Analytic and arithmetic solutions of algebraic equations

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Solutions of algebraic equations (degrees 2 and 3)

Solution of quadratic equations (Babylonians): $ax^2 + bx + c = 0$

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Solution of cubic equations (Niccolo Tartaglia, Gerolamo Cardano):

$x^3 + px + q = 0$ has 3 complex solutions $x_k = j^k u + \bar{j}^k v$, $k = 1, 2, 3$

$$\text{where } j = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \; u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \quad v = -\frac{p/3}{u}.$$

Tartaglia
$\sim$ 1535

Cardano

Solutions of algebraic equations (higher degrees)

Solution of quartic equations (Lodovico Ferrari, 1540):

The solution of $ax^4 + bx^3 + cx^2 + dx + e = 0$ can be reduced to solving consecutively one cubic and two quadratic equations.

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However, Paolo Ruffini, Niels Abel and finally Évariste Galois ($\sim$ 1831) show the impossibility of solving equations of degree ≥ 5 by using only radicals.

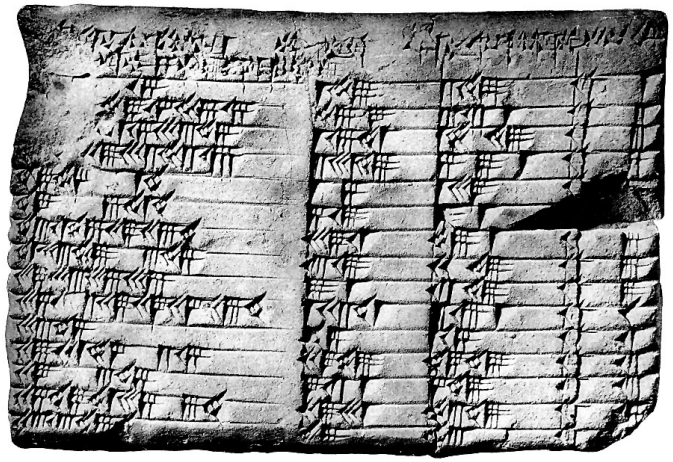
Ferrari

Galois

Galois's proof relies on the fact that the symmetric group $\mathfrak{S}_n$ of permutation of roots is not solvable for $n \geq 5$.

Diophantine equations

Babylonian tablet ($\sim$ 1800 BC) showing Pythagorean triples



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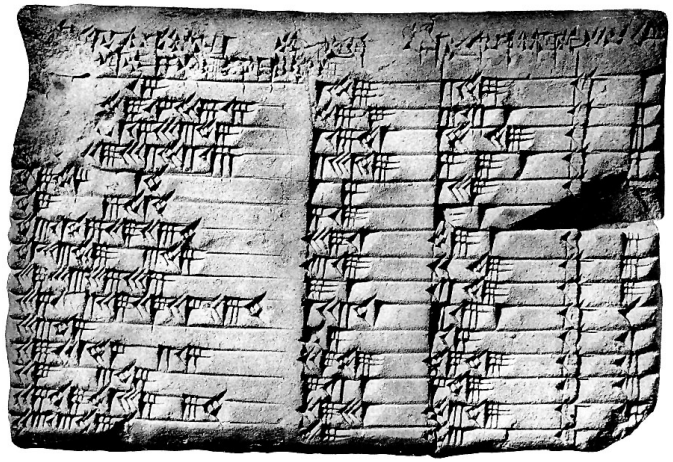

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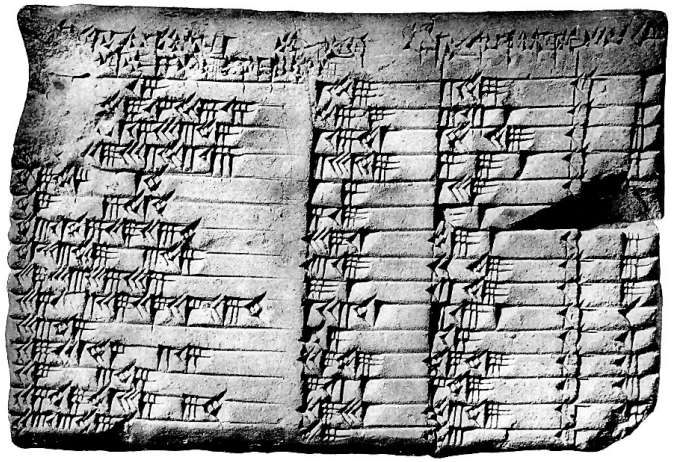

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The general solution attributed to Euclid is

$$a = k(p^2 - q^2), \quad b = 2kpq, \quad c = k(p^2 + q^2)$$

where $k > 0, p > q > 0$ are integers.

Interpretation of Euclid's solution

The equation $a^2 + b^2 = c^2$ is equivalent to

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \quad \Leftrightarrow \quad x^2 + y^2 = 1$$

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$$x = \frac{a}{c} = \frac{p^2 - q^2}{p^2 + q^2}, \quad y = \frac{b}{c} = \frac{2pq}{p^2 + q^2}$$

which is equivalent to

$$(*) \qquad x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}, \quad t = \frac{q}{p}.$$

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Clearly, any rational value of $t \in \mathbb{Q}$ yields a rational point $(x, y) \in \mathbb{Q}^2$, which itself corresponds to a Pythagorean triple (a, b, c). The important point is that $(*)$ is a rational parametrization of the circle.

Rational curves

More generally, consider a polynomial equation $P(x, y) = 0$ with integer coefficients, and assume that there is a parametrization

$$x = \frac{R(t)}{Q(t)}, \quad y = \frac{V(t)}{U(t)}$$

as rational functions with integer coefficients, then we get infinitely many rational solutions by taking $t \in \mathbb{Q}$.

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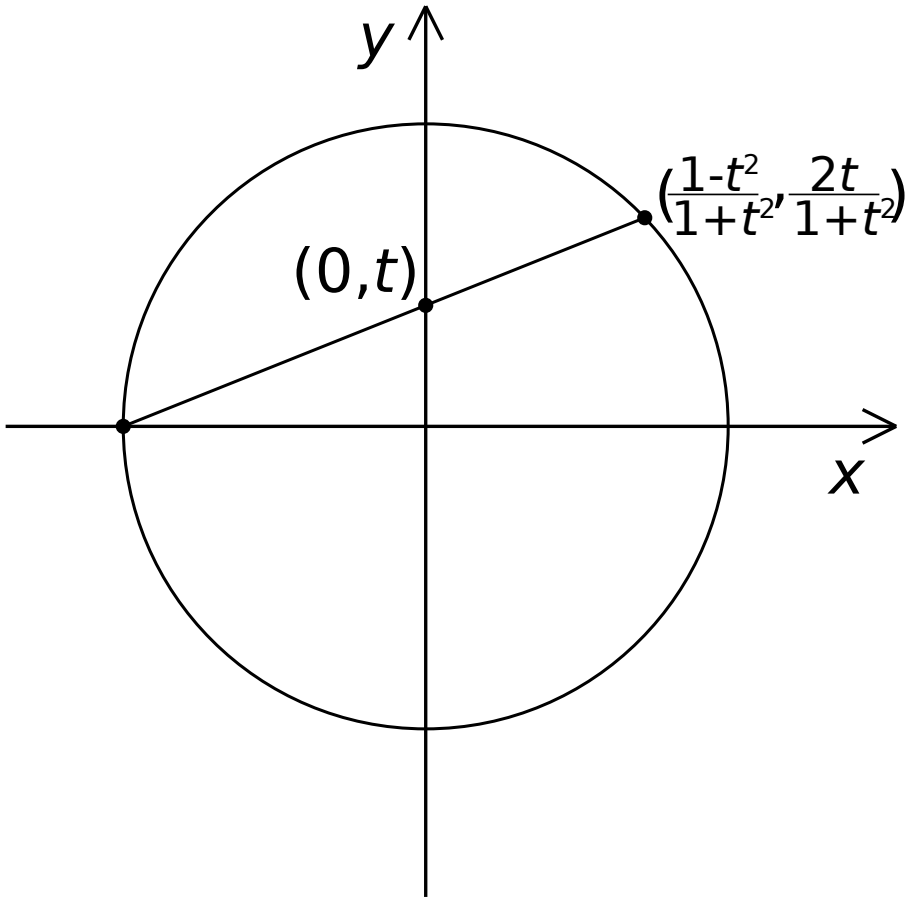
It is convenient to work with complex numbers and consider more generally homogeneous polynomial equations

$$P(x, y, z) = 0 \quad \Leftrightarrow \quad P(x/z, y/z, 1) = 0,$$

which amounts to work in the complex projective plane of points $[x : y : z] \simeq (x/z, y/z)$, with a line $z = 0$ of "points at infinity".

Conics



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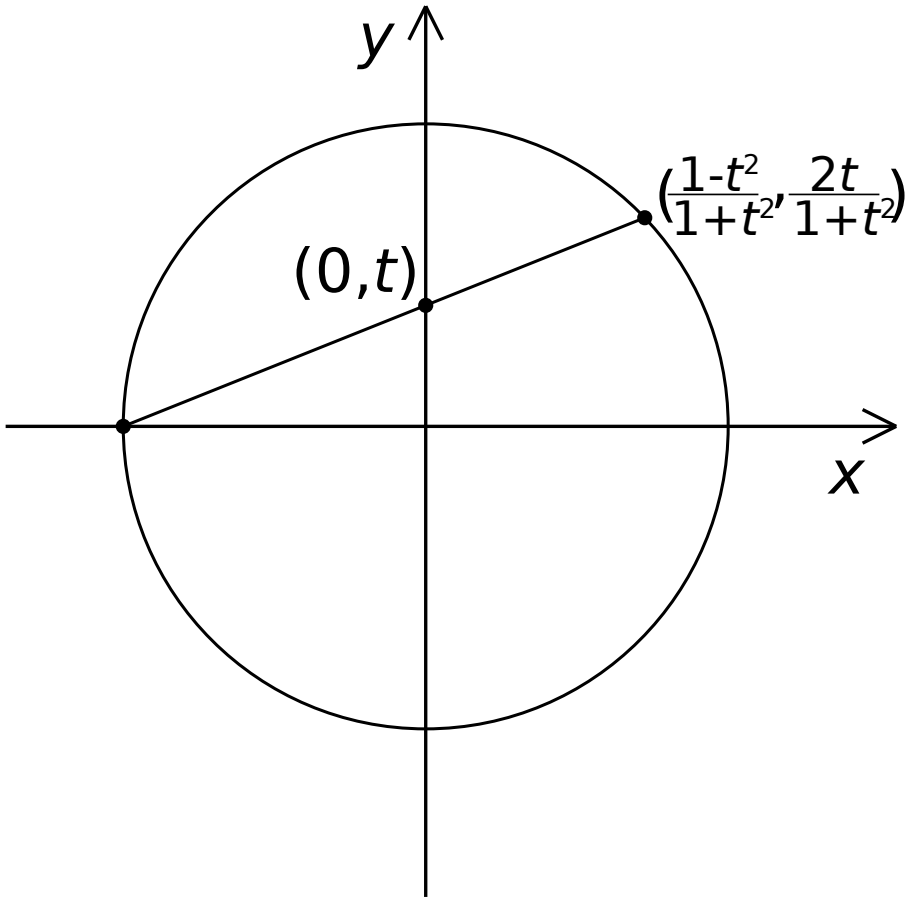
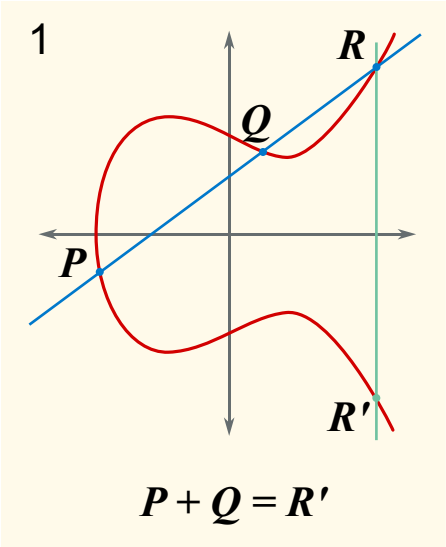


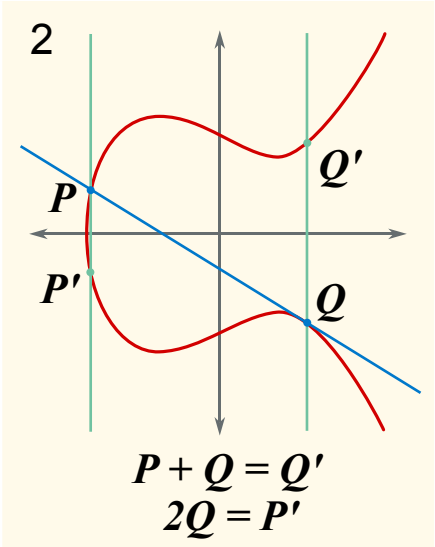
In fact, all plane conics $C = \{ax^2 + bx^2 + cxy + dx + ey + f = 0\}$ can be parametrized in that way, and if the coefficients are rational (with $C(\mathbb{Q}) \neq \emptyset$ and $C(\mathbb{R})$ not reduced to a point), then C contains infinitely many rational points.

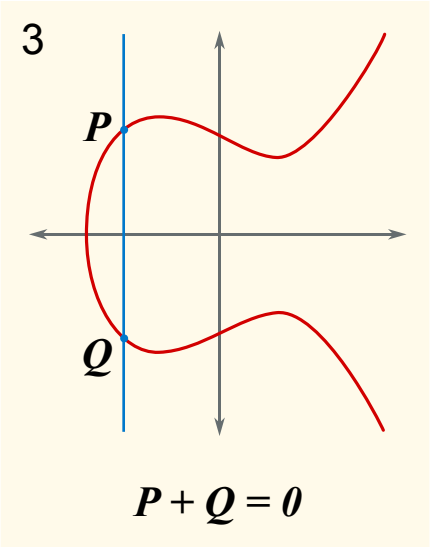
Elliptic curves

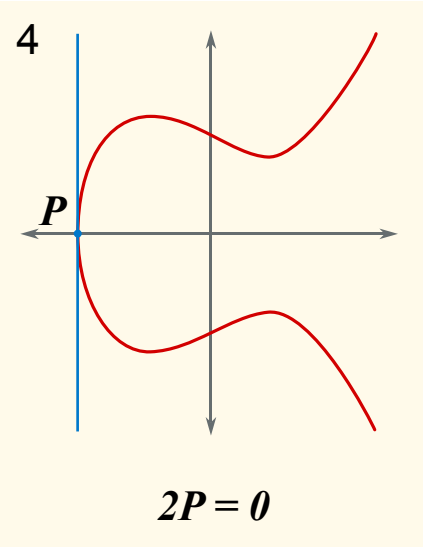
Elliptic curves can be defined by plane equations of the form

$$y^2 = x^3 + ax + b \quad \text{(below, } y^2 = x^3 - x + 1 \text{ is shown)}$$









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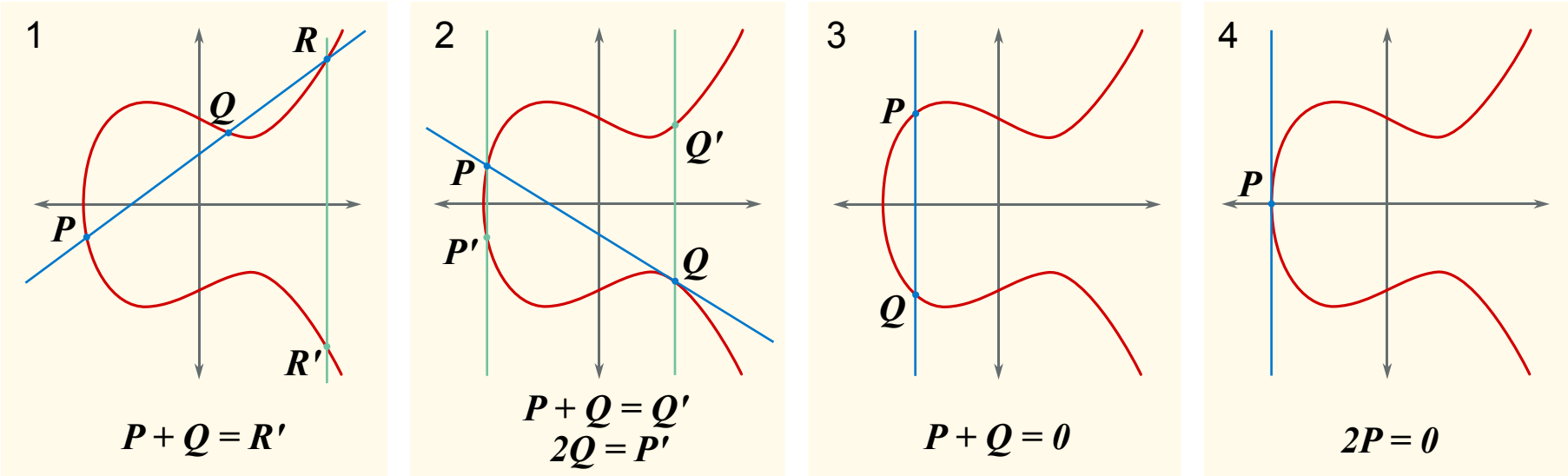
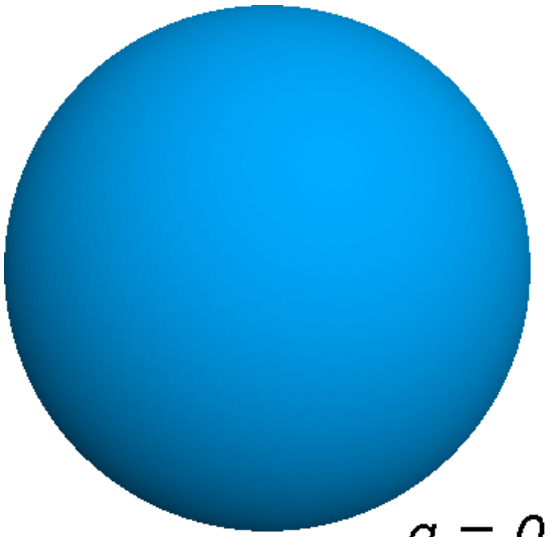


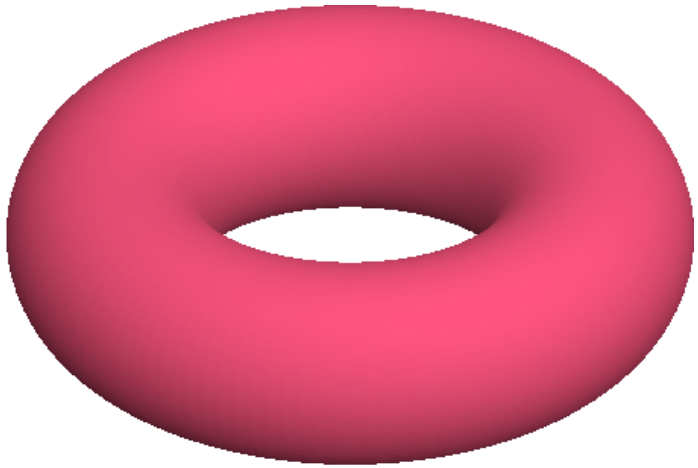
A fundamental property is that are isomorphic to an additive group $(\mathbb{C}/\Lambda, +)$ (2-torus) where Λ is a lattice, and they can be parametrized as $(x, y) = (\wp(t), \wp'(t))$ where $\wp$ is the Weierstrass $\wp$ function (a transcendental, Λ-periodic, meromorphic function of $t \in \mathbb{C}$).

Classification of curves ("Riemann surfaces")



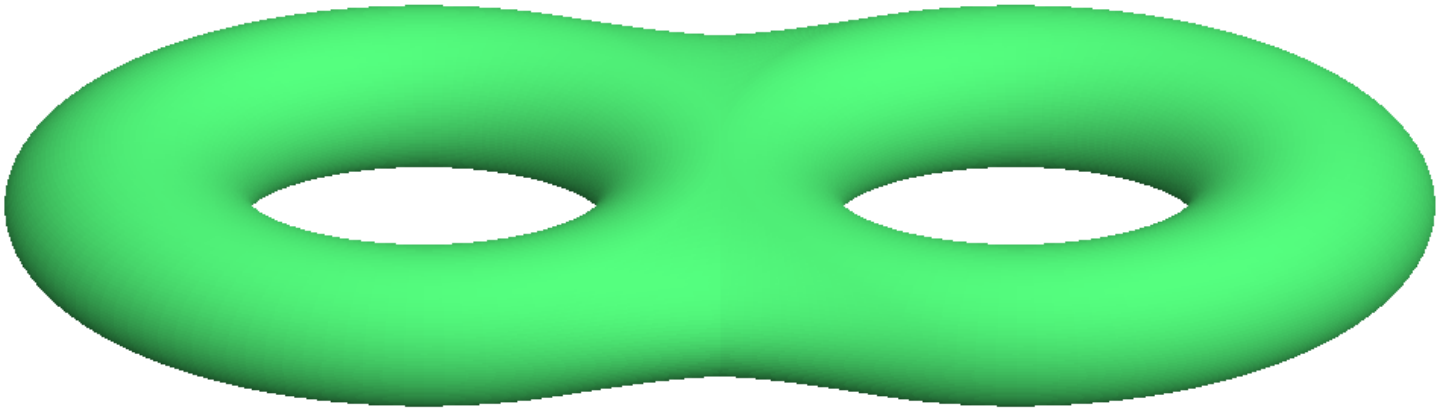
$g = 0\,,\ K_X < 0$

(positive curvature)



$g = 1\,,\ K_X = 0$

(zero curvature)



$g > 1\,,\ K_X > 0$

(negative curvature)

Over $\mathbb{C}$, curves are surfaces!

Rational points on curves

L. Mordell

G. Faltings

Gerd Faltings, Fields medal 1986 for the solution of a conjecture by Louis Mordell (1922):

Theorem (Faltings, 1983)

Let C be a smooth curve of genus $g \geq 2$ in the complex projective plane, defined as $\{P(x, y, z) = 0\}$ for some homogeneous polynomial $P(x, y, z) \in \mathbb{Q}[x, y, z]$. Then the curve C contains finitely many rational points $[x : y : z]$, i.e. $x, y, z \in \mathbb{Q}$.

General Diophantine equations

General diophantine equations $P_j(x_0, x_1, \ldots, x_n) = 0$, $1 \leq j \leq m$ are much harder.

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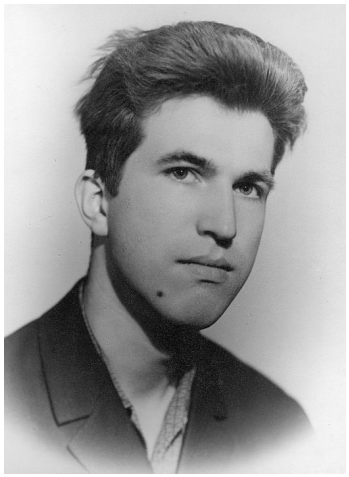

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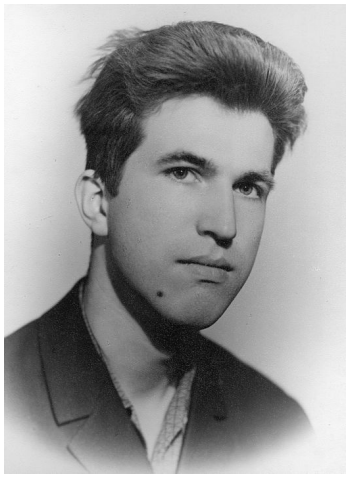

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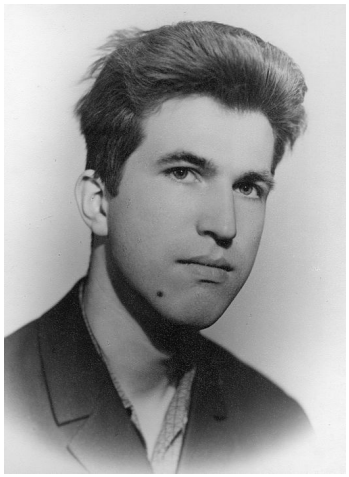

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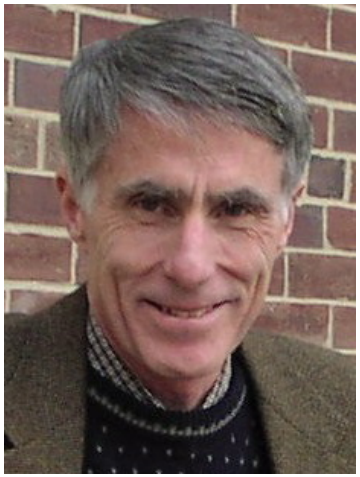
In fact, Diophantine equations can encompass many mathematical algorithms, e.g. there exists a polynomial $P \in \mathbb{Z}[X_1, \ldots, X_{26}]$ whose set of positive values is the set of prime numbers !

(Non) existence of rational & elliptic curves



H. Clemens

C. Voisin

Theorem (Herb Clemens $\sim$ 1986, Claire Voisin $\sim$ 1996)

Let $X = \{P(z_0, \ldots, z_{n+1}) = 0\}$ be a generic n-dimensional algebraic hypersurface in complex projective space (i.e. with general enough "random" coefficients) of degree $d \geq 2n + 1$, $n \geq 2$. Then X does not contain rational or elliptic curves.

In other words, there are no entire curves $f : \mathbb{C} \to X$ solving the equation $P(f(t)) = 0$, where f is either rational or a Weierstrass type transcendental function.

The Kobayashi conjecture

S. Kobayashi

Conjecture (Shoshichi Kobayashi $\sim$ 1970)

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Definition

X is said to be Kobayashi hyperbolic if it contains no entire holomorphic curve $f : \mathbb{C} \to X$.

Solution of Kobayashi conjecture (Brotbek 2016)

Theorem (Brotbek 2016)

The Kobayashi conjecture holds true for $d_n \gg 1$ (non explicit), i.e. a generic polynomial equation $P(z) = 0$ of degree $d \geq d_n$ contains no entire analytic solution $z = f(t)$, $t \in \mathbb{C}$.

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The proof is based on finding certain algebraic differential operators

$$D(f) = \sum a_{\alpha_1 \ldots \alpha_k}(f)(f')^{\alpha_1}(f'')^{\alpha_2} \ldots (f^{(k)})^{\alpha_k}$$

whose coefficients a_α vanish along some hyperplane section $H \subset X$.

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Theorem (Green-Griffiths 1979, D– 1995, Siu-Yeung 1996)

If such a global operator D exists on X, then all entire curves $f : \mathbb{C} \to X$ must satisfy the differential equation $D(f) \equiv 0$.

Simplified proof of Kobayashi conjecture

The "simplified proof" consists of finding sufficiently many global differential operators $D(f) = 0$ in the form of Wronskians

$$W(s_0, \ldots, s_k)(f) = \begin{vmatrix} s_0(f) & s_1(f) & \ldots & s_k(f) \\ \nabla(s_0(f)) & \nabla(s_1(f)) & \ldots & \nabla(s_k(f)) \\ \vdots & & & \vdots \\ \nabla^k(s_0(f)) & \nabla^k(s_1(f)) & \ldots & \nabla^k(s_k(f)) \end{vmatrix}$$

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Moreover, by using the geometric structure of jet bundles elaborated in (D–, 1995), one can show that these operators have "deformations" to generic hypersurfaces of high degree.

Green-Griffiths conjecture

M. Green

P. Griffiths

Conjecture (Mark Green & Phillip Griffiths 1979)

Let $X = \{P(z_0, \ldots, z_{n+1}) = 0\}$ be a smooth n-dimensional algebraic hypersurface in complex projective space. Assume $d = \deg P \geq n + 2$. Then X possesses an algebraic subvariety $Y = \{Q(z) = 0\}$ containg all entire curves $f : \mathbb{C} \to X$, i.e. $Q(f) = 0$.

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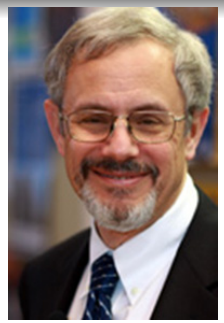
Green-Griffiths conjecture



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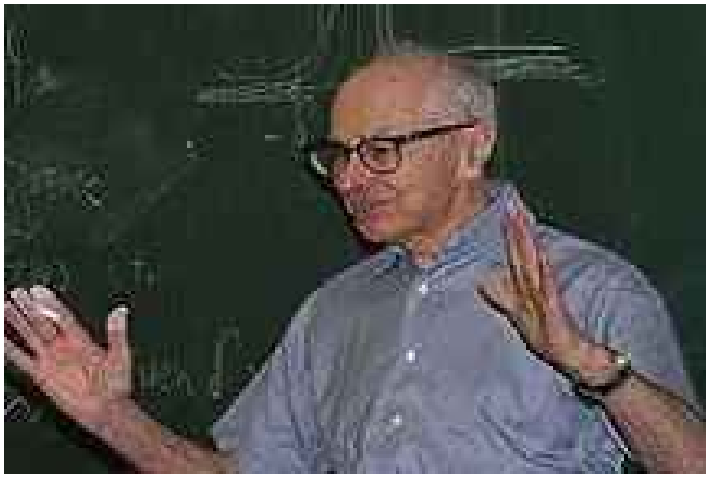
Theorem (Demailly 2010)

Rather than an algebraic equation $Q(f) = 0$, such entire curves $f : \mathbb{C} \to X = \{P = 0\}$, $\deg P \geq n + 2$, satisfy at least one (and in fact many) algebraic differential equations $D(f) = 0$.

Lang-Vojta program



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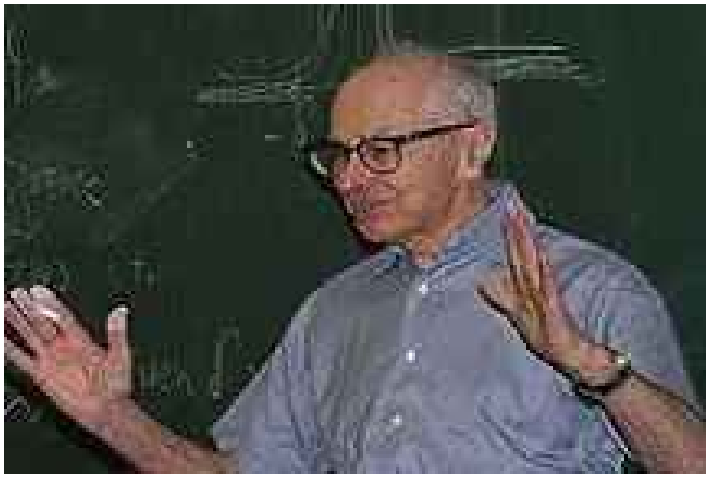

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This is a fascinating program that tries to relate the algebro-geometric properties of algebraic varieties with their arithmetic properties.

Lang's conjecture 1987 – very optimistic ?

For X projective defined over a number field $\mathbb{K}_0$, the Green-Griffiths locus $\mathrm{GG}(X)$ in GG conjecture equals $\mathrm{Mordell}(X) =$ smallest Y such that $X(\mathbb{K}) \setminus Y$ is finite, $\forall \mathbb{K}$ number field $\supset \mathbb{K}_0$.

The end