Report on the manuscript "Degenerate complex Monge-Ampère equations over compact Kähler manifolds" by Demailly and Pali

The manuscript deals with solving and applications of the degenerate Monge-Ampère equation

$$(\omega + i\partial\overline{\partial}\phi)^n = v$$

on a compact n-dim Kähler manifold X, in the case when ω is a smooth closed semipositive (1,1) form and $v \geq 0$ is a (n, n) form, $\int_X v = \int_X \omega^n$ and v has density (w.r.t. some fixed volume form) in an Orlicz space $L \log^{n+\epsilon} L$.

Previous results on this equation are by Yau, [Kol1,2] (see the bibliography), and [EGZ]. [Kol1,2] solves the equation in the nondegenerate case ω Kähler, but with more general right hand side. [EGZ] solve it in the degenerate case, and for v with L^p, $p > 1$ density.

The referee finds that the manuscript lacks a lot in exposition, which is disorganized and many times unclear and difficult to follow. The results of the first 4 sections were carefully checked, and are correct, despite some mistakes that should not be present in any submitted manuscript (see comments 1 and 6 below)! The manuscript needs however more than extensive revisions, in fact the rewriting of some sections. In the referee's opinion the manuscript cannot be accepted for publication.

Some comments:

1. The limit in Remark 2.2 is equal to 0, not ∞, so the Orlicz space L^Ψ of Kolodziej is larger than $L \log^{n+\epsilon} L$. Hence when γ is Kähler the result of Kolodziej is more general.

2. Statement of Theorem 2.1: define Osc.

3. The proof of Theorem 2.1 starts with a series of 9 preliminary lemmas, claims, a theorem and corollary, some included within others, which makes the proof hard to follow (e.g. claims 2.5, 2.6 within the proof of Lemma 2.4, Theorem 2.10 and Corollary 2.11 within the proof of Lemma 2.9; in Corollary 2.11 one needs to assume K is nonpluripolar). The authors should explain after the statement of Theorem 2.1 what are the main ideas and steps of the long proof, then proceed with preliminary lemmas organized in a better way.

4. Most of the preliminary lemmas mentioned above concern known facts

from global or local pluripotential theory. The capacity Cap_γ, extremal function Ψ_K, are introduced and studied in Guedj-Zeriahi 2005, and many of their properties reproved in the manuscript can be found there. The authors should refer to that paper, while of course noting how the constants involved depend on their parameter t.

5. Claim 2.6: the fact that $\gamma_\psi \wedge T^l$ is commutative follows simply from the local theory Bedford-Taylor 1982, and should not be reproved here. In the proof of $C_l < \infty$, inequalities on top of p.8 use that $\psi_{c,\epsilon} \leq 0$. This is not the case as $\max \psi = 0$. (It is easy to get around this, but something should be said.)

6. End of proof of lemma 2.9, p.13: It is claimed that by a "basic fact about measure theory", $\int_K \Omega = \int_{K^0} \Omega$. This is false, just take K a compact with positive volume and empty interior! However, the inequality the authors claim is still true, since negligible sets have measure 0.

7. Proof of part B, p.15: a is used to denote both a constant and a function.

8. Theorem 3.2: The authors should emphasize after the statement what is the issue about the formula $\gamma_\phi^k \wedge T = T \wedge \gamma_\phi^k$, namely that LHS is defined because of their theorem part A, while RHS because of the classical Bedford-Taylor definition, and the issue is whether the 2 are equal.

9. Proof of Thr 3.2, bottom of p.23: (3.10) does not follow from (3.9) by continuity of $\partial\overline{\partial}$, but (3.10) "$k+1$" follows from (3.9) and (3.10) "k".

10. Proof of Thr 3.2 p.25-27. Since $S \wedge \gamma_\phi^k \wedge T$ is not a-priori defined one has to use parentheses to indicate order when such currents are involved (e.g. in formulas on p.26). Alternatively, it is natural to prove that

$$S \wedge (\gamma_\phi^k \wedge T) = \gamma_\phi^k \wedge (S \wedge T),$$

where S, T are (1,1) currents with bounded local potentials, LHS defined by Bedford-Taylor, while RHS by Thr 3.2 A. This will simplify the rest of the proof.

11. End of proof of Lemma 4.1, p.31, is very unclear. It is stated that "Ψ approximates Φ faster and faster..." What does it mean, exactly? Clarify! Also "By well-known properties of complex potential theory Ψ extends...". What properties do you use, exactly?

12. Proof of Thr 4.2 (B), p. 33-34. This is exactly the same argument as in [EGZ], but wrongly reproduced. It is proved on top of p. 34 that $\gamma_\phi^n \geq \gamma_\psi^n$.

This is an inequality of measures, not some pointwise γ^n_ψ-a.e. inequality, as the manuscript tries to explain...!

13. The proof of Lemma 5.2 is not given, but is referred to the "proof of Theorem 3.4 in [De-Pa]". Some words should be said about the proof.

14. The authors include in "Acknowledgements" p.50 some critiques to other manuscripts (unpublished, but available on arxiv) about similar problems. This is not the right place for such critiques.