting. In the case of [Zaid93], the meeting was held in 1992; there may be more similar cases. Papers published in journals have some similarity in the sense that they have the meta data of the receied dates and the final revision dates besides the publication years; but it is usuaaly to use the publication years for the citations. Therefore, this inconsistency may give a misleading impression to the reaers and may cause a confusion in references in future.

It is also strongly recommended to check the accuracy of the reference data; e.g., [Nog83] (83 is the volume number) should be [Nog81b] (need to change [Nog81] to [Nog81a]), since the present article is expected to serve as the standard reference of the topics, once it is published.

P.S.: Althoug most of (3)—(13) were mentioned in the 1st Report, item (2) (a) and (b) were not and should have been mentioned in the first report, but the referee did not notice them; for this, he expresses an apology.